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Vertex links and the Grushko decomposition (*)

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Abstract. — We develop an algorithm of polynomial time complexity to construct the Grushko decomposition of fundamental groups of graphs of free groups with cyclic edge groups. Our methods rely on analysing vertex links of certain CAT(0) square complexes naturally associated with a special class of the above groups. Our main result transforms a one-ended CAT(0) square complex of the above type to one whose vertex links satisfy a strong connectivity condition, as first studied by Brady and Meier.

Résumé. — Nous développons un algorithme polynomial pour construire la décomposition de Grushko du groupe fondamental d'un graphe de groupes libres à sous-groupes d'arêtes cycliques. Notre méthode repose sur l'analyse du link des sommets d'un complexe carré naturellement associé à une classe spéciale des groupes ci-dessus. Notre résultat principal transforme un complexe carré CAT(0) avec un seul bout en un autre dont le link des sommets vérifie une condition de connectivité forte, étudiée pour la première fois par Brady et Meier.

1. Introduction

The Grushko decomposition theorem [8] states that a finitely generated group is a free product of finitely many freely indecomposable non-free groups and a finite rank free group. This decomposition is unique in the sense that the freely indecomposable groups appearing in this decomposition are unique up to reordering and conjugation, and the rank of the free group is invariant.

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Article proposé par Jean-Pierre Otal.

Given a finitely presented group, there is no algorithm in general to compute its Grushko decomposition. Even when algorithms do exist, they are often not tractable. In this article, we develop an algorithm, with polynomial time complexity, to compute the Grushko decomposition of fundamental groups of graphs of free groups with cyclic edge groups (see Theorem B).

Our algorithm is obtained as a consequence of Corollary 1.3, which gives an algorithm to obtain the Grushko decomposition of fundamental groups of compact nonpositively curved square complexes that we call *tubular graphs of graphs*. A tubular graph of graphs (see Definition 2.8 for the precise definition) is a square complex obtained by attaching finitely many tubes (a *tube* is the Cartesian product of a circle and the unit interval) to a finite collection of finite graphs. Tubular graphs of graphs are thus nonpositively curved $\mathcal{VH}$-complexes (in the sense of Wise [24]) in which vertical hyperplanes are homeomorphic to circles (see Section 2).

In [20], Stallings built on work by Whitehead [22] and developed an algorithm that takes a free group and finitely many cyclic subgroups as input and decides whether or not the free group splits freely relative to the cyclic subgroups. We use the machinery of tubular graphs of graphs in this article to develop an alternative algorithm.

In fact, we also obtain an analogue for tubular graphs of graphs of a result by Jaco [11] in 3-manifold theory which states that if the fundamental group of a 3-manifold splits as a free product, then each free factor is itself the fundamental group of a 3-manifold.

Our approach is geometric and in fact we are more interested in nonpositively curved square complexes than groups. We show how to construct the Grushko decomposition by cutting along contractible subspaces of tubular graphs of graphs which induce free splittings.

Crucial to our methods is the following key result by Brady and Meier:

Theorem 1.1 ([1]). — *Let X be a finite connected nonpositively curved cube complex. Suppose that*

- (BM1) *for each vertex $v \in X$, the link of v is connected and*
- (BM2) *for each vertex $v \in X$ and each simplex σ in $\operatorname{link}(v)$, $\operatorname{link}(v) \setminus \sigma$ is (non-empty and) connected.*

Then $\widetilde{X}$ is one-ended.

We say that a cube complex is *Brady–Meier* if it satisfies the conditions (BM1) and (BM2) above. Note that the Brady–Meier conditions are local and hence preserved by covering maps. The converse of Theorem 1.1 is not true in general. The main result of the article gives a geometric/combinatorial procedure that modifies a given tubular graph of graphs to a homotopy equivalent tubular graph of graphs which is Brady–Meier if and only if the fundamental group is one-ended:

Theorem A (Theorem 5.3). — *There is an algorithm of polynomial time complexity which takes a tubular graph of graphs as input and returns a homotopy equivalent tubular graph of graphs which is either a Brady–Meier complex or contains a locally disconnecting vertex which splits the fundamental group as a free product.*

For this and the corollaries below, the running times of the algorithms are polynomial in the number of squares of the input square complex. A major application of the above theorem is in the development of an algorithm to obtain the JSJ decomposition of one-ended hyperbolic fundamental groups of tubular graphs of graphs [13]. The main point is that one can work with a Brady–Meier tubular graph of graphs (obtained in polynomial time) when the fundamental group is one-ended, thanks to Theorem A.

The key step in the construction of our algorithm involves a simplification of the input tubular graph of graphs by "opening-up" at a vertex which does not satisfy (BM2). This opening-up keeps the number of squares in the complex constant, while simplifying certain vertex links. We call such an opening procedure as an *SL-move* ("SL" stands for simplified link). We thus obtain a partial converse to Theorem 1.1:

Corollary 1.2. — *A tubular graph of graphs has a one-ended universal cover if and only if it can be simplified in finitely many SL-moves to a Brady–Meier tubular graph of graphs with isomorphic fundamental group.*

We also obtain the Grushko decomposition of these groups using Theorem A. In fact, we obtain a stronger result:

Corollary 1.3 (Corollary 5.4). — *There is an algorithm of polynomial time complexity which takes as input a tubular graph of graphs and returns a homotopy equivalent tubular graph of graphs obtained by gluing together certain vertices of a finite collection of Brady–Meier tubular graphs of graphs and finite graphs. Further, the free product decomposition induced by cutting along the glued vertices is the Grushko decomposition of the fundamental group of the input tubular graph of graphs.*

We point out that our proof neither uses Stallings' theorem on ends of finitely generated groups nor assumes the existence of a Grushko decomposition. In fact, our procedure yields a new proof of Stallings' theorem for fundamental groups of tubular graphs of graphs as well as the existence of a Grushko decomposition for these groups.

The analogue of Jaco's result immediately follows:

Corollary 1.4 (Corollary 5.6). — *Let X be a tubular graph of graphs with fundamental group G. If $G = A * B$, then there exist tubular graphs of graphs X_1 and X_2 such that A and B are fundamental groups of X_1 and X_2 respectively. Moreover, X_1 and X_2 can be so chosen such that the total number of squares in X_1 and X_2 is bounded by the number of squares in X.*

The Grushko decomposition may be found algorithmically in other situations. Jaco, Letscher and Rubinstein [12] gave an algorithm of polynomial time complexity to compute the prime decomposition of a 3-manifold from a triangulation. Gerasimov [6] showed that the Grushko decomposition can be computed for hyperbolic groups. But there is no known bound on the complexity of his algorithm. Dahmani and Groves [3] extended Gerasimov's ideas to groups which are hyperbolic relative to abelian subgroups. Diao and Feighn [4] gave an algorithm for graphs of free groups. Their algorithm relies on certain simplifications of a given graph of free groups which depend on finding Gersten representatives of the incident edge groups in each vertex group, which is algorithmic. More recently, Touikan [21] presented an important algorithm which returns the Grushko decomposition of finitely presented groups with solvable word problem and no 2-torsion. However, the time complexity of his algorithm is not known.

We note that our algorithm is explicit and neither uses the Rips machine nor requires solutions of equations in free groups, as some of the above algorithms do.

We mention another application of our algorithm, before stating our result for general graphs of free groups with cyclic edge groups. As defined by Stallings in [20], a finite set of words W of a finite rank free group F is *separable* if there exists a nontrivial free splitting of F such that each word of W conjugates into a free factor.

Stallings obtained an algorithm to detect separability in [20]. In a related result, Roig, Ventura and Weil [15] obtained a polynomial time algorithm to solve the Whitehead minimization problem and therefore the primitivity problem. We give an alternate version of Stallings' algorithm using Theorem A:

Corollary 1.5 (Corollary 6.2). — *There exists an algorithm of polynomial time complexity that takes a finite set of words in a finite rank free group as input and decides whether it is separable.*

Stallings obtains his algorithm by constructing a Whitehead graph for the given set of words in a chosen basis. He then uses a Whitehead automorphism to modify the basis whenever there is a cut vertex in the Whitehead graph to reduce the total length of the given set of words. We first give a new proof of Whitehead's cut vertex theorem (Proposition 6.4). We then obtain our algorithm by first constructing the tubular graph of graphs associated to a "double" of the free group with the given set of words. We then apply the algorithm of Theorem A.

By combining a result of Wilton [23] (see Theorem 6.7) with Corollary 1.3, we obtain the advertised result:

Theorem B (Theorem 7.1). — *There exists an algorithm of polynomial time complexity which takes a graph of free groups with cyclic edge groups as input and returns as output the Grushko decomposition of its fundamental group.*

The running time of the algorithm is polynomial in the sum of the lengths of the words induced by the generators of the edge groups in the respective vertex groups.

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2. The setup

2.1. $\mathcal{VH}$-complexes

The notion of $\mathcal{VH}$-complexes was first introduced in [24].

Definition 2.1. — *A* square complex *is a two dimensional CW complex in which each 2-cell is attached to a combinatorial loop of length 4 and is isometric to the standard Euclidean unit square $I^2 = [0,1]^2$.*

All our square complexes will be locally finite.

Definition 2.2 (Vertex links). — *Let $v \in X$ be a vertex of a square complex. The link of v, denoted by* $\mathrm{link}(v)$ *is a graph whose vertex set is the set $\{e \mid e$ is a half-edge incident to $v\}$. The number of edges between two vertices e, f is the number of squares of X in which e, f are adjacent half-edges.*

Definition 2.3. — *A square complex is* nonpositively curved *if the length of a closed path in the link of any of its vertices is at least four.*

By a result of Gromov [7], a simply connected nonpositively curved square complex is CAT(0) in the metric sense.

Definition 2.4 ([16]). — *Let X be a square complex. A* mid-edge *of a square s in X is an edge (after subdivision of s) running through the center of s and parallel to two of the edges of s. Declare two edges e and f to be equivalent if there exists a sequence $e = e_1, \ldots, e_n = f$ of edges such that e_i and e_{i+1} are opposite edges of some square of X. Given an equivalence class $[e]$ of edges, the* hyperplane dual to e*, denoted by h_e, is the collection of mid-edges which intersect edges in $[e]$.*

Definition 2.5 ([24]). — *A* $\mathcal{VH}$-complex *is a square complex in which every 1-cell is labelled as either vertical or horizontal in such a way that each 2-cell is attached to a loop which alternates between horizontal and vertical 1-cells.*

The labelling of the edges of a $\mathcal{VH}$-complex as horizontal and vertical induces a labelling of the vertices in the link of any vertex as horizontal and vertical, thus making the link a bipartite graph. Similarly, the hyperplanes of a $\mathcal{VH}$ complex are also labelled as vertical and horizontal, with a vertical hyperplane being dual to an equivalence class of horizontal edges and a horizontal hyperplane being dual to an equivalence class of vertical edges.

Remark 2.6. — Since the link of any vertex of a $\mathcal{VH}$-complex is bipartite, the length of a closed path is even. Thus a $\mathcal{VH}$-complex is nonpositively curved if there exists no bigon in any vertex link.

2.2. Graphs of spaces

Graphs of groups are the basic objects of study in Bass–Serre theory [18]. It was studied from a topological perspective in [17] by looking at graphs of spaces instead of graphs of groups. We will adopt this point of view.

Definition 2.7. — *By a* graph of spaces*, we mean the following data: Γ is a connected graph, called the underlying graph. For each vertex s (edge a)*

of Γ*,* X_s *(*X_a*) is a topological space. Further, whenever* a *is incident to* s*,* $\partial_{a,s} : X_a \to X_s$ *is a* π_1*-injective continuous map. The* geometric realisation *of the above graph of spaces is the space* $X = (\bigsqcup_{s\in\Gamma^{(0)}} X_s \sqcup \bigsqcup_{a\in\Gamma^{(1)}} X_a \times [0,1])/\sim$*, where* $(x,0)$ *and* $(x,1)$ *are identified respectively with* $\partial_{a,s}(x)$ *and* $\partial_{a,s'}(x)$*. Here,* s *and* s' *are the two endpoints of* a*.*

Note that the universal cover of X has the structure of a *tree of spaces*, a graph of spaces whose underlying graph is the Bass–Serre tree of the associated graph of groups structure of X [17].

2.3. Tubular graphs of graphs

Definition 2.8. — *A* tubular graph of graphs *is a finite graph of spaces in which each vertex space is a finite connected simplicial graph and each edge space is a simplicial graph homeomorphic to a circle. Further, the attaching maps are simplicial immersions. We will always assume that the underlying graph is connected.*

We note that no vertex graph is a tree, as a consequence of the definition. We also remark that asking for each vertex graph to be simplicial is not a serious restriction as every one dimensional CW complex is a simplicial graph after subdivision.

Remark 2.9. — Tubular graphs of graphs are topological versions of certain graphs of free groups with cyclic edge groups (see Section 7 for a definition). We note that any graph of free groups with cyclic edge groups in which the underlying graph is a tree can be realised topologically as a tubular graph of graphs. If the underlying graph contains a loop such that the generator of the edge group is attached to words of different lengths in the incident vertex group, then such a graph of free groups with cyclic edge groups cannot be realised as a tubular graph of graphs. In particular, this rules out non-Euclidean Baumslag–Solitar groups.

We also have

Proposition 2.10 ([24]). — *The geometric realisation of a tubular graph of graphs is a finite (hence compact), connected nonpositively curved* $\mathcal{VH}$*-complex whose vertical hyperplanes are circles.*

Convention. — *Throughout this text, we will use the same notation for a graph of graphs and the* $\mathcal{VH}$*-complex which is its geometric realisation.* X *will denote a tubular graph of graphs with underlying graph* Γ_X*. Let* $s \in \Gamma_X$ *be a vertex. Then* X_s *will denote the vertex graph at* s *and if* a *is an edge*

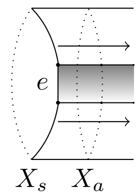


Figure 2.1. Removing rudimentary edges.

of Γ_X*, we will denote the edge graph at* a *by* X_a*. Thus, every edge of any vertex graph* X_s *will be a vertical edge in the* $\mathcal{VH}$*-complex* X *while horizontal edges in* X *are the edges of the form* $\{v\} \times [0,1]$*, for vertices* v *in the edge graphs* X_a*.*

Definition 2.11 (Thickness). — *For an edge* e *in* X*, the* thickness *of* e *is the number of squares of* X *which contain* e*.*

Observe that a horizontal edge of X always has thickness equal to two.

Definition 2.12. — *Let* X_s *be a vertex graph of a tubular graph of graphs* X*. We say that* X_s *(and hence* X*) has a* hanging tree *if* X_s *is a wedge of two subgraphs* A *and* B *such that one of them, say* A*, is a tree. Here,* A *is called a hanging (sub)tree of* X_s*.*

Remark 2.13. — Since the attaching maps of edge graphs are immersions, an edge in a hanging tree of X has thickness zero.

We thus have that

Lemma 2.14. — *A tubular graph of graphs is homotopy equivalent to a tubular graph of graphs with no hanging trees.*

Definition 2.15. — *An edge* e *in* X_s *is a* rudimentary edge *if it is of thickness one and moreover* X_s *is a circle.*

Lemma 2.16. — *A tubular graph of graphs is homotopically equivalent to a tubular graph of graphs with no rudimentary edges.*

Proof. — Let e be a rudimentary graph in a vertex graph X_s of a tubular graph of graphs X. Since X_s is a circle and attaching maps of edge graphs to X_s are graph immersions, there exists exactly one edge a incident to s in the underlying graph Γ_X and the attaching map from X_a to X_s is a graph isomorphism (see Figure 2.1).

Thus X is homotopic to X' obtained by removing X_a and the open tube containing X_a. $\Gamma_{X'}$ is the graph obtained from Γ_X by collapsing $a = (s, s')$ to s'. Repeating this procedure at each rudimentary edge gives the result. □

2.4. Ends

The theory of ends of a topological space was first studied by Freudenthal [5]. The notion we require is that of "one-endedness". We will use the following definition due to Specker (see [14] or [19]).

Definition 2.17. — *A locally finite CW complex X is* one-ended *if for every compact set K, $X \setminus K$ has exactly one unbounded component.*

It is a well-known fact that being one-ended is a quasi-isometry invariant (see [2, Proposition I.8.29], for instance). Then by an application of the Švarc–Milnor Lemma (see [2, Proposition I.8.19]), for instance), we have the following definition of one-endedness of a finitely presented group.

Proposition 2.18. — *Let G be a finitely presented group and X be a finite connected CW complex such that $G \cong \pi_1(X)$. G is one-ended if and only if $\widetilde{X}$ is one-ended.*

3. Not one-ended

In this section, we will collect a few results that help determine when the fundamental group of a tubular graph of graphs is not one-ended.

Definition 3.1. — *Let Z be a CW complex and $v \in Z$ be a vertex. Let $\widetilde{v}$ denote a lift of v in the universal cover $\widetilde{Z}$ of Z. The* star of v, *denoted by* $\operatorname{star}(v)$, *is the smallest subcomplex of $\widetilde{Z}$ which contains all cells σ such that $\widetilde{v} \in \sigma$. The* open star of v, *denoted by* $\mathring{\operatorname{star}}(v)$, *is the interior of* $\operatorname{star}(v)$.

We first recall a classical result due to Hopf:

Lemma 3.2 ([10]). — *Let G be a torsion-free finitely generated group such that either*

1. *$G = H * K$ is a nontrivial free splitting of G, or*
2. *$G = H *_1$ is an HNN extension of a finitely generated group H over its trivial subgroup.*

Then G is not one-ended.

We remark that the two conditions above are cases of an edge of groups with trivial edge group. We make a distinction between them because of the following standard lemma.

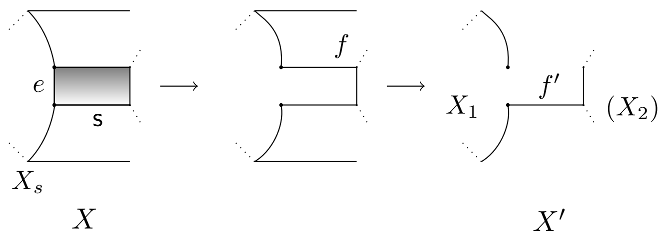


Figure 3.1. Removing squares containing thickness-one edges.

Lemma 3.3. — *Let Z be a connected CW complex. Let $c \in Z$ be a vertex. Suppose that either*

(1) $Z = Z_1 \vee_{\{c\}} Z_2$, *with neither Z_1 nor Z_2 simply connected, or*
(2) $\mathrm{star}(c) \setminus \{c\}$ *is not connected, but $Z \setminus \{c\}$ is connected.*

Then $\widetilde{Z}$ is not one-ended.

Proposition 3.4. — *Let X be a tubular graph of graphs with no hanging trees. Suppose there exists an edge of thickness zero. Then $\widetilde{X}$ is not one-ended.*

Proof. — Since all horizontal edges have thickness two, an edge of thickness zero has to be vertical. Let e in X_s be such an edge with midpoint c. Subdivide e so that c is a vertex of X. Either $X \setminus \{c\}$ is connected, or $X = X_1 \vee_c X_2$ is a wedge of two subcomplexes. Let $X_s = A \vee_c B$ be the induced decomposition of X_s. Since e is not in a hanging tree, neither A nor B is a tree. Thus, X_1 and X_2 are not simply connected (as $\pi_1(A) \hookrightarrow \pi_1(X_1)$ and $\pi_1(B) \hookrightarrow \pi_1(X_2)$ in the graph of groups setup [18]). Lemma 3.3 then gives the result. □

Proposition 3.5. — *Let X be a tubular graph of graphs with no hanging trees and no rudimentary edges. Suppose there exists an edge of thickness one. Then $\widetilde{X}$ is not one-ended.*

Proof. — Let e be an edge in $X_s \hookrightarrow X$ of thickness one and s be the lone square containing e. We will show that X is homotopic to a wedge of two non-simply connected square complexes.

Note that X is homotopic to a complex obtained by removing the open square s and the open edge e (Figure 3.1). Removing s decreases the thickness of a horizontal edge f adjacent to e. We remove the only square that contains f, which in turn creates another horizontal edge of thickness one. We continue removing until we end up with a horizontal edge f' (adjacent to e, see Figure 3.1) of thickness zero.

Call the resulting subcomplex as X'. Either the midpoint of f' does not disconnect X', or X' is a wedge of two subcomplexes X_1 and X_2, say (Figure 3.1). Note that X_1 is not simply connected as X had neither hanging trees nor rudimentary edges. Also X_2 is not simply connected as X_2 is a subcomplex of X which is not simply connected. By Lemma 3.3, $\widetilde{X}$ is not one-ended. $\square$

Proposition 3.6. — *Suppose that every edge of X has thickness at least two. If $\widetilde{X}$ is one-ended, then the link of every vertex of X is connected.*

We will need the following result.

Lemma 3.7. — *Let Z be a compact, connected nonpositively curved square complex which has at least one edge. If each edge of Z is contained in at least two squares, then $\pi_1(Z)$ is infinite.*

Proof. — Let e be an edge of Z. The hyperplane h_e dual to e is a finite connected graph in which every vertex is of valence at least two, by the hypothesis on Z. This implies that $\pi_1(\mathsf{h}_e)$ is a free group of positive rank. Any lift of h_e embeds as a hyperplane in $\widetilde{Z}$, since $\widetilde{Z}$ is CAT(0) ([16, Theorem 4.10]). This implies that $\pi_1(\mathsf{h}_e) \hookrightarrow \pi_1(Z)$. Hence the result. $\square$

Proof of Proposition 3.6. — Let $u \in X_s$ be a vertex whose link is not connected. This implies that $\mathrm{star}(u) \setminus \{u\}$ is not connected. The result then follows from Lemma 3.3. Indeed, if $X = X_1 \vee_u X_2$, then X_i is not simply connected by Lemma 3.7. $\square$

4. The second Brady–Meier criterion

In this section, we will assume that each edge of X has thickness at least two and every vertex link is connected, but X does not satisfy the second Brady–Meier criterion (BM2). We will explain how to simplify X in this case by defining an opening of the complex at a vertex whose link does not satisfy (BM2). Fix one such vertex $u \in X_s \subset X$.

A vertex of $\mathrm{link}(u)$ is *vertical (horizontal)* if it is a vertical (horizontal) half-edge incident to u in X. Observe that the horizontal vertices have valence exactly two.

Lemma 4.1. — *X does not satisfy* (BM2) *at u if and only if a vertical vertex of* $\mathrm{link}(u)$ *disconnects* $\mathrm{link}(u)$.

Proof. — One direction is clear. For the converse, there are two cases to consider: either a horizontal vertex h or an edge e of $\mathrm{link}(u)$ disconnects $\mathrm{link}(u)$.

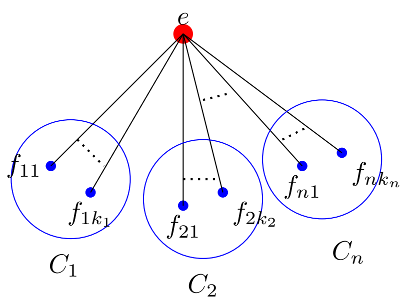


Figure 4.1. link(u)

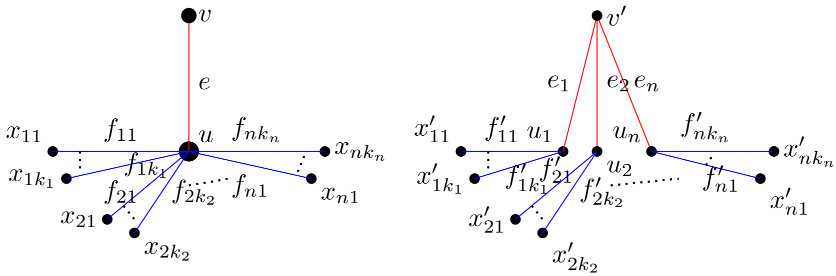
In the first case, let v_1 and v_2 be the two vertical vertices adjacent to h. Let C be the component of $\operatorname{link}(u) \setminus \{h\}$ that contains v_1. Let $x \neq v_1 \in C$ be a vertex. Such a vertex exists as every edge of X has thickness at least two. Then any path in link(u) from x to v_2 meets h, and hence meets v_1. Thus v_1 disconnects link(u). In the second case, since horizontal vertices have valence two, it is easy to see that the unique vertical vertex incident on e disconnects link(u). □

The opening procedure

Throughout, we will denote an edge incident to u and the corresponding vertex in link(u) by the same notation. Let e be a vertical edge incident to u which disconnects link(u). Let $C_1, \ldots, C_n$ denote the maximal connected subgraphs of $\operatorname{link}(u) \setminus e$ (Figure 4.1), where maximality is by inclusion. We will denote by $f_{i1}, \ldots, f_{ik_i}$ the vertical vertices in C_i so that the vertical edges $f_{i1}, \ldots, f_{ik_i}$ incident on u belong to star(u, X_s) (Figure 4.2). Let x_{ij} denote the other endpoint of f_{ij}. We will now explain how to open star(u, X_s):

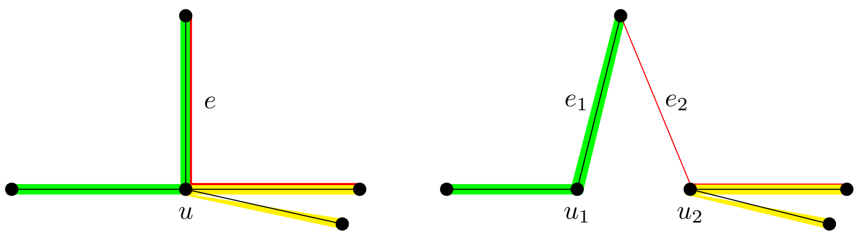
Definition 4.2. — *The* opening of star(u, X_s) *along e is a tree T_u (Figure 4.2) defined as follows: There is one "primary" vertex v' out of which emit n edges $e_1, \ldots, e_n$, one for each C_i. For each i, we label the other endpoint of e_i as u_i. From each u_i, we have k_i edges to the vertices $x'_{i1}, \ldots, x'_{ik_i}$ (compare with* star(u, X_s)*). We label these edges as $f'_{i1}, \ldots, f'_{ik_i}$. The* opened-up graph of X_s *along e is a graph X'_s obtained by replacing* $\mathring{\operatorname{star}}(u, X_s)$ *in X_s by T_u, with the obvious identifications.*

Clearly, X'_s is connected, $X_s \setminus \mathring{\operatorname{star}}(u, X_s) \hookrightarrow X'_s$ and $T_u \hookrightarrow X'_s$. There is a natural surjective map from X'_s to X_s which sends each e_i in T_u to e. Note that the graphs X'_s and X_s are homotopy equivalent.



(a) $\mathrm{star}(u, X_s)$ (b) T_u

Figure 4.2. Opening $\mathrm{star}(u, X_s)$ to T_u



(a) $\mathrm{star}(u, X_s)$ (b) T_u

Figure 4.3. A highlighted path of each colour indicates a part of the image of an attaching map

Construction

We now construct a new tubular graph of graphs X' with the same underlying graph Γ_X as X and the only change is that X'_s replaces X_s. An attaching map of an edge graph is unchanged if u is not in the image, as $X_s \setminus \mathring{\mathrm{star}}(u, X_s)$ embeds in X'_s. If u is in the image, we do the obvious modification (see Figure 4.3 for an illustration).

Notation. — The tubular graph of graphs X' is called an *SL-complex* (simplified link complex) of X, simplified link in the sense that the vertex u has been replaced by $u_1, \ldots, u_n$ where for each i, $\mathrm{link}(u_i)$ is simpler than $\mathrm{link}(u)$.

There exists a natural map from X' to X. Further, Proposition 4.3. — *The tubular graphs of graphs X and X' are homotopy equivalent.* □

Since the number of edges of X'_s is strictly greater than the number of edges of X_s, we have

Lemma 4.4. — *X' is not isomorphic to X as square complexes.* □

Lemma 4.5. — *Every edge of X' belongs to at least one square and the number of squares in X' is the same as the number of squares in X.*

Proof. — The number of squares in X' is equal to the total number of edges in the cyclic edge graphs, which is equal to the number of squares of X.

If a vertical edge does not belong to T_u, then since the attaching maps are unchanged from X, the vertical edge belongs to at least two squares. Similarly, each f'_{ij} belongs to at least two squares. The edge e_i belongs to a square if and only if a pair of adjacent edges $y_{\lambda_1}, y_{\lambda_2}$ in some edge graph is mapped to a pair e, f_{ij}. Such a pair exists as there is an edge between e and f_{ij} in link(u). □

Remark 4.6. — It is possible that an edge e_i incident to a u_i in T_u is of thickness one in X'. However, e_i cannot be rudimentary as X'_s is not a circle. In that case, by Proposition 3.5, $\widetilde{X}$ is not one-ended.

5. The Algorithm

Definition 5.1. — *A tubular graph of graphs is* wedge-like *if*

1. *it has no hanging trees or rudimentary edges, and*
2. *either there exists a vertex whose link is not connected or there exists an edge of thickness one.*

Remark 5.2. — By Proposition 3.4, Proposition 3.5 and Proposition 3.6, the fundamental group of a wedge-like tubular graph of graphs is not one-ended.

Theorem 5.3 (Main Theorem; Theorem A). — *There is an explicit algorithm of polynomial time complexity that takes a tubular graph of graphs as input and returns a homotopy equivalent tubular graph of graphs which is either Brady–Meier, or is a point, or is wedge-like.*

Proof. — We will prove the theorem by constructing the algorithm. Let $X = X_0$ be the input tubular graph of graphs. Let $k \in \mathbb{N} \cup \{0\}$.

Step 1 Check if X_k has hanging subtrees. If yes, collapse each hanging subtree to a point and call the new complex also as X_k. Go to the next step.

Step 2 Check if X_k has a rudimentary edge. If yes, remove tubes attached to rudimentary edges (Lemma 2.16) and call the resulting complex also as X_k. Go to the next step.

Step 3 Check if X_k has at least one square. If yes, go to the next step. Otherwise, X_k is either a point or wedge-like. Stop.

Step 4 Check if X_k has an edge of thickness zero or one. If yes, X_k is wedge-like. Stop. If not, go to the next step.

Step 5 Check if the link of every vertex of X_k is connected. If yes, go to the next step. If not, X_k is wedge-like. Stop.

Step 6 Check if X_k satisfies (BM2). If yes, X_k is Brady–Meier. Stop. If not, go to the next step.

Step 7 Replace X_k by $X_{k+1} = X'_k$, an SL-complex of X_k, and go to Step 4.

We observe that for $l \gneq k \geqslant 1$,

(i) $X_k \ncong X_l$, as each opening increases the number of edges (Lemma 4.4).
(ii) X_k and X_l have the same number of squares (Lemma 4.5).
(iii) There is no edge of thickness zero in X_k (Lemma 4.5).

(i) implies that the procedure above does not return a tubular graph of graphs from an earlier step. Since there are only a finite number of connected square complexes with a fixed number of squares (ii) and no thickness zero edges (iii), the procedure stops in finite time.

Checking if a graph has hanging trees takes linear time in the number of vertices and edges of X. Similarly, checking for edges of thickness zero or one or for rudimentary edges takes linear time in the number of edges and squares of X. Thus Steps 1 through 4 run in linear time in the number of vertices, edges and squares of X.

From Step 5 onwards, the number of vertices and edges of X_k is bounded by the number of squares of X_k, as each edge is contained in a square. Steps 5 and 6 run in polynomial (quadratic) time in the number of squares: indeed, the size of a vertex link in X_k is bounded by the number of squares of X_k and checking for connectedness and disconnecting vertices in a vertex link is linear in the number of vertices and edges of the link (see [9] for details).

If n is the number of squares in X, we claim that the number of times the algorithm goes back to Step 4 is at most n. Indeed, the algorithm performs the k^{th} opening-up only if every square of X_{k-1} is of thickness at least two. When each edge is of thickness at least two, the number of vertical edges (as well as horizontal edges) of a tubular graph of graphs can be at most equal to the number of squares. Observe that the opening procedure in Step 7 increases the number of vertical edges of X_k by at least one, while decreasing the thickness of certain vertical edges. Thus, the algorithm continues at most until each vertical edge is contained in exactly two squares. □

As an immediate consequence, we have:

Corollary 5.4 (Corollary 1.3). — *There is an algorithm of polynomial time complexity which takes as input a tubular graph of graphs and returns a homotopy equivalent tubular graph of graphs obtained by gluing together certain vertices of a finite collection of Brady–Meier tubular graphs of graphs and finite graphs. Further, the free product decomposition induced by cutting along the glued vertices is the Grushko decomposition of the fundamental group of the input tubular graph of graphs.*

Proof. — Let X be the input tubular graph of graphs with fundamental group G. Apply the algorithm of Theorem 5.3 to X. Let X_N be the output. If X_N is a point, then G is trivial. If X_N is Brady–Meier, G has trivial Grushko decomposition. If X_N is wedge-like, G is a free product (Remark 5.2). We first remove every edge of thickness one in X_N by the procedure of Figure 3.1.

Cut X_N along an edge of thickness zero or a locally disconnecting vertex (in the obvious way). Either we get a connected tubular graph of graphs X'_1 or we get a disconnected space with components $X'_1, \ldots, X'_n$, where each X'_i is a tubular graph of graphs. In the first case, $G = G_1 * \mathbb{Z}$. In the latter case, $G = G_1 * \cdots * G_n$. Apply the algorithm again to each X'_i. If each G_i is one-ended, we are done. Otherwise, cut again at an X'_i with a many-ended G_i and repeat. This procedure terminates in polynomial time. Indeed, at each step we get tubular graphs of graphs whose total number of squares is bounded by the number of squares of X. □

Remark 5.5. — We point out that we do not use Stallings' theorem on ends for our proof. In fact, our procedure yields an alternate proof of Stallings' theorem for fundamental groups of tubular graphs of graphs. Similarly, we do not assume the existence of the Grushko decomposition either. Our algorithm proves its existence for the groups under consideration.

By the uniqueness of the Grushko decomposition, we have:

Corollary 5.6 (Corollary 1.4). — *Let X be a tubular graph of graphs with fundamental group G. Suppose that G admits a free splitting as $G = A * B$. Then there exist tubular graphs of graphs X_1 and X_2 such that A and B are fundamental groups of X_1 and X_2 respectively. Moreover, X_1 and X_2 can be so chosen such that the total number of squares in X_1 and X_2 is bounded by the number of squares in X.* □

6. Whitehead graphs and separability

The goal of this section is to give an alternative proof of Stallings' algorithm to detect whether a finite set of words in a free group is separable (Corollary 6.2). We also give a new proof of Whitehead's cut vertex theorem (Proposition 6.4).

Let F_n be a free group of rank $n \geqslant 2$ and let W be a finite set of non-trivial elements of F_n.

Definition 6.1 ([20]). — *W is* separable *if there exists a non-trivial free splitting of $F_n = H * K$ such that each element of W is either a conjugate of an element of H or a conjugate of an element of K.*

In [20], Stallings developed an algorithm that detects whether W is separable. We will use Theorem 5.3 to obtain an alternative algorithm:

Corollary 6.2 (Stallings; Corollary 1.5). — *There exists an algorithm of polynomial time complexity that detects whether a given finite set of words in a finite rank free goup is separable.*

Our method is closely related to Stallings', which uses Whitehead graphs [22] (defined below). Let H_n denote the orientable 3 dimensional handlebody of genus n. Fix an identification of F_n with the fundamental group of H_n. A basis B of F_n corresponds to a system of embedded disks $D = \{d_1, \dots, d_n\}$ such that for an element $b_i \in B$, b_i is represented by a closed path in H_n which starts from the chosen basepoint, intersects d_i transversely and returns to the basepoint without touching any other d_j. Cutting open H_n along these disks results in a 3-ball with 2n disks $d_i^{\pm}$ (such that the chosen representative b_i enters along d_i^+ and leaves along d_i^-). W is represented by a set of curves in H_n. After cutting, the set of curves is now a set of arcs between these discs.

Definition 6.3 ([22]). — *The* Whitehead graph $\Gamma_{F_n,B}(W)$ *is the graph with* $2n$ *vertices labelled $\{b_1^{\pm}, \dots, b_n^{\pm}\}$ and an edge between two vertices b_i^+ (respectively, b_i^-) and b_j^+ (b_i^-) for every arc coming from W between the corresponding discs d_i^+ (d_i^-) and d_j^+ (d_j^-) in the cut up handlebody.*

Figure 6.1 illustrates an example when $n = 2$, $B = \{b_1, b_2\}$ and $W = \{b_1 b_2 b_1\}$.

Recall that if Y is a topological space, then a *cut point* $y \in Y$ is a point such that $Y \setminus \{y\}$ is not connected. There is a well-known result about the separability of W.

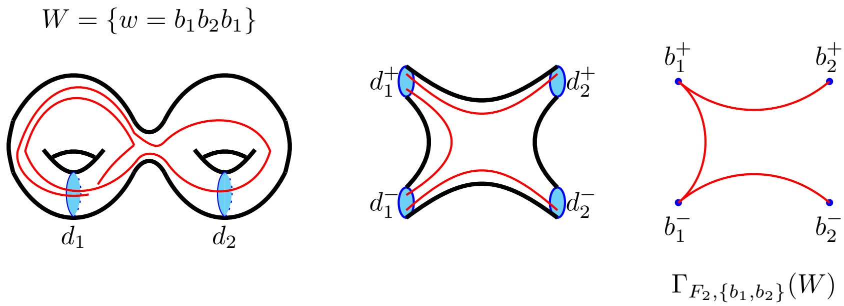


Figure 6.1. A Whitehead graph

Proposition 6.4 ([22]). — *If W is separable, then for any basis B, the Whitehead graph $\Gamma_{F_n,B}(W)$ is either disconnected, or has a cut vertex.*

More details can be found in [20]. We will re-prove Proposition 6.4 above using tubular graphs of graphs. Stallings constructs his algorithm to detect separability by choosing a Whitehead automorphism whenever there is a cut vertex in a Whitehead graph. Our strategy is to use the machinery of Theorem 5.3 when a Whitehead graph contains a cut vertex.

6.1. Construction of a double

Let R_n denote an oriented rose with petals $\{a_1, \ldots, a_n\}$. Fix an identification of F_n with the fundamental group of R_n such that each petal of R_n in the positive direction represents a distinct element of the basis $B = \{b_1, \ldots, b_n\}$. For each element $w_j \in W = \{w_1, \ldots, w_k\}$, let $\phi_j : C_j \to R_n$ denote a cycle from the circle C_j such that ϕ_j induces the word w_j in F_n. We assume that w_j is cyclically reduced so that ϕ_j is an immersion. Subdivide R_n and each C_j so that each ϕ_j is a simplicial immersion between simplicial graphs. Denote the subdivided R_n by X_s.

We call the descendant v in X_s of the unique vertex of R_n as the *special vertex* of X_s.

We define the *double* of X_s along W to be the tubular graph of graphs X such that X_s is a vertical graph of X with exactly k tubes attached to X_s via the attaching maps ϕ_j. Further, the underlying graph of X consists of exactly two vertices and k edges between them, where each vertex graph is isomorphic to X_s and each edge graph is isomorphic to a unique C_j with attaching map ϕ_j on both sides.

Lemma 6.5. — *The vertex link of every vertex of $X_s \subset X$ is connected with no cut vertices if and only if the link of the special vertex v is connected with no cut vertices.*

Proof. — The main case to consider is that of a vertex $u \neq v \in X_s$ whose vertex link is either not connected or has a cut vertex. Then there exists at most one path between the two vertical vertices e_1, e_2 in $\mathrm{link}(u)$. There is a bijective correspondence between paths in $\mathrm{link}(u)$ between e_1 and e_2 and paths in $\mathrm{link}(v)$ between vertices that induce the petal in R_n that contains u. Hence the result. ☐

The main lemma of this section is the following:

Lemma 6.6. — *The link in X of the special vertex v is isomorphic as graphs to the first subdivision of the Whitehead graph $\Gamma_{F_n,B}(W)$.*

Proof. — Each petal of R_n induces two vertical vertices in $\mathrm{link}(v)$ and hence there are $2n$ vertical vertices. Paths of length two in $\mathrm{link}(v)$ between these vertices correspond to the occurrence of the respective letters in a word w_j of W. The isomorphism is then clear. ☐

Proof of Proposition 6.4. — Let X be the double of (X_s, W), with fundamental group G. G is not one-ended as W is separable. By Proposition 3.6, the link of some vertex of X_s is either not connected or has a cut vertex. Lemma 6.5 and Lemma 6.6 then give the result. ☐

We will need the following result by Wilton [23, Theorem 18]:

Theorem 6.7 (Wilton). — *The fundamental group of a graph of free groups with cyclic edge groups is freely indecomposable if and only if every vertex group is freely indecomposable relative to the incident edge groups.*

Note that a vertex group is freely indecomposable relative to the incident edge groups if and only if the set of words induced by the generators of these edge groups is not separable.

Proof of Corollary 6.2. — Let W be the given set of words of the free group F. Let X be the double of (X_s, W) and G its fundamental group. Apply the algorithm of Corollary 5.4 to detect whether G is one-ended. By Theorem 6.7 above, G is one-ended if and only if W is not separable. ☐

7. The general case

We recall that by a *graph of groups $\mathcal{G}$*, we mean the following data: Γ is a connected graph, called the underlying graph. For each vertex s (edge a) of

Γ, G_s (G_a) is a group. Further, whenever a is incident to s, $\partial_{a,s} : G_a \to G_s$ is an injective homomorphism. Given a graph of groups as above, one can naturally associate with it a graph of spaces $\mathcal{X}_\mathcal{G}$ with the same underlying graph Γ such that for each vertex s (edge a) of Γ, X_a (X_s) is a connected topological space such that $\pi_1(X_a) \cong G_a$ ($\pi_1(X_s) \cong G_s$). The *fundamental group* of the graph of groups $\mathcal{G}$ is the fundamental group of the geometric realisation of $\mathcal{X}_\mathcal{G}$ (Section 2.2). We refer the reader to [17] for details. In this section, we freely switch between graphs of groups and graphs of spaces as required.

Theorem 7.1 (Theorem B). — *There is an algorithm of polynomial time complexity which takes a graph of free groups with cyclic edge groups as input and returns the Grushko decomposition of its fundamental group.*

Before going into the proof, we recall the definitions of collapse and blow up. Given a graph of groups $\mathcal{G}$, and an edge a of the underlying graph Γ with endpoints s_1, s_2, by a *collapse* of the edge s we mean a graph of groups $\mathcal{G}'$ with the following data: the underlying graph Γ' is a quotient of Γ with the edge s collapsed to a point (vertex) a_s. The only change in vertex (edge) groups is the new vertex group G_{a_s}. G_{a_s} is the fundamental group of the graph of groups whose underlying graph is the edge s, the vertex groups are G_{s_1} and G_{s_2} respectively and the edge group is G_a, with the injective homomorphisms from the edge group remaining the same as in $\mathcal{G}$.

An *elementary blow up* is the inverse operation of a collapse of an edge. A *blow up* is an iterated process of finitely many elementary blow up operations. Note that a collapse or blow up operation does not change the fundamental group.

We will also recall the notion of a relative Grushko decomposition. Let F be a finite rank free group and let W be a finite set of words in F. The *relative Grushko decomposition* of the pair (F, W) is a free splitting of F such that each element of W conjugates into a free factor of the splitting. Further, each free factor of the splitting is itself freely indecomposable relative to W.

Proof. — Given a graph of free groups with cyclic edge groups $\mathcal{G}$, for each vertex s of the underlying graph Γ, we have a pair (G_s, W_s), where G_s denotes the free vertex group at s and W_s is the set of words induced by generators of the incident edge groups. The Grushko decomposition of the fundamental group of $\mathcal{G}$ can be obtained in two steps:

(1) Obtain for each pair (G_s, W_s) its *relative Grushko decomposition* $\mathcal{G}_s^{Gr}$ in the following way: For the pair (G_s, W_s), let X_s denote a suitable subdivision of an oriented rose with fundamental group G_s (see Section 6.1) and let Y_s denote its double. Apply Corollary 5.4

to obtain a tubular graph of graphs Y_s^{Gr} which induces the Grushko decomposition of $\pi_1(Y_s)$. Note that by Theorem 6.7, any free splitting of $\pi_1(Y_s)$ is a free splitting of G_s relative to W_s. Thus the obtained Grushko decomposition of $\pi_1(Y_s)$ is a double over W_s of the relative Grushko decomposition $\mathcal{G}_s^{Gr}$ of (G_s, W_s). Let Γ_s denote the underlying graph of $\mathcal{G}_s^{Gr}$.

(2) Modify $\mathcal{G}$ by blowing-up each vertex s to $\mathcal{G}_s^{Gr}$. Let $\mathcal{G}'$ be the new graph of groups. Note that the edge groups of $\mathcal{G}'$ are either cyclic or trivial (trivial if and only if the edge belongs to some Γ_s). Collapse all edges with nontrivial edge group to obtain a new graph of groups $\mathcal{G}^{Gr}$. By Theorem 6.7, the fundamental group of $\mathcal{G}$ admits no other free splitting and hence $\mathcal{G}^{Gr}$ is the required Grushko decomposition.

The running time of the above algorithm is of the order of the sum of the running times of the algorithms to obtain the relative Grushko decompositions in Step (1). But each such algorithm runs in polynomial time in the number of squares of Y_s. The number of squares of Y_s is the sum of the lengths of the words of W_s in X_s. Thus, the algorithm runs in polynomial time in the lengths of the words defined by the incident edge groups (in the respective vertex groups) of the input graph of free groups with cyclic edge groups. □

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