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Abelian varieties as automorphism groups of smooth projective varieties in arbitrary characteristics (*)

Jérémy Blanc (1) and Michel Brion (2)

Abstract. — Let A be an abelian variety over an algebraically closed field. We show that A is the automorphism group scheme of some smooth projective variety if and only if A has only finitely many automorphisms as an algebraic group. This generalizes a result of Lombardo and Maffei for complex abelian varieties.

Résumé. — Soit A une variété abélienne sur un corps algébriquement clos. Nous montrons que A est le groupe d'automorphismes d'une variété projective lisse si et seulement si A n'a qu'un nombre fini d'automorphismes en tant que groupe algébrique. Ceci généralise un résultat de Lombardo et Maffei pour les variétés abéliennes complexes.

1. Introduction

Let X be a projective algebraic variety over an algebraically closed field. The automorphism group functor of X is represented by a group scheme Aut_X, locally of finite type (see [3, p. 268] or [7, Thm. 3.7]). Thus, the automorphism group $\mathrm{Aut}(X)$ is the group of k-rational points of a smooth group scheme that we will still denote by $\mathrm{Aut}(X)$ for simplicity. One may ask which smooth group schemes are obtained in this way, possibly imposing some additional conditions on X such as smoothness or normality. It is

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known that every finite group G is the automorphism group scheme of some smooth projective curve X (see e.g. the main result of [5]). The case of a complex abelian variety A was treated recently by Lombardo and Maffei in [4]; they showed that $A = \mathrm{Aut}(X)$ for some complex projective manifold X if and only if A has only finitely many automorphisms as an algebraic group. In this note, we generalize their result as follows:

Theorem A. — *Let A be an abelian variety over an algebraically closed field. Denote by $\mathrm{Aut}_{\mathrm{gp}}(A)$ the group of automorphisms of A as an algebraic group.*

(1) *If $A = \mathrm{Aut}(X)$ for some projective variety X, then $\mathrm{Aut}_{\mathrm{gp}}(A)$ is finite.*
(2) *If $\mathrm{Aut}_{\mathrm{gp}}(A)$ is finite, then there exists a smooth projective variety X such that $A = \mathrm{Aut}_X$.*

Like in [4], the proof of the first assertion is easy, and the second one is obtained by constructing X as a quotient $(A \times Y)/G$, where $G \subset A$ is a finite subgroup, Y is a smooth projective variety such that $G = \mathrm{Aut}_Y$, and the quotient is taken for the diagonal action of G on $A \times Y$. In [4], G is a cyclic group of prime order ℓ, and Y a surface of degree ℓ in $\mathbb{P}^3$ equipped with a free action of G. As the construction of Y does not extend readily to prime characteristics, we take for G the n-torsion subgroup scheme $A[n]$ for an appropriate integer n, and for Y an appropriate rational variety.

A different construction of a variety X satisfying the second assertion has been obtained independently by Mathieu Florence, see [2]; it works over an arbitrary field.

Let us briefly describe the structure of this note. Section 2 is a short introduction to basic notation and reminders on abelian varieties. In Section 3, we take an abelian variety A with $\mathrm{Aut}_{\mathrm{gp}}(A)$ infinite, assume that $A = \mathrm{Aut}(X)$ for some projective variety X, and derive a contradiction. In Section 4, we take an abelian variety A with $\mathrm{Aut}_{\mathrm{gp}}(A)$ finite and prove that for each prime number ℓ different from the characteristic of the ground field, for each $m \geqslant 1$ big enough, and for each smooth rational projective variety Y with $\mathrm{Aut}_Y \simeq A[\ell^m]$, one has

$$\mathrm{Aut}_X = A$$

where X is the smooth projective variety $(A \times Y)/A[\ell^m]$. Then, Section 5 is devoted to an explicit construction of Y.

2. Preliminaries and notation

We begin by fixing some notation and conventions which will be used throughout this note. The ground field $\mathbf{k}$ is algebraically closed, of characteristic $p \geqslant 0$. A variety X is a separated integral scheme of finite type over $\mathbf{k}$. By a point of X, we mean a $\mathbf{k}$-rational point.

We use [8] as a general reference for abelian varieties. We denote by A such a variety of dimension $g \geqslant 1$, with group law $+$ and neutral element 0. Then

$$\mathrm{Aut}(A) = A \rtimes \mathrm{Aut}_{\mathrm{gp}}(A),$$

where A acts on itself by translations. Moreover, $\mathrm{Aut}_{\mathrm{gp}}(A) = \mathrm{Aut}(A, 0)$ (the group of automorphisms fixing the neutral element), see [8, §4, Cor. 1].

For any positive integer n, we denote by $A[n]$ the n-torsion subgroup scheme of A, i.e., the schematic kernel of the multiplication map

$$n_A \colon A \longrightarrow A, \quad a \longmapsto na.$$

Clearly, $A[n]$ is stable by $\mathrm{Aut}_{\mathrm{gp}}(A)$. Also, recall from [8, §6, Prop.] that $A[n]$ is finite; moreover, $A[n]$ is the constant group scheme $(\mathbb{Z}/n)^{2g}$ if n is prime to p.

We denote by

$$q \colon A \longrightarrow A/A[n], \quad a \longmapsto \bar{a}$$

the quotient morphism. Then n_A factors as q followed by an isomorphism $A/A[n] \xrightarrow{\simeq} A$.

3. Proof of Theorem A (1)

In this section, we choose an abelian variety A such that $\mathrm{Aut}_{\mathrm{gp}}(A)$ is infinite, and proceed to the proof of Theorem A (1). We will need:

Lemma 3.1. — *For any positive integer n, the kernel of the restriction map*

$$\rho_n \colon \mathrm{Aut}_{\mathrm{gp}}(A) \longrightarrow \mathrm{Aut}_{\mathrm{gp}}(A[n])$$

is infinite.

Proof. — Note that ρ_n extends to a ring homomorphism

$$\sigma_n \colon \mathrm{End}_{\mathrm{gp}}(A) \longrightarrow \mathrm{End}_{\mathrm{gp}}(A[n])$$

with an obvious notation. Moreover, the image of σ_n is a finitely generated abelian group (as a quotient of $\mathrm{End}_{\mathrm{gp}}(A)$) and is killed by n; thus, this image is finite. So the image of ρ_n is finite as well. □

We assume, for contradiction, the existence of a projective variety X such that $A = \mathrm{Aut}(X)$; in particular, X is equipped with a faithful action of A. By [1, Lem. 3.2], there exist a finite subgroup scheme G of A and an A-equivariant morphism $f\colon X \to A/G$, where A acts on A/G via the quotient map. Denote by n the order of G; then G is a subgroup scheme of $A[n]$. By composing f with the natural map $A/G \to A/A[n]$, we may thus assume that $G = A[n]$.

We now adapt the proof of [4, Thm. 2.2]. Let Y be the schematic fiber of f at $\bar{0}$. Then Y is a closed subscheme of X, stable by the action of $A[n]$. Form the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{\ f'\ } & A \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\ f\ } & A/A[n]. \end{array}$$

Then X' is a projective scheme equipped with an action of A; moreover, f' is an A-equivariant morphism and its fiber at 0 may be identified to Y. It follows that the morphism

$$A \times Y \longrightarrow X', \quad (a, y) \longmapsto a \cdot y$$

is an isomorphism with inverse

$$X' \longrightarrow A \times Y, \quad x' \longmapsto (f'(x'), -f'(x') \cdot x').$$

So we may identify X' with $A \times Y$; then r is invariant under the action of $A[n]$ via $g\cdot(a, y) = (a-g, g\cdot y)$. Since q is an $A[n]$-torsor, so is r. In particular, $X = (A\times Y)/A[n]$ and the stabilizer in A of any $y \in Y$ is a subgroup scheme of $A[n]$.

By Lemma 3.1, we may choose a nontrivial $v \in \mathrm{Aut}_{\mathrm{gp}}(A)$ which restricts to the identity on $A[n]$. Then $v \times \mathrm{id}$ is an automorphism of $A \times Y$ that commutes with the action of $A[n]$. Since r is an $A[n]$-torsor and hence a categorical quotient, it follows that $v \times \mathrm{id} \in \mathrm{Aut}(A \times Y)$ factors through a unique $u \in \mathrm{Aut}(X)$, which satisfies $u(a\cdot y) = v(a)\cdot y$ for all $a \in A$ and $y \in Y$.

As $\mathrm{Aut}(X) = A$, we have $u \in A$. For any $a, b \in A$ and $y \in Y$, we have $(a + b) \cdot y = b \cdot (a \cdot y)$. Choosing $b = u$ in the above formula yields $(a + u) \cdot y = u \cdot (a \cdot y) = v(a) \cdot y$. Thus, $v(a) - a - u$ fixes every point of Y for any $a \in A$. Taking $a = 0$, it follows that u fixes Y pointwise, and hence $u \in A[n]$. So $v(a) - a \in A[n]$ for any $a \in A$, i.e., $v - \mathrm{id}$ factors through a homomorphism $A \to A[n]$.

Since A is smooth and connected, it follows that $v - \mathrm{id} = 0$, a contradiction.

4. Proof of Theorem A (2): first steps

We assume from now on that the group $\mathrm{Aut}_{\mathrm{gp}}(A)$ is finite. Recall that $q\colon A \to A/A[n]$ is the quotient morphism (see Section 2).

Lemma 4.1. —

(1) *The map* $q_*\colon \mathrm{Aut}_{\mathrm{gp}}(A) \to \mathrm{Aut}_{\mathrm{gp}}(A/A[n])$ *is an isomorphism for any integer* $n \geqslant 1$.
(2) *Let* $\ell \neq p$ *be a prime number. Then* $\rho_{\ell^m}\colon \mathrm{Aut}_{\mathrm{gp}}(A) \to \mathrm{Aut}_{\mathrm{gp}}(A[\ell^m])$ *is injective for* $m \gg 0$.

Proof. —

(1). — Since $\mathrm{Aut}_{\mathrm{gp}}(A/A[n]) \simeq \mathrm{Aut}_{\mathrm{gp}}(A)$ is finite, it suffices to show that q_* is injective. Let $u \in \mathrm{Aut}_{\mathrm{gp}}(A)$ such that $q_*(u) = \mathrm{id}$. Then we have $u(a) - a \in A[n]$ for any $a \in A$, that is, $u - \mathrm{id}$ factors through a homomorphism $A \to A[n]$. As in the very end of the proof of Theorem A (1) the smoothness and connectedness of A yield $u = \mathrm{id}$.

(2). — Let $T_\ell(A) = \lim_{\leftarrow} A[\ell^m]$; then $T_\ell(A)$ is a $\mathbb{Z}_\ell$-module and the natural map $\mathrm{Aut}_{\mathrm{gp}}(A) \to \mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell(A))$ is injective (see [8, §19, Thm. 3]). Thus, $\bigcap_{m\geqslant 1} \mathrm{Ker}(\rho_{\ell^m}) = \{\mathrm{id}\}$. Since the $\mathrm{Ker}(\rho_{\ell^m})$ form a decreasing sequence, we get $\mathrm{Ker}(\rho_{\ell^m}) = \{\mathrm{id}\}$ for $m \gg 0$. □

Next, consider a smooth projective variety Y equipped with an action of the finite group $G = A[n]$, for some integer n prime to p. Then G acts freely on $A \times Y$ via $g \cdot (a, y) = (a - g, g \cdot y)$. The quotient $X = (A \times Y)/G$ exists and is a smooth projective variety (see [8, §7, Thm.]). The A-action on $A \times Y$ via translation on itself yields an action on X. The projection $\mathrm{pr}_A\colon A \times Y \to A$ yields a morphism

$$f\colon X \longrightarrow A/G$$

which is A-equivariant, where A acts on A/G via the quotient map q. Moreover, f is smooth and its schematic fiber at $\bar{0}$ is G-equivariantly isomorphic to Y.

Lemma 4.2. — *Assume that* Y *is rational.*

(1) *The map* f *is the Albanese morphism of* X.
(2) *The neutral component* $\mathrm{Aut}^0(Y)$ *is a linear algebraic group.*

Proof. —

(1). — Let B be an abelian variety, and $u\colon X \to B$ a morphism. Composing u with the quotient morphism $A \times Y \to X$ yields a G-invariant morphism $v\colon A \times Y \to B$. As Y is rational, v factors through a morphism $A \to B$, which must be G-invariant. So u factors through a morphism $A/G \to B$.

(2). — By a theorem of Nishi and Matsumura (see [1] for a modern proof), there exist a closed affine subgroup scheme $H \subset \mathrm{Aut}^0(Y)$ such that the homogeneous space $\mathrm{Aut}^0(Y)/H$ is an abelian variety, and an $\mathrm{Aut}^0(Y)$-equivariant morphism $u\colon Y \to \mathrm{Aut}^0(Y)/H$. As Y is rational and u is surjective, this forces $H = \mathrm{Aut}^0(Y)$. □

As a consequence of Lemma 4.2, if Y is rational then f induces a homomorphism

$$f_*\colon \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(A/G),$$

and hence an exact sequence

$$1 \longrightarrow \mathrm{Aut}_{A/G}(X) \longrightarrow \mathrm{Aut}(X) \xrightarrow{f_*} A/G \rtimes \mathrm{Aut}_{\mathrm{gp}}(A/G),$$

where $\mathrm{Aut}_{A/G}(X)$ denotes the group of relative automorphisms. The A-action on X yields a homomorphism $G \to \mathrm{Aut}_{A/G}(X)$. Moreover, the image of f_* contains the group A/G of translations, and hence equals $A/G \rtimes \Gamma$, where Γ denotes the subgroup of $\mathrm{Aut}_{\mathrm{gp}}(A/G)$ consisting of automorphisms which lift to X.

Lemma 4.3. — *Let $G = A[\ell^m]$, where ℓ, m satisfy the assumptions of Lemma 4.1 (2).*

Let Y be a smooth projective rational G-variety such that $\mathrm{Aut}(Y) = G$.

(1) *The map $G \to \mathrm{Aut}_{A/G}(X)$ is an isomorphism.*
(2) *The group Γ is trivial.*

Proof. —

(1). — Let $u \in \mathrm{Aut}_{A/G}(X)$. Then u restricts to an automorphism of Y (the fiber of f at 0), and hence to a unique $g \in G$. Replacing u with $g^{-1}u$, we may assume that u fixes Y pointwise. For any $a \in A$ and $y \in Y$, we have $f(u(\overline{(a,y)})) = f(\overline{(a,y)}) = \bar{a}$, where $\overline{(a,y)}$ denotes the image of (a,y) in X. As f is A-equivariant, it follows that $(-a) \cdot u(\overline{(a,y)}) \in Y$. This defines a morphism

$$F\colon A \times Y \longrightarrow Y, \quad (a,y) \longmapsto (-a) \cdot u(\overline{(a,y)})$$

such that $F(0,y) = u(y) = y$ for all $y \in Y$. As A is connected, this defines in turn a morphism (of varieties) $A \to \mathrm{Aut}^0(Y)$, which must be constant by Lemma 4.2 (2). So $u(\overline{(a,y)}) = a \cdot y = \overline{(a,y)}$ identically, i.e., $u = \mathrm{id}$.

(2). — Let $\gamma \in \Gamma$; then there exists $u \in \mathrm{Aut}(X)$ such that $f_*(u) = \gamma$. Since $\gamma(\bar{0}) = \bar{0}$, we see that u stabilizes Y; thus, $u|_Y = g$ for a unique $g \in G$. Also, there exists $v \in \mathrm{Aut}_{\mathrm{gp}}(A)$ such that $q_*(v) = \gamma$ (Lemma 4.1 (1)). Thus, we have $f(u(\overline{a,y})) = \gamma f(\overline{(a,y)}) = \overline{v(a)}$, i.e., $(-v(a)) \cdot u(\overline{(a,y)}) \in Y$ for all $a \in A$ and $y \in Y$. Arguing as in the proof of 1, it follows that

$$u(\overline{(a,y)}) = v(a) \cdot g(y)$$

identically. In particular, $g(a \cdot y) = v(a) \cdot g(y)$ for all $a \in G$ and $y \in Y$. Since G is commutative, we obtain $v(a) = a$ for all $a \in G$. Thus, $v = \mathrm{id}$ by Lemma 4.1 (2). So $\gamma = \mathrm{id}$ as well. □

Proposition 4.4. — *Under the assumptions of Lemma 4.3, the A-action on X yields an isomorphism $A \to \mathrm{Aut}(X)$. If in addition $G = \mathrm{Aut}_Y$, then $A \to \mathrm{Aut}_X$ is an isomorphism as well.*

Proof. — We have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & A & \longrightarrow & A/G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Aut}_{A/G}(X) & \longrightarrow & \mathrm{Aut}(X) & \xrightarrow{f_*} & \mathrm{Aut}(A/G). & & \end{array}$$

By Lemma 4.3, the left vertical map is an isomorphism and the image of f_* is the group A/G of translations. This yields the first assertion.

To show the second assertion, it suffices to show that the induced homomorphism of Lie algebras $\mathrm{Lie}(A) \to \mathrm{Lie}(\mathrm{Aut}_X)$ is an isomorphism when $G = \mathrm{Aut}_Y$. Recall that $\mathrm{Lie}(\mathrm{Aut}_X)$ is the space of global sections of the tangent bundle T_X (see e.g. [7, Lem. 3.4]). Moreover, as f is smooth, we have an exact sequence

$$0 \longrightarrow T_f \longrightarrow T_X \xrightarrow{df} f^*(T_{A/G}) \longrightarrow 0,$$

where T_f denotes the relative tangent bundle. Since $T_{A/G}$ is the trivial bundle with fiber $\mathrm{Lie}(A/G)$, this yields an exact sequence

$$0 \longrightarrow H^0(X, T_f) \longrightarrow H^0(X, T_X) \longrightarrow \mathrm{Lie}(A/G)$$

such that the composition $\mathrm{Lie}(A) \to H^0(X, T_X) \to \mathrm{Lie}(A/G)$ is $\mathrm{Lie}(q)$. So it suffices in turn to show that $H^0(X, T_f) = 0$.

We have a cartesian diagram

$$\begin{array}{ccc} A \times Y & \xrightarrow{\mathrm{pr}_A} & A \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & A/G, \end{array}$$

where the vertical arrows are G-torsors. This yields an isomorphism

$$H^0(X, T_f) \simeq H^0(A \times Y, T_{\mathrm{pr}_A})^G$$

and hence

$$H^0(X, T_f) \simeq H^0(A \times Y, \mathrm{pr}_Y^*(T_Y))^G \simeq (\mathcal{O}_A(A) \otimes H^0(Y, T_Y))^G \simeq H^0(Y, T_Y)^G.$$

As $G = \mathrm{Aut}_Y$, we have $H^0(Y, T_Y) = \mathrm{Lie}(G) = 0$; this completes the proof. □

5. Proof of Theorem A (2): the construction of Y

In this section, we fix integers $n, r \geqslant 2$, where p does not divide n, and construct a smooth projective rational variety Y of dimension r such that $\mathrm{Aut}_Y = (\mathbb{Z}/n)^r$.

We define

$$G = \{(\mu_1, \ldots, \mu_r) \in \mathbf{k}^r \mid \mu_i^n = 1 \text{ for each } i \in \{1, \ldots, r\}\} \simeq (\mathbb{Z}/n)^r$$

and let G act on $(\mathbb{P}^1)^r$ by

$$\begin{array}{ccc} G \times (\mathbb{P}^1)^r & \longrightarrow & (\mathbb{P}^1)^r \\ ((\mu_1, \ldots, \mu_r), ([u_1 : v_1], \ldots, [u_r : v_r])) & \longmapsto & ([u_1 : \mu_1 v_1], \ldots, [u_r : \mu_r v_r]) \end{array}$$

For each $i \in \{1, \ldots, r\}$, we denote by $\ell_i \subset (\mathbb{P}^1)^r$ the closed curve isomorphic to $\mathbb{P}^1$ given by the image of

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & (\mathbb{P}^1)^r \\ ([u : v]) & \longmapsto & ([0 : 1], \ldots, [0 : 1], [u : v], [0 : 1], \ldots, [0 : 1]) \end{array}$$

where the $[u : v]$ is at the place i. The curves $\ell_1, \ldots, \ell_r \subset (\mathbb{P}^1)^r$ generate the cone of curves of $(\mathbb{P}^1)^r$.

For each $i \in \{1, \ldots, r\}$, the curve ℓ_i is stable by G and the action of G on ℓ_i corresponds to a cyclic action of order n on $\mathbb{P}^1$, given by $[u : v] \mapsto [\mu u : v]$, where $\mu \in \mathbf{k}$, $\mu^n = 1$. All orbits are of size n, except the two fixed points $[0 : 1]$ and $[1 : 0]$.

We choose $s = (s_1, \ldots, s_r)$ to be a sequence of positive integers, all distinct, such that $s_i \cdot n \geqslant 3$ for each i if $r = 2$, and consider a finite subset

$$\Delta \subset \ell_1 \cup \cdots \cup \ell_r \subset (\mathbb{P}^1)^r,$$

stable by G, given by a union of orbits of size n. For each $i \in \{1, \ldots, r\}$, we define $\Delta_i \subset \ell_i$ to be a union of exactly $s_i \geqslant 1$ orbits of size n, and choose then $\Delta = \bigcup_{i=1}^r \Delta_i$. We moreover choose the points such that the group $H = \{h \in \mathrm{Aut}(\mathbb{P}^1) \mid h(\Delta_i) = \Delta_i, h([0 : 1]) = [0 : 1]\}$ only consists of $\{[u : v] \mapsto [\mu u : v] \mid u^n = 1\}$. As the unique point of intersection of any two distinct ℓ_i is fixed by G, each point of Δ lies on exactly one of the curves ℓ_i. This gives

$$\Delta = \biguplus_{i=1}^{r} \Delta_i$$

Let $\pi \colon Y \to (\mathbb{P}^1)^r$ be the blow-up of Δ. As Δ is G-invariant, the action of G lifts to an action on Y. We want to prove that the resulting homomorphism $G \to \mathrm{Aut}_Y$ is an isomorphism.

5.1. Intersection on $(\mathbb{P}^1)^r$

For $i = 1, \dots, r$, we denote by $H_i \subset (\mathbb{P}^1)^r$ the hypersurface given by

$$H_i = \{([u_1 : v_1], \dots, [u_r : v_r]) \in (\mathbb{P}^1)^r \mid u_i = 0\}.$$

Then $H_1, \dots, H_r$ generate the cone of effective divisors on $(\mathbb{P}^1)^r$, and we have

$$H_i \cdot \ell_i = 1, H_i \cdot \ell_j = 0$$

for all $i, j \in \{1, \dots, r\}$ with $i \neq j$. Moreover, the canonical divisor class of $(\mathbb{P}^1)^r$ satisfies $K_{(\mathbb{P}^1)^r} = -2H_1 - 2H_2 - \cdots - 2H_r$, so $K_{(\mathbb{P}^1)^r} \cdot \ell_i = -2$ for each $i \in \{1, \dots, r\}$.

We also observe that $\ell_i \subset H_j$ for all $i, j \in \{1, \dots, r\}$ with $i \neq j$ and that $\ell_i \not\subset H_i$.

5.2. Intersection on Y

For $i = 1, \dots, r$, denote by $\widetilde{\ell}_i, \widetilde{H}_i \subset Y$ the strict transforms of ℓ_i and H_i.

For each $p \in \Delta$, we denote by $E_p = \pi^{-1}(p)$ the exceptional divisor, isomorphic to $\mathbb{P}^{r-1}$, and choose a line $e_p \subset E_p$.

A basis of the Picard group of Y is given by the union of $\widetilde{H}_1, \dots, \widetilde{H}_r$ and of all exceptional divisors E_p, with $p \in \Delta$. A basis of the vector space of curves (up to numerical equivalence) is given by $\widetilde{\ell}_1, \dots, \widetilde{\ell}_r$ and by all e_p with $p \in \Delta$. We have

$$e_p \cdot E_p = -1, e_p \cdot E_q = 0$$

for all $p, q \in \Delta$, $p \neq q$.

LEMMA 5.1. — *For all $i, j \in \{1, \dots, r\}$ with $i \neq j$, the following hold:*

(1) $\widetilde{H}_i = \pi^*(H_i) - \sum_{p \in \Delta \cap H_i} E_p = \pi^*(H_i) - \sum_{s \neq i} \sum_{p \in \Delta_s} E_p$.
(2) $\widetilde{\ell}_i \cdot E_p = 1$ *if* $p \in \Delta_i$ *and* $\widetilde{\ell}_i \cdot E_p = 0$ *if* $p \in \Delta \setminus \Delta_i$.
(3) $\widetilde{H}_i \cdot \widetilde{\ell}_i = 1$.
(4) $\widetilde{H}_i \cdot \widetilde{\ell}_j = -|\Delta_j| = -ns_j$.

Proof. —

(1). — It follows from the fact that H_i is a smooth hypersurface of $(\mathbb{P}^1)^r$ and that $\Delta \cap H_i = \bigcup_{s \neq i} \Delta_s$.

(2). — It follows from the fact that ℓ_i is a smooth curve, passing through all points of Δ_i and not through any point of $\Delta \setminus \Delta_i$.

(3). — With (1) and (2), we get $\widetilde{H}_i \cdot \widetilde{\ell}_i = H_i \cdot \ell_i = 1$.

(4). — With (1) and (2), we get $\widetilde{H}_i \cdot \widetilde{\ell}_j = H_i \cdot \ell_j - |\Delta_j| = -|\Delta_j| = -ns_j$. $\square$

Lemma 5.2. — *For all $i \in \{1, \dots, r\}$ and each $p \in \Delta \setminus \Delta_i$, we take the irreducible curve $\gamma_{p,i} \subset (\mathbb{P}^1)^r$ passing through p and being numerically equivalent to ℓ_i.*

(1) *Let $j \in \{1, \dots, r\}$ be such that $p \in \Delta_j$. The j-th coordinate of $\gamma_{p,i}$ is the one of p, its i-th coordinate is free, and all others are $[0:1]$.*
(2) *The strict transform $\widetilde{\gamma}_{p,i}$ of $\gamma_{p,i}$ on Y is numerically equivalent to $\widetilde{\ell}_i + \sum_{q \in \Delta_i} e_q - e_p$ and satisfies $\widetilde{\gamma}_{p,i} \cdot E_p = 1$ and $\widetilde{\gamma}_{p,i} \cdot E_q = 0$ for all $q \in \Delta \setminus \{p\}$.*

Proof. —

(1). — We write $p = (p_1, \dots, p_r) \in (\mathbb{P}^1)^r$. Since $\gamma_{p,i} \subset (\mathbb{P}^1)^r$ is a curve equivalent to ℓ_i and passing through p, it has to be

$$\gamma_{p,i} = \{(p_1, \dots, p_{i-1}, t, p_{i+1}, \dots, p_r) \in (\mathbb{P}^1)^r \mid t \in \mathbb{P}^1\} \simeq \mathbb{P}^1.$$

Moreover, for each $s \in \{1, \dots, r\} \setminus \{j\}$, we have $p_s = [0:1]$, as $p \in \Delta_j \subset \ell_j$. This completes the proof of 1.

(2). — We want to prove that $\widetilde{\gamma}_{p,i} \equiv \widetilde{\ell}_i + \sum_{q \in \Delta_i} e_q - e_p$. For each divisor D on $(\mathbb{P}^1)^r$, we have

$$\widetilde{\gamma}_{p,i} \cdot \pi^*(D) = \pi(\widetilde{\gamma}_{p,i}) \cdot D = \gamma_{p,i} \cdot D$$
$$(\widetilde{\ell}_i - e_p) \cdot \pi^*(D) = \pi(\widetilde{\ell}_i) \cdot D = \ell_i \cdot D = \gamma_{p,i} \cdot D$$

We moreover have (with Lemma 5.1 (2))

$$\widetilde{\gamma}_{p,i} \cdot E_p = 1 = E_p \cdot \left(\widetilde{\ell}_i + \sum_{q \in \Delta_i} e_q - e_p \right),$$
$$\widetilde{\gamma}_{p,i} \cdot E_{p'} = 0 = E_{p'} \cdot \left(\widetilde{\ell}_i + \sum_{q \in \Delta_i} e_q - e_p \right), \text{ for all } p' \in \Delta \setminus \{p\}. \qquad \square$$

Lemma 5.3. — *Let $\gamma \subset Y$ be an irreducible curve. Then, one of the following holds:*

(1) *We have $\gamma \equiv de_p$ for some $d \geqslant 1$ and some $p \in \Delta$ (where $\equiv$ denotes numerical equivalence);*
(2) *There are non-negative integers $a_1, \dots, a_r$ and $\{\mu_p\}_{p \in \Delta}$ such that*

$$\gamma \equiv \sum_{i=1}^{r} a_i \widetilde{\ell}_i + \sum_{p \in \Delta} \mu_p e_p$$

and such that $a_1 + \cdots + a_r \geqslant 1$.

(3) *There are $j \in \{1, \dots, r\}$, $q \in \Delta_j$ and integers $a_1, \dots, a_r \geqslant 0$ such that*
$$\gamma \equiv a_j e_q + \sum_{i \neq j} a_i \widetilde{\gamma}_{q,i}$$
and such that $\sum_{i \neq j} a_i \geqslant 1$.

Proof. — Suppose first that γ is contained in some E_p, where $p \in \Delta$. In this case, γ is a curve of degree $d \geqslant 1$ in the projective space $E_p \simeq \mathbb{P}^{r-1}$ (if $r = 2$, then $\gamma = e_p = E_p$ and $d = 1$), and thus $\gamma \equiv d e_p$. This gives Case (1).

We may now assume that γ is not contained in E_p for any $p \in \Delta$. Hence, γ is the strict transform of the irreducible curve $\pi(\gamma) \subset (\mathbb{P}^1)^r$, numerically equivalent to $\sum_{i=1}^r a_i \ell_i$, with $a_1, \dots, a_r \geqslant 0$ and $\sum_{i=1}^r a_i \geqslant 1$. For each $p \in \Delta$, we write $\epsilon_p = E_p \cdot \gamma \geqslant 0$.

We first prove that
$$\gamma \equiv \sum_{i=1}^{r} a_i \widetilde{\ell}_i + \sum_{i=1}^{r} \sum_{p \in \Delta_i} (a_i - \epsilon_p) e_p. \tag{$\spadesuit$}$$
Intersecting both sides of ($\spadesuit$) with the divisor $\pi^*(D)$, for any divisor D on $(\mathbb{P}^1)^r$, gives $\pi(\gamma) \cdot D = \sum a_i \ell_i \cdot D$. Moreover, for each $p \in \Delta$, there is $j \in \{1, \dots, r\}$ such that $p \in \Delta_j$. Intersecting E_p with both sides of ($\spadesuit$), we obtain $E_p \cdot \gamma = \epsilon_p \overset{\text{Lemma 5.1 (2)}}{=} E_p \cdot (\sum_{i=1}^r a_i \widetilde{\ell}_i + \sum_{i=1}^r \sum_{p \in \Delta_i} (a_i - \epsilon_p) e_p)$. This completes the proof of ($\spadesuit$).

For each $p \in \Delta$, we denote by $i \in \{1, \dots, r\}$ the integer such that $p \in \Delta_i$ and by $H_p \subset (\mathbb{P}^1)^r$ the hypersurface consisting of points $q \in (\mathbb{P}^1)^r$ having the same i-th coordinate as p. Hence $p_i \in H_p$, $H_p \cap \Delta = \{p\}$ and $H_p \sim H_i$. The strict transform of H_p, that we write $\widetilde{H}_p$, satisfies $\widetilde{H}_p \sim \pi^*(H_i) - E_p$. This gives
$$\widetilde{H}_p \cdot \gamma = a_i - E_p \cdot \gamma = a_i - \epsilon_p. \tag{$\heartsuit$}$$

Suppose first that $\widetilde{H}_p \cdot \gamma \geqslant 0$ for each $p \in \Delta$. This means (with ($\heartsuit$)), that $a_i - \epsilon_p \geqslant 0$ for each $i \in \{1, \dots, r\}$ and each $p \in \Delta_i$. Hence all coefficients in ($\spadesuit$) are non-negative, so we obtain (2).

Suppose now that $\widetilde{H}_q \cdot \gamma < 0$ for some $q \in \Delta$. This implies that $\gamma \subset \widetilde{H}_q$. As $H_q \cap \Delta = \{q\}$, we obtain $E_p \cap \widetilde{H}_q = \emptyset$ for each $p \in \Delta \setminus \{q\}$, which yields $\epsilon_p = E_p \cdot \gamma = 0$. Writing $j \in \{1, \dots, r\}$ the element such that $q \in \Delta_j$, the j-th component of $\pi(\gamma) \subset (\mathbb{P}^1)^r$ is constant, so $a_j = \pi^*(H_j) \cdot \gamma = H_j \cdot \pi(\gamma) = 0$. We now prove that
$$\gamma \equiv \left(-\epsilon_q + \sum_{i \neq j} a_i \right) e_q + \sum_{i \neq j} a_i \widetilde{\gamma}_{q,i} \tag{$\diamondsuit$}$$

Intersecting both sides of ($\diamondsuit$) with the divisor $\pi^*(D)$, for any divisor D on $(\mathbb{P}^1)^r$, gives $\pi(\gamma) \cdot D = \sum a_i \ell_i \cdot D$. Intersecting E_q with both sides gives $\epsilon_q = \epsilon_q$, since $E_q \cdot \widetilde{\gamma}_{q,i} = 1$ for each $i \neq j$ (Lemma 5.2 (2)). Intersecting with E_p for $p \in \Delta \setminus \{q\}$ gives $\epsilon_p = 0$. This completes the proof of ($\diamondsuit$).

As the j-th component of $\pi(\gamma) \subset (\mathbb{P}^1)^r$ is constant, there is an integer $i \in \{1, \ldots, r\} \setminus \{j\}$ such that the i-th component of $\pi(\gamma)$ is not constant. This implies that $\pi(\gamma) \not\subset H_i$, so $\widetilde{\gamma} \not\subset \widetilde{H}_i$. We obtain

$$0 \leqslant \widetilde{H}_i \cdot \gamma \overset{\text{Lemma 5.1 (1)}}{=} \left(\pi^*(H_i) - \sum_{s \neq i} \sum_{p \in \Delta_s} E_p\right) \cdot \gamma = a_i - \epsilon_q.$$

Hence, the coefficents of ($\diamondsuit$) are non-negative, giving (3). ☐

Proposition 5.4. — *Let $\gamma \subset Y$ be an irreducible curve. Then, the following are equivalent:*

1. *For all effective 1-cycles γ_1, γ_2 on Y such that $\gamma \equiv \gamma_1 + \gamma_2$, we have $\gamma_1 = 0$ or $\gamma_2 = 0$.*
2. *γ is numerically equivalent to $\widetilde{\ell}_i$ for some $i \in \{1, \ldots, r\}$, to $\widetilde{\gamma}_{p,i}$ for some $i \in \{1, \ldots, r\}, p \in \Delta \setminus \Delta_i$, or to e_p for some $p \in \Delta$.*
3. *γ is either equal to $\widetilde{\ell}_i$ for some $i \in \{1, \ldots, r\}$, or equal to $\widetilde{\gamma}_{p,i}$ for some $i \in \{1, \ldots, r\}, p \in \Delta \setminus \Delta_i$, or is a line in E_p, for some $p \in \Delta$.*

Proof. —

(1) $\Rightarrow$ (2). — By Lemma 5.3, $\gamma \equiv \gamma_1 + \cdots + \gamma_s$ where $s \geqslant 1$ and where the points $\gamma_1, \ldots, \gamma_s$ belong to $\{\widetilde{\ell}_i \mid i \in \{1, \ldots, r\}\} \cup \{e_p \mid p \in \Delta\} \cup \{\widetilde{\gamma}_{p,i} \mid i \in \{1, \ldots, r\}, p \in \Delta \setminus \Delta_i\}$. As (1) is satisfied, we have $s = 1$, which implies (2).

(2) $\Rightarrow$ (3). — Suppose first that $\gamma \equiv e_p$ for some $p \in \Delta$. For an ample divisor D on $(\mathbb{P}^1)^r$, we have $0 = e_p \cdot \pi^*(D) = \pi_*(\gamma) \cdot D$, which implies that γ is contracted by π. Hence, γ is a curve of degree $d \geqslant 1$ in some E_q, $q \in \Delta$, and is thus equivalent to de_q. As $-1 = E_p \cdot e_p = E_p \cdot \gamma$, we have $q = p$ and $d = 1$.

Suppose now that $\gamma \equiv \widetilde{\ell}_i$ for some $i \in \{1, \ldots, r\}$. For each $j \in \{1, \ldots, r\}$ with $j \neq i$, we have $\widetilde{H}_i \cdot \gamma = \widetilde{H}_i \cdot \widetilde{\ell}_j \overset{\text{Lemma 5.1 (4)}}{=} -ns_j < 0$. Hence, $\pi(\gamma) \subset \bigcap_{j \neq i} H_j = \ell_i$. As $\pi(\gamma) \cdot H_i = \pi^*(H_i) \cdot \gamma = \pi^*(H_i) \cdot \widetilde{\ell}_i = 1$, we have $\pi(\gamma) = \ell_i$ and $\widetilde{\gamma} = \widetilde{\ell}_i$.

In the remaining case, $\gamma \equiv \widetilde{\gamma}_{p,i}$ for some $i \in \{1, \ldots, r\}$ and some $p \in \Delta \setminus \Delta_i$. Hence, $\pi(\gamma)$ is numerically equivalent to $\pi(\widetilde{\gamma}_{p,i})$, which is equivalent to ℓ_i (Lemma 5.2 (2)). Hence, all coordinates of $\pi(\gamma)$ except the i-th one are constant. As $\gamma \cdot E_p = \widetilde{\gamma}_{p,i} \cdot E_p = 1$ (again by Lemma 5.2 (2)), the point p belongs to both $\pi(\gamma)$ and $\gamma_{p,i}$, which yields $\pi(\gamma) = \gamma_{p,i}$ and thus $\gamma = \widetilde{\gamma}_{p,i}$.

(3) ⇒ (1). — We take effective 1-cycles γ_1, γ_2 on Y such that $\gamma \equiv \gamma_1 + \gamma_2$ and prove that one of the two is zero, using (3).

For each $i \in \{1, \dots, r\}$, we write $a_i = \pi^*(H_i) \cdot \gamma$, $b_i = \pi^*(H_i) \cdot \gamma_1$ and $c_i = \pi^*(H_i) \cdot \gamma_2$ and obtain $a_i = b_i + c_i$. As H_i is nef, $\pi^*(H_i)$ is nef, so $a_i, b_i, c_i \geqslant 0$. Moreover, γ satisfying (3), we have $\sum_{i=1}^r a_i = 1$, which implies that, up to exchanging γ_1 and γ_2, we may assume that $\sum_{i=1}^r a_i = \sum_{i=1}^r b_i$ and $c_i = 0$ for $i = 1, \dots, r$. In particular, γ_2 is a sum of irreducible curves contained in the exceptional divisors E_p, $p \in \Delta$.

Suppose first that $\gamma = e_q$ for some $q \in \Delta$. This gives $\sum_{i=1}^r a_i = \sum_{i=1}^r b_i = 0$, which implies that both γ_1 and γ_2 are sums of irreducible curves contained in the exceptional divisors E_p, $p \in \Delta$. For each $p' \in \Delta$ and each irreducible curve $c \subset E_{p'}$ of degree $d \geqslant 1$ we get $\sum_{p \in \Delta} E_p \cdot c = -d$. As $\sum_{p \in \Delta} E_p \cdot \gamma = -1$, this gives $\gamma_1 = 0$ or $\gamma_2 = 0$.

We may now take $s \in \{1, \dots, r\}$ and either $\gamma = \widetilde{\ell}_s$ or $\gamma = \widetilde{\gamma}_{p,s}$ for some $p \in \Delta \setminus \Delta_s$. This gives $b_s = 1$ and $b_i = 0$ for all $i \in \{1, \dots, r\} \setminus \{s\}$. Lemma 5.3 implies that γ_1 is equivalent to a sum of curves contained in $\{\widetilde{\ell}_i \mid i \in \{1, \dots, r\}\} \cup \{e_p \mid p \in \Delta\} \cup \{\widetilde{\gamma}_{p,i} \mid i \in \{1, \dots, r\}, p \in \Delta \setminus \Delta_i\}$. As $b_s = 1$ and $b_i = 0$ for all $i \in \{1, \dots, r\} \setminus \{s\}$, we have $\gamma_1 \equiv \alpha + \beta$, where α is either equal to $\widetilde{\ell}_s$ or $\widetilde{\gamma}_{p,s}$ for some $p \in \Delta \setminus \Delta_s$ and where β is a non-negative sum of $e_p, p \in \Delta$. For each $p \in \Delta$, we obtain

$$E_p \cdot \gamma = E_p \cdot \alpha + E_p \cdot \beta + E_p \cdot \gamma_2 \leqslant E_p \cdot \alpha.$$

We now use the fact that we know the intersection of α and γ with E_p (which is given either by Lemma 5.1 (2) or by Lemma 5.2 (2), depending if the curve is equal to $\widetilde{\ell}_s$ or $\widetilde{\gamma}_{p,s}$).

If $\gamma = \widetilde{\gamma}_{p,s}$ for some $p \in \Delta \setminus \Delta_s$, then $1 = E_p \cdot \gamma \leqslant E_p \cdot \alpha$, which implies that $\alpha = \widetilde{\gamma}_{p,s}$. If $\gamma = \widetilde{\gamma}_s$, then $1 = E_q \cdot \gamma \leqslant E_q \cdot \alpha$ for each $q \in \Delta_s$, which implies that $\alpha = \widetilde{\gamma}_s$. In both cases, we get $\alpha = \gamma$, which implies that $E_p \cdot \gamma_2 = 0$ for each $p \in \Delta$, and thus that $\gamma_2 = 0$, as desired. □

Theorem 5.5. — *The map $G \to \mathrm{Aut}_Y$ is an isomorphism.*

Proof. — We first show that $G \overset{\sim}{\to} \mathrm{Aut}(Y)$. Let $\alpha \in \mathrm{Aut}(Y)$. For each irreducible curve $\gamma \subset Y$ that satisfies Proposition 5.4 (1), the curve $\alpha(\gamma)$ also satisfies Proposition 5.4 (1). Hence, the union $F \subset Y$ of all curves satisfying this assertion is also stable by $\mathrm{Aut}(Y)$.

By Proposition 5.4, we have

$$F = \left(\bigcup_{p\in\Delta} E_p\right) \cup \left(\bigcup_{i=1}^{r} \widetilde{\ell}_i\right) \cup \left(\bigcup_{i=1}^{r}\left(\bigcup_{p\in\Delta\setminus\Delta_i} \widetilde{\gamma}_{p,i}\right)\right).$$

We observe that the above union is the decomposition of F into irreducible components. Hence, α permutes the irreducible components. We now make the following observations:

(1) For each $i \in \{1,\ldots,r\}$, $\widetilde{\ell}_i$ intersects exactly $n \cdot s_i$ other irreducible components of F, namely the E_p with $p \in \Delta$.
(2) For each $p \in \Delta_i$, the divisor E_p intersects exactly r other irreducible components of F, namely the curve $\widetilde{\ell}_i$ and the curves $\widetilde{\gamma}_{p,j}$ with $j \in \{1,\ldots,r\} \setminus \{i\}$.
(3) For each $i \in \{1,\ldots,r\}$ and $p \in \Delta \setminus \Delta_i$, the curve $\widetilde{\gamma}_{p,i}$ intersects exactly $n \cdot s_i + 1$ other irreducible components of F. Writing $j \in \{1,\ldots,r\}$ the element such that $p \in \Delta_j$, the curve intersects E_p and all curves $\widetilde{\gamma}_{q,j}$ for each $q \in \Delta_i$.

If $r \geqslant 3$, the exceptional divisors E_p are the irreducible components of maximal dimension of F, so g permutes them. If $r = 2$, then g also permutes the E_p, as these are the only irreducible components of F that intersect exactly 2 other irreducible components of F (we assumed $n \cdot s_i \geqslant 3$ for each i in the case $r = 2$). In any case, g permutes the exceptional divisors E_p and is thus the lift of an automorphism $\widehat{g}$ of $(\mathbb{P}^1)^r$: we observe that the birational self-map $\widehat{g} = \pi g \pi^{-1}$ of $(\mathbb{P}^1)^r$ restricts to an automorphism on the complement of Δ, and as Δ has codimension $\geqslant 2$, $\widehat{g}$ is an automorphism. We then use again the three observations above to see that $g(\widetilde{\ell}_i) = \widetilde{\ell}_i$ for each $i \in \{1,\ldots,r\}$, as the s_i are all distinct. Hence, $\widehat{g}(\ell_i) = \widehat{g}(\ell_i)$ for each i. This implies that $\widehat{g}$ is of the form

$$\begin{array}{ccc} (\mathbb{P}^1)^r & \longrightarrow & (\mathbb{P}^1)^r \\ ((\mu_1,\ldots,\mu_r),) & \longmapsto & ([u_1 : \mu_1 v_1 + \kappa_1 u_1],\ldots,[u_r : \mu_r v_r + \kappa_r u_r]) \end{array}$$

for some $\mu_1,\ldots,\mu_r \in \mathbf{k}^*$ and $\kappa_1,\ldots,\kappa_r \in \mathbf{k}$.

For each $i \in \{1,\ldots,r\}$, the restriction of $\widehat{g}$ to ℓ_i corresponds to the automorphism $[u : v] \mapsto [u_i : \mu_i v_1 + \kappa_i u_i]$. As it has to stabilize the set Δ_i, we have $\kappa_i = 0$ and $\mu_i \in \mathbf{k}^*$ is of order n. This yields the isomorphism $G \simeq \mathrm{Aut}(Y)$.

To complete the proof, it suffices to show that Aut_Y is constant, or equivalently that its Lie algebra is trivial. (We refer to [6, §2.1] for background on infinitesimal automorphisms and vector fields). Recall that $\mathrm{Lie}(\mathrm{Aut}_Y) = H^0(Y, \mathcal{T}_Y)$, where $\mathcal{T}_Y$ denotes the tangent sheaf. In other terms, $\mathrm{Lie}(\mathrm{Aut}_Y)$

consists of the global vector fields on Y. Denoting by $E = \biguplus_{p\in\Delta} E_p$ the exceptional divisor, we have an exact sequence of sheaves on Y

$$0 \longrightarrow \mathcal{T}_{Y,E} \longrightarrow \mathcal{T}_Y \longrightarrow \bigoplus_{p\in\Delta} (i_{E_p})_*(\mathcal{N}_{E_p/Y}) \longrightarrow 0,$$

where $\mathcal{T}_{Y,E}$ is the sheaf of vector fields that are tangent to E, and $\mathcal{N}_{E_p/Y}$ denotes the normal sheaf. Moreover, for any $p \in \Delta$, we have $E_p \simeq \mathbb{P}^{r-1}$ and this identifies $\mathcal{N}_{E_p/Y}$ with $\mathcal{O}_{\mathbb{P}^{r-1}}(-1)$; thus, $H^0(E_p, \mathcal{N}_{E_p/Y}) = 0$. As a consequence, $H^0(Y, \mathcal{T}_{Y,E}) \xrightarrow{\sim} H^0(Y, \mathcal{T}_Y)$. Viewing vector fields as derivations of the structure sheaf $\mathcal{O}_Y$, this yields

$$\mathrm{Der}(\mathcal{O}_Y, \mathcal{O}_Y(-E)) \xrightarrow{\sim} \mathrm{Der}(\mathcal{O}_Y),$$

where the left-hand side denotes the Lie algebra of derivations which stabilize the ideal sheaf of E.

The blow-up $\pi : Y \to (\mathbb{P}^1)^r$ contracts E to Δ and satisfies $\pi_*(\mathcal{O}_Y) = \mathcal{O}_{(\mathbb{P}^1)^r}$; also, $\pi_*(\mathcal{O}_Y(-E)) = \mathcal{I}_\Delta$ (the ideal sheaf of Δ). So π induces a homomorphism of Lie algebras $\pi_* : \mathrm{Der}(\mathcal{O}_Y) \to \mathrm{Der}(\mathcal{O}_{(\mathbb{P}^1)^r})$, which is injective as π is birational. Moreover, π_* sends $\mathrm{Der}(\mathcal{O}_Y, \mathcal{O}_Y(-E))$ into $\mathrm{Der}(\mathcal{O}_{(\mathbb{P}^1)^r}, \mathcal{I}_\Delta)$, the Lie algebra of vector fields on $(\mathbb{P}^1)^r$ which vanish at each $p \in \Delta$. So it suffices to show that each such vector field is zero.

We have

$$\mathrm{Der}(\mathcal{O}_{(\mathbb{P}^1)^r}) = H^0((\mathbb{P}^1)^r, \mathcal{T}_{(\mathbb{P}^1)^r}) = \bigoplus_{i=1}^{r} H^0(\mathbb{P}^1, \mathcal{T}_{\mathbb{P}^1}) = \mathrm{Lie}(\mathrm{Aut}_{\mathbb{P}^1})^r.$$

Moreover, $\mathrm{Lie}(\mathrm{Aut}_{\mathbb{P}^1}) = M_2(\mathbf{k})/\mathbf{k}\,\mathrm{id}$, the quotient of the Lie algebra of 2×2 matrices by the scalar matrices. Let $\xi = (\xi_1, \dots, \xi_r) \in \mathrm{Der}(\mathcal{O}_{(\mathbb{P}^1)^r})$, with representative $(A_1, \dots, A_r) \in M_2(\mathbf{k})^r$. Then ξ vanishes at $p = ([x_1 : y_1], \dots, [x_r : y_r])$ if and only if (x_i, y_i) is an eigenvector of A_i for each $i \in \{1, \dots, r\}$. Thus, if $\xi \in \mathrm{Der}(\mathcal{O}_{(\mathbb{P}^1)^r}, \mathcal{I}_\Delta)$, then $(0, 1)$ is an eigenvector of each A_i, i.e., A_i is lower triangular. In addition, each point of Δ_i yields an eigenvector of A_i. So each A_i is scalar, and $\xi = 0$ as desired. $\square$

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