

ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE Mathématiques

HIDEKAZU FURUSHO AND NAO KOMIYAMA

Kashiwara–Vergne and dihedral bigraded Lie algebras in mould theory

Tome XXXII, n° 4 (2023), p. 655–725.

<https://doi.org/10.5802/afst.1749>

© les auteurs, 2023.

Les articles des *Annales de la Faculté des Sciences de Toulouse* sont mis à disposition sous la licence Creative Commons Attribution (CC-BY) 4.0
<http://creativecommons.org/licenses/by/4.0/>



Kashiwara–Vergne and dihedral bigraded Lie algebras in mould theory ^(*)

HIDEKAZU FURUSHO ⁽¹⁾ AND NAO KOMIYAMA ⁽²⁾

ABSTRACT. — We introduce the Kashiwara–Vergne bigraded Lie algebra associated with a finite abelian group and give its mould theoretic reformulation. By using the mould theory, we show that it includes Goncharov’s dihedral Lie algebra, which generalizes the result of Raphael and Schneps.

RÉSUMÉ. — Nous introduisons l’algèbre de Lie bi-graduée de Kashiwara–Vergne associée à un groupe fini abélien, et donnons sa reformulation moule-théorique. En utilisant la théorie des moules, nous démontrons que celle-ci inclut l’algèbre de Lie diédrale de Goncharov, généralisant ainsi un résultat de Raphael et Schneps.

Introduction

The dihedral Lie algebra $D(\Gamma)_{\bullet\bullet}$ is the bigraded Lie algebra introduced in [12] which is associated with a finite abelian group Γ . It reflects the double shuffle and distribution relations among multiple polylogarithms evaluated at roots of unity. Its relation with a certain bigraded variant of motivic Lie algebra is discussed in loc. cit.

Kashiwara–Vergne Lie algebra \mathfrak{kv}_{\bullet} is the filtered graded Lie algebra introduced in [1] and [3]. It acts on the set of solutions of “a formal version” of Kashiwara–Vergne conjecture. Related to conjectures on mixed Tate motives, it is expected to be isomorphic to the motivic Lie algebra (cf. [11]).

(*) Reçu le 7 décembre 2020, accepté le 20 juillet 2021.

Keywords: Kashiwara–Verge Lie algebra, dihedral Lie algebra, mould theory.

2020 Mathematics Subject Classification: 17B05, 11M32, 16S30, 17B40.

⁽¹⁾ Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, 464-8602, Japan — furusho@math.nagoya-u.ac.jp

⁽²⁾ Department of Mathematics, Graduate School of Science, Osaka University Toyonaka, Osaka 560-0043, Japan — komiyama.nao.aww@osaka-u.ac.jp

H.F. and N.K. have been supported by grants JSPS KAKENHI JP18H01110 and JP18J14774 respectively.

Article proposé par Damian Rössler.

A bigraded variant $\mathfrak{f}\mathfrak{r}\mathfrak{v}_{\bullet\bullet}$ of $\mathfrak{f}\mathfrak{r}\mathfrak{v}_{\bullet}$ is introduced in [17] where they give its interpretation in terms of Ecalle’s mould theory ([7, 8, 9]). The results in [14, 17] give an inclusion of bigraded Lie algebras

$$D(\{e\})_{\bullet\bullet} \hookrightarrow \mathfrak{f}\mathfrak{r}\mathfrak{v}_{\bullet\bullet}. \tag{0.1}$$

Our objective of this paper is to extend it to any Γ by exploiting Ecalle’s mould theory with self-contained proofs. Our results are exhibited as follows:

- (1) In Definition 2.1, we introduce the filtered graded \mathbb{Q} -linear space $\mathfrak{f}\mathfrak{r}\mathfrak{v}(\Gamma)_{\bullet}$ which generalizes $\mathfrak{f}\mathfrak{r}\mathfrak{v}_{\bullet}$. In Theorem 2.15, we show that $\mathfrak{f}\mathfrak{r}\mathfrak{v}(\Gamma)_{\bullet}$ is identified with the \mathbb{Q} -linear space of finite polynomial-valued alternal moulds satisfying Ecalle’s senary relation (2.14) and whose swap’s (1.8) are pus-neutral (1.9), that is, there is an isomorphism of \mathbb{Q} -linear spaces

$$\mathfrak{f}\mathfrak{r}\mathfrak{v}(\Gamma)_{\bullet} \simeq \text{ARI}(\Gamma)_{\text{sena/pusnu}} \cap \text{ARI}(\Gamma)_{\text{al}}^{\text{fin,pol}}.$$

- (2) In Definition 2.17, we introduce a bigraded \mathbb{Q} -linear space $\mathfrak{f}\mathfrak{r}\mathfrak{v}(\Gamma)_{\bullet\bullet}$ which is defined by “the leading terms” of the defining equations of $\mathfrak{f}\mathfrak{r}\mathfrak{v}(\Gamma)_{\bullet}$. We also consider its subspace $\mathfrak{f}\mathfrak{r}\mathfrak{v}\mathfrak{d}(\Gamma)_{\bullet\bullet}$ by imposing the distribution relation in Definition 2.25. Both of them recover $\mathfrak{f}\mathfrak{r}\mathfrak{v}_{\bullet\bullet}$ when $\Gamma = \{e\}$. It is shown that they form Lie algebras in Theorem 2.23 and Corollary 2.26. An inclusion of bigraded \mathbb{Q} -linear spaces

$$\text{gr}_{\mathcal{D}}\mathfrak{f}\mathfrak{r}\mathfrak{v}(\Gamma)_{\bullet} \hookrightarrow \mathfrak{f}\mathfrak{r}\mathfrak{v}(\Gamma)_{\bullet\bullet}$$

is presented in (2.31), where the first term means the associated bigraded of the filtered graded linear space $\mathfrak{f}\mathfrak{r}\mathfrak{v}(\Gamma)_{\bullet}$.

- (3) In Theorem 2.22, we show that $\mathfrak{f}\mathfrak{r}\mathfrak{v}(\Gamma)_{\bullet\bullet}$ is identified with the Lie algebra (cf. Theorem 1.32) of finite polynomial-valued alternal moulds which are push-invariant (1.5) and whose swap’s are pus-neutral (1.9), that is, there is an isomorphism of Lie algebras

$$\mathfrak{f}\mathfrak{r}\mathfrak{v}(\Gamma)_{\bullet\bullet} \simeq \text{ARI}(\Gamma)_{\text{push/pusnu}} \cap \text{ARI}(\Gamma)_{\text{al}}^{\text{fin,pol}}.$$

- (4) In Section 3, we consider Goncharov’s dihedral bigraded Lie algebra $D(\Gamma)_{\bullet\bullet}$ and its related Lie algebra $\mathbb{D}(\Gamma)_{\bullet\bullet}$ with the dihedral symmetry which contains $D(\Gamma)_{\bullet\bullet}$. It is explained in Theorem 3.5 that its depth $>$ 1-part of $\mathbb{D}(\Gamma)_{\bullet\bullet}$ coincides with the depth $>$ 1-part of the Lie algebra of finite polynomial-valued part of the set of moulds $\text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$ (cf. Definition 1.22), namely

$$\text{Fil}_{\mathcal{D}}^2\mathbb{D}(\Gamma)_{\bullet\bullet} \simeq \text{Fil}_{\mathcal{D}}^2\text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}^{\text{fin,pol}}.$$

In Theorem 3.13, we show that there is an inclusion of graded Lie algebras

$$\text{Fil}_{\mathcal{D}}^2\text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}} \hookrightarrow \text{ARI}(\Gamma)_{\text{push/pusnu}}.$$

In Corollary 3.14, by taking an intersection with $\text{ARI}(\Gamma)_{\text{al}}^{\text{fin, pol}}$ we obtain the inclusion of bigraded Lie algebras from the depth > 1 part $\text{Fil}_{\mathcal{D}}^2 \mathbb{D}(\Gamma)_{\bullet\bullet}$ of $\mathbb{D}(\Gamma)_{\bullet\bullet}$ to $\mathfrak{ltrv}(\Gamma)_{\bullet\bullet}$

$$\text{Fil}_{\mathcal{D}}^2 \mathbb{D}(\Gamma)_{\bullet\bullet} \hookrightarrow \mathfrak{ltrv}(\Gamma)_{\bullet\bullet},$$

which extends (0.1). By imposing the distribution relation there the inclusion $\text{Fil}_{\mathcal{D}}^2 D(\Gamma)_{\bullet\bullet} \hookrightarrow \mathfrak{ltrv} \mathfrak{d}(\Gamma)_{\bullet\bullet}$ is similarly obtained in Corollary 3.15.

In Appendix A, we give self-contained proofs of several fundamental properties on the ari-bracket of the Lie algebra $\text{ARI}(\Gamma)$ of moulds associated with Γ . In Appendix B, we discuss moulds arising from the multiple polylogarithms evaluated at roots of unity.

Acknowledgements

We thank for L. Schneps who gave comments on the first version of the paper and informing us [17]. We are grateful to the referee whose comments helped us to improve the paper in a better form.

1. Preparation on mould theory

We prepare several techniques of moulds which will be employed in our later sections. The notion of moulds, the alternality, flexions and the ari-bracket associated with a finite abelian group Γ are explained in Sections 1.1 and 1.2. In Section 1.3, we explain that the set $\text{ARI}(\Gamma)_{\text{al}/\text{al}}$ of bialternal moulds forms a Lie algebra under the ari-bracket (whose self-contained proof is given in Appendix A). In Section 1.4, we introduce the set $\text{ARI}_{\text{push}/\text{pusnu}}(\Gamma)$ of push-invariant and pus-neutral moulds and show that it forms a Lie algebra under the ari-bracket in Theorem 1.32.

1.1. Moulds and alternality

We introduce and discuss moulds associated with a finite abelian group Γ .

The notion of moulds was invented by Ecalle (cf. [7, Tome I. pp. 12-13]). For our convenience we employ the following formulation influenced by [21] which is different from the one employed in [5, Définition 1 or Définition II.1] and [19, §4.1].

Let Γ be a finite abelian group. We set $\mathcal{F} := \bigcup_{m \geq 1} \mathbb{Q}(x_1, \dots, x_m)$.

DEFINITION 1.1. — *A mould on $\mathbb{Z}_{\geq 0}$ with values in \mathcal{F} and indexed by Γ in a lower layer is a collection*

$$M = (M^m(\begin{smallmatrix} x_1, \dots, x_m \\ \sigma_1, \dots, \sigma_m \end{smallmatrix}))_{m \in \mathbb{Z}_{\geq 0}, \sigma_i \in \Gamma}$$

with $M^m(\begin{smallmatrix} x_1, \dots, x_m \\ \sigma_1, \dots, \sigma_m \end{smallmatrix}) \in \mathbb{Q}(x_1, \dots, x_m)$ for $m \geq 0$.⁽¹⁾ We denote the set of all such moulds with values in \mathcal{F} by $\mathcal{M}(\mathcal{F}; \Gamma)$. The set $\mathcal{M}(\mathcal{F}; \Gamma)$ forms a \mathbb{Q} -linear space by

$$\begin{aligned} A + B &:= (A^m(\begin{smallmatrix} x_1, \dots, x_m \\ \sigma_1, \dots, \sigma_m \end{smallmatrix}) + B^m(\begin{smallmatrix} x_1, \dots, x_m \\ \sigma_1, \dots, \sigma_m \end{smallmatrix}))_{m \in \mathbb{Z}_{\geq 0}, \sigma_i \in \Gamma}, \\ cA &:= (cA^m(\begin{smallmatrix} x_1, \dots, x_m \\ \sigma_1, \dots, \sigma_m \end{smallmatrix}))_{m \in \mathbb{Z}_{\geq 0}, \sigma_i \in \Gamma}, \end{aligned}$$

for $A, B \in \mathcal{M}(\mathcal{F}; \Gamma)$ and $c \in \mathbb{Q}$, namely the addition and the scalar are taken componentwise. We define a product on $\mathcal{M}(\mathcal{F}; \Gamma)$ by

$$(A \times B)^m(\begin{smallmatrix} x_1, \dots, x_m \\ \sigma_1, \dots, \sigma_m \end{smallmatrix}) := \sum_{i=0}^m A^i(\begin{smallmatrix} x_1, \dots, x_i \\ \sigma_1, \dots, \sigma_i \end{smallmatrix}) B^{m-i}(\begin{smallmatrix} x_{i+1}, \dots, x_m \\ \sigma_{i+1}, \dots, \sigma_m \end{smallmatrix}),$$

for $A, B \in \mathcal{M}(\mathcal{F}; \Gamma)$ and for $m \geq 0$ and for $(\sigma_1, \dots, \sigma_m) \in \Gamma^{\oplus m}$. Then the pair $(\mathcal{M}(\mathcal{F}; \Gamma), \times)$ is a non-commutative, associative, unital \mathbb{Q} -algebra. Here, the unit $I \in \mathcal{M}(\mathcal{F}; \Gamma)$ is given by $I := (1, 0, 0, \dots)$.

By the regular action of Γ on $\mathbb{Q}[\Gamma]$, $\mathcal{M}(\mathcal{F}; \Gamma)$ admits the action of Γ . It is described by

$$(\gamma M)^m(\begin{smallmatrix} x_1, \dots, x_m \\ \sigma_1, \dots, \sigma_m \end{smallmatrix}) = M^m\left(\begin{smallmatrix} x_1, \dots, x_m \\ \gamma^{-1}\sigma_1, \dots, \gamma^{-1}\sigma_m \end{smallmatrix}\right)$$

for $\gamma \in \Gamma$.

The set $\mathcal{M}(\mathcal{F}; \Gamma)$ is encoded with the depth filtration $\{\text{Fil}_D^m \mathcal{M}(\mathcal{F}; \Gamma)\}_{m \geq 0}$ where $\text{Fil}_D^m \mathcal{M}(\mathcal{F}; \Gamma)$ is the collection of moulds with $M^r(\begin{smallmatrix} x_1, \dots, x_r \\ \sigma_1, \dots, \sigma_r \end{smallmatrix}) = 0$ for $r < m$. It is clear that the algebra structure of $\mathcal{M}(\mathcal{F}; \Gamma)$ is compatible with the depth filtration. Put

$$\text{ARI}(\Gamma) := \{M \in \mathcal{M}(\mathcal{F}; \Gamma) \mid M^0(\emptyset) = 0\}.$$

It is a filtered (non-unital) subalgebra.

DEFINITION 1.2. — *A mould $M \in \mathcal{M}(\mathcal{F}; \Gamma)$ is called finite when $M^m(\begin{smallmatrix} x_1, \dots, x_m \\ \sigma_1, \dots, \sigma_m \end{smallmatrix}) = 0$ except for finitely many m . It is called polynomial-valued when $M^m(\begin{smallmatrix} x_1, \dots, x_m \\ \sigma_1, \dots, \sigma_m \end{smallmatrix}) \in \mathbb{Q}[x_1, \dots, x_m]$ for all $(\sigma_1, \dots, \sigma_m) \in \Gamma^{\oplus m}$ and m . We denote $\mathcal{M}(\mathcal{F}; \Gamma)^{\text{fin, pol}}$ (resp. $\text{ARI}(\Gamma)^{\text{fin, pol}}$) to be the subset of all finite polynomial-valued moulds in $\mathcal{M}(\mathcal{F}; \Gamma)$ (resp. $\text{ARI}(\Gamma)$).*

⁽¹⁾ In this case, we have $M^0(\emptyset) \in \mathbb{Q}$ for $m = 0$.

We prepare the following algebraic formulation which is useful to present the notion of the alternality of mould: Put $X := \left\{ \binom{x_i}{\sigma} \right\}_{i \in \mathbb{N}, \sigma \in \Gamma}$. Let $X_{\mathbb{Z}}$ be the set such that

$$X_{\mathbb{Z}} := \left\{ \binom{u}{\sigma} \mid u = a_1 x_1 + \cdots + a_k x_k, k \in \mathbb{N}, a_j \in \mathbb{Z}, \sigma \in \Gamma \right\},$$

and let $X_{\mathbb{Z}}^{\bullet}$ be the non-commutative free monoid generated by all elements of $X_{\mathbb{Z}}$ with the empty word \emptyset as the unit. Occasionally we denote each element $\omega = u_1 \dots u_m \in X_{\mathbb{Z}}^{\bullet}$ with $u_1, \dots, u_m \in X_{\mathbb{Z}}$ by $\omega = (u_1, \dots, u_m)$ as a sequence. The *length* of $\omega = u_1 \dots u_m$ is defined to be $l(\omega) := m$.

For our simplicity we occasionally denote $M \in \mathcal{M}(\mathcal{F}; \Gamma)$ by

$$M = (M^m(\mathbf{x}_m))_{m \in \mathbb{Z}_{\geq 0}} \quad \text{or} \quad M = (M(\mathbf{x}_m))_{m \in \mathbb{Z}_{\geq 0}},$$

where $\mathbf{x}_0 := \emptyset$ and $\mathbf{x}_m := \left(\binom{x_1}{\sigma_1}, \dots, \binom{x_m}{\sigma_m} \right)$ for $m \geq 1$. Under the notations, the product of $A, B \in \text{ARI}(\Gamma)$ is expressed as

$$A \times B = \left(\sum_{\mathbf{x}_m = \alpha\beta} A^{l(\alpha)}(\alpha) B^{l(\beta)}(\beta) \right)_{m \in \mathbb{Z}_{\geq 0}}$$

where α and β run over $X_{\mathbb{Z}}^{\bullet}$.

We set $\mathcal{A}_X := \mathbb{Q}\langle X_{\mathbb{Z}} \rangle$ to be the non-commutative polynomial algebra generated by $X_{\mathbb{Z}}$ (i.e. \mathcal{A}_X is the \mathbb{Q} -linear space generated by $X_{\mathbb{Z}}^{\bullet}$).⁽²⁾ We equip \mathcal{A}_X with a product $\text{III} : \mathcal{A}_X^{\otimes 2} \rightarrow \mathcal{A}_X$ (called the *shuffle product*) which is linearly defined by $\emptyset \text{ III } \omega := \omega \text{ III } \emptyset := w$ and

$$u\omega \text{ III } v\eta := u(\omega \text{ III } v\eta) + v(u\omega \text{ III } \eta), \tag{1.1}$$

for $u, v \in X_{\mathbb{Z}}$ and $\omega, \eta \in X_{\mathbb{Z}}^{\bullet}$. Then the pair $(\mathcal{A}_X, \text{III})$ forms a commutative, associative, unital \mathbb{Q} -algebra.

We define the family $\left\{ \text{Sh} \binom{\omega; \eta}{\alpha} \right\}_{\omega, \eta, \alpha \in X_{\mathbb{Z}}^{\bullet}}$ in \mathbb{Z} by

$$\omega \text{ III } \eta = \sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \binom{\omega; \eta}{\alpha} \alpha.$$

Particularly for $p, q \in \mathbb{N}$ and $u_1, \dots, u_{p+q} \in X_{\mathbb{Z}}$, we rewrite the shuffle product by

$$\begin{aligned} (u_1, \dots, u_p) \text{ III } (u_{p+1}, \dots, u_{p+q}) \\ = \sum_{\sigma \in \text{III}_{p,q}} (u_{\sigma(1)}, \dots, u_{\sigma(p)}, u_{\sigma(p+1)}, \dots, u_{\sigma(p+q)}). \end{aligned}$$

Here the set $\text{III}_{p,q}$ is defined by

$$\{ \sigma \in S_{p+q} \mid \sigma^{-1}(1) < \cdots < \sigma^{-1}(p), \sigma^{-1}(p+1) < \cdots < \sigma^{-1}(p+q) \}, \tag{1.2}$$

⁽²⁾ We should beware of the inequality $\binom{x_1+x_2}{\sigma} \neq \binom{x_1}{\sigma} + \binom{x_2}{\sigma}$ and $\binom{0}{\sigma} \neq 0$.

where S_{p+q} is the symmetry group with degree $p + q$.

DEFINITION 1.3. — *A mould $M \in \text{ARI}(\Gamma)$ is called alternal (cf. [7, I-p.118]) if*

$$\sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\binom{x_1, \dots, x_p}{\sigma_1, \dots, \sigma_p}; \binom{x_{p+1}, \dots, x_{p+q}}{\sigma_{p+1}, \dots, \sigma_{p+q}} \right) M^{p+q}(\alpha) = 0, \quad (1.3)$$

for all $p, q \geq 1$. The \mathbb{Q} -linear space $\text{ARI}(\Gamma)_{\text{al}}$ is defined to be the subset of moulds $M \in \text{ARI}(\Gamma)$ which are alternal (cf. [8, 9]).

We encode it with the induced depth filtrations. We exhibit a couple of examples of alternal moulds for $\Gamma = \{e\}$ below.

Example 1.4. —

(a) For $f(x) \in \mathbb{Q}(x)$, we define $M_f \in \text{ARI}(\Gamma)$ by

$$M_f(\mathbf{x}_m) := \begin{cases} f(x_1) & m = 1, \\ 0 & m \neq 1, \end{cases}$$

that is,

$$M_f = (0, f(x_1), 0, 0, \dots).$$

(b) We define $A \in \text{ARI}(\Gamma)$ by⁽³⁾

$$A(\mathbf{x}_m) := \begin{cases} 0 & m = 0, 1, \\ \frac{1}{x_2 - x_1} \cdots \frac{1}{x_m - x_{m-1}} & m \geq 2, \end{cases}$$

that is,

$$A = \left(0, 0, \frac{1}{x_2 - x_1}, \frac{1}{(x_2 - x_1)(x_3 - x_2)}, \frac{1}{(x_2 - x_1)(x_3 - x_2)(x_4 - x_3)}, \dots \right).$$

Proof of their alternalities. —

(a). — By definition, we have $M_f(\emptyset) = 0$. Put $\omega, \eta \in X^{\bullet}$ with $l(\omega), l(\eta) \geq 1$. Note that for $\alpha \in X^{\bullet}$ with $\text{Sh} \binom{\omega; \eta}{\alpha} \neq 0$, we have $l(\alpha) = l(\omega) + l(\eta) \geq 2$ and we get

$$M_f(\alpha) = 0.$$

Therefore, we obtain

$$\sum_{\alpha \in X^{\bullet}} \text{Sh} \binom{\omega; \eta}{\alpha} M_f(\alpha) = \sum_{\substack{\alpha \in X^{\bullet} \\ \text{Sh} \binom{\omega; \eta}{\alpha} \neq 0}} \text{Sh} \binom{\omega; \eta}{\alpha} M_f(\alpha) = 0.$$

Hence, M_f is an alternal mould.

⁽³⁾ This mould is presented in [7, p. 124] and [5, (II.64)].

(b). — The proof is given in [5, Lemme II.5], but we prove the alternality of A by using the following lemma:

LEMMA 1.5. — For $r \geq 2$ and $s \geq 0$, we have

$$\sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\begin{matrix} (\omega_1, \dots, \omega_{r-1}); (\omega_{r+1}, \dots, \omega_{r+s}) \\ \alpha \end{matrix} \right) A(\alpha, \omega_r) \\ = (-1)^s A(\omega_1, \dots, \omega_{r-1}, \omega_r, \omega_{r+s}, \dots, \omega_{r+1}),$$

where $\omega_i := \left(\begin{smallmatrix} x_i \\ e \end{smallmatrix} \right)$ for $i \in \mathbb{N}$.

Proof. — When $s = 0$, the claim is clear. Assume $s \geq 1$. We prove by induction on $r + s$. By the definition of the mould A , we easily see the case of $(r, s) = (2, 1)$. For $r \geq 2$ and $s \geq 1$, we have

$$\sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\begin{matrix} (\omega_1, \dots, \omega_{r-1}); (\omega_{r+1}, \dots, \omega_{r+s}) \\ \alpha \end{matrix} \right) A(\alpha, \omega_r) \\ = \sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \left\{ \text{Sh} \left(\begin{matrix} (\omega_1, \dots, \omega_{r-2}); (\omega_{r+1}, \dots, \omega_{r+s}) \\ \alpha \end{matrix} \right) A(\alpha, \omega_{r-1}, \omega_r) \right. \\ \left. + \text{Sh} \left(\begin{matrix} (\omega_1, \dots, \omega_{r-1}); (\omega_{r+1}, \dots, \omega_{r+s-1}) \\ \alpha \end{matrix} \right) A(\alpha, \omega_{r+s}, \omega_r) \right\}.$$

Because $A(\omega_1, \dots, \omega_{r-1}, \omega_r) = \frac{1}{x_r - x_{r-1}} A(\omega_1, \dots, \omega_{r-1})$ for $r \geq 3$, we get

$$= \frac{1}{x_r - x_{r-1}} \sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\begin{matrix} (\omega_1, \dots, \omega_{r-2}); (\omega_{r+1}, \dots, \omega_{r+s}) \\ \alpha \end{matrix} \right) A(\alpha, \omega_{r-1}) \\ + \frac{1}{x_r - x_{r+s}} \sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\begin{matrix} (\omega_1, \dots, \omega_{r-1}); (\omega_{r+1}, \dots, \omega_{r+s-1}) \\ \alpha \end{matrix} \right) A(\alpha, \omega_{r+s}).$$

By our induction hypothesis, we calculate

$$= (-1)^s \frac{1}{x_r - x_{r-1}} A(\omega_1, \dots, \omega_{r-2}, \omega_{r-1}, \omega_{r+s}, \dots, \omega_{r+1}) \\ + (-1)^{s-1} \frac{1}{x_r - x_{r+s}} A(\omega_1, \dots, \omega_{r-1}, \omega_{r+s}, \omega_{r+s-1}, \dots, \omega_{r+1}) \\ = (-1)^s \left\{ \frac{1}{x_r - x_{r-1}} - \frac{1}{x_r - x_{r+s}} \right\} A(\omega_1, \dots, \omega_{r-2}, \omega_{r-1}, \omega_{r+s}, \dots, \omega_{r+1}) \\ = (-1)^s \frac{x_{r-1} - x_{r+s}}{(x_r - x_{r-1})(x_r - x_{r+s})} A(\omega_1, \dots, \omega_{r-2}, \omega_{r-1}, \omega_{r+s}, \dots, \omega_{r+1}) \\ = (-1)^s \frac{x_{r-1} - x_{r+s}}{(x_r - x_{r-1})(x_r - x_{r+s})} A(\omega_1, \dots, \omega_{r-1}) \frac{1}{x_{r+s} - x_{r-1}} \frac{1}{x_{r+s-1} - x_{r+s}} \\ \times \cdots \times \frac{1}{x_{r+1} - x_{r+2}}$$

$$\begin{aligned}
 &= (-1)^s A(\omega_1, \dots, \omega_{r-1}) \frac{1}{x_r - x_{r-1}} \frac{1}{x_{r+s} - x_r} \frac{1}{x_{r+s-1} - x_{r+s}} \dots \frac{1}{x_{r+1} - x_{r+2}} \\
 &= (-1)^s A(\omega_1, \dots, \omega_{r-1}, \omega_r, \omega_{r+s}, \dots, \omega_{r+1}).
 \end{aligned}$$

Hence, we obtain the claim. □

By using the above lemma, we prove the alternality of A . By the definition, we have $A(\emptyset) = 0$, so it is sufficient to prove

$$\sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\begin{matrix} (\omega_1, \dots, \omega_r); (\omega_{r+1}, \dots, \omega_{r+s}) \\ \alpha \end{matrix} \right) A(\alpha) = 0, \tag{1.4}$$

for $r, s \geq 1$. Assume $r \geq s$ without loss of generality. When $r = s = 1$, the left hand side of (1.4) is equal to

$$A(\omega_1, \omega_2) + A(\omega_2, \omega_1) = \frac{1}{x_2 - x_1} + \frac{1}{x_1 - x_2} = 0.$$

When $r \geq 2$ and $s \geq 1$, the left hand side of (1.4) is equal to

$$\begin{aligned}
 &\sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\begin{matrix} (\omega_1, \dots, \omega_{r-1}); (\omega_{r+1}, \dots, \omega_{r+s}) \\ \alpha \end{matrix} \right) A(\alpha, \omega_r) \\
 &\quad + \sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\begin{matrix} (\omega_1, \dots, \omega_r); (\omega_{r+1}, \dots, \omega_{r+s-1}) \\ \alpha \end{matrix} \right) A(\alpha, \omega_{r+s}).
 \end{aligned}$$

By using Lemma 1.5, we get

$$\begin{aligned}
 &= (-1)^s A(\omega_1, \dots, \omega_{r-1}, \omega_r, \omega_{r+s}, \dots, \omega_{r+1}) \\
 &\quad + (-1)^{s-1} A(\omega_1, \dots, \omega_r, \omega_{r+s}, \omega_{r+s-1}, \dots, \omega_{r+1}) \\
 &= 0.
 \end{aligned}$$

Hence, the mould A is alternal. □

Remark 1.6. — Assume that $u_1, \dots, u_m \in \mathcal{F}$ are algebraically independent over \mathbb{Q} . For $M \in \text{ARI}(\Gamma)$ we denote $M^m(\begin{smallmatrix} u_1, \dots, u_m \\ \sigma_1, \dots, \sigma_m \end{smallmatrix})$ to be the image of $M^m(\begin{smallmatrix} x_1, \dots, x_m \\ \sigma_1, \dots, \sigma_m \end{smallmatrix})$ under the field embedding $\mathbb{Q}(x_1, \dots, x_m) \hookrightarrow \mathcal{F}$ sending $x_i \mapsto u_i$.

For our later use, we prepare more notations:

Notation 1.7 ([9, §2.1]). — For any mould $M = (M^m(\frac{u_1}{\sigma_1}, \dots, \frac{u_m}{\sigma_m}))_m \in \text{ARI}(\Gamma)$, we define

$$\begin{aligned} \text{mantar}(M)^m(\frac{u_1}{\sigma_1}, \dots, \frac{u_m}{\sigma_m}) &= (-1)^{m-1} M^m(\frac{u_m}{\sigma_m}, \dots, \frac{u_1}{\sigma_1}), \\ \text{push}(M)^m(\frac{u_1}{\sigma_1}, \dots, \frac{u_m}{\sigma_m}) &= M^m\left(\frac{-u_1 - \dots - u_m}{\sigma_m^{-1}}, \frac{u_1}{\sigma_1 \sigma_m^{-1}}, \dots, \frac{u_{m-1}}{\sigma_{m-1} \sigma_m^{-1}}\right), \\ \text{neg}(M)^m(\frac{u_1}{\sigma_1}, \dots, \frac{u_m}{\sigma_m}) &= M^m\left(\frac{-u_1}{\sigma_1^{-1}}, \dots, \frac{-u_m}{\sigma_m^{-1}}\right), \\ \text{teru}(M)^m(\frac{u_1}{\sigma_1}, \dots, \frac{u_m}{\sigma_m}) &= M^m\left(\frac{u_1}{\sigma_1}, \dots, \frac{u_m}{\sigma_m}\right) \\ &\quad + \frac{1}{u_m} \left\{ M^{m-1}\left(\frac{u_1}{\sigma_1}, \dots, \frac{u_{m-2}}{\sigma_{m-2}}, \frac{u_{m-1}+u_m}{\sigma_{m-1}}\right) - M^{m-1}\left(\frac{u_1}{\sigma_1}, \dots, \frac{u_{m-1}}{\sigma_{m-1}}\right) \right\}. \end{aligned}$$

Note that they are all \mathbb{Q} -linear endomorphisms on $\text{ARI}(\Gamma)$. We remark that $\text{neg} \circ \text{neg} = \text{id}$ and $\text{mantar} \circ \text{mantar} = \text{id}$.

DEFINITION 1.8. — We call a mould $M \in \text{ARI}(\Gamma)$ push-invariant when we have

$$\text{push}(M) = M. \tag{1.5}$$

We define $\text{ARI}(\Gamma)_{\text{push}}$ ([9, §2.5]) to be the set of moulds M in $\text{ARI}(\Gamma)$ which is push-invariant (1.5).

Example 1.9. — By direct computations we observe that the mould $P \in \text{ARI}(\{e\})$ defined by

$$P(\mathbf{x}_m) := \begin{cases} 0 & m = 0, 1, \\ \frac{1}{x_1} + \dots + \frac{1}{x_m} - \frac{1}{x_1 + \dots + x_m} & m \geq 2 \end{cases}$$

is push-invariant, but is not alternal.

1.2. Flexions and ari-bracket

We explain an ari-bracket ari_u on $\text{ARI}(\Gamma)$ by using flexions.

The notion of flexions is introduced by Ecalle in [9, §2.1] for bimoulds (cf. [21, §2.2]). Here we consider those for moulds in $\text{ARI}(\Gamma)$.

DEFINITION 1.10. — *The flexions are the four binary operators $_*[\ast, \ast]_*$, $_*[\ast, \ast]_* : X_{\mathbb{Z}}^\bullet \times X_{\mathbb{Z}}^\bullet \rightarrow X_{\mathbb{Z}}^\bullet$ which are defined by*

$$\begin{aligned} {}_\beta[\alpha] &:= (b_1 + \dots + b_n + a_1, \frac{a_2}{\sigma_1}, \dots, \frac{a_m}{\sigma_m}), \\ \alpha]_\beta &:= (\frac{a_1}{\sigma_1}, \dots, \frac{a_{m-1}}{\sigma_{m-1}}, \frac{a_m + b_1 + \dots + b_n}{\sigma_m}), \\ {}_\beta[\alpha] &:= (\frac{a_1}{\tau_n^{-1}\sigma_1}, \dots, \frac{a_m}{\tau_n^{-1}\sigma_m}), \\ \alpha]_\beta &:= (\frac{a_1}{\sigma_1\tau_1^{-1}}, \dots, \frac{a_m}{\sigma_m\tau_1^{-1}}), \\ {}_\emptyset[\gamma := \gamma]_\emptyset &:= {}_\emptyset[\gamma := \gamma]_\emptyset := \gamma, \\ {}_\gamma[\emptyset := \emptyset]_\gamma &:= {}_\gamma[\emptyset := \emptyset]_\gamma := \emptyset, \end{aligned}$$

for $\alpha = (\frac{a_1, \dots, a_m}{\sigma_1, \dots, \sigma_m})$, $\beta = (\frac{b_1, \dots, b_n}{\tau_1, \dots, \tau_n}) \in X_{\mathbb{Z}}^\bullet$ ($m, n \geq 1$) and $\gamma \in X_{\mathbb{Z}}^\bullet$.

Note that we have $l({}_\beta[\alpha]) = l(\alpha]_\beta) = l({}_\beta[\alpha) = l(\alpha]_\beta) = l(\alpha)$ and $l(\alpha, \beta) = l(\alpha) + l(\beta)$ for $\alpha, \beta \in X_{\mathbb{Z}}^\bullet$.

The derivation *arit* and bracket *ari* are introduced for bimoulds in terms of flexions in [9, §2.2] (cf. [21, §2.2]) and here we consider those for $\text{ARI}(\Gamma)$ as follows.

DEFINITION 1.11. — *Let $B \in \text{ARI}(\Gamma)$. The linear map⁽⁴⁾ $\text{arit}_u(B) : \text{ARI}(\Gamma) \rightarrow \text{ARI}(\Gamma)$ is defined by⁽⁵⁾*

$$\begin{aligned} (\text{arit}_u(B)(A))^m(\mathbf{x}_m) &= (\text{arit}_u(B)(A))^m(\frac{x_1}{\sigma_1}, \dots, \frac{x_m}{\sigma_m}) \\ &:= \begin{cases} \sum_{\substack{\mathbf{x}_m = \alpha\beta\gamma \\ \beta, \gamma \neq \emptyset}} A^{l(\alpha, \gamma)}(\alpha]_\beta[\gamma)B^{l(\beta)}(\beta]_\gamma) - \sum_{\substack{\mathbf{x}_m = \alpha\beta\gamma \\ \alpha, \beta \neq \emptyset}} A^{l(\alpha, \gamma)}(\alpha]_\beta\gamma)B^{l(\beta)}({}_\alpha[\beta) & (m \geq 2), \\ 0 & (m = 0, 1), \end{cases} \end{aligned}$$

for $A \in \text{ARI}(\Gamma)$.

It is shown in [21, Appendix A.1] that *arit* forms a derivation on the set of bimoulds. The same holds for $\text{ARI}(\Gamma)$ as follows, where we present an alternative proof to hers.

LEMMA 1.12. — *For any $A \in \text{ARI}(\Gamma)$, $\text{arit}_u(A)$ forms a derivation of $\text{ARI}(\Gamma)$ with respect to the product \times , that is, for any $B, C \in \text{ARI}(\Gamma)$, we have*

$$\text{arit}_u(A)(B \times C) = \text{arit}_u(A)(B) \times C + B \times \text{arit}_u(A)(C).$$

⁽⁴⁾ The lower suffix *u* is reflected by the notion of *u*-moulds in [21].

⁽⁵⁾ For $m \geq 2$ and $\alpha, \beta, \gamma \in X_{\mathbb{Z}}^\bullet$ with $\mathbf{x}_m = \alpha\beta\gamma$, all letters appearing in the word $\alpha]_\beta[\gamma$ (resp. ${}_\beta[\gamma$) are algebraically independent over \mathbb{Q} . So by Remark 1.6, the component $A^{l(\alpha, \gamma)}(\alpha]_\beta[\gamma)$ (resp. $B^{l(\beta)}(\beta]_\gamma)$) is well-defined. Similarly, $A^{l(\alpha, \gamma)}(\alpha]_\beta\gamma)$ and $B^{l(\beta)}({}_\alpha[\beta)$ are also well-defined.

Proof. — Let $m \geq 0$. We have

$$\begin{aligned} & (\text{arit}(A)(B \times C))(\mathbf{x}_m) \\ &= \sum_{\substack{\mathbf{x}_m=abc \\ b,c \neq \emptyset}} (B \times C)(a_b \lceil c) A(b \rfloor_c) - \sum_{\substack{\mathbf{x}_m=abc \\ a,b \neq \emptyset}} (B \times C)(a \lceil_b c) A(a \rfloor_b) \\ &= \sum_{\substack{\mathbf{x}_m=abc \\ b,c \neq \emptyset}} \left\{ \sum_{a_b \lceil c=de} B(d) C(e) \right\} A(b \rfloor_c) - \sum_{\substack{\mathbf{x}_m=abc \\ a,b \neq \emptyset}} \left\{ \sum_{a \lceil_b c=ef} B(e) C(f) \right\} A(a \rfloor_b) \end{aligned}$$

By applying Lemma A.2(1) for $c \neq \emptyset, f = \emptyset$ to the first term and by applying Lemma A.2(2) for $a \neq \emptyset, d = \emptyset$ to the second term, we have

$$\begin{aligned} &= \sum_{\substack{\mathbf{x}_m=abc \\ b,c \neq \emptyset}} \left\{ \sum_{a=a_1 a_2} B(a_1) C(a_{2_b} \lceil c) + \sum_{\substack{c=c_1 c_2 \\ c_1 \neq \emptyset}} B(a_b \lceil_{c_1}) C(c_2) \right\} A(b \rfloor_c) \\ &\quad - \sum_{\substack{\mathbf{x}_m=abc \\ a,b \neq \emptyset}} \left\{ \sum_{\substack{a=a_1 a_2 \\ a_2 \neq \emptyset}} B(a_1) C(a_2 \rfloor_b c) + \sum_{c=c_1 c_2} B(a \rfloor_b c_1) C(c_2) \right\} A(a \rfloor_b) \\ &= \sum_{\substack{\mathbf{x}_m=a_1 a_2 bc \\ b,c \neq \emptyset}} B(a_1) C(a_{2_b} \lceil c) A(b \rfloor_c) + \sum_{\substack{\mathbf{x}_m=abc_1 c_2 \\ b,c_1 \neq \emptyset}} B(a_b \lceil_{c_1}) C(c_2) A(b \rfloor_{c_1 c_2}) \\ &\quad - \sum_{\substack{\mathbf{x}_m=a_1 a_2 bc \\ a_2, b \neq \emptyset}} B(a_1) C(a_2 \rfloor_b c) A(a_{1 a_2} \lceil b) - \sum_{\substack{\mathbf{x}_m=abc_1 c_2 \\ a,b \neq \emptyset}} B(a \rfloor_b c_1) C(c_2) A(a \rfloor_b). \end{aligned}$$

By applying Lemma A.1 to the second term, we get $A(b \rfloor_{c_1 c_2}) = A(b \rfloor_{c_1})$. Similarly, for the third term, we get $A(a_{1 a_2} \lceil b) = A(a_{a_2} \lceil b)$. Therefore, we calculate

$$\begin{aligned} &= \sum_{\mathbf{x}_m=dc_2} \left\{ \sum_{\substack{d=abc_1 \\ b,c_1 \neq \emptyset}} B(a_b \lceil_{c_1}) A(b \rfloor_{c_1}) - \sum_{\substack{d=abc_1 \\ a,b \neq \emptyset}} B(a \rfloor_b c_1) A(a \rfloor_b) \right\} C(c_2) \\ &\quad + \sum_{\mathbf{x}_m=a_1 d} B(a_1) \left\{ \sum_{\substack{d=a_2 bc \\ b,c \neq \emptyset}} C(a_{2_b} \lceil c) A(b \rfloor_c) - \sum_{\substack{d=a_2 bc \\ a_2, b \neq \emptyset}} C(a_2 \rfloor_b c) A(a_{a_2} \lceil b) \right\} \\ &= \sum_{\mathbf{x}_m=dc_2} (\text{arit}(A)(B))(d) C(c_2) + \sum_{\mathbf{w}_n=a_1 d} B(a_1) (\text{arit}(A)(C))(d) \\ &= (\text{arit}(A)(B) \times C)(\mathbf{x}_m) + (B \times \text{arit}(A)(C))(\mathbf{x}_m). \end{aligned}$$

Hence, we obtain the claim. □

We define the following bracket as with the bracket ari introduced in [9, (2.40)].

DEFINITION 1.13. — *The ari_u -bracket means the bilinear map $\text{ari}_u : \text{ARI}(\Gamma)^{\otimes 2} \rightarrow \text{ARI}(\Gamma)$ which is defined by*

$$\text{ari}_u(A, B) := \text{arit}_u(B)(A) - \text{arit}_u(A)(B) + [A, B] \quad (1.6)$$

for $A, B \in \text{ARI}(\Gamma)$. Here⁽⁶⁾, we have $[A, B] := A \times B - B \times A$.

We note that the bracket $\text{ari}_u(A, B)$ in the case when $\Gamma = \{e\}$ also appears in [15, (A.3)] and is denoted by $[A, B]_{\text{ari}}$.

The following is also stated for moulds and bimoulds in [21, Proposition 2.2.2] where her key formula (2.2.10) looks unproven and containing a signature error.

PROPOSITION 1.14. — *The \mathbb{Q} -linear space $\text{ARI}(\Gamma)$ forms a filtered Lie algebra under the ari_u -bracket.*

Proof. — We give a self-contained proof in Appendix A.1. □

The following proposition for $\Gamma = \{e\}$ is shown in [18, Appendix A].

PROPOSITION 1.15. — *The \mathbb{Q} -linear space $\text{ARI}(\Gamma)_{\text{al}}$ forms a filtered Lie subalgebra of $\text{ARI}(\Gamma)$ under the ari_u -bracket.*

Proof. — We prove this in Appendix A.2. □

1.3. Swap and bialternality

We encode $\text{ARI}(\Gamma)$ with another Lie algebra structure introduced by ari_v . We prepare $\overline{\text{ARI}}(\Gamma)$, a copy of $\text{ARI}(\Gamma)$. We denote $M_{\sigma_1, \dots, \sigma_m}^m(x_1, \dots, x_m)$ by $M^m(\frac{\sigma_1}{x_1}, \dots, \frac{\sigma_m}{x_m})$ ⁽⁷⁾ for each element M in $\overline{\text{ARI}}(\Gamma)$ to distinguish it from an element in $\text{ARI}(\Gamma)$.

Similarly to our previous sections, we work over the following algebraic formulation: Put $Y := \{(\frac{\sigma}{y_i})\}_{i \in \mathbb{N}, \sigma \in \Gamma}$. Let $Y_{\mathbb{Z}}$ be the set such that

$$Y_{\mathbb{Z}} := \{(\frac{\sigma}{v}) \mid v = a_1 y_1 + \dots + a_k y_k, k \in \mathbb{N}, a_j \in \mathbb{Z}, \sigma \in \Gamma\},$$

and let $Y_{\mathbb{Z}}^{\bullet}$ be the non-commutative free monoid generated by all elements of $Y_{\mathbb{Z}}$ with the empty word \emptyset as the unit. We set $\mathcal{A}_Y := \mathbb{Q}\langle Y_{\mathbb{Z}} \rangle$ to be the non-commutative polynomial algebra generated by $Y_{\mathbb{Z}}$. In the same way to

⁽⁶⁾ In the papers [9, 17, 21], the product $A \times B$ (resp. the bracket $[A, B]$) is denoted by $mu(A, B)$ (resp. $lu(A, B)$).

⁽⁷⁾ We note that the top and bottom rows are switched.

\mathcal{A}_X , it is equipped with a structure of a commutative, associative, unital \mathbb{Q} -algebra with the shuffle product $\text{III} : \mathcal{A}_Y^{\otimes 2} \rightarrow \mathcal{A}_Y$. The flexions are also introduced in this setting.

DEFINITION 1.16. — *The flexions on $Y_{\mathbb{Z}}^{\bullet}$ are the four binary operators ${}_{\ast}[\ast, \ast]_{\ast}, {}_{\ast}[\ast, \ast]_{\ast} : Y_{\mathbb{Z}}^{\bullet} \times Y_{\mathbb{Z}}^{\bullet} \rightarrow Y_{\mathbb{Z}}^{\bullet}$ which are defined by*

$$\begin{aligned} {}_{\beta}[\alpha] &:= (\sigma_1 \tau_1 \dots \tau_n, \sigma_2, \dots, \sigma_m), \\ \alpha]_{\beta} &:= (\sigma_1, \dots, \sigma_{m-1}, \sigma_m \tau_1 \dots \tau_n), \\ {}_{\beta}[\alpha] &:= (\sigma_1, \dots, \sigma_m), \\ \alpha]_{\beta} &:= (\sigma_1, \dots, \sigma_m), \\ {}_{\emptyset}[\gamma := \gamma]_{\emptyset} &:= {}_{\emptyset}[\gamma := \gamma]_{\emptyset} := \gamma, \\ {}_{\gamma}[\emptyset := \emptyset]_{\gamma} &:= {}_{\gamma}[\emptyset := \emptyset]_{\gamma} := \emptyset, \end{aligned}$$

for $\alpha = (\sigma_1, \dots, \sigma_m), \beta = (\tau_1, \dots, \tau_n) \in Y_{\mathbb{Z}}^{\bullet}$ ($m, n \geq 1$) and $\gamma \in Y_{\mathbb{Z}}^{\bullet}$.

We denote $\mathbf{y}_0 := \emptyset$ and $\mathbf{y}_m := (y_1, \dots, y_m)$ for $m \geq 1$.

DEFINITION 1.17. — *Let $B \in \overline{\text{ARI}}(\Gamma)$. The linear map $\text{arit}_v(B) : \overline{\text{ARI}}(\Gamma) \rightarrow \overline{\text{ARI}}(\Gamma)$ is defined by⁽⁸⁾*

$$\begin{aligned} &(\text{arit}_v(B)(A))^m(\mathbf{y}_m) = (\text{arit}_v(B)(A))^m(y_1, \dots, y_m) \\ &:= \begin{cases} \sum_{\substack{\mathbf{y}_m = \alpha\beta\gamma \\ \beta, \gamma \neq \emptyset}} A^{l(\alpha, \gamma)}(\alpha_{\beta}[\gamma]B^{l(\beta)}(\beta]_{\gamma}) - \sum_{\substack{\mathbf{y}_m = \alpha\beta\gamma \\ \alpha, \beta \neq \emptyset}} A^{l(\alpha, \gamma)}(\alpha]_{\beta}\gamma)B^{l(\beta)}(\alpha]_{\beta}) \\ 0 \end{cases} \begin{matrix} (m \geq 2), \\ (m = 0, 1), \end{matrix} \end{aligned}$$

for $A \in \overline{\text{ARI}}(\Gamma)$.

Similarly to Lemma 1.12, the following holds.

LEMMA 1.18. — *For any $A \in \overline{\text{ARI}}(\Gamma)$, $\text{arit}_v(A)$ forms a derivation of $\overline{\text{ARI}}(\Gamma)$ with respect to the product \times .*

Proof. — The arit_v -bracket can be expressed by the exactly same formula as the arit_u -bracket in terms of flexions. Therefore it can be proved in the same way to the one of Lemma 1.12. \square

⁽⁸⁾ The lower suffix v is reflected by the notion of v -moulds in [21].

DEFINITION 1.19. — *The ari_v -bracket means the bilinear map $\text{ari}_v : \overline{\text{ARI}}(\Gamma)^{\otimes 2} \rightarrow \overline{\text{ARI}}(\Gamma)$ which is defined by⁽⁹⁾*

$$\text{ari}_v(A, B) := \text{arit}_v(B)(A) - \text{arit}_v(A)(B) + [A, B] \tag{1.7}$$

for $A, B \in \overline{\text{ARI}}(\Gamma)$. Here $[A, B] := A \times B - B \times A$.

Similarly to Proposition 1.14, the following holds.

PROPOSITION 1.20. — *The \mathbb{Q} -linear space $\overline{\text{ARI}}(\Gamma)$ forms a filtered Lie algebra under the ari_v -bracket.*

Proof. — It can be also proved in the same way to the one of Proposition 1.14. □

In the same way to Definition 1.3, alternal moulds in $\overline{\text{ARI}}(\Gamma)$ can be introduced and we denote $\overline{\text{ARI}}(\Gamma)_{\text{al}}$ to be its subset consisting of alternal moulds. Similarly to Proposition 1.15, the following holds.

PROPOSITION 1.21. — *$\overline{\text{ARI}}(\Gamma)_{\text{al}}$ forms a Lie algebra under the ari_v -bracket.*

Proof. — It can be proved in the same way to the one of Proposition 1.15. □

We define the \mathbb{Q} -linear map $\text{swap} : \text{ARI}(\Gamma) \rightarrow \overline{\text{ARI}}(\Gamma)$ by

$$\text{swap}(M)^m \left(\begin{smallmatrix} \sigma_1, & \dots, & \sigma_m \\ v_1, & \dots, & v_m \end{smallmatrix} \right) = M^m \left(\begin{smallmatrix} v_m, & v_{m-1}-v_m, & \dots, & v_2-v_3, & v_1-v_2 \\ \sigma_1 \dots \sigma_m, & \sigma_1 \dots \sigma_{m-1}, & \dots, & \sigma_1 \sigma_2, & \sigma_1 \end{smallmatrix} \right) \tag{1.8}$$

for any mould $M = (M^m \left(\begin{smallmatrix} u_1, & \dots, & u_m \\ \sigma_1, & \dots, & \sigma_m \end{smallmatrix} \right)) \in \text{ARI}(\Gamma)$.

DEFINITION 1.22 (cf. [8, 9]). — *The subset $\text{ARI}(\Gamma)_{\text{al}/\text{al}}$ of bialternal moulds is defined to be*

$$\text{ARI}(\Gamma)_{\text{al}/\text{al}} := \left\{ M \in \text{ARI}(\Gamma)_{\text{al}} \mid \text{swap}(M) \in \overline{\text{ARI}}(\Gamma)_{\text{al}}, M^1 \left(\begin{smallmatrix} x_1 \\ \sigma_1 \end{smallmatrix} \right) = M^1 \left(\begin{smallmatrix} -x_1 \\ \sigma_1 \end{smallmatrix} \right) \right\}.$$

Here are observations on Example 1.4.

Example 1.23. —

- (a) When $f(x) \in \mathbb{Q}(x)$ is an even function, the mould M_f in Example 1.4 (a) is in $\text{ARI}(\Gamma)_{\text{al}/\text{al}}$.
- (b) However, the mould A in Example 1.4 (b) is not in $\text{ARI}(\Gamma)_{\text{al}/\text{al}}$. In fact, we have

$$\text{swap}(A)^2 \left(\begin{smallmatrix} e, & e \\ v_1, & v_2 \end{smallmatrix} \right) = A^2 \left(\begin{smallmatrix} v_2, & v_1-v_2 \\ e, & e \end{smallmatrix} \right) = \frac{1}{v_1 - 2v_2},$$

⁽⁹⁾ In [17], the brackets $\text{arit}_v(A, B)$ and $\text{ari}_v(A, B)$ are denoted by $\overline{\text{arit}}$ and $\overline{\text{ari}}$ respectively.

and we get

$$\begin{aligned} \sum_{\alpha \in Y_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\binom{\binom{e}{v_1}; \binom{e}{v_2}}{\alpha} \right) \text{swap}(A)^2(\alpha) &= \text{swap}(A)^2 \binom{e, e}{v_1, v_2} + \text{swap}(A)^2 \binom{e, e}{v_2, v_1} \\ &= \frac{1}{v_1 - 2v_2} + \frac{1}{v_2 - 2v_1} = -\frac{v_1 + v_2}{(v_1 - 2v_2)(v_2 - 2v_1)} \neq 0. \end{aligned}$$

Hence, $\text{swap}(A)$ is not alternal, which says $A \notin \text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$.

PROPOSITION 1.24. — *The \mathbb{Q} -linear space $\text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$ forms a filtered Lie subalgebra of $\text{ARI}(\Gamma)_{\text{al}}$ under the ari_u -bracket.*

Proof. — We prove this in Appendix A.3. □

1.4. Push-invariance and pus-neutrality

We introduce the set $\text{ARI}(\Gamma)_{\text{push}/\text{pusnu}}$ of moulds which are push-invariant and whose swap are pus-neutral and show that it forms a Lie algebra under the ari-bracket.

By abuse of notation, we introduce the following notations for $\overline{\text{ARI}}(\Gamma)$ similarly to Notation 1.7.

Notation 1.25 ([9, §2.1]). — For any mould $M = (M^m(\sigma_1, \dots, \sigma_m)_{v_1, \dots, v_m})_m \in \overline{\text{ARI}}(\Gamma)$, we define

$$\begin{aligned} \text{mantar}(M)^m(\sigma_1, \dots, \sigma_m)_{v_1, \dots, v_m} &= (-1)^{m-1} M^m(\sigma_m, \dots, \sigma_1)_{v_m, \dots, v_1}, \\ \text{pus}(M)^m(\sigma_1, \dots, \sigma_m)_{v_1, \dots, v_m} &= M^m(\sigma_m, \sigma_1, \dots, \sigma_{m-1})_{v_m, v_1, \dots, v_{m-1}}, \\ \text{neg}(M)^m(\sigma_1, \dots, \sigma_m)_{v_1, \dots, v_m} &= M^m(\sigma_1^{-1}, \dots, \sigma_m^{-1})_{-u_1, \dots, -u_m}. \end{aligned}$$

DEFINITION 1.26. — *We call a mould $N \in \overline{\text{ARI}}(\Gamma)$ pus-neutral ([9, (2.73)]⁽¹⁰⁾ when we have*

$$\sum_{i=1}^m \text{pus}^i(N)^m(\sigma_1, \dots, \sigma_m)_{v_1, \dots, v_m} \left(= \sum_{i \in \mathbb{Z}/m\mathbb{Z}} N^m(\sigma_{i+1}, \dots, \sigma_{i+m})_{v_{i+1}, \dots, v_{i+m}} \right) = 0 \quad (1.9)$$

for all $m \geq 1$ and $\sigma_1, \dots, \sigma_m \in \Gamma$. We define $\text{ARI}(\Gamma)_{\text{push}/\text{pusnu}}$ ⁽¹¹⁾ to be the set of moulds $M \in \text{ARI}(\Gamma)$ which is push-invariant (1.5) and whose $\text{swap}(M)$ is pus-neutral (1.9).

⁽¹⁰⁾ It is not push-neutral but pus-neutral. In [17, Definition 5’], it is called *circ-neutral* when (1.9) holds for $m > 1$ when $\Gamma = \{e\}$.

⁽¹¹⁾ We expect that our $\text{ARI}(\Gamma)_{\text{push}/\text{pusnu}}$ might be related to Ecalle’s $\text{ARI}^{*/\overline{\text{pusnu}}}$ ([9, §2.5]), whose precise definition looks missing.

Example 1.27. — We consider the mould $Q \in \overline{\text{ARI}}(\{e\})$ defined by

$$Q(\overset{e}{v_1}, \dots, \overset{e}{v_m}) := \begin{cases} 0 & m = 0, 1, \\ \frac{1}{v_2 - v_1} + \dots + \frac{1}{v_m - v_1} & m \geq 2. \end{cases}$$

Then this mould Q is pus-neutral. In fact, for $1 \leq i \leq m$, we have

$$Q(\overset{e}{v_i}, \dots, \overset{e}{v_m}, \overset{e}{v_1}, \dots, \overset{e}{v_{i-1}}) = \sum_{j=i+1}^m \frac{1}{v_j - v_i} + \sum_{j=1}^{i-1} \frac{1}{v_j - v_i},$$

and so we calculate

$$\begin{aligned} \sum_{i=1}^m \text{pus}^i(Q)(\overset{e}{v_1}, \dots, \overset{e}{v_m}) &= \sum_{i=1}^m Q(\overset{e}{v_i}, \dots, \overset{e}{v_m}, \overset{e}{v_1}, \dots, \overset{e}{v_{i-1}}) \\ &= \sum_{i=1}^m \left\{ \sum_{j=i+1}^m \frac{1}{v_j - v_i} + \sum_{j=1}^{i-1} \frac{1}{v_j - v_i} \right\} \\ &= \sum_{\substack{i, j \in \{1, \dots, m\} \\ i < j}} \frac{1}{v_j - v_i} + \sum_{\substack{i, j \in \{1, \dots, m\} \\ i > j}} \frac{1}{v_j - v_i} = 0. \end{aligned}$$

Hence, the mould Q is pus-neutral.

Firstly we show that the set ARI_{push} forms a Lie algebra under the ari_u -bracket which was stated in [9, §2.5] without a detailed proof.

PROPOSITION 1.28. — *If $A, B \in \text{ARI}(\Gamma)$ are push-invariant, then $\text{ari}_u(A, B)$ is push-invariant.*

Proof. — Let $m \geq 1$. We put $\omega = (\overset{u_1}{\sigma_1}, \dots, \overset{u_m}{\sigma_m})$. We have

$$\begin{aligned} \text{push}(A \times B)(\omega) &= (A \times B) \left(\begin{array}{c} -u_1 \cdots -u_m, \quad u_1, \quad \dots, \quad u_{m-1} \\ \sigma_m^{-1}, \quad \sigma_1 \sigma_m^{-1}, \quad \dots, \quad \sigma_{m-1} \sigma_m^{-1} \end{array} \right) \\ &= A(\emptyset) B \left(\begin{array}{c} -u_1 \cdots -u_m, \quad u_1, \quad \dots, \quad u_{m-1} \\ \sigma_m^{-1}, \quad \sigma_1 \sigma_m^{-1}, \quad \dots, \quad \sigma_{m-1} \sigma_m^{-1} \end{array} \right) \\ &\quad + \sum_{i=1}^m A \left(\begin{array}{c} -u_1 \cdots -u_m, \quad u_1, \quad \dots, \quad u_{i-1} \\ \sigma_m^{-1}, \quad \sigma_1 \sigma_m^{-1}, \quad \dots, \quad \sigma_{i-1} \sigma_m^{-1} \end{array} \right) B \left(\begin{array}{c} u_i, \quad \dots, \quad u_{m-1} \\ \sigma_i \sigma_m^{-1}, \quad \dots, \quad \sigma_{m-1} \sigma_m^{-1} \end{array} \right). \end{aligned}$$

Because A, B are push-invariant, we get

$$\begin{aligned} \text{push}(A \times B)(\omega) &= A(\emptyset)B(\omega) + A(\omega)B(\emptyset) \\ &\quad + \sum_{i=1}^{m-1} A \left(\begin{array}{c} u_1, \dots, u_{i-1}, u_i + \dots + u_m \\ \sigma_1, \dots, \sigma_{i-1}, \sigma_m \end{array} \right) B \left(\begin{array}{c} u_i, \dots, u_{m-1} \\ \sigma_i \sigma_m^{-1}, \dots, \sigma_{m-1} \sigma_m^{-1} \end{array} \right). \quad (1.10) \end{aligned}$$

On the other hand, for $\eta = \begin{pmatrix} u_0, u_1, \dots, u_{m-1} \\ \tau_0, \tau_1, \dots, \tau_{m-1} \end{pmatrix}$, we have

$$\begin{aligned}
 & \text{arit}_u(B)(A)(\eta) \\
 &= \sum_{\substack{\eta=\alpha\beta\gamma \\ \beta, \gamma \neq \emptyset}} A(\alpha_\beta[\gamma])B(\beta]_\gamma) - \sum_{\substack{\eta=\alpha\beta\gamma \\ \alpha, \beta \neq \emptyset}} A(\alpha]_\beta\gamma)B(\alpha_\beta] \\
 &= \sum_{0 \leq i < j < m-1} A\left(\begin{matrix} u_0, \dots, u_{i-1}, u_i+\dots+u_{j+1}, u_{j+2}, \dots, u_{m-1} \\ \tau_0, \dots, \tau_{i-1}, \tau_{j+1}, \tau_{j+2}, \dots, \tau_{m-1} \end{matrix}\right) B\left(\begin{matrix} u_i, \dots, u_j \\ \tau_i\tau_{j+1}^{-1}, \dots, \tau_j\tau_{j+1}^{-1} \end{matrix}\right) \\
 &\quad - \sum_{1 \leq i \leq j \leq m-1} A\left(\begin{matrix} u_0, \dots, u_{i-2}, u_{i-1}+\dots+u_j, u_{j+1}, \dots, u_{m-1} \\ \tau_0, \dots, \tau_{i-2}, \tau_{i-1}, \tau_{j+1}, \dots, \tau_{m-1} \end{matrix}\right) B\left(\begin{matrix} u_i, \dots, u_j \\ \tau_i\tau_{i-1}^{-1}, \dots, \tau_j\tau_{i-1}^{-1} \end{matrix}\right) \\
 &= \sum_{0 \leq j < m-1} A\left(\begin{matrix} u_0+\dots+u_{j+1}, u_{j+2}, \dots, u_{m-1} \\ \tau_{j+1}, \tau_{j+2}, \dots, \tau_{m-1} \end{matrix}\right) B\left(\begin{matrix} u_0, \dots, u_j \\ \tau_0\tau_{j+1}^{-1}, \dots, \tau_j\tau_{j+1}^{-1} \end{matrix}\right) \\
 &\quad - \sum_{1 \leq j \leq m-1} A\left(\begin{matrix} u_0+\dots+u_j, u_{j+1}, \dots, u_{m-1} \\ \tau_0, \tau_{j+1}, \dots, \tau_{m-1} \end{matrix}\right) B\left(\begin{matrix} u_1, \dots, u_j \\ \tau_1\tau_0^{-1}, \dots, \tau_j\tau_0^{-1} \end{matrix}\right) \\
 &\quad + \sum_{0 < i \leq j < m-1} A\left(\begin{matrix} u_0, \dots, u_{i-1}, u_i+\dots+u_{j+1}, u_{j+2}, \dots, u_{m-1} \\ \tau_0, \dots, \tau_{i-1}, \tau_{j+1}, \tau_{j+2}, \dots, \tau_{m-1} \end{matrix}\right) B\left(\begin{matrix} u_i, \dots, u_j \\ \tau_i\tau_{j+1}^{-1}, \dots, \tau_j\tau_{j+1}^{-1} \end{matrix}\right) \\
 &\quad - \sum_{1 < i \leq j \leq m-1} A\left(\begin{matrix} u_0, \dots, u_{i-2}, u_{i-1}+\dots+u_j, u_{j+1}, \dots, u_{m-1} \\ \tau_0, \dots, \tau_{i-2}, \tau_{i-1}, \tau_{j+1}, \dots, \tau_{m-1} \end{matrix}\right) B\left(\begin{matrix} u_i, \dots, u_j \\ \tau_i\tau_{i-1}^{-1}, \dots, \tau_j\tau_{i-1}^{-1} \end{matrix}\right).
 \end{aligned}$$

So by putting $u_0 = -u_1 - \dots - u_m$ and $\tau_0 = \sigma_m^{-1}$ and $\tau_i = \sigma_i\sigma_m^{-1}$ for $1 \leq i \leq m-1$, we calculate

$$\begin{aligned}
 & \text{push}(\text{arit}_u(B)(A))(\omega) \\
 &= (\text{arit}_u(B)(A))\left(\begin{matrix} -u_1-\dots-u_m, u_1, \dots, u_{m-1} \\ \sigma_m^{-1}, \sigma_1\sigma_m^{-1}, \dots, \sigma_{m-1}\sigma_m^{-1} \end{matrix}\right) \\
 &= \sum_{0 \leq j < m-1} A\left(\begin{matrix} -u_{j+2}-\dots-u_m, u_{j+2}, \dots, u_{m-1} \\ \sigma_{j+1}\sigma_m^{-1}, \sigma_{j+2}\sigma_m^{-1}, \dots, \sigma_{m-1}\sigma_m^{-1} \end{matrix}\right) \\
 &\quad \cdot B\left(\begin{matrix} -u_1-\dots-u_m, u_1, \dots, u_j \\ \sigma_{j+1}^{-1}, \sigma_1\sigma_{j+1}^{-1}, \dots, \sigma_j\sigma_{j+1}^{-1} \end{matrix}\right) \\
 &\quad - \sum_{1 \leq j \leq m-1} A\left(\begin{matrix} -u_{j+1}-\dots-u_m, u_{j+1}, \dots, u_{m-1} \\ \sigma_m^{-1}, \sigma_{j+1}\sigma_m^{-1}, \dots, \sigma_{m-1}\sigma_m^{-1} \end{matrix}\right) B\left(\begin{matrix} u_1, \dots, u_j \\ \sigma_1, \dots, \sigma_j \end{matrix}\right) \\
 &\quad + \sum_{0 < i \leq j < m-1} A\left(\begin{matrix} -u_1-\dots-u_m, u_1, \dots, u_{i-1}, u_i+\dots+u_{j+1}, u_{j+2}, \dots, u_{m-1} \\ \sigma_m^{-1}, \sigma_1\sigma_m^{-1}, \dots, \sigma_{i-1}\sigma_m^{-1}, \sigma_{j+1}\sigma_m^{-1}, \sigma_{j+2}\sigma_m^{-1}, \dots, \sigma_{m-1}\sigma_m^{-1} \end{matrix}\right) \\
 &\quad \cdot B\left(\begin{matrix} u_i, \dots, u_j \\ \sigma_i\sigma_{j+1}^{-1}, \dots, \sigma_j\sigma_{j+1}^{-1} \end{matrix}\right) \\
 &\quad - \sum_{1 < i \leq j \leq m-1} A\left(\begin{matrix} -u_1-\dots-u_m, u_1, \dots, u_{i-2}, u_{i-1}+\dots+u_j, u_{j+1}, \dots, u_{m-1} \\ \sigma_m^{-1}, \sigma_1\sigma_m^{-1}, \dots, \sigma_{i-2}\sigma_m^{-1}, \sigma_{i-1}\sigma_m^{-1}, \sigma_{j+1}\sigma_m^{-1}, \dots, \sigma_{m-1}\sigma_m^{-1} \end{matrix}\right) \\
 &\quad \cdot B\left(\begin{matrix} u_i, \dots, u_j \\ \sigma_i\sigma_{i-1}^{-1}, \dots, \sigma_j\sigma_{i-1}^{-1} \end{matrix}\right).
 \end{aligned}$$

Because A, B are push-invariant, we get

$$\begin{aligned}
 &= \sum_{0 \leq j < m-1} A\left(\begin{smallmatrix} u_{j+2}, & \dots, & u_m \\ \sigma_{j+2}\sigma_{j+1}^{-1}, & \dots, & \sigma_m\sigma_{j+1}^{-1} \end{smallmatrix}\right) B\left(\begin{smallmatrix} u_1, \dots, u_j, u_{j+1}+\dots+u_m \\ \sigma_1, \dots, \sigma_j, \sigma_{j+1} \end{smallmatrix}\right) \\
 &\quad - \sum_{1 \leq j \leq m-1} A\left(\begin{smallmatrix} u_{j+1}, \dots, u_m \\ \sigma_{j+1}, \dots, \sigma_m \end{smallmatrix}\right) B\left(\begin{smallmatrix} u_1, \dots, u_j \\ \sigma_1, \dots, \sigma_j \end{smallmatrix}\right) \\
 &\quad + \sum_{0 < i \leq j < m-1} A\left(\begin{smallmatrix} u_1, \dots, u_{i-1}, u_i+\dots+u_{j+1}, u_{j+2}, \dots, u_m \\ \sigma_1, \dots, \sigma_{i-1}, \sigma_{j+1}, \sigma_{j+2}, \dots, \sigma_m \end{smallmatrix}\right) B\left(\begin{smallmatrix} u_i, \dots, u_j \\ \sigma_i\sigma_{j+1}^{-1}, \dots, \sigma_j\sigma_{j+1}^{-1} \end{smallmatrix}\right) \\
 &\quad - \sum_{1 < i \leq j \leq m-1} A\left(\begin{smallmatrix} u_1, \dots, u_{i-2}, u_{i-1}+\dots+u_j, u_{j+1}, \dots, u_m \\ \sigma_1, \dots, \sigma_{i-2}, \sigma_{j-1}, \sigma_{j+1}, \dots, \sigma_m \end{smallmatrix}\right) B\left(\begin{smallmatrix} u_i, \dots, u_j \\ \sigma_i\sigma_{i-1}^{-1}, \dots, \sigma_j\sigma_{i-1}^{-1} \end{smallmatrix}\right) \\
 &= \sum_{\substack{\omega=\alpha\beta \\ \alpha, \beta \neq \emptyset}} A(\alpha \lfloor \beta) B(\alpha \rfloor \beta) - (B \times A)\left(\begin{smallmatrix} u_1, \dots, u_m \\ \sigma_1, \dots, \sigma_m \end{smallmatrix}\right) \\
 &\quad + \sum_{\substack{\omega=\alpha\beta\gamma \\ \beta, \gamma \neq \emptyset}} A(\alpha_\beta \lceil \gamma) B(\beta \rfloor \gamma) \\
 &\quad - \sum_{0 < i \leq m-1} A\left(\begin{smallmatrix} u_1, \dots, u_{i-1}, u_i+\dots+u_m \\ \sigma_1, \dots, \sigma_{i-1}, \sigma_m \end{smallmatrix}\right) B\left(\begin{smallmatrix} u_i, \dots, u_{m-1} \\ \sigma_i\sigma_m^{-1}, \dots, \sigma_j\sigma_m^{-1} \end{smallmatrix}\right) \\
 &\quad - \left\{ \sum_{\substack{\omega=\alpha\beta\gamma \\ \alpha, \beta \neq \emptyset}} A(\alpha \rfloor \beta \lceil \gamma) B(\alpha \lfloor \beta) - \sum_{\substack{\omega=\alpha\beta \\ \alpha, \beta \neq \emptyset}} A(\alpha \rfloor \beta) B(\alpha \lfloor \beta) \right\}.
 \end{aligned}$$

So we get

$$\begin{aligned}
 \text{push}(\text{arit}_u(B)(A))(\omega) &= (\text{arit}_u(B)(A))(\omega) - (B \times A)(\omega) \\
 &\quad - \sum_{i=1}^{m-1} A\left(\begin{smallmatrix} u_1, \dots, u_{i-1}, u_i+\dots+u_m \\ \sigma_1, \dots, \sigma_{i-1}, \sigma_m \end{smallmatrix}\right) B\left(\begin{smallmatrix} u_i, \dots, u_{m-1} \\ \sigma_i\sigma_m^{-1}, \dots, \sigma_j\sigma_m^{-1} \end{smallmatrix}\right) \\
 &\quad + \sum_{\substack{\omega=\alpha\beta \\ \alpha, \beta \neq \emptyset}} \{A(\alpha \rfloor \beta) B(\alpha \lfloor \beta) + A(\alpha \lfloor \beta) B(\alpha \rfloor \beta)\}. \quad (1.11)
 \end{aligned}$$

Therefore, by using (1.10) and (1.11), we have

$$\begin{aligned}
 \text{push}(\text{ari}_u(A, B))(\omega) &= \text{push}(\text{arit}_u(B)(A))(\omega) - \text{push}(\text{arit}_u(A)(B))(\omega) \\
 &\quad + \text{push}(A \times B)(\omega) - \text{push}(B \times A)(\omega) \\
 &= (\text{arit}_u(B)(A))(\omega) - (\text{arit}_u(A)(B))(\omega) \\
 &\quad + (A \times B)(\omega) - (B \times A)(\omega) \\
 &= \text{ari}_u(A, B)(\omega).
 \end{aligned}$$

Thus $\text{ari}_u(A, B)$ is push-invariant. \square

Remark 1.29. — This proposition improves the results of [17, §4.1.3], which shows that the intersection $\text{ARI}_{\text{push}}(\Gamma) \cap \text{ARI}_{\text{al}}^{\text{pol}}(\Gamma)$ forms a Lie algebra under the ari_u -bracket when $\Gamma = \{e\}$.

Secondly we show that the set of pus-neutral moulds is closed under the ari_v -bracket. The following proposition is a generalization of [17, Lemma 21] which treats the case when $\Gamma = \{e\}$.

PROPOSITION 1.30. — *If $A, B \in \overline{\text{ARI}}(\Gamma)$ are pus-neutral, then $\text{ari}_v(A, B)$ is pus-neutral.*

Proof. — The proof goes in the same way to that of [17]. Let $m \geq 1$. Since the algebra $\overline{\text{ARI}}(\Gamma)$ is graded by depth, it is enough to prove A and $B \in \text{ARI}(\Gamma)$ with depth k and $l \in \mathbb{N}$ with $m = k + l$.

Firstly, we prove that $[A, B]$ is pus-neutral. We calculate

$$\begin{aligned} \sum_{i=1}^m \text{pus}^i([A, B])^m(\sigma_1, \dots, \sigma_m) &= \sum_{i \in \mathbb{Z}/m\mathbb{Z}} (A \times B - B \times A)^m(\sigma_{1+i}, \dots, \sigma_{m+i}) \\ &= \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \left\{ A^k(\sigma_{1+i}, \dots, \sigma_{k+i}) B^l(\sigma_{k+1+i}, \dots, \sigma_{m+i}) \right. \\ &\quad \left. - B^l(\sigma_{1+i}, \dots, \sigma_{l+i}) A^k(\sigma_{l+1+i}, \dots, \sigma_{m+i}) \right\} \\ &= \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \left\{ A^k(\sigma_{1+i}, \dots, \sigma_{k+i}) B^l(\sigma_{k+1+i}, \dots, \sigma_{m+i}) \right. \\ &\quad \left. - B^l(\sigma_{1+i-l}, \dots, \sigma_i) A^k(\sigma_{1+i}, \dots, \sigma_{k+i}) \right\} \\ &= \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \left\{ A^k(\sigma_{1+i}, \dots, \sigma_{k+i}) B^l(\sigma_{k+1+i}, \dots, \sigma_{m+i}) \right. \\ &\quad \left. - B^l(\sigma_{k+1+i}, \dots, \sigma_{m+i}) A^k(\sigma_{1+i}, \dots, \sigma_{k+i}) \right\} \end{aligned}$$

The first and the second terms are equal, so $[A, B]$ is pus-neutral.

Secondly, we prove that $\text{arit}_v(B)(A)$ is pus-neutral. Then we have

$$\begin{aligned} (\text{arit}_v(B)(A))^m(\sigma_1, \dots, \sigma_m) &= \sum_{\substack{\omega = \alpha\beta\gamma \\ \beta, \gamma \neq \emptyset}} A^k(\alpha_\beta \lceil \gamma \rceil) B^l(\beta \lfloor \gamma \rfloor) - \sum_{\substack{\omega = \alpha\beta\gamma \\ \alpha, \beta \neq \emptyset}} A^k(\alpha \lceil \beta \rceil) B^l(\alpha \lfloor \beta \rfloor). \end{aligned}$$

Because $l \geq 1$ and B is with depth l , we may put⁽¹²⁾ $\alpha = (\sigma_1, \dots, \sigma_j)$, $\beta = (\sigma_{j+1}, \dots, \sigma_{j+l})$ and $\gamma = (\sigma_{j+l+1}, \dots, \sigma_m)$ and we get

$$\begin{aligned} & (\text{arit}_v(B)(A))^m(\sigma_1, \dots, \sigma_m) \\ &= \sum_{j=0}^{k-1} A^k(\sigma_1, \dots, \sigma_j, \sigma_{j+1} \dots \sigma_{j+l+1}, \sigma_{j+l+2}, \dots, \sigma_m) B^l(\sigma_{j+1}, \dots, \sigma_{j+l}) \\ & \quad - \sum_{j=1}^k A^k(\sigma_1, \dots, \sigma_{j-1}, \sigma_j \dots \sigma_{j+l}, \sigma_{j+l+1}, \dots, \sigma_m) B^l(\sigma_{j+1}, \dots, \sigma_{j+l}). \end{aligned}$$

Here, for $0 \leq j \leq k-1$, we have

$$\begin{aligned} & \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \sum_{j=0}^{k-1} A^k(\sigma_{i+1}, \dots, \sigma_{i+j}, \sigma_{i+j+1} \dots \sigma_{i+j+l+1}, \sigma_{i+j+l+2}, \dots, \sigma_{i+m}) \\ & \quad \cdot B^l(\sigma_{i+j+1}, \dots, \sigma_{i+j+l}) \\ &= \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \sum_{j=0}^{k-1} A^k(\sigma_{i+1-j}, \dots, \sigma_i, \sigma_{i+1} \dots \sigma_{i+l+1}, \sigma_{i+l+2}, \dots, \sigma_{i+m-j}) \\ & \quad \cdot B^l(\sigma_{i+1}, \dots, \sigma_{i+l}) \\ &= \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \sum_{j=0}^{k-1} (\text{pus}^j(A))^k(\sigma_{i+1} \dots \sigma_{i+l+1}, \sigma_{i+l+2}, \dots, \sigma_{i+m}) \\ & \quad \cdot B^l(\sigma_{i+1}, \dots, \sigma_{i+l}). \end{aligned}$$

Since A is pus-neutral, this is 0. Similarly by using the pus-neutrality of A , we obtain

$$\begin{aligned} & \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \sum_{j=1}^k A^k(\sigma_{i+1}, \dots, \sigma_{i+j-1}, \sigma_{i+j} \dots \sigma_{i+j+l}, \sigma_{i+j+l+1}, \dots, \sigma_{i+m}) \\ & \quad \cdot B^l(\sigma_{i+j+1}, \dots, \sigma_{i+j+l}) = 0 \end{aligned}$$

for all $1 \leq j \leq k$. Therefore, we get

$$\begin{aligned} & \sum_{i=1}^m \text{pus}^i(\text{arit}_v(B)(A))^m(\sigma_1, \dots, \sigma_m) \\ & \quad = \sum_{i \in \mathbb{Z}/m\mathbb{Z}} (\text{arit}_v(B)(A))^m(\sigma_{1+i}, \dots, \sigma_{m+i}) = 0. \end{aligned}$$

Whence $\text{arit}_v(B)(A)$ is pus-neutral.

⁽¹²⁾ For the first (resp. second) term, the word α (resp. γ) can be \emptyset and the word γ (resp. α) is not \emptyset . So the index j must run from 0 (resp. 1) to $k-1 = m-l-1$ (resp. $k = m-l$).

Thirdly, in the same way, we can show that $\text{arit}_v(A)(B)$ is pus-neutral by using the pus-neutrality of B .

Thus we complete the proof because $\text{ari}_v(A, B) = [A, B] + \text{arit}_v(B)(A) - \text{arit}_v(A)(B)$. \square

We need the following lemma for the proof of Theorem 1.32.

LEMMA 1.31. — *If $A, B \in \text{ARI}(\Gamma)$ are push-invariant, then we have*

$$\text{swap}(\text{ari}_u(A, B)) = \text{ari}_v(\text{swap}(A), \text{swap}(B)).$$

Proof. — This follows from the proof of [21, Lemma 2.4.1], which actually works for BARI. \square

THEOREM 1.32. — *The set $\text{ARI}(\Gamma)_{\text{push}/\text{pusnu}}$ forms a Lie subalgebra of $\text{ARI}(\Gamma)_{\text{push}}$ under the ari_u -bracket.*

Proof. — Let $A, B \in \text{ARI}(\Gamma)_{\text{push}/\text{pusnu}}$. Then by Proposition 1.28, $\text{ari}_u(A, B) \in \text{ARI}(\Gamma)_{\text{push}}$. Let $m \geq 1$. Because A and B are push-invariant, by Lemma 1.31, we have

$$\begin{aligned} \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \text{pus}^i \circ \text{swap}(\text{ari}_u(A, B))^m(\begin{smallmatrix} \sigma_1, \dots, \sigma_m \\ v_1, \dots, v_m \end{smallmatrix}) \\ = \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \text{pus}^i(\text{ari}_v(\text{swap}(A), \text{swap}(B)))^m(\begin{smallmatrix} \sigma_1, \dots, \sigma_m \\ v_1, \dots, v_m \end{smallmatrix}). \end{aligned}$$

By Proposition 1.30, the right hand side is equal to 0. Hence $\text{swap}(\text{ari}_u(A, B))$ is pus-neutral. Thus we obtain $\text{ari}_u(A, B) \in \text{ARI}(\Gamma)_{\text{push}/\text{pusnu}}$. \square

Remark 1.33. — The above theorem improves [17, Corollary 22] which shows that $\text{ARI}(\Gamma)_{\text{push}/\text{pusnu}} \cap \text{ARI}_{\text{al}}^{\text{pol}}(\Gamma)$ forms a Lie algebra when $\Gamma = \{e\}$.

Our Lie algebra $\text{ARI}(\Gamma)_{\text{push}/\text{pusnu}}$ will play an important role in the following sections.

DEFINITION 1.34. — *For $N \geq 1$, put $\Gamma^N := \{g^N \in \Gamma \mid g \in \Gamma\}$. We consider the map $i_N : \text{ARI}(\Gamma) \rightarrow \text{ARI}(\Gamma^N)$ which is given by*

$$i_N(M)^m(\begin{smallmatrix} x_1, \dots, x_m \\ \sigma_1, \dots, \sigma_m \end{smallmatrix}) = M^m(\begin{smallmatrix} x_1, \dots, x_m \\ \sigma_1, \dots, \sigma_m \end{smallmatrix}) \tag{1.12}$$

and also the map $m_N : \text{ARI}(\Gamma) \rightarrow \text{ARI}(\Gamma^N)$ which is given by

$$m_N(M)^m(\begin{smallmatrix} x_1, \dots, x_m \\ \sigma_1, \dots, \sigma_m \end{smallmatrix}) = \sum_{\tau_i^N = \sigma_i} M^m(\begin{smallmatrix} Nx_1, \dots, Nx_m \\ \tau_1, \dots, \tau_m \end{smallmatrix}). \tag{1.13}$$

We define the following \mathbb{Q} -linear subspaces which are subject to the distribution relations:

$$\text{ARID}(\Gamma) := \{M \in \text{ARI}(\Gamma) \mid i_N(M) = m_N(M) \text{ for all } N \geq 1 \text{ with } N \mid |\Gamma|\},$$

$$\text{ARID}(\Gamma)_{\text{al}/\text{al}} := \text{ARID}(\Gamma) \cap \text{ARI}(\Gamma)_{\text{al}/\text{al}}.$$

PROPOSITION 1.35. — *The \mathbb{Q} -linear subspace $\text{ARID}(\Gamma)$ forms a filtered Lie subalgebra of $\text{ARI}(\Gamma)$ under the ari_u -bracket.*

Proof. — First we prove that m_N is a Lie algebra homomorphism, that is,

$$m_N(\text{ari}_u(A, B)) = \text{ari}_u(m_N(A), m_N(B))$$

for $A, B \in \text{ARI}(\Gamma)$. Let $m \geq 0$. We have

$$\begin{aligned} m_N(A \times B)^m(\mathbf{x}_m) &= \sum_{\substack{\tau_i^N = \sigma_i \\ 1 \leq i \leq m}} (A \times B)^m(Nx_1, \dots, Nx_m) \\ &= \sum_{\substack{\tau_i^N = \sigma_i \\ 1 \leq i \leq m}} \sum_{j=0}^m A^i(Nx_1, \dots, Nx_j) B^{m-i}(Nx_{j+1}, \dots, Nx_m) \\ &= \sum_{j=0}^m \left\{ \sum_{\substack{\tau_i^N = \sigma_i \\ 1 \leq i \leq j}} A^i(Nx_1, \dots, Nx_j) \right\} \left\{ \sum_{\substack{\tau_i^N = \sigma_i \\ j+1 \leq i \leq m}} B^{m-i}(Nx_{j+1}, \dots, Nx_m) \right\} \\ &= \sum_{j=0}^m m_N(A)^i(\sigma_1, \dots, \sigma_j) m_N(B)^{m-i}(\sigma_{j+1}, \dots, \sigma_m) \\ &= (m_N(A) \times m_N(B))^m(\mathbf{x}_m). \end{aligned}$$

Similarly, we have

$$\begin{aligned} m_N(\text{arit}_u(B)(A))^m(\mathbf{x}_m) &= \sum_{\substack{\nu_i^N = \sigma_i \\ 1 \leq i \leq m}} \text{arit}(B)(A)^m(Nx_1, \dots, Nx_m) \\ &= \sum_{\substack{\nu_i^N = \sigma_i \\ 1 \leq i \leq m}} \left\{ \sum_{1 \leq k \leq l < m} A \left(\begin{array}{c} Nx_1, \dots, Nx_{k-1}, N(x_k + \dots + x_{l+1}), Nx_{l+2}, \dots, Nx_m \\ \nu_1, \dots, \nu_{k-1}, \nu_{l+1}, \nu_{l+2}, \dots, \nu_m \end{array} \right) \right. \\ &\quad \left. \cdot B \left(\begin{array}{c} Nx_k, \dots, Nx_l \\ \nu_k \nu_{l+1}, \dots, \nu_l \nu_{l+1} \end{array} \right) \right\} \\ &\quad - \sum_{\substack{\nu_i^N = \sigma_i \\ 1 \leq i \leq m}} \left\{ \sum_{1 < k \leq l \leq m} A \left(\begin{array}{c} Nx_1, \dots, Nx_{k-2}, N(x_{k-1} + \dots + x_l), Nx_{l+1}, \dots, Nx_m \\ \nu_1, \dots, \nu_{k-2}, \tau_{k-1}, \nu_{l+1}, \dots, \nu_m \end{array} \right) \right. \\ &\quad \left. \cdot B \left(\begin{array}{c} Nx_k, \dots, Nx_l \\ \nu_k \nu_{k-1}, \dots, \nu_l \nu_{k-1} \end{array} \right) \right\}. \end{aligned}$$

Here, we put $S_{k,l} := \{i \in \mathbb{N} \mid 1 \leq i \leq k-1\} \cup \{i \in \mathbb{N} \mid l+1 \leq i \leq m\}$ for $1 \leq k \leq l \leq m$. Then we can divide the set of variables of the summation in the following :

$$\{\nu_i \mid 1 \leq i \leq m\} = \{\nu_i \mid i \in S_{k,l}\} \cup \{\nu_i \mid k \leq i \leq l\}.$$

So by substituting $\nu_i := \tau_i$ for $i \in S_{k,l}$ and $\nu_i := \tau_i \tau_{l+1}$ for $k \leq i \leq l$ for the first summation, and by substituting $\nu_i := \tau_i$ for $i \in S_{k,l}$ and $\nu_i := \tau_i \tau_{k-1}$ for $k \leq i \leq l$ for the second summation, we calculate

$$\begin{aligned} & m_N(\text{arit}_u(B)(A))^m(\mathbf{x}_m) \\ &= \sum_{1 \leq k \leq l < m} \left\{ \sum_{\substack{\tau_i^N = \sigma_i \\ i \in S_{k,l}}} A \left(\begin{matrix} Nx_1, \dots, Nx_{k-1}, N(x_k + \dots + x_{l+1}), Nx_{l+2}, \dots, Nx_m \\ \tau_1, \dots, \tau_{k-1}, \tau_{l+1}, \tau_{l+2}, \dots, \tau_m \end{matrix} \right) \right\} \\ & \quad \cdot \left\{ \sum_{\substack{\tau_i^N = \sigma_i \sigma_{l+1}^{-1} \\ k \leq i \leq l}} B \left(\begin{matrix} Nx_k, \dots, Nx_l \\ \tau_k, \dots, \tau_l \end{matrix} \right) \right\} \\ & - \sum_{1 < k \leq l \leq m} \left\{ \sum_{\substack{\tau_i^N = \sigma_i \\ i \in S_{k,l}}} A \left(\begin{matrix} Nx_1, \dots, Nx_{k-2}, N(x_{k-1} + \dots + x_l), Nx_{l+1}, \dots, Nx_m \\ \tau_1, \dots, \tau_{k-2}, \tau_{k-1}, \tau_{l+1}, \dots, \tau_m \end{matrix} \right) \right\} \\ & \quad \cdot \left\{ \sum_{\substack{\tau_i^N = \sigma_i \sigma_{k-1}^{-1} \\ k \leq i \leq l}} B \left(\begin{matrix} Nx_k, \dots, Nx_l \\ \tau_k, \dots, \tau_l \end{matrix} \right) \right\} \\ &= \sum_{1 \leq k \leq l < m} m_N(A) \left(\begin{matrix} x_1, \dots, x_{k-1}, x_k + \dots + x_{l+1}, x_{l+2}, \dots, x_m \\ \sigma_1, \dots, \sigma_{k-1}, \sigma_{l+1}, \sigma_{l+2}, \dots, \sigma_m \end{matrix} \right) \\ & \quad \cdot m_N(B) \left(\begin{matrix} x_k, \dots, x_l \\ \sigma_k \sigma_{l+1}^{-1}, \dots, \sigma_l \sigma_{l+1}^{-1} \end{matrix} \right) \\ & - \sum_{1 < k \leq l \leq m} m_N(A) \left(\begin{matrix} x_1, \dots, x_{k-2}, x_{k-1} + \dots + x_l, x_{l+1}, \dots, x_m \\ \sigma_1, \dots, \sigma_{k-2}, \sigma_{k-1}, \sigma_{l+1}, \dots, \sigma_m \end{matrix} \right) \\ & \quad \cdot m_N(B) \left(\begin{matrix} x_k, \dots, x_l \\ \sigma_k \sigma_{k-1}^{-1}, \dots, \sigma_l \sigma_{k-1}^{-1} \end{matrix} \right). \end{aligned}$$

By putting $\alpha = \begin{pmatrix} x_1, \dots, x_{k-1} \\ \sigma_1, \dots, \sigma_{k-1} \end{pmatrix}$, $\beta = \begin{pmatrix} x_k, \dots, x_l \\ \sigma_k, \dots, \sigma_l \end{pmatrix}$ and $\gamma = \begin{pmatrix} x_{l+1}, \dots, x_m \\ \sigma_{l+1}, \dots, \sigma_m \end{pmatrix}$, we get

$$\begin{aligned} &= \sum_{\substack{\mathbf{x}_m = \alpha\beta\gamma \\ \beta, \gamma \neq \emptyset}} m_N(A)(\alpha_\beta[\gamma]m_N(B)(\beta)_\gamma) - \sum_{\substack{\mathbf{x}_m = \alpha\beta\gamma \\ \alpha, \beta \neq \emptyset}} m_N(A)(\alpha)_\beta\gamma m_N(B)(\alpha)_\beta \\ &= \text{arit}_u(m_N(B))(m_N(A))^m(\mathbf{x}_m). \end{aligned}$$

Hence we get

$$\begin{aligned} &m_N(\text{ari}_u(A, B)) \\ &= m_N(\text{arit}_u(B)(A)) - m_N(\text{arit}_u(A)(B)) + m_N(A \times B) - m_N(B \times A) \\ &= \text{arit}_u(m_N(B))(m_N(A)) - \text{arit}_u(m_N(A))(m_N(B)) \\ &\quad + m_N(A) \times m_N(B) - m_N(B) \times m_N(A) \\ &= \text{ari}_u(m_N(A), m_N(B)). \end{aligned}$$

Therefore, m_N is a Lie algebra homomorphism.

It is obvious that the map i_N is a Lie algebra homomorphism. Since both m_N and i_N are Lie algebra homomorphisms, $\ker(i_N - m_N)$ forms Lie algebra.

By the definition of $\text{ARID}(\Gamma)$, we have $\text{ARID}(\Gamma) = \bigcap_{N \mid |\Gamma|} \ker(i_N - m_N)$. Therefore $\text{ARID}(\Gamma)$ forms a Lie subalgebra under the ari_u -bracket. \square

COROLLARY 1.36. — *The \mathbb{Q} -linear subspace $\text{ARID}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$ forms a filtered Lie subalgebra of $\text{ARI}(\Gamma)$ under the ari_u -bracket.*

Proof. — It is a direct consequence of Proposition 1.24 and Proposition 1.35. \square

2. Kashiwara–Vergne Lie algebra

We introduce the Kashiwara–Vergne bigraded Lie algebra $\mathfrak{k}\mathfrak{t}\mathfrak{v}(\Gamma)_{\bullet\bullet}$ associated with a finite abelian group Γ and give its mould theoretical interpretation by using $\text{ARI}(\Gamma)_{\text{push}/\text{pusnu}}$.

2.1. Γ -variant of the KV condition

We investigate a variant of the defining conditions of Kashiwara–Vergne graded Lie algebra associated with a finite abelian group Γ (cf. Definition 2.1) and explain its mould theoretical interpretation in Theorem 2.15.

Let $\mathbb{L} = \bigoplus_{w \geq 1} \mathbb{L}_w$ be the free graded Lie \mathbb{Q} -algebra generated by $N + 1$ variables x and y_σ ($\sigma \in \Gamma$) with $\deg x = \deg y_\sigma = 1$. Here \mathbb{L}_w is the \mathbb{Q} -linear space generated by Lie monomials whose total degree is w . Occasionally we regard \mathbb{L} as a bigraded Lie algebra $\mathbb{L}_{\bullet\bullet} = \bigoplus_{w,d} \mathbb{L}_{w,d}$, where $\mathbb{L}_{w,d}$ is the \mathbb{Q} -linear space generated by Lie monomials whose *weight* (the total degree) is w and *depth* (the degree with respect to all y_σ) is d . We encode \mathbb{L} with a structure of filtered graded Lie algebra by the filtration $\text{Fil}_{\mathcal{D}}^d \mathbb{L}_w := \bigoplus_{N \geq d} \mathbb{L}_{w,N}$ for $d > 0$ and denote the associated bigraded Lie algebra by $\text{gr}_{\mathcal{D}} \mathbb{L} = \bigoplus_{w,d} \text{gr}_{\mathcal{D}}^d \mathbb{L}_w$ with $\text{gr}_{\mathcal{D}}^d \mathbb{L}_w = \text{Fil}_{\mathcal{D}}^d \mathbb{L}_w / \text{Fil}_{\mathcal{D}}^{d+1} \mathbb{L}_w$.

The $N + 1$ -variable non-commutative polynomial algebra $\mathbb{A} = \mathbb{Q}\langle x, y_\sigma; \sigma \in \Gamma \rangle$ is regarded as the universal enveloping algebra of \mathbb{L} and is encoded with the induced degree. Similarly \mathbb{A} is encoded with a structure of bigraded algebra; $\mathbb{A}_{\bullet\bullet} = \bigoplus_{w,d} \mathbb{A}_{w,d}$. By putting $\text{Fil}_{\mathcal{D}}^d \mathbb{A}_w := \bigoplus_{N \geq d} \mathbb{A}_{w,N}$ for $d > 0$, we also encode \mathbb{A} with a structure of filtered graded algebra. We define the action of $\tau \in \Gamma$ on \mathbb{A} (hence on \mathbb{L}) by

$$\tau(x) = x \text{ and } \tau(y_\sigma) = y_{\tau\sigma}.$$

For any $h \in \mathbb{A}$, we denote

$$h = h_x x + \sum_{\alpha} h_{y_\alpha} y_\alpha = x h^x + \sum_{\alpha} y_\alpha h^{y_\alpha}.$$

We denote π_Y to be the composition of the natural projection and inclusion:

$$\pi_Y : \mathbb{A} \rightarrow \mathbb{A}/\mathbb{A} \cdot x \simeq \mathbb{Q} \oplus \left(\bigoplus_{\sigma \in \Gamma} \mathbb{A} y_\sigma \right) \hookrightarrow \mathbb{A}$$

and the \mathbb{Q} -linear isomorphism \mathbf{q} on \mathbb{A} defined by

$$\mathbf{q}(x^{e_0} y_{\sigma_1} x^{e_1} y_{\sigma_2} \dots x^{e_{r-1}} y_{\sigma_r} x^{e_r}) = x^{e_0} y_{\sigma_1} x^{e_1} y_{\sigma_2 \sigma_1^{-1}} \dots x^{e_{r-1}} y_{\sigma_r \sigma_{r-1}^{-1}} x^{e_r}$$

(cf. [16]). The \mathbb{Q} -linear endomorphism

$$\text{anti} : \mathbb{A} \rightarrow \mathbb{A}$$

is the palindrome (backwards-writing) operator in \mathbb{A}_w (cf. [20, Def. 1.3]).

We put $\text{Cyc}(\mathbb{A})$ to be \mathbb{Q} -linear space generated by cyclic words of \mathbb{A} and $\text{tr} : \mathbb{A} \rightarrow \text{Cyc}(\mathbb{A})$ to be the trace map, the natural projection to $\text{Cyc}(\mathbb{A})$ (cf. [3]).

DEFINITION 2.1. — We define the graded \mathbb{Q} -linear space $\mathfrak{kv}(\Gamma)_\bullet = \bigoplus_{w>1} \mathfrak{kv}(\Gamma)_w$, where its degree w -part $\mathfrak{kv}(\Gamma)_w$ is defined to be the set of Lie elements $F \in \mathbb{L}_w$ such that there exists $G = G(F)$ in \mathbb{L}_w with

$$[x, G] + \sum_{\tau \in \Gamma} [y_\tau, \tau(F)] = 0, \tag{KV1}$$

$$\mathrm{tr} \circ \mathbf{q} \circ \pi_Y(F(z; (y_\sigma))) = 0 \tag{KV2}$$

with $z = -x - \sum_{\sigma \in \Gamma} y_\sigma$.

We note that such $G = G(F)$ uniquely exists when $w > 1$. For $d \geq 1$, we put $\mathrm{Fil}_D^d \mathfrak{kv}(\Gamma)_w$ to be the subspace of $\mathfrak{kv}(\Gamma)_w$ consisting of $F \in \mathrm{Fil}_D^d \mathbb{L}_w$. By (KV2),

$$\mathfrak{kv}(\Gamma)_w = \mathrm{Fil}_D^2 \mathfrak{kv}(\Gamma)_w.$$

LEMMA 2.2. — Assume that $\Gamma = \{e\}$. Let $F \in \mathbb{L}_w$ satisfying (KV1) with $w \geq 2$. Then (KV2) for F is equivalent to

$$\mathrm{tr}(G_x x + F_y y) = 0. \tag{2.1}$$

Proof. — By $\mathrm{tr} \circ \mathbf{q} \circ \pi_Y(F(z, y)) = \mathrm{tr} \circ \pi_Y(F(z, y)) = \mathrm{tr}(-F_x(z, y)y + F_y(z, y)y)$, the condition (KV2) is equivalent to

$$\mathrm{tr}((F_y - F_x)y) = 0.$$

By (KV1), we have $Gx - xG = yF - Fy$. So $F_y = F^y$ and $G_y = F^x$. By $F \in \mathbb{L}_w$, $F_x = (-1)^{w-1} \mathrm{anti}(F^x)$ and $F_y = (-1)^{w-1} \mathrm{anti}(F^y)$. By $G \in \mathbb{L}_w$, we have $\mathrm{tr}(G_x x + G_y y) = 0$. Therefore

$$\begin{aligned} \mathrm{tr}((F_y - F_x)y) &= (-1)^{w-1} \mathrm{tr}((F^y - F^x)y) \\ &= (-1)^{w-1} \mathrm{tr}(F_y y - G_y y) = (-1)^{w-1} \mathrm{tr}(F_y y + G_x x), \end{aligned}$$

whence we get the claim. \square

Remark 2.3. — Since (2.1) agrees with original defining condition (KV2) in [3], we see that our $\mathfrak{kv}(\Gamma)_\bullet$ with $\Gamma = \{e\}$ recovers the original Kashiwara–Vergne Lie algebra denoted by \mathfrak{kv}_2^0 in [3] and the depth >1 -part of \mathfrak{kv} in [17, Definition 3]. We do not know how their Lie algebras \mathfrak{kv}_{n+1}^0 ($n \geq 0$) in [3] and also krv_{n+1} in [2] are related to our $\mathfrak{kv}(\Gamma)_\bullet$. It is also not clear if our $\mathfrak{kv}(\Gamma)_\bullet$ forms a Lie algebra or not.

Let $h \in \mathbb{A}_w$ be a degree w homogeneous polynomial with $h = \sum_{r=0}^w h^r$ and

$$h^r = \sum_{(\sigma_1, \dots, \sigma_r) \in \Gamma^{\oplus r}} \sum_{(e_0, \dots, e_r) \in \mathcal{E}_w^r} a(h :_{\sigma_1, \dots, \sigma_r}^{e_0, \dots, e_r}) x^{e_0} y_{\sigma_1} \dots y_{\sigma_r} x^{e_r} \in \mathbb{A}_{w,r} \tag{2.2}$$

where $\mathcal{E}_w^r = \{(e_0, \dots, e_r) \in \mathbb{N}_0^{r+1} \mid \sum_{i=0}^r e_i = w - r\}$.

DEFINITION 2.4. — *By following [20, Appendix A], we associate a mould*

$$\mathrm{ma}_h = (\mathrm{ma}_h^0, \mathrm{ma}_h^1, \mathrm{ma}_h^2, \dots, \mathrm{ma}_h^w, 0, 0, \dots) \in \mathcal{M}(\mathcal{F}; \Gamma)$$

which is defined by $\mathrm{ma}_h^r = \{\mathrm{ma}_h^r(u_1, \dots, u_r)\}_{(\sigma_1, \dots, \sigma_r) \in \Gamma^{\oplus r}}$ with

$$\begin{aligned} \mathrm{ma}_h^r(u_1, \dots, u_r) &= \mathrm{vimo}_h^r(0, u_1, u_1 + u_2, \dots, u_1 + \dots + u_r), \\ \mathrm{vimo}_h^r(z_0, \dots, z_r) &= \sum_{(e_0, \dots, e_r) \in \mathcal{E}_w^r} a\left(h : \begin{matrix} e_0, & \dots, & e_r \\ \sigma_1^{-1}, & \dots, & \sigma_r^{-1} \end{matrix}\right) z_0^{e_0} z_1^{e_1} z_2^{e_2} \dots z_r^{e_r}. \end{aligned}$$

We start with the following technical lemma which is required to our later arguments.

LEMMA 2.5. — *When $h \in \mathbb{L}_{w,r}$, we have*

$$\mathrm{vimo}_h^r(z_0, \dots, z_r) = \mathrm{vimo}_h^r(0, z_1 - z_0, \dots, z_r - z_0), \quad (2.3)$$

$$\mathrm{mantar} \circ \mathrm{ma}_h^r = \mathrm{ma}_h^r. \quad (2.4)$$

Proof. — The proof of (2.3) can be done by induction on degree in the same way to the arguments in [20] p. 71. Assume $h \in \mathbb{L}_{w,r}$ is given by $[f, g]$ for some f and $g \in \mathbb{L}$ with depth s and t respectively. Then by definition, we have

$$\begin{aligned} \mathrm{vimo}_h^r(z_0, \dots, z_r) &= \mathrm{vimo}_f^s(z_0, \dots, z_s) \mathrm{vimo}_g^t(z_s, \dots, z_{s+t}) \\ &\quad - \mathrm{vimo}_g^t(z_0, \dots, z_t) \mathrm{vimo}_f^s(z_t, \dots, z_{s+t}). \end{aligned}$$

While by our induction assumption we have

$$\begin{aligned} &\mathrm{vimo}_h^r(0, z_1 - z_0, \dots, z_r - z_0) \\ &= \mathrm{vimo}_f^s(0, z_1 - z_0, \dots, z_s - z_0) \mathrm{vimo}_g^t(z_s - z_0, \dots, z_{s+t} - z_0) \\ &\quad - \mathrm{vimo}_g^t(0, z_1 - z_0, \dots, z_t - z_0) \mathrm{vimo}_f^s(z_s - z_0, \dots, z_{s+t} - z_0) \\ &= \mathrm{vimo}_f^s(z_0, \dots, z_s) \mathrm{vimo}_g^t(z_s, \dots, z_{s+t}) - \mathrm{vimo}_g^t(z_0, \dots, z_t) \mathrm{vimo}_f^s(z_t, \dots, z_{s+t}) \\ &= \mathrm{vimo}_h^r(z_0, \dots, z_r). \end{aligned}$$

The equation (2.4) is a formal generalization of [20, Lemma A.2]. We give a short proof below. Since $h \in \mathbb{L}_w$, we have $h = (-1)^{w-1} \mathrm{anti}(h)$. Thus by definition, we have $\mathrm{vimo}_h^r(z_0, \dots, z_r) = (-1)^{w-1} \mathrm{vimo}_h^r(z_r, \dots, z_0)$. Therefore

$$\begin{aligned} \mathrm{ma}_h^r(z_1, \dots, z_r) &= \mathrm{vimo}_h^r(0, z_1, z_1 + z_2, \dots, z_1 + \dots + z_r) \\ &= (-1)^{w-1} \mathrm{vimo}_h^r(z_1 + \dots + z_r, \dots, z_1 + z_2, z_1, 0) \\ &= (-1)^{r-1} \mathrm{vimo}_h^r(-z_1 - \dots - z_r, \dots, -z_1 - z_2, -z_1, 0) \\ &= (-1)^{r-1} \mathrm{vimo}_h^r(0, z_r, z_r + z_{r-1}, \dots, z_r + \dots + z_1) \\ &= (-1)^{r-1} \mathrm{ma}_h^r(z_r, \dots, z_1) = \mathrm{mantar} \circ \mathrm{ma}_h^r(z_1, \dots, z_r). \end{aligned}$$

Here in the third equality we use that vimo_h^r is homogeneous with degree $w - r$ and the fourth equality is by (2.3). \square

DEFINITION 2.6. — Let \mathfrak{tdet} be the set of tangential derivation of \mathbb{L} , the derivation $D_{\{F_\sigma\}_\sigma, G}$ of \mathbb{L} defined by $x \mapsto [x, G]$ and $y_\sigma \mapsto [y_\sigma, F_\sigma]$ for some $F_\sigma, G \in \mathbb{L}$. It forms a Lie algebra by the bracket

$$\begin{aligned} & [D_{\{F_\sigma^{(1)}\}, G^{(1)}}, D_{\{F_\sigma^{(2)}\}, G^{(2)}}] \\ &= D_{\{F_\sigma^{(1)}\}, G^{(1)}} \circ D_{\{F_\sigma^{(2)}\}, G^{(2)}} - D_{\{F_\sigma^{(2)}\}, G^{(2)}} \circ D_{\{F_\sigma^{(1)}\}, G^{(1)}}. \end{aligned} \quad (2.5)$$

The action of Γ on \mathbb{L} induces the Γ -action on \mathfrak{tdet} . We denote its invariant part by \mathfrak{tdet}^Γ . We mean \mathfrak{sdet} to be the set of special derivations, tangential derivations such that $D_{\{F_\sigma\}, G}(x + \sum_\sigma y_\sigma) = 0$ and \mathfrak{sdet}^Γ to be its intersection with \mathfrak{tdet}^Γ , both of which forms a Lie subalgebra of \mathfrak{tdet} . We put $\mathfrak{mt} := \bigoplus_{(w,d) \neq (1,0)} \mathbb{L}_{w,d}$. It forms a Lie algebra by the bracket

$$\{f_1, f_2\} = D_{\{\sigma(f_1)\}, 0}(f_2) - D_{\{\sigma(f_2)\}, 0}(f_1) + [f_1, f_2], \quad (2.6)$$

in other words, $D_{\{\sigma(\{f_1, f_2\})\}, 0} := [D_{\{\sigma(f_1)\}, 0}, D_{\{\sigma(f_2)\}, 0}]$.

We occasionally regard \mathfrak{mt} as a Lie subalgebra of \mathfrak{tdet} by $f \mapsto D_{\{\sigma(f)\}, 0}$. We note that the condition (KV1) is equivalent to $D_{\{\sigma(F)\}, G} \in \mathfrak{sdet}^\Gamma$.

We regard \mathfrak{sdet}^Γ and \mathfrak{mt} as filtered graded Lie algebras by encoding them with $\text{Fil}_{\mathcal{D}}^d \mathfrak{sdet}_w^\Gamma = \{D_{\{\sigma(F)\}, G} \in \mathfrak{sdet}^\Gamma \mid F \in \text{Fil}_{\mathcal{D}}^d \mathbb{L}_w\}$ and $\text{Fil}_{\mathcal{D}}^d \mathfrak{mt} = \{D_{\{\sigma(F)\}, 0} \in \mathfrak{mt} \mid F \in \text{Fil}_{\mathcal{D}}^d \mathbb{L}_w\}$. The following is required in the next section.

LEMMA 2.7. — We have a natural graded Lie algebra homomorphism

$$\text{res} : \text{gr}_{\mathcal{D}}(\mathfrak{sdet}^\Gamma) \rightarrow \mathfrak{mt} \quad (2.7)$$

sending $D_{\{\tau(F)\}, G} \mapsto D_{\{\tau(\bar{F})\}, 0}$.

Proof. — Let $D_i = D_{\{\tau(F_i)\}, G_i} \in \text{Fil}_{\mathcal{D}}^{d_i} \mathfrak{sdet}_{w_i}^\Gamma$ for $i = 1, 2$. Put $D_3 = [D_1, D_2]$. Since it belongs to \mathfrak{sdet}^Γ , D_3 is expressed as $D_{\{\tau(F_3)\}, G_3}$ for some $F_3 \in \mathbb{L}$ and $G_3 \in \mathbb{L}$.

By (KV1), we have $\tau(F_i) \in \text{Fil}_{\mathcal{D}}^{d_i} \mathbb{L}_{w_i}$, $G_i \in \text{Fil}_{\mathcal{D}}^{d_i+1} \mathbb{L}_{w_i}$ ($i = 1, 2$). Since F_3 is calculated to be

$$F_3 = D_{\{\tau(F_1)\}, G_1}(F_2) - D_{\{\tau(F_2)\}, G_2}(F_1) + [F_1, F_2]$$

by (2.5), we see that $F_3 \in \text{Fil}_{\mathcal{D}}^{d_1+d_2} \mathbb{L}_{w_1+w_2}$. The residue class $\bar{F}_3 \in \text{gr}_{\mathcal{D}}^{d_1+d_2} \mathbb{L}_{w_1+w_2}$ is calculated as

$$\bar{F}_3 = D_{\{\tau(\bar{F}_1)\}, 0}(\bar{F}_2) - D_{\{\tau(\bar{F}_2)\}, 0}(\bar{F}_1) + [\bar{F}_1, \bar{F}_2], \quad (2.8)$$

where $\bar{F}_1 \in \text{gr}_{\mathcal{D}}^{d_1} \mathbb{L}_{w_1}$ and $\bar{F}_2 \in \text{gr}_{\mathcal{D}}^{d_2} \mathbb{L}_{w_2}$ are the residue classes of F_1 and F_2 . Therefore our map is a Lie algebra homomorphism. \square

PROPOSITION 2.8. — *The map ma sending $h \rightarrow \text{ma}_h$ induces a filtered graded Lie algebra isomorphism*

$$\text{ma} : \text{mt} \simeq \text{ARI}(\Gamma)_{\text{al}}^{\text{fin}, \text{pol}} \tag{2.9}$$

where $\text{ARI}(\Gamma)_{\text{al}}^{\text{fin}, \text{pol}}$ (cf. Definition 1.2) is the Lie algebra (cf. Proposition 1.15) equipped with the ari_u -bracket.

Proof. — It can be proved completely in a same way to that of [21, Theorem 3.4.2]. \square

We prepare the following technical lemma which is required to the proof of a reformulation of (KV1) in Lemma 2.10.

LEMMA 2.9. — *Let $H \in \mathbb{L}_w$ with $w \geq 2$. Assume that H has no words starting with any y_σ and ending in any y_τ . Then there exists $G \in \mathbb{L}_{w-1}$ such that $H = [x, G]$.*

Proof. — The proof goes on the same way to the proof of [20, Proposition 2.2]. Define the derivation ∂_x of \mathbb{A} sending $x \mapsto 1$ and $y_\sigma \mapsto 0$ and the \mathbb{Q} -linear endomorphism sec of \mathbb{A} by $\text{sec}(h) := \sum_{i \geq 0} \frac{(-1)^i}{i!} \partial_x^i(h) x^i$ for $h \in \mathbb{A}$. Let us write $H = H_x x + \sum_\sigma H_{y_\sigma} y_\sigma$. Then by our assumption, we have $P \in \mathbb{A}$ such that $xP = \sum_\sigma H_{y_\sigma} y_\sigma$. Then by [16, Proposition 4.2.2] and $\partial_x^i(xP) = i \partial_x^{i-1}(P) + x \partial_x^i(P)$, we have

$$\begin{aligned} H &= \text{sec}\left(\sum_\sigma H_{y_\sigma} y_\sigma\right) = \text{sec}(xP) = \sum_{i \geq 0} \frac{(-1)^i}{i!} \partial_x^i(xP) x^i \\ &= \sum_{i \geq 1} \frac{(-1)^i}{(i-1)!} \partial_x^{i-1}(P) x^i + x \sum_{i \geq 0} \frac{(-1)^i}{i!} \partial_x^i(P) x^i \\ &= -\text{sec}(P)x + x \text{sec}(P) = [x, \text{sec}(P)]. \end{aligned}$$

It remains to show that $G = \text{sec}(P)$ is in \mathbb{L} , which follows from exactly the same arguments to the last half of the proof of [20, Proposition 2.2]. \square

Let $F = F(x; (y_\sigma)) \in \mathbb{L}$. As in [20], we put

$$f(x; (y_\sigma)) = F(z; (y_\sigma)) \quad \text{and} \quad \tilde{f}(x; (y_\sigma)) = f(x; (-y_\sigma)) \tag{2.10}$$

with $z = -x - \sum_\sigma y_\sigma$.

The following generalizes the equivalence between (i) and (v) in [20, Theorem 2.1].

LEMMA 2.10. — *Let $F \in \mathbb{L}_w$ with $w \geq 1$. Then saying (KV1) for F is equivalent to saying that*

$$\{\tilde{f}_{y_\gamma} + \tilde{f}_x\} = (-1)^{w-1} \gamma \cdot \text{anti}\{\tilde{f}_{y_{\gamma^{-1}}} + \tilde{f}_x\} \tag{2.11}$$

for all $\gamma \in \Gamma$.

Proof. — Firstly, we show that (KV1) for F is equivalent to $\alpha(F)_{y_\beta} = \beta(F)^{y_\alpha}$ for any $\alpha, \beta \in \Gamma$. Set $H = \sum_{\sigma} [y_{\sigma}, \sigma(F)]$. Then we have

$$H = \sum_{\alpha, \beta} y_{\alpha} \{ \alpha(F)_{y_{\beta}} - \beta(F)^{y_{\alpha}} \} y_{\beta} + \sum_{\alpha} y_{\alpha} \alpha(F)_{x} x - \sum_{\beta} x \beta(F)^x y_{\beta}. \quad (2.12)$$

- Assume (KV1) for F . Then

$$H = Gx - xG. \quad (2.13)$$

So H has no words starting with any y_{σ} and ending in any y_{τ} . By (2.12), we have $\alpha(F)_{y_{\beta}} = \beta(F)^{y_{\alpha}}$.

- Conversely assume $\alpha(F)_{y_{\beta}} = \beta(F)^{y_{\alpha}}$. Then H has no words starting with any y_{σ} and ending in any y_{τ} . By Lemma 2.9, there is a $G \in \mathbb{L}_w$ such that $H = [G, x]$. Whence we get (KV1).

Secondly, by $\alpha(F) \in \mathbb{L}_w$, we have $\alpha(F)^{y_{\beta}} = (-1)^{w-1} \text{anti}(\alpha(F)_{y_{\beta}})$. Therefore $\alpha(F)_{y_{\beta}} = \beta(F)^{y_{\alpha}}$ is equivalent to $\alpha(F)_{y_{\beta}} = (-1)^{w-1} \text{anti}(\beta(F)_{y_{\alpha}})$.

Lastly, by (2.10) we have $f_{y_{\alpha}}(z; (y_{\sigma})) - f_x(z; (y_{\sigma})) = F_{y_{\alpha}}$. Hence

$$\alpha(F)_{y_{\beta}} = \alpha(F_{y_{\alpha-1\beta}}) = \alpha \left\{ f_{y_{\alpha-1\beta}}(z; (y_{\sigma})) - f_x(z; (y_{\sigma})) \right\}$$

Whence $\alpha(F)_{y_{\beta}} = (-1)^{w-1} \text{anti}(\beta(F)_{y_{\alpha}})$ is equivalent to

$$\begin{aligned} \alpha \left\{ f_{y_{\alpha-1\beta}}(z; (y_{\sigma})) - f_x(z; (y_{\sigma})) \right\} \\ = (-1)^{w-1} \text{anti} \left(\beta \left\{ f_{y_{\alpha\beta-1}}(z; (y_{\sigma})) - f_x(z; (y_{\sigma})) \right\} \right) \end{aligned}$$

from which our claim follows. \square

The following reformulation is suggested by the arguments in [20, Appendix A].

LEMMA 2.11. — *Let $\tilde{f} \in \mathbb{L}_w$ with $w > 1$. Then (2.11) for all $\gamma \in \Gamma$ is equivalent to the following senary relation⁽¹³⁾ (cf. [9, (3.64)])*

$$\text{teru}(M)^r = \text{push} \circ \text{mantar} \circ \text{teru} \circ \text{mantar}(M)^r \quad (2.14)$$

for $1 \leq r \leq w$ with $M = \text{ma}_{\tilde{f}}$.

Proof. — Because $M = \text{ma}_{\tilde{f}}$ is mantar invariant for $\tilde{f} \in \mathbb{L}_w$ by (2.4), the senary relation (2.14) is equivalent to

$$\text{swap} \circ \text{teru}(M)^r = \text{swap} \circ \text{push} \circ \text{mantar} \circ \text{teru}(M)^r. \quad (2.15)$$

⁽¹³⁾ It is because it consists of 6 terms, 3 terms on each hand side.

For simplicity, we denote $\sigma_i \dots \sigma_j$ by $\sigma_{i,j}$ for $1 \leq i \leq j \leq r$. By definition, its left hand side is calculated to be

$$\begin{aligned} & \text{vimo}_{\tilde{f}}^r \left(\begin{matrix} 0, & v_r, & \dots, & v_2, & v_1 \\ \sigma_{1,r}, & \dots, & \sigma_{1,2}, & \sigma_1 \end{matrix} \right) \\ & + \frac{1}{v_1 - v_2} \left\{ \text{vimo}_{\tilde{f}}^{r-1} \left(\begin{matrix} 0, & v_r, & \dots, & v_3, & v_1 \\ \sigma_{1,r}, & \dots, & \sigma_{1,3}, & \sigma_{1,2} \end{matrix} \right) - \text{vimo}_{\tilde{f}}^{r-1} \left(\begin{matrix} 0, & v_r, & \dots, & v_3, & v_2 \\ \sigma_{1,r}, & \dots, & \sigma_{1,3}, & \sigma_{1,2} \end{matrix} \right) \right\}. \end{aligned} \tag{2.16}$$

By $\text{vimo}_{\tilde{f}_r}^r \left(\begin{matrix} z_0, \dots, z_r \\ \sigma_1, \dots, \sigma_r \end{matrix} \right) = (-1)^{w-r} \text{vimo}_{\tilde{f}_r}^r \left(\begin{matrix} -z_0, \dots, -z_r \\ \sigma_1, \dots, \sigma_r \end{matrix} \right)$ and definition, its right hand side is calculated to be

$$\begin{aligned} & (-1)^{w-1} \left[\text{vimo}_{\tilde{f}}^r \left(\begin{matrix} 0, & v_3 - v_2, & \dots, & v_r - v_2, & -v_2, & v_1 - v_2 \\ \sigma_{2,2}, & \dots, & \sigma_{2,r-1}, & \sigma_{2,r}, & \sigma_1^{-1} \end{matrix} \right) \right. \\ & \left. + \frac{1}{v_1} \left\{ \text{vimo}_{\tilde{f}}^{r-1} \left(\begin{matrix} 0, & v_3 - v_2, & \dots, & v_r - v_2, & v_1 - v_2 \\ \sigma_{2,2}, & \dots, & \sigma_{2,r-1}, & \sigma_{2,r} \end{matrix} \right) - \text{vimo}_{\tilde{f}}^{r-1} \left(\begin{matrix} 0, & v_3 - v_2, & \dots, & v_r - v_2, & -v_2 \\ \sigma_{2,2}, & \dots, & \sigma_{2,r-1}, & \sigma_{2,r} \end{matrix} \right) \right\} \right]. \end{aligned} \tag{2.17}$$

By $\text{vimo}_{\tilde{f}_r}^r \left(\begin{matrix} z_0, \dots, z_r \\ \sigma_1, \dots, \sigma_r \end{matrix} \right) = \text{vimo}_{\tilde{f}_r}^r \left(\begin{matrix} 0, z_1 - z_0, \dots, z_r - z_0 \\ \sigma_1, \dots, \sigma_r \end{matrix} \right)$ in (2.3) and the change of variables $z_0 = -v_1$, $z_1 = v_r - v_1$, \dots , $z_{r-1} = v_2 - v_1$ and $\gamma = \sigma_1$, $\tau_1 = \sigma_{1,r}$, \dots , $\tau_{r-1} = \sigma_{1,2}$, the formula (2.16) is equal to

$$\begin{aligned} & \text{vimo}_{\tilde{f}_r}^r \left(\begin{matrix} z_0, \dots, z_{r-1}, 0 \\ \tau_1, \dots, \tau_{r-1}, \gamma \end{matrix} \right) \\ & + \frac{1}{z_{r-1}} \left\{ \text{vimo}_{\tilde{f}_r}^{r-1} \left(\begin{matrix} z_0, \dots, z_{r-1} \\ \tau_1, \dots, \tau_{r-1} \end{matrix} \right) - \text{vimo}_{\tilde{f}_r}^{r-1} \left(\begin{matrix} z_0, \dots, z_{r-2}, 0 \\ \tau_1, \dots, \tau_{r-2}, \tau_{r-1} \end{matrix} \right) \right\}, \end{aligned} \tag{2.18}$$

and (2.17) is equal to

$$\begin{aligned} & (-1)^{w-1} \left[\text{vimo}_{\tilde{f}_r}^r \left(\begin{matrix} z_{r-1}, \dots, z_0, 0 \\ \gamma^{-1} \tau_{r-1}, \dots, \gamma^{-1} \tau_1, \gamma^{-1} \end{matrix} \right) \right. \\ & \left. + \frac{1}{z_0} \left\{ \text{vimo}_{\tilde{f}_r}^{r-1} \left(\begin{matrix} z_{r-1}, \dots, z_1, z_0 \\ \gamma^{-1} \tau_{r-1}, \dots, \gamma^{-1} \tau_1 \end{matrix} \right) - \text{vimo}_{\tilde{f}_r}^{r-1} \left(\begin{matrix} z_{r-1}, \dots, z_1, 0 \\ \gamma^{-1} \tau_{r-1}, \dots, \gamma^{-1} \tau_1 \end{matrix} \right) \right\} \right]. \end{aligned} \tag{2.19}$$

Whence (2.14) is equivalent to (2.18)=(2.19) for $1 \leq r \leq w$ and $\tau_1, \dots, \tau_{r-1}, \gamma \in \Gamma$.

On the other hand, (2.11) for all $\gamma \in \Gamma$ is equivalent to⁽¹⁴⁾

$$\begin{aligned} & \text{vimo}_{\tilde{f}_y}^r \left(\begin{matrix} z_0, \dots, z_r \\ \tau_1, \dots, \tau_r \end{matrix} \right) + \text{vimo}_{\tilde{f}_x}^r \left(\begin{matrix} z_0, \dots, z_r \\ \tau_1, \dots, \tau_r \end{matrix} \right) \\ & = (-1)^{w-1} \left[\text{vimo}_{\gamma\text{-anti}(\tilde{f}_{y_{\gamma^{-1}}})}^r \left(\begin{matrix} z_0, \dots, z_r \\ \tau_1, \dots, \tau_r \end{matrix} \right) + \text{vimo}_{\gamma\text{-anti}(\tilde{f}_x)}^r \left(\begin{matrix} z_0, \dots, z_r \\ \tau_1, \dots, \tau_r \end{matrix} \right) \right] \end{aligned} \tag{2.20}$$

for $\gamma \in \Gamma$ and $1 \leq r \leq w$. By definition, we have

$$\text{vimo}_{(\tilde{f}_y)^r y_\gamma}^{r+1} \left(\begin{matrix} z_0, \dots, z_{r+1} \\ \tau_1, \dots, \tau_{r+1} \end{matrix} \right) = \delta_{\tau_{r+1}, \gamma} \text{vimo}_{\tilde{f}_x}^{r+1} \left(\begin{matrix} z_0, \dots, z_r, 0 \\ \tau_1, \dots, \tau_r, \gamma \end{matrix} \right)$$

⁽¹⁴⁾ Here \tilde{f}_x^r and \tilde{f}_y^r mean $(\tilde{f}_x)^r$ and $(\tilde{f}_y)^r$.

where $\delta_{\tau_{r+1}, \gamma} := \begin{cases} 1 & \tau_{r+1} = \gamma, \\ 0 & \text{otherwise.} \end{cases}$ So by using this, we get

$$\text{vimo}_{\tilde{f}_{y_\gamma}^r} (z_0, \dots, z_r) = \text{vimo}_{(\tilde{f}_{y_\gamma})^r y_\gamma}^{r+1} (z_0, \dots, z_r, 0) = \text{vimo}_{\tilde{f}_{r+1}^{r+1}} (z_0, \dots, z_r, 0). \quad (2.21)$$

By

$$\tilde{f}^r = (\tilde{f}_x)^r x + \sum_{\gamma \in \Gamma} (\tilde{f}_{y_\gamma})^{r-1} y_\gamma \text{ and } \text{vimo}_{(\tilde{f}_x)^r x} (z_0, \dots, z_r) = z_r \cdot \text{vimo}_{\tilde{f}_x^r} (z_0, \dots, z_r),$$

we have

$$\text{vimo}_{\tilde{f}_x^r} (z_0, \dots, z_r) = \frac{1}{z_r} \left\{ \text{vimo}_{\tilde{f}_r^r} (z_0, \dots, z_r) - \text{vimo}_{\tilde{f}_r^r} (z_0, \dots, z_{r-1}, 0) \right\}. \quad (2.22)$$

Hence, by (2.21) and (2.22), the left hand side of (2.20) is calculated to be

$$\text{vimo}_{\tilde{f}_{r+1}^{r+1}} (z_0, \dots, z_r, 0) + \frac{1}{z_r} \left\{ \text{vimo}_{\tilde{f}_r^r} (z_0, \dots, z_r) - \text{vimo}_{\tilde{f}_r^r} (z_0, \dots, z_{r-1}, 0) \right\}, \quad (2.23)$$

and the left hand side of (2.20) is calculated to be

$$\begin{aligned} & (-1)^{w-1} \left[\text{vimo}_{\tilde{f}_{r+1}^{r+1}} \left(\gamma^{-1} \tau_r, \dots, \gamma^{-1} \tau_1, \gamma^{-1} \right) \right. \\ & \quad \left. + \frac{1}{z_0} \left\{ \text{vimo}_{\tilde{f}_r^r} \left(\gamma^{-1} \tau_r, \dots, \gamma^{-1} \tau_1 \right) - \text{vimo}_{\tilde{f}_r^r} \left(\gamma^{-1} \tau_r, \dots, \gamma^{-1} \tau_1 \right) \right\} \right]. \quad (2.24) \end{aligned}$$

So (2.11) for all $\gamma \in \Gamma$ is equivalent to (2.23)=(2.24) for $1 \leq r+1 \leq w$ and $\tau_1, \dots, \tau_r, \gamma \in \Gamma$. This is nothing but (2.18)=(2.19) for $r-1$ instead of r . Therefore we get the equivalence between (2.11) and (2.14). \square

Next we consider the condition (KV2).

LEMMA 2.12. — *Let $F \in \mathbb{L}_w$. Then (KV2) for F is equivalent to*

$$\sum_{i \in \mathbb{Z}/r\mathbb{Z}} a(\mathbf{q} \circ \pi_Y(f) : \sigma_{i+1, \sigma_{i+2}, \dots, \sigma_{i+r}}^{e_i, e_{i+1}, \dots, e_{i+r-1}, 0}) = 0 \quad (2.25)$$

for each $1 \leq r \leq w$, $(\sigma_1, \dots, \sigma_r) \in \Gamma^r$ and $(e_0, \dots, e_r) \in \mathcal{E}_w^r$ with f as in (2.10).

Proof. — It is immediate that (KV2) for F is equivalent to

$$\text{tr}(\mathbf{q} \circ \pi_Y(f)) = 0. \quad (2.26)$$

For $W = u_1 \dots u_w \in \mathbb{A}_w$ with $u_i = x$ or y_σ ($\sigma \in \Gamma$), we define its cyclic permutation by $\text{cy}(W) := u_2 \dots u_w u_1 \in \mathbb{A}_w$. We put $c(W)$ to be the number of its cycles. It divides w . Then the natural pairing in $\text{Cyc}(\mathbb{A})$ is calculated

by the one in \mathbb{A} as follows: When $W = x^{e_0}y_{\sigma_1} \dots x^{e_{r-1}}y_{\sigma_r}$, we have

$$\begin{aligned} \left\langle \operatorname{tr}(\mathbf{q} \circ \pi_Y(f)) \mid \operatorname{tr}(W) \right\rangle_{\operatorname{Cyc}(\mathbb{A})} &= \left\langle \mathbf{q} \circ \pi_Y(f) \mid \sum_{i=0}^{c(W)-1} \mathbf{c}y^i(W) \right\rangle_{\mathbb{A}} \\ &= \frac{c(W)}{w} \left\langle \mathbf{q} \circ \pi_Y(f) \mid \sum_{i=0}^{w-1} \mathbf{c}y^i(W) \right\rangle_{\mathbb{A}} \\ &= \frac{c(W)}{w} \sum_{i \in \mathbb{Z}/r\mathbb{Z}} a(\mathbf{q} \circ \pi_Y(f) : \begin{smallmatrix} e_i, \dots, e_{i+r-1}, 0 \\ \sigma_{i+1}, \dots, \sigma_{i+r} \end{smallmatrix}). \end{aligned}$$

So we get the claim. \square

LEMMA 2.13. — *Let $f \in \mathbb{L}_w$. Then (2.25) for f is equivalent to the pus-neutrality (1.9) for $\operatorname{swap}(\operatorname{ma}_{\tilde{f}})$, i.e.*

$$\sum_{i \in \mathbb{Z}/r\mathbb{Z}} \operatorname{pus}^i \circ \operatorname{swap}(M)^r(\begin{smallmatrix} \sigma_1, \dots, \sigma_r \\ v_1, \dots, v_r \end{smallmatrix}) = 0 \quad (2.27)$$

with $M = \operatorname{ma}_{\tilde{f}}$ for all $1 \leq r \leq w$ and $\sigma_i \in \Gamma$ ($1 \leq i \leq r$).

Proof. — We decompose $\tilde{f} = \sum_r \tilde{f}^r$ and describe $\tilde{f}^r \in \mathbb{L}_{w,r}$ as in (2.2). Then

$$\operatorname{ma}_{\tilde{f}}^r(\begin{smallmatrix} u_1, \dots, u_r \\ \sigma_1, \dots, \sigma_r \end{smallmatrix}) = \sum_{\substack{(e_0, \dots, e_r) \in \mathcal{E}_w^r \\ e_0=0}} a(\tilde{f} : \begin{smallmatrix} e_0, \dots, e_r \\ \sigma_1^{-1}, \dots, \sigma_r^{-1} \end{smallmatrix}) u_1^{e_1} (u_1 + u_2)^{e_2} \dots (u_1 + \dots + u_r)^{e_r}.$$

By $\tilde{f}^r \in \mathbb{L}_{w,r}$, we have

$$a(\tilde{f} : \begin{smallmatrix} e_0, \dots, e_r \\ \sigma_1, \dots, \sigma_r \end{smallmatrix}) = (-1)^{w+1} a(\tilde{f} : \begin{smallmatrix} e_r, \dots, e_0 \\ \sigma_r, \dots, \sigma_1 \end{smallmatrix})$$

because $\operatorname{anti}(\tilde{f}^r) = (-1)^{w+1} \tilde{f}^r$. So

$$\begin{aligned} &\operatorname{swap} \circ \operatorname{ma}_{\tilde{f}}^r(\begin{smallmatrix} \sigma_1, \dots, \sigma_r \\ v_1, \dots, v_r \end{smallmatrix}) \\ &= \sum_{\substack{(e_0, \dots, e_r) \in \mathcal{E}_w^r \\ e_0=0}} a(\tilde{f} : \begin{smallmatrix} e_0, e_1, \dots, e_r \\ (\sigma_1 \dots \sigma_r)^{-1}, \dots, (\sigma_1 \sigma_2)^{-1}, \sigma_1^{-1} \end{smallmatrix}) v_1^{e_1} v_{r-1}^{e_2} \dots v_1^{e_r} \\ &= (-1)^{w+1} \sum_{\substack{(e_0, \dots, e_r) \in \mathcal{E}_w^r \\ e_0=0}} a(\tilde{f} : \begin{smallmatrix} e_r, \dots, e_1, e_0 \\ \sigma_1^{-1}, (\sigma_1 \sigma_2)^{-1}, \dots, (\sigma_1 \dots \sigma_r)^{-1} \end{smallmatrix}) v_1^{e_r} v_2^{e_{r-1}} \dots v_r^{e_1} \\ &= (-1)^{w+1} \sum_{\substack{(e_0, \dots, e_r) \in \mathcal{E}_w^r \\ e_r=0}} a(\tilde{f} : \begin{smallmatrix} e_0, \dots, e_{r-1}, e_r \\ \sigma_1^{-1}, (\sigma_1 \sigma_2)^{-1}, \dots, (\sigma_1 \dots \sigma_r)^{-1} \end{smallmatrix}) v_1^{e_0} v_2^{e_1} \dots v_r^{e_{r-1}} \end{aligned}$$

$$\begin{aligned}
 &= (-1)^{w+r+1} \sum_{\substack{(e_0, \dots, e_r) \in \mathcal{E}_w^r \\ e_r=0}} a(f : \sigma_1^{-1}, (\sigma_1 \sigma_2)^{-1}, \dots, (\sigma_1 \dots \sigma_r)^{-1}) v_1^{e_0} v_2^{e_1} \dots v_r^{e_{r-1}} \\
 &= (-1)^{w+r+1} \sum_{\substack{(e_0, \dots, e_r) \in \mathcal{E}_w^r \\ e_r=0}} a(\mathbf{q} \circ \pi_Y(f) : \sigma_1^{-1}, \sigma_2^{-1}, \dots, \sigma_r^{-1}) v_1^{e_0} v_2^{e_1} \dots v_r^{e_{r-1}}.
 \end{aligned}$$

Here in the last equality we use

$$a(\mathbf{q} \circ \pi_Y(h) : \tau_1, \dots, \tau_r, 0) = a(h : \tau_1, \tau_1 \tau_2, \dots, \tau_1 \dots \tau_r)$$

for any $h \in \pi(\mathbb{A})$. Therefore we obtain

$$\begin{aligned}
 &\sum_{i \in \mathbb{Z}/r\mathbb{Z}} \text{pus}^i \circ \text{swap} \circ \text{ma}_{\bar{f}}^r(\sigma_1, \dots, \sigma_r) \\
 &= (-1)^{w+r+1} \sum_{i \in \mathbb{Z}/r\mathbb{Z}} \sum_{\substack{(e_0, \dots, e_r) \in \mathcal{E}_w^r \\ e_r=0}} a(\mathbf{q} \circ \pi_Y(f) : \sigma_{i+1}^{-1}, \dots, \sigma_{i+r}^{-1}, 0) v_1^{e_0} v_2^{e_1} \dots v_r^{e_{r-1}}.
 \end{aligned}$$

It is immediate to see that it is equivalent to (2.25). □

The following definition of the mould version of $\mathfrak{f}\mathfrak{v}_\bullet$ is suggested by our previous lemmas.

DEFINITION 2.14. — *We define the \mathbb{Q} -linear space $\text{ARI}(\Gamma)_{\text{sena}/\text{pusnu}}$ to be the subset of moulds M in ARI which satisfy the senary relation (2.14) and whose swap satisfy the pus-neutrality (2.27).*

The \mathbb{Q} -linear space $\text{ARI}(\Gamma)_{\text{sena}/\text{pusnu}}$ is filtered by the depth filtration $\{\text{Fil}_{\mathcal{D}}^m\}_m$. We note that $\text{Fil}_{\mathcal{D}}^2 \text{ARI}(\Gamma)_{\text{sena}/\text{pusnu}} \cap \text{ARI}(\Gamma)_{\text{al}}^{\text{fin}, \text{pol}}$ is identified with the finite and depth > 1 -part of $\text{ARI}_{\text{al}+\text{sen}*\text{circonst}}^{\text{pol}}$ in [17, Proposition 28] when $\Gamma = \{e\}$. It is because their formula (79), actually (81), agrees with (2.14) for a mould in $\text{ARI}(\Gamma)_{\text{al}}^{\text{fin}, \text{pol}}$ by Lemma 3.7.

There is an embedding of \mathbb{Q} -linear space

$$\text{gr}_{\mathcal{D}} \text{ARI}(\Gamma)_{\text{sena}/\text{pusnu}} \hookrightarrow \text{ARI}(\Gamma)_{\text{push}/\text{pusnu}} \tag{2.28}$$

that is, the associated graded quotient $\text{gr}_{\mathcal{D}} \text{ARI}(\Gamma)_{\text{sena}/\text{pusnu}}$ of the filtered \mathbb{Q} -linear space $\text{ARI}(\Gamma)_{\text{sena}/\text{pusnu}}$ is embedded to $\text{ARI}(\Gamma)_{\text{push}/\text{pusnu}}$ introduced in Definition 1.26. It is because the push-invariance (1.5) is associated graded with (2.14) by $\text{gr}_{\mathcal{D}}(\text{teru}) = \text{id}$ and $\text{mantar} \circ \text{mantar} = \text{id}$.

THEOREM 2.15. — *The map sending $F \in \mathbb{A} \mapsto \text{ma}_{\bar{f}} \in \mathcal{M}(\mathcal{F}; \Gamma)$ induces an isomorphism of filtered \mathbb{Q} -linear spaces*

$$\mathfrak{f}\mathfrak{v}(\Gamma)_\bullet \simeq \text{ARI}(\Gamma)_{\text{sena}/\text{pusnu}} \cap \text{ARI}(\Gamma)_{\text{al}}^{\text{fin}, \text{pol}}. \tag{2.29}$$

Proof. — The restriction of our map decomposes as

$$F \in \mathfrak{ktv}(\Gamma)_\bullet \mapsto \tilde{f} \in \mathfrak{mt} \mapsto \text{ma}_{\tilde{f}} \in \text{ARI}(\Gamma)_{\text{al}}^{\text{fin,pol}}. \quad (2.30)$$

By Proposition 2.8, we see that it gives an isomorphism $\mathfrak{mt} \simeq \text{ARI}(\Gamma)_{\text{al}}^{\text{fin,pol}}$ as (actually filtered) \mathbb{Q} -linear spaces. Thus we get the claim by our previous lemmas. \square

Remark 2.16. — When $\Gamma = \{e\}$, the above isomorphism (2.29) can be recovered by the composition of the isomorphisms (29), (71) and Proposition 28⁽¹⁵⁾ in [17].

The authors are not sure if the bigger space $\text{ARI}(\Gamma)_{\text{sena/pusnu}}$ is equipped with a structure of Lie algebra under the ari_u -bracket or not although we show that a related space $\text{ARI}(\Gamma)_{\text{push/pusnu}}$ forms a Lie algebra in Theorem 1.32.

2.2. Kashiwara–Vergne bigraded Lie algebra

Based on our arguments in the previous subsection, we introduce a Γ -variant $\mathfrak{ktv}(\Gamma)_{\bullet\bullet}$ of the Kashiwara–Vergne bigraded Lie algebra \mathfrak{ktv} ([17]) in Definition 2.17 and give a mould theoretical interpretation in Theorem 2.22. As a corollary we show that it forms a Lie algebra in Theorem 2.23.

DEFINITION 2.17. — *Kashiwara–Vergne bigraded Lie algebra is defined to be the bigraded \mathbb{Q} -linear space $\mathfrak{ktv}(\Gamma)_{\bullet\bullet} = \bigoplus_{w>1, d>0} \mathfrak{ktv}(\Gamma)_{w,d}$, where $\mathfrak{ktv}(\Gamma)_{w,d}$ is the \mathbb{Q} -linear space consisting of $\bar{F} \in \text{gr}_{\mathcal{D}}^d \mathbb{L}_w$ whose lift $F \in \text{Fil}_{\mathcal{D}}^d \mathbb{L}_w$ satisfies the following relations, “the leading-terms” of (KV1) and (KV2),*

$$[x, G] + \sum_{\tau \in \Gamma} [y_\tau, \tau(F)] \equiv 0 \pmod{\text{Fil}_{\mathcal{D}}^{d+2} \mathbb{L}_{w+1}}, \quad (\text{LKV1})$$

$$\text{tr} \circ \mathbf{q} \circ \pi_Y(F) \equiv 0 \pmod{\text{tr}(\text{Fil}_{\mathcal{D}}^{d+1} \mathbb{A}_w)} \quad (\text{LKV2})$$

for a certain $G = G(F) \in \mathbb{L}_w$.

We note that such $G \in \mathbb{L}_w$ is in $\text{Fil}_{\mathcal{D}}^{d+1} \mathbb{L}_w$ and is uniquely determined modulo $\text{Fil}_{\mathcal{D}}^{d+2} \mathbb{L}_w$ by (LKV1). We note that, by (LKV2) and $\dim \mathbb{L}_{w,1} = 1$, we have

$$\mathfrak{ktv}_{w,d} = \{0\} \quad \text{for } d = 1.$$

⁽¹⁵⁾ It looks that there is a slip on the isomorphism on [17, Proposition 28] because its right hand side is completed by *depth* while its left hand side is not.

Remark 2.18. — When $\Gamma = \{e\}$, our definition of $\mathfrak{ltrv}(\Gamma)_{\bullet\bullet}$ does not agree with that of \mathfrak{ltrv} in [17, Definition 5 and 10], which is constructed as a Lie polynomial version of $\text{ARI}(\Gamma)_{\text{push}/\text{pusnu}}$. However their depth >1 -parts are eventually isomorphic to our depth >1 -parts via Theorem 2.22.

By definition, there is an inclusion of bigraded \mathbb{Q} -linear spaces

$$\text{gr}_{\mathcal{D}}\mathfrak{ltrv}(\Gamma)_{\bullet} \hookrightarrow \mathfrak{ltrv}(\Gamma)_{\bullet\bullet}, \tag{2.31}$$

which generalizes [17, Proposition 2], that is, the associated graded \mathbb{Q} -linear space $\text{gr}_{\mathcal{D}}\mathfrak{ltrv}(\Gamma)_{\bullet}$ of the filtered Lie algebra $\mathfrak{ltrv}(\Gamma)_{\bullet}$ is embedded to $\mathfrak{ltrv}(\Gamma)_{\bullet\bullet}$. We do not know if it is an isomorphism.

For $F \in \text{Fil}_{\mathcal{D}}^d\mathbb{L}_w$, we put \mathfrak{f} to be the element in $\mathbb{L}_{w,d}$ corresponding to $(-1)^{w-d}\bar{F} \in \text{gr}_{\mathcal{D}}^m\mathbb{L}_w$ under the natural identification $\text{gr}_{\mathcal{D}}^d\mathbb{L}_w \simeq \mathbb{L}_{w,d}$. By abuse of notation,

$$\mathfrak{f} = (-1)^{w-d}\bar{F}. \tag{2.32}$$

We write $\mathfrak{f} = \mathfrak{f}_x x + \sum_{\sigma} \mathfrak{f}_{y_{\sigma}} y_{\sigma}$. We also put

$$\tilde{\mathfrak{f}} = \mathfrak{f}(x, (-y_{\sigma})_{\sigma})$$

and write $\tilde{\mathfrak{f}} = \tilde{\mathfrak{f}}_x x + \sum_{\sigma} \tilde{\mathfrak{f}}_{y_{\sigma}} y_{\sigma}$.

The following is a bigraded variant of Lemma 2.10.

LEMMA 2.19. — *Let $F \in \text{Fil}_{\mathcal{D}}^d\mathbb{L}_w$ for $w \geq 1$. Then (LKV1) for F is equivalent to*

$$\tilde{\mathfrak{f}}_{y_{\gamma}} = (-1)^{w-1}\gamma(\text{anti}(\tilde{\mathfrak{f}}_{y_{\gamma^{-1}}})) \tag{2.33}$$

for any $\gamma \in \Gamma$.

Proof. — The proof goes similarly to Lemma 2.10.

Firstly, we show that (LKV1) for F is equivalent to $\alpha(F)_{y_{\beta}} \equiv \beta(F)^{y_{\alpha}} \pmod{\text{Fil}_{\mathcal{D}}^d\mathbb{L}_{w-1}}$. Set $H = \sum_{\sigma} [y_{\sigma}, \sigma(F)] \in \text{Fil}_{\mathcal{D}}^{d+1}\mathbb{L}_{w+1}$.

Assume (LKV1) for F . Then $H \equiv Gx - xG \pmod{\text{Fil}_{\mathcal{D}}^{d+2}\mathbb{L}_{w+1}}$. So the depth $(d+1)$ -part of H has no words starting and ending in any y_{σ} . By (2.12), we have $\alpha(F)_{y_{\beta}} \equiv \beta(F)^{y_{\alpha}} \pmod{\text{Fil}_{\mathcal{D}}^d\mathbb{L}_{w-1}}$.

Conversely assume $\alpha(F)_{y_{\beta}} \equiv \beta(F)^{y_{\alpha}} \pmod{\text{Fil}_{\mathcal{D}}^d\mathbb{L}_{w-1}}$. Then the depth $(d+1)$ -part of H has no words starting and ending in any y_{σ} . By Lemma 2.9, there is a $G \in \mathbb{L}_w$ which express this part as $[G, x]$, i.e. $H \equiv [G, x] \pmod{\text{Fil}_{\mathcal{D}}^{d+2}\mathbb{L}_{w+1}}$. Whence we get (LKV1).

Secondly, by $\alpha(F) \in \mathbb{L}$, we have $\alpha(F)^{y_{\beta}} = (-1)^{w-1}\text{anti}(\alpha(F)_{y_{\beta}})$. Therefore $\alpha(F)_{y_{\beta}} \equiv \beta(F)^{y_{\alpha}} \pmod{\text{Fil}_{\mathcal{D}}^d\mathbb{L}_{w-1}}$ is equivalent to

$$\alpha(F)_{y_{\beta}} \equiv (-1)^{w-1}\text{anti}(\beta(F)_{y_{\alpha}}) \pmod{\text{Fil}_{\mathcal{D}}^d\mathbb{L}_{w-1}}.$$

Lastly, by (2.32) and $\alpha(F)_{y_\beta} = \alpha(F_{y_{\alpha^{-1}\beta}})$, it is equivalent to $\alpha(\mathfrak{f}_{y_{\alpha^{-1}\beta}}) = (-1)^{w-1} \text{anti}(\beta(\mathfrak{f}_{y_{\alpha\beta^{-1}}}))$, so for $\tilde{\mathfrak{f}}_y$. \square

The following might be regarded as a bigraded variant of Lemma 2.11.

LEMMA 2.20. — *Let $\tilde{\mathfrak{f}} \in \mathbb{L}_{w,d}$ with $w > 1$. Then (2.33) for any $\gamma \in \Gamma$ is equivalent to push-invariance (1.5) for $M = \text{ma}_{\tilde{\mathfrak{f}}}$.*

Proof. — The proof goes similarly to Lemma 2.11. The condition (2.33) for $\gamma \in \Gamma$ is reformulated to

$$\text{vimo}_{\tilde{\mathfrak{f}}_\gamma}^r \left(\begin{smallmatrix} z_0, \dots, z_r \\ \tau_1, \dots, \tau_r \end{smallmatrix} \right) = (-1)^{w-1} \text{vimo}_{\gamma \cdot \text{anti}(\tilde{\mathfrak{f}}_{y_{\gamma^{-1}}})}^r \left(\begin{smallmatrix} z_0, \dots, z_r \\ \tau_1, \dots, \tau_r \end{smallmatrix} \right)$$

for $1 \leq r \leq w$, which is equivalent to

$$\text{vimo}_{\tilde{\mathfrak{f}}_{r+1}}^{r+1} \left(\begin{smallmatrix} z_0, \dots, z_r, 0 \\ \tau_1, \dots, \tau_r, \gamma \end{smallmatrix} \right) = (-1)^{w-1} \text{vimo}_{\tilde{\mathfrak{f}}_{r+1}}^{r+1} \left(\begin{smallmatrix} z_r, \dots, z_0, 0 \\ \gamma^{-1}\tau_r, \dots, \gamma^{-1}\tau_1, \gamma^{-1} \end{smallmatrix} \right) \quad (2.34)$$

for $1 \leq r+1 \leq w$.

On the other hand, (1.5) can be shown to be equivalent to

$$\text{vimo}_{\tilde{\mathfrak{f}}_r}^r \left(\begin{smallmatrix} z_0, \dots, z_{r-1}, 0 \\ \tau_1, \dots, \tau_{r-1}, \gamma \end{smallmatrix} \right) = (-1)^{w-1} \text{vimo}_{\tilde{\mathfrak{f}}_r}^r \left(\begin{smallmatrix} z_{r-1}, \dots, z_0, 0 \\ \gamma^{-1}\tau_{r-1}, \dots, \gamma^{-1}\tau_1, \gamma^{-1} \end{smallmatrix} \right)$$

for $1 \leq r \leq w$, which turns to be equivalent to (2.34). \square

The following is a bigraded variant of Lemma 2.13.

LEMMA 2.21. — *Then (LKV2) for $F \in \text{Fil}_D^d \mathbb{L}_w$ is equivalent to the pus-neutrality (1.9) for $M = \text{swap}(\text{ma}_{\tilde{\mathfrak{f}}})$, i.e. (2.27) for all $r \geq 1$,*

Proof. — We note that actually only the terms for $r = d$ contributes in the above equation. Decompose $\tilde{\mathfrak{f}}$ as in (2.2). Then by the arguments in the proof of Lemma 2.13, (2.27) is equivalent to

$$\sum_{i \in \mathbb{Z}/r\mathbb{Z}} a \left(\mathbf{q} \circ \pi_Y(\mathfrak{f}) : \begin{smallmatrix} e_i, \dots, e_{i+r-1}, 0 \\ \sigma_{i+1}^{-1}, \dots, \sigma_{i+r}^{-1} \end{smallmatrix} \right) = 0$$

for all $(\sigma_1, \dots, \sigma_r) \in \Gamma^r$ and $(e_0, \dots, e_{r-1}, 0) \in \mathcal{E}_w^r$. It is nothing but

$$\text{tr} \circ \mathbf{q} \circ \pi_Y(\mathfrak{f}) = 0,$$

which is equivalent to (LKV2). \square

THEOREM 2.22. — *The map sending $\bar{F} \in \text{gr}_D \mathbb{L} \mapsto \text{ma}_{\tilde{\mathfrak{f}}} \in \mathcal{M}(\mathcal{F}; \Gamma)$ induces an isomorphism of bigraded \mathbb{Q} -linear spaces*

$$\mathfrak{ltrv}(\Gamma)_{\bullet\bullet} \simeq \text{ARI}(\Gamma)_{\text{push/pusnu}} \cap \text{ARI}(\Gamma)_{\text{al}}^{\text{fin, pol}}.$$

Proof. — Our map decomposes as

$$\bar{F} \in \mathfrak{ltrv}(\Gamma)_{\bullet\bullet} \mapsto \tilde{\mathfrak{f}} \in \mathfrak{mt} \mapsto \text{ma}_{\tilde{\mathfrak{f}}} \in \text{ARI}(\Gamma)_{\text{al}}^{\text{fin, pol}}. \quad (2.35)$$

The first map is injective and the second one is isomorphic by Proposition 2.8. Our claim follows by our previous lemmas. \square

As a generalization of [17, Proposition 1], we obtain the following.

THEOREM 2.23. — *The space $\mathfrak{ltrv}(\Gamma)_{\bullet\bullet}$ forms a bigraded Lie algebra under the bracket (2.6).*

Proof. — As for the morphism (2.35), the second map is Lie algebra homomorphism by Proposition 2.8. It is easy to see that the first map forms a Lie algebra homomorphism when we encode $\mathfrak{ltrv}(\Gamma)_{\bullet\bullet}$ with the bracket (2.6). Thus our claim follows because it is shown that $\text{ARI}(\Gamma)_{\text{push/pusnu}}$ forms a Lie algebra by Theorem 1.32 and so $\text{ARI}(\Gamma)_{\text{al}}$ does by Lemma 1.15. \square

Remark 2.24. — By (2.28), (2.31), Theorem 2.15 and Theorem 2.22, we obtain the following commutative diagram of Lie algebras.

$$\begin{array}{ccc} \text{gr}_{\mathcal{D}}\mathfrak{ltrv}(\Gamma)_{\bullet} & \xrightarrow{\cong} & \text{gr}_{\mathcal{D}}(\text{ARI}(\Gamma)_{\text{sena/pusnu}} \cap \text{ARI}(\Gamma)_{\text{al}}^{\text{fin,pol}}) \\ \downarrow & & \downarrow \\ \mathfrak{ltrv}(\Gamma)_{\bullet\bullet} & \xrightarrow{\cong} & \text{ARI}(\Gamma)_{\text{push/pusnu}} \cap \text{ARI}(\Gamma)_{\text{al}}^{\text{fin,pol}} \end{array} \quad (2.36)$$

The diagram is commutative because (2.35) is associated graded with (2.30).

Similarly to Definition 1.34, we impose a distribution relation on $\mathfrak{ltrv}(\Gamma)$.

DEFINITION 2.25. — *For $N \geq 1$ and $\Gamma^N = \{g^N \in \Gamma \mid g \in \Gamma\}$, we consider the map $i_N : \mathfrak{ltrv}(\Gamma)_{\bullet\bullet} \rightarrow \mathfrak{ltrv}(\Gamma^N)_{\bullet\bullet}$ which is induced by*

$$i_N(x) = x, \quad i_N(y_\tau) = \begin{cases} y_\tau & \text{when } \tau \in \Gamma^N, \\ 0 & \text{when } \tau \notin \Gamma^N \end{cases}$$

and also the map $m_N : \mathfrak{ltrv}(\Gamma)_{\bullet\bullet} \rightarrow \mathfrak{ltrv}(\Gamma^N)_{\bullet\bullet}$ which is induced by

$$m_N(x) = Nx, \quad m_N(y_\sigma) = y_{\sigma^N}$$

for $\sigma \in \Gamma$. We define the following \mathbb{Q} -linear subspace

$$\mathfrak{ltrv}\mathfrak{d}(\Gamma)_{\bullet\bullet} := \{\bar{F} \in \mathfrak{ltrv}(\Gamma)_{\bullet\bullet} \mid i_N(\bar{F}) = m_N(\bar{F}) \text{ for all } N \geq 1\}.$$

As a corollary of Theorems 2.22 and 2.23, we obtain the following corollary.

COROLLARY 2.26. — *The space $\mathfrak{ltrv}\mathfrak{d}(\Gamma)$ forms a bigraded Lie algebra and we have the following isomorphism of bigraded Lie algebras*

$$\mathfrak{ltrv}\mathfrak{d}(\Gamma)_{\bullet\bullet} \simeq \text{ARI}(\Gamma)_{\text{push/pusnu}} \cap \text{ARID}(\Gamma)_{\text{al}}^{\text{fin,pol}}.$$

Proof. — By definition we see that both i_N and m_N form Lie algebra homomorphisms and by Proposition 1.14 $\text{ARI}(\Gamma)$ forms a Lie algebra.

Therefore our claim follows from the commutativity of the following diagrams and Proposition 1.35:

$$\begin{array}{ccc}
 \mathfrak{ftrv}(\Gamma)_{\bullet\bullet} & \longrightarrow & \text{ARI}(\Gamma) \\
 \downarrow i_N & & \downarrow i_N \\
 \mathfrak{ftrv}(\Gamma^N)_{\bullet\bullet} & \longrightarrow & \text{ARI}(\Gamma^N),
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathfrak{ftrv}(\Gamma)_{\bullet\bullet} & \longrightarrow & \text{ARI}(\Gamma) \\
 \downarrow m_N & & \downarrow m_N \\
 \mathfrak{ftrv}(\Gamma^N)_{\bullet\bullet} & \longrightarrow & \text{ARI}(\Gamma^N).
 \end{array}
 \quad \square$$

3. Dihedral Lie algebra

By using mould theoretic interpretations of the bigraded Lie algebra $\mathbb{D}(\Gamma)_{\bullet\bullet}$ (Γ : a finite abelian group) with a dihedral symmetry and of the Kashiwara–Vergne bigraded Lie algebra $\mathfrak{ftrv}(\Gamma)_{\bullet\bullet}$, we realize an embedding $\text{Fil}_{\mathcal{D}}^2 \mathbb{D}(\Gamma)_{\bullet\bullet} \hookrightarrow \mathfrak{ftrv}(\Gamma)_{\bullet\bullet}$ which extends the result of [17].

3.1. Dihedral bigraded Lie algebra

We recall the definition of the dihedral bigraded Lie algebra $D(\Gamma)_{\bullet\bullet}$ introduced by Goncharov [12] and introduce a related Lie algebra $\mathbb{D}(\Gamma)_{\bullet\bullet}$ which contains $D(\Gamma)_{\bullet\bullet}$ in Definitions 3.1 and 3.4.

We call an element $f = f(t_1, \dots, t_{m+1})$ in $\mathbb{Q}[t_1, \dots, t_{m+1}]$ *translation invariant* when $f(t_1, \dots, t_{m+1}) = f(t_1 + c, \dots, t_{m+1} + c)$ for any $c \in \mathbb{Q}$. We often denote this by $f(t_1 : \dots : t_{m+1})$. We consider a set of collections

$$\mathcal{Z} = \{ \mathcal{Z}(g_1, \dots, g_m, g_{m+1} \mid t_1 : \dots : t_{m+1}) \}_{g_1, \dots, g_m \in \Gamma} \quad (3.1)$$

with

$$g_{m+1} = (g_1 \dots g_m)^{-1}$$

of translation invariant element in $\mathbb{Q}[t_1, \dots, t_{m+1}]$. For such \mathcal{Z} , we associate

$$\begin{aligned}
 & \tilde{\mathcal{Z}}(g_1 : \dots : g_{m+1} \mid t_1, \dots, t_{m+1}) \\
 & := \mathcal{Z}(g_1^{-1} g_2, g_2^{-1} g_3, \dots, g_m^{-1} g_{m+1}, g_{m+1}^{-1} g_1 \mid t_1 : t_1 + t_2 : \dots : t_1 + \dots + t_m : 0)
 \end{aligned} \quad (3.2)$$

with any $g_1, \dots, g_{m+1} \in \Gamma$ and $t_1 + \dots + t_{m+1} = 0$ and also

$$\begin{aligned}
 & \mathcal{Z}(g_1 : \dots : g_{m+1} \mid t_1 : \dots : t_{m+1}) \\
 & := \tilde{\mathcal{Z}}(g_1 : \dots : g_{m+1} \mid t_1 - t_{m+1}, t_2 - t_1, \dots, t_m - t_{m-1}, t_{m+1} - t_m)
 \end{aligned} \quad (3.3)$$

$$= \mathcal{Z}(g_1^{-1} g_2, g_2^{-1} g_3, \dots, g_m^{-1} g_{m+1}, g_{m+1}^{-1} g_1 \mid t_1 : t_2 : \dots : t_m : t_{m+1}) \quad (3.4)$$

with any $g_1, \dots, g_{m+1} \in \Gamma$ and any t_1, \dots, t_{m+1} .

DEFINITION 3.1 ([12]). — *Set-theoretically the dihedral bigraded Lie algebra means the \mathbb{Q} -linear bigraded space*

$$D(\Gamma)_{\bullet\bullet} = \bigoplus_{w,m} D(\Gamma)_{w,m},$$

where the bidegree (w, m) -part $D(\Gamma)_{w,m}$ is defined to be the set \mathcal{Z} in (3.1) of translation invariant elements in $\mathbb{Q}[t_1, \dots, t_{m+1}]$ with total degree $w - m$ satisfying

- (a) the double shuffle relation, that is,
 - (i) the harmonic product

$$\sum_{\sigma \in \text{III}_{p,q}} \mathcal{Z}(g_{\sigma(1)}, \dots, g_{\sigma(m)}, g_{m+1} \mid t_{\sigma(1)} : \dots : t_{\sigma(m)} : t_{m+1}) = 0 \quad (3.5)$$

- (ii) the shuffle product

$$\sum_{\sigma \in \text{III}_{p,q}} \tilde{\mathcal{Z}}(g_{\sigma(1)} : \dots : g_{\sigma(m)} : g_{m+1} \mid t_{\sigma(1)}, \dots, t_{\sigma(m)}, t_{m+1}) = 0 \quad (3.6)$$

for any $p, q \geq 1$ with $p + q = m$.

- (b) the distribution relation for $N \in \mathbb{Z}$ such that $|N|$ divides $|\Gamma|$,

$$\begin{aligned} & Z(g_1 : \dots : g_{m+1} \mid t_1 : \dots : t_{m+1}) \\ &= \frac{1}{|\Gamma_N|} \sum_{h_i^N = g_i} Z(h_1 : \dots : h_{m+1} \mid Nt_1 : \dots : Nt_{m+1}) \end{aligned} \quad (3.7)$$

except that a constant in t is allowed when $m = 1$ and $g_1 = g_2$. Here $|\Gamma_N|$ is the order of the N -torsion subgroup of Γ and $\text{III}_{p,q}$ is defined by (1.2).

- (c) Additionally we put

$$Z(e : e \mid 0 : 0) = 0 \quad (3.8)$$

for a technical reason.

Its Lie algebra structure is explained in [12, §§4-5]. In [12, Theorem 4.1], it is shown that the double shuffle relation implies the dihedral symmetry relations:

THEOREM 3.2 ([12, Theorem 4.1]). — *The double shuffle relation implies the dihedral symmetry relations, which consist of the cyclic symmetry relation*

$$\begin{aligned} & Z(g_1 : g_2 : \dots : g_{m+1} \mid t_1 : t_2 : \dots : t_{m+1}) \\ &= Z(g_2 : \dots : g_{m+1} : g_1 \mid t_2 : \dots : t_{m+1} : t_1), \end{aligned} \quad (3.9)$$

the inversion relation (“the distribution relation for $N = -1$ ”)

$$Z(g_1 : \cdots : g_{m+1} \mid t_1 : \cdots : t_{m+1}) = Z(g_1^{-1} : \cdots : g_{m+1}^{-1} \mid -t_1 : \cdots : -t_{m+1}).$$

the reflection relation

$$\begin{aligned} Z(g_1 : g_2 : \cdots : g_{m+1} \mid t_1 : \cdots : t_m : t_{m+1}) \\ = (-1)^{m+1} Z(g_{m+1} : \cdots : g_1 \mid -t_m : \cdots : -t_1 : -t_{m+1}), \end{aligned}$$

for $m \geq 2$.

The following reformulation of the above dihedral symmetry relations is useful in our later arguments.

Remark 3.3. — The transformation (3.4) allows us to rewrite the above dihedral symmetry relations as follows:

the cyclic symmetry relation

$$\begin{aligned} \underline{Z}(g_1, g_2, \dots, g_{m+1} \mid t_1 : t_2 : \cdots : t_{m+1}) \\ = \underline{Z}(g_2, \dots, g_{m+1}, g_1 \mid t_2 : \cdots : t_{m+1} : t_1), \end{aligned} \quad (3.10)$$

the inversion relation

$$\begin{aligned} \underline{Z}(g_1, g_2, \dots, g_{m+1} \mid t_1 : t_2 : \cdots : t_{m+1}) \\ = \underline{Z}(g_1^{-1}, g_2^{-1}, \dots, g_{m+1}^{-1} \mid -t_1 : -t_2 : \cdots : -t_{m+1}), \end{aligned} \quad (3.11)$$

the reflection relation

$$\begin{aligned} \underline{Z}(g_1, \dots, g_m, g_{m+1} \mid t_1 : \cdots : t_m : t_{m+1}) \\ = (-1)^{m+1} \underline{Z}(g_m^{-1}, \dots, g_1^{-1}, g_{m+1}^{-1} \mid -t_m : \cdots : -t_1 : -t_{m+1}) \end{aligned} \quad (3.12)$$

with $g_1 g_2 \dots g_{m+1} = 1$.

DEFINITION 3.4. — Goncharov [12, §4.5 and §5.2] also introduced a related Lie algebra

$$D'(\Gamma)_{\bullet\bullet} = \bigoplus_{w,m} D'(\Gamma)_{w,m}$$

which consists of the collections \underline{Z} satisfying the shuffle product (3.6), the cyclic symmetry relation (3.9) and the additional condition (3.8) (cf. [12, Proposition 4.6]). For our purpose, we consider its \mathbb{Q} -linear subspace

$$\mathbb{D}(\Gamma)_{\bullet\bullet} = \bigoplus_{w,m} \mathbb{D}(\Gamma)_{w,m}$$

which consists of elements in $D'(\Gamma)_{\bullet\bullet}$ satisfying the double shuffle relations (a).

By Theorem 3.2, we have

$$D'(\Gamma)_{\bullet\bullet} \supset \mathbb{D}(\Gamma)_{\bullet\bullet} \supset D(\Gamma)_{\bullet\bullet}.$$

3.2. Mould theoretic reformulation

We explain a reformulation of $\mathbb{D}(\Gamma)\bullet\bullet$ and $D(\Gamma)\bullet\bullet$ in terms of moulds in Theorem 3.5 and Corollary 3.6 respectively.

Let \underline{Z} be a collection (3.1) of translation invariant elements. We associate a mould

$$M_{\underline{Z}} = (M_{\underline{Z}}^i)_{i \in \mathbb{Z}_{\geq 0}} \in \mathcal{M}(\mathcal{F}; \Gamma)$$

by $M_{\underline{Z}}^i = 0$ for $i \neq m$ and

$$M_{\underline{Z}}^m(u_1, \dots, u_m) = \tilde{Z}(g_1 : \dots : g_m : 1 \mid u_1, \dots, u_{m+1}).$$

The following is a generalization of the results in [14, 21] which treat the case of $\Gamma = \{e\}$.

THEOREM 3.5. — *The map sending $M : \underline{Z} \in D'(\Gamma)\bullet\bullet \mapsto M_{\underline{Z}} \in \text{ARI}(\Gamma)$ forms a Lie algebra homomorphism and it induces an isomorphism between*

$$\text{Fil}_{\mathcal{D}}^2 \mathbb{D}(\Gamma)\bullet\bullet \simeq \text{Fil}_{\mathcal{D}}^2 \text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}^{\text{fin, pol}}. \tag{3.13}$$

Here the left hand side means the depth > 1 -part of $\mathbb{D}(\Gamma)\bullet\bullet$ and the right hand side means the finite polynomial-valued part of the subset of $\text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$ (cf. Definition 1.22) consisting of M with depth > 1 .

Proof. — It is immediate that (3.6) is equivalent to the condition for $M_{\underline{Z}}$ being in $\text{ARI}(\Gamma)_{\underline{\text{al}}}$. By (3.2), we have

$$\begin{aligned} & \text{swap}(M_{\underline{Z}}^m)_{(v_1, \dots, v_m)}^{(g_1, \dots, g_m)} \\ &= \tilde{Z}(g_1 \cdots g_m : g_1 \cdots g_{m-1} : \dots : g_1 : 1 \mid v_m, v_{m-1} - v_m, \dots, v_2 - v_3, v_1 - v_2, -v_1) \\ &= \underline{Z}(g_m^{-1}, g_{m-1}^{-1}, \dots, g_1^{-1}, g_1 \cdots g_m \mid v_m : \dots : v_2 : v_1 : 0). \end{aligned}$$

Thus we see that (3.5) is equivalent to the condition for $\text{swap}(M_{\underline{Z}})$ being in $\overline{\text{ARI}}(\Gamma)_{\underline{\text{al}}}$. Therefore our map forms a \mathbb{Q} -linear isomorphism (3.13) by Theorem 3.2.

Since $M_{\underline{Z}}$ is in $\text{ARI}(\Gamma)_{\underline{\text{al}}}^{\text{fin, pol}}$ when $\underline{Z} \in D'(\Gamma)\bullet\bullet$, by Proposition 2.8 there is an $h \in \mathbb{L}$ with depth m such that

$$\text{ma}(h) = M_{\underline{Z}},$$

that is,

$$\text{ma}_h^m(u_1, \dots, u_m) = \tilde{Z}(g_1 : \dots : g_m : 1 \mid u_1, \dots, u_{m+1})$$

with $u_1 + \dots + u_m + u_{m+1} = 0$. By (2.3) and (3.3), we have

$$\text{vimo}_h^m(u_0, \dots, u_m) = Z(g_1 : \dots : g_m : 1 \mid u_1 : \dots : u_m : u_0).$$

In [12, Theorem 5.2], $D'(\Gamma)\bullet\bullet$ is realized as a Lie subalgebra of \mathfrak{sdet}^Γ under the morphism

$$\xi'_\Gamma : D'(\Gamma)\bullet\bullet \rightarrow \text{gr}_{\mathcal{D}}\mathfrak{sdet}^\Gamma \tag{3.14}$$

sending each $Z = \{Z(g_1, \dots, g_{m+1} \mid t_1 : \dots : t_{m+1})\}_{g_1, \dots, g_{m+1} \in \Gamma} \in D_{w,m}(\Gamma)$ to the residue class of $D_{\{\sigma(F)\}_\sigma, G(F)} \in \mathfrak{sdet}^\Gamma$ in $\text{gr}_{\mathcal{D}}^m\mathfrak{sdet}^\Gamma$ with

$$\begin{aligned} F &= |\Gamma|^{-1} \sum_{\substack{n_1, \dots, n_{m+1} > 0 \\ g_1, \dots, g_{m+1} \in \Gamma}} I_{n_1, \dots, n_{m+1}}(g_1 : \dots : g_{m+1}) \\ &\quad \cdot X^{n_1-1} Y_{g_1^{-1}g_2}^{-1} \dots X^{n_m-1} Y_{g_1^{-1}g_{m+1}}^{-1} X^{n_{m+1}-1} \\ &= \sum_{\substack{n_0, \dots, n_m > 0 \\ g_1, \dots, g_m \in \Gamma}} I_{n_0, \dots, n_m}(1 : g_1 : \dots : g_m) X^{n_0-1} Y_{g_1} \dots X^{n_{m-1}-1} Y_{g_m} X^{n_m-1} \in \mathbb{L}_{w,m} \end{aligned}$$

when the associated element $Z = \{Z(g_1 : \dots : g_{m+1} \mid t_1 : \dots : t_{m+1})\}_{g_1, \dots, g_{m+1} \in \Gamma}$ given by (3.4) is expressed as

$$\begin{aligned} Z(g_1 : \dots : g_{m+1} \mid t_1 : \dots : t_{m+1}) \\ &:= \sum_{n_1, \dots, n_{m+1} > 0} I_{n_1, \dots, n_{m+1}}(g_1 : \dots : g_{m+1}) t_1^{n_1-1} \dots t_{m+1}^{n_{m+1}-1} \end{aligned}$$

in $\mathbb{Q}[t_1, \dots, t_{m+1}]$. We note that $I_{n_1, \dots, n_{m+1}}(g_1 : \dots : g_{m+1})$ is uniquely determined. By combining the Lie algebra homomorphisms ξ'_Γ with (2.7) and (2.9), we obtain

$$\text{ma} \circ \text{res} \circ \xi'_\Gamma(Z) = \text{ma}(F).$$

By definition, we have

$$\begin{aligned} \text{vimo}_F^m \left(\begin{matrix} u_1, \dots, u_m \\ g_1^{-1}, \dots, g_m^{-1} \end{matrix} \right) &= \sum_{n_1, \dots, n_{m+1} > 0} I_{n_0, \dots, n_m}(1 : g_1 : \dots : g_m) u_0^{n_0-1} \dots u_m^{n_m-1} \\ &= Z(1 : g_1 : \dots : g_m \mid u_0 : \dots : u_m). \end{aligned}$$

By the cyclic symmetry relation (3.9),

$$= Z(g_1 : \dots : g_m : 1 \mid u_1 : \dots : u_m : u_0) = \text{vimo}_h^m \left(\begin{matrix} u_1, \dots, u_m \\ g_1, \dots, g_m \end{matrix} \right).$$

Therefore we have $F(x, (y_\sigma)_\sigma) = h(x, (y_{\sigma^{-1}})_\sigma)$. So

$$M(Z) = \text{ma}(h) = \iota \circ \text{ma} \circ \text{res} \circ \xi'_\Gamma(Z)$$

where ι is the map defined by $(\iota M)^m \left(\begin{matrix} u_1, \dots, u_m \\ g_1, \dots, g_m \end{matrix} \right) = M^m \left(\begin{matrix} u_1, \dots, u_m \\ g_1^{-1}, \dots, g_m^{-1} \end{matrix} \right)$ which forms a Lie algebra homomorphism. Since ι , ma in (2.9), res in (2.7) and ξ'_Γ in (3.14) are all Lie algebra homomorphisms, we see that M is so. Whence we get our claims. \square

Since $\text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$, and hence the right hand side of (3.13), forms a Lie algebra by Proposition 1.24, we learn that $\text{Fil}_{\mathcal{D}}^2 \mathbb{D}(\Gamma)_{\bullet\bullet}$ forms a Lie subalgebra of $D'(\Gamma)_{\bullet\bullet}$.

COROLLARY 3.6. — *The map M in Theorem 3.5 induces a Lie algebra isomorphism between*

$$\text{Fil}_{\mathcal{D}}^2 D(\Gamma)_{\bullet\bullet} \simeq \text{Fil}_{\mathcal{D}}^2 \text{ARID}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}^{\text{fin, pol}}. \tag{3.15}$$

Here the left hand side means the depth >1 -part of $D(\Gamma)_{\bullet\bullet}$ and the right hand side means the finite polynomial-valued part of the subset of $\text{ARID}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$ (cf. Definition 1.34) consisting of M with depth >1 .

Proof. — Since the distribution relation (3.7) is equivalent to

$$\mathbb{Z}(g_1, \dots, g_{m+1} \mid t_1 : \dots : t_{m+1}) = \sum_{h_i^N = g_i} \mathbb{Z}(h_1, \dots, h_{m+1} \mid Nt_1 : \dots : Nt_{m+1}),$$

which corresponds to $i_N(M_{\mathbb{Z}}) = m_N(M_{\mathbb{Z}})$, that is, $M_{\mathbb{Z}} \in \text{ARID}(\Gamma)$. Since $\text{ARID}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$, and hence the right hand side of (3.15), forms a Lie algebra by Corollary 1.36, and $\text{Fil}_{\mathcal{D}}^2 \mathbb{D}(\Gamma)_{\bullet\bullet}$ forms a Lie subalgebra of $D(\Gamma)_{\bullet\bullet}$ by [12, Theorem 5.2], our claim follows. \square

3.3. Embedding

In this subsection, we construct an embedding $\text{Fil}_{\mathcal{D}}^2 \text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}} \hookrightarrow \text{ARI}(\Gamma)_{\text{push}/\text{pusnu}}$ in Theorem 3.13. As a corollary, we get an embedding $\text{Fil}_{\mathcal{D}}^2 \mathbb{D}(\Gamma)_{\bullet\bullet} \hookrightarrow \text{ftv}(\Gamma)_{\bullet\bullet}$ in Corollary 3.14.

The following generalizes [21, Lemma 2.5.3].

LEMMA 3.7. — *Any mould $M \in \text{ARI}(\Gamma)_{\text{al}}$ is mantar-invariant (cf. Notation 1.7), namely, for $m \geq 1$ and $\sigma_1, \dots, \sigma_m \in \Gamma$, we have*

$$M^m \left(\begin{matrix} x_1, \dots, x_m \\ \sigma_1, \dots, \sigma_m \end{matrix} \right) = (-1)^{m-1} M^m \left(\begin{matrix} x_m, \dots, x_1 \\ \sigma_m, \dots, \sigma_1 \end{matrix} \right). \tag{3.16}$$

Proof. — For simplicity, we denote $\omega_i := \begin{pmatrix} x_i \\ \sigma_i \end{pmatrix}$. By using alternality of M , we have

$$\sum_{i=1}^{m-1} (-1)^{i-1} \left\{ \sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\begin{matrix} (\omega_i, \dots, \omega_1); (\omega_{i+1}, \dots, \omega_m) \\ \alpha \end{matrix} \right) M^m(\alpha) \right\} = 0.$$

Here, we calculate the left hand side as follows:

$$\begin{aligned}
 & \sum_{i=1}^{m-1} (-1)^{i-1} \left\{ \sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\begin{matrix} (\omega_i, \dots, \omega_1); (\omega_{i+1}, \dots, \omega_m) \\ \alpha \end{matrix} \right) M^m(\alpha) \right\} \\
 &= \sum_{i=1}^{m-1} (-1)^{i-1} \left\{ \sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\begin{matrix} (\omega_{i-1}, \dots, \omega_1); (\omega_{i+1}, \dots, \omega_m) \\ \alpha \end{matrix} \right) M^m(\omega_i, \alpha) \right. \\
 &\quad \left. + \sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\begin{matrix} (\omega_i, \dots, \omega_1); (\omega_{i+2}, \dots, \omega_m) \\ \alpha \end{matrix} \right) M^m(\omega_{i+1}, \alpha) \right\} \\
 &= \sum_{i=1}^{m-1} (-1)^{i-1} \sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\begin{matrix} (\omega_{i-1}, \dots, \omega_1); (\omega_{i+1}, \dots, \omega_m) \\ \alpha \end{matrix} \right) M^m(\omega_i, \alpha) \\
 &\quad - \sum_{i=2}^m (-1)^{i-1} \sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\begin{matrix} (\omega_{i-1}, \dots, \omega_1); (\omega_{i+1}, \dots, \omega_m) \\ \alpha \end{matrix} \right) M^m(\omega_i, \alpha) \\
 &= M^m(\omega_1, \omega_2, \dots, \omega_m) - (-1)^{m-1} M^m(\omega_m, \omega_{m-1}, \dots, \omega_1).
 \end{aligned}$$

So we obtain (3.16). □

We define the parallel translation map $\underline{t} : \overline{\text{ARI}}(\Gamma) \rightarrow \overline{\text{ARI}}(\Gamma)$ by

$$\underline{t}(M)^m(\mathbf{y}_m) := \begin{cases} M^m(\mathbf{y}_m) & m = 0, 1, \\ M^{m-1} \begin{pmatrix} \sigma_2, & \dots, & \sigma_m \\ y_2 - y_1, & \dots, & y_m - y_1 \end{pmatrix} & m \geq 2, \end{cases}$$

for $M \in \overline{\text{ARI}}(\Gamma)$. For our simplicity, we put

$$\underline{y} := \underline{t} \circ \text{swap}.$$

LEMMA 3.8. — *Let $M \in \text{ARI}(\Gamma)_{\text{al/al}}$ and $m \geq 2$ and $\sigma_1, \dots, \sigma_m \in \Gamma$ with $\sigma_1 \dots \sigma_m = 1$. Then we have*

$$\underline{y}(M)^m \begin{pmatrix} \sigma_1, & \dots, & \sigma_m \\ x_1, & \dots, & x_m \end{pmatrix} = \underline{y}(M)^m \begin{pmatrix} \sigma_2^{-1}, & \dots, & \sigma_m^{-1}, & \sigma_1^{-1} \\ -x_2, & \dots, & -x_m, & -x_1 \end{pmatrix}. \quad (3.17)$$

Proof. — We have

$$\begin{aligned}
 \underline{y}(M)^m \begin{pmatrix} \sigma_1, & \dots, & \sigma_m \\ x_1, & \dots, & x_m \end{pmatrix} &= \text{swap}(M)^{m-1} \begin{pmatrix} \sigma_2, & \sigma_3, & \dots, & \sigma_{m-1}, & \sigma_m \\ x_2 - x_1, & x_3 - x_1, & \dots, & x_{m-1} - x_1, & x_m - x_1 \end{pmatrix} \\
 &= M^{m-1} \begin{pmatrix} x_m - x_1, & x_{m-1} - x_m, & \dots, & x_2 - x_3 \\ \sigma_2 \dots \sigma_m, & \sigma_2 \dots \sigma_{m-1}, & \dots, & \sigma_2 \end{pmatrix}.
 \end{aligned}$$

By using (3.16), we get

$$\begin{aligned}
 &= (-1)^{m-2} M^{m-1} \begin{pmatrix} x_2 - x_3, & \dots, & x_{m-1} - x_m, & x_m - x_1 \\ \sigma_2, & \dots, & \sigma_2 \dots \sigma_{m-1}, & \sigma_2 \dots \sigma_m \end{pmatrix} \\
 &= (-1)^{m-2} \text{swap}(M)^{m-1} \begin{pmatrix} \sigma_2 \dots \sigma_m, & \sigma_m^{-1}, & \dots, & \sigma_3^{-1} \\ x_2 - x_1, & x_2 - x_m, & \dots, & x_2 - x_3 \end{pmatrix}.
 \end{aligned}$$

By (3.16), $\sigma_1 \dots \sigma_m = 1$ and $M \in \text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$, we calculate

$$\begin{aligned} &= \text{swap}(M)^{m-1} \begin{pmatrix} \sigma_3^{-1}, \dots, \sigma_m^{-1}, \sigma_1^{-1} \\ x_2 - x_3, \dots, x_2 - x_m, x_2 - x_1 \end{pmatrix} \\ &= \underline{u}(M)^m \begin{pmatrix} \sigma_2^{-1}, \sigma_3^{-1}, \dots, \sigma_m^{-1}, \sigma_1^{-1} \\ -x_2, -x_3, \dots, -x_m, -x_1 \end{pmatrix}. \end{aligned}$$

Therefore, we obtain the claim. \square

The following three relations should be called as the cyclic symmetry relation, the inversion relation and the reflection relation respectively (compare them with (3.10), (3.11) and (3.12) with $m+1$ replaced with m).

LEMMA 3.9. — *Let $m \geq 3$ and $M \in \text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$. Then, for $\sigma_1, \dots, \sigma_m \in \Gamma$ with $\sigma_1 \dots \sigma_m = 1$, we have the following:*

$$\underline{u}(M)^m \begin{pmatrix} \sigma_1, \sigma_2, \dots, \sigma_m \\ x_1, x_2, \dots, x_m \end{pmatrix} = \underline{u}(M)^m \begin{pmatrix} \sigma_2, \dots, \sigma_m, \sigma_1 \\ x_2, \dots, x_m, x_1 \end{pmatrix}, \quad (3.18)$$

$$\underline{u}(M)^m \begin{pmatrix} \sigma_1, \sigma_2, \dots, \sigma_m \\ x_1, x_2, \dots, x_m \end{pmatrix} = \underline{u}(M)^m \begin{pmatrix} \sigma_1^{-1}, \sigma_2^{-1}, \dots, \sigma_m^{-1} \\ -x_1, -x_2, \dots, -x_m \end{pmatrix}, \quad (3.19)$$

$$\underline{u}(M)^m \begin{pmatrix} \sigma_1, \dots, \sigma_{m-1}, \sigma_m \\ x_1, \dots, x_{m-1}, x_m \end{pmatrix} = (-1)^m \underline{u}(M)^m \begin{pmatrix} \sigma_{m-1}^{-1}, \dots, \sigma_1^{-1}, \sigma_m^{-1} \\ -x_{m-1}, \dots, -x_1, -x_m \end{pmatrix}. \quad (3.20)$$

Proof. — It is easy to get (3.19) from (3.17) and (3.18) and to get (3.20) from (3.16), (3.18) and (3.19). So it is enough to prove (3.18).

Firstly, we have

$$\begin{aligned} &\sum_{\alpha \in Y_{\mathbb{Z}}^\bullet} \text{Sh} \left(\begin{pmatrix} \sigma_1 \\ x_1 \end{pmatrix}; \begin{pmatrix} \sigma_2, \dots, \sigma_m \\ x_2, \dots, x_m \end{pmatrix} \right) \underline{u}(M)^m(\alpha) \\ &= \underline{u}(M)^m \begin{pmatrix} \sigma_1, \sigma_2, \dots, \sigma_m \\ x_1, x_2, \dots, x_m \end{pmatrix} + \sum_{\alpha \in Y_{\mathbb{Z}}^\bullet} \text{Sh} \left(\begin{pmatrix} \sigma_1 \\ x_1 \end{pmatrix}; \begin{pmatrix} \sigma_3, \dots, \sigma_m \\ x_3, \dots, x_m \end{pmatrix} \right) \underline{u}(M)^m \left(\begin{pmatrix} \sigma_2 \\ x_2 \end{pmatrix}, \alpha \right) \\ &= \underline{u}(M)^m \begin{pmatrix} \sigma_1, \dots, \sigma_m \\ x_1, \dots, x_m \end{pmatrix} \\ &\quad + \sum_{\alpha \in Y_{\mathbb{Z}}^\bullet} \text{Sh} \left(\begin{pmatrix} \sigma_1 \\ x_1 - x_2 \end{pmatrix}; \begin{pmatrix} \sigma_3, \dots, \sigma_m \\ x_3 - x_2, \dots, x_m - x_2 \end{pmatrix} \right) \text{swap}(M)^{m-1}(\alpha). \end{aligned}$$

So by using alternality of $\text{swap}(M)$, we obtain

$$\sum_{\alpha \in Y_{\mathbb{Z}}^\bullet} \text{Sh} \left(\begin{pmatrix} \sigma_1 \\ x_1 \end{pmatrix}; \begin{pmatrix} \sigma_2, \dots, \sigma_m \\ x_2, \dots, x_m \end{pmatrix} \right) \underline{u}(M)^m(\alpha) = \underline{u}(M)^m \begin{pmatrix} \sigma_1, \dots, \sigma_m \\ x_1, \dots, x_m \end{pmatrix}. \quad (3.21)$$

Secondly, we have

$$\begin{aligned} &\sum_{\alpha \in Y_{\mathbb{Z}}^\bullet} \text{Sh} \left(\begin{pmatrix} \sigma_1 \\ x_1 \end{pmatrix}; \begin{pmatrix} \sigma_2, \dots, \sigma_m \\ x_2, \dots, x_m \end{pmatrix} \right) \underline{u}(M)^m(\alpha) \\ &= \underline{u}(M)^m \begin{pmatrix} \sigma_2, \dots, \sigma_m, \sigma_1 \\ x_2, \dots, x_m, x_1 \end{pmatrix} + \sum_{\alpha \in Y_{\mathbb{Z}}^\bullet} \text{Sh} \left(\begin{pmatrix} \sigma_1 \\ x_1 \end{pmatrix}; \begin{pmatrix} \sigma_2, \dots, \sigma_{m-1} \\ x_2, \dots, x_{m-1} \end{pmatrix} \right) \underline{u}(M)^m(\alpha, \begin{pmatrix} \sigma_m \\ x_m \end{pmatrix}). \end{aligned}$$

By applying Lemma 3.8 to the second term, we get

$$\begin{aligned}
 &= \underline{y}(M)^m(\sigma_2, \dots, \sigma_m, \sigma_1) \\
 &\quad + \sum_{\alpha \in Y_{\mathbb{Z}}^{\bullet}} \text{Sh}\left(\begin{pmatrix} \sigma_1 \\ x_1 \end{pmatrix}; \begin{pmatrix} \sigma_2, \dots, \sigma_{m-1} \\ x_2, \dots, x_{m-1} \end{pmatrix}; \alpha\right) \underline{y}(M)^m\left(\begin{pmatrix} \sigma_m^{-1} \\ -x_m \end{pmatrix}, \alpha_{-}\right) \\
 &= \underline{y}(M)^m(\sigma_2, \dots, \sigma_m, \sigma_1) \\
 &\quad + \sum_{\alpha \in Y_{\mathbb{Z}}^{\bullet}} \text{Sh}\left(\begin{pmatrix} \sigma_1 \\ x_1 - x_m \end{pmatrix}; \begin{pmatrix} \sigma_2, \dots, \sigma_{m-1} \\ x_2 - x_m, \dots, x_{m-1} - x_m \end{pmatrix}; \alpha\right) \text{swap}(M)^{m-1}(\alpha_{-}).
 \end{aligned}$$

Here, the symbol α_{-} is $\begin{pmatrix} \tau_1^{-1}, \dots, \tau_m^{-1} \\ -u_1, \dots, -u_m \end{pmatrix}$ for $\alpha = \begin{pmatrix} \tau_1, \dots, \tau_m \\ u_1, \dots, u_m \end{pmatrix}$ and for $\begin{pmatrix} \tau_1 \\ u_1 \end{pmatrix}, \dots, \begin{pmatrix} \tau_m \\ u_m \end{pmatrix} \in Y_{\mathbb{Z}}$. So by using alternality of $\text{swap}(M)$, we obtain

$$\sum_{\alpha \in Y_{\mathbb{Z}}^{\bullet}} \text{Sh}\left(\begin{pmatrix} \sigma_1 \\ x_1 \end{pmatrix}; \begin{pmatrix} \sigma_2, \dots, \sigma_m \\ x_2, \dots, x_m \end{pmatrix}; \alpha\right) \underline{y}(M)^m(\alpha) = \underline{y}(M)^m(\sigma_2, \dots, \sigma_m, \sigma_1). \quad (3.22)$$

Therefore, by combining (3.21) and (3.22), we obtain (3.18). \square

LEMMA 3.10. — *For any mould $M \in \text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$, its $\text{swap}(M)$ is neg-invariant and mantar-invariant (cf. Notation 1.7) namely we have*

$$\text{swap}(M)^m(\sigma_1, \dots, \sigma_m) = \text{swap}(M)^m\left(\begin{pmatrix} \sigma_1^{-1}, \dots, \sigma_m^{-1} \\ -x_1, \dots, -x_m \end{pmatrix}\right), \quad (3.23)$$

$$\text{swap}(M)^m(\sigma_1, \dots, \sigma_m) = (-1)^{m-1} \text{swap}(M)^m\left(\begin{pmatrix} \sigma_m^{-1}, \dots, \sigma_1^{-1} \\ -x_m, \dots, -x_1 \end{pmatrix}\right), \quad (3.24)$$

for $m \geq 0$.

Proof. — For $m \geq 2$, the first equation follows from (3.19) and the second one follows from (3.18) and (3.20). For $m = 0, 1$, they follow from the definition of $\text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$. \square

The following can be also found in [21, Lemma 2.5.5] but we give a different proof below.

PROPOSITION 3.11. — *Any mould $M \in \text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$ is push-invariant (1.5).*

Proof. — For $m = 0, 1$, it is obvious because we have $M^1\left(\begin{smallmatrix} x_1 \\ \sigma_1 \end{smallmatrix}\right) = M^1\left(\begin{smallmatrix} -x_1 \\ \sigma_1^{-1} \end{smallmatrix}\right)$. Assume $m \geq 2$. By using (3.18), we have

$$\underline{y}(M)^{m+1}\left(\begin{pmatrix} \tau_1, \tau_2, \dots, \tau_{m+1} \\ y_1, y_2, \dots, y_{m+1} \end{pmatrix}\right) = \underline{y}(M)^{m+1}\left(\begin{pmatrix} \tau_2, \dots, \tau_{m+1}, \tau_1 \\ y_2, \dots, y_{m+1}, y_1 \end{pmatrix}\right),$$

for $\begin{pmatrix} \tau_1 \\ y_1 \end{pmatrix}, \dots, \begin{pmatrix} \tau_{m+1} \\ y_{m+1} \end{pmatrix} \in Y_{\mathbb{Z}}$ with $\tau_1 \cdots \tau_{m+1} = 1$. We calculate the left hand side

$$\begin{aligned}
 \underline{y}(M)^{m+1}\left(\begin{pmatrix} \tau_1, \tau_2, \dots, \tau_{m+1} \\ y_1, y_2, \dots, y_{m+1} \end{pmatrix}\right) &= \text{swap}(M)^m\left(\begin{pmatrix} \tau_2, \dots, \tau_{m+1} \\ y_2 - y_1, \dots, y_{m+1} - y_1 \end{pmatrix}\right) \\
 &= M^m\left(\begin{pmatrix} y_{m+1} - y_1, y_m - y_{m+1}, \dots, y_2 - y_3 \\ \tau_2 \cdots \tau_{m+1}, \tau_2 \cdots \tau_m, \dots, \tau_2 \end{pmatrix}\right).
 \end{aligned}$$

Similarly, we calculate the right hand side

$$\underline{y}(M)^{m+1} \begin{pmatrix} \tau_2, \dots, \tau_{m+1}, \tau_1 \\ y_2, \dots, y_{m+1}, y_1 \end{pmatrix} = M^m \begin{pmatrix} y_1 - y_2, & y_{m+1} - y_1, & y_m - y_{m+1}, & \dots, & y_3 - y_4 \\ \tau_3 \dots \tau_{m+1} \tau_1, & \tau_3 \dots \tau_{m+1}, & \tau_3 \dots \tau_m, & \dots, & \tau_3 \end{pmatrix}.$$

By substituting $y_1 := 0$ and $y_i := x_1 + \dots + x_{m+2-i}$ ($2 \leq i \leq m+1$) and $\tau_1 := \sigma_1^{-1}$ and $\tau_2 := \sigma_m$ and $\tau_j := \sigma_{m+2-j} \sigma_{m+3-j}^{-1}$ ($3 \leq j \leq m+1$) for any $\sigma_1, \dots, \sigma_m \in \Gamma$, we have

$$\begin{aligned} M^m \begin{pmatrix} x_1, x_2, \dots, x_m \\ \sigma_1, \sigma_2, \dots, \sigma_m \end{pmatrix} &= M^m \begin{pmatrix} -x_1 - \dots - x_m, & x_1, & \dots, & x_{m-1} \\ \sigma_m^{-1}, & \sigma_1 \sigma_m^{-1}, & \dots, & \sigma_{m-1} \sigma_m^{-1} \end{pmatrix} \\ &= \text{push}(M)^m \begin{pmatrix} x_1, x_2, \dots, x_m \\ \sigma_1, \sigma_2, \dots, \sigma_m \end{pmatrix}. \end{aligned}$$

Whence we obtain the claim. □

The following generalizes [17, Theorem 14].

PROPOSITION 3.12. — *For any mould $M \in \text{Fil}_{\mathcal{D}}^2 \text{ARI}(\Gamma)_{\text{al}/\text{al}}$, its swap $\text{swap}(M)$ is pus-neutral (1.9), that is, for all $m \geq 1$ and $\sigma_1, \dots, \sigma_m \in \Gamma$,*

$$\sum_{i \in \mathbb{Z}/m\mathbb{Z}} \text{pus}^i \circ \text{swap}(M)^m (\sigma_1, \dots, \sigma_m) = 0.$$

Proof. — For $m = 0, 1$, it is obvious because we have $M^1(x_1) = 0$. Assume $m \geq 2$. Let $\binom{\tau_0}{u_0}, \binom{\tau_1}{u_1}, \dots, \binom{\tau_m}{u_m} \in Y_{\mathbb{Z}}$ with $\tau_0 \tau_1 \dots \tau_m = 1$. By using alternality of $\text{swap}(M)$, we have

$$\sum_{\alpha \in Y_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\binom{\tau_2, \dots, \tau_m}{u_2 - u_1, \dots, u_m - u_1}; \binom{\tau_0}{u_0 - u_1} \right)_{\alpha} \text{swap}(M)^m(\alpha) = 0.$$

On the other hand, we calculate

$$\begin{aligned} &\sum_{\alpha \in Y_{\mathbb{Z}}^{\bullet}} \text{Sh} \left(\binom{\tau_2, \dots, \tau_m}{u_2 - u_1, \dots, u_m - u_1}; \binom{\tau_0}{u_0 - u_1} \right)_{\alpha} \text{swap}(M)^m(\alpha) \\ &= \underline{y}(M)^{m+1} \begin{pmatrix} \tau_1, \tau_2, \dots, \tau_m, \tau_0 \\ u_1, u_2, \dots, u_m, u_0 \end{pmatrix} + \underline{y}(M)^{m+1} \begin{pmatrix} \tau_1, \tau_2, \dots, \tau_{m-1}, \tau_0, \tau_m \\ u_1, u_2, \dots, u_{m-1}, u_0, u_m \end{pmatrix} \\ &\quad + \dots + \underline{y}(M)^{m+1} \begin{pmatrix} \tau_1, \tau_2, \tau_0, \tau_3, \dots, \tau_m \\ u_1, u_2, u_0, u_3, \dots, u_m \end{pmatrix} + \underline{y}(M)^{m+1} \begin{pmatrix} \tau_1, \tau_0, \tau_2, \dots, \tau_m \\ u_1, u_0, u_2, \dots, u_m \end{pmatrix}. \end{aligned}$$

Repeated applications of (3.18) yield

$$\begin{aligned} &= \underline{y}(M)^{m+1} \begin{pmatrix} \tau_0, \tau_1, \tau_2, \dots, \tau_m \\ u_0, u_1, u_2, \dots, u_m \end{pmatrix} + \underline{y}(M)^{m+1} \begin{pmatrix} \tau_0, \tau_m, \tau_1, \tau_2, \dots, \tau_{m-1} \\ u_0, u_m, u_1, u_2, \dots, u_{m-1} \end{pmatrix} \\ &\quad + \dots + \underline{y}(M)^{m+1} \begin{pmatrix} \tau_0, \tau_3, \dots, \tau_m, \tau_1, \tau_2 \\ u_0, u_3, \dots, u_m, u_1, u_2 \end{pmatrix} + \underline{y}(M)^{m+1} \begin{pmatrix} \tau_0, \tau_2, \dots, \tau_m, \tau_1 \\ u_0, u_2, \dots, u_m, u_1 \end{pmatrix} \\ &= \sum_{i \in \mathbb{Z}/m\mathbb{Z}} \underline{y}(M)^{m+1} \begin{pmatrix} \tau_0, \tau_{i+1}, \tau_{i+2}, \dots, \tau_{i+m} \\ u_0, u_{i+1}, u_{i+2}, \dots, u_{i+m} \end{pmatrix}. \end{aligned}$$

Therefore, by substituting $u_0 = 0$ and $u_i = x_i$ ($1 \leq i \leq m$) and $\tau_0 = (\sigma_1 \dots \sigma_m)^{-1}$ and $\tau_i = \sigma_i$ ($1 \leq i \leq m$), we get the claim. □

THEOREM 3.13. — *There is an embedding*

$$\mathrm{Fil}_{\mathcal{D}}^2 \mathrm{ARI}(\Gamma)_{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \hookrightarrow \mathrm{ARI}(\Gamma)_{\mathrm{push}/\mathrm{pusnu}}$$

of graded Lie algebras.

Proof. — Let $M \in \mathrm{Fil}_{\mathcal{D}}^2 \mathrm{ARI}(\Gamma)_{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$. Then we have $M \in \mathrm{ARI}(\Gamma)_{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$. So by Proposition 3.11 and Proposition 3.12, M is push-invariant and $\mathrm{swap}(M)$ is pus-neutral. Therefore, we obtain $M \in \mathrm{ARI}(\Gamma)_{\mathrm{push}/\mathrm{pusnu}}$. To see that it is a Lie algebra homomorphism is immediate. \square

As a corollary, by taking an intersection with $\mathrm{ARI}(\Gamma)_{\underline{\mathrm{al}}}^{\mathrm{fin},\mathrm{pol}}$ in the embedding of the above theorem, we obtain the following inclusion which generalizes [17, Theorem 3].

COROLLARY 3.14. — *There is an embedding*

$$\mathrm{Fil}_{\mathcal{D}}^2 \mathbb{D}(\Gamma)_{\bullet\bullet} \hookrightarrow \mathfrak{ftv}(\Gamma)_{\bullet\bullet}$$

of bigraded Lie algebras.

Proof. — It follows from Theorem 2.22, (3.13) and Theorem 3.13. \square

By imposing the distribution relation, we also obtain the following.

COROLLARY 3.15. — *There is an embedding*

$$\mathrm{Fil}_{\mathcal{D}}^2 D(\Gamma)_{\bullet\bullet} \hookrightarrow \mathfrak{ftv}\mathfrak{d}(\Gamma)_{\bullet\bullet}$$

of bigraded Lie algebras.

Proof. — It follows from Corollary 2.26, Corollary 3.6 and Theorem 3.13. \square

It looks interesting to see if the map is an isomorphism.

Remark 3.16. — By (3.13), Theorem 2.22, Theorem 3.13 and Corollary 3.14, we obtain the commutative diagram (3.25) of Lie algebras:

$$\begin{array}{ccc} \mathrm{Fil}_{\mathcal{D}}^2 \mathbb{D}(\Gamma)_{\bullet\bullet} & \xrightarrow{\simeq} & \mathrm{Fil}_{\mathcal{D}}^2 \mathrm{ARI}(\Gamma)_{\underline{\mathrm{al}}/\underline{\mathrm{al}}}^{\mathrm{fin},\mathrm{pol}} \\ \downarrow & & \downarrow \\ \mathfrak{ftv}(\Gamma)_{\bullet\bullet} & \xrightarrow{\simeq} & \mathrm{ARI}(\Gamma)_{\mathrm{push}/\mathrm{pusnu}} \cap \mathrm{ARI}(\Gamma)_{\underline{\mathrm{al}}}^{\mathrm{fin},\mathrm{pol}} \end{array} \quad (3.25)$$

Appendix A. On the ari-bracket of $\mathrm{ARI}(\Gamma)$

In this appendix, we give self-contained proofs of fundamental properties of the ari-bracket of $\mathrm{ARI}(\Gamma)$, that is, Propositions 1.14, 1.15 and 1.24, which are required in this paper.

A.1. Proof of Proposition 1.14

We give a proof Proposition 1.14 which claims that $\text{ARI}(\Gamma)$ forms a Lie algebra, by showing that it actually forms a pre-Lie algebra.

We start with three fundamental lemmas which can be proved directly by simple computations.

LEMMA A.1. — For $\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \in X_{\mathbb{Z}}^{\bullet}$, we have

$$\begin{aligned} {}_{\alpha_1\alpha_2}[\beta &= {}_{\alpha_1}[({}_{\alpha_2}[\beta]), & \beta]_{\alpha_1\alpha_2} &= (\beta]_{\alpha_1})_{\alpha_2}, \\ {}_{\alpha}[(\beta_1\beta_2) &= {}_{\alpha}[\beta_1]_{\alpha}[\beta_2, & (\beta_1\beta_2)]_{\alpha} &= \beta_1]_{\alpha}[\beta_2]_{\alpha}. \end{aligned}$$

Especially, if $\beta_1 \neq \emptyset$, we have

$${}_{\alpha}[(\beta_1\beta_2) = ({}_{\alpha}[\beta_1])\beta_2, \quad (\beta_2\beta_1)]_{\alpha} = \beta_2(\beta_1]_{\alpha}),$$

and if $\alpha_2 \neq \emptyset$, we have

$${}_{\alpha_1\alpha_2}[\beta = {}_{\alpha_2}[\beta, \quad \beta]_{\alpha_2\alpha_1} = \beta]_{\alpha_2}.$$

Proof. — Let $\alpha_1 = (\begin{smallmatrix} a_1, \dots, a_1 \\ \sigma_1, \dots, \sigma_1 \end{smallmatrix})$, $\alpha_2 = (\begin{smallmatrix} a_{l+1}, \dots, a_{l+m} \\ \sigma_{l+1}, \dots, \sigma_{l+m} \end{smallmatrix})$, $\beta = (\begin{smallmatrix} b_1, \dots, b_n \\ \tau_1, \dots, \tau_n \end{smallmatrix})$. By $\alpha_1\alpha_2 = (\begin{smallmatrix} a_1, \dots, a_{l+m} \\ \sigma_1, \dots, \sigma_{l+m} \end{smallmatrix})$, we have

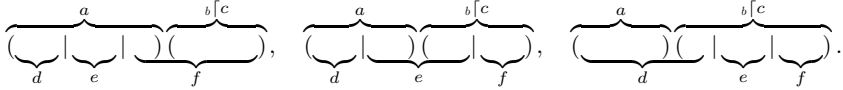
$$\begin{aligned} {}_{\alpha_1\alpha_2}[\beta &= (\begin{smallmatrix} a_1 + \dots + a_{l+m} + b_1, & b_2, \dots, & b_n \\ \tau_1, & \tau_2, \dots, & \tau_n \end{smallmatrix}) = {}_{\alpha_1}[({}_{\alpha_2}[\beta]), \\ &= {}_{\alpha_1}[({}_{\alpha_2}[\beta]). \end{aligned}$$

All the other cases can shown similarly. \square

LEMMA A.2. — Let $a, b, c, d, e, f \in X_{\mathbb{Z}}^{\bullet}$.

- (1) If we have $a_b]c = def$, one of the following holds:
 - (I) There exists $a_1, a_2, a_3 \in X_{\mathbb{Z}}^{\bullet}$ such that $a = a_1a_2a_3$ and $(d, e, f) = (a_1, a_2, a_3]_b]c)$,
 - (II) There exists $a_1, a_2, c_1, c_2 \in X_{\mathbb{Z}}^{\bullet}$ such that $a = a_1a_2$ and $c = c_1c_2$ and $(d, e, f) = (a_1, a_2]_b]c_1, c_2)$ and $c_1 \neq \emptyset$,
 - (III) There exists $c_1, c_2, c_3 \in X_{\mathbb{Z}}^{\bullet}$ such that $c = c_1c_2c_3$ and $(d, e, f) = (a_b]c_1, c_2, c_3)$ and $c_1 \neq \emptyset$.
- (2) If we have $a]_b]c = def$, one of the followings holds:
 - (I) There exists $a_1, a_2, a_3 \in X_{\mathbb{Z}}^{\bullet}$ such that $a = a_1a_2a_3$ and $(d, e, f) = (a_1, a_2, a_3]_b]c)$ and $a_3 \neq \emptyset$,
 - (II) There exists $a_1, a_2, c_1, c_2 \in X_{\mathbb{Z}}^{\bullet}$ such that $a = a_1a_2$ and $c = c_1c_2$ and $(d, e, f) = (a_1, a_2]_b]c_1, c_2)$ and $a_2 \neq \emptyset$,
 - (III) There exists $c_1, c_2, c_3 \in X_{\mathbb{Z}}^{\bullet}$ such that $c = c_1c_2c_3$ and $(d, e, f) = (a]_b]c_1, c_2, c_3)$.

Proof. — We present a proof for (1). When $a_b \lceil c = def$, the following depict all the possible cases of the positions of a , $b \lceil c$, d , e and f .



The first, the second and the third cases correspond (I), (II) and (III) in (1) respectively⁽¹⁶⁾. The claim for (2) can be proved in the same way. \square

LEMMA A.3. — For $a, b, c \in X_{\mathbb{Z}}^{\bullet}$, the following hold:

(1) (commutativity)

$$\begin{aligned} a \lceil (b \lceil c) &= b \lceil (a \lceil c), & (c \lceil a) \lceil b &= (c \lceil b) \lceil a, & a \lceil (b \lceil c) &= b \lceil (a \lceil c), & (c \lceil a) \lceil b &= (c \lceil b) \lceil a, \\ a \lceil (c \lceil b) &= (a \lceil c) \lceil b, & a \lceil (b \lceil c) &= b \lceil (a \lceil c), & a \lceil (c \lceil b) &= (a \lceil c) \lceil b, & (b \lceil c) \lceil a &= b \lceil (c \lceil a), \\ (c \lceil b) \lceil a &= (c \lceil a) \lceil b, & a \lceil (c \lceil b) &= (a \lceil c) \lceil b. \end{aligned}$$

(2) (composition)

$$\begin{aligned} (b \lceil a) \lceil c &= (a \lceil b) \lceil c = ab \lceil c, & (b \lceil a) \lceil (c \lceil a) &= (a \lceil b) \lceil (a \lceil c) = b \lceil c, \\ c \lceil (b \lceil a) &= c \lceil (a \lceil b) = c \lceil ab, & (c \lceil a) \lceil (b \lceil a) &= (a \lceil c) \lceil (a \lceil b) = c \lceil b. \end{aligned}$$

(3) (independence)

$$\begin{aligned} (b \lceil a) \lceil c &= (a \lceil b) \lceil c = b \lceil c, & (b \lceil a) \lceil c &= (a \lceil b) \lceil c = b \lceil c, \\ c \lceil (b \lceil a) &= c \lceil (a \lceil b) = c \lceil b, & c \lceil (b \lceil a) &= c \lceil (a \lceil b) = c \lceil b. \end{aligned}$$

Proof. — Let $a = \begin{pmatrix} a_1, \dots, a_l \\ \sigma_1, \dots, \sigma_l \end{pmatrix}$, $b = \begin{pmatrix} b_1, \dots, b_m \\ \tau_1, \dots, \tau_m \end{pmatrix}$, $c = \begin{pmatrix} c_1, \dots, c_n \\ \mu_1, \dots, \mu_n \end{pmatrix}$. We give proofs for specific cases because all the other cases can be proved in a similar way.

(1). — We calculate

$$\begin{aligned} a \lceil (b \lceil c) &= a \lceil \left(\begin{matrix} b_1 + \dots + b_m + c_1, & c_2, & \dots, & c_n \\ \mu_1, & \mu_2, & \dots, & \mu_n \end{matrix} \right) = \left(\begin{matrix} a_1 + \dots + a_l + b_1 + \dots + b_m + c_1, & c_2, & \dots, & c_n \\ \mu_1, & \mu_2, & \dots, & \mu_n \end{matrix} \right) \\ &= b \lceil \left(\begin{matrix} a_1 + \dots + a_l + c_1, & c_2, & \dots, & c_n \\ \mu_1, & \mu_2, & \dots, & \mu_n \end{matrix} \right) = b \lceil (a \lceil c). \end{aligned}$$

(2). — By $b \lceil a = \begin{pmatrix} b_1, \dots, b_{m-1}, b_m + a_1, \dots, a_l \\ \tau_1, \dots, \tau_{m-1}, \tau_m \end{pmatrix}$ and $ab = \begin{pmatrix} a_1, \dots, a_l, b_1, \dots, b_m \\ \sigma_1, \dots, \sigma_l, \tau_1, \dots, \tau_m \end{pmatrix}$, we get

$$(b \lceil a) \lceil c = \begin{pmatrix} a_1 + \dots + a_l + b_1 + \dots + b_m + c_1, & c_2, & \dots, & c_n \\ \mu_1, & \mu_2, & \dots, & \mu_n \end{pmatrix} = ab \lceil c.$$

On the other hand, by $b \lceil a = \begin{pmatrix} b_1, & \dots, & b_m \\ \tau_1 \sigma_1^{-1}, & \dots, & \tau_m \sigma_1^{-1} \end{pmatrix}$ and $c \lceil a = \begin{pmatrix} c_1, & \dots, & c_n \\ \mu_1 \sigma_1^{-1}, & \dots, & \mu_n \sigma_1^{-1} \end{pmatrix}$, we get

$$(b \lceil a) \lceil (c \lceil a) = \begin{pmatrix} c_1, & \dots, & c_n \\ \mu_1 \tau_m^{-1}, & \dots, & \mu_n \tau_m^{-1} \end{pmatrix} = b \lceil c.$$

(3). — By the above expression of $b \lceil a$, we get

$$(b \lceil a) \lceil c = \begin{pmatrix} b_1 + \dots + b_m + c_1, & c_2, & \dots, & c_n \\ \mu_1, & \mu_2, & \dots, & \mu_n \end{pmatrix} = b \lceil c. \quad \square$$

⁽¹⁶⁾ Note that, in order to decompose $b \lceil c$, we need the condition $c_1 \neq \emptyset$ for the second and third cases.

The following formula is essential to prove Proposition 1.14. It is stated in [21, (2.2.10)] (where it looks that there is an error on the signature) without a proof.

PROPOSITION A.4. — *For any $A, B \in \text{ARI}(\Gamma)$, we have*

$$\text{arit}_u(B) \circ \text{arit}_u(A) - \text{arit}_u(A) \circ \text{arit}_u(B) = \text{arit}_u(\text{arit}_u(A, B)).$$

Proof. — Let $m \geq 0$. Then we have

$$\begin{aligned} & (\text{arit}_u(B) \circ \text{arit}_u(A) - \text{arit}_u(A) \circ \text{arit}_u(B))(C)(\mathbf{x}_m) \\ &= \text{arit}_u(B)(\text{arit}_u(A)(C))(\mathbf{x}_m) - \text{arit}_u(A)(\text{arit}_u(B)(C))(\mathbf{x}_m) \\ &= \sum_{\substack{\mathbf{x}_m=abc \\ b,c \neq \emptyset}} \text{arit}_u(A)(C)(a_b \lceil c \rceil B(b)_c) - \sum_{\substack{\mathbf{x}_m=abc \\ a,b \neq \emptyset}} \text{arit}_u(A)(C)(a \lceil b \rceil c) B_a \lceil b \rceil \\ &\quad - \sum_{\substack{\mathbf{x}_m=abc \\ b,c \neq \emptyset}} \text{arit}_u(B)(C)(a_b \lceil c \rceil A(b)_c) + \sum_{\substack{\mathbf{x}_m=abc \\ a,b \neq \emptyset}} \text{arit}_u(B)(C)(a \lceil b \rceil c) A_a \lceil b \rceil \\ &= \sum_{\substack{\mathbf{x}_m=abc \\ b,c \neq \emptyset}} \left\{ \sum_{\substack{a_b \lceil c=def \\ e,f \neq \emptyset}} C(d_e \lceil f \rceil A(e)_f) - \sum_{\substack{a_b \lceil c=def \\ d,e \neq \emptyset}} C(d \lceil_e f \rceil A_d \lceil e \rceil) \right\} B(b)_c \\ &\quad - \sum_{\substack{\mathbf{x}_m=abc \\ a,b \neq \emptyset}} \left\{ \sum_{\substack{a \lceil_b c=def \\ e,f \neq \emptyset}} C(d_e \lceil f \rceil A(e)_f) - \sum_{\substack{a \lceil_b c=def \\ d,e \neq \emptyset}} C(d \lceil_e f \rceil A_d \lceil e \rceil) \right\} B_a \lceil b \rceil \\ &\quad - \sum_{\substack{\mathbf{x}_m=abc \\ b,c \neq \emptyset}} \left\{ \sum_{\substack{a_b \lceil c=def \\ e,f \neq \emptyset}} C(d_e \lceil f \rceil B(e)_f) - \sum_{\substack{a_b \lceil c=def \\ d,e \neq \emptyset}} C(d \lceil_e f \rceil B_d \lceil e \rceil) \right\} A(b)_c \\ &\quad + \sum_{\substack{\mathbf{x}_m=abc \\ a,b \neq \emptyset}} \left\{ \sum_{\substack{a \lceil_b c=def \\ e,f \neq \emptyset}} C(d_e \lceil f \rceil B(e)_f) - \sum_{\substack{a \lceil_b c=def \\ d,e \neq \emptyset}} C(d \lceil_e f \rceil B_d \lceil e \rceil) \right\} A_a \lceil b \rceil. \end{aligned}$$

By using Lemma A.2, we have

$$\begin{aligned} &= \sum_{\substack{\mathbf{x}_m=abc \\ b,c \neq \emptyset}} \left\{ \sum_{\substack{a=a_1 a_2 a_3 \\ a_2 \neq \emptyset}} C(a_{1a_2} \lceil (a_{3b} \lceil c \rceil) \rceil A(a_2)_{a_3 \lceil c \rceil}) + \sum_{\substack{a=a_1 a_2 \\ c=c_1 c_2 \\ c_1, c_2 \neq \emptyset}} C(a_{1a_2} \lceil_{c_1} \lceil c_2 \rceil A((a_{2b} \lceil c_1 \rceil)_{c_2})) \right\} \\ &\quad + \sum_{\substack{c=c_1 c_2 c_3 \\ c_1, c_2, c_3 \neq \emptyset}} C(a_b \lceil_{c_1 c_2} \lceil c_3 \rceil A(c_2)_{c_3}) - \sum_{\substack{c=c_1 c_2 c_3 \\ c_1, c_2 \neq \emptyset}} C((a_b \lceil_{c_1} \rceil)_{c_2} c_3) A_{a \lceil_{c_1} \rceil} c_2 \end{aligned}$$

$$\begin{aligned}
 & \left. \begin{aligned}
 & - \sum_{\substack{a=a_1 a_2 \\ c=c_1 c_2 \\ a_1, c_1 \neq \emptyset}} C(a_1]_{a_2]c_1} c_2) A_{(a_1} [(a_2]_b c_1)) - \sum_{\substack{a=a_1 a_2 a_3 \\ a_1, a_2 \neq \emptyset}} C(a_1]_{a_2} a_3]_b c) A_{(a_1} [a_2) \Big\} B(b]_c) \\
 & - \sum_{\substack{\mathbf{x}_m=abc \\ a, b \neq \emptyset}} \left\{ \sum_{\substack{a=a_1 a_2 a_3 \\ a_2, a_3 \neq \emptyset}} C(a_1 a_2 [(a_3]_b c)) A(a_2]_{a_3]c} + \sum_{\substack{a=a_1 a_2 \\ c=c_1 c_2 \\ a_2, c_2 \neq \emptyset}} C(a_1 a_2]_{c_1} [c_2) A((a_2]_b c_1)]_{c_2}) \right. \\
 & + \sum_{\substack{c=c_1 c_2 c_3 \\ c_2, c_3 \neq \emptyset}} C(a]_b c_1 c_2 [c_3) A(c_2]_{c_3}) - \sum_{\substack{c=c_1 c_2 c_3 \\ c_2 \neq \emptyset}} C((a]_b c_1)]_{c_2} c_3) A_{(a]c_1} [c_2) \\
 & - \sum_{\substack{a=a_1 a_2 \\ c=c_1 c_2 \\ a_1, a_2 \neq \emptyset}} C(a_1]_{a_2]c_1} c_2) A_{(a_1} [(a_2]_b c_1)) - \sum_{\substack{a=a_1 a_2 a_3 \\ a_1, a_2, a_3 \neq \emptyset}} C(a_1]_{a_2} a_3]_b c) A_{(a_1} [a_2) \Big\} B(a]_b) \\
 & - \sum_{\substack{\mathbf{x}_m=abc \\ b, c \neq \emptyset}} \left\{ \sum_{\substack{a=a_1 a_2 a_3 \\ a_2 \neq \emptyset}} C(a_1 a_2 [(a_3]_b c)) B(a_2]_{a_3]c} + \sum_{\substack{a=a_1 a_2 \\ c=c_1 c_2 \\ c_1, c_2 \neq \emptyset}} C(a_1 a_2]_{c_1} [c_2) B((a_2]_b c_1)]_{c_2}) \right. \\
 & + \sum_{\substack{c=c_1 c_2 c_3 \\ c_1, c_2, c_3 \neq \emptyset}} C(a_b [c_1 c_2 [c_3) B(c_2]_{c_3}) - \sum_{\substack{c=c_1 c_2 c_3 \\ c_1, c_2 \neq \emptyset}} C((a_b [c_1)]_{c_2} c_3) B_{(a]c_1} [c_2) \\
 & - \sum_{\substack{a=a_1 a_2 \\ c=c_1 c_2 \\ a_1, c_1 \neq \emptyset}} C(a_1]_{a_2]c_1} c_2) B_{(a_1} [(a_2]_b c_1)) - \sum_{\substack{a=a_1 a_2 a_3 \\ a_1, a_2 \neq \emptyset}} C(a_1]_{a_2} a_3]_b c) B_{(a_1} [a_2) \Big\} A(b]_c) \\
 & + \sum_{\substack{\mathbf{x}_m=abc \\ a, b \neq \emptyset}} \left\{ \sum_{\substack{a=a_1 a_2 a_3 \\ a_2, a_3 \neq \emptyset}} C(a_1 a_2 [(a_3]_b c)) B(a_2]_{a_3]c} + \sum_{\substack{a=a_1 a_2 \\ c=c_1 c_2 \\ a_2, c_2 \neq \emptyset}} C(a_1 a_2]_{c_1} [c_2) B((a_2]_b c_1)]_{c_2}) \right. \\
 & + \sum_{\substack{c=c_1 c_2 c_3 \\ c_2, c_3 \neq \emptyset}} C(a]_b c_1 c_2 [c_3) B(c_2]_{c_3}) - \sum_{\substack{c=c_1 c_2 c_3 \\ c_2 \neq \emptyset}} C((a]_b c_1)]_{c_2} c_3) B_{(a]c_1} [c_2) \\
 & - \sum_{\substack{a=a_1 a_2 \\ c=c_1 c_2 \\ a_1, a_2 \neq \emptyset}} C(a_1]_{a_2]c_1} c_2) B_{(a_1} [(a_2]_b c_1)) - \sum_{\substack{a=a_1 a_2 a_3 \\ a_1, a_2, a_3 \neq \emptyset}} C(a_1]_{a_2} a_3]_b c) B_{(a_1} [a_2) \Big\} A(a]_b).
 \end{aligned}
 \end{aligned}$$

By using Lemma A.1⁽¹⁷⁾ and Lemma A.3(2) and (3)⁽¹⁸⁾ and changing variables, we calculate

$$\begin{aligned}
 &= \sum_{\substack{\mathbf{x}_m=abcde \\ b,d,e \neq \emptyset}} C(a_b[(c_d[e])A(b]_{ce})B(d]_e) + \sum_{\substack{\mathbf{x}_m=abcde \\ c,d,e \neq \emptyset}} C(a_{bcd}[e]A((b_c[d])_e)B(c]_{de}) \\
 &+ \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,d,e \neq \emptyset}} C(a_b[c_d[e]A(d]_e)B(b]_{cde}) - \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,d \neq \emptyset}} C(a_b[c]_d e)A(ac[d]B(b]_{cde}) \\
 &- \sum_{\substack{\mathbf{x}_m=abcde \\ a,c,d \neq \emptyset}} C(a]_{bcd} e)A(a[(b_c[d])B(c]_{de}) - \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,d,e \neq \emptyset}} C(a]_b c_d [e)A(a]_b B(d]_e) \\
 &- \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,d \neq \emptyset}} C(a_b[c]_d e)A(b]_{ce})B(abc[d] - \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,e \neq \emptyset}} C(a_{bcd}[e]A((b]_c d])_e)B(ab]_c) \\
 &- \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,d,e \neq \emptyset}} C(a]_b c_d [e)A(d]_e)B(a]_b) + \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,d \neq \emptyset}} C((a]_b c]_d e)A(ac[d]B(a]_b) \\
 &+ \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,c \neq \emptyset}} C(a]_{bcd} e)A(a]_b [(b]_c d])B(ab]_c) + \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,c,d \neq \emptyset}} C(a]_b c]_d e)A(a]_b B(abc[d] \\
 &- \sum_{\substack{\mathbf{x}_m=abcde \\ b,d,e \neq \emptyset}} C(a_b[(c_d[e])B(b]_{ce})A(d]_e) - \sum_{\substack{\mathbf{x}_m=abcde \\ c,d,e \neq \emptyset}} C(a_{bcd}[e]B((b_c[d])_e)A(c]_{de}) \\
 &- \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,d,e \neq \emptyset}} C(a_b[c]_d e)B(d]_e)A(b]_{cde}) + \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,d \neq \emptyset}} C(a_b[c]_d e)B(ac[d]A(b]_{cde}) \\
 &+ \sum_{\substack{\mathbf{x}_m=abcde \\ a,c,d \neq \emptyset}} C(a]_{bcd} e)B(a]_b [(b_c[d])A(c]_{de}) + \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,d,e \neq \emptyset}} C(a]_b c_d [e)B(a]_b A(d]_e) \\
 &+ \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,d \neq \emptyset}} C(a_b[c]_d e)B(b]_{ce})A(abc[d] + \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,e \neq \emptyset}} C(a_{bcd}[e]B((b]_c d])_e)A(ab]_c) \\
 &+ \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,d,e \neq \emptyset}} C(a]_b c_d [e)B(d]_e)A(a]_b) - \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,d \neq \emptyset}} C((a]_b c]_d e)B(ac[d]A(a]_b) \\
 &- \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,c \neq \emptyset}} C(a]_{bcd} e)B(a]_b [(b]_c d])A(ab]_c) - \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,c,d \neq \emptyset}} C(a]_b c]_d e)B(a]_b A(abc[d].
 \end{aligned}$$

⁽¹⁷⁾ We apply Lemma A.1 to the 4th, 7th, 16th and 19th terms.

⁽¹⁸⁾ Especially, we apply Lemma A.3(2) to the middle terms of each lines, and apply Lemma A.3(3) to the first terms of each lines.

Cancellation⁽¹⁹⁾ yields

$$\begin{aligned}
 &= \sum_{\substack{\mathbf{x}_m=abcde \\ b,d,e \neq \emptyset}} C(a_b[(c_d[e])A(b)_{ce}]B(d)_e) + \sum_{\substack{\mathbf{x}_m=abcde \\ c,d,e \neq \emptyset}} C(a_{bcd}[e]A((b_c[d])_e)B(c)_{de}) \\
 &+ \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,d,e \neq \emptyset}} C(a_b[c_d[e]A(d)_e]B(b)_{cde}) - \sum_{\substack{\mathbf{x}_m=abcde \\ a,c,d \neq \emptyset}} C(a_{bcd}e]A_a[(b_c[d])B(c)_{de}) \\
 &- \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,e \neq \emptyset}} C(a_{bcd}[e]A((b)_c d)_e]B(a_b[c]) + \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,d \neq \emptyset}} C((a]_b c]_d e)A(ac]_d)B(a]_b) \\
 &+ \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,c \neq \emptyset}} C(a]_{bcd} e)A_a[(b]_c d)B(a_b[c]) + \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,c,d \neq \emptyset}} C(a]_b c]_d e)A_a[b]B(abc]_d) \\
 &- \sum_{\substack{\mathbf{x}_m=abcde \\ b,d,e \neq \emptyset}} C(a_b[(c_d[e])B(b)_{ce}]A(d)_e) - \sum_{\substack{\mathbf{x}_m=abcde \\ c,d,e \neq \emptyset}} C(a_{bcd}[e]B((b_c[d])_e)A(c)_{de}) \\
 &- \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,d,e \neq \emptyset}} C(a_b[c_d[e]B(d)_e]A(b)_{cde}) + \sum_{\substack{\mathbf{x}_m=abcde \\ a,c,d \neq \emptyset}} C(a_{bcd}e]B_a[(b_c[d])A(c)_{de}) \\
 &+ \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,e \neq \emptyset}} C(a_{bcd}[e]B((b)_c d)_e]A(a_b[c]) - \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,d \neq \emptyset}} C((a]_b c]_d e)B(ac]_d)A_a[b] \\
 &- \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,c \neq \emptyset}} C(a]_{bcd} e)B_a[(b]_c d)A(a_b[c]) - \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,c,d \neq \emptyset}} C(a]_b c]_d e)B_a[b]A(abc]_d).
 \end{aligned}$$

Here, for the first term, we have

$$\begin{aligned}
 &\sum_{\substack{\mathbf{x}_m=abcde \\ b,d,e \neq \emptyset}} C(a_b[(c_d[e])A(b)_{ce}]B(d)_e) \\
 &= \sum_{\substack{\mathbf{x}_m=abcde \\ b,d,e \neq \emptyset \\ c=\emptyset}} C(a_b[(c_d[e])A(b)_{ce}]B(d)_e) + \sum_{\substack{\mathbf{x}_m=abcde \\ b,d,e \neq \emptyset \\ c \neq \emptyset}} C(a_b[(c_d[e])A(b)_{ce}]B(d)_e).
 \end{aligned}$$

By using Lemma A.1, we get

$$= \sum_{\substack{\mathbf{x}_m=abcde \\ b,d,e \neq \emptyset \\ c=\emptyset}} C(a_b[(c_d[e])A(b)_{ce}]B(d)_e) + \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,d,e \neq \emptyset}} C(a_b[c_d[e]A(b)_c]B(d)_e).$$

⁽¹⁹⁾ The cancellations occur on the four pairs: 4th and 19th, 6th and 21th, 7th and 16th, 9th and 18th.

Therefore, cancellation⁽²⁰⁾ yields

$$\begin{aligned}
 & (\text{arit}_u(B) \circ \text{arit}_u(A) - \text{arit}_u(A) \circ \text{arit}_u(B))(C)(\mathbf{x}_m) \\
 &= \sum_{\substack{\mathbf{x}_m=abcde \\ b,d,e \neq \emptyset \\ c=\emptyset}} C(a_b \lceil c_d \lceil e) A(b \lceil_{ce}) B(d \lceil_e) + \sum_{\substack{\mathbf{x}_m=abcde \\ c,d,e \neq \emptyset}} C(a_{bcd} \lceil e) A((b \lceil_d) \lceil_e) B(c \lceil_{de}) \\
 & - \sum_{\substack{\mathbf{x}_m=abcde \\ a,c,d \neq \emptyset}} C(a \lceil_{bcd} e) A(a \lfloor (b_c \lceil_d)) B(c \lceil_{de}) - \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,e \neq \emptyset}} C(a_{bcd} \lceil e) A((b \lceil_c d) \lceil_e) B(ab \lceil_c) \\
 & + \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,d \neq \emptyset \\ c=\emptyset}} C((a \lceil_b c) \lceil_a e) A(ac \lceil_d) B(a \lfloor b) + \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,c \neq \emptyset}} C(a \lceil_{bcd} e) A(a \lfloor (b \lceil_c d)) B(ab \lceil_c) \\
 & - \sum_{\substack{\mathbf{x}_m=abcde \\ b,d,e \neq \emptyset \\ c=\emptyset}} C(a_b \lceil (c_d \lceil e)) B(b \lceil_{ce}) A(d \lceil_e) - \sum_{\substack{\mathbf{x}_m=abcde \\ c,d,e \neq \emptyset}} C(a_{bcd} \lceil e) B((b \lceil_d) \lceil_e) A(c \lceil_{de}) \\
 & + \sum_{\substack{\mathbf{x}_m=abcde \\ a,c,d \neq \emptyset}} C(a \lceil_{bcd} e) B(a \lfloor (b_c \lceil_d)) A(c \lceil_{de}) + \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,e \neq \emptyset}} C(a_{bcd} \lceil e) B((b \lceil_c d) \lceil_e) A(ab \lceil_c) \\
 & - \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,d \neq \emptyset \\ c=\emptyset}} C((a \lceil_b c) \lceil_a e) B(ac \lceil_d) A(a \lfloor b) - \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,c \neq \emptyset}} C(a \lceil_{bcd} e) B(a \lfloor (b \lceil_c d)) A(ab \lceil_c).
 \end{aligned}$$

By rearranging each terms and by calculating the terms with $c = \emptyset$, we have

$$\begin{aligned}
 &= \sum_{\substack{\mathbf{x}_m=abde \\ b,d,e \neq \emptyset}} C(a_{bd} \lceil e) A(b \lceil_e) B(d \lceil_e) - \sum_{\substack{\mathbf{x}_m=abde \\ b,d,e \neq \emptyset}} C(a_{bd} \lceil e) B(b \lceil_e) A(d \lceil_e) \\
 & + \sum_{\substack{\mathbf{x}_m=abde \\ a,b,d \neq \emptyset}} C(a \lceil_{bd} e) A(a \lceil_d) B(a \lfloor b) - \sum_{\substack{\mathbf{x}_m=abde \\ a,b,d \neq \emptyset}} C(a \lceil_{bd} e) B(a \lceil_d) A(a \lfloor b) \\
 & + \sum_{\substack{\mathbf{x}_m=abcde \\ c,d,e \neq \emptyset}} C(a_{bcd} \lceil e) A((b_c \lceil_d) \lceil_e) B(c \lceil_{de}) - \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,e \neq \emptyset}} C(a_{bcd} \lceil e) A((b \lceil_c d) \lceil_e) B(ab \lceil_c) \\
 & - \sum_{\substack{\mathbf{x}_m=abcde \\ c,d,e \neq \emptyset}} C(a_{bcd} \lceil e) B((b_c \lceil_d) \lceil_e) A(c \lceil_{de}) + \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,e \neq \emptyset}} C(a_{bcd} \lceil e) B((b \lceil_c d) \lceil_e) A(ab \lceil_c)
 \end{aligned}$$

⁽²⁰⁾ The cancellations occur on the four pairs when $c \neq \emptyset$: 1st and 11th, 3rd and 9th, 6th and 16th, 8th and 14th.

$$\begin{aligned}
 & + \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,c \neq \emptyset}} C(a]_{bcd}e)A(a[(b]_c d))B(ab]_c) - \sum_{\substack{\mathbf{x}_m=abcde \\ a,c,d \neq \emptyset}} C(a]_{bcd}e)A(a[(b_c]d))B(c]_{de}) \\
 & - \sum_{\substack{\mathbf{x}_m=abcde \\ a,b,c \neq \emptyset}} C(a]_{bcd}e)B(a[(b]_c d))A(ab]_c) + \sum_{\substack{\mathbf{x}_m=abcde \\ a,c,d \neq \emptyset}} C(a]_{bcd}e)B(a[(b_c]d))A(c]_{de}).
 \end{aligned}$$

Here, the first term is calculated such that

$$\begin{aligned}
 \sum_{\substack{\mathbf{x}_m=abde \\ b,d,e \neq \emptyset}} C(a_{bd}]e)A(b]_e)B(d]_e) &= \sum_{\substack{\mathbf{x}_m=abce \\ b,c,e \neq \emptyset}} C(a_{bc}]e)A(b]_e)B(c]_e) \\
 &= \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,e \neq \emptyset \\ d=\emptyset}} C(a_{bcd}]e)A((b_c]d]_e)B(c]_{de}),
 \end{aligned}$$

The second, third and fourth terms are also calculated respectively as

$$\begin{aligned}
 - \sum_{\substack{\mathbf{x}_m=abde \\ b,d,e \neq \emptyset}} C(a_{bd}]e)B(b]_e)A(d]_e) &= - \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,e \neq \emptyset \\ d=\emptyset}} C(a_{bcd}]e)B((b_c]d]_e)A(c]_{de}), \\
 \sum_{\substack{\mathbf{x}_m=abde \\ a,b,d \neq \emptyset}} C(a]_{bd}e)A(a]_d)B(a]_b) &= \sum_{\substack{\mathbf{x}_m=abcde \\ a,c,d \neq \emptyset \\ b=\emptyset}} C(a]_{bcd}e)A(a[(b]_c d))B(ab]_c), \\
 - \sum_{\substack{\mathbf{x}_m=abde \\ a,b,d \neq \emptyset}} C(a]_{bd}e)B(a]_d)A(a]_b) &= - \sum_{\substack{\mathbf{x}_m=abcde \\ a,c,d \neq \emptyset \\ b=\emptyset}} C(a]_{bcd}e)B(a[(b]_c d))A(ab]_c).
 \end{aligned}$$

These computations yield

$$\begin{aligned}
 & (\text{arit}_u(B) \circ \text{arit}_u(A) - \text{arit}_u(A) \circ \text{arit}_u(B))(C)(\mathbf{x}_m) \\
 &= \sum_{\substack{\mathbf{x}_m=abcde \\ c,e \neq \emptyset}} C(a_{bcd}]e)A((b_c]d]_e)B(c]_{de}) - \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,e \neq \emptyset}} C(a_{bcd}]e)A((b]_c d]_e)B(ab]_c) \\
 & - \sum_{\substack{\mathbf{x}_m=abcde \\ c,e \neq \emptyset}} C(a_{bcd}]e)B((b_c]d]_e)A(c]_{de}) + \sum_{\substack{\mathbf{x}_m=abcde \\ b,c,e \neq \emptyset}} C(a_{bcd}]e)B((b]_c d]_e)A(ab]_c) \\
 & + \sum_{\substack{\mathbf{x}_m=abcde \\ a,c \neq \emptyset}} C(a]_{bcd}e)A(a[(b]_c d))B(ab]_c) - \sum_{\substack{\mathbf{x}_m=abcde \\ a,c,d \neq \emptyset}} C(a]_{bcd}e)A(a[(b_c]d))B(c]_{de}) \\
 & - \sum_{\substack{\mathbf{x}_m=abcde \\ a,c \neq \emptyset}} C(a]_{bcd}e)B(a[(b]_c d))A(ab]_c) + \sum_{\substack{\mathbf{x}_m=abcde \\ a,c,d \neq \emptyset}} C(a]_{bcd}e)B(a[(b_c]d))A(c]_{de})
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{\mathbf{x}_m = afe \\ e \neq \emptyset}} C(a_f \lceil e) \left\{ \sum_{\substack{f=bcd \\ c \neq \emptyset}} A((b_c \lceil d]_e) B(c]_{de}) - \sum_{\substack{f=bcd \\ b, c \neq \emptyset}} A((b]_c d]_e) B(ab]_c) \right\} \\
 &\quad - \sum_{\substack{\mathbf{x}_m = afe \\ e \neq \emptyset}} C(a_f \lceil e) \left\{ \sum_{\substack{f=bcd \\ c \neq \emptyset}} B((b_c \lceil d]_e) A(c]_{de}) - \sum_{\substack{f=bcd \\ b, c \neq \emptyset}} B((b]_c d]_e) A(ab]_c) \right\} \\
 &\quad + \sum_{\substack{\mathbf{x}_m = afe \\ a \neq \emptyset}} C(a]_f e) \left\{ \sum_{\substack{f=bcd \\ c \neq \emptyset}} A_a \lfloor (b]_c d) B(ab]_c) - \sum_{\substack{f=bcd \\ c, d \neq \emptyset}} A_a \lfloor (b_c \lceil d) B(c]_{de}) \right\} \\
 &\quad - \sum_{\substack{\mathbf{x}_m = afe \\ a \neq \emptyset}} C(a]_f e) \left\{ \sum_{\substack{f=bcd \\ c \neq \emptyset}} B_a \lfloor (b]_c d) A(ab]_c) - \sum_{\substack{f=bcd \\ c, d \neq \emptyset}} B_a \lfloor (b_c \lceil d) A(c]_{de}) \right\} \\
 &= \sum_{\substack{\mathbf{x}_m = afe \\ e \neq \emptyset}} C(a_f \lceil e) \left\{ \sum_{\substack{f=bcd \\ c, d \neq \emptyset}} A((b_c \lceil d]_e) B(c]_d) - \sum_{\substack{f=bcd \\ b, c \neq \emptyset}} A((b]_c d]_e) B_b]_c \right. \\
 &\quad \left. + \sum_{\substack{f=bc \\ c \neq \emptyset}} A(b]_e) B(c]_e) \right\} \\
 &\quad - \sum_{\substack{\mathbf{x}_m = afe \\ e \neq \emptyset}} C(a_f \lceil e) \left\{ \sum_{\substack{f=bcd \\ c, d \neq \emptyset}} B((b_c \lceil d]_e) A(c]_d) - \sum_{\substack{f=bcd \\ b, c \neq \emptyset}} B((b]_c d]_e) A_b]_c \right. \\
 &\quad \left. + \sum_{\substack{f=bc \\ c \neq \emptyset}} B(b]_e) A(c]_e) \right\} \\
 &\quad + \sum_{\substack{\mathbf{x}_m = afe \\ a \neq \emptyset}} C(a]_f e) \left\{ \sum_{\substack{f=bcd \\ b, c \neq \emptyset}} A_a \lfloor (b]_c d) B_b]_c) - \sum_{\substack{f=bcd \\ c, d \neq \emptyset}} A_a \lfloor (b_c \lceil d) B(c]_d) \right. \\
 &\quad \left. + \sum_{\substack{f=cd \\ c \neq \emptyset}} A_a \lfloor d) B_a]_c) \right\}
 \end{aligned}$$

$$- \sum_{\substack{\mathbf{x}_m = afe \\ a \neq \emptyset}} C(a]_f e) \left\{ \sum_{\substack{f = bcd \\ b, c \neq \emptyset}} B({}_a[(b]_c d)) A({}_b]_c) - \sum_{\substack{f = bcd \\ c, d \neq \emptyset}} B({}_a]_c (b_c[d]) A({}_c]_d) + \sum_{\substack{f = cd \\ c \neq \emptyset}} B({}_a]_c d) A({}_a]_c) \right\}.$$

By using Lemma A.3(2), (3) and Lemma A.1, we have

$$\begin{aligned} &= \sum_{\substack{\mathbf{x}_m = afe \\ e \neq \emptyset}} C(a_f]_f e) \{ (\text{arit}_u(B)(A))(f]_c) + (A \times B)(f]_c) \\ &\quad - (\text{arit}_u(A)(B))(f]_c) - (B \times A)(f]_c) \} \\ &+ \sum_{\substack{\mathbf{x}_m = afe \\ a \neq \emptyset}} C(a]_f e) \{ -(\text{arit}_u(B)(A))({}_a]_f) + (B \times A)({}_a]_f) \\ &\quad + (\text{arit}_u(A)(B))({}_a]_f) - (A \times B)({}_a]_f) \} \\ &= \sum_{\substack{\mathbf{x}_m = afe \\ e \neq \emptyset}} C(a_f]_f e) \text{ari}_u(A, B)(f]_c) - \sum_{\substack{\mathbf{x}_m = afe \\ a \neq \emptyset}} C(a]_f e) \text{ari}_u(A, B)({}_a]_f) \\ &= (\text{arit}_u(\text{ari}_u(A, B))(C))(\mathbf{x}_m). \end{aligned}$$

Therefore, we obtain the claim. \square

DEFINITION A.5 ([9, (2.46)]). — We consider a binary operation $\text{preari}_u : \text{ARI}(\Gamma)^{\otimes 2} \rightarrow \text{ARI}(\Gamma)$ which is defined by

$$\text{preari}_u(A, B) := \text{arit}_u(B)(A) + A \times B$$

for $A, B \in \text{ARI}(\Gamma)$.

Then we have $\text{ari}_u(A, B) = \text{preari}_u(A, B) - \text{preari}_u(B, A)$.

PROPOSITION A.6. — The pair $(\text{ARI}(\Gamma), \text{preari}_u)$ forms a pre-Lie algebra, i.e. the following formula holds:

$$\begin{aligned} &\text{preari}_u(A, \text{preari}_u(B, C)) - \text{preari}_u(\text{preari}_u(A, B), C) \\ &= \text{preari}_u(A, \text{preari}_u(C, B)) - \text{preari}_u(\text{preari}_u(A, C), B) \end{aligned}$$

for $A, B, C \in \text{ARI}(\Gamma)$.

Proof. — We have

$$\begin{aligned}
 & \{\text{preari}_u(A, \text{preari}_u(B, C)) - \text{preari}_u(\text{preari}_u(A, B), C)\} \\
 & \quad - \{\text{preari}_u(A, \text{preari}_u(C, B)) - \text{preari}_u(\text{preari}_u(A, C), B)\} \\
 & = \text{arit}_u(\text{preari}_u(B, C))(A) + A \times \text{preari}_u(B, C) - \text{arit}_u(C)(\text{preari}_u(A, B)) \\
 & \quad - \text{preari}_u(A, B) \times C - \text{arit}_u(\text{preari}_u(C, B))(A) - A \times \text{preari}_u(C, B) \\
 & \quad + \text{arit}_u(B)(\text{preari}_u(A, C)) + \text{preari}_u(A, C) \times B \\
 & = \text{arit}_u(\text{arit}_u(C)(B))(A) + \text{arit}_u(B \times C)(A) + A \times (\text{arit}_u(C)(B)) \\
 & \quad + A \times (B \times C) - \text{arit}_u(C)(\text{arit}_u(B)(A)) - \text{arit}_u(C)(A \times B) \\
 & \quad - \text{arit}_u(B)(A) \times C - (A \times B) \times C - \text{arit}_u(\text{arit}_u(B)(C))(A) \\
 & \quad - \text{arit}_u(C \times B)(A) - A \times (\text{arit}_u(B)(C)) - A \times (C \times B) \\
 & \quad + \text{arit}_u(B)(\text{arit}_u(C)(A)) + \text{arit}_u(B)(A \times C) + \text{arit}_u(C)(A) \times B \\
 & \quad + (A \times C) \times B.
 \end{aligned}$$

By using associativity of $(\text{ARI}(\Gamma), \times)$ and using Lemma 1.12, we get

$$\begin{aligned}
 & = \text{arit}_u(\text{arit}_u(C)(B))(A) + \text{arit}_u(B \times C)(A) - \text{arit}_u(C)(\text{arit}_u(B)(A)) \\
 & \quad - \text{arit}_u(\text{arit}_u(B)(C))(A) - \text{arit}_u(C \times B)(A) + \text{arit}_u(B)(\text{arit}_u(C)(A)) \\
 & = \text{arit}_u(\text{ari}_u(B, C))(A) - \{\text{arit}_u(C) \circ \text{arit}_u(B)(A) - \text{arit}_u(B) \circ \text{arit}_u(C)(A)\}.
 \end{aligned}$$

Therefore, by using Proposition A.4, we obtain the claim. \square

Proof of Proposition 1.14. — It is sufficient to prove the Jacobi identity

$$\text{ari}_u(\text{ari}_u(A, B), C) + \text{ari}_u(\text{ari}_u(B, C), A) + \text{ari}_u(\text{ari}_u(C, A), B) = 0$$

for $A, B, C \in \text{ARI}(\Gamma)$. By the relationship between ari_u and preari_u , we calculate

$$\begin{aligned}
 & \text{ari}_u(\text{ari}_u(A, B), C) + \text{ari}_u(\text{ari}_u(B, C), A) + \text{ari}_u(\text{ari}_u(C, A), B) \\
 & = \text{preari}_u(\text{ari}_u(A, B), C) - \text{preari}_u(C, \text{ari}_u(A, B)) \\
 & \quad + \text{preari}_u(\text{ari}_u(B, C), A) - \text{preari}_u(A, \text{ari}_u(B, C)) \\
 & \quad + \text{preari}_u(\text{ari}_u(C, A), B) - \text{preari}_u(B, \text{ari}_u(C, A)) \\
 & = \text{preari}_u(\text{preari}_u(A, B) - \text{preari}_u(B, A), C) \\
 & \quad - \text{preari}_u(C, \text{preari}_u(A, B) - \text{preari}_u(B, A)) \\
 & \quad + \text{preari}_u(\text{preari}_u(B, C) - \text{preari}_u(C, B), A) \\
 & \quad - \text{preari}_u(A, \text{preari}_u(B, C) - \text{preari}_u(C, B)) \\
 & \quad + \text{preari}_u(\text{preari}_u(C, A) - \text{preari}_u(A, C), B) \\
 & \quad - \text{preari}_u(B, \text{preari}_u(C, A) - \text{preari}_u(A, C)).
 \end{aligned}$$

By Proposition A.6, it is equal to 0. So we obtain the Jacobi identity. \square

A.2. Proof of Proposition 1.15

We give a proof of Proposition 1.15 which claims that $\text{ARI}(\Gamma)_{\text{al}}$ forms a Lie algebra.

We show the following key lemma in this section.

LEMMA A.7. — For $\omega, \eta, \alpha_1, \dots, \alpha_r \in X_{\mathbb{Z}}^{\bullet}$, we have

$$\text{Sh}\left(\begin{array}{c} \omega; \eta \\ \alpha_1 \dots \alpha_r \end{array}\right) = \sum_{\substack{\omega = \omega_1 \dots \omega_r \\ \eta = \eta_1 \dots \eta_r}} \text{Sh}\left(\begin{array}{c} \omega_1; \eta_1 \\ \alpha_1 \end{array}\right) \dots \text{Sh}\left(\begin{array}{c} \omega_r; \eta_r \\ \alpha_r \end{array}\right). \quad (\text{A.1})$$

where $\omega_1, \dots, \omega_r, \eta_1, \dots, \eta_r$ run over $X_{\mathbb{Z}}^{\bullet}$.

Proof. — We consider the deconcatenation coproduct $\Delta : \mathcal{A}_X \rightarrow \mathcal{A}_X^{\otimes 2}$ defined by

$$\Delta(\omega) := \sum_{\substack{\omega = \omega_1 \omega_2 \\ \omega_1, \omega_2 \in X_{\mathbb{Z}}^{\bullet}}} \omega_1 \otimes \omega_2$$

for $\omega \in X_{\mathbb{Z}}^{\bullet}$. We recursively define \mathbb{Q} -linear maps $\Delta_r : \mathcal{A}_X \rightarrow \mathcal{A}_X^{\otimes r}$ by $\Delta_2 := \Delta$ and for $r \geq 3$

$$\Delta_r := \underbrace{(\text{Id} \otimes \dots \otimes \text{Id} \otimes \Delta)}_{r-2} \circ \Delta_{r-1}.$$

It is clear that Δ_r is an algebra homomorphism⁽²¹⁾, i.e. for $\omega, \eta \in X_{\mathbb{Z}}^{\bullet}$, we have

$$\Delta_r(\omega \text{ III } \eta) = \Delta_r(\omega) \text{ III } \Delta_r(\eta).$$

By expanding the above both sides and taking the coefficient of $\alpha_1 \otimes \dots \otimes \alpha_r$ for $\alpha_1, \dots, \alpha_r \in X_{\mathbb{Z}}^{\bullet}$, we obtain the claim. \square

Proof of Proposition 1.15. — Because we have

$$\text{ari}(A, B) = \text{arit}(B)(A) - \text{arit}(A)(B) + [A, B],$$

it is sufficient to prove the following two formulae for $A, B \in \text{ARI}(\Gamma)_{\text{al}}$:

$$[A, B] \in \text{ARI}(\Gamma)_{\text{al}}, \quad (\text{A.2})$$

$$\text{arit}(B)(A) \in \text{ARI}(\Gamma)_{\text{al}}. \quad (\text{A.3})$$

⁽²¹⁾ Note that the product III of \mathcal{A}_X induces the product of $\mathcal{A}_X^{\otimes r}$ (we also denote this product to the same symbol III) as

$$(\omega_1 \otimes \dots \otimes \omega_r) \text{ III } (\eta_1 \otimes \dots \otimes \eta_r) := (\omega_1 \text{ III } \eta_1) \otimes \dots \otimes (\omega_r \text{ III } \eta_r)$$

for any $\omega_i, \eta_i \in X_{\mathbb{Z}}^{\bullet}$.

Firstly, we prove (A.2). Let $p, q \geq 1$ and put $\omega = (\frac{x_1}{\sigma_1}, \dots, \frac{x_p}{\sigma_p})$ and $\eta = (\frac{x_{p+1}}{\sigma_{p+1}}, \dots, \frac{x_{p+q}}{\sigma_{p+q}})$. For our simplicity, we denote

$$\mathcal{S}h(M)(\omega; \eta) := \sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \begin{pmatrix} \omega; \eta \\ \alpha \end{pmatrix} M^{p+q}(\alpha). \quad (\text{A.4})$$

Then we have

$$\begin{aligned} & \mathcal{S}h(A \times B)(\omega; \eta) \\ &= \sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \begin{pmatrix} \omega; \eta \\ \alpha \end{pmatrix} \sum_{\alpha = \alpha_1 \alpha_2} A(\alpha_1) B(\alpha_2) = \sum_{\alpha_1, \alpha_2 \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \begin{pmatrix} \omega; \eta \\ \alpha_1 \alpha_2 \end{pmatrix} A(\alpha_1) B(\alpha_2). \end{aligned}$$

By using Lemma A.7 for $r = 2$, we get

$$\begin{aligned} &= \sum_{\alpha_1, \alpha_2 \in X_{\mathbb{Z}}^{\bullet}} \sum_{\substack{\omega = \omega_1 \omega_2 \\ \eta = \eta_1 \eta_2}} \text{Sh} \begin{pmatrix} \omega_1; \eta_1 \\ \alpha_1 \end{pmatrix} \text{Sh} \begin{pmatrix} \omega_2; \eta_2 \\ \alpha_2 \end{pmatrix} A(\alpha_1) B(\alpha_2) \\ &= \sum_{\substack{\omega = \omega_1 \omega_2 \\ \eta = \eta_1 \eta_2}} \left\{ \sum_{\alpha_1 \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \begin{pmatrix} \omega_1; \eta_1 \\ \alpha_1 \end{pmatrix} A(\alpha_1) \right\} \left\{ \sum_{\alpha_2 \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \begin{pmatrix} \omega_2; \eta_2 \\ \alpha_2 \end{pmatrix} B(\alpha_2) \right\}. \end{aligned}$$

By using alternality of A , we calculate

$$\begin{aligned} &= \sum_{\substack{\omega = \omega_1 \omega_2 \\ \omega_1 \neq \emptyset}} A(\omega_1) \left\{ \sum_{\alpha_2 \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \begin{pmatrix} \omega_2; \eta \\ \alpha_2 \end{pmatrix} B(\alpha_2) \right\} \\ &\quad + \sum_{\substack{\eta = \eta_1 \eta_2 \\ \eta_1 \neq \emptyset}} A(\eta_1) \left\{ \sum_{\alpha_2 \in X_{\mathbb{Z}}^{\bullet}} \text{Sh} \begin{pmatrix} \omega; \eta_2 \\ \alpha_2 \end{pmatrix} B(\alpha_2) \right\}. \end{aligned}$$

By using alternality of B , we obtain

$$= A(\omega) B(\eta) + A(\eta) B(\omega).$$

Because we have $[A, B] = A \times B - B \times A$, we get $\mathcal{S}h([A, B])(\omega; \eta) = 0$, that is, we obtain (A.2).

Secondly, we prove (A.3). We remark that, for $\alpha \in X_{\mathbb{Z}}^{\bullet}$ with $l(\alpha) \geq 2$, we have

$$\begin{aligned}
 & \text{arit}(A)(B)(\alpha) \\
 &= \sum_{\substack{\alpha=\alpha_1\alpha_2\alpha_3 \\ \alpha_2, \alpha_3 \neq \emptyset}} A(\alpha_{1\alpha_2}[\alpha_3])B(\alpha_2]_{\alpha_3}) - \sum_{\substack{\alpha=\alpha_1\alpha_2\alpha_3 \\ \alpha_1, \alpha_2 \neq \emptyset}} A(\alpha_1]_{\alpha_2}\alpha_3)B(\alpha_1[\alpha_2) \\
 &= \sum_{\substack{\alpha=\alpha_1\alpha_2\alpha_3 \\ \alpha_3 \neq \emptyset}} A(\alpha_{1\alpha_2}[\alpha_3])B(\alpha_2]_{\alpha_3}) - \sum_{\substack{\alpha=\alpha_1\alpha_2\alpha_3 \\ \alpha_1 \neq \emptyset}} A(\alpha_1]_{\alpha_2}\alpha_3)B(\alpha_1[\alpha_2) \\
 &= \sum_{\substack{\alpha=\alpha_1\alpha_2x\alpha_3 \\ \alpha_i \in X_{\mathbb{Z}}^{\bullet}, x \in X_{\mathbb{Z}}}} A(\alpha_{1\alpha_2}[x\alpha_3])B(\alpha_2]_x) - \sum_{\substack{\alpha=\alpha_1x\alpha_2\alpha_3 \\ \alpha_i \in X_{\mathbb{Z}}^{\bullet}, x \in X_{\mathbb{Z}}}} A(\alpha_1x]_{\alpha_2}\alpha_3)B(x[\alpha_2). \quad (\text{A.5})
 \end{aligned}$$

Then by the definition (A.4) of the map \mathcal{Sh} , we calculate

$$\mathcal{Sh}(\text{arit}(B)(A))(\omega; \eta) = \sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh}\left(\begin{matrix} \omega; \eta \\ \alpha \end{matrix}\right) \text{arit}(A)(B)(\alpha).$$

Then by (A.5), we have

$$\begin{aligned}
 &= \sum_{\alpha \in X_{\mathbb{Z}}^{\bullet}} \text{Sh}\left(\begin{matrix} \omega; \eta \\ \alpha \end{matrix}\right) \left\{ \sum_{\substack{\alpha=\alpha_1\alpha_2x\alpha_3 \\ \alpha_i \in X_{\mathbb{Z}}^{\bullet}, x \in X_{\mathbb{Z}}}} A(\alpha_{1\alpha_2}[x\alpha_3])B(\alpha_2]_x) \right. \\
 &\qquad\qquad\qquad \left. - \sum_{\substack{\alpha=\alpha_1x\alpha_2\alpha_3 \\ \alpha_i \in X_{\mathbb{Z}}^{\bullet}, x \in X_{\mathbb{Z}}}} A(\alpha_1x]_{\alpha_2}\alpha_3)B(x[\alpha_2) \right\} \\
 &= \sum_{\substack{\alpha_1, \alpha_2, \alpha_3 \in X_{\mathbb{Z}}^{\bullet} \\ x \in X_{\mathbb{Z}}}} \left\{ \text{Sh}\left(\begin{matrix} \omega; \eta \\ \alpha_1\alpha_2x\alpha_3 \end{matrix}\right) A(\alpha_{1\alpha_2}[x\alpha_3])B(\alpha_2]_x) \right. \\
 &\qquad\qquad\qquad \left. - \text{Sh}\left(\begin{matrix} \omega; \eta \\ \alpha_1x\alpha_2\alpha_3 \end{matrix}\right) A(\alpha_1x]_{\alpha_2}\alpha_3)B(x[\alpha_2) \right\}.
 \end{aligned}$$

By using Lemma A.7, we have

$$\begin{aligned}
 &= \sum_{\substack{\alpha_1, \alpha_2, \alpha_3 \in X_{\mathbb{Z}}^{\bullet} \\ x \in X_{\mathbb{Z}}}} \left\{ \sum_{\substack{\omega=\omega_1\omega_2\omega'_2\omega_3 \\ \eta=\eta_1\eta_2\eta'_2\eta_3}} \text{Sh}\left(\begin{matrix} \omega_1; \eta_1 \\ \alpha_1 \end{matrix}\right) \text{Sh}\left(\begin{matrix} \omega_2; \eta_2 \\ \alpha_2 \end{matrix}\right) \right. \\
 &\qquad\qquad\qquad \left. \cdot \text{Sh}\left(\begin{matrix} \omega'_2; \eta'_2 \\ x \end{matrix}\right) \text{Sh}\left(\begin{matrix} \omega_3; \eta_3 \\ \alpha_3 \end{matrix}\right) A(\alpha_{1\alpha_2}[x\alpha_3])B(\alpha_2]_x) \right\}
 \end{aligned}$$

$$\left. \begin{aligned} & - \sum_{\substack{\omega=\omega_1\omega_2'\omega_3 \\ \eta=\eta_1\eta_2'\eta_3}} \text{Sh}\left(\begin{smallmatrix} \omega_1; \eta_1 \\ \alpha_1 \end{smallmatrix}\right) \text{Sh}\left(\begin{smallmatrix} \omega_2'; \eta_2' \\ x \end{smallmatrix}\right) \\ & \qquad \qquad \qquad \cdot \text{Sh}\left(\begin{smallmatrix} \omega_2; \eta_2 \\ \alpha_2 \end{smallmatrix}\right) \text{Sh}\left(\begin{smallmatrix} \omega_3; \eta_3 \\ \alpha_3 \end{smallmatrix}\right) A(\alpha_1 x]_{\alpha_2} \alpha_3) B(x[\alpha_2) \end{aligned} \right\}.$$

Here, if $\text{Sh}\left(\begin{smallmatrix} \omega_2; \eta_2 \\ \alpha_2 \end{smallmatrix}\right) \neq 0$ holds for $\alpha_2 \in X_{\mathbb{Z}}^{\bullet}$, then all letters appearing in α_2 match with all ones appearing in ω_2 and η_2 . So we have $\alpha_2[x =_{(\omega_2, \eta_2)}[x$ and $x]_{\alpha_2} = x]_{(\omega_2, \eta_2)}$ for $\alpha_2 \in X_{\mathbb{Z}}^{\bullet}$ and we continue

$$\begin{aligned} & = \sum_{\substack{\alpha_1, \alpha_3 \in X_{\mathbb{Z}}^{\bullet} \\ x \in X_{\mathbb{Z}}}} \left\{ \sum_{\substack{\omega=\omega_1\omega_2'\omega_3 \\ \eta=\eta_1\eta_2'\eta_3}} \text{Sh}\left(\begin{smallmatrix} \omega_1; \eta_1 \\ \alpha_1 \end{smallmatrix}\right) \text{Sh}\left(\begin{smallmatrix} \omega_2'; \eta_2' \\ x \end{smallmatrix}\right) \right. \\ & \qquad \qquad \qquad \cdot \text{Sh}\left(\begin{smallmatrix} \omega_3; \eta_3 \\ \alpha_3 \end{smallmatrix}\right) A(\alpha_1(\omega_2, \eta_2)[x\alpha_3) \mathcal{S}h(B)(\omega_2]_x; \eta_2]_x) \\ & - \sum_{\substack{\omega=\omega_1\omega_2'\omega_3 \\ \eta=\eta_1\eta_2'\eta_3}} \text{Sh}\left(\begin{smallmatrix} \omega_1; \eta_1 \\ \alpha_1 \end{smallmatrix}\right) \text{Sh}\left(\begin{smallmatrix} \omega_2'; \eta_2' \\ x \end{smallmatrix}\right) \\ & \qquad \qquad \qquad \cdot \text{Sh}\left(\begin{smallmatrix} \omega_3; \eta_3 \\ \alpha_3 \end{smallmatrix}\right) A(\alpha_1 x]_{(\omega_2, \eta_2)} \alpha_3) \mathcal{S}h(B)(x[\omega_2; x[\eta_2) \left. \right\}. \end{aligned}$$

Because $B \in \text{ARI}(\Gamma)_{\text{al}}$, we have $\mathcal{S}h(B)(\emptyset; \emptyset) = 0$ and $\mathcal{S}h(B)(\omega_2]_x; \eta_2]_x) = \mathcal{S}h(B)(x[\omega_2; x[\eta_2) = 0$ for $\omega_2, \eta_2 \neq \emptyset$. So we have

$$\begin{aligned} & = \sum_{\substack{\alpha_1, \alpha_3 \in X_{\mathbb{Z}}^{\bullet} \\ x \in X_{\mathbb{Z}}}} \left\{ \sum_{\substack{\omega=\omega_1\omega_2'\omega_3 \\ \eta=\eta_1\eta_2'\eta_3 \\ \omega_2 \neq \emptyset}} \text{Sh}\left(\begin{smallmatrix} \omega_1; \eta_1 \\ \alpha_1 \end{smallmatrix}\right) \text{Sh}\left(\begin{smallmatrix} \omega_2'; \eta_2' \\ x \end{smallmatrix}\right) \text{Sh}\left(\begin{smallmatrix} \omega_3; \eta_3 \\ \alpha_3 \end{smallmatrix}\right) A(\alpha_1 \omega_2[x\alpha_3) B(\omega_2]_x) \right. \\ & - \sum_{\substack{\omega=\omega_1\omega_2'\omega_3 \\ \eta=\eta_1\eta_2'\eta_3 \\ \omega_2 \neq \emptyset}} \text{Sh}\left(\begin{smallmatrix} \omega_1; \eta_1 \\ \alpha_1 \end{smallmatrix}\right) \text{Sh}\left(\begin{smallmatrix} \omega_2'; \eta_2' \\ x \end{smallmatrix}\right) \text{Sh}\left(\begin{smallmatrix} \omega_3; \eta_3 \\ \alpha_3 \end{smallmatrix}\right) A(\alpha_1 x]_{\omega_2} \alpha_3) B(x[\omega_2) \left. \right\} \end{aligned}$$

$$\begin{aligned}
 & + \sum_{\substack{\omega=\omega_1\omega'_2\omega_3 \\ \eta=\eta_1\eta_2\eta'_2\eta_3 \\ \eta_2 \neq \emptyset}} \text{Sh}\left(\begin{matrix} \omega_1; \eta_1 \\ \alpha_1 \end{matrix}\right) \text{Sh}\left(\begin{matrix} \omega'_2; \eta'_2 \\ x \end{matrix}\right) \text{Sh}\left(\begin{matrix} \omega_3; \eta_3 \\ \alpha_3 \end{matrix}\right) A(\alpha_{1\eta_2} \lceil x\alpha_3 \rceil B(\eta_2 \lfloor x)) \\
 & - \sum_{\substack{\omega=\omega_1\omega'_2\omega_3 \\ \eta=\eta_1\eta'_2\eta_2\eta_3 \\ \eta_2 \neq \emptyset}} \text{Sh}\left(\begin{matrix} \omega_1; \eta_1 \\ \alpha_1 \end{matrix}\right) \text{Sh}\left(\begin{matrix} \omega'_2; \eta'_2 \\ x \end{matrix}\right) \text{Sh}\left(\begin{matrix} \omega_3; \eta_3 \\ \alpha_3 \end{matrix}\right) A(\alpha_1 x \rfloor_{\eta_2} \alpha_3) B(x \lfloor \eta_2) \Bigg\}.
 \end{aligned}$$

Here, $\text{Sh}\left(\begin{matrix} \omega'_2; \eta'_2 \\ x \end{matrix}\right) \neq 0$ holds for $x \in X_{\mathbb{Z}}$ if and only if $(\omega'_2, \eta'_2) = (x, \emptyset)$ or (\emptyset, x) . So we get

$$\begin{aligned}
 & = \sum_{\substack{\alpha_1, \alpha_3 \in X_{\mathbb{Z}}^{\bullet} \\ x \in X_{\mathbb{Z}}}} \left[\sum_{\substack{\omega=\omega_1\omega_2x\omega_3 \\ \eta=\eta_1\eta_3 \\ \omega_2 \neq \emptyset}} \text{Sh}\left(\begin{matrix} \omega_1; \eta_1 \\ \alpha_1 \end{matrix}\right) \text{Sh}\left(\begin{matrix} \omega_3; \eta_3 \\ \alpha_3 \end{matrix}\right) A(\alpha_{1\omega_2} \lceil x\alpha_3 \rceil B(\omega_2 \lfloor x)) \right. \\
 & \quad - \sum_{\substack{\omega=\omega_1x\omega_2\omega_3 \\ \eta=\eta_1\eta_3 \\ \omega_2 \neq \emptyset}} \text{Sh}\left(\begin{matrix} \omega_1; \eta_1 \\ \alpha_1 \end{matrix}\right) \text{Sh}\left(\begin{matrix} \omega_3; \eta_3 \\ \alpha_3 \end{matrix}\right) A(\alpha_1 x \rfloor_{\omega_2} \alpha_3) B(x \lfloor \omega_2) \\
 & \quad \left. + \sum_{\substack{\omega=\omega_1x\omega_3 \\ \eta=\eta_1\eta_2\eta_3 \\ \eta_2 \neq \emptyset}} \text{Sh}\left(\begin{matrix} \omega_1; \eta_1 \\ \alpha_1 \end{matrix}\right) \text{Sh}\left(\begin{matrix} \omega_3; \eta_3 \\ \alpha_3 \end{matrix}\right) \left\{ A(\alpha_{1\eta_2} \lceil x\alpha_3 \rceil B(\eta_2 \lfloor x)) \right. \right. \\
 & \quad \left. \left. - A(\alpha_1 x \rfloor_{\eta_2} \alpha_3) B(x \lfloor \eta_2) \right\} \right] \\
 & + \sum_{\substack{\alpha_1, \alpha_3 \in X_{\mathbb{Z}}^{\bullet} \\ x \in X_{\mathbb{Z}}}} \left[\sum_{\substack{\omega=\omega_1\omega_2x\omega_3 \\ \eta=\eta_1x\eta_3 \\ \omega_2 \neq \emptyset}} \text{Sh}\left(\begin{matrix} \omega_1; \eta_1 \\ \alpha_1 \end{matrix}\right) \text{Sh}\left(\begin{matrix} \omega_3; \eta_3 \\ \alpha_3 \end{matrix}\right) \left\{ A(\alpha_{1\omega_2} \lceil x\alpha_3 \rceil B(\omega_2 \lfloor x)) \right. \right. \\
 & \quad \left. \left. - A(\alpha_1 x \rfloor_{\omega_2} \alpha_3) B(x \lfloor \omega_2) \right\} \right. \\
 & \quad + \sum_{\substack{\omega=\omega_1\omega_3 \\ \eta=\eta_1\eta_2x\eta_3 \\ \eta_2 \neq \emptyset}} \text{Sh}\left(\begin{matrix} \omega_1; \eta_1 \\ \alpha_1 \end{matrix}\right) \text{Sh}\left(\begin{matrix} \omega_3; \eta_3 \\ \alpha_3 \end{matrix}\right) A(\alpha_{1\eta_2} \lceil x\alpha_3 \rceil B(\eta_2 \lfloor x)) \\
 & \quad \left. - \sum_{\substack{\omega=\omega_1\omega_3 \\ \eta=\eta_1x\eta_2\eta_3 \\ \eta_2 \neq \emptyset}} \text{Sh}\left(\begin{matrix} \omega_1; \eta_1 \\ \alpha_1 \end{matrix}\right) \text{Sh}\left(\begin{matrix} \omega_3; \eta_3 \\ \alpha_3 \end{matrix}\right) A(\alpha_1 x \rfloor_{\eta_2} \alpha_3) B(x \lfloor \eta_2) \right].
 \end{aligned}$$

Because x runs over $X_{\mathbb{Z}}$, we get ${}_{\omega_2}[x = x]_{\omega_2}$ and ${}_{\eta_2}[x = x]_{\eta_2}$ and $\eta_2]_x = {}_x[\eta_2$ and $\omega_2]_x = {}_x[\omega_2$. Hence we calculate

$$\begin{aligned}
 &= \sum_{\substack{\alpha_1, \alpha_3 \in X_{\mathbb{Z}}^{\bullet} \\ x \in X_{\mathbb{Z}}}} \left\{ \sum_{\substack{\omega = \omega_1 \omega_2 x \omega_3 \\ \eta = \eta_1 \eta_3 \\ \omega_2 \neq \emptyset}} \text{Sh} \begin{pmatrix} \omega_1; \eta_1 \\ \alpha_1 \end{pmatrix} \text{Sh} \begin{pmatrix} \omega_3; \eta_3 \\ \alpha_3 \end{pmatrix} A(\alpha_{1\omega_2}[x\alpha_3]B(\omega_2]_x) \right. \\
 &\quad \left. - \sum_{\substack{\omega = \omega_1 x \omega_2 \omega_3 \\ \eta = \eta_1 \eta_3 \\ \omega_2 \neq \emptyset}} \text{Sh} \begin{pmatrix} \omega_1; \eta_1 \\ \alpha_1 \end{pmatrix} \text{Sh} \begin{pmatrix} \omega_3; \eta_3 \\ \alpha_3 \end{pmatrix} A(\alpha_{1\omega_2}[x\alpha_3]B(\omega_2]_x) \right\} \\
 &+ \sum_{\substack{\alpha_1, \alpha_3 \in X_{\mathbb{Z}}^{\bullet} \\ x \in X_{\mathbb{Z}}}} \left\{ \sum_{\substack{\omega = \omega_1 \omega_3 \\ \eta = \eta_1 \eta_2 x \eta_3 \\ \eta_2 \neq \emptyset}} \text{Sh} \begin{pmatrix} \omega_1; \eta_1 \\ \alpha_1 \end{pmatrix} \text{Sh} \begin{pmatrix} \omega_3; \eta_3 \\ \alpha_3 \end{pmatrix} A(\alpha_{1\eta_2}[x\alpha_3]B(\eta_2]_x) \right. \\
 &\quad \left. - \sum_{\substack{\omega = \omega_1 \omega_3 \\ \eta = \eta_1 x \eta_2 \eta_3 \\ \eta_2 \neq \emptyset}} \text{Sh} \begin{pmatrix} \omega_1; \eta_1 \\ \alpha_1 \end{pmatrix} \text{Sh} \begin{pmatrix} \omega_3; \eta_3 \\ \alpha_3 \end{pmatrix} A(\alpha_{1\eta_2}[x\alpha_3]B(\eta_2]_x) \right\}.
 \end{aligned}$$

For $\omega_1, \omega_2, \omega_3, x$ with $\omega = \omega_1 \omega_2 x \omega_3$, by using Lemma A.7 with $r = 3$ and $\omega = \omega_{1\omega_2}[x\omega_3$ and $\alpha_2 = {}_{\omega_2}[x$, we have

$$\text{Sh} \begin{pmatrix} \omega_{1\omega_2}[x\omega_3; \eta \\ \alpha_{1\omega_2}[x\alpha_3] \end{pmatrix} = \sum_{\substack{\omega_1 \omega_2 [x\omega_3 = \omega'_1 \omega'_2 \omega'_3 \\ \eta = \eta_1 \eta_2 \eta_3}} \text{Sh} \begin{pmatrix} \omega'_1; \eta_1 \\ \alpha_1 \end{pmatrix} \text{Sh} \begin{pmatrix} \omega'_2; \eta_2 \\ {}_{\omega_2}[x \end{pmatrix} \text{Sh} \begin{pmatrix} \omega'_3; \eta_3 \\ \alpha_3 \end{pmatrix}.$$

Because $\eta = (\sigma_{p+1}, \dots, \sigma_{p+q})$ and $\omega_2 \neq \emptyset$, the letter ${}_{\omega_2}[x \in X_{\mathbb{Z}}$ does not appear in η . So for any word η_2 such that $\eta = \eta_1 \eta_2 \eta_3$, we get $\eta_2 \neq {}_{\omega_2}[x$. Hence, $\text{Sh} \begin{pmatrix} \omega'_2; \eta_2 \\ {}_{\omega_2}[x \end{pmatrix} \neq 0$ holds if and only if $\omega'_2 = {}_{\omega_2}[x$ and $\eta_2 = \emptyset$. So we have

$$= \sum_{\eta = \eta_1 \eta_3} \text{Sh} \begin{pmatrix} \omega_1; \eta_1 \\ \alpha_1 \end{pmatrix} \text{Sh} \begin{pmatrix} \omega_3; \eta_3 \\ \alpha_3 \end{pmatrix}.$$

Similarly, we get

$$\sum_{\omega = \omega_1 \omega_3} \text{Sh} \begin{pmatrix} \omega_1; \eta_1 \\ \alpha_1 \end{pmatrix} \text{Sh} \begin{pmatrix} \omega_3; \eta_3 \\ \alpha_3 \end{pmatrix} = \text{Sh} \begin{pmatrix} \omega; \eta_{1\eta_2}[x\eta_3 \\ \alpha_{1\omega_2}[x\alpha_3] \end{pmatrix}.$$

So by using these, we calculate

$$\begin{aligned}
 & \mathit{Sh}(\text{arit}(B)(A))(\omega; \eta) \\
 &= \sum_{x \in X_{\mathbb{Z}}} \left\{ \sum_{\substack{\omega = \omega_1 \omega_2 x \omega_3 \\ \omega_2 \neq \emptyset}} \mathit{Sh}(A)(\omega_{1\omega_2} \lceil x \omega_3 \rceil \eta) B(\omega_2 \rfloor_x) \right. \\
 & \qquad \qquad \qquad \left. - \sum_{\substack{\omega = \omega_1 x \omega_2 \omega_3 \\ \omega_2 \neq \emptyset}} \mathit{Sh}(A)(\omega_{1\omega_2} \lceil x \omega_3 \rceil \eta) B(\omega_2 \rfloor_x) \right\} \\
 &+ \sum_{x \in X_{\mathbb{Z}}} \left\{ \sum_{\substack{\eta = \eta_1 \eta_2 x \eta_3 \\ \eta_2 \neq \emptyset}} \mathit{Sh}(A)(\omega; \eta_{1\eta_2} \lceil x \eta_3 \rceil B(\eta_2 \rfloor_x)) \right. \\
 & \qquad \qquad \qquad \left. - \sum_{\substack{\eta = \eta_1 x \eta_2 \eta_3 \\ \eta_2 \neq \emptyset}} \mathit{Sh}(A)(\omega; \eta_{1\eta_2} \lceil x \eta_3 \rceil B(\eta_2 \rfloor_x)) \right\}.
 \end{aligned}$$

Lastly, similarly to footnote 5 in Definition 1.11, all letters appearing in two words $\omega_{1\omega_2} \lceil x \omega_3 \rceil$ and η (resp. ω and $\eta_{1\eta_2} \lceil x \eta_3 \rceil$) are algebraically independent over \mathbb{Q} , so by Remark 1.6, the component $\mathit{Sh}(A)(\omega_{1\omega_2} \lceil x \omega_3 \rceil \eta)$ (resp. $\mathit{Sh}(A)(\omega; \eta_{1\eta_2} \lceil x \eta_3 \rceil)$) is well-defined. Hence, by using alternality of A , we obtain (A.3). \square

A.3. Proof of Proposition 1.24

We give a proof Proposition 1.24 which claims that $\text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$ forms a filtered Lie subalgebra of $\text{ARI}(\Gamma)_{\text{al}}$ under the ari_u -bracket.

For $A, B \in \text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$, it is enough to prove $\text{ari}_u(A, B) \in \text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$, that is,

$$\text{ari}_u(A, B) \in \text{ARI}(\Gamma)_{\text{al}}, \tag{A.6}$$

$$\text{swap}(\text{ari}_u(A, B)) \in \overline{\text{ARI}}(\Gamma)_{\text{al}}, \tag{A.7}$$

$$\text{ari}_u(A, B)^1 \begin{pmatrix} x_1 \\ \sigma_1 \end{pmatrix} = \text{ari}_u(A, B)^1 \begin{pmatrix} -x_1 \\ \sigma_1^{-1} \end{pmatrix}. \tag{A.8}$$

Because $\text{ARI}(\Gamma)_{\text{al}}$ forms a Lie algebra under the ari_u -bracket, (A.6) is obvious. By the definition of ari_u , (A.8) is also clear. By Lemma 1.31 and Proposition 3.11, we have

$$\text{swap}(\text{ari}_u(A, B)) = \text{ari}_v(\text{swap}(A), \text{swap}(B)).$$

Since we have $\text{swap}(A), \text{swap}(B) \in \overline{\text{ARI}}(\Gamma)_{\text{al}}$ for $A, B \in \text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$, we get $\text{swap}(\text{ari}_u(A, B)) \in \overline{\text{ARI}}(\Gamma)_{\text{al}}$ by Proposition 1.21. Thus we obtain (A.7). \square

Appendix B. Multiple polylogarithms at roots of unity

In this appendix we recall how $\text{ARI}(\Gamma)_{\underline{\text{al}}/\underline{\text{al}}}$ and $\mathbb{D}(\Gamma)_{\bullet\bullet}$ are related to multiple polylogarithms at roots of unity.

Multiple polylogarithm is the several variable complex function which is defined by the following power series

$$\text{Li}_{n_1, \dots, n_r}(z_1, \dots, z_r) := \sum_{0 < k_1 < \dots < k_r} \frac{z_1^{k_1} \dots z_r^{k_r}}{k_1^{n_1} \dots k_r^{n_r}}$$

for $n_1, \dots, n_r, r \in \mathbb{N}$. For $N \in \mathbb{N}$, we denote μ_N to be the group of N -th roots of unity in \mathbb{C} . Its limit value at r -tuple $(\epsilon_1, \dots, \epsilon_r) \in \mu_N^{\oplus r}$ makes sense if and only if $(n_r, \epsilon_r) \neq (1, 1)$. They are called *multiple L -values* (particularly multiple zeta values when $N = 1$).

In [8, §8] and [9, §1.2] (it is also recalled in [15, §A.3] in the case when $\Gamma = \{e\}$), the following moulds in $\mathcal{M}(\mathcal{F}; \Gamma)$ with $\mathcal{F} = \cup_m \mathbb{C}[[u_1, \dots, u_m]]$ and $\Gamma = \mu_N$

$$\text{Zag} = \{\text{Zag}(\begin{smallmatrix} u_1, \dots, u_m \\ \epsilon_1, \dots, \epsilon_m \end{smallmatrix})\}_m \quad \text{and} \quad \text{Zig} = \{\text{Zig}(\begin{smallmatrix} \epsilon_1, \dots, \epsilon_m \\ v_1, \dots, v_m \end{smallmatrix})\}_m$$

are introduced and defined by

$$\text{Zag}(\begin{smallmatrix} u_1, \dots, u_m \\ \epsilon_1, \dots, \epsilon_m \end{smallmatrix}) = \sum_{n_1, \dots, n_m > 0} \text{Li}_{n_1, \dots, n_m}^{\text{III}}\left(\frac{\epsilon_1}{\epsilon_2}, \dots, \frac{\epsilon_{m-1}}{\epsilon_m}, \epsilon_m\right) u_1^{n_1-1} (u_1 + u_2)^{n_2-1} \dots (u_1 + \dots + u_m)^{n_m-1}$$

$$\text{Zig}(\begin{smallmatrix} \epsilon_1, \dots, \epsilon_m \\ v_1, \dots, v_m \end{smallmatrix}) = \sum_{n_1, \dots, n_m > 0} \text{Li}_{n_m, \dots, n_1}^*\left(\epsilon_m, \dots, \epsilon_1\right) v_1^{n_1-1} \dots v_m^{n_m-1}$$

where $\text{Li}_{n_1, \dots, n_m}^{\text{III}}(\epsilon_1, \dots, \epsilon_m)$ and $\text{Li}_{n_1, \dots, n_m}^*(\epsilon_1, \dots, \epsilon_m)$ mean the shuffle regularization and the harmonic (stuffle) regularization of $\text{Li}_{n_1, \dots, n_m}(\epsilon_1, \dots, \epsilon_m)$ respectively (cf. [4]). Particularly

$$\text{Li}_{n_1, \dots, n_m}^{\text{III}}(\epsilon_1, \dots, \epsilon_m) = \text{Li}_{n_1, \dots, n_m}^*(\epsilon_1, \dots, \epsilon_m) = \text{Li}_{n_1, \dots, n_m}(\epsilon_1, \dots, \epsilon_m)$$

when $(n_m, \epsilon_m) \neq (1, 1)$. In [8, (37)] and [9, (1.27)], it is explained that they are related as follows

$$\text{Zig} = \text{Mini} \times \text{swap}(\text{Zag}) \tag{B.1}$$

(see (1.8) for swap). Here $\text{Mini} = \{\text{Mini}(\begin{smallmatrix} u_1, \dots, u_m \\ \epsilon_1, \dots, \epsilon_m \end{smallmatrix})\}_m$ is the mould defined by

$$\text{Mini}(\begin{smallmatrix} \epsilon_1, \dots, \epsilon_m \\ v_1, \dots, v_m \end{smallmatrix}) = \begin{cases} \text{Mono}_m & \text{when } (\epsilon_1, \dots, \epsilon_m) = (1, \dots, 1), \\ 0 & \text{otherwise} \end{cases}$$

with $1 + \sum_{r=2}^{\infty} \text{Mono}_r \cdot t^r := \exp\{\sum_{k=2}^{\infty} (-1)^{k-1} \frac{\zeta(k)}{k} t^k\}$ (cf. [9, (1.30)]).

Goncharov’s arguments in [13] suggest us to express them as

$$\begin{aligned} \text{Zag}(\epsilon_1^{u_1}, \dots, \epsilon_m^{u_m}) &= \text{reg}^{\text{III}} \int_{0 < s_1 < \dots < s_m < 1} \frac{\epsilon_1 s_1^{-u_1}}{1 - \epsilon_1 s_1} ds_1 \wedge \dots \wedge \frac{\epsilon_m s_m^{-u_m}}{1 - \epsilon_m s_m} ds_m, \\ \text{Zig}(\epsilon_1^{v_1}, \dots, \epsilon_m^{v_m}) &= \text{reg}^* \sum_{0 < k_m < \dots < k_1} \frac{\epsilon_m^{k_m} \dots \epsilon_1^{k_1}}{(k_m - v_m) \dots (k_1 - v_1)} \end{aligned}$$

by using his regularizations. These might also helpful to understand [9, (9.5)] saying that Zag belongs to $\text{GARI}(\Gamma)_{\text{as/is}}$. The authors are not aware of its definition but expect that it means the symmetry for Zag and the symmetry for Zig ([9, §§1.1–1.2]), which looks corresponding to the shuffle and the harmonic product among multiple L -values. In [9, §4.7] it is directed to combine a related group $\text{GARI}(\Gamma)_{\text{as/is}}$ and its Lie algebra $\text{ARI}(\Gamma)_{\text{al/il}}$ with a bigraded variant $\text{GARI}(\Gamma)_{\text{as/as}}$ and $\text{ARI}(\Gamma)_{\text{al/al}}$ (cf. Definition 1.22) under maps $\text{adgari}(\text{pal})$ and $\text{adari}(\text{pal})$ (cf. [9, (2.54), (2.55), and §4.2]):

$$\begin{array}{ccc} \text{GARI}(\Gamma)_{\text{as/as}} & \xrightarrow{\text{adgari}(\text{pal})} & \text{GARI}(\Gamma)_{\text{as/is}} \\ \downarrow \text{logari} & & \downarrow \text{logari} \\ \text{ARI}(\Gamma)_{\text{al/al}} & \xrightarrow{\text{adari}(\text{pal})} & \text{ARI}(\Gamma)_{\text{al/il}}. \end{array} \tag{B.2}$$

While the cyclotomic analogue of Drinfeld’s KZ-associator ([6]) which is constructed from the KZ-like differential equation in $\widehat{\mathbb{A}}_{\mathbb{C}}$ over $\mathbb{P}^1(\mathbb{C}) \setminus \{0, \mu_N, \infty\}$ with a complex variable s

$$\frac{d}{ds} H(s) = \left(\frac{x}{s} + \sum_{\xi \in \mu_N} \frac{y\xi}{\xi s - 1} \right) H(s)$$

is introduced and discussed in [10, 16]. It is a non-commutative formal power series $\Phi_{\text{KZ}}^N \in \widehat{\mathbb{A}}_{\mathbb{C}}$ with $\Gamma = \mu_N$ whose coefficients are given by multiple L -values. In particular, the coefficient of

$$x^{n_r-1} y_{\epsilon_r} x^{n_{r-1}-1} y_{\epsilon_{r-1}\epsilon_r} \dots x^{n_1-1} y_{\epsilon_1 \dots \epsilon_r}$$

is $(-1)^r \text{Li}_{n_1, \dots, n_r}^{\text{III}}(\epsilon_1, \dots, \epsilon_r)$. Definition 2.4 enables us to calculate a relation between the associator Φ_{KZ}^N and the mould Zag as follows

$$\text{ma}_{(\Phi_{\text{KZ}}^N)^{-1}} = \left\{ \text{Zag} \left(\begin{matrix} -u_1, \dots, -u_m \\ \epsilon_1^{-1}, \dots, \epsilon_m^{-1} \end{matrix} \right) \right\}_m \in \mathcal{M}(\mathcal{F}; \mu_N). \tag{B.3}$$

Put $\widetilde{\Phi}_{\text{KZ}}^N = \Phi_{\text{KZ}}^N(x, (-y_\sigma))$ and $\widetilde{\Phi}_{\text{KZ,corr}} = \exp\{\sum_{n \geq 2} \frac{(-1)^{n-1}}{n} \text{Li}_n(1) y_1^n\} \in \widehat{\mathbb{A}}_{\mathbb{C}}$ and define

$$\widetilde{\Phi}_{\text{KZ,*}}^N := \widetilde{\Phi}_{\text{KZ,corr}} \cdot \mathbf{q}(\pi_Y(\widetilde{\Phi}_{\text{KZ}}^N)) \tag{B.4}$$

(for π_Y and \mathbf{q} , see Section 2.1). It is shown in [16] that $\tilde{\Phi}_{KZ}^N$ is group-like with respect to the shuffle (deconcatenation) coproduct of $\widehat{\mathbb{A}}_{\mathbb{C}}$ and $\tilde{\Phi}_{KZ,*}^N$ is so with respect to the harmonic coproduct of $\text{Im } \pi_Y$. They correspond to the shuffle and the harmonic product among multiple L -values respectively. The regularized double shuffle relations (the shuffle and the harmonic products and the regularization relations (B.4)) are the defining equations of his torsor $\text{DMR}_{\mu}^{(22)}$ for $\mu \in \mathbb{C}^{\times}$ which contains $\tilde{\Phi}_{KZ}^N$ as a specific point when $\mu = 2\pi\sqrt{-1}$. It is equipped with a free and transitive action of the group DMR_0 . Its associated Lie algebra, which he denotes by $\mathfrak{d}\text{mr}_0$, is a filtered graded Lie algebra defined by the regularized double shuffle relations modulo products. Our dihedral Lie algebra $\mathbb{D}(\mu_N)_{\bullet\bullet}$ should be called its bigraded variant, defined by their highest depth parts” of the relations. By definition, it contains the associated graded $\text{gr}\mathfrak{d}\text{mr}_0$. By translating Ecalle’s pictures including the diagram (B.2) to this setting, we might learn more enriched structures on these Lie algebras.

Bibliography

- [1] A. ALEKSEEV, B. ENRIQUEZ & C. TOROSSIAN, “Drinfeld associators, braid groups and explicit solutions of the Kashiwara-Vergne equations”, *Publ. Math., Inst. Hautes Étud. Sci.* **112** (2010), no. 112, p. 143-189.
- [2] A. ALEKSEEV, N. KAWAZUMI, Y. KUNO & F. NAEF, “The Goldman-Turaev Lie bialgebra in genus zero and the Kashiwara-Vergne problem”, *Adv. Math.* **326** (2018), p. 1-53.
- [3] A. ALEKSEEV & C. TOROSSIAN, “The Kashiwara-Vergne conjecture and Drinfeld’s associators”, *Ann. Math.* **175** (2012), no. 2, p. 415-463.
- [4] T. ARAKAWA & M. KANEKO, “On multiple L -values”, *J. Math. Soc. Japan* **56** (2004), no. 4, p. 967-991.
- [5] J. CRESSON, “Calcul moulien”, *Ann. Fac. Sci. Toulouse, Math.* **18** (2009), no. 2, p. 307-395.
- [6] V. G. DRINFEL’D, “On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ ”, *Leningr. Math. J.* **2** (1991), no. 4, p. 829-860.
- [7] J. ECALLE, *Les fonctions résurgentes. Tome I et II*, Publications Mathématiques d’Orsay, vol. 81, Université de Paris-Sud, 1981.
- [8] ———, “ARI/GARI, la dimorphie et l’arithmétique des multizêtas: un premier bilan”, *J. Théor. Nombres Bordeaux* **15** (2003), no. 2, p. 411-478.
- [9] ———, “The flexion structure and dimorphy: flexion units, singulators, generators, and the enumeration of multizeta irreducibles”, in *Asymptotics in dynamics, geometry and PDEs. Generalized Borel summation. Vol. II*, Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series (Nuova Serie), vol. 12, Edizioni della Normale, 2011, p. 27-211.
- [10] B. ENRIQUEZ, “Quasi-reflection algebras and cyclotomic associators”, *Sel. Math., New Ser.* **13** (2007), no. 3, p. 391-463.

(22) The word DMR stands for the French “double mélange et régularisation”.

- [11] H. FURUSHO, “Around associators”, in *Automorphic forms and Galois representations*, London Mathematical Society Lecture Note Series, vol. 415, Cambridge University Press, 2014, p. 105-117.
- [12] A. B. GONCHAROV, “The dihedral Lie algebras and Galois symmetries of $\pi_1^{(l)}(\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N))$ ”, *Duke Math. J.* **110** (2001), no. 3, p. 397-487.
- [13] ———, “Multiple polylogarithms and mixed Tate motives”, 2001, <https://arxiv.org/abs/math/0103059>.
- [14] M. MAASSARANI, “Bigraded Lie algebras related to multiple zeta values”, *Publ. Res. Inst. Math. Sci.* **58** (2022), no. 4, p. 757-791.
- [15] G. RACINET, “Séries génératrices non-commutatives de polyzêtas et associateurs de Drinfeld”, PhD Thesis, Université de Picardie Jules Verne, 2000.
- [16] ———, “Doubles mélanges des polylogarithmes multiples aux racines de l’unité”, *Publ. Math., Inst. Hautes Étud. Sci.* **95** (2002), p. 185-231.
- [17] E. RAPHAEL & L. SCHNEPS, “On linearised and elliptic versions of the Kashiwara-Vergne Lie algebra”, 2017, <https://arxiv.org/abs/1706.08299v1>.
- [18] A. SALERNO & L. SCHNEPS, “Mould theory and the double shuffle Lie algebra structure”, in *Periods in quantum field theory and arithmetic*, Springer Proceedings in Mathematics & Statistics, vol. 314, Springer, 2020, p. 399-430.
- [19] D. SAUZIN, “Mould expansions for the saddle-node and resurgence monomials”, in *Renormalization and Galois theories*, IRMA Lectures in Mathematics and Theoretical Physics, vol. 15, European Mathematical Society, 2009, p. 83-163.
- [20] L. SCHNEPS, “Double shuffle and Kashiwara-Vergne Lie algebras”, *J. Algebra* **367** (2012), p. 54-74.
- [21] ———, “ARI, GARI, ZIG and ZAG: An introduction to Ecalle’s theory of multiple zeta values”, 2015, <https://arxiv.org/abs/1507.01534>.