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An explicit estimate of the Bergman kernel for positive line bundles (*)

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Abstract. — We shall give an explicit estimate of the lower bound of the Bergman kernel associated to a positive line bundle. In the compact Riemann surface case, our result can be seen as an explicit version of Tian's partial C^0-estimate.

Résumé. — Nous donnerons une estimation explicite de la borne inférieure du noyau de Bergman associé à un fibré de droites positif. Dans le cas de la surface compacte de Riemann, notre résultat peut être vu comme une version explicite de l'estimation partielle C^0 de Tian.

1. Introduction

Let $(L, e^{-\phi})$ be a positive line bundle over an n-dimensional complex manifold X. Let m be a positive integer. Let K_X be the canonical line bundle over X. We call

$$\mathrm{K}_{m\phi}(x) := \sup_{u \in H^0(X, K_X + mL)} \frac{u(x) \wedge \overline{u(x)}\, e^{-m\phi(x)}}{\int_X u \wedge \bar{u}\, e^{-m\phi}}, \tag{1.1}$$

the *Bergman kernel forms* and

$$\mathrm{B}_{m\phi}(x) := \sup_{u \in H^0(X, mL)} \frac{|u(x)|^2 e^{-m\phi(x)}}{\int_X |u|^2 e^{-m\phi}\, \mathrm{MA}_{m\phi}}, \quad \mathrm{MA}_{m\phi} := \frac{(i\partial\bar{\partial}(m\phi))^n}{n!}, \tag{1.2}$$

the *Bergman kernel functions*. In [21] Tian proved that if X is compact then (rigorously speaking, Tian only proved the identity for the Bergman kernel

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function, but the Hörmander method used in his proof also applies to the Bergman kernel form)

$$\lim_{m\to\infty} \frac{\mathrm{K}_{m\phi}}{\mathrm{MA}_{m\phi}} = \lim_{m\to\infty} \mathrm{B}_{m\phi} = \frac{1}{(2\pi)^n}. \tag{1.3}$$

Effective lower bound estimate (with Ricci curvature, diameter and volume assumptions) for $\mathrm{B}_{m\phi}$ is known as Tian's *partial C^0-estimate* [22]. The first general result is obtained by Donaldson–Sun [10] using proof by contradiction. Our main results are the followings:

Theorem A. — *Let $(L, e^{-\phi})$ be a positive line bundle over a compact Riemann surface X. Put $\omega := \mathrm{MA}_\phi = i\partial\bar\partial\phi$. Denote by $\mathrm{Ric}\,\omega := i\bar\partial\partial\log\omega$ the Ricci form of ω. Assume that*

$$\mathrm{Ric}\,\omega \leqslant \omega, \quad L_0 \geqslant 2\pi,$$

where L_0 denotes the infimum of the length of closed geodesics in X, then

$$\mathrm{K}_\phi/\mathrm{MA}_\phi \geqslant \frac{1}{8\pi}.$$

Theorem B. — *Let $(L, e^{-\phi})$ be a positive line bundle over a compact Riemann surface X. If*

$$-\omega/2 \leqslant \mathrm{Ric}\,\omega \leqslant \omega/2, \quad L_0 \geqslant 2\pi\sqrt{2},$$

then $\mathrm{B}_\phi \geqslant \frac{1}{16\pi}$.

Remark. — There always exist closed geodesics on a compact Riemann surface. In case $X = \mathbb{P}^1$ and $\omega = 2\cdot i\partial\bar\partial\log(1+|z|^2)$ we have

$$\mathrm{Ric}\,\omega = \omega, \quad L_0 = 2\pi,$$

a direct computation gives $L = -K_X$ and $\mathrm{K}_\phi/\mathrm{MA}_\phi = \frac{1}{4\pi}$. We do not know whether

$$\mathrm{K}_\phi/\mathrm{MA}_\phi \geqslant \frac{1}{4\pi}$$

is always true with the assumptions in Theorem A. On the other hand, Theorem A implies $K_{m\phi}/\mathrm{MA}_{m\phi} \geqslant 1/(8\pi)$ for every positive integer m. This is also nearly optimal since by (1.3)

$$\lim_{m\to\infty} K_{m\phi}/\mathrm{MA}_{m\phi} = 1/(2\pi).$$

In case $\mathrm{Ric}\,\omega \leqslant 0$, $L_0/2$ is equal to the injectivity radius by Klingenberg's estimate (let c go to infinity in (3.4)). For example if $X = \mathbb{C}/\Gamma$ is a torus and $\omega = i\partial\bar\partial(|z|^2/2)$ then $\mathrm{Ric}\,\omega = 0$ and $L_0 = \inf_{0\neq\gamma\in\Gamma}|\gamma|$.

In the first version of this paper, a weaker version of the above theorems is proved using an Ohsawa–Takegoshi type theorem, a variant of the Blocki–Zwonek estimate [6] and the isoperimetric inequality. Later we find that one may use the Hessian comparison theorem to simplify the proof and generalize the above theorems to the following higher dimensional cases.

THEOREM An. — *Let* $(L, e^{-\phi})$ *be a positive line bundle over an* n*-dimensional compact complex manifold* X*. Assume that the sectional curvature of* $\omega := i\partial\bar{\partial}\phi$ *is bounded above by* $1/(4n)$ *and* $L_0 \geqslant 2\pi\sqrt{n}$ *then* $\mathrm{K}_\phi/\mathrm{MA}_\phi \geqslant \frac{1}{2}\frac{n!}{(4\pi n)^n}$.

THEOREM Bn. — *Let* $(L, e^{-\phi})$ *be a positive line bundle over an* n*-dimensional compact manifold* X*. Assume that the sectional curvature of* $\omega := i\partial\bar{\partial}\phi$ *is bounded above by* $1/(8n)$*,* $L_0 \geqslant 2\pi\sqrt{2n}$ *and* $\mathrm{Ric}\,\omega \geqslant -\omega/2$*. Then* $\mathrm{B}_\phi \geqslant \frac{1}{2}\frac{n!}{(8\pi n)^n}$.

Remark. — Since the Ricci curvature is certain sum of sectional curvature, the curvature assumptions in Theorem Bn also imply a lower bound of the sectional curvature. Hence one may use [13, Corollary 2.3.2] to find a lower bound of L_0 in terms of the lower bound of the volume and the upper bound of the diameter. Thus, except for the upper bound of the sectional curvature, the assumptions in Theorem Bn follow from the standard assumptions in Tian's partial C^0-estimate (for results on Tian's partial C^0-estimate, see [1, 7, 8, 10, 14, 15, 16, 20, 23, 24, 26], etc). Our main contribution is the explicit constant in the estimate. Moreover, our estimate implies that

$$\mathrm{B}_{m\phi} \geqslant \frac{1}{2}\frac{n!}{(8\pi n)^n} \tag{$\star$}$$

for all positive integers m. From the last section in [10], it seems that for general positive line bundles over higher dimensional manifolds, the Ricci curvature assumptions might not enough to derive ($\star$).

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2. Hessian comparison theorem

DEFINITION 2.1. — *Let* X *be a Riemann manifold. Denote by* $K(V, W)$ *the sectional curvature of the tangent plane spanned by* V, W*. Fix* $x \in X$*, the injectivity radius at* $x \in X$ *is defined as*

$$\mathrm{inj}(x) := \sup\{r > 0 : \exp_x|_{B(0,r)} \text{ is diffeomorphism}\}$$

where $\exp_x : T_xX \to X$ *denotes the exponential map at* x *and* $B(0,r)$ *denotes the ball of radius* r *around* $0 \in T_xX$*. We call* $\mathrm{inj}_X := \inf_{x\in X} \mathrm{inj}(x)$ *the injectivity radius of* X*.*

The following $\partial\bar\partial$-comparison theorem is a direct consequence of the Hessian comparison theorem (see [11, Lemma 1.13 in p. 14 and Theorem A in p. 19]).

Theorem 2.2. — *Let* X_1, X_2 *be Kähler manifolds. Let* $\gamma_1 : [0,b] \to X_1$ *and* $\gamma_2 : [0,b] \to X_2$ *be unit speed geodesics. With the definition above, suppose that*

$$b \leqslant \min\{\mathrm{inj}(\gamma_1(0)), \mathrm{inj}(\gamma_2(0))\} \tag{2.1}$$

and for all $t \in [0,b]$, $v_1 \perp \dot\gamma_1(t)$ *and* $v_2 \perp \dot\gamma_2(t)$,

$$K(\dot\gamma_1(t), v_1) \leqslant K(\dot\gamma_2(t), v_2). \tag{2.2}$$

Let $d_j := d(\,\cdot\,, \gamma_j(0))$ *be distance functions. If* $f : (0,b) \to \mathbb{R}$ *is smooth and increasing then*

$$i\partial\bar\partial(f\circ d_1)(V_1,V_1) \geqslant i\partial\bar\partial(f\circ d_2)(V_2,V_2) \tag{2.3}$$

for all $t \in (0,b)$, $V_j \in T_{\gamma_j(t)}X_j$, $j = 1,2$, *such that* $|V_1| = |V_2|$ *and*

$$(\dot\gamma_1(t), V_1) = (\dot\gamma_2(t), V_2), \quad (\dot\gamma_1(t), JV_1) = (\dot\gamma_2(t), JV_2).$$

We shall apply the above theorem to $X_2 = \mathbb{P}^n$ with the Fubini study metric form $\omega_2 = 2\,i\partial\bar\partial \log(1+|z|^2)$. A direct computation gives

$$K(V,W) = \frac{1}{4}\left(1 + 3(V,JW)^2\right), \ \forall\, V \perp W, \ \ \mathrm{inj}(x) = \pi. \tag{2.4}$$

In particular, $K(V,W) = 1$ in case $n = 1$ and $K(V,W) \geqslant 1/4$ in case $n \geqslant 2$. We also need the following distance function formula on $\mathbb{C}^n \subset \mathbb{P}^n$

$$d_2 := d(0,z) = 2\int_0^{|z|} \frac{\mathrm{d}x}{1+x^2} = 2\arctan|z|.$$

Put

$$\psi = \log \sin^2(d_2/2)$$

we have

$$\psi = \log \frac{\tan^2(d_2/2)}{1+\tan^2(d_2/2)} = \log|z|^2 - \log(1+|z|^2) \leqslant 0$$

and

$$i\partial\bar\partial\psi \geqslant -i\partial\bar\partial \log(1+|z|^2) = -\omega_2/2.$$

Apply the above theorem to $f(x) = \log\sin^2(x/2)$ and $b = \pi$ we get:

Corollary 2.3. — *Let (X_1,ω_1) be an n-dimension Kähler manifold. Fix $x\in X_1$. Put*

$$\psi(z) := \begin{cases} \log\sin^2(d(z,x)/2) & d(z,x)\leqslant \pi \\ 0 & d(z,x)>\pi. \end{cases}$$

Assume that $\operatorname{inj}(x)>\pi$ *and*

$$\begin{cases} \text{the sectional curvature of } \omega_1 \text{ is no bigger than } 1, & n=1; \\ \text{the sectional curvature of } \omega_1 \text{ is no bigger than } 1/4, & n\geqslant 2, \end{cases}$$

then $i\partial\bar\partial\psi\geqslant -\omega_1/2$ on X.

Proof. — By the lemma below, we can write

$$\psi(z)=2f(d(z,x)/2).$$

Since $\operatorname{inj}(x)>\pi$, we know that $d(z,x)$ is smooth on a neighborhood of $\{d(z,x)\leqslant\pi\}$, hence the lemma below implies that ψ is $C^{1,1}$ on X_1. Thus it suffices to verify positivity (in the sense of current) of $i\partial\bar\partial\psi+\omega_1/2$ on $\{d(z,x)\neq\pi\}$, which follows directly by the theorem above. □

Lemma 2.4. — *The following function*

$$f(t)=\begin{cases} \log\sin t & 0<t\leqslant\pi/2 \\ 0 & t>\pi/2 \end{cases}$$

is $C^{1,1}$ in $t>0$.

Proof. — By a direct computation, we have

$$f'(t)=\begin{cases} \cos t/\sin t & 0<t\leqslant\pi/2 \\ 0 & t>\pi/2 \end{cases},\qquad f''(t)=\begin{cases} -\sin^{-2}t & 0<t\leqslant\pi/2 \\ 0 & t>\pi/2. \end{cases}$$

Hence f' is continuous and f'' is locally bounded and the lemma follows. □

3. An Ohsawa–Takegoshi type theorem

We shall use the following Ohsawa–Takegoshi type theorem [4, 18, 19], which is a special case of the main theorem in [12].

Theorem 3.1. — *Let $(L,e^{-\phi})$ be a positive line bundle on an n-dimensional compact complex manifold. Fix $x\in X$. Assume that there is a non-positive function G smooth outside x such that $G(z)-\log|z-x|^{2n}$ is smooth near x and*

$$i\partial\bar\partial\phi+\lambda\, i\partial\bar\partial G\geqslant 0$$

on X for some constant $\lambda > 1$. Then K_ϕ *in* (1.1) *satisfies*

$$\frac{\mathrm{K}_\phi(x)}{\mathrm{MA}_\phi(x)} \geqslant \frac{\lambda-1}{\lambda} \lim_{t\to-\infty} \frac{1}{e^{-t}\int_{G<t}\mathrm{MA}_\phi}, \tag{3.1}$$

where MA_ϕ *is defined in* (1.2).

Proof. — Let us rephrase the proof in [18]. Our curvature assumption implies that

$$\phi^{t+is} := \phi + \lambda \max\{G-t, 0\}$$

defines a singular metric on $\mathbb{C}_{t+is} \times L$ with non-negative curvature current. Hence Berndtsson's theorem [2] implies that (notice that $\phi^t(x) = \phi(x)$)

$$\log \frac{\mathrm{K}_{\phi^t}(x)}{\mathrm{MA}_\phi(x)}$$

is a convex function of t. By a direct computation (see the lemma below, see also the appendix in [18] or Theorem 3.8 in [3]) we find that

$$\lim_{t\to-\infty} e^t \frac{\mathrm{K}_{\phi^t}(x)}{\mathrm{MA}_\phi(x)} = \frac{\lambda-1}{\lambda} \lim_{t\to-\infty} \frac{1}{e^{-t}\int_{G<t}\mathrm{MA}_\phi} \tag{3.2}$$

is finite since $G(z) - \log|z-x|^{2n}$ is smooth near x. Hence $e^t\mathrm{K}_{\phi^t}(x)/\mathrm{MA}_\phi(x)$, as a convex function of t bounded near $-\infty$, must be increasing. Thus

$$\mathrm{K}_\phi(x) = e^0\mathrm{K}_{\phi^0}(x) \geqslant \lim_{t\to-\infty} e^t\mathrm{K}_{\phi^t}(x) = \frac{\lambda-1}{\lambda} \lim_{t\to-\infty} \frac{\mathrm{MA}_\phi(x)}{e^{-t}\int_{G<t}\mathrm{MA}_\phi}$$

gives our estimate. □

Lemma 3.2. — *Put*

$$\|F\|_t^2 := \int_X i^{n^2} F \wedge \overline{F} e^{-\phi-\lambda\max\{G-t,0\}}, \quad F \in H^0(X, K_X + L),$$

then

$$\lim_{t\to-\infty} e^{-t}\|F\|_t^2 = \frac{\lambda}{\lambda-1} \frac{i^{n^2} F(x) \wedge \overline{F(x)} e^{-\phi(s)}}{\mathrm{MA}_\phi(x)} \lim_{t\to-\infty} e^{-t} \int_{G<t} \mathrm{MA}_\phi.$$

Proof. — Consider a positive Borel measure μ on X defined by

$$\mathrm{d}\mu := i^{n^2} F \wedge \overline{F}\, e^{-\phi}.$$

Then

$$\mu(X) = \int_X i^{n^2} F \wedge \overline{F}\, e^{-\phi}, \quad \mu(G<s) = \int_{G<s} i^{n^2} F \wedge \overline{F}\, e^{-\phi},$$

and we have

$$\|F\|_t^2 = \int_X e^{-\chi(G-t)}\,\mathrm{d}\mu, \quad \chi(s) := \lambda\max\{s, 0\}.$$

Since $G \leqslant 0$ on X, using the Lebesgue–Stieltjes integral we can write

$$\int_X e^{-\chi(G-t)}\, \mathrm{d}\mu = e^{\lambda t}\mu(X) - \int_{-\infty}^{0} \mu(G<s)\, \mathrm{d}e^{-\chi(s-t)}, \tag{3.3}$$

Since $\mu(X)$ is finite (note that X is compact) and $\lambda > 1$, we have

$$\begin{aligned}
\lim_{t\to-\infty} e^{-t}\|F\|_t^2 &= \lim_{t\to-\infty}\left(-e^{-t}\int_{-\infty}^{0}\mu(G<s)\,\mathrm{d}e^{-\chi(s-t)}\right)\\
&= \lim_{t\to-\infty}\left(-e^{-t}\int_{t}^{0}\mu(G<s)\,\mathrm{d}e^{-\lambda(s-t)}\right)\\
&= \lim_{t\to-\infty}\left(\lambda\int_{t}^{0}\mu(G<s)e^{-s}e^{-(\lambda-1)(s-t)}\,\mathrm{d}s\right)\\
&= \lim_{t\to-\infty}\left(\lambda\int_{0}^{-t}\mu(G<a+t)e^{-(a+t)}e^{-(\lambda-1)a}\,\mathrm{d}a\right)\\
&= \left(\lambda\int_{0}^{\infty}e^{-(\lambda-1)a}\,\mathrm{d}a\right)\lim_{s\to-\infty}\left(\mu(G<s)e^{-s}\right)\\
&= \frac{\lambda}{\lambda-1}\lim_{s\to-\infty} e^{-s}\int_{G<s} i^{n^2}F\wedge\overline{F}e^{-\phi},
\end{aligned}$$

which gives the lemma (note that $\{G<s\}$ converges to the single point x). $\square$

3.1. Proof of Theorem A, B, An, Bn

Proof of Theorem A, An. — By Klingenberg's estimate (see [25, Corollary 1.2]), if the sectional curvature is no bigger than $1/c$ then we have

$$\min\{L_0/2, \sqrt{c}\,\pi\} \leqslant \mathrm{inj}_X \leqslant L_0/2. \tag{3.4}$$

Hence our assumptions implies that the injectivity radius of $(X, \omega/n)$, $\omega := i\partial\overline{\partial}\phi$, is no less than π. Thus one may apply Corollary 2.3 to $(X_1,\omega_1) = (X,\omega/n)$. Put $G = n\psi$. Corollary 2.3 implies that

$$i\partial\overline{\partial}\phi + 2\, i\partial\overline{\partial}G \geqslant 0.$$

By Theorem 3.1 ($\lambda = 2$), we get

$$\frac{\mathrm{K}_\phi(x)}{\mathrm{MA}_\phi(x)} \geqslant \frac{1}{2}\lim_{t\to-\infty}\frac{1}{e^{-t}\int_{n\psi<t}\mathrm{MA}_\phi} = \frac{1}{2}\lim_{t\to-\infty}\frac{1}{e^{-t}\int_{\log\sin^{2n}(d(z,x)/2)<t}\frac{\omega^n}{n!}}.$$

Note that $\lim_{s\to 0}\frac{\sin s}{s}=1$ gives

$$\lim_{t\to-\infty}\frac{1}{e^{-t}\int_{\log\sin^{2n}(d(z,x)/2)<t}\frac{\omega^n}{n!}}=\lim_{t\to-\infty}\frac{1}{e^{-t}\int_{d(z,x)<2e^{t/(2n)}}\frac{n^n\omega_1^n}{n!}}$$
$$=\frac{1}{\frac{\pi^n}{n!}\,2^{2n}n^n},$$

from which Theorem A, An follows. □

Proof of Theorem B, Bn. — By the Ricci curvature assumption, ϕ and $i\partial\bar\partial\phi$ defines a metric on $L-K_X$ with curvature

$$i\Theta=\omega+\operatorname{Ric}\omega\geqslant\omega/2.$$

Thus one may apply Corollary 2.3 to $(X_1,\omega_1)=(X,\omega/(2n))$. Put $G=n\psi$ then

$$i\Theta+2\,i\partial\bar\partial G\geqslant 0.$$

Apply Theorem 3.1 to $L-K_X$, we get Theorem B, Bn. □

4. Another proof of a weaker version of Theorem A, B

In this section we shall give another proof of Theorem A, B with an extra volume assumption.

Theorem C. — *Let $(L,e^{-\phi})$ be a positive line bundle over a compact Riemann surface X. If*

$$\int_X\omega\geqslant 8\pi,\quad \operatorname{Ric}\omega\leqslant\omega,\quad L_0\geqslant 2\pi,$$

then $\mathrm{K}_\phi/\mathrm{MA}_\phi\geqslant\frac{1}{8\pi}$.

Theorem D. — *Let $(L,e^{-\phi})$ be a positive line bundle over a compact Riemann surface X. If*

$$\int_X\omega\geqslant 16\pi,\quad -\omega/2\leqslant\operatorname{Ric}\omega\leqslant\omega/2,\quad L_0\geqslant 2\pi\sqrt{2},$$

then $\mathrm{B}_\phi\geqslant\frac{1}{16\pi}$.

4.1. The Blocki–Zwonek estimate

We shall study the right hand side of (3.1) using a variant of Blocki–Zwonek's estimate [6, Proof of Theorem 3] (see also [5, Section 10] for related results).

LEMMA 4.1. — *With the notation in Theorem 3.1. Let ω be an arbitrary Kähler form on X. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{G<t}\omega_n \geqslant \frac{\sigma(G=t)^2}{2\int_{G<t} i\partial\bar{\partial}G\wedge\omega_{n-1}}, \quad \omega_q := \omega^q/q!,$$

where

$$\sigma(G=t) := \int_{G=t}\mathrm{d}\sigma, \quad \mathrm{d}\sigma := \sqrt{2}\sum \frac{G_{\bar{\alpha}}\omega^{\bar{\alpha}\beta}}{|\bar{\partial}G|_\omega}\frac{\partial}{\partial z^\beta}\rfloor\,\omega_n,$$

is the measure of the hypersurface $\{G=t\}$ with respect to ω.

Proof. — Notice that

$$V := \frac{\partial}{\partial t} + \sum \frac{G_{\bar{\alpha}}\omega^{\bar{\alpha}\beta}}{|\bar{\partial}G|^2_\omega}\frac{\partial}{\partial z^\beta}$$

satisfies $V(G-t)=0$, hence it can be used to compute $\frac{\mathrm{d}}{\mathrm{d}t}\int_{G<t}$, in particular, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{G<t}\omega_n = \int_{G<t}L_V\omega_n = \int_{G=t}V\rfloor\,\omega_n = \frac{1}{\sqrt{2}}\int_{G=t}\frac{\mathrm{d}\sigma}{|\bar{\partial}G|_\omega}.$$

Hence the Cauchy–Schwarz inequality gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{G<t}\omega_n &\geqslant \frac{\sigma(G=t)^2}{\sqrt{2}\int_{G=t}|\bar{\partial}G|_\omega\,\mathrm{d}\sigma}\\ &= \frac{\sigma(G=t)^2}{2\int_{G=t} i\bar{\partial}G\wedge\omega_{n-1}}\\ &= \frac{\sigma(G=t)^2}{2\int_{G<t} i\partial\bar{\partial}G\wedge\omega_{n-1}},\end{aligned}$$

where we use the Stokes theorem in the last equality. □

Since

$$G_1 \leqslant G_2 \Longrightarrow \int_{G_2<t}\omega_n \leqslant \int_{G_1<t}\omega_n,$$

in order to get the best estimate from (3.1), one should choose G to be the following *envelope*, say $g_{\phi,x,\lambda}$, defined by

$$\sup\{G\leqslant 0 : G(z)-\log|z-x|^{2n} \text{ smooth near } x,\ i\partial\bar{\partial}\phi+\lambda\, i\partial\bar{\partial}G\geqslant 0\}. \quad (4.1)$$

It is known that (see [9])

$$\epsilon_x(L) := \sup\{\lambda\geqslant 0 : \exists\, G\leqslant 0 \text{ on } X \text{ with the blue part in (4.1) holds}\} \quad (4.2)$$

is equal to the *Seshadri constant* up to a constant factor n. If $0<\lambda<\epsilon_x(L)$ then $g_{\phi,x,\lambda}$, as an envelope, must satisfy

$$(i\partial\bar{\partial}\phi+\lambda\, i\partial\bar{\partial}g_{\phi,x,\lambda})^n = (2\pi n\lambda)^n\delta_x \quad (4.3)$$

on $\{g_{\phi,x,\lambda} < 0\}$, where δ_x is the Dirac measure defined by $\int_X f\,\delta_x = f(x)$. Thus if we choose $G = g_{\phi,x,\lambda}$ and $\omega = i\partial\bar{\partial}\phi$ then

$$\int_{G<t} i\partial\bar{\partial}G \wedge \omega_{n-1} = \int_{G<t} (i\partial\bar{\partial}G + \omega/\lambda - \omega/\lambda) \wedge \omega_{n-1} \tag{4.4}$$

$$= \int_{G<t} (i\partial\bar{\partial}G + \omega/\lambda) \wedge \omega_{n-1} - \frac{n}{\lambda}\int_{G<t} \omega_n \tag{4.5}$$

$$\leqslant \int_{G<0} (i\partial\bar{\partial}G + \omega/\lambda) \wedge \omega_{n-1} - \frac{n}{\lambda}\int_{G<t} \omega_n \tag{4.6}$$

$$= \frac{n}{\lambda}\left(\int_{G<0} \omega_n - \int_{G<t} \omega_n\right), \tag{4.7}$$

where we use $\int_{G<0} i\partial\bar{\partial}G \wedge \omega_{n-1} = 0$ (since $G = 0$ outside $\{G < 0\}$) in (4.7). In case $n = 1$, (4.3) directly gives

$$\int_{G<t} i\partial\bar{\partial}G = \int_{G<t} i\partial\bar{\partial}G + \omega/\lambda - \omega/\lambda = 2\pi - \int_{G<t} \omega/\lambda. \tag{4.8}$$

Hence Lemma 4.1 implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{G<t} \omega \geqslant \frac{\sigma(G=t)^2}{4\pi - \frac{2}{\lambda}\int_{G<t}\omega}. \tag{4.9}$$

4.2. Isoperimetric inequality

We shall use the following result (see inequality (5.4) in [17, Proposition 5.2]).

Lemma 4.2 (Isoperimetric inequality). — *Let U be an open subset of a compact Riemann surface (X, ω). Assume that*

$$A := \int_U \omega \leqslant \frac{1}{2}\int_X \omega, \quad \mathrm{Ric}\,\omega \leqslant k\,\omega.$$

Then

$$\sigma(\partial U)^2 \geqslant \min\{L_0^2, A(4\pi - kA)\}, \tag{4.10}$$

where L_0 denotes the infimum of the length of simple closed geodesics in X.

Proof. — By the definition of the Seshadri constant in (4.2), we know that in case $n = 1$, the Hodge decomposition gives

$$\epsilon_x = \deg(L) := \int_X c_1(L) = \frac{1}{2\pi}\int_X \omega. \tag{4.11}$$

Hence if $\int_X \omega \geqslant 8\pi$ then $\epsilon_x \geqslant 4$. Hence we can take $\lambda = 2$ in (4.9), which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{G<t} \omega \geqslant \frac{\sigma(G=t)^2}{4\pi - \int_{G<t}\omega}. \tag{4.12}$$

Now, since $\operatorname{Ric}\omega \leqslant \omega$, $L_0^2 \geqslant 4\pi^2$ and

$$\int_{G<t} \omega \leqslant \int_{G<0} \omega = 2\pi\lambda = 4\pi \leqslant \frac{1}{2}\int_X \omega,$$

one may apply the above lemma to $U = \{G < t\}$. Then by (4.10) we have

$$\sigma(G=t)^2 \geqslant \min\{4\pi^2, A(4\pi - A)\} = A(4\pi - A), \quad A := \int_{G<t} \omega.$$

Thus (4.12) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{G<t} \omega \geqslant \int_{G<t} \omega,$$

which implies that $e^{-t}\int_{G<t}\omega$ is increasing with respect to $t < 0$. Hence

$$\lim_{t\to-\infty} e^{-t}\int_{G<t} \omega \leqslant e^{-0}\int_{G<0} \omega = 2\pi\lambda = 4\pi.$$

Then by (3.1) we have $K_\phi \geqslant \frac{\omega}{8\pi}$. □

Theorem D follows by a similar argument (see the difference between the proof of Theorem A, B above).

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