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A Nonvanishing Conjecture for Cotangent Bundles (*)

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Abstract. — In this paper we study the positivity of the cotangent bundle of projective manifolds. We conjecture that the cotangent bundle is pseudoeffective if and only the manifold has non-zero symmetric differentials. We confirm this conjecture for most projective surfaces that are not of general type.

Résumé. — Dans ce papier nous étudions la positivité du fibré cotangent des variétés projective lisses. Nous conjecturons que le fibré cotangent est pseudoeffectif si et seulement si la variété possède des formes holomorphes symétriques non-nulles. Nous montrons cette conjecture pour la plupart des surfaces projectives lisses qui ne sont pas de type général.

1. Introduction

1.1. Main result

A central part in the minimal model program in algebraic geometry is the so-called nonvanishing conjecture: given a projective manifold or, more generally, a variety with klt singularities, X, whose canonical class K_X is pseudoeffective, one has

$$H^0(X, \mathcal{O}_X(mK_X)) \neq 0$$

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for some positive integer m. This conjecture has been proven some time ago in dimension at most three, but is wide open in higher dimensions.

In analogy to the nonvanishing conjecture, one might ask for

Conjecture 1.1. — *Let X be a normal projective variety with klt singularities. Let $1 \leqslant q \leqslant \dim X$. Then $\Omega^{[q]}_X$, the sheaf of reflexive holomorphic differentials in degree q, is pseudoeffective, (see Definition 3.5), if and only if for some positive integer m one has*

$$H^0(X, S^{[m]}\Omega^{[q]}_X) \neq 0.$$

In the case $q = \dim X$, this is of course the nonvanishing conjecture stated above. There is an important difference with the classical case of the canonical bundle: while the pseudoeffectivity of K_X is equivalent to the non-uniruledness of the manifold, we do not know many examples where Ω^q_X is pseudoeffective, but not big. We expect that this property is actually quite restrictive, our Theorem 1.2 confirms this intuition in the first non-trivial case.

In this paper we are mainly interested in the case $q = 1$. The only general result confirming Conjecture 1.1 is given in [29, Thm. 1.6], [23, Thm. 1.2]: Suppose X is klt with $K_X \equiv 0$. If $\Omega^{[1]}_X$ is pseudoeffective, there is a quasi-étale cover $\widetilde{X} \to X$ such that $q(\widetilde{X}) > 0$. In particular one has $H^0(X, S^{[m]}\Omega^{[1]}_X) \neq 0$ for some positive integer m ([3, Prop. 2.2], see also Lemma 4.6). If $K_X \not\equiv 0$, Conjecture 1.1 is delicate, even for smooth surfaces. In this case we can assume without loss of generality that X is minimal, see Proposition 4.1. By surface classification, see Corollary 4.14, the problem starts with $\kappa(X) = 1$. We basically settle this case:

Theorem 1.2. — *Let $f : X \to B$ be a (minimal) smooth elliptic surface with $\kappa(X) = 1$ such that Ω^1_X is pseudoeffective. Suppose one of the following.*

(1) *f is not isotrivial;*
(2) *f is isotrivial and the general fibre has trivial group automorphism group $\mathbb{Z}_2$;*
(3) *the tautological class on $\mathbb{P}(\Omega^1_X)$ is nef in codimension one.*

Then $\widetilde{q}(X) > 0$ (see Definition 2.2), so there is a positive integer m such that

$$H^0(X, S^m\Omega^1_X) \neq 0.$$

Note that each of the cases requires a different proof, Theorem 1.2 is obtained as the union of Corollary 5.5, Corollary 6.8 and Corollary 6.13.

In general, the pseudoeffectivity of $\Omega_X^{[1]}$ does not imply $\widetilde{q}(X) > 0$: a smooth complete intersection surface $X \subset \mathbb{P}^N$ is simply connected, so $\widetilde{q}(X) = 0$. However, if $N \geqslant 4$ and the multidegrees are sufficiently high, the cotangent bundle Ω_X^1 is ample [10].

For surfaces of general type, Conjecture 1.1 is open. If $c_1^2(X) > c_2(X)$, then by Bogomolov's vanishing theorem

$$h^0(X, S^m\Omega_X^1)) \sim \left(\frac{c_1^2(X) - c_2(X)}{6}\right)m^3,$$

but already the boundary case $c_1^2(X) = c_2(X)$ is unclear.

In higher dimension, things get worse due to the singularities of minimal models. For example, we know Conjecture 1.1 for terminal threefolds with numerically trivial canonical class, but we cannot deduce easily Conjecture 1.1 for smooth threefolds X with $\kappa(X) = 0$, although X has a terminal minimal model as above.

We would finally like to point out the connection to a question posed by H.Esnault, see [12]: let X be a projective (or compact Kähler) manifold whose fundamental group $\pi_1(X)$ is infinite. Does there exist a positive integer m such that

$$H^0(X, S^m\Omega_X^1) \neq 0?$$

An intermediate step might be to prove that Ω_X^1 is pseudoeffective. Brunebarbe–Klingler–Totaro confirm Esnault's conjecture if there is a representation

$$\pi_1(X) \longrightarrow \mathrm{GL}(N, \mathbb{C})$$

with infinite image. The key point of their proof is to show that in many cases, the cotangent bundle Ω_X^1 is even big.

Theorem 1.2 fits in this framework: combining Theorem 1.2 and a well-known argument (see Appendix A) one sees that an elliptic surface should have pseudoeffective cotangent bundle if and only if the fundamental group $\pi_1(X)$ is infinite. Thus the nonvanishing would be a consequence of Esnault's conjecture. Unfortunately it is a priori not clear if the fundamental group is infinite, in particular Theorem 1.2 can not be deduced from [12].

1.2. Strategy of the proof

Let X be a smooth projective surface such that K_X is nef and $c_1(K_X)^2 = 0$. Then K_X is semiample, so we have the Iitaka fibration

$$f : X \longrightarrow B$$

such that the general fibre F is elliptic and $mK_X \simeq f^*A$ with A an ample divisor. Assume now that $\Omega_X := \Omega^1_X$ is pseudoeffective, then one expects that there exists a pseudoeffective subsheaf of Ω_X that is induced by a pull-back from the base B. Let $f^*\Omega_B \to \Omega_X$ be the cotangent map, and denote by

$$f^*\Omega_B(D) \subset \Omega_X$$

the saturation (cf. Setup 5.1 for the definition of D). If $f^*\Omega_B(D)$ is pseudoeffective, Proposition 5.2 shows that $\widetilde{q}(X) > 0$. Thus the main issue in Theorem 1.2 is to show that $f^*\Omega_B(D)$ is pseudoeffective. A natural approach is to show that the sheaf $\Omega_X \to \omega_{X/B}(-D)$ is not pseudoeffective if f is not almost smooth[(1)]. However, by a theorem of Brunella [13], the line bundle $\omega_{X/B}(-D)$ is always pseudoeffective! This leads us to considering the more refined quotient sheaf

$$\Omega_X \longrightarrow \mathcal{I}_Z \otimes \omega_{X/B}(-D) \longrightarrow 0,$$

where Z has support in the singular points of the reduction of the fibres. The basic idea of the proof of Theorem 1.2 is to show that the torsion-free sheaf $\mathcal{I}_Z \otimes \omega_{X/B}(-D)$ is not strongly pseudoeffective (see Definition 3.7), although its bidual is a pseudoeffective line bundle. Thanks to a result of Demailly–Peternell–Schneider [20] this idea leads immediately to the result in the non-isotrivial case, see Section 5. For an isotrivial fibration this approach only yields the weaker statement appearing as part c) of the main theorem, see Subsection 6.4. On the other hand we know that X is birational to a quotient $(C \times E)/G$, so we aim to compute explicitly the spaces of global sections

$$H^0(X, S^i\Omega_X \otimes \mathcal{O}_X(jA)),$$

following a strategy introduced by Sakai [36]. Apart from the technical setup, the main difficulty is to understand the local obstruction near the fixed points of the group action. For A_1-singularities this information is provided by [11, Prop. 3.2]. We expect that a similar description of the local obstruction for klt singularities would allow to handle the case when the elliptic curve has non-trivial group automorphisms.

1.3. Structure of the paper

In Section 3 we introduce a positivity notion ("strongly pseudoeffective") that is adapted for this type of torsion-free sheaf, and present material on pseudoeffective torsion free sheaves which will be used in later sections.

(1) See the introduction of Section 5 for the almost smooth case.

Section 4 is concerned with some general results on varieties with pseudoeffective cotangent sheaves. In particular, generalised Kodaira dimensions are introduced and a relation to the MRC fibration is studied.

The last two sections are devoted to the proof of Theorem 1.2. Section 5 gives the general setup and settles the case that the elliptic fibration is not isotrivial. The surprisingly difficult isotrivial case finally is studied in Section 6.

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2. Basic notations

We work over the complex numbers, for general definitions we refer to [27]. We use the terminology of [19] and [31] for birational geometry and notions from the minimal model program and [32] for notions of positivity. Manifolds and varieties will always be supposed to be irreducible and reduced.

Notation 2.1. — Let X be a normal complex variety. As usual, Ω^1_X denotes the sheaf of Kähler differentials, and we set

$$\Omega^{[q]}_X := \left(\bigwedge^q \Omega^1_X\right)^{**}.$$

If X is klt and $\mu : \widehat{X} \to X$ a is desingularization, then by [25, Thm. 1.4],

$$\Omega^{[q]}_X = \mu_*(\Omega^q_{\widehat{X}}).$$

If $q = 1$ and X smooth, we simply set $\Omega_X := \Omega^1_X$. Finally, for any normal variety X, we denote by $T_X := (\Omega^1_X)^*$ its tangent sheaf.

A finite surjective map $\gamma : X' \to X$ between normal varieties is *quasi-étale* if its ramification divisor is empty (or equivalently, by purity of the branch locus, γ is étale over the smooth locus of X).

Definition 2.2. — *Let X be a normal projective variety with klt singularities. Then, as usual,*

$$q(X) = h^1(X, \mathcal{O}_X) = h^0(X, \Omega^{[1]}_X)$$

is the irregularity of X. Further, we denote by $\widetilde{q}(X)$ the maximal irregularity $q(\widetilde{X})$, where $\widetilde{X} \to X$ is any quasi-étale cover.

While the irregularity $q(X)$ is a birational invariant of projective varieties with klt singularities, this is not the case for $\widetilde{q}(X)$:

Example 2.3. — Let $\tau : E_1 \to \mathbb{P}^1$ be a (hyper)elliptic curve, and denote by i_{E_1} the involution induced by the double cover. Let E_2 be an elliptic curve, and denote by i_{E_2} the involution defined by $z \mapsto -z$. The surface

$$X' := (E_1 \times E_2)/\langle i_{E_1} \times i_{E_2} \rangle$$

is normal and has A_1-singularities in the branch points of the quasi-étale map $E_1 \times E_2 \to X'$. The projection on the first factor induces an isotrivial elliptic fibration

$$f' : X' \longrightarrow \mathbb{P}^1 = E_1/\langle i_{E_1} \rangle$$

that has exactly $2g(E_1) + 2$ singular fibres, all of them are multiple fibres of multiplicity 2 such that the reduction is isomorphic to $\mathbb{P}^1 = E_2/\langle i_{E_2} \rangle$. By construction we have $\widetilde{q}(X') \geqslant g(E_1) + g(E_2) > 0$.

Denote by $\mu : X \to X'$ the minimal resolution, then the induced elliptic fibration $f : X \to \mathbb{P}^1$ is relatively minimal, isotrivial and has exactly $2g(E_1) + 2$ singular fibres, all of them of type I_0^* (in Kodaira's terminology, see [4, V, Table 3]).

In the classical case where E_1 is an elliptic curve, the surface X is a K3 surface of Kummer type. In particular we have $\widetilde{q}(X) = 0$.

3. Pseudoeffective sheaves

Notation 3.1. — Let $\mathcal{G}$ be a coherent sheaf on a variety X, and let $\mathcal{T} \subset \mathcal{G}$ its torsion subsheaf. Then we denote by $\mathcal{G}/\operatorname{Tor}$ the quotient $\mathcal{G}/\mathcal{T}$. Furthermore, we set

$$S^{[m]}(\mathcal{G}) := (S^m \mathcal{G})^{**}.$$

3.1. Projectivization of sheaves

Definition 3.2. — *Let $\mathcal{F}$ be a coherent sheaf on a variety X. Then we denote by $\pi : \mathbb{P}(\mathcal{F}) \to X$ the projectivisation of $\mathcal{F}$ in the sense of* [1, II, §2, Sect. 2].

We denote by $\zeta_{\mathbb{P}(\mathcal{F})}$ (or ζ when no confusion is possible) the Cartier divisor class associated to the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$.

Remark. — The reduction of any fibre of π is a projective space, and π is locally trivial if and only if $\mathcal{F}$ is locally free [1, p. 27].

For locally free sheaves, the following definition of pseudoeffectivity is now in common.

Definition 3.3. — *Let X be a projective variety, and let $\mathcal{F}$ be a locally free sheaf on X. Denote by $\pi : \mathbb{P}(\mathcal{F}) \to X$ the projectivisation, and by ζ the tautological class on $\mathbb{P}(\mathcal{F})$. We say that $\mathcal{F}$ is pseudoeffective if ζ is a pseudoeffective Cartier divisor class.*

Remark. — By [21, Lem. 2.7], the locally free sheaf $\mathcal{F}$ is pseudoeffective if and only if for some ample Cartier divisor H on X and for all $c > 0$ there exist numbers $j \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $i > cj$ and

$$H^0(X, S^i\mathcal{F} \otimes \mathcal{O}_X(jH)) \neq 0.$$

It is not difficult to generalise Druel's argument to the case where H is big, see [28, Lem. 2.2.] for a proof.

We will use the following lemma, which will be generalised below.

Lemma 3.4. — *Let $f : X \to Y$ be a surjective morphism of projective varieties and $\mathcal{F}$ a locally free sheaf on Y. Then $\mathcal{F}$ is pseudoeffective if and only if $f^*(\mathcal{F})$ is pseudoeffective.*

Remark. — Note that the statement applies in particular to the normalisation, so for locally free sheaves pseudoeffectivity can be verified on the normalisation.

Proof. — Recall the pull-back formula for the tautological classes

$$\zeta_{\mathbb{P}(f^*(\mathcal{F}))} = p^*(\zeta_{\mathbb{P}(\mathcal{F})}),$$

where $p : \mathbb{P}(f^*(\mathcal{F})) = \mathbb{P}(\mathcal{F}) \times_Y X \to \mathbb{P}(\mathcal{F})$ is the canonical projection. Thus we are reduced to the case where $\mathcal{F}$ has rank one, which is immediate by [32, Thm. 2.2.26, Prop. 2.2.43]. □

3.2. Strongly pseudoeffective torsion-free sheaves

For the purpose of this paper it is not sufficient to discuss the positivity of locally free sheaves, in fact we will need the more subtle positivity properties of torsion-free sheaves. It will suffice to consider normal varieties.

We first recall the definition of pseudoeffectivity for reflexive sheaves from [21] and [29, Def. 2.1].

Definition 3.5. — *Let X be a normal projective variety, and let $\mathcal{F}$ be a reflexive sheaf on X. Then $\mathcal{F}$ is pseudoeffective if for some ample Cartier divisor H on X and for all $c > 0$ there exist numbers $j \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $i > cj$ and*

$$H^0(X, S^{[i]}\mathcal{F} \otimes \mathcal{O}_X(jH)) \neq 0.$$

Remark. — An equivalent definition using an adapted resolution of singularities of $\mathbb{P}(\mathcal{F})$ is given in [29].

Example 3.6. — Our definition of pseudoeffectivity is less restrictive than [9, Def. 7.1]: if $\mathcal{G} \subset \mathcal{F}$ is a pseudoeffective reflexive subsheaf, then $\mathcal{F}$ is pseudoeffective. In particular if $\mathcal{F} = L \oplus H$ where L is pseudoeffective and H an antiample reflexive sheaf, then $\mathcal{F}$ is pseudoeffective in the sense of Definition 3.5, but not in the sense of [9, Def. 7.1].

Definition 3.5 makes also sense for torsion-free sheaves, but would not be very useful: by definition a torsion-free sheaf would be pseudoeffective if and only if its bidual is pseudoeffective. The following definition takes this difference into account:

Definition 3.7. — *Let X be a normal projective variety, and let $\mathcal{F}$ be a torsion free sheaf on X. We say that $\mathcal{F}$ is strongly pseudoeffective if for some ample Cartier divisor H on X and for all $c > 0$ there exist numbers $j \in \mathbb{N}$ and $i \in \mathbb{N}$ such that $i > cj$ and*

$$H^0(X, (S^i\mathcal{F})/\mathrm{Tor} \otimes \mathcal{O}_X(jH)) \neq 0.$$

Remark 3.8. — For locally free sheaves, the Definitions 3.3, 3.5, 3.7 obviously coincide. Even for reflexive sheaves, Definition 3.7 is more restrictive than Definition 3.5: in general $(S^i\mathcal{F})/\mathrm{Tor}$ is not reflexive and has less global sections than its bidual, so we might have

$$H^0(X, (S^i\mathcal{F})/\mathrm{Tor} \otimes \mathcal{O}_X(jH)) = 0,$$

although $\mathcal{F}$ is pseudoeffective in the sense of Definition 3.5. We thank C. Gachet for the following example:

Example 3.9. — Let

$$\mathbb{C}^2 \longrightarrow X = \{(x, y, z) \in \mathbb{C}^3 \mid xy - z^2 = 0\}, \quad (u, v) \longmapsto (u^2, v^2, uv)$$

be the double cover of the A_1-singularity, we identify the polynomial ring of X to its image $\mathbb{C}[u^2, v^2, uv]$ in $\mathbb{C}[u, v]$. The invariant elements under the involution

$$j : \mathbb{C}[u, v] \longrightarrow \mathbb{C}[u, v], \quad f(p) \longmapsto -f(-p)$$

are exactly the odd polynomials, i.e. the polynomials that can be written as $uf + vg$ with $f, g \in \mathbb{C}[u^2, v^2, uv]$. Denote this set by $\mathbb{C}[u, v]^{\mathbb{Z}_2}$, then $\mathbb{C}[u, v]^{\mathbb{Z}_2}$ has a natural structure of $\mathbb{C}[u^2, v^2, uv]$-module that is reflexive, but not (locally) free. The tensor power $(\mathbb{C}[u, v]^{\mathbb{Z}_2})^{\otimes 2}$ is generated by u^2, v^2, uv, so it naturally embeds into $\mathbb{C}[u^2, v^2, uv]$. Remembering that this ring is actually the function ring of the A_1-singularity, we see that $(\mathbb{C}[u, v]^{\mathbb{Z}_2})^{\otimes 2}$ is isomorphic to the maximal ideal defining the origin.

Let now $q : A \to X$ be the quotient of an abelian surface under the involution $z \mapsto -z$, so X is the singular Kummer surface. Since $S^{[2]}\Omega_X$ is

globally generated, the sheaf of reflexive differentials $\Omega_X^{[1]}$ is pseudoeffective. Let us see that it is not strongly pseudoeffective: we have $\Omega_X^{[1]} \simeq \mathcal{F} \oplus \mathcal{F}$, where $\mathcal{F}$ is the sheaf of $\mathbb{Z}_2$-invariants for the natural action on $\Omega_A \simeq \mathcal{O}_A \mathrm{d}z_1 \oplus \mathcal{O}_A \mathrm{d}z_2$. It is immediate to see that, near the fixed points, the $\mathbb{Z}_2$-action on $\mathcal{O}_A \mathrm{d}z_l$ identifies to the action j in the paragraph above. Thus, using the local computation, we see that

$$\mathcal{F}^{\otimes i} \simeq \begin{cases} \mathcal{I}_{X_{\mathrm{sing}}}^{\frac{i}{2}} & \text{if } i \text{ even,} \\ \mathcal{I}_{X_{\mathrm{sing}}}^{\frac{i-1}{2}} \otimes \mathcal{F} & \text{if } i \text{ odd.} \end{cases}$$

Combined with Example 3.12 this shows that $\mathcal{F}$ is not strongly pseudoeffective.

In general it is not clear if one can check strong pseudoeffectivity by looking at a tautological class on (a modification of) the projectivisation. However there is a natural construction in a special case:

Setup 3.10. — Let $\mathcal{F}$ be a torsion-free sheaf on a normal projective variety X such that

$$\mathcal{F} \simeq \mathcal{I}_Z \otimes \mathcal{E}$$

where $\mathcal{I}_Z$ is an ideal sheaf and $\mathcal{E}$ is a locally free sheaf.

Let $\mu : \widehat{X} \to X$ be the blow-up of the ideal sheaf $\mathcal{I}_Z$, then $\widehat{X}$ is a (not necessarily normal) variety [27, II, Prop. 7.16]. We denote by

$$\mathcal{O}_{\hat{X}}(1) := \mu^{-1}(\mathcal{I}_Z)\mathcal{O}_{\hat{X}}$$

the tautological sheaf on $\widehat{X}$. Recall that by the definition of the blow-up [1, II, §3] one has

$$\mu_*(\mathcal{O}_{\hat{X}}(i)) = \mathcal{I}_Z^i \qquad \forall\, i \geqslant 0. \tag{3.1}$$

Note also that if Z is locally generated by a regular sequence, one has $S^i\mathcal{I}_Z \simeq \mathcal{I}_Z^i$ for all $i \geqslant 0$ (e.g. [6, Prop. 2.2.8]). In particular the blowup $\mathrm{Bl}_{\mathcal{I}_Z}(X)$ coincides with the projectivisation $\mathbb{P}(\mathcal{I}_Z)$.

Lemma 3.11. — *In the situation of Setup 3.10, the torsion-free sheaf $\mathcal{F}$ is strongly pseudoeffective if and only if the locally free sheaf $\mathcal{O}_{\hat{X}}(1) \otimes \mu^*\mathcal{E}$ on the variety $\widehat{X}$ is pseudoeffective.*

Proof. — Let H be an ample Cartier divisor on X. By the projection formula and (3.1) one has

$$\mu_*(\mu^*(\mathcal{O}_X(jH) \otimes S^i\mathcal{E}) \otimes \mathcal{O}_{\hat{X}}(i)) \simeq \mathcal{O}_X(jH) \otimes S^i\mathcal{E} \otimes \mathcal{I}_Z^i$$

for all $i \geqslant 0$. Moreover we know by [33] that

$$(S^i\mathcal{I}_Z)/\operatorname{Tor} \simeq \mathcal{I}_Z^i.$$

Thus we obtain that

$$H^0(\widehat{X}, S^i(\mathcal{O}_{\hat{X}}(1) \otimes \mu^*\mathcal{E}) \otimes \mathcal{O}_{\hat{X}}(j\mu^*H)) \simeq H^0(X, (S^i\mathcal{F})/\operatorname{Tor} \otimes \mathcal{O}_X(jH))$$

for all $i \geqslant 0$. Now we apply [21, Lem. 2.7], cf. also the Remark after Definition 3.3. □

Example 3.12. — Let X be a normal projective variety, and let $\mathcal{I}_Z \subset \mathcal{O}_X$ an ideal sheaf. Then $\mathcal{I}_Z$ is not strongly pseudoeffective by Lemma 3.11.

On the other hand let $Z \subset \mathbb{P}^2$ be a point, then $\mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^2}(1)$ is strongly pseudoeffective. Indeed the locally free sheaf $\mathcal{O}_{\hat{X}}(1) \otimes \mu^*\mathcal{O}_{\mathbb{P}^2}(1)$ on the blowup $\widehat{X} \simeq \mathbb{F}_1$ is pseudoeffective, since it is isomorphic to $\mathcal{O}_{\mathbb{F}_1}(F)$ where F is the strict transform of a line through Z.

Corollary 3.13. — *In the situation of Setup 3.10, suppose that the ideal sheaf $\mathcal{I}_Z$ is locally generated by a regular sequence (e.g. if Z is a locally complete intersection scheme). Then the following statements are equivalent:*

(1) *The sheaf $\mathcal{F}$ is strongly pseudoeffective;*
(2) *The locally free sheaf $\mathcal{O}_{\hat{X}}(1) \otimes \mu^*\mathcal{E}$ on the blow-up*

$$\mu : \widehat{X} = \operatorname{Bl}_{\mathcal{I}_Z} X \longrightarrow X,$$

is pseudoeffective;
(3) *The tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ on the projectivisation $\mathbb{P}(\mathcal{F})$ is pseudoeffective.*

Proof. — The equivalence between (1) and (2) is shown in Lemma 3.11.

Denote by $\widehat{\pi} : \mathbb{P}(\mu^*\mathcal{E}) \to \widehat{X}$ the projectivisation, and by $\mathcal{O}_{\mathbb{P}(\mu^*\mathcal{E})}(1)$ its tautological sheaf. Since $\mathcal{O}_{\hat{X}}(1)$ is μ-ample and $\mathcal{O}_{\mathbb{P}(\mu^*\mathcal{E})}(1)$ is $\widehat{\pi}$-ample and $\mu \circ \widehat{\pi}$-nef, we know that $\widehat{\pi}^*(\mathcal{O}_{\hat{X}}(1)) \otimes \mathcal{O}_{\mathbb{P}(\mu^*\mathcal{E})}(1)$ is $\mu \circ \widehat{\pi}$-ample. Since $\mathcal{I}_Z$ is locally generated by a regular sequence, we have $S^i\mathcal{I}_Z \simeq \mathcal{I}_Z^i$ for all $i \geqslant 0$ (e.g. [6, Prop. 2.2.8]). Thus for all $i \geqslant 0$ we have

$$\begin{aligned}(\mu \circ \widehat{\pi})_*((\widehat{\pi})^*(\mathcal{O}_{\hat{X}}(i)) \otimes \mathcal{O}_{\mathbb{P}(\mu^*\mathcal{E})}(i)) \\ \simeq \mu_*(\mathcal{O}_{\hat{X}}(i) \otimes \mu^*S^i\mathcal{E}) \simeq \mathcal{I}_Z^i \otimes S^i\mathcal{E} \simeq S^i(\mathcal{I}_Z \otimes \mathcal{E}).\end{aligned}$$

Therefore we have an isomorphism $\psi : \mathbb{P}(\mu^*\mathcal{E}) \to \mathbb{P}(\mathcal{F})$ such that $\psi^*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) = \widehat{\pi}^*(\mathcal{O}_{\hat{X}}(1)) \otimes \mathcal{O}_{\mathbb{P}(\mu^*\mathcal{E})}(1)$. Thus (2) and (3) are equivalent. □

3.3. Generalised Kodaira–Iitaka dimensions and functoriality

In this subsection we introduce a Kodaira–Iitaka dimension for reflexive sheaves and establish functoriality.

Definition 3.14. — *Let X be a normal projective variety, and let $\mathcal{F}$ be a reflexive sheaf on X. Let $\pi : P \to X$ be a desingularization of the normalization of the unique component $\mathbb{P}'(\mathcal{F})$ of $\mathbb{P}(\mathcal{F})$ dominating X such that the preimage of the singular locus of X and of the singular locus of the sheaf $\mathcal{F}$ is a divisor in P. Let ζ be a tautological class on P,* [29, Def. 2.2]. *Then we define*

$$\kappa(X, \mathcal{F}) = \kappa(P, \zeta).$$

By construction $\kappa(P, \zeta) \geqslant 0$ if and only if $H^0(X, S^{[m]}(\mathcal{F})) \neq 0$ for some $m \in \mathbb{N}$.

Remark. — If $\pi : P \to X$ denotes the projection, then

$$H^0(P, \mathcal{O}_P(m\zeta)) = H^0(X, S^{[m]}(\mathcal{F}))$$

for all positive numbers m, and therefore the definition is independent on the choices made.

We will use Definition 3.14 in Section 4 to introduce a generalised Kodaira dimenion of X (Definition 4.5).

The next result generalizes Lemma 3.4 for finite morphisms.

Lemma 3.15. — *Let $f : \widetilde{X} \to X$ be a finite morphism of normal projective varieties. Let $\mathcal{F}$ be a reflexive sheaf on X. Then the following holds:*

(1) *The reflexive pullback $f^{[*]}(\mathcal{F})$ is pseudoeffective if and only if $\mathcal{F}$ is pseudoeffective.*
(2) *One has $\kappa(\widetilde{X}, f^{[*]}(\mathcal{F})) = \kappa(X, \mathcal{F})$.*

Proof. — Let $\pi : P \to X$ be the projective manifold from Definition 3.14, and ζ a tautological class on P. By the construction of ζ (see [29, Def. 2.2]), we have

$$\pi_*(\mathcal{O}_P(m\zeta)) \simeq S^{[m]}(\mathcal{F}) \tag{3.2}$$

for all positive integers m. We introduce the fibre product

$$\widetilde{P} := P \times_X \widetilde{X}.$$

Let $\sigma : \widehat{P} \to \widetilde{P}$ be a desingularization, which is an isomorphism outside the singular locus of $\widetilde{P}$ and set $\widehat{\zeta} := \sigma^* p_1^*(\zeta)$, where $p_1 : \widetilde{P} \to P$ denotes the projection. Let $\widetilde{\pi} : \widetilde{P} \to \widetilde{X}$ be the projection and $\widehat{\pi} := \widetilde{\pi} \circ \sigma$.

Claim. — There exists a $\widehat{\pi}$-exceptional divisor $\widehat{D}$ on $\widehat{P}$, such that

$$\widehat{\pi}_*(\mathcal{O}_{\widehat{P}}(m(\widehat{\zeta} + \widehat{D}))) \simeq S^{[m]} f^{[*]}(\mathcal{F})$$

for all $m \in \mathbb{N}$.

Proof of the claim. — By [35, III.5.10.3] there exists $\widehat{\pi}$-exceptional divisor $\widehat{D}$ on $\widehat{P}$ such that

$$\widehat{\pi}_*(\mathcal{O}_{\widehat{P}}(m(\widehat{\zeta}+\widehat{D})))$$

is reflexive for all $m \in \mathbb{N}$.

Let $X_0 \subset X$ be the locus where X is smooth and $\mathcal{F}$ is locally free. Since X is normal and $\mathcal{F}$ is reflexive, the complement of X_0 has codimension at least two. Set now $\widetilde{X}_0 = f^{-1}(X_0)$. Since X_0 is smooth, the restriction $f|_{\widetilde{X}_0}$ is flat. Thus by flat base change [27, III, Prop. 9.3] we have an isomorphism

$$\widetilde{\pi}_*(p_1^*(\mathcal{O}_P(m\zeta)) \simeq f^* S^{[m]}(\mathcal{F})$$

over $\widetilde{X}_0$. Since $\mathcal{F}$ is locally free on X_0, we have

$$f^* S^{[m]}(\mathcal{F}) \simeq f^{[*]} S^{[m]}(\mathcal{F})$$

over $\widetilde{X}_0$. Thus $\widehat{\pi}_*(\mathcal{O}_{\widehat{P}}(m(\widehat{\zeta}+\widehat{D})))$ and $f^{[*]}S^{[m]}(\mathcal{F})$ are isomorphic over $\widetilde{X}_0$. Since they are both reflexive, they are isomorphic : indeed the complement of $\widetilde{X}_0$ has codimension at least two, since f is finite. For the same reason we have

$$f^{[*]} S^{[m]}(\mathcal{F}) \simeq S^{[m]} f^{[*]}(\mathcal{F}),$$

which shows the claim. □

Proof of (1). — If $\mathcal{F}$ is pseudoeffective, fix an ample Cartier divisor H on X. Since f is finite, a section of $S^{[i]}\mathcal{F} \otimes \mathcal{O}_X(jH)$ pulls back to a section $S^{[i]}(f^{[*]}(\mathcal{F})) \otimes \mathcal{O}_X(jf^*H)$. Thus $f^{[*]}(\mathcal{F})$ is pseudoeffective, by Definition 3.5.

Assume now that $f^{[*]}(\mathcal{F})$ is pseudoeffective, so the divisor class $\widehat{\zeta}+\widehat{D}$ is pseudoeffective by [29, Lem. 2.3]. Let now $\mu : \widehat{P} \to P'$ and $p_1' : P' \to P$ be the Stein factorisation of the generically finite morphism $p_1 \circ \sigma$, i.e. μ is birational onto the normal variety P' and p_1' is finite. Then $\widehat{\zeta} = (p_1 \circ \sigma)^* \zeta = (p_1' \circ \mu)^* \zeta$, so

$$\mu_*(\widehat{\zeta}+\widehat{D}) = (p_1')^*\zeta + \mu_*(\widehat{D})$$

is a pseudoeffective Weil divisor class (cf. [35, II, Defn.5.5]). Setting $D = (p_1')_* \mu_*(\widehat{D})$, we have an inclusion of Weil divisors $\mu_*(\widehat{D}) \subset (p_1')^* D$. Thus we have an inclusion of Weil divisor classes

$$(p_1')^*\zeta + \mu_*(\widehat{D}) \subset (p_1')^*(\zeta + D),$$

which shows that $(p_1')^*(\zeta + D)$ is pseudoeffective. Since $\zeta + D$ is Cartier, this shows that $\zeta + D$ is pseudoeffective. Since $\widehat{D}$ is $\widehat{\pi}$-exceptional, the effective divisor D is π-exceptional. By (3.2) we thus have

$$\pi_*(\mathcal{O}_P(m(\zeta + D)))^{**} = \pi_*(\mathcal{O}_P(m\zeta)^{**} \simeq S^{[m]}(\mathcal{F})$$

for all $m \in \mathbb{N}$. This shows that

$$S^{[m]}(\mathcal{F}) \simeq \pi_*(\mathcal{O}_P(m\zeta) \hookrightarrow \pi_*(\mathcal{O}_P(m(\zeta + D))) \hookrightarrow \pi_*(\mathcal{O}_P(m(\zeta + D)))^{**} \simeq S^{[m]}(\mathcal{F})$$

is a chain of isomorphisms for all $m \in \mathbb{N}$. Hence the reflexive sheaf $\mathcal{F}$ is pseudoeffective by [29, Lem. 2.3].

Proof of (2). — As for the first statement, the inequality $\kappa(\widetilde{X}, f^{[*]}(\mathcal{F})) \geqslant \kappa(X, \mathcal{F})$ is immediate. Let us show the other inequality: by the claim we know that $\widehat{\zeta} + \widehat{D}$ is a tautological class on $\widehat{P}$. Thus by assumption, one has

$$\kappa(\widehat{P}, \widehat{\zeta} + \widehat{D}) = \kappa((\widetilde{X}, f^{[*]}(\mathcal{F}))) \geqslant 0.$$

Hence $\kappa(P', (p_1')^*(\zeta) + \mu_*(\widehat{D})) = \kappa((\widetilde{X}, f^{[*]}(\mathcal{F}))) \geqslant 0$, and,

$$\kappa(P', (p_1')^*(\zeta + D)) \geqslant \kappa(P', (p_1')^*(\zeta) + \mu_*(\widehat{D})),$$

where we use again the inclusion $(p_1')^*\zeta + \mu_*(\widehat{D}) \subset (p_1')^*(\zeta + D)$. Since

$$\kappa(P', (p_1')^*(\zeta + D)) = \kappa(P, \zeta + D)),$$

by [38, Thm. 5.13], the statement follows. □

In Section 4 this will be applied to sheaves of reflexive differentials, cf. Corollary 4.6.

3.4. Pseudoeffective sheaves on fibered surfaces

The results of this section will be relevant to the study of elliptic surfaces. Let us recall the Zariski decomposition on surfaces [5, Thm.], [32, Thm. 2.3.19]:

- Let D be an effective $\mathbb{Q}$-divisor on a smooth surface. Then there exist uniquely determined effective $\mathbb{Q}$-divisors P and N with
$$D = P + N$$
such that P is nef, the divisor $N = \sum a_j N_j$ is zero or has negative definite intersection matrix and $P \cdot N_j = 0$ for all j.
- The same statement holds if D is pseudoeffective, except that in this case P is not necessarily effective.

The following lemma is well-known to experts, we give the details in order to prepare the proof of its singular version in Lemma 3.17.

Lemma 3.16. — *Let X be a smooth projective surface, and let $f : X \to B$ be a fibration over a smooth curve B. Let L be a pseudoeffective line bundle on X such that $L_F \simeq \mathcal{O}_F$ for the general fibre F of f. Let D be a Cartier divisor such that $L \simeq \mathcal{O}_X(D)$, and let*

$$D = P + N$$

its Zariski decomposition. Then the following holds:

(1) *Up to taking multiples, one has $\mathcal{O}_X(P) \simeq f^*M$ with M a nef line bundle.*
(2) *If $P \not\equiv 0$, one has $\kappa(L) \geqslant 1$.*
(3) *If $P \equiv 0$, there exists $m \in \mathbb{N}$ and a numerically trivial line bundle M on B such that $\kappa(X, L^{\otimes m} \otimes f^*M^*) = 0$.*

Proof. — Note that the statements are invariant under taking multiples.

Up to replacing L by some multiple, we can assume that P and N are Cartier divisors. Since N is effective and

$$\mathcal{O}_F \simeq \mathcal{O}_F(D) \simeq \mathcal{O}_F(P) \otimes \mathcal{O}_F(N)$$

we see that N has support in the fibres of f and $\mathcal{O}_F(N) \simeq \mathcal{O}_F(P) \simeq \mathcal{O}_F$. Thus the direct image sheaf $f_*(\mathcal{O}_X(P))$ is locally free of rank one, and we have

$$\mathcal{O}_X(P) = f^*(f_*\mathcal{O}_X(P)) \otimes \mathcal{O}_X(E)$$

where E is an effective divisor, supported in fibers of f. Since $P^2 \geqslant 0$ and $E^2 \leqslant 0$ [4, III, Lem. 8.2], it follows $E^2 = 0$. Hence by [4, III, Lem. 8.2] there exists a number k such that $kE = f^*(H)$ with some effective divisor H. Thus, again up to replacing L by some multiple, we have $\mathcal{O}_X(P) = f^*M$ for some line bundle M on B.

If $M \not\equiv 0$, it is ample on B, so

$$1 = \kappa(M) = \kappa(P) = \kappa(L).$$

If $P \equiv 0$, then $M \equiv 0$, so $\kappa(L \otimes f^*M^*) = \kappa(N) = 0$. □

Lemma 3.17. — *Let $\widehat{X}$ be an irreducible reduced projective surface, and let $\widehat{f} : \widehat{X} \to B$ be a fibration over a smooth curve B such that the general fibre F is smooth. Let $\widehat{L}$ be a pseudoeffective line bundle on $\widehat{X}$ such that $\widehat{L}|_F \simeq \mathcal{O}_F$.*

(1) *There exists $m \in \mathbb{N}$ and a numerically trivial line bundle M on B such that $h^0(\widehat{X}, \widehat{L}^{\otimes m} \otimes \widehat{f}^*M) > 0$.*
(2) *If $h^0(\widehat{X}, \widehat{L}^{\otimes m} \otimes \widehat{f}^*M) > 1$ for some numerically trivial line bundle M, then $h^0(\widehat{X}, \widehat{L}^{\otimes k}) > 1$ for some $k \in \mathbb{N}$.*

Remark 3.18. — We will frequently apply the lemma in the case where B is a rational curve. In this case one obtains $h^0(\widehat{X}, \widehat{L}^{\otimes m}) > 0$ for some $m \in \mathbb{N}$.

Since the dimension of the space of global sections is not necessarily invariant under normalisation, the statement requires some work:

Proof. — Note that the statement is invariant under taking multiples.

Let $\mu : X \to \widehat{X}$ be the composition of normalisation and desingularisation, set $f := \widehat{f} \circ \mu$. Then $\mu^* \widehat{L} =: \mathcal{O}_X(D)$ is pseudoeffective, and we denote by

$$D = P + N$$

the Zariski decomposition. By Lemma 3.16 (1) we have, up to taking multiples, that $\mathcal{O}_X(P) \simeq f^* M$ for some line bundle on M. Thus we see that $\widehat{L} \simeq \widehat{f}^* M \otimes \widehat{N}$, where $\widehat{N}$ is a Cartier divisor on $\widehat{X}$ such that $N \equiv \mu^* \widehat{N}$. By Lemma 3.16 (2) and (3) it is sufficient to show that we can choose $\widehat{N}$ to be effective.

Since

$$\widehat{N} \simeq \widehat{L} \otimes (\widehat{f})^* M^*,$$

we have $\mathcal{O}_F(\widehat{N}) \simeq \mathcal{O}_F$. Thus $(\widehat{f})_* \mathcal{O}_{\hat{X}}(\widehat{N})$ is locally free of rank one, hence for some $m \gg 0$, we have

$$H^0(\widehat{X}, \mathcal{O}_{\hat{X}}(mF) \otimes \mathcal{O}_{\hat{X}}(\widehat{N})) \simeq H^0(B, \mathcal{O}_B(m) \otimes (\widehat{f})_* \mathcal{O}_{\hat{X}}(\widehat{N})) \neq 0.$$

Thus we can fix an effective Cartier divisor E on $\widehat{X}$ such that $\mathcal{O}_{\hat{X}}(E) \simeq \mathcal{O}_X(mF) \otimes \mathcal{O}_{\hat{X}}(\widehat{N})$. We can decompose

$$E = f^* E_B + R$$

where E_B is an effective $\mathbb{Q}$-divisor on B and R is an effective divisor such that for every connected component $R' \subset R$ we have a strict set-theoretical inclusion $R' \subsetneq \widehat{f}^{-1}(\widehat{f}(R'))$. Up to taking multiples and replacing E_B by a linearly equivalent divisor, we can also suppose that $\mathrm{supp}(f^* E_B) \subset X_{nons}$. Now observe that

$$\mu^* E = \mu^* f^* E_B + \mu^* R$$

is a Zariski decomposition. Since

$$\mu^* E \equiv mF + N.$$

is also a Zariski decomposition and the negative part of the Zariski decomposition is unique in the numerical equivalence class, we finally obtain $N = \mu^* R$. □

Corollary 3.19. — *Let X be a smooth projective surface, and let $f : X \to B$ a fibration over a smooth* rational *curve B. Let $Z \subset X$ be a local complete intersection of codimension 2. Let L be a line bundle on X such that $L_F \simeq \mathcal{O}_F$ for the general fibre F of f. Then $\mathcal{I}_Z \otimes L$ is strongly pseudoeffective if and only if $\kappa(X, \mathcal{I}_Z \otimes L) \geqslant 0$, i.e., if there exists a positive integer m such that*

$$H^0(X, \mathcal{I}_Z^m \otimes L^m) \neq 0.$$

Proof. — One direction being obvious, so assume that $\mathcal{I}_Z \otimes L$ is pseudoeffective. Let

$$\mu : \widehat{X} \longrightarrow X$$

be the blow-up of X along Z and denote by E the exceptional divisor. Since Z does not surject onto B, the general fibre of $f \circ \mu$ is smooth.

By Corollary 3.13 the line bundle $\mathcal{O}_{\widehat{X}}(1) \otimes \mu^*(L) \simeq \mathcal{O}_{\widehat{X}}(-E) \otimes \mu^*(L)$ is pseudoeffective. By Remark 3.18 we see that there exists some $m \in \mathbb{N}$ such that

$$H^0(\widehat{X}, \mathcal{O}_{\widehat{X}}(-mE) \otimes \mu^*(L^{\otimes m})) \neq 0.$$

Since Z is a local complete intersection, we have $\mu_*(\mathcal{O}_{\widehat{X}}(-mE)) = \mathcal{I}_Z^m$. Thus we conclude by the projection formula. □

4. Pseudoeffective cotangent sheaves and the Nonvanishing Conjecture

In this section we gather some basic facts on the behaviour of pseudoeffective cotangent sheaves under birational maps and finite covers. We also establish a relation with the MRC fibration of the variety.

Proposition 4.1. — *Let $\mu : \widehat{X} \to X$ be a birational morphism of normal projective varieties.*

(1) *If $\Omega^{[q]}_{\widehat{X}}$ is pseudoeffective, so is $\Omega^{[q]}_X$.*
(2) *Suppose that X is smooth. Then the converse also holds.*

Proof.

(1). — We choose ample divisors $\widehat{H}$ on $\widehat{X}$ and H on X such that

$$\widehat{H} + E = \mu^*(H)$$

with E an effective divisor supported on the exceptional locus of μ. By assumption, for all $c > 0$, there are numbers i and j with $i > cj$ such that

$$H^0(\widehat{X}, S^{[i]}\Omega^{[q]}_{\widehat{X}} \otimes \mathcal{O}_{\widehat{X}}(j\widehat{H})) \neq 0.$$

In particular,

$$0 \neq H^0(\widehat{X}, S^{[i]}\Omega^{[q]}_{\hat{X}} \otimes \mathcal{O}_{\hat{X}}(j\mu^* H)) = H^0(X, \mu_*(S^{[i]}\Omega^{[q]}_{\hat{X}}) \otimes \mathcal{O}_X(jH)).$$

Since $\mu_*(S^{[i]}\Omega^{[q]}_{\hat{X}}) \subset S^{[i]}\Omega^{[q]}_X$, we conclude.

(2). — Suppose that X is smooth. By (1), we may assume $\widehat{X}$ to be smooth as well. Then all the involved sheaves are locally free, in particular

$$S^i\mu^*(\Omega^q_X) = \mu^*(S^i\Omega^q_X) \subset S^i\Omega^q_{\hat{X}},$$

and the claim follows. □

Example 4.2. — Assertion 4.1 (2) fails in general if X is singular, even if X has only canonical singularities. In fact, the paper [26] constructs a normal projective surface X with the following properties.

- X has only ADE singularities ;
- the minimal desingularization $\widehat{X}$ is rationally connected ;
- $H^0(X, S^{[2]}\Omega^1_X) \neq 0$.

Thus $\Omega^1_{\hat{X}}$ is not pseudoeffective in contrast to $\Omega^{[1]}_X$.

Another example is a K3 surface of Kummer type, see Example 2.3.

Corollary 4.3. — *Let X be a normal projective variety with klt singularities, and let $X \dashrightarrow X'$ be a composition of divisorial contractions and flips. If $\Omega^{[q]}_X$ is pseudoeffective, so is $\Omega^{[q]}_{X'}$.*

Proof. — By Proposition 4.1 (1) it suffices to treat the case of a flip $\mu : X \dashrightarrow X'$. Since a flip is an isomorphism in codimension two, one has

$$H^0(X, S^{[i]}\Omega^{[q]}_X \otimes \mathcal{O}_X(jH)) \simeq H^0(X', S^{[i]}\Omega^{[q]}_{X'} \otimes \mathcal{O}_{X'}(j\mu_* H))$$

for all $i, j \in \mathbb{N}$. Thus the condition in Definition 3.5 holds for a big $\mathbb{Q}$-Cartier divisor, which is sufficient (see [28, Lem. 2.2]). □

Proposition 4.4. — *Let $f : \widetilde{X} \to X$ be a finite surjective morphism of normal projective varieties. If $\Omega^{[q]}_X$ is pseudoeffective, so is $\Omega^{[q]}_{\widetilde{X}}$. If f is quasi-étale, the converse also holds.*

Proof. — Over the smooth locus of X we have a injective morphism

$$f^*(S^i\Omega^q_{X_{\mathrm{nons}}}) \longrightarrow S^{[i]}\Omega^{[q]}_{\widetilde{X}}.$$

Since the complement of $f^{-1}(X_{\mathrm{nons}})$ has codimension at least two, the morphism extends to

$$f^{[*]}(S^{[i]}\Omega^{[q]}_X) \longrightarrow S^{[i]}(\Omega^{[q]}_{\widetilde{X}}).$$

which gives the first claim.

Assume now that f is quasi-étale and that $\Omega^{[q]}_{\widetilde{X}}$ is pseudoeffective. Then $f^{[*]}(\Omega^{[q]}_X) \simeq \Omega^{[q]}_{\widetilde{X}}$ is pseudoeffective. Now we conclude by Lemma 3.15. □

At this point we introduce generalised Kodaira dimension:

Definition 4.5. — *Let X be a normal projective variety with klt singularities and $1 \leqslant q \leqslant n$. Then we define*

$$\kappa_q(X) = \kappa(X, \Omega^{[q]}_X).$$

In case $q = \dim X$, we have of course $\kappa_q(X) = \kappa(X)$.

As a corollary to Lemma 3.15 we obtain a generalisation of [3, Prop. 2.2]:

Proposition 4.6. — *Let $\widetilde{X} \to X$ be a quasi-étale morphism of projective varieties with klt singularities. Then*

$$\kappa_q(\widetilde{X}) = \kappa_q(X)$$

for all $1 \leqslant q \leqslant \dim X$.

Proof. — This follows from Lemma 3.15, since for a quasi-étale morphism

$$f^{[*]}(\Omega^{[q]}_X) = \Omega^{[q]}_{\widetilde{X}}.$$ □

Remark 4.7. — Let $\mu : \widehat{X} \to X$ be a birational morphism of normal projective varieties with klt singularities. Then $\kappa_q(\widehat{X}) \leqslant \kappa_q(X)$ with equality if X is smooth. The same inequality holds if $\mu : \widehat{X} \dashrightarrow X$ is a composition of divisorial contractions and flips.

Although Conjecture 1.1 can be formulated for any p, we are mainly interested in the case $p = 1$. We next confirm the conjecture for $p = 1$ in case $K_X \equiv 0$.

Proposition 4.8. — *Let X be a normal projective variety with klt singularities such that $K_X \equiv 0$. Assume that X is smooth in codimension two, e.g., X has terminal singularities. Then the following are equivalent:*

(1) *$\Omega^{[1]}_X$ is pseudoeffective ;*
(2) *we have $\widetilde{q}(X) > 0$, i.e., there exists a quasi-étale cover $\widetilde{X} \to X$ such that $H^0(\widetilde{X}, \Omega^{[1]}_{\widetilde{X}}) \neq 0$;*
(3) *we have $H^0(X, S^{[m]}\Omega^1_X)) \neq 0$ for some positive integer m, i.e., $\kappa_1(X) \geqslant 0$.*

Proof. — By [29, Thm. 1.6] we know that (1) implies (2). By Proposition 4.6 we know that (2) implies (3) which obviously implies (1). □

We will now discuss the relation between the pseudoeffectivity of Ω^q_X and the MRC fibration. First, the rational connectedness criterion given in [17] can be stated as follows.

THEOREM 4.9. — *Let X be a projective manifold of dimension n. Then X is rationally connected if and only if for all $1 \leqslant q \leqslant n$ the vector bundle Ω^q_X is not pseudoeffective.*

Proof. — If X is rationally connected, it is dominated by very free rational curves [30, IV.3.9]. It is then straightforward to verify the vanishing condition in Definition 3.5.

Assume that Ω^q_X is not pseudoeffective for all $1 \leqslant q \leqslant n$. Let $\mathcal{F} \subset \Omega^q_X$ be an invertible subsheaf, then $\mathcal{F}$ is not pseudoeffective (see Example 3.6). Hence by [17, Thm. 1.1], X is rationally connected. □

Theorem 4.9 can be generalised as follows.

THEOREM 4.10. — *Let X be a projective variety of dimension n. Fix some $r \in \{1, \ldots, n\}$ and assume that $\Omega^{[q]}_X$ is not pseudoeffective for all $r \leqslant q \leqslant n$.*

Then X is uniruled, and the base Z of the MRC fibration satifies $\dim Z \leqslant r - 1$.

Proof. — By Proposition 4.1 we can assume without loss of generality that X is smooth. Since $K_X = \Omega^n_X$ is not pseudoeffective, the manifold X is uniruled by [9]. Hence we consider the MRC fibration

$$f : X \dashrightarrow Z.$$

Up to replacing Z by a resolution and X by a blow-up, we may assume, by Proposition 4.1, that Z is smooth and f is a morphism.

Arguing by contradiction we suppose that $d := \dim Z \geqslant r$. By [24], the variety Z is not uniruled, hence K_Z is pseudoeffective. Choose $\mathcal{H}_Z$ ample on Z and set $\mathcal{H}_X = f^*(\mathcal{H}_Z)$. Since K_Z is pseudoeffective, for all $c > 0$ there exist integers i and j with $i > cj$ such that

$$H^0(Z, \mathcal{O}_Z(iK_Z) \otimes \mathcal{O}_Z(j\mathcal{H}_Z)) \neq 0.$$

Since $0 \neq f^*(\Omega^d_Z) \subset \Omega^d_X$, it follows that

$$H^0(X, S^i\Omega^d_X \otimes \mathcal{O}_X(j\mathcal{H}_X)) \neq 0.$$

Thus Ω^d_X is pseudoeffective and $d \geqslant r$, a contradiction to our assumption. □

If X is smooth, the converse to Theorem 4.10 is also true:

PROPOSITION 4.11. — *Let X be a uniruled projective manifold of dimension n, and let $f : X \dashrightarrow Z$ be the MRC fibration. If $d = \dim Z$, then Ω^q_X is not pseudoeffective for all $d + 1 \leqslant q \leqslant n$.*

Proof. — Let F be a general fibre of f. Then F is rationally connected of dimension $\dim F = n - d$. Let $C \subset F$ be a general very free rational curve [30, IV.3.9], so $T_{F|C}$ is ample. Then

$$T_X|_C \simeq \mathcal{O}_C^{\oplus d} \oplus T_F|_C.$$

Thus for every $q \geqslant d+1$, the exterior power $\bigwedge^q T_X|_C$ is ample. Hence Ω_X^q is not pseudoeffective. □

For smooth varieties, the MRC-fibration should allow to reduce Conjecture 1.1 to non-uniruled varieties:

Conjecture 4.12. — *Let X be a uniruled projective manifold, and let $f : X \dashrightarrow Z$ be the MRC fibration to the projective manifold Z. Let $1 \leqslant q \leqslant n$. Then Ω_X^q is pseudoeffective if and only if Ω_Z^q is pseudoeffective.*

Note that, by Proposition 4.1, we may assume f holomorphic. Then one direction is clear: if Ω_Z^q is pseudoeffective, then so is $f^*(\Omega_Z^q)$ by Lemma 3.4. Hence Ω_X^q is pseudoeffective, see Example 3.6. Vice versa assume that Ω_X^q is pseudoeffective. Applying [7, Cor. 1.3.2], f factors into a sequence of divisorial contractions and flips, ending with a Mori fibre space $f' : X' \to Z$ of relative Picard number one. By Corollary 4.3, we may therefore assume that f is a Mori contraction, but now X may have terminal singularities instead of being smooth.

Proposition 4.13. — *Conjecture 4.12 is true in dimension three.*

Proof. — As just noticed it suffices to treat Mori contractions $f : X \to Z$ where X has terminal singularities. Since Z is not uniruled, we may assume that $\dim Z = 2$; otherwise Z is a curve of genus $g \geqslant 1$ and there is nothing to prove. Further, since K_Z is pseudoeffective, only the case $q = 1$ needs to be treated. Now f is a generically a conic bundle [2, 4.1]. More precisely, the singular locus of X being finite, there is a finite set $B = f(\mathrm{Sing}(X))$ in Z such that, setting $Z_0 = Z \setminus B$ and $X_0 = f^{-1}(Z_0)$, the map $f_0 : X_0 \to Z_0$ is a conic bundle with only finitely many non-reduced fibers. Furthermore, $f^{-1}(B)$ is one-dimensional.

Since $-K_X$ is relatively ample and relatively globally generated on X_0, we can choose a very ample Cartier divisor H on Z such that $-K_{X/Z} + f^*H =: A$ is ample and satisfies $H^0(X, \mathcal{O}_X(A)) \neq 0$. In particular there is an injection

$$\omega_{X/Z} \hookrightarrow \mathcal{O}_X(f^*H). \tag{4.1}$$

Claim. — For every $c > 1$ there exist positive integers k, j such that $k \geqslant cj$ such that

$$H^0(X_0, f^*S^k\Omega_Z \otimes \mathcal{O}_X(j(A + f^*H))) \neq 0.$$

Since $f^{-1}(B)$ has codimension at least two, this shows that $f^*\Omega_Z$ is pseudoeffective. Thus Ω_Z is pseudoeffective by Lemma 3.4.

Proof of the claim. — Since Ω_X is pseudoeffective, there exist positive integers i, j such that $i \geqslant 2cj$ such that

$$H^0(X_0, S^i\Omega_X \otimes \mathcal{O}_X(jA)) \neq 0.$$

We consider the canonical exact sequence

$$0 \longrightarrow f^*(\Omega_{Z_0}) \xrightarrow{\mathrm{d}f} \Omega_{X_0} \longrightarrow \Omega_{X_0/Z_0} \longrightarrow 0. \tag{4.2}$$

Since $X_0 \to Z_0$ is a conic bundle, we know that $\mathrm{d}f$ cannot vanish along a divisor D. Thus $\Omega_{X/Z}$ is torsion free and the singular locus of $\Omega_{X/Z}$ is at most one-dimensional. Thus we get

$$H^0(X_0, f^*S^k\Omega_{Z_0} \otimes \omega_{X_0/Z_0}^{[i-k]} \otimes \mathcal{O}_X(jA)) \neq 0.$$

for some $k \in \{0, \ldots, i\}$. Since $\omega_{X_0/Z_0}^{[i-k]} \otimes \mathcal{O}_X(jA)$ has negative degree on the fibres of f if $i - k > j$ we see that $k \in \{i - j, \ldots, j\}$. Note that since $i \geqslant 2cj$ and $c > 1$ this implies that $k \geqslant j$. Using the morphism (4.1) obtain

$$H^0(X_0, f^*S^k\Omega_{Z_0} \otimes \mathcal{O}_X(jA + (i-k)f^*H)) \neq 0.$$

Since $i - k \leqslant j$ we finally obtain the claim. □

Remark. — The key point of the proof above is that the morphism $\mathrm{d}f$ does not vanish along a divisor D_0. In higher dimension, since the total space of the Mori fibre space is not necessarily smooth, this might very well happen. Then this type of argument only shows that $f^*\Omega_Z(D)$ is pseudoeffective where D has support inside the support of D_0. At least in dimension three we also see that

$$H^0(X, S^m\Omega_X) = H^0(Z, S^m\Omega_Z)$$

where $X \dashrightarrow Z$ is the MRC fibration of a uniruled smooth threefold X and Z is smooth.

Corollary 4.14. — *Let X be a smooth projective surface such that Ω_X is pseudoeffective. If $\kappa(X) \leqslant 0$, then Conjecture 1.1 holds.*

Proof. — If $\kappa(X) = -\infty$, the surface X is uniruled. Since Ω_X is pseudoeffective, by Proposition 4.11 the base of the MRC fibration is a curve of genus at least one. Thus we have $q(X) > 0$.

If $\kappa(X) = 0$, let $X_{\min}$ be the minimal model of X. Then $\Omega_{X_{\min}}$ is pseudoeffective by Proposition 4.1. Thus by Proposition 4.8 one has $H^0(X_{\min}, S^m\Omega^1_{X_{\min}})) \neq 0$ for some positive integer m. Since $X_{\min}$ is smooth, we have an isomorphism

$$H^0(X, S^m\Omega^1_X)) \simeq H^0(X_{\min}, S^m\Omega^1_{X_{\min}})).$$ □

In the next two sections we will deal with surfaces X of Kodaira dimension $\kappa(X) = 1$.

5. Elliptic surfaces: general set-up and the non-isotrivial case

We are starting here to study elliptic fibrations $f : X \to B$ with $\kappa(X) = 1$ towards Conjecture 1.1. If f is almost smooth, i.e., the only singular fibers are multiples of elliptic curves, then $c_2(X) = e(X) = 0$ [4, III, Prop. 11.4]. Thus by Noether's formula $\chi(X, \mathcal{O}_X) \leqslant 0$, and therefore $q(X) > 0$, so there is nothing to prove. We first fix notations.

5.1. Elliptic fibrations: the setup

Setup 5.1. — Let X be a smooth projective surface, and let $f : X \to B$ be an elliptic fibration onto a smooth curve B. We set

$$D = \sum_{b \in B} f^*b - (f^*b)_{\mathrm{red}},$$

so D is an effective divisor having support exactly on the irreducible components of a fibre that are not reduced. The exact sequence

$$0 \longrightarrow f^*\Omega_B \longrightarrow \Omega_X \longrightarrow \Omega_{X/B} \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow f^*\Omega_B(D) \longrightarrow \Omega_X \longrightarrow \mathcal{I}_Z \otimes \omega_{X/B}(-D) \longrightarrow 0, \tag{5.1}$$

where Z is a local complete intersection scheme of codimension two whose support coincides with the singular points of the reduction $(f^*b)_{\mathrm{red}}$ of the fibres [37, Prop. 3.1 (iii)].

We denote by $\pi : \mathbb{P}(\Omega_X) \to X$ the projectivisation, and by $\zeta \to \mathbb{P}(\Omega_X)$ the tautological class.

We set

$$Y := \mathbb{P}(\mathcal{I}_Z \otimes \omega_{X/B}(-D)) \subset \mathbb{P}(\Omega_X). \tag{5.2}$$

Since $\mathcal{I}_Z$ is a local complete intersection of codimension two, the projectivisation coincides with the blow-up of the ideal sheaf $\mathcal{I}_Z$ (see Setup 3.10). In particular Y is a prime divisor in $\mathbb{P}(\Omega_X)$ and

$$[Y] = \zeta - \pi^* c_1(f^*\Omega_B(D)). \tag{5.3}$$

Denote by $K \subset B$ the finite set of points such that the fibre f^*b is not multiple and not reduced. The divisor D can be decomposed as

$$D = \sum_{i=1}^{s}(m_i - 1)F_i + \sum_{b\in K} D_{0,b}, \tag{5.4}$$

where the F_i are the reductions of multiple f-fibres and

$$D_0 := \sum_{b\in K} D_{0,b}$$

is simply the remainder, i.e. the part of D coming from non-multiple, non-reduced fibres. It follows from Kodaira's classification [4, V, §7, Table 3] that the support of D_0 does not contain any fibre, so the intersection matrix of D_0 is negative definite by Zariski's lemma [4, III, Lem. 8.2]. It is now straightforward to check that (5.4) is the Zariski decomposition of D (see Section 3.4) with $P = \sum_{i=1}^{s}(m_i - 1)F_i$.

If f is relatively minimal, the canonical bundle formula [4, V, Thm. 12.1, Prop. 12.2] holds:

$$\omega_X \simeq f^*(\omega_B \otimes (R^1 f_*\mathcal{O}_X)^*) \otimes \mathcal{O}_X\left(\sum_{i=1}^{s}(m_i - 1)F_i\right), \tag{5.5}$$

and $\deg(R^1 f_*\mathcal{O}_X)^* = \chi(X, \mathcal{O}_X)$.

Finally assume that f is relatively minimal and $B = \mathbb{P}^1$. Then (5.5) implies that

$$K_{X/B} - D \sim aF - D_0 \tag{5.6}$$

where $a = \chi(X, \mathcal{O}_X)$ and F is a general fibre.

Proposition 5.2. — *In the situation of Setup 5.1, suppose that $f^*\Omega_B(D)$ is pseudoeffective. Then we have $\widetilde{q}(X) > 0$.*

Proof. — The statement is trivial if $g(B) \geqslant 1$, so assume $B \simeq \mathbb{P}^1$. We follow the philosophy of [15, §3.5]. By Remark 3.18 one has $\kappa(f^*\Omega_B(D)) \geqslant 0$. Choose a positive integer m and a non-zero section $s \in H^0(X, (f^*\Omega_B(D))^{\otimes m})$. Let E be the divisor defined by s. Note that E is supported on fibers of f and that

$$E \sim m(D - 2F),$$

where F is a general fibre of f. By the discussion in Setup 5.1 we know that the nef part of D is represented by $\sum_{i=1}^{s}(m_i - 1)F_i \equiv \lambda F$ where the F_i are the reductions of multiple f-fibres. Since

$$D \sim_{\mathbb{Q}} \frac{1}{m}E + 2F,$$

is a decomposition in effective divisors and $2F$ is nef, we obtain $\lambda \geqslant 2$.

Introducing the $\mathbb{Q}$-divisor $\Delta = \sum_{i=1}^{s}(1-\frac{1}{m_i})p_i$ with $p_i = f(F_i)$, we have $f^*\Delta = \sum_{i=1}^{s}(m_i-1)F_i$, so

$$\sum_{i=1}^{s}\left(1-\frac{1}{m_i}\right) = \lambda \geqslant 2.$$

Thus f has at least three multiple fibers. Then there is a ramified base change $\widetilde{B} \to B$ inducing an étale map

$$\widetilde{X} \longrightarrow X$$

such that $\widetilde{f} : \widetilde{X} \to \widetilde{B}$ has no multiple fibers and $g(\widetilde{B}) \geqslant 1$, see e.g. [22, IV.9.12]. Thus $\widetilde{q}(X) \neq 0$. □

5.2. The non-isotrivial case

In the situation of Setup 5.1, assume furthermore that the elliptic fibration f is not isotrivial. Let F be a general fibre, then its Kodaira–Spencer class is not zero. Thus we have a non-split extension defined by the restriction of (5.1) to F

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \Omega_X|_F \longrightarrow \Omega_F \simeq \mathcal{O}_F \longrightarrow 0.$$

Denote by $\zeta_F \to \mathbb{P}(\Omega_X|_F)$ the tautological class. Then the (1,1)-class ζ_F is nef with $\zeta_F^2 = 0$, moreover it is represented by the current of integration over the curve C defined by the quotient $\Omega_X|_F \to \mathcal{O}_F$. We recall the following result:

Lemma 5.3 ([20, Ex. 1.7]). — *Let h_F be any singular metric on ζ_F such that the curvature current $\Theta_{h_F}(\zeta_F)$ is positive. Then*

$$\Theta_{h_F}(\zeta_F) = [C]$$

where $[C]$ is the current of integration over C.

Proposition 5.4. — *In the situation of Setup 5.1, assume that f is not isotrivial. If ζ is pseudoeffective, the line bundle $f^*\Omega_B(D)$ is pseudoeffective.*

Proof. — By assumption there exists a singular metric h on ζ such that $\Theta_h(\zeta) \geqslant 0$. Then we can consider the Siu decomposition

$$\Theta_h(\zeta) = \sum_k \nu(\Theta_h(\zeta), Y_k)[Y_k] + P$$

where $Y_k \subset \mathbb{P}(\Omega_X)$ are prime divisors and P is a positive closed current such that the (countably many) irreducible components of $E_+(P)$ [8, §2.2.1] have codimension at least two. Let now F be a very general f-fibre, so that $\mathbb{P}(\Omega_X|_F)$ does not contain any positive dimensional irreducible component

of $E_+(P)$ then the restriction h_F of the metric h to $\mathbb{P}(\Omega_X|_F)$ is well-defined and yields a singular metric on ζ_F. Moreover we have a decomposition

$$\Theta_{h_F}(\zeta_F) = (\Theta_h(\zeta))|_F = \sum \nu(\Theta_h(\zeta), Y_k)[Y_k \cap F] + P|_F.$$

By Lemma 5.3 we see that $P|_F = 0$, and there exists a unique Y_k (say Y_1) such that $Y_1 \cap F = C$ and $\nu(\Theta_h(\zeta), Y_1) = 1$.

Since the intersection $Y_1 \cap F$ coincides with C on all the general fibres and the quotient $\Omega_X|_F \to \Omega_F$ is the restriction of the quotient $\Omega^1_X \to \mathcal{I}_Z \otimes \omega_{X/B}(-D)$ we obtain that $Y_1 = Y$. Thus we have

$$\Theta_h(\zeta) - [Y] = \sum_{k \geqslant 2} \nu(\Theta_h(\zeta), Y_k)[Y_k] + P$$

which is a positive current. By (5.3) this implies the statement. □

Remark. — The non-isotriviality of f is crucial for the preceding argument: if f is isotrivial, the restriction $\Omega_X|_F$ of the cotangent bundle to F is isomorphic to $\mathcal{O}_F^{\oplus 2}$. Thus Lemma 5.3 does not apply and in fact there are infinitely many possibilities for the positive current $\Theta_{h_F}(\zeta_F)$. This makes it impossible to reconstruct the singular locus of $\Theta_h(\zeta)$ from its restriction over a general fibre.

By Proposition 5.2 and 4.6, we conclude

Corollary 5.5. — *Let X be a smooth projective surface, and let $f : X \to B$ be a non-isotrivial elliptic fibration Then the following are equivalent:*

(1) *Ω^1_X is pseudoeffective ;*
(2) *we have $\widetilde{q}(X) > 0$;*
(3) *we have $H^0(X, S^m\Omega^1_X)) \neq 0$ for some positive integer m*

In summary, Conjecture 1.1 holds for non-isotrivial elliptic surfaces.

6. Elliptic surfaces: the isotrivial case

In this section we treat isotrivial elliptic fibrations and prove parts (2) and (3) of Theorem 1.2.

6.1. Notation

In the situation of Setup 5.1, assume that

$$f : X \longrightarrow B$$

is relatively minimal and isotrivial. Denote by E the elliptic curve such that a general f-fibre is isomorphic to E. By [37, §2] there exists a smooth curve C and a finite group G acting diagonally on the product $C \times E$ such that X is birational to the quotient $(C \times E)/G$ and the fibration f corresponds to the fibration $(C \times E)/G \to C/G \simeq B$ induced by projection on the first factor.

More precisely, denote by $q : C \times E \to (C \times E)/G$ the quotient map, and by $\bar{p}_C : (C \times E)/G \to C/G$ the map induced by the projection p_C. Denote by $\lambda : R \to (C \times E)/G$ the minimal resolution of singularities, then the exceptional divisors are Hirzebruch–Jung strings [37, 2.0.2] and the singular fibres of

$$f_R := \bar{p}_C \circ \lambda : R \longrightarrow B$$

are described in [37, Thm. 2.1]. Following Serrano, we call f_R the standard model of the isotrivial fibration f.

The standard isotrivial fibration f_R factors through its relative minimal model. Since we assumed that f is relatively minimal and the relative minimal model of an elliptic surface is unique, we have a birational morphism $\mu : R \to X$ such that $f_R = f \circ \mu$. We summarise the construction in a commutative diagram:

$$\begin{array}{ccc} R & & C \times E \\ {\scriptstyle\mu}\downarrow & \searrow^{\lambda} & \downarrow{\scriptstyle q} \\ X & & (C \times E)/G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{p}_C} \\ B & = & C/G \end{array} \tag{6.1}$$

In general the birational map μ is not an isomorphism [37, (2.4)], but as shown in Lemma 6.2 below, it is an isomorphism unless E has non-trivial group automorphisms. The following lemma is well-known, we recall the proof for the convenience of the reader:

Lemma 6.1. — *Using the Notation 6.1, let $x \in C$ be a point with non-trivial stabiliser $G_x \simeq \mathbb{Z}_2$ on C. Then G_x acts either by translation or as the involution $z \mapsto -z$ on the elliptic curve $x \times E$.*

In particular the corresponding singular fiber of f is either a multiple elliptic curve or a curve of type I_0^* in Kodaira's classification, see e.g., [4, V, §7, Table 3].

Proof. — We follow the description of the singularities of $(C \times E)/G$ in [37, 2.0.2]: if $G_x \simeq \mathbb{Z}_2$ acts freely on $x \times E$, the quotient $(C \times E)/G$ is

smooth near the fibre which is multiple elliptic. Assume now that there exists a point $e \in E$ with non-trivial stabiliser $G_{x,e} \subset G_x$. Then $G_{x,e}$ is a subgroup of the group automorphism group $\mathrm{Aut}(E, e)$. In view of [27, IV, Cor. 4.7] this strongly limits the possibilities. Since we assumed that $G_x \simeq \mathbb{Z}_2$, the group $G_{x,e} = G_x$ acts as $z \mapsto -z$ and $(C \times E)/G$ has four singularities of type A_1 along the fibre. Thus the fibre in the minimal resolution has type I_0^*. □

Lemma 6.2. — *Let $f : X \to B$ be a minimal isotrivial elliptic fibration as in Notation 6.1. If E trivial group automorphism group $\mathbb{Z}_2$, the fibration $f : X \to B$ is standard. Moreover all the singular fibres are multiple elliptic curves or of type I_0^*.*

Proof. — Since the fibration is isotrivial the J-function is constant for all the smooth fibres, hence extends as a constant function to B. Thus by [34, Table IV.3.1] this excludes singular fibres of type I_N and I_N^*. Since E has trivial group automorphism group $\mathbb{Z}_2$, its J-invariant is distinct from 0 and 1. Hence again by [34, Table IV.3.1] this leaves only the possibility of singular fibres that are either multiple elliptic curves or of type I_0^*. But these fibres are standard, cp. the proof of Lemma 6.1 above. □

6.2. Relatively minimal standard isotrivial fibrations

Assumption 6.3. — *In the* whole subsection *we assume that*

$$f : X \longrightarrow B$$

is a relatively minimal, isotrivial elliptic fibration such that all the singular fibres are multiple elliptic curves or of type I_0^*. In particular the fibration is standard, for simplicity of the exposition we thus identify $X = R$.

We have the following crucial

Lemma 6.4. — *Using the Notation 6.1, let Z denote the set of points in $x \in C$ such that G_x acts as the involution $z \mapsto -z$ on the elliptic curve $x \times E$.*

If $f^*\Omega_B(D)$ is not pseudoeffective, then Z has at least $2g(C)-1$ elements.

Proof. — Since $f^*\Omega_B(D)$ is not pseudoeffective, we have $B \simeq \mathbb{P}^1$.

The action of G on the curve C defines a Galois cover $\psi : C \to C/G \simeq \mathbb{P}^1$ of degree $d = |G|$. By the Hurwitz formula we have

$$2g(C) - 2 = \deg K_C = -2d + \deg R,$$

where R is the ramification divisor of ψ. If $x \in Z$, then it is a point with stabiliser $G_x \simeq \mathbb{Z}_2$, hence ψ ramifies with order 2 in x. Thus $\deg R_x = 1$ and

$$\#(Z) = 2g(C) - 2 + 2d - \deg R_t,$$

where $R_t \subset R$ is the ramification corresponding to points $y \in C$ such that $G_y \simeq \mathbb{Z}_{m_y}$ acts by translation on $y \times E$ (see Lemma 6.1). If $y \in C$ is a point with this property, every point in its orbit $G.y$ has the same property. Since the orbit has length $\frac{d}{m_y}$, this induces a ramification divisor of order $d\frac{m_y-1}{m_y}$. Now note that the corresponding f- fibre over $\bar{y} = \psi(y)$ is multiple elliptic with multiplicity m_y, so it defines a component of D that is numerically equivalent to $\frac{m_y-1}{m_y}F$ where F is a general fibre of f. Thus we see that $f^*\Omega_{\mathbb{P}^1}(D)$ is pseudoeffective if

$$\sum_{\bar{y}\in\mathbb{P}^1,\ G_y \text{ acts by translation}} \frac{m_y-1}{m_y} \geqslant 2.$$

Suppose now that this is not the case: then we have

$$\deg R_t = \sum_{y\in C,\ G_y \text{ acts by translation}} d\left(\frac{m_y-1}{m_y}\right) < 2d,$$

hence $\#(Z) = 2g(C) - 2 + 2d - \deg R_t > 2g(C) - 2$. □

Construction 6.5. — We again use the Notation 6.1. Let $\bar{A}$ be an ample Cartier divisor on $(C\times E)/G$, and set $A_X = \lambda^*\bar{A}$. Then by definition (and [28, Lem. 2.2]) the vector bundle Ω_X is not pseudoeffective if and only if there exists a $c > 0$ such that for all $i, j \in \mathbb{N}$ such that $i > cj$ one has

$$H^0(X, S^i\Omega_X \otimes \mathcal{O}_X(jA_X)) = 0.$$

Denote by $D \subset X$ the exceptional locus of λ, then we have morphisms

$$\begin{array}{c} H^0(X, S^i\Omega_X \otimes \mathcal{O}_X(jA_X)) \\ \big\downarrow \\ H^0(X \setminus D, S^i\Omega_X \otimes \mathcal{O}_X(jA_X)) \\ \big\downarrow\simeq \\ H^0((C\times E)/G \setminus \lambda(D), S^i\Omega_{(C\times E)/G} \otimes \mathcal{O}_{(C\times E)/G}(j\bar{A})) \\ \big\downarrow \\ H^0(C\times E \setminus q^{-1}(\lambda(D)), S^i\Omega_{C\times E} \otimes \mathcal{O}_{C\times E}(jq^*\bar{A})) \\ \big\downarrow\simeq \\ H^0(C\times E, S^i\Omega_{C\times E} \otimes \mathcal{O}_{C\times E}(jq^*\bar{A})) \end{array}$$

where the last isomorphism is due to the fact that $S^i\Omega_{C\times E}$ is reflexive, since $C \times E$ is smooth.

Finally let A_C be an ample divisor on C and A_E an ample divisor of degree one on E. Set

$$A := p_C^*(A_C) \otimes p_E^*(A_E)$$

with the canonical projections $p_C : C \times E \to C$ and $p_E : C \times E \to E$. Then for some $l \in \mathbb{N}$ sufficiently high we have an inclusion

$$\mathcal{O}_{C\times E}(q^* \bar{A}) \hookrightarrow \mathcal{O}_{C\times E}(lA),$$

so for all $i, j \in \mathbb{N}$ we obtain an inclusion

$$\Phi = \Phi_{i,j,l} : H^0(X, S^i\Omega_X \otimes \mathcal{O}_X(jA_X)) \hookrightarrow H^0(C \times E, S^i\Omega_{C\times E} \otimes \mathcal{O}_{C\times E}(jlA)). \quad (6.2)$$

Definition 6.6. — *We say that a $\eta \in H^0(C\times E, S^i\Omega_{C\times E}\otimes\mathcal{O}_{C\times E}(jA))$ induces a holomorphic symmetric differential with values in A_X on X if it is in the image of an inclusion Φ in* (6.2).

Remarks. — The definition is a slight abuse of terminology, since the inclusion Φ in (6.2) is only defined for j divisible by l. Since in the definition of pseudoeffectivity we can always replace i and j by il and jl, we will, for the simplicity of notation, ignore this point.

Note also that the chain of inclusions does not use that C is proper, so the terminology also applies to an analytic open subset $\Delta \times E \subset C \times E$ with some subgroup $G_x \subset G$ acting on $\Delta \times E$.

6.3. The nonvanishing conjecture for standard isotrivial fibrations

The goal of this subsection is to prove the following

Theorem 6.7. — *Let X be a smooth projective surface that admits relatively minimal, isotrivial elliptic fibration*

$$f : X \longrightarrow B$$

such that all the singular fibres are multiple elliptic curves or of type I_0^*. If $f^*\Omega_B(D)$ is not pseudoeffective, then Ω_X is not pseudoeffective.

By Proposition 5.2 and 4.6 and Lemma 6.2 we thus obtain:

Corollary 6.8. — *Let $f : X \to B$ be a relatively minimal isotrivial elliptic fibration such that the general fibre has trivial group automorphism group $\mathbb{Z}_2$. Then the following are equivalent:*

(1) *Ω^1_X is pseudoeffective ;*
(2) *we have $\widetilde{q}(X) > 0$;*
(3) *we have $H^0(X, S^m\Omega^1_X)) \neq 0$ for some positive integer m*

The proof of Theorem 6.7 is done by showing that if $i \gg j$, a non-zero

$$\eta \in H^0(C \times E, S^i\Omega_{C\times E} \otimes \mathcal{O}_{C\times E}(jA))$$

does not induce a holomorphic symmetric differential on X. In other words, $\Phi = 0$. Since the details are somewhat technical, let us first recall and reprove a result of Sakai.

Example 6.9 ([36, §4, (D)]*).* — Let C be a hyperelliptic curve, and let $\tau = (i_C, i_E)$ be the involution on $C\times E$ defined by the hyperelliptic involution i_C on C and the map $i_E : E \to E, z \mapsto -z$ on an elliptic curve E. The minimal resolution $X \to (C \times E)/\langle\tau\rangle$, has an isotrivial fibration $f : X \to \mathbb{P}^1$ such that all the singular fibres are of type I_0^*, in particular it is relatively minimal and standard. Then one has

$$H^0(X, S^i\Omega_X) = 0 \qquad \forall\, i \in \mathbb{N}.$$

Proof. — Let

$$p_C^*\alpha \otimes p_E^*\beta \in H^0(C \times E, S^i\Omega_{C\times E}) = \bigoplus_{l=0}^{i} H^0(C \times E, p_C^*\omega_C^{\otimes l} \otimes p_E^*\omega_E^{\otimes i-l})$$

be a rank one tensor, i.e. $\alpha \in H^0(C, \omega_C^{\otimes l})$ and $\beta \in H^0(E, \omega_E^{\otimes i-l})$ for some $l \in 0, \ldots, i$.

Assume that $\alpha \otimes \beta$ induces a holomorphic symmetric differential on X. Arguing by contradiction we assume that $\alpha \neq 0$ and $\beta \neq 0$.

If $(x, 0)$ is a fixed point of τ, choose local coordinates z_1 on C and z_2 on E such that locally near $(x, 0)$, the involution is given by $(z_1, z_2) \mapsto (-z_1, -z_2)$. In these local coordinates we write $\alpha = f_\alpha(z_1)\mathrm{d}z_1^l$ and $\beta = f_\beta(z_2)\mathrm{d}z_2^{i-l}$. For a general point in the exceptional divisor over the point $\overline{(x, 0)}$, we can choose local coordinates (u, v) on X such that $z_1 = u^2, z_2 = uv$. In these coordinates the exceptional divisor is given by $u = 0$. Substituting (z_1, z_2) by these coordinates we see that $\alpha \otimes \beta$ induces the meromorphic symmetric differential

$$f_\alpha(\sqrt{u}) \cdot f_\beta(\sqrt{u}) \cdot \mathrm{d}u^l \frac{(u\mathrm{d}v + v\mathrm{d}u)^{i-l}}{(2\sqrt{u})^i}.$$

on X. Looking at the term of $\mathrm{d}u^i$ we see that the differential is holomorphic along the exceptional divisor if and only if

$$\frac{f_\alpha(\sqrt{u}) \cdot f_\beta(\sqrt{u})}{(\sqrt{u})^i}$$

is holomorphic, i.e. if and only if $\alpha \otimes \beta$ vanishes with order at least i in the fixed point. Since $0 \neq \beta \in H^0(E, \omega_E^{\otimes i-l}) \simeq H^0(E, \mathcal{O}_E)$ does not vanish, this shows that $\alpha \in H^0(C, \omega_C^{\otimes l})$ vanishes with order i in x. Since the involution i_C has $2g(C) + 2$ fixed points, we obtain that α vanishes along a divisor of

degree at least $i\cdot(2g(C)+2)$. Since $\deg \omega_C^{\otimes l} = l\cdot(2g(C)-2) < i\cdot(2g(C)+2)$ we obtain that $\alpha = 0$. Since $H^0(E,\omega_E^{\otimes i-l})$ has dimension one for every $i-l$ one can reduce the general case to rank one tensors, so the statement follows. This settles the proof of the example. □

In the proof of Theorem 6.7 we have $h^0(E,\omega_E^{\otimes i-l}\otimes\mathcal{O}_E(jA_E)) > 1$, so the symmetric differentials are not global rank one tensors. Somewhat surprisingly, this leads to a much weaker local obstruction (cf. [11, Prop. 3.2]), i.e., the vanishing order of the symmetric differential in a fixed point can be strictly smaller than i. We will improve this local estimate by taking into account that the vanishing order along E is bounded by $j\deg A_E$.

The local obstruction – Setup. — Let us describe the local obstruction for a holomorphic symmetric differential on $C\times E$ to induce a holomorphic symmetric differential on X: using the notation of Lemma 6.4, fix a point $x\in Z\subset C$, and let τ_x be the generator of $G_x\simeq\mathbb{Z}_2$. Let $0\in E$ be a fixed point of τ_x, for this local computation we choose $A_E := 0$ to be the corresponding ample divisor of degree one[(2)].

Let $x\in\Delta\subset C$ be a small disc and choose a local coordinate z_1 such that the action of G_x is given by $z_1\mapsto -z_1$ (in particular $x=0$). We have

$$H^0(\Delta\times E,\mathcal{O}_{\Delta\times E}(jA)) \simeq \mathbb{C}\{z_1\}\otimes H^0(E,\mathcal{O}_E(jA_E)),$$

where z_1 is a local coordinate on Δ and $\mathbb{C}\{z_1\}$ denotes the algebra of convergent power series in z_1.

Since E is an elliptic curve, we have $h^0(E,\mathcal{O}_E(jA_E)) = j$, and we may choose a basis of sections $s_{j,0},\dots,s_{j,j-2},s_{j,j}$ such that $s_{j,k}$ vanishes with order exactly k in the neutral element 0 of the elliptic curve. In particular we have a $\mathbb{C}$-basis of $H^0(\Delta\times E,\mathcal{O}_{\Delta\times E}(jA))$

$$z_1^{n-k}s_{j,k}\qquad n\in\mathbb{N},\ k=0,\dots,j-2,j.$$

Note that by construction $z_1^{n-k}s_{j,k}$ vanishes with order exactly n in $(0,0)$.

Since E is an elliptic curve, its cotangent bundle is trivial and we denote by $\mathrm{d}z_2$ a global generator of Ω_E. Let now

$$H^0(\Delta\times E,S^i\Omega_{\Delta\times E}\otimes\mathcal{O}_{\Delta\times E}(jA)) \simeq \bigoplus_{l=0}^{i} H^0(\Delta\times E,\mathcal{O}_{\Delta\times E}(jA))\otimes \mathrm{d}z_1^l\mathrm{d}z_2^{i-l}$$

(2) We make this choice in order to simplify the notation. Since the obstruction depends only on a small neighbourhood of $0\in E$, the statement in Corollary 6.11 is independent of this choice.

be the space of symmetric differentials with values in $\mathcal{O}_{\Delta\times E}(jA)$. Then we can decompose

$$H^0(\Delta\times E, S^i\Omega_{\Delta\times E}\otimes\mathcal{O}_{\Delta\times E}(jA)) = \bigoplus_{n\in\mathbb{N}} V_n$$

where V_n is generated by

$$z_1^{n-k}s_{j,k}\mathrm{d}z_1^l\mathrm{d}z_2^{i-l} \qquad k=0,\dots,\min\{j,n\}, \qquad l=0\dots i.$$

Let now $\omega\in H^0(\Delta\times E, S^i\Omega_{\Delta\times E}\otimes\mathcal{O}_{\Delta\times E}(jA))$, and let

$$\omega = \sum_{n\in\mathbb{N}}\omega_n$$

be the decomposition such that $\omega_n\in V_n$. Finally let

$$M := z_1\mathrm{d}z_2 - s_{1,1}\mathrm{d}z_1$$

be the holomorphic 1-form with values in A giving in local coordinates the form $z_1\mathrm{d}z_2 - z_2\mathrm{d}z_1$.

In the setup just introduced, the following holds :

Lemma 6.10. — *Let S be the minimal resolution of $(\Delta\times E)/G_x$. If ω induces a holomorphic symmetric differential with values in A_S on S (see Definition 6.6), then for all $n\in\mathbb{N}$ the form ω_n induces a holomorphic symmetric differential with values in A_S. Moreover, if $n<i$, there exists a differential*

$$\eta_n\in H^0\left(\Delta\times E, S^{\frac{i+n}{2}}\Omega_{\Delta\times E}\otimes\mathcal{O}_{\Delta\times E}\left(\left(j-\frac{i-n}{2}\right)A\right)\right)$$

such that $\omega_n = \eta_n\times M^{\frac{i-n}{2}}$.

Remark. — Since $G_x\simeq\mathbb{Z}_2$ acts locally as $(z_1,z_2)\mapsto(-z_1,-z_2)$, one has $n=i \mod 2$ [11, §3], so $\frac{i+n}{2}$ and $\frac{i-n}{2}$ are positive integers.

Proof. — The property of being holomorphic is local, so we can just apply the proof of [11, Prop. 3.2]. □

Since $H^0(E,\mathcal{O}_E((j-\frac{i-n}{2})A_E))=0$ for $j-\frac{i-n}{2}<0$ this implies :

Corollary 6.11. — *If $\omega\in H^0(\Delta\times E, S^i\Omega_{\Delta\times E}\otimes\mathcal{O}_{\Delta\times E}(jA))$ induces a holomorphic symmetric differential with values in A_S on S, then*

$$\omega\in H^0(\Delta\times E, I_{0,0}^n\otimes S^i\Omega_{\Delta\times E}\otimes\mathcal{O}_{\Delta\times E}(jA))$$

where $I_{0,0}$ is the ideal sheaf of the origin and $n\geqslant i-2j$.

Proof of Theorem 6.7. — We use the notation of Subsection 6.1 and Construction 6.5. Denote by $Z \subset C$ the set of points in $x \in C$ such that the stabiliser $G_x = \langle \tau_x \rangle$ acts as the involution $z \mapsto -z$ on the elliptic curve $x \times E$. Since $f^*\Omega_B(D)$ is not pseudoeffective and all the fibres are multiple elliptic or of type I_0^*, we know by Lemma 6.4 that Z has at least $2g-1$ element where $g = g(C)$.

For every $x \in Z$ we fix a point $x' \in (x \times E)$ such that (x, x') is a fixed point of τ_x.

Fix some rational $\epsilon > 0$ such that $\frac{2g-2}{2g-1} + \epsilon < 1$ and fix a positive integer N such that

$$\left(\frac{2g-2}{2g-1} + \epsilon\right) N \in \mathbb{N}.$$

Assume now that

$$\eta \in H^0(C \times E, S^i\Omega_{C\times E} \otimes \mathcal{O}_{C\times E}(jA))$$

induces a holomorphic symmetric differential with values in A_X on X (see Definition 6.6). Then for every $x \in Z$, the restriction to some neighbourhood $\Delta \times E$ induces a holomorphic symmetric differential on the minimal resolution S of $(\Delta \times E)/G_x$. Thus by Corollary 6.11 we have for every $i \in \mathbb{N}$ that

$$\eta \in H^0\left(C \times E, \left(\bigotimes_{x\in Z} I^n_{(x,x')}\right) \otimes S^i\Omega_{C\times E} \otimes \mathcal{O}_{\Delta\times E}(jA).\right)$$

with $n \geqslant i - 2j$.

Observe that if $i \geqslant \frac{2}{1-\frac{2g-2}{2g-1}-\epsilon} j$ one has

$$n \geqslant \left(\frac{2g-2}{2g-1} + \epsilon\right) i.$$

Thus we get an inclusion

$$\eta^N \in H^0\left(C \times E, \left(\bigotimes_{x\in Z} I^{(\frac{2g-2}{2g-1}+\epsilon)Ni}_{(x,x')}\right) \otimes S^{Ni}\Omega_{C\times E} \otimes \mathcal{O}_{C\times E}(NjA)\right).$$

We claim that this last vector space is zero for $i \geqslant \frac{2}{1-\frac{2g-2}{2g-1}-\epsilon} j$. As explained after Theorem 6.7, this shows the statement.

Proof of the claim. — Recall that

$$S^i\Omega_{C\times E} = \bigoplus_{l=0}^{i} p_C^*\omega_C^l \otimes p_E^*\omega_E^{i-l} \simeq \bigoplus_{l=0}^{i} p_C^*\omega_C^l$$

since $\omega_E \simeq \mathcal{O}_E$. Since $\omega_C^l \hookrightarrow \omega_C^i$ for $l < i$ we are reduced to showing that

$$H^0\left(C \times E, \left(\bigotimes_{x\in Z} I_{(x,x')}^{(\frac{2g-2}{2g-1}+\epsilon)Ni}\right) \otimes p_C^*\omega_C^{Ni} \otimes \mathcal{O}_{C\times E}(NjA)\right) = 0$$

for $i \gg j$. By Definition 3.7 this is the same as proving that

$$\bigotimes_{x\in Z} I_{(x,x')}^{(\frac{2g-2}{2g-1}+\epsilon)N} \otimes p_C^*\omega_C^N$$

is not strongly pseudoeffective. Arguing by contradiction assume that the sheaf is strongly pseudoeffective.

Let $\mu : S \to C \times E$ be the blowup of the (reduced) set $\bigcup_{x\in Z}(x, x') \subset C \times E$ and denote by $\mathcal{O}_S(1)$ the tautological sheaf, i.e. the ideal sheaf of the exceptional divisor of μ. Recall that S is isomorphic to the blowup of the ideal sheaf $\bigotimes_{x\in Z} I_{(x,x')}^{(\frac{2g-2}{2g-1}+\epsilon)N}$ (e.g. [27, II, Ex. 7.11]), so by Corollary 3.13 the line bundle

$$\mathcal{O}_S\left(\left(\frac{2g-2}{2g-1}+\epsilon\right)N\right) \otimes \mu^* p_C^*\omega_C^N$$

is pseudoeffective. By Lemma 3.16, applied to the elliptic fibration $p_C \circ \mu : S \to C$, there exists $m \in \mathbb{N}$ and a numerically trivial line bundle M on C such that

$$H^0\left(S, \mathcal{O}_S\left(\left(\frac{2g-2}{2g-1}+\epsilon\right)mN\right) \otimes \mu^* p_C^*(\omega_C^{mN} \otimes M^*)\right) \neq 0.$$

Thus we have

$$H^0\left(C \times E, \bigotimes_{x\in Z} I_{(x,x')}^{(\frac{2g-2}{2g-1}+\epsilon)mN} \otimes \mu^* p_C^*(\omega_C^{mN} \otimes M^*)\right) \neq 0. \tag{6.3}$$

Yet by Lemma 6.4 we know that Z has at least $2g-1$ elements, hence

$$\deg\left(\bigotimes_{x\in Z} I_x^{(\frac{2g-2}{2g-1}+\epsilon)mN}\right) \geqslant mN\left(\frac{2g-2}{2g-1}+\epsilon\right)\cdot 2g - 1 > mN(2g-2).$$

Since $\deg(\omega_C^{mN} \otimes M^*) = mN(2g-2)$, we obtain a contradiction to (6.3). □

6.4. Isotrivial fibrations and the Zariski decomposition

Let $f : X \to \mathbb{P}^1$ be an isotrivial elliptic fibration over a rational curve, and assume that Ω_X is pseudoeffective. Since the proof of Theorem 6.7 is a bit tedious, we present here a more conceptual approach based on the ideas of Subsection 5.2. The considerations of this section are independent of whether f is standard or not.

We use the notations of the setup 5.1. Let ζ be the tautological class of $\mathbb{P}(\Omega_X)$. If the elliptic fibration f is not almost smooth, we would like to show that the subvariety

$$Y := \mathbb{P}(\mathcal{I}_Z \otimes \omega_{X/B}(-D)) \subset \mathbb{P}(\Omega_X)$$

defined by f is in the negative part of the Zariski decomposition of ζ. If f is isotrivial, the restriction $\mathbb{P}(\Omega_X|_F)$ over a general fibre F is isomorphic to $\mathbb{P}^1 \times F$, so the proof of Proposition 5.4 does not apply. We therefore have to use some global information to explicitly compute the restriction $\zeta|_Y$.

Proposition 6.12. — *In the situation of Setup 5.1, assume that f is isotrivial, relatively minimal and not almost smooth. Then $\zeta|_Y$ is not pseudoeffective.*

Proof. — By Corollary 3.13 the statement is equivalent to showing that $\mathcal{I}_Z \otimes \omega_{X/B}(-D)$ is not strongly pseudoeffective. By Corollary 3.19 this is equivalent to showing that

$$H^0(X, \mathcal{I}_Z^k \otimes (\omega_{X/B}(-D))^{\otimes k}) = 0$$

for all $k \in \mathbb{N}$. Recall the canonical bundle formula (5.5), formula (5.6) and [4, III,Prop. 11.4, Rem. 11.5]:

$$\chi(X, \mathcal{O}_X) = \frac{e(X)}{12} = \frac{1}{12} \sum_{b \in K} e(X_b).$$

Here the set $K \subset B$ consists of all the points such that the reduction of the scheme-theoretic fibre X_b is not an elliptic curve. From these facts, we obtain,

$$K_{X/B} - D \sim_{\mathbb{Q}} \sum_{b \in K} (\frac{e(X_b)}{12} F - D_{0,b}).$$

Now the proof is finished by observing that

$$\kappa\left(X, \mathcal{I}_{Z,b} \otimes \frac{e(X_b)}{12} F\right) = -\infty$$

for all $b \in K$. This is done by a tedious case by case calculation, using that, by the proof of Lemma 6.1, the non-multiple fibres are of Kodaira's type II, III, IV, II^*, III^*, IV^* or I_0^*. The details are left to the interested reader. □

As an application, we obtain

Corollary 6.13. — *Let X be a smooth projective surface such that K_X is nef and $\kappa(X) = 1$. Denote by ζ the tautological divisor on $\pi : \mathbb{P}(\Omega_X) \to X$. If ζ is pseudoeffective and nef in codimension one, then we have $\widetilde{q}(X) > 0$. In particular, by* [3, Prop. 2.2]*, we have $H^0(X, S^m\Omega_X^1) \neq 0$ for some positive integer m.*

Proof. — If the Iitaka fibration is isotrivial and not almost smooth, we see by Proposition 6.12 that the restriction $\zeta|_Y$ is not pseudoeffective. Thus Y is in the negative part of the divisorial Zariski decomposition, hence ζ is not nef in codimension one.

Thus we know that f is almost smooth or not isotrivial. If the Iitaka fibration is almost smooth, then $c_2(X) = 0$ [4, III, Prop. 11.4] and thus $\widetilde{q}(X) > 0$ as already noticed at the beginning of Section 5.

If the Iitaka fibration $\varphi : X \to B$ is not isotrivial, we conclude by Corollary 5.5. □

Remark. — By Proposition 6.12 the restriction $\zeta|_Y$ is not pseudoeffective, so by divisorial Zariski decomposition there exists a $c > 0$ such that $\zeta - cY$ is pseudoeffective. If we prove that $\zeta - Y$ is pseudoeffective, we obtain the nonvanishing conjecture as in Corollary 5.5. However this is not obvious, even in simple situations.

Appendix A. Fundamental group of elliptic surfaces

The following statement is essentially a consequence of [15, §3.5]

Lemma A.1. — *Let $f : X \to B$ be a smooth elliptic surface such that the cotangent bundle Ω_X is not pseudoeffective. Then the fundamental group of X is finite.*

Proof. — By Proposition 4.1 we can assume without loss of generality that X is minimal. Since Ω_X is not pseudoeffective, we have $B \simeq \mathbb{P}^1$. If f is almost smooth, there exists an étale cover $B' \times E \to X$ with E an elliptic curve (see introduction to Section 5). In particular $\Omega_{B' \times E}$ is pseudoeffective, a contradiction to Proposition 4.4. Thus f is not almost smooth, hence by [18, Lem. 1.39] and [16, Cor. 12.10] one has $\pi_1(X) \simeq \pi_1(B, \Delta)$, where (B, Δ) is the orbifold structure defined by the multiple fibres. Since B is a curve, by [14, App. C], we can find a finite étale cover $X' \to X$ such that the orbifold divisor of $X' \to B'$ is empty. Since $\Omega_{X'}$ is not pseudoeffective by Proposition 4.4, we have $B' \simeq \mathbb{P}^1$. Thus $\pi_1(X') \simeq \pi_1(B') \simeq \{1\}$. □

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