ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

Patrick Graf

A decomposition theorem for singular Kähler spaces with trivial first Chern class of dimension at most four

Tome XXXII, nº 5 (2023), p. 893–909.

https://doi.org/10.5802/afst.1757

Publication membre du centre
Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2258-7519

Annales de la faculté des sciences de Toulouse Volume XXXII, n° 5, 2023
pp. 893-909

A decomposition theorem for singular Kähler spaces with trivial first Chern class of dimension at most four (*)

Patrick Graf (1)

Nemo:
Ad litteram nemo:
Ne anima quidem ulla:
Graf: Cum non inutile sit eandem veritatem per methodos diversas perscrutari ...

Abstract. — Let X be a compact Kähler fourfold with klt singularities and vanishing first Chern class, smooth in codimension two. We show that X admits a Beauville–Bogomolov decomposition: a finite quasi-étale cover of X splits as a product of a complex torus and singular Calabi–Yau and irreducible holomorphic symplectic varieties. We also prove that X has small projective deformations and the fundamental group of X is projective. To obtain these results, we propose and study a new version of the Lipman–Zariski conjecture.

Résumé. — Soit X une variété kählérienne compacte de dimension quatre, avec des singularités klt et première classe de Chern nulle, lisse en codimension deux. Nous montrons que X admet une decomposition de Beauville–Bogomolov: à un revêtement quasi-étale fini près, X est un produit d'un tore complexe et des variétés singulières de Calabi–Yau et holomorphes symplectiques irréductibles. Nous prouvons aussi que X admet des deformations projectives petites et que le groupe fondamental de X est projective. Pour obtenir ces resultats, nous proposons et étudions une nouvelle version de la conjecture de Lipman–Zariski.

1. Introduction

Let X be a compact Kähler manifold such that $c_1(X) = 0 \in H^2(X, \mathbb{R})$. The structure of such X is in many aspects well-understood [6, 7, 26, 27]:

(*) Reçu le 15 avril 2021, accepté le 27 octobre 2021.

Keywords: Kähler spaces, klt singularities, vanishing first Chern class, unobstructed deformations, decomposition theorem.

2010 *Mathematics Subject Classification:* 32J27, 14E30, 14J32.

(1) Fachbereich 08, Johannes-Gutenberg-Universität Mainz, Staudingerweg 9, 55099 Mainz, Germany — patrick.graf@uni-bayreuth.de — www.pgraf.uni-bayreuth.de/en/

Article proposé par Vincent Guedj.

- Geometry: a finite étale cover of X splits as a product of a complex torus, simply connected Calabi–Yau manifolds and irreducible holomorphic symplectic manifolds (Beauville–Bogomolov decomposition).
- Deformation theory: The local deformation space of X is smooth (Bogomolov–Tian–Todorov theorem), and X admits small projective deformations (Kodaira problem).
- Topology: the fundamental group $\pi_1(X)$ is projective, virtually abelian, and finite if the augmented irregularity $\widetilde{q}(X)$ of X vanishes.

The BB (= Beauville–Bogomolov) decomposition is a cornerstone in the classification of compact Kähler manifolds up to biholomorphic maps. The subject of birational geometry, however, is rather a structure theory up to *bimeromorphic* maps. As is well-known, in this context it is necessary to consider singular analogues of the above manifolds. By this, we mean compact Kähler spaces with klt singularities and $\mathrm{c}_1(X) = 0$. In the projective case, the BB decomposition has been established by [12, 16, 18]. This result is commonly referred to as the BBDGGHKP decomposition. But most of the other properties listed in the beginning remain elusive even for projective varieties.

Very recently, the decomposition theorem has been extended to the Kähler case [3, 9, 15]. The final statement may therefore be called the

BBBCDGGGHKLNPS decomposition.

The goal of this paper is to give an independent proof of the decomposition in dimension four. Our principal result, however, is a generalization of the BTT (= Bogomolov–Tian–Todorov) theorem.

THEOREM 1.1 (Singular BTT theorem in dimension four). — *Let X be a normal compact Kähler space of dimension $\leqslant 4$, with klt singularities and such that $\mathrm{c}_1(X) = 0 \in \mathrm{H}^2(X, \mathbb{R})$. Assume that $\dim X_{\mathrm{sg}} \leqslant 1$. Then the semiuniversal locally trivial deformation space $\mathrm{Def}^{\mathrm{lt}}(X)$ is smooth, unless possibly if X is projective.*

We remark that the assumption on the codimension of the singular locus is satisfied e.g. if X has terminal singularities. In all cases where Theorem 1.1 does not apply, the projectivity of X comes from the vanishing of global holomorphic 2-forms. Since the latter property is stable under locally trivial deformations, we may draw the following consequence.

COROLLARY 1.2 (Kodaira problem in dimension four). — *Let X be as in Theorem 1.1. Then the semiuniversal family $\mathfrak{X} \to \mathrm{Def}^{\mathrm{lt}}(X)$ is a strong locally trivial algebraic approximation. In particular, X is locally algebraic.*

This puts [9, Thm. H] in its final form: we remove both the local algebraicity assumption and the necessity to take a quasi-étale cover before obtaining an algebraic approximation of X. Note also that Corollary 1.2 confirms [9, Conj. K] in this particular case.

We now turn to fundamental groups. In [9, Thm. G], we showed that in dimension four $\pi_1(X)$ is virtually abelian. In particular, this group is "virtually projective", i.e. it contains a normal subgroup of finite index which is isomorphic to $\pi_1(Y)$ for some projective manifold Y. Corollary 1.3 below shows that passing to a subgroup is in fact not necessary. It also implies that π_1 of any Kähler fourfold of Kodaira dimension zero admitting a good minimal model is projective. This gives some new evidence towards the conjecture that every Kähler group is projective [2, (1.26)].

Campana's Abelianity Conjecture [8, Conj. 7.3] also makes similar predictions for the fundamental group of the smooth locus $\pi_1(X_{\mathrm{reg}})$, but this is currently out of reach even for X projective. Using Corollary 1.2, we can at least confirm the conjecture for the profinite completion $\widehat{\pi}_1(X_{\mathrm{reg}})$.

Corollary 1.3 (Fundamental groups). — *Let X be as in Theorem 1.1. Then:*

(1) *The fundamental group $\pi_1(X)$ is projective.*
(2) *The algebraic fundamental group of the smooth locus $\widehat{\pi}_1(X_{\mathrm{reg}})$ is virtually abelian, and finite if $\widetilde{q}(X) = 0$.*

Finally, we have the decomposition theorem mentioned in the title. For the definition of (singular) Calabi–Yau and irreducible holomorphic symplectic varieties, we refer to [9, Def. 6.11].

Corollary 1.4 (BB decomposition). — *Let X be as in Theorem 1.1. Then some quasi-étale cover of X splits as a product of a complex torus, Calabi–Yau and irreducible holomorphic symplectic varieties.*

The unitarily flat factor of $\mathscr{T}_X$

Let us briefly comment on the proof of Theorem 1.1. It relies on an analysis of the unitarily flat factor $\mathscr{F}$ in the holonomy decomposition of $\mathscr{T}_X$. To explain this, and also to fix notation, recall that in [9, Thm. C] we proved in particular the following.

Setup and Notation 1.5 (Standard setting). — Let X be a normal compact Kähler space with klt singularities such that $\mathrm{c}_1(X) = 0 \in \mathrm{H}^2(X, \mathbb{R})$.

Then after replacing X by a finite quasi-étale cover, the so-called *holonomy cover*, the tangent sheaf of X decomposes as

$$\mathscr{T}_X = \mathscr{F} \oplus \bigoplus_{k \in K} \mathscr{E}_k,$$

where the sheaves $\mathscr{F}$ and $\mathscr{E}_k$ satisfy the following:

- The restriction $\mathscr{F}\big|_{X_{\mathrm{reg}}}$ is unitarily flat, i.e. given by a representation $\pi_1(X_{\mathrm{reg}}) \to \mathrm{SU}(r)$, where $r = \mathrm{rk}(\mathscr{F})$.
- Each summand $\mathscr{E}_k\big|_{X_{\mathrm{reg}}}$ has full holonomy group either $\mathrm{SU}(n_k)$ or $\mathrm{Sp}(n_k/2)$, with respect to a suitable singular Ricci-flat metric. Here $n_k = \mathrm{rk}(\mathscr{E}_k) \geqslant 2$.

The natural conjecture concerning the unitarily flat factor $\mathscr{F}$ is that it corresponds to the torus factor in the (conjectural) Beauville–Bogomolov decomposition of X. It is convenient to rephrase this in different but equivalent ways:

- The rank of $\mathscr{F}$ should equal the augmented irregularity of X, i.e. $r = \widetilde{q}(X)$.
- If $\widetilde{q}(X)$ vanishes, then $\mathscr{F}$ should be the zero sheaf.

The conjecture follows from [3], but for the purpose of giving a logically independent proof of Theorem 1.1, the following partial result is key. It bounds the rank of $\mathscr{F}$ from above in terms of the dimension of the singular locus of X and in fact holds in arbitrary dimension.

Theorem 1.6 (Bounding the flat factor). — *In the standard setting 1.5, assume that the augmented irregularity of X vanishes, $\widetilde{q}(X) = 0$. Then:*

(1) *The unitarily flat factor $\mathscr{F}$ satisfies* $\mathrm{rk}(\mathscr{F}) \leqslant \dim X_{\mathrm{sg}}$.
(2) *If* $\dim X_{\mathrm{sg}} \leqslant 1$*, then* $\mathscr{F} = 0$.

Remark. — In (1), we need to adopt the convention that the empty set has dimension zero (as opposed to -1 or $-\infty$) in order for the conclusion to hold also if X is smooth.

The Lipman–Zariski Conjecture for direct summands

To prove Theorem 1.6, we propose and study a new variant of the Lipman–Zariski Conjecture, which we explain now. The classical Lipman–Zariski Conjecture (which is still open) states that if the tangent sheaf $\mathscr{T}_X$ of a complex algebraic variety or complex space is locally free, then X is smooth. Here we ask what happens if $\mathscr{T}_X$ is not necessarily locally free, but contains a locally free direct summand.

Question 1.7 (Lipman–Zariski Conjecture for direct summands). — *Let X be a complex space. Assume that the tangent sheaf of X admits a direct sum decomposition*

$$\mathscr{T}_X = \mathscr{E} \oplus \mathscr{F},$$

where $\mathscr{F}$ is locally free. Under what assumptions on the rank of $\mathscr{F}$ and on the singularities of X can we conclude that X is smooth?

Question 1.7 is formulated in a deliberately vague way. In this work, we will concentrate on the case of klt singularities, but other classes of singularities would be equally interesting.

Theorem 1.8 (Question 1.7 for klt singularities). — *Let X be a normal complex space with klt singularities such that the tangent sheaf admits a direct sum decomposition*

$$\mathscr{T}_X = \mathscr{E} \oplus \mathscr{F},$$

where $\mathscr{F}$ is locally free of rank r. If $\dim X_{\mathrm{sg}} \leqslant r - 1$, then X is smooth.

Easy examples show that the bound on r in Theorem 1.8 is sharp (Remark 3.2).

Acknowledgements

I would like to thank the anonymous referee for a quick but thorough review of my paper, and also for helping me to improve the exposition.

2. Notation and basic facts

Unless otherwise stated, complex spaces are assumed to be countable at infinity, separated, reduced and connected. Algebraic varieties and schemes are always assumed to be defined over the complex numbers.

Definition 2.1 (Torsion-free differentials). — *Let X be a reduced complex space and $p \in \mathbb{N}$ a non-negative integer. The sheaf of* torsion-free differential p-forms *on X is defined to be*

$$\check{\Omega}^p_X := \Omega^p_X \Big/ \operatorname{tor} \Omega^p_X,$$

where $\Omega^p_X := \bigwedge^p \Omega^1_X$ is the sheaf of Kähler differentials and $\operatorname{tor} \Omega^p_X$ is the subsheaf of Ω^p_X consisting of the sections vanishing on some dense open subset $U \subset X$. Equivalently, $\operatorname{tor} \Omega^p_X$ consists of those sections whose support is contained in the singular locus X_{sg}.

Definition 2.2 (Quasi-étale covers). — *A* cover *is a finite, surjective morphism $\gamma\colon Y \to X$ of normal, connected complex spaces. A cover γ is called* quasi-étale *if there exists a closed subset $Z \subset Y$ with* $\operatorname{codim}_Y(Z) \geqslant 2$ *such that $\gamma\big|_{Y\setminus Z}\colon Y \setminus Z \to X$ is étale.*

Definition 2.3. — *Let X be a normal complex space. A* maximally quasi-étale cover *of X is a quasi-étale Galois cover $\gamma\colon Y \to X$ satisfying the following equivalent conditions:*

(1) *Any étale cover of Y_{reg} extends to an étale cover of Y.*
(2) *Any quasi-étale cover of Y is étale.*
(3) *The natural map of étale fundamental groups $\widehat{\pi}_1(Y_{\mathrm{reg}}) \to \widehat{\pi}_1(Y)$ induced by the inclusion $Y_{\mathrm{reg}} \hookrightarrow Y$ is an isomorphism.*

Definition 2.4 (Irregularity). — *The* irregularity *of a compact complex space X is $q(X) \coloneqq h^1(Y, \mathscr{O}_Y)$, where $Y \to X$ is any resolution of singularities. The* augmented irregularity *of X is*

$$\widetilde{q}(X) \coloneqq \max\left\{ q\big(\widetilde{X}\big) \;\Big|\; \widetilde{X} \to X \text{ quasi-étale} \right\} \in \mathbb{N}_0 \cup \{\infty\}.$$

Vector fields

Let X be a reduced complex space. A *vector field* on X is a (local) section of the *tangent sheaf* $\mathscr{T}_X \coloneqq \mathscr{H}om\big(\Omega_X^1, \mathscr{O}_X\big)$, where Ω_X^1 is the sheaf of Kähler differentials.

Let v be a vector field. For any point $x \in X$, the germ of v at x is a $\mathbb{C}$-linear derivation $v_x\colon \mathscr{O}_{X,x} \to \mathscr{O}_{X,x}$ and hence it can be restricted to an element of the Zariski tangent space of X at x:

$$v(x) \in \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^{\vee} \eqqcolon T_x X.$$

Note, however, that in general not every Zariski tangent vector at x is of the form $v(x)$ for some local vector field v.

Lemma 2.5. — *Let $Z \subset X$ be an analytic subset that is fixed by every local automorphism of X. Then $v(z) \in T_z Z \subset T_z X$ for every vector field v on X defined near $z \in Z$.*

Proof. — We will use the correspondence between derivations, vector fields and local $\mathbb{C}$-actions as described in [1, §1.4, §1.5]. The vector field v induces a local $\mathbb{C}$-action $\Phi\colon \mathbb{C} \times X \to X$. By the definition of local group action, $\Phi(t, -)$ is an automorphism of germs $(X, z) \xrightarrow{\sim} \big(X, \Phi(t, z)\big)$ for every sufficiently small $t \in \mathbb{C}$. It then follows from the assumption that $\Phi(t, z) \in Z$

for every $t \in \mathbb{C}$. Now, we can recover the derivation δ corresponding to v from Φ by the formula

$$\delta(f)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f\big(\Phi(t,x)\big) \tag{2.1}$$

for every $f \in \mathscr{O}_{X,z}$. Plugging the above statement into (2.1), we see that δ stabilizes the ideal of Z, i.e. $\delta(I_{Z,z}) \subset I_{Z,z}$. Hence δ induces a derivation of $\mathscr{O}_{Z,z}$ and then also an element of $T_z Z$. □

Example. — Let $X = \mathbb{C}^2$ and let $0 \in C \subset X$ be a curve such that 0 is a singular point of C. If v is a vector field on X that is tangent to $C \setminus \{0\}$, then its local flows restrict to automorphisms of C. These automorphisms necessarily fix the singular point $0 \in C_{\mathrm{sg}}$ and hence v vanishes at the origin, $v(0) = 0$.

Deformation theory

This is just a very quick reminder. For more details, see for example [15].

Definition 2.6 (Deformations of complex spaces). — *A* deformation *of a (reduced) compact complex space X is a proper flat morphism $\pi\colon \mathfrak{X} \to (S,0)$ from a (not necessarily reduced) complex space $\mathfrak{X}$ to a complex space germ $(S,0)$, equipped with a fixed isomorphism $\mathfrak{X}_0 := \pi^{-1}(0) \cong X$.*

Definition 2.7 (Algebraic approximations). — *Let X be a compact complex space and $\pi\colon \mathfrak{X} \to S$ a deformation of X. Consider the set of projective fibres*

$$S^{\mathrm{alg}} := \big\{ s \in S \bigm| \mathfrak{X}_s \text{ is projective} \big\} \subset S$$

and its closure $\overline{S^{\mathrm{alg}}} \subset S$. We say that $\mathfrak{X} \to S$ is an algebraic approximation of *X if $0 \in \overline{S^{\mathrm{alg}}}$. We say that $\mathfrak{X} \to S$ is a* strong algebraic approximation of *X if $\overline{S^{\mathrm{alg}}} = S$ as germs, i.e. S^{alg} is dense near $0 \in S$.*

Definition 2.8 (Locally trivial deformations). — *A deformation $\pi : \mathfrak{X} \to S$ is called* locally trivial *if for every $x \in \mathfrak{X}_0$ there exist open subsets $0 \in S^\circ \subset S$ and $x \in U \subset \pi^{-1}(S^\circ)$ and an isomorphism*

$$\begin{array}{ccc} U & \xrightarrow{\ \sim\ } & (\mathfrak{X}_0 \cap U) \times S^\circ \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\ & S^\circ. & \end{array}$$

3. The Lipman–Zariski Conjecture for direct summands

In this section, we prove Theorem 1.8 from the introduction and the following corollary. We then deduce Theorem 1.6.

Corollary 3.1 (Spaces with large flat summands). — *Let X be a normal complex space with klt singularities such that*

$$\mathscr{T}_X = \mathscr{E} \oplus \mathscr{F},$$

where $\mathscr{F}\big|_{X_{\mathrm{reg}}}$ is flat (in the differential-geometric sense) of rank r, i.e. given by a representation $\pi_1(X_{\mathrm{reg}}) \to \mathrm{GL}(r, \mathbb{C})$. If $\dim X_{\mathrm{sg}} \leqslant r-1$, then X has only quotient singularities.

Remark 3.2 (Sharpness of Theorem 1.8). — The bound on $\dim X_{\mathrm{sg}}$ in Theorem 1.8 is sharp, as shown by the (easy) example $X = Y \times \mathbb{C}$, where Y is a (non-smooth) isolated klt singularity. In this case $\mathscr{T}_X$ has a rank one free summand and $\dim X_{\mathrm{sg}} = 1$, but X is not smooth.

Remark 3.3 (Reformulation of Theorem 1.8). — The conclusion of Theorem 1.8 could also be formulated in a somewhat oblique manner as follows: every irreducible component of X_{sg} has dimension at least r.

Remark 3.4 (Comparison to previous results). — The "usual" Lipman–Zariski Conjecture is well-known for spaces with klt, or even log canonical, singularities [11, 14]. If $r = n \coloneqq \dim X$ in Theorem 1.8, the statement reduces to this result.

In general, $\dim X_{\mathrm{sg}} = n-2$ and then the only case left where Theorem 1.8 applies is $r = n-1$. In this case, we may consider a (local) index one cover $X_1 \to X$. There, also the rank one sheaf $\mathscr{E}$ will become locally free, hence X_1 is smooth. Summing up, we see that [11, 14] only yield the weaker statement that X has quotient singularities (instead of being smooth).

3.1. Proof of Theorem 1.8

Let $n \coloneqq \dim X$. Assuming that X_{sg} is non-empty, we will derive a contradiction. Let $f\colon Y \to X$ be the functorial resolution. Pick an irreducible component $Z \subset X_{\mathrm{sg}}$ and a sufficiently general point $z \in Z$. Then Z is smooth at z, i.e. $z \in Z_{\mathrm{reg}}$. Let $\{E_i\}_{i \in I}$ be the set of exceptional divisors satisfying $f(E_i) \subset Z$. For any nonempty subset $J \subset I$, denote $E_J \coloneqq \bigcap_{i \in J} E_i \subset Y$. Note that each E_J is smooth of dimension $n - |J|$ (or empty). Hence by Generic Smoothness, for any J with $f(E_J) = Z$, the restriction $f\big|_{E_J}\colon E_J \to Z$ is smooth of relative dimension $n - |J| - \dim Z$ near

z. Consider the fibre $F := f^{-1}(z) \subset Y$ and its decomposition into irreducible components $F = \bigcup_{\lambda \in \Lambda} F_\lambda$. As before, for any nonempty subset $\Lambda' \subset \Lambda$ denote $F_{\Lambda'} := \bigcap_{\lambda \in \Lambda'} F_\lambda$. By the above observation, each $F_{\Lambda'}$ is smooth of dimension $\dim F + 1 - |\Lambda'|$, where $\dim F = n - 1 - \dim Z$. In other words, all components of F are smooth and intersect transversely of the correct codimension. Therefore F will be a simple normal crossings variety, albeit not necessarily a divisor in Y. (See also [28, Prop. 4.0.4] for a similar, but somewhat stronger statement.) Shrinking X around z, we may without loss of generality make the following

Additional Assumption 3.5. — The singular locus Z of X is smooth. The sheaf $\mathscr{F}$ is free, isomorphic to $\mathscr{O}_X^{\oplus r}$.

Let $\{v_1, \dots, v_r\}$ be a basis of $\mathscr{F}$ and let $\{\alpha_1, \dots, \alpha_r\}$ be the dual basis of $\mathscr{F}^\vee$, defined by $\alpha_i(v_j) = \delta_{ij}$. Since $\mathscr{F} \subset \mathscr{T}_X$ is a direct summand, so is $\mathscr{F}^\vee \subset \Omega_X^{[1]}$. This enables us to consider the sections α_i as reflexive 1-forms on X. Evaluating the vector fields v_i at the point z and taking into account that Z is stabilized by their flows, we obtain $v_i(z) \in T_z Z$. Since $\dim T_z Z = \dim Z \leqslant r - 1$, by Lemma 2.5 there is a non-trivial relation

$$\sum_{i=1}^{r} \lambda_i v_i(z) = 0 \in T_z Z \subset T_z X, \qquad \lambda_i \in \mathbb{C}. \tag{3.1}$$

Some coefficient in (3.1), say λ_1, will be non-zero. Replacing v_1 by $\sum_{i=1}^r \lambda_i v_i$, we arrive at the

Additional Assumption 3.6. — The free sheaf $\mathscr{F}$ has a basis $\{v_1, \dots, v_r\}$ with the property that $v_1(z) = 0$.

This means that z is stabilized by the flow of v_1. Let $\widetilde{v}_1$ be the lift of v_1 to Y. Then the flow of $\widetilde{v}_1$ stabilizes the fibre $F = f^{-1}(z)$, i.e. $\widetilde{v}_1$ restricts to a vector field on F. The same is then true of any irreducible component of F. Fix one such component $P \subset F$, and note that P is smooth because F is an snc variety.

On the other hand, let $\widetilde{\alpha}_1$ be the lift of α_1 to Y, which is a holomorphic 1-form by [23]. Since the function $\widetilde{\alpha}_1(\widetilde{v}_1)$ is identically one, the restricted form $\widetilde{\alpha}_1\big|_P$ cannot be zero. Indeed, if $p \in P$ is arbitrary then $\widetilde{\alpha}_1$ evaluated on $\widetilde{v}_1(p) \in T_p P$ is non-zero. On the other hand, the restriction map factors as

$$\mathrm{H}^0\big(Y, \Omega_Y^1\big) \longrightarrow \underbrace{\mathrm{H}^0\big(F, \check{\Omega}_F^1\big)}_{=0} \longrightarrow \mathrm{H}^0\big(P, \Omega_P^1\big) \tag{3.2}$$

because P is smooth and not contained in the singular locus of F. The middle term vanishes due to [17, Cor. 1.5] and [22, Thm. 4.1]. Hence (3.2) implies

$\widetilde{\alpha}_1\big|_P = 0$, contradicting our previous observation that this form is non-zero and thus ending the proof. □

3.2. Proof of Corollary 3.1

Since the problem is local, we may assume X to be a germ. We argue by induction on $\dim X_{\mathrm{sg}}$. If $X_{\mathrm{sg}} = \emptyset$, there is nothing to show. Otherwise, by [9, Prop. 5.8] there exists an open subset $X_{\mathrm{reg}} \subset X^\circ \subset X$ admitting a maximally quasi-étale cover $\gamma^\circ \colon Y^\circ \to X^\circ$ and satisfying $\dim(X \setminus X^\circ) \leqslant \dim X_{\mathrm{sg}} - 1$. We may extend γ° to a quasi-étale cover $\gamma \colon Y \to X$, by [10, Thm. 3.4].

$$\begin{array}{ccc} Y^\circ & \hookrightarrow & Y \\ {\scriptstyle \gamma^\circ}\downarrow & & \downarrow{\scriptstyle \gamma} \\ X^\circ & \hookrightarrow & X \end{array}$$

Note that Y reproduces all assumptions of Corollary 3.1. In particular, taking the reflexive pullback of the decomposition $\mathscr{T}_X = \mathscr{E} \oplus \mathscr{F}$, we get

$$\mathscr{T}_Y = \mathscr{E}_Y \oplus \mathscr{F}_Y$$

where $\mathscr{F}_Y\big|_{Y_{\mathrm{reg}}}$ is flat. Restricting $\mathscr{F}_Y$ to Y°_{reg} and using property (3) combined with [25], it follows that $\mathscr{F}_Y\big|_{Y^\circ}$ is flat, in particular locally free. Thus Y° is smooth thanks to Theorem 1.8. This implies

$$\dim Y_{\mathrm{sg}} \leqslant \dim(Y \setminus Y^\circ) = \dim(X \setminus X^\circ) \leqslant \dim X_{\mathrm{sg}} - 1.$$

By the induction hypothesis, Y has only quotient singularities and hence so does X. □

3.3. Proof of Theorem 1.6

Recall that $\mathscr{F}$ is the unitarily flat factor of the tangent sheaf of X (or rather, of its holonomy cover). Assume that $\mathrm{rk}(\mathscr{F}) \geqslant \dim X_{\mathrm{sg}} + 1$. By Corollary 3.1, the space X has only quotient singularities. In particular, it is locally algebraic and hence $\mathrm{rk}(\mathscr{F}) = \widetilde{q}(X) = 0$ by [9, Thm. C]. This contradiction proves (1).

For (2), we only need to exclude the case that $\mathrm{rk}(\mathscr{F}) = 1$. In this case, $\mathscr{F}\big|_{X_{\mathrm{reg}}}$ is given by a representation $\pi_1(X_{\mathrm{reg}}) \to \mathrm{SU}(1) = \{1\}$, so it is the trivial line bundle. By reflexivity, this implies $\mathscr{F} \cong \mathscr{O}_X$. In particular, $\mathrm{H}^0(X, \mathscr{F}^\vee) \neq 0$. Since $\mathscr{F}^\vee \subset \Omega_X^{[1]}$, this contradicts the assumption that $\widetilde{q}(X) = 0$. □

4. Proof of main results

4.1. Torus covers revisited

In the standard setting 1.5, recall the notion of *torus cover* from [9, Thm. B]: this is a quasi-étale cover $\gamma\colon T \times Z \to X$, where T is a complex torus of dimension $\widetilde{q}(X)$, while Z satisfies $\omega_Z \cong \mathscr{O}_Z$ as well as $\widetilde{q}(Z) = 0$. We do not know if the map γ can always be chosen to be Galois. Indeed, this is not obvious from the construction and taking the Galois closure of a given γ might destroy the splitting property. The following weaker statement is however sufficient for our purposes:

Proposition 4.1 (Torus covers revisited). — *In the standard setting 1.5, the torus cover can be chosen in such a way that it is a composition of quasi-étale Galois morphisms:*

$$T \times Z \xrightarrow{\text{Galois}} X' \xrightarrow{\text{Galois}} X.$$

Proof. — By [9, Cor. 4.2 (1)], the augmented irregularity $\widetilde{q}(X)$ is finite. Consider the index one cover $X_1 \to X$ and choose a quasi-étale cover $X_2 \to X_1$ with $q(X_2) = \widetilde{q}(X)$. Replacing $X_2 \to X$ by its Galois closure $X' \to X$ yields the first map in the statement to be proven. Cf. [9, Lem. 2.8] for the existence of Galois closures in the analytic category.

For the second map, we know from [9, Thm. 4.1] that the Albanese map $X' \to A := \operatorname{Alb}(X')$ becomes trivial after a finite étale base change $A_1 \to A$. Note that the latter map is automatically Galois because $\pi_1(A)$ is abelian.

$$\begin{array}{ccc} F \times A_1 & \longrightarrow & X' \\ \downarrow & & \downarrow \\ A_1 & \longrightarrow & A. \end{array}$$

Furthermore, $F \times A_1 \to X'$ is Galois, being the pullback of the Galois morphism $A_1 \to A$ along $X' \to A$. We now set $T := A_1$ and $Z := F$. The proof that this is indeed a torus cover of X is the same as in [9, proof of Cor. 4.2]. □

Lemma 4.2. — *Notation as above.*

(1) *The tangent space $T_0 \operatorname{Def}^{\mathrm{lt}}(T \times Z)$ can be calculated as follows:*

$$\mathrm{H}^1(T \times Z, \mathscr{T}_{T \times Z}) = \mathrm{H}^1(T, \mathscr{T}_T) \oplus \mathrm{H}^1(Z, \mathscr{T}_Z)\,.$$

(2) *If $\operatorname{Def}^{\mathrm{lt}}(Z)$ is smooth, then so is $\operatorname{Def}^{\mathrm{lt}}(T \times Z)$. More precisely, in this case $\operatorname{Def}^{\mathrm{lt}}(T \times Z) = \operatorname{Def}^{\mathrm{lt}}(T) \times \operatorname{Def}^{\mathrm{lt}}(Z)$.*

Proof. — Let $p\colon T \times Z \to T$ and $r\colon T \times Z \to Z$ be the projections. The proof is in a series of claims.

Claim 4.3. — The tangent sheaf of $T \times Z$ decomposes as

$$\mathscr{T}_{T\times Z} = p^* \mathscr{T}_T \oplus r^* \mathscr{T}_Z.$$

Proof. — Clearly the decomposition exists on the smooth locus of $T \times Z$. By reflexivity, it extends to a decomposition $\mathscr{T}_{T\times Z} = p^{[*]} \mathscr{T}_T \oplus r^{[*]} \mathscr{T}_Z$, where $p^{[*]} \mathscr{T}_T := (p^* \mathscr{T}_T)^{\vee\vee}$ and $r^{[*]} \mathscr{T}_Z$ denote the reflexive pullback. Hence it suffices to show that $p^* \mathscr{T}_T$ and $r^* \mathscr{T}_Z$ are already reflexive. For $p^* \mathscr{T}_T$, this is obvious because $\mathscr{T}_T$ is even (locally) free.

For $r^* \mathscr{T}_Z$, we use the characterization of reflexive sheaves as locally 2nd syzygy sheaves, [24, Ch. I, Lemma 1.1.16 and proof of Lemma 1.1.10](1). That is, on sufficiently small open sets $U \subset Z$ there exists an exact sequence

$$0 \longrightarrow \mathscr{T}_Z\big|_U \longrightarrow \mathscr{O}_U^{\oplus n} \longrightarrow \mathscr{O}_U^{\oplus m}.$$

The sequence stays exact when pulled back along the flat morphism r, showing that also $r^* \mathscr{T}_Z\big|_{T\times U}$ is reflexive. Of course, this argument shows quite generally that the pullback of a reflexive sheaf via a flat map remains reflexive. □

Claim 4.4. — For any $i \geqslant 0$, we have

$$R^i p_* p^* \mathscr{T}_T = \mathscr{T}_T \otimes R^i p_* \mathscr{O}_{T\times Z}$$

and

$$R^i r_* r^* \mathscr{T}_Z = \mathscr{T}_Z \otimes R^i r_* \mathscr{O}_{T\times Z}.$$

Proof. — For $\mathscr{T}_T$, this is simply the projection formula. Regarding $\mathscr{T}_Z$, some care is required because that sheaf is not locally free. But $R^i r_* \mathscr{O}_{T\times Z}$ is locally free. Even better, this is the trivial vector bundle with fibre $\mathrm{H}^i(T, \mathscr{O}_T)$. So we need to check that for any Stein open subset $U \subset Z$,

$$\mathrm{H}^i\big(r^{-1}(U), r^* \mathscr{T}_Z\big) = \mathrm{H}^0(U, \mathscr{T}_Z) \otimes_{\mathbb{C}} \mathrm{H}^i(T, \mathscr{O}_T)\,. \tag{4.1}$$

(1) The cited reference assumes the underlying space to be smooth, but the arguments work verbatim for sheaves on normal complex spaces.

But $r^{-1}(U) = T \times U$, so the Künneth formula [19, Thm. I] tells us that the left-hand side equals[(2)]

$$\bigoplus_{k+\ell=i} \mathrm{H}^k(U, \mathscr{T}_Z) \otimes_{\mathbb{C}} \mathrm{H}^\ell(T, \mathscr{O}_T) = \mathrm{H}^0(U, \mathscr{T}_Z) \otimes_{\mathbb{C}} \mathrm{H}^i(T, \mathscr{O}_T) \tag{4.2}$$

since the higher cohomology groups on the Stein space U vanish. Claim 4.4 now follows by comparing (4.1) and (4.2). □

By Claim 4.3, statement (1) is reduced to the following claim.

Claim 4.5. — $\mathrm{H}^1(T \times Z, p^* \mathscr{T}_T) = \mathrm{H}^1(T, \mathscr{T}_T)$ and $\mathrm{H}^1(T \times Z, r^* \mathscr{T}_Z) = \mathrm{H}^1(Z, \mathscr{T}_Z)$.

Proof. — Concerning the first factor, the Leray spectral sequence combined with Claim 4.4 gives

$$0 \longrightarrow \mathrm{H}^1(T, \mathscr{T}_T) \longrightarrow \mathrm{H}^1(T \times Z, p^* \mathscr{T}_T) \longrightarrow \mathrm{H}^0\big(T, \mathscr{T}_T \otimes R^1 p_* \mathscr{O}_{T\times Z}\big),$$

where the last term vanishes because $q(Z) = 0$. We obtain that

$$\mathrm{H}^1(T \times Z, p^* \mathscr{T}_T) = \mathrm{H}^1(T, \mathscr{T}_T).$$

For the second factor, again by Claim 4.4 we have a similar exact sequence

$$0 \longrightarrow \mathrm{H}^1(Z, \mathscr{T}_Z) \longrightarrow \mathrm{H}^1(T \times Z, r^* \mathscr{T}_Z) \longrightarrow \mathrm{H}^0\big(Z, \mathscr{T}_Z \otimes R^1 r_* \mathscr{O}_{T\times Z}\big)$$

but here $R^1 r_* \mathscr{O}_{T\times Z}$ is a trivial vector bundle (of rank equal to $\dim T$). To conclude as before, we therefore need to know that $\mathrm{H}^0(Z, \mathscr{T}_Z) = 0$. This can be seen as follows, where $m = \dim Z$ and $\widetilde{Z} \to Z$ is a resolution:

$$\begin{aligned}
h^0(Z, \mathscr{T}_Z) &= h^0\Big(Z, \Omega_Z^{[m-1]}\Big) && \text{contraction and } \omega_Z \cong \mathscr{O}_Z \\
&= h^0\Big(\widetilde{Z}, \Omega_{\widetilde{Z}}^{m-1}\Big) && \text{[23, Cor. 1.8]} \\
&= h^{m-1}\Big(\widetilde{Z}, \mathscr{O}_{\widetilde{Z}}\Big) && \text{Hodge theory on } \widetilde{Z} \\
&= h^{m-1}(Z, \mathscr{O}_Z) && Z \text{ has rational singularities} \\
&= h^1(Z, \mathscr{O}_Z) && \text{Serre duality [5, Ch. VII, Thm. 3.10]} \\
&= 0 && \text{because } q(Z) = \widetilde{q}(Z) = 0.
\end{aligned}$$

This ends the proof of Claim 4.5. □

---

(2) The assumptions of [19, Thm. I] are satisfied: $\mathscr{T}_U$ and $\mathscr{O}_T$ being Fréchet-n-sheaves for all $n \geqslant 0$ means that their cohomology groups are Hausdorff. This is clear except for $\mathrm{H}^0(U, \mathscr{T}_U)$, since these are finite-dimensional. But $\mathrm{H}^0(U, \mathscr{T}_U)$ is also Hausdorff because any coherent analytic sheaf is a Fréchet sheaf [21, Thm. 55.5]. Concerning the conclusion, the Fréchet space tensor product considered by Kaup boils down to the usual algebraic one, the cohomology groups of the compact space T being finite-dimensional. Also, by [20, Thm. III] the Fréchet sheaf tensor product $\mathscr{F} \varepsilon \mathscr{G}$ actually is the usual one for coherent analytic sheaves.

For (2), assume that $\mathrm{Def}^{\mathrm{lt}}(Z)$ is smooth. Recall that $\mathrm{Def}^{\mathrm{lt}}(T)$ is smooth in any case because T is a complex torus. Hence also $B \coloneqq \mathrm{Def}^{\mathrm{lt}}(T) \times \mathrm{Def}^{\mathrm{lt}}(Z)$ is smooth. Consider the product deformation of $T \times Z$ over B, i.e. the fibre over a point $(t, s) \in B$ is $T_t \times Z_s$. The Kodaira–Spencer map of this deformation,

$$\kappa \colon T_0 B \longrightarrow \mathrm{H}^1(T \times Z, \mathscr{T}_{T \times Z}),$$

is an isomorphism by (1). In particular, it is surjective and this implies by Lemma 4.6 below that $\mathrm{Def}^{\mathrm{lt}}(T \times Z)$ is smooth. Furthermore, our deformation over B is pulled back from the semiuniversal deformation via a map

$$B \longrightarrow \mathrm{Def}^{\mathrm{lt}}(T \times Z),$$

which on tangent spaces induces the isomorphism κ. Hence the map itself is likewise an isomorphism. This ends the proof. □

Lemma 4.6. — *Let $(x \in X), (y \in Y)$ be germs of complex spaces and $f \colon X \to Y$ a holomorphic map (i.e. $f(x) = y$). Assume that X is smooth and that the induced map on Zariski tangent spaces $\mathrm{d}_x f \colon T_x X \to T_y Y$ is surjective. Then Y is smooth.*

Proof. — Choose a closed embedding $i \colon (y \in Y) \hookrightarrow (0 \in \mathbb{C}^N)$, where $N = \dim T_y Y$. Consider the induced map $g = i \circ f \colon X \to \mathbb{C}^N$. This is a submersion of complex manifolds, hence surjective (as germs). In particular, i is surjective. Since i is an embedding, it is an isomorphism and in particular Y is smooth. □

4.2. Proof of Theorem 1.1

Keeping notation, the smoothness of $\mathrm{Def}^{\mathrm{lt}}(Z)$ implies the smoothness of $\mathrm{Def}^{\mathrm{lt}}(X)$ by Lemma 4.2, Proposition 4.1 and [15, Prop. 5.4]. We therefore only need to show that $\mathrm{Def}^{\mathrm{lt}}(Z)$ is smooth. Note that $\dim Z_{\mathrm{sg}} \leqslant \dim X_{\mathrm{sg}} \leqslant 1$ by assumption.

If $\dim Z \leqslant 2$, then Z has only quotient singularities and hence the claim follows directly from [15, Cor. 1.7]. If $\dim Z \in \{3, 4\}$, we apply Theorem 1.6 to conclude that the unitarily flat factor of $\mathscr{T}_Z$ vanishes, $\mathscr{F} = 0$. Note that here we have implicitly replaced Z by its holonomy cover $\widetilde{Z}$, but this is harmless by [15, Prop. 5.4] again because $\widetilde{Z} \to Z$ can be taken to be Galois and hence smoothness of $\mathrm{Def}^{\mathrm{lt}}(\widetilde{Z})$ implies smoothness of $\mathrm{Def}^{\mathrm{lt}}(Z)$. Consequently, there are only four possibilities for the holonomy decomposition of $\mathscr{T}_Z$. In slightly abused notation, they are:

- $\mathrm{SU}(3)$,
- $\mathrm{SU}(4)$,

- $\mathrm{Sp}(2)$, and
- $\mathrm{SU}(2) \oplus \mathrm{SU}(2)$.

In the first two cases, Z is projective: we have $\mathrm{H}^0\big(Z, \Omega_Z^{[2]}\big) = 0$ by the Bochner principle [9, Thm. A], hence also $\mathrm{H}^2(Z, \mathscr{O}_Z) = 0$ by the same argument as in the proof of Claim 4.5 (pass to a resolution $\widetilde{Z} \to Z$ and use Hodge symmetry on $\widetilde{Z}$). The projectivity of Z now follows from Kodaira's Embedding Theorem as explained e.g. in [13, Prop. 4.10]. Since the torus factor T, being of dimension $\leqslant 1$, is likewise projective, also X is projective. But we have excluded this situation.

In the last two cases, Z carries a holomorphic symplectic form since $\mathrm{SU}(2) = \mathrm{Sp}(1)$. This implies smoothness of $\mathrm{Def}^{\mathrm{lt}}(Z)$ by [4, Thm. 4.7], cf. also the footnote in the proof of [9, Thm. 8.4]. □

4.3. Proof of Corollary 1.2

Again let $T \times Z \to X$ be a torus cover. If Theorem 1.1 applies, then $\mathrm{Def}^{\mathrm{lt}}(X)$ is smooth and Corollary 1.2 follows directly from [15, Thm. 1.2]. The only case where Theorem 1.1 does not apply is if the holonomy cover of Z is $\mathrm{SU}(3)$ or $\mathrm{SU}(4)$. But in this case, $\mathrm{H}^2(Z, \mathscr{O}_Z) = 0$ as we explained in the proof of Theorem 1.1 above. Since also $\mathrm{H}^2(T, \mathscr{O}_T) = 0$ for dimension reasons, we conclude from the Künneth formula that $\mathrm{H}^2(T \times Z, \mathscr{O}_{T\times Z}) = 0$. This in turn implies $\mathrm{H}^2(X, \mathscr{O}_X) = 0$. This Hodge number is constant in locally trivial families, as one can see e.g. by performing a simultaneous resolution and using that X has rational singularities. That is, every locally trivial deformation X_t of X still satisfies $\mathrm{H}^2(X_t, \mathscr{O}_{X_t}) = 0$. By Kodaira's Embedding Theorem (cf. proof of Theorem 1.1), X_t is projective for any t (including $X_t = X$). In particular, any locally trivial deformation of X is a strong algebraic approximation. □

4.4. Proof of Corollaries 1.3 and 1.4

The projectivity of $\pi_1(X)$ follows from Thom's First Isotopy Lemma, which in our situation says that X and X_t are homeomorphic. For more details, cf. the proof of [13, Cor. 1.8]. Thus (1) is proved.

Regarding (2), we know from Corollary 1.2 that X is locally algebraic and hence it admits a maximally quasi-étale cover $\gamma\colon Y \to X$ by [9, Prop. 5.9]. By (3), the map $\widehat{\pi}_1(Y_{\mathrm{reg}}) \to \widehat{\pi}_1(Y)$ is an isomorphism. Also, $\widehat{\pi}_1(Y_{\mathrm{reg}}) =$

$\widehat{\pi}_1(\gamma^{-1}(X_{\mathrm{reg}})) \to \widehat{\pi}_1(X_{\mathrm{reg}})$ is injective with image of finite index. It is therefore sufficient to prove the claim for $\widehat{\pi}_1(Y_{\mathrm{reg}})$. But this reduces to [9, Thm. G] applied to Y.

We already remarked above that we now know that X is locally algebraic. Therefore Corollary 1.4 follows immediately from [9, Thm. H]. □

Bibliography

[1] D. N. Akhiezer, *Lie group actions in complex analysis*, Aspects of Mathematics, vol. E27, Vieweg & Sohn, 1995.

[2] J. Amorós, M. Burger, A. Corlette, D. Kotschick & D. Toledo, *Fundamental Groups of Compact Kähler Manifolds*, Mathematical Surveys and Monographs, vol. 44, American Mathematical Society, 1996.

[3] B. Bakker, H. Guenancia & C. Lehn, "Algebraic approximation and the decomposition theorem for Kähler Calabi–Yau varieties", *Invent. Math.* **228** (2022), p. 1255-1308.

[4] B. Bakker & C. Lehn, "The global moduli theory of symplectic varieties", 2020, `https://arxiv.org/abs/1812.09748v3`.

[5] C. Bănică & O. Stănăşilă, *Algebraic methods in the global theory of complex spaces*, John Wiley & Sons, 1976.

[6] A. Beauville, "Variétés Kähleriennes dont la première classe de Chern est nulle", *J. Differ. Geom.* **18** (1983), no. 4, p. 755-782.

[7] F. A. Bogomolov, "Hamiltonian Kählerian manifolds", *Dokl. Akad. Nauk SSSR* **243** (1978), no. 5, p. 1101-1104.

[8] F. Campana, "Orbifolds, special varieties and classification theory", *Ann. Inst. Fourier* **54** (2004), no. 3, p. 499-630.

[9] B. Claudon, P. Graf, H. Guenancia & P. Naumann, "Kähler spaces with zero first Chern class: Bochner principle, fundamental groups, and the Kodaira problem", *J. Reine Angew. Math.* **2022** (2022), no. 786, p. 245-275.

[10] G.-E. Dethloff & H. Grauert, "Seminormal complex spaces", in *Several complex variables VII. Sheaf-theoretical methods in complex analysis*, Encyclopaedia of Mathematical Sciences, vol. 74, Springer, 1994, p. 183-220.

[11] S. Druel, "The Zariski–Lipman conjecture for log canonical spaces", *Bull. Lond. Math. Soc.* **46** (2014), no. 4, p. 827-835.

[12] ———, "A decomposition theorem for singular spaces with trivial canonical class of dimension at most five", *Invent. Math.* **211** (2018), no. 1, p. 245-296.

[13] P. Graf, "Algebraic approximation of Kähler threefolds of Kodaira dimension zero", *Math. Ann.* **371** (2018), no. 1-2, p. 487-516.

[14] P. Graf & S. J. Kovács, "An optimal extension theorem for 1-forms and the Lipman–Zariski Conjecture", *Doc. Math.* **19** (2014), p. 815-830.

[15] P. Graf & M. Schwald, "The Kodaira problem for Kähler spaces with vanishing first Chern class", *Forum Math. Sigma* **9** (2021), article no. e24 (15 pages).

[16] D. Greb, H. Guenancia & S. Kebekus, "klt varieties with trivial canonical class: holonomy, differential forms, and fundamental groups", *Geom. Topol.* **23** (2019), no. 4, p. 2051-2124.

[17] C. D. Hacon & J. McKernan, "On Shokurov's rational connectedness conjecture", *Duke Math. J.* **138** (2007), no. 1, p. 119-136.

[18] A. Höring & T. Peternell, "Algebraic integrability of foliations with numerically trivial canonical bundle", *Invent. Math.* **216** (2019), no. 2, p. 395-419.

[19] L. Kaup, "Eine Künnethformel für Fréchetgarben", *Math. Z.* **97** (1967), p. 158-168.

[20] ———, "Das topologische Tensorprodukt kohärenter analytischer Garben", *Math. Z.* **106** (1968), p. 273-292.

[21] L. Kaup & B. Kaup, *Holomorphic functions of several variables. An introduction to the fundamental theory*, De Gruyter Studies in Mathematics, vol. 3, Walter de Gruyter, 1983, with the assistance of Gottfried Barthel, translated from the German by Michael Bridgland.

[22] S. Kebekus, "Pull-back morphisms for reflexive differential forms", *Adv. Math.* **245** (2013), p. 78-112.

[23] S. Kebekus & C. Schnell, "Extending holomorphic forms from the regular locus of a complex space to a resolution of singularities", *J. Am. Math. Soc.* **34** (2021), no. 2, p. 315-368.

[24] C. Okonek, M. Schneider & H. Spindler, *Vector bundles on complex projective spaces*, Progress in Mathematics, vol. 3, Birkhäuser, 1980.

[25] V. P. Platonov, "A certain problem for finitely generated groups", *Dokl. Akad. Nauk SSSR* **12** (1968), p. 492-494.

[26] G. Tian, "Smoothness of the universal deformation space of compact Calabi–Yau manifolds and its Petersson–Weil metric", in *Mathematical aspects of string theory (San Diego, 1986)*, Advanced Series in Mathematical Physics, vol. 1, World Scientific, 1987, p. 629-646.

[27] A. N. Todorov, "The Weil–Petersson geometry of the moduli space of SU($n \geqslant 3$) (Calabi–Yau) manifolds. I", *Commun. Math. Phys.* **126** (1989), no. 2, p. 325-346.

[28] J. Włodarczyk, "Equisingular resolution with snc fibers and combinatorial type of varieties", 2016, `https://arxiv.org/abs/1602.01535`.