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BENOÎT JUBIN

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## Intrinsic volumes of sublevel sets <sup>(\*)</sup>

BENOÎT JUBIN <sup>(1)</sup>

**ABSTRACT.** — We establish formulas that give the intrinsic volumes, or curvature measures or Lipschitz–Killing curvatures, of sublevel sets of functions defined on Riemannian manifolds as integrals of functionals of the function and its derivatives up to order 3. For instance, in the Euclidean case, if  $f \in \mathcal{C}^3(\mathbb{R}^n, \mathbb{R})$  and 0 is a regular value of  $f$ , then the intrinsic volume of degree  $n - k$  of the sublevel set  $M_f^0 := f^{-1}([-\infty, 0])$ , if the latter is compact, is given by

$$\mathcal{L}_{n-k}(M_f^0) = \frac{\Gamma(k/2)}{2\pi^{k/2}(k-1)!} \int_{M_f^0} \operatorname{div} \left( \frac{P_{n,k}(\operatorname{Hess}(f), \nabla f)}{\sqrt{f^{2(3k-2)} + \|\nabla f\|^{2(3k-2)}}} \nabla f \right) \operatorname{vol}_n$$

for  $1 \leq k \leq n$ , where the  $P_{n,k}$ 's are polynomials given in the text.

This includes as special cases the Euler–Poincaré characteristic of sublevel sets and the nodal volumes of functions defined on Riemannian manifolds. Therefore, these formulas give generalizations of the Kac–Rice formula.

Finally, we use these formulas to prove the Lipschitz continuity of the intrinsic volumes of sublevel sets.

**RÉSUMÉ.** — Nous établissons des formules donnant les volumes intrinsèques, ou mesures de courbure ou courbures de Lipschitz–Killing, des ensembles de sous-niveau de fonctions définies sur des variétés riemanniennes, comme intégrales de fonctionnelles de la fonction et de ses dérivées jusqu'à l'ordre 3. Par exemple, dans le cas euclidien, si  $f \in \mathcal{C}^3(\mathbb{R}^n, \mathbb{R})$  et 0 est une valeur régulière de  $f$ , alors le volume intrinsèque de degré  $n - k$  de l'ensemble de sous-niveau  $M_f^0 := f^{-1}([-\infty, 0])$ , lorsque ce dernier est compact, est donné par

$$\mathcal{L}_{n-k}(M_f^0) = \frac{\Gamma(k/2)}{2\pi^{k/2}(k-1)!} \int_{M_f^0} \operatorname{div} \left( \frac{P_{n,k}(\operatorname{Hess}(f), \nabla f)}{\sqrt{f^{2(3k-2)} + \|\nabla f\|^{2(3k-2)}}} \nabla f \right) \operatorname{vol}_n$$

pour  $1 \leq k \leq n$ , où les  $P_{n,k}$  sont des polynômes donnés dans l'article.

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<sup>(1)</sup> Sorbonne Université, Institut de Mathématiques de Jussieu, F-75005 Paris  
France — benoit.jubin@imj-prg.fr

Article proposé par Xavier Tolsa.

Ces formules incluent comme cas particuliers la caractéristique d’Euler–Poincaré des ensembles de sous-niveau et les volumes nodaux des fonctions définies sur des variétés riemanniennes. Elles peuvent donc être vues comme des généralisations de la formule de Kac–Rice.

Finalement, nous utilisons ces formules pour prouver la continuité de Lipschitz des volumes intrinsèques des ensembles de sous-niveau.

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## Introduction

Intrinsic volumes are geometric invariants attached to well-behaved subsets of Riemannian manifolds. They include the volume and the Euler–Poincaré characteristic. Among their applications in the field of integral geometry are Weyl’s tube formula ([22]), that gives the volumes of tubular neighborhoods of submanifolds, and the kinematic formula of Blaschke, Chern and Santaló ([7, 8]), that gives the average intrinsic volumes of the intersection of two submanifolds over all rigid motions of one of them. They were introduced in their modern form by Herbert Federer in the seminal article [9], where they are called curvature measures, after special cases in convex geometry were treated by Hermann Minkowski. Among the vast literature on their subject, we only mention the book [18], the survey on a related topic [21], and the articles [11, 12, 24].

In this article, we study the intrinsic volumes of sublevel sets of functions defined on Riemannian manifolds. These were already studied from the point of view of Morse theory in [10]. Since intrinsic volumes include the volume of the boundary, this study encompasses volumes of level sets, and in particular of zero sets, also called nodal sets. The first closed explicit formulas computing nodal volumes appeared in [2], which was a motivation for the present article. These formulas can be seen as generalizations of the Kac–Rice formula (see for instance [19]).

Sublevel sets are also studied in probability theory, where superlevel sets of random fields are called excursion sets; see for instance the books [1, 4] and the articles [2, 17, 23]. The importance of the formulas obtained in this paper for the study of random fields (as studied in [2]), compared to existing Kac–Rice formulas, stems from the fact that they are in closed form as opposed to being limits of integrals depending on a parameter.

## Main results

We now describe the contents of this article in more detail. In this introduction, we restrict ourselves to the flat case. In Section 1, we recall

the definition and main properties of intrinsic volumes. If  $N$  is a flat compact  $n$ -dimensional Riemannian manifold with boundary, they take the form  $\mathcal{L}_n(N) = \text{vol}(N)$  and

$$\mathcal{L}_{n-k}(N) = b_k \int_{\partial N} \text{tr} \left( \bigwedge^{k-1} S \right) \text{vol}_{\partial N} \quad (0.1)$$

for  $1 \leq k \leq n$ , where  $b_k \in \mathbb{R}$  and  $S$  is the second fundamental form of  $\partial N$  in  $N$ .

In Section 2, we specialize our study to the case where  $N$  is a sublevel set. Namely, let  $M$  be a flat  $n$ -dimensional Riemannian manifold (without boundary), let  $f \in \mathcal{C}^3(M, \mathbb{R})$ , and assume that  $a \in \mathbb{R}$  is a regular value of  $f$  and that the sublevel set  $M^a := f^{-1}([-\infty, a])$  is compact. The second fundamental form of  $\partial M^a$  in  $M$  can be expressed in terms of the gradient and the Hessian of  $f$ . An important lemma (Lemma 2.2) establishes that the above integrand is then a polynomial in  $\nabla f$  and  $\text{Hess}(f)$  divided by  $\|\nabla f\|^{2(k-1)}$ . We then use the divergence theorem to transform the above integral over  $\partial M^a$  into an integral over  $M^a$ . This leads to our main formula which, in the flat case, reads

$$\mathcal{L}_{n-k}(M^a) = b_k \int_{M^a} \text{div} \left( \frac{P_{n,k}(\text{Hess}(f), \nabla f)}{\sqrt{(f-a)^{2(3k-2)} + \|\nabla f\|^{2(3k-2)}}} \nabla f \right) \text{vol}_M \quad (0.2)$$

for  $1 \leq k \leq n$ , where the  $P_{n,k}$ 's are polynomials given in the text (Theorem 2.9). The main advantage of this formula is that it is an explicit integral over  $M^a$  (and not  $\partial M^a$ ) of a continuous functional in  $f$  and its derivatives up to order 3.

Since the intrinsic volume of degree  $n-1$  is half the volume of the boundary, this formula can be used to compute the volume of level sets. If the ambient manifold  $M$  is compact, one can use the intrinsic volume of either the sublevel or the superlevel set, yielding for the volume of the zero set  $\mathcal{Z}_f$  of  $f$  the formula

$$\text{vol}(\mathcal{Z}_f) = \frac{1}{2} \int_M \frac{\sigma_f}{\eta_f^3} (f \|\nabla f\|^2 + \text{Hess}(f)(\nabla f, \nabla f) - \eta_f^2 \Delta f) \text{vol}_M \quad (0.3)$$

where  $\sigma_f$  is the sign of  $f$  and  $\eta_f := \sqrt{f^2 + \|\nabla f\|^2}$ . This formula was obtained in the case of a flat torus in [2]. As in [2], one can do an integration by parts to eliminate  $\sigma_f$  and obtain an integral over  $M$  where the integrand is a Lipschitz continuous functional of  $f$  and its derivatives up to order 2 (Equation (2.38)). This regularity allows one to apply techniques of the Malliavin calculus to obtain results about the expected value, variance and higher moments of the nodal volumes of certain families of random fields (see [2]), and more generally of the intrinsic volumes of their excursion sets.

In Section 3, after recalling basic facts on natural topologies on  $\mathcal{C}^p(M, \mathbb{R})$ , the uniform and the (Whitney) strong  $\mathcal{C}^p$ -topologies, we prove that the condition of validity of our formulas (regularity of the considered value of the function) is generic. Then, we prove the continuity of intrinsic volumes of sublevel sets. For instance, if  $0 \leq k \leq n$ , then the function

$$\begin{aligned} \mathcal{L}_{n-k}^{\text{sub}} : \text{Reg}_{\text{pb}}^3(M, \mathbb{R})_U &\longrightarrow \mathbb{R} \\ (f, a) &\longmapsto \mathcal{L}_{n-k}(M_f^a) \end{aligned} \tag{0.4}$$

is Lipschitz continuous, where the domain is the set of couples  $(f, a)$  where  $f \in \mathcal{C}^3(M, \mathbb{R})$  is proper bounded below and  $a \in \mathbb{R}$  is a regular value of  $f$ , and is equipped with the uniform  $\mathcal{C}^3$ -topology (Theorem 3.9). In particular, the Euler–Poincaré characteristic of sublevel sets is locally constant.

### Conventions and notation

- If  $P$  is a proposition, then  $[P] := 1$  if  $P$  else 0.
- We denote by  $\text{pr}_i$  the projection on the  $i^{\text{th}}$  factor of a direct product.
- The bracket  $\lfloor - \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$  denotes the floor function.
- If  $a, b \in \mathbb{R}$ , then  $\llbracket a, b \rrbracket := \mathbb{Z} \cap [a, b]$ .
- If  $a \in \mathbb{R}$ , then  $\overline{\mathbb{N}}_{\geq a} := \{n \in \mathbb{N} \mid n \geq a\} \cup \{\infty\}$  and similarly for similar symbols.
- The symbol  $\odot$  (resp.  $\wedge$ ) denotes the symmetric (resp. exterior) product or power of vector spaces.
- Unless otherwise specified, manifolds are Hausdorff, paracompact, real, finite-dimensional, and smooth, that is, of class  $\mathcal{C}^\infty$ .
- The space of smooth sections of the vector bundle  $E \rightarrow M$  is denoted by  $\Gamma(E \rightarrow M)$  or  $\Gamma^{(p)}(E \rightarrow M)$  if the differentiability class  $p \in \overline{\mathbb{N}}$  need be specified. For instance, the metric tensor of a Riemannian manifold  $M$  is an element of  $\Gamma(\odot^2 T^*M \rightarrow M)$ .

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## 1. Intrinsic volumes

Let  $(M, g)$  be an  $n$ -dimensional compact Riemannian manifold with boundary. Its metric will also be denoted by  $\langle -, - \rangle$  and its associated norm

by  $\|-\|$ . This metric induces inner products on all tensor, symmetric and exterior powers of tangent and cotangent spaces. We denote by  $\nabla^{\text{LC}}$  its Levi-Civita connection. Let  $\text{vol}_M$  be the Riemannian density on  $M$  and  $\text{vol}_{\partial M}$  be the induced density on  $\partial M$ . The symbol  $\text{vol}$  will also denote the associated volume of (sub)manifolds.

Let  $R \in \Gamma(\odot^2 \wedge^2 T^*M \rightarrow M)$  be the covariant curvature tensor of  $M$ . Let  $S := ((\nabla^{\text{LC}} \nu|_{T\partial M})^T)^\flat \in \Gamma(\odot^2 T^*\partial M \rightarrow \partial M)$  be the second fundamental form of  $\partial M$  in  $M$ , where  $\nu \in \Gamma(TM|_{\partial M} \rightarrow \partial M)$  is the outward unit normal vectorfield on  $\partial M$  and  $(-)^T: TM|_{\partial M} \rightarrow T\partial M$  denotes the tangential component, and  $\flat: T\partial M \rightarrow T^*\partial M$  denotes the isomorphism induced by the metric. The symbol  $\text{tr}$  denotes the trace of a bilinear form on a Euclidean vector space.

For the exterior product of symmetric bilinear forms,  $\wedge: \odot^2 \wedge^p \mathbb{V} \times \odot^2 \wedge^q \mathbb{V} \rightarrow \odot^2 \wedge^{p+q} \mathbb{V}$ , also called in differential geometry the Kulkarni–Nomizu product, see for instance [9, §2].

For  $0 \leq k \leq n$ , the *intrinsic volume of degree  $n - k$  of  $M$*  is defined as

$$\begin{aligned} \mathcal{L}_{n-k}(M) &:= a_k \int_M \text{tr} \left( \wedge^{k/2} R \right) \text{vol}_M \\ &\quad + \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{k,m} \int_{\partial M} \text{tr} \left( \wedge^m R|_{\partial M} \wedge \wedge^{k-1-2m} S \right) \text{vol}_{\partial M} \end{aligned} \quad (1.1)$$

where

$$a_k := \frac{[k \text{ even}]}{(-2\pi)^{k/2} (k/2)!} \quad \text{for } 0 \leq k, \quad (1.2)$$

$$b_{k,m} := \frac{(-1)^m \Gamma(k/2 - m)}{2^{m+1} \pi^{k/2} m! (k-1-2m)!} \quad \text{for } 0 \leq m \leq \left\lfloor \frac{k-1}{2} \right\rfloor. \quad (1.3)$$

We also set

$$b_k := b_{k,0} = \frac{\Gamma(k/2)}{2\pi^{k/2} (k-1)!} \quad \text{for } 1 \leq k. \quad (1.4)$$

Note that  $b_k$  can also be written  $b_k = \frac{\Gamma(1+k/2)}{k! \pi^{k/2}} = (k! \text{vol}(B_k))^{-1}$  where  $B_k$  is the  $k$ -dimensional Euclidean unit ball.

*Remark 1.1.* — Formula (1.1) was obtained by specializing the general definition [1, Def. 10.7.2], which holds for Whitney stratified spaces of positive reach in Riemannian manifolds (including Riemannian manifolds with

corners), to the case of Riemannian manifolds with boundary.<sup>(1)</sup> The integrands, which are contractions of the curvature tensor and the second fundamental form, are called the *Lipschitz–Killing curvatures* of  $\partial M$  in  $M$ . More general versions  $\mathcal{L}_{n-k}(M, A)$  can be defined for Borel subsets  $A \subseteq M$  and are called *curvature measures* in  $M$ . The intrinsic volumes are the total measures of these curvature measures, that is,  $\mathcal{L}_{n-k}(M) = \mathcal{L}_{n-k}(M, M)$ .

One has

$$\mathcal{L}_n(M) = \text{vol}(M), \tag{1.5}$$

$$\mathcal{L}_{n-1}(M) = \frac{1}{2} \text{vol}(\partial M), \tag{1.6}$$

$$\mathcal{L}_0(M) = \chi(M). \tag{1.7}$$

The first two equalities immediately follow from  $a_0 = 1$  and from  $a_1 = 0$  and  $b_1 = \frac{1}{2}$  respectively (indeed,  $\text{tr}(\bigwedge^0 R_x)$  is the trace of the identity of  $\bigwedge^0 T_x^* M \simeq \mathbb{R}$ , which is 1, and similarly for  $\text{tr}(\bigwedge^0 S_x)$ ). The third equality is the Gauss–Bonnet–Chern theorem (see [5] for the original article, and [6, 20] for manifolds with boundary), where the right-hand side is the Euler–Poincaré characteristic of  $M$ , and in particular is an integer and is zero in the odd-dimensional boundaryless case.

Since  $a_2 = -b_2 = -(2\pi)^{-1}$ , one has

$$\mathcal{L}_{n-2}(M) = -\frac{1}{2\pi} \int_M \text{scal} \text{vol}_M + \frac{1}{2\pi} \int_{\partial M} (\text{tr } S) \text{vol}_{\partial M} \tag{1.8}$$

where  $\text{scal}$  denotes the scalar curvature of  $M$ . Note that  $\text{tr } S$  is  $(n-1)$  times the mean curvature of  $\partial M$  in  $M$ .

If  $M$  is flat, that is,  $R = 0$ , then Formula (1.1) simplifies, since only the summand corresponding to  $m = 0$  may be nonzero, giving

$$\mathcal{L}_{n-k}(M) = b_k \int_{\partial M} \text{tr} \left( \bigwedge^{k-1} S \right) \text{vol}_{\partial M} \tag{1.9}$$

for  $1 \leq k \leq n$ .

The trace of an exterior (actually, Kulkarni–Nomizu) power can be computed using minors of the matrix of  $S$  (see [1, (7.2.7)]). For the top power, this gives  $\text{tr}(\bigwedge^{n-1} S) = (n-1)! \det S$ . Therefore,

$$\chi(M) = b_n (n-1)! \int_{\partial M} (\det S) \text{vol}_{\partial M} \quad \text{if } n \geq 1, \tag{1.10}$$

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<sup>(1)</sup> The general formula of [1, Def. 10.7.2] takes a simpler form in the case of manifolds with boundary. In particular, Equation (1.2) above giving the values of the  $a_k$ 's agrees with [1, (7.6.1)]. As for the  $b_{k,m}$ 's, we recover [1, (10.7.5)] when  $m = 0$ .

where  $\det S$  is “the” Lipschitz–Killing, or Gauss–Kronecker, curvature of  $\partial M$  in  $M$ .

*Example 1.2.* — We compute as an example the intrinsic volumes of the  $n$ -dimensional Euclidean unit ball  $B_n$  using Equation (1.9). We have  $\mathcal{L}_{n-k}(B_n) = b_k \int_{\partial B_n} \text{tr}(\bigwedge^{k-1} S) \text{vol}_{\partial B_n}$ . Note that  $\text{vol}(\partial B_n) = n \text{vol}(B_n)$  and  $\text{tr}(\bigwedge^{k-1} S) = \frac{(n-1)!}{(n-k)!}$  by [1, (7.2.11)]. Therefore,  $\mathcal{L}_{n-k}(B_n) = \frac{n \text{vol}(B_n)}{k! \text{vol}(B_k)} \frac{(n-1)!}{(n-k)!} = \binom{n}{k} \frac{\text{vol}(B_n)}{\text{vol}(B_k)}$ , in agreement with [1, (6.3.7)] and [16, Thm. 9.2.4]. This is essentially the computation in [1, Footnote 20, p. 247].

## 2. Intrinsic volumes of sublevel sets

### 2.1. Sublevel sets and level sets

If  $f: M \rightarrow \mathbb{R}$  is a function on a set and  $a \in \mathbb{R}$ , then the  $a$ -sublevel set of  $f$  is defined by

$$M_f^a := f^{-1}([-\infty, a]) \tag{2.1}$$

also written  $M^a$  if there is no risk of confusion, and the  $a$ -level set of  $f$  is  $f^{-1}(a)$ .

Let  $M$  be a manifold, let  $p \in \overline{\mathbb{N}_{\geq 1}}$ , and let  $f \in \mathcal{C}^p(M, \mathbb{R})$ . The real number  $a \in \mathbb{R}$  is a *regular value* of  $f$  if  $f(x) = a$  implies  $\text{d}f(x) \neq 0$  for all  $x \in M$ . We define the sets

$$\text{Reg}^p(M, \mathbb{R}) := \{(f, a) \in \mathcal{C}^p(M, \mathbb{R}) \times \mathbb{R} \mid a \text{ is a regular value of } f\}, \tag{2.2}$$

$$\text{Reg}_c^p(M, \mathbb{R}) := \{(f, a) \in \text{Reg}^p(M, \mathbb{R}) \mid M_f^a \text{ is compact}\}. \tag{2.3}$$

We also set  $\mathcal{C}_{a\text{-reg}}^p(M, \mathbb{R}) := \text{pr}_1(\text{Reg}^p(M, \mathbb{R}) \cap (\mathcal{C}^p(M, \mathbb{R}) \times \{a\}))$  and similarly for  $\mathcal{C}_{a\text{-reg},c}^p(M, \mathbb{R})$ .

**PROPOSITION 2.1.** — *Let  $M$  be a manifold, let  $p \in \overline{\mathbb{N}_{\geq 1}}$ , and let  $(f, a) \in \text{Reg}^p(M, \mathbb{R})$ . Then,  $M_f^a$  is a full-dimensional  $\mathcal{C}^p$ -submanifold with boundary of  $M$ . Its manifold boundary, equal to its topological boundary, is the  $\mathcal{C}^p$ -hypersurface  $\partial M_f^a = f^{-1}(a)$ .*

*Proof.* — By the submersion theorem,  $f^{-1}(a)$  is a  $\mathcal{C}^p$ -hypersurface of  $M$ . By considering separately points  $x \in M$  such that  $f(x) < a$  and such that  $f(x) = a$ , one checks that  $M_f^a$  is a full-dimensional  $\mathcal{C}^p$ -submanifold with boundary of  $M$ , and that  $\partial M_f^a = f^{-1}(a)$ .  $\square$

## 2.2. Intrinsic volumes of sublevel sets

Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold (not necessarily compact, but without boundary). Let  $f \in \mathcal{C}^2(M, \mathbb{R})$ . Its *gradient* is defined by  $\nabla f := (df)^\sharp$ . Its *Hessian* is defined by  $\text{Hess}(f) := \nabla^{\text{LC}} df = (\nabla^{\text{LC}} \nabla f)^\flat$ . Its *Laplacian* is the trace of its Hessian,  $\Delta f := \text{tr}(\text{Hess}(f))$ .

Let  $(f, a) \in \text{Reg}^2(M, \mathbb{R})$ . By Proposition 2.1, the set  $M^a$  is a full-dimensional  $\mathcal{C}^2$ -submanifold with boundary of  $M$  and its boundary is the  $\mathcal{C}^2$ -hypersurface  $\partial M^a$ . The outward unit normal vectorfield of  $\partial M^a$  is  $\nu = \frac{\nabla f}{\|\nabla f\|}$ . Therefore, one has  $\nabla^{\text{LC}} \nu = \|\nabla f\|^{-1} \nabla^{\text{LC}} \nabla f + d\|\nabla f\|^{-1} \odot \nabla f$ , which has tangential component  $\|\nabla f\|^{-1} \nabla^{\text{LC}} \nabla f$ . Therefore, the second fundamental form of  $\partial M^a$  in  $M$  is given by

$$S = \frac{\text{Hess}(f)|_{\nabla f^\perp}}{\|\nabla f\|}. \quad (2.4)$$

We briefly explain the idea underlying the rest of this subsection. By Formula (2.4), the integrals in the sum on  $m$  in Formula (1.1) are equal to

$$\int_{\partial M^a} \|\nabla f\|^{2m+1-k} \text{tr} \left( \bigwedge^m R|_{\nabla f^\perp} \wedge \bigwedge^{k-1-2m} \text{Hess}(f)|_{\nabla f^\perp} \right) \text{vol}_{\partial M^a}. \quad (2.5)$$

We will convert these integrals on  $\partial M^a$  into integrals on  $M^a$  by using the divergence theorem. To do this, we need to find a vectorfield  $X \in \mathfrak{X}(M^a)$  such that

$$X|_{\partial M^a} = \nu = \frac{\nabla f}{\|\nabla f\|}$$

and  $\|\nabla f\|^{2m+1-k} \text{tr} \left( \bigwedge^m R|_{\nabla f^\perp} \wedge \bigwedge^{k-1-2m} \text{Hess}(f)|_{\nabla f^\perp} \right) X$

has a divergence which is integrable on  $M^a$ . Besides the boundary, the possibly problematic points are the points where  $\nabla f = 0$ , first because of the factor  $\|\nabla f\|^{2m+1-k}$ , and also because of the restriction to  $\nabla f^\perp$ . Since the two regions of interest are at  $f = a$  and at  $\nabla f = 0$ , it makes sense to look for a vectorfield of the form  $X = F \circ \frac{\nabla f}{a-f}$  where  $F \in \mathcal{C}^1(TM, TM)$  is fiber-preserving such that  $F(u) \sim_\infty \frac{u}{\|u\|}$  and  $F$  vanishes sufficiently fast at 0 for the divergence to be integrable.

We now make this idea precise.

**LEMMA 2.2.** — *For all  $n, k, m \in \mathbb{N}$  such that  $1 \leq k \leq n$  and  $0 \leq m \leq \lfloor \frac{k-1}{2} \rfloor$ , there exists a homogeneous polynomial  $P_{n,k,m}$  with integer coefficients such that for any  $n$ -dimensional Euclidean vector space with orthonormal*

basis  $(\mathbb{V}, \mathfrak{B})$ , any  $v \in \mathbb{V} \setminus \{0\}$ , and any symmetric bilinear forms  $H \in \odot^2 \mathbb{V}^*$  and  $R \in \odot^2 \wedge^2 \mathbb{V}^*$ , one has

$$\mathrm{tr} \left( \wedge^m R|_{v^\perp} \wedge \wedge^{k-1-2m} H|_{v^\perp} \right) = \frac{P_{n,k,m}((r_{ijkl}), (h_{ij}), (v_i))}{\|v\|^{2(k-1)}} \quad (2.6)$$

where  $(r_{ijkl})$  (resp.  $(h_{ij})$  and  $(v_i)$ ) are the coefficients of  $R$  (resp.  $H$  and  $v$ ) in  $\mathfrak{B}$ .

For the sake of definiteness, if  $\mathfrak{B} = (e_i)_{1 \leq i \leq n}$  is an orthonormal basis of  $\mathbb{V}$ , we consider the basis  $(e_i \wedge e_j)_{1 \leq i < j \leq n}$  of  $\wedge^2 \mathbb{V}$ . The coefficients of  $R$  can be written  $(r_{ijkl})_{1 \leq i, j, k, l \leq n}$  with  $i < j$  and  $k < l$  and  $(i, j) \leq (k, l)$  in the lexicographic order.

*Proof.* — Let  $n, k, m, (\mathbb{V}, \mathfrak{B}), v, H, R$  be as in the statement. Set  $a_i := \sqrt{\sum_{j=1}^i v_j^2}$  for  $1 \leq i \leq n$ . One has  $|v_1| = a_1 \leq \dots \leq a_n = \|v\|$ .

We first assume that  $v_1 \neq 0$ . Let  $U$  be the change of basis matrix defined by  $U_{i1} := v_i/\|v\|$ , and if  $j \geq 2$ , then  $U_{ij} := \beta_{ij}/(a_{j-1}a_j)$  with  $\beta_{ij} := -v_i v_j$  if  $i < j$ , or  $a_{j-1}^2$  if  $i = j$ , or 0 if  $j < i$ . It is an orthogonal matrix and the matrix  $U^{-1} H U$  restricted to the rows and columns  $2 \leq i, j \leq n$  is the matrix of  $H|_{v^\perp}$  in an orthonormal basis. Since the  $\beta_{ij}$ 's are polynomials in the  $v_k$ 's and  $a_k$ 's, one has

$$a_{i-1} a_i a_{j-1} a_j (U^{-1} H U)_{ij} = \sum_{k=1}^n \sum_{l=1}^n \beta_{ki} \beta_{lj} h_{kl} \in \mathbb{Z}[(v_i), (a_i), (h_{ij})]$$

for  $2 \leq i, j \leq n$ .

Let  $V$  be the change of basis matrix in  $\wedge^2 \mathbb{V}$  associated with  $U$ . Its coefficients are given by  $V_{ijkl} = U_{ik} U_{jl} - U_{il} U_{jk}$  for  $1 \leq i, j, k, l \leq n$  with  $i < j$  and  $k < l$ . As with  $U$ , the matrix  $V^{-1} R V$  restricted to the rows and columns  $2 \leq i, j, k, l \leq n$  (with  $i < j$  and  $k < l$ ) is the matrix of  $R|_{v^\perp}$  in an orthonormal basis. As above, one has

$$a_{i-1} a_i a_{j-1} a_j a_{k-1} a_k a_{l-1} a_l (V^{-1} R V)_{ijkl} \in \mathbb{Z}[(v_i), (a_i), (r_{ijkl})]$$

for  $2 \leq i, j, k, l \leq n$  (with  $i < j$  and  $k < l$ ).

The coefficients of an exterior power of a form are minors of the coefficients of that form. The coefficients of an exterior product and of a trace are similarly obtained using polynomial operations. Therefore, denoting the left-hand side of Equation (2.6) by  $T(v, H, R)$ , we obtain that for every  $n, k, m$  as in the statement of the lemma, there exists a homogeneous polynomial  $P \in \mathbb{Z}[(v_i), (a_i), (h_{ij}), (r_{ijkl})]$  and a function  $\alpha: \llbracket 1, n \rrbracket \rightarrow \mathbb{N}$  such that for all

$(\mathbb{V}, \mathfrak{B}), v, H, R$  as in the statement of the lemma, if  $v_1 \neq 0$ , then one has

$$T(v, H, R) = P((v_i), (a_i), (h_{ij}), (r_{ijkl})) \prod_{i=1}^n a_i^{-\alpha_i}. \quad (2.7)$$

We can furthermore assume that this rational function is irreducible, that is, if  $\alpha_i > 0$ , then  $a_i$  is not a factor of  $P$ . By counting the number of products involved in the computation of  $T(v, H, R)$ , we obtain  $\alpha_n \leq 2(2m + (k - 1 - 2m)) = 2(k - 1)$ .

We now prove that  $\alpha_i = 0$  for  $i < n$  (that is, only  $a_n$  can occur at the denominator). For any fixed forms  $H$  and  $R$ , the function  $v \mapsto T(H, R, v)$  is bounded on  $\mathbb{V} \setminus \{0\}$ . Therefore, the rational function on the right-hand side of Equation (2.7) has no pole on  $\mathbb{V} \setminus \{0\}$ , so one has  $\alpha_i = 0$  for  $i < n$ .

Equation (2.7) holds for  $v_1 \neq 0$  and both sides are continuous functions of  $v \in \mathbb{V} \setminus \{0\}$ , so the equality holds for all  $v \in \mathbb{V} \setminus \{0\}$ .

Finally, we prove that  $P$  is a polynomial function of the  $v_k$ 's, which is equivalent to the fact that the  $a_k$ 's occur in  $P$  only with even exponents. The polynomial  $P$  is a linear combination of monomials in  $(h_{ij}), (r_{ijkl})$ , where the coefficient of each such monomial is a polynomial in  $(v_i), (a_i)$ . Since each monomial is uniquely determined by its multidegree, the arguments of the next two paragraphs (the first dealing with the  $a_i$ 's with  $i < n$  and the second with  $a_n$ ) can be applied separately to each monomial, for instance by considering partial differentiations of the function  $(v, H, R) \mapsto T(v, H, R)$  with respect to the  $h_{ij}$ 's and the  $r_{ijkl}$ 's.

First, note that  $T(v, H, R)$  is invariant under orthogonal transformations of  $\mathbb{V}$ , that is, if  $f \in \text{SO}(\mathbb{V})$ , then  $T(f_*v, f_*H, f_*R) = T(v, H, R)$ . All the  $a_i$ 's as functions of the  $v_k$ 's are smooth at  $e_n$ , so the function  $(v, H, R) \mapsto T(v, H, R)$  is smooth at any  $(e_n, H, R)$ . Therefore, by invariance under orthogonal transformations, it is also smooth at any  $(e_2, H, R)$ . Among the  $a_i$ 's seen as functions of the  $v_k$ 's, only  $a_1$  is not smooth at  $e_2$ . Therefore,  $a_1$  only occurs in  $P$  with even exponents. Proceeding similarly, since  $a_2$  is the only of the remaining  $a_i$ 's that is not smooth at  $e_3$ , it only occurs in  $P$  with even exponents. Proceeding by steps, all  $a_i$ 's with  $1 \leq i \leq n-1$  only occur in  $P$  with even exponents.

As for  $a_n$ , we compute its exponent in each monomial in  $(h_{ij}), (r_{ijkl})$  (taking into account the denominator  $a_n^{-\alpha_n}$ , so that the exponent is in  $\mathbb{Z}$ ). By invariance of  $T(v, H, R)$  under the homothety  $v \mapsto \lambda v$  for  $\lambda \neq 0$ , the exponent of  $a_n$ , say  $d$ , has to be compensated by a total degree in the  $v_i$ 's equal to  $-d$ . The map  $v \mapsto -v$  leaves  $T(v, H, R)$  invariant, and multiplies the monomial by  $(-1)^d$ . Therefore,  $d$  is even.  $\square$

DEFINITION 2.3. — For all  $n, k, m \in \mathbb{N}$  such that  $1 \leq k \leq n$  and  $0 \leq m \leq \lfloor \frac{k-1}{2} \rfloor$ , we define  $P_{n,k,m}$  to be the (unique) polynomial whose existence is asserted in Lemma 2.2. For other values of the indices, we set  $P_{n,k,m} := 0$ . We set  $P_{n,k} := P_{n,k,0}$ .

Remark 2.4. — One has

$$\dim \bigcirc^2 \mathbb{V}^* = \frac{n(n+1)}{2}$$

and  $\dim \bigcirc^2 \bigwedge^2 \mathbb{V}^* = \frac{n(n-1)(n(n-1)+2)}{8}$ .

Therefore,  $P_{n,k,m}$  has  $n + [k-1-2m \geq 1] \frac{n(n+1)}{2} + [m \geq 1] \frac{n(n-1)(n(n-1)+2)}{8}$  variables. By homogeneity considerations,  $P_{n,k,m}$  has degree  $2(k-1)$  in the coefficients of  $v$ , degree  $k-1-2m$  in the coefficients of  $H$ , and degree  $m$  in the coefficients of  $R$ .

The proof shows how to compute the  $P_{n,k,m}$ 's. For instance, with the notation of the proof, one has

$$(U^{-1}HU)_{ij} = \frac{1}{a_{i-1}a_i a_{j-1}a_j} \times \left( a_{i-1}^2 h_{ij} a_{j-1}^2 + v_i v_j \sum_{k=1}^{i-1} \sum_{l=1}^{j-1} v_k h_{kl} v_l - a_{i-1}^2 v_j \sum_{l=1}^{j-1} h_{il} v_l - a_{j-1}^2 v_i \sum_{k=1}^{i-1} v_k h_{kj} \right)$$

for  $2 \leq i, j \leq n$ , from which one can compute the coefficients of any exterior power of  $H|_{v^\perp}$ , and similarly for  $R|_{v^\perp}$ , for the exterior product, and finally for the trace. We consider a few special cases:

- If  $k = 1$  (hence  $m = 0$ ), then the left-hand side of (2.6) is the trace of the identity on  $\bigwedge^0 v^\perp \simeq \mathbb{R}$ , so  $P_{n,1} = 1$ .
- For the case  $k = 2$  (hence  $m = 0$ ), note that  $\text{tr}(H|_{v^\perp}) = \text{tr} H - \|v\|^{-2} H(v, v)$ . This gives

$$P_{n,2} = \sum_{i=1}^n \left( \sum_{j \neq i} v_j^2 \right) h_{ii} - 2 \sum_{i < j} v_i v_j h_{ij}. \tag{2.8}$$

The first such polynomial is  $P_{2,2} = v_2^2 h_{11} - 2v_1 v_2 h_{12} + v_1^2 h_{22}$ . This is simply  $H(u, u)$  where  $u$  is any of the two unit vectors orthogonal to  $v$ .

- For the case  $(n, k, m) = (n, n, 0)$ , one has  $\det H = \det(H|_{v^\perp}) \|v\|^{-2} \times H(v, v^H)$  where  $v^H$  is the projection of  $v$  onto  $(v^\perp)^{\perp H}$  parallel to  $v^\perp$ , provided that  $H|_{v^\perp}$  is nondegenerate. Indeed, considering an orthonormal basis  $(f_i)_{1 \leq i \leq n}$  of  $\mathbb{V}$  with  $f_1 = \frac{v}{\|v\|}$  and  $(f_i)_{2 \leq i \leq n}$  a

diagonalizing basis for  $H|_{v^\perp}$ , one has

$$\det H = \det \begin{pmatrix} H\left(\frac{v}{\|v\|}, \frac{v}{\|v\|}\right) & \dots & H\left(\frac{v}{\|v\|}, f_i\right) & \dots \\ \vdots & \ddots & 0 & 0 \\ H\left(\frac{v}{\|v\|}, f_i\right) & 0 & \lambda_i & 0 \\ \vdots & 0 & 0 & \ddots \end{pmatrix}.$$

A double expansion of this determinant yields

$$\det H = (\det H|_{v^\perp}) \|v\|^{-2} H\left(v, v - \sum_{i=2}^n H\left(v, \frac{f_i}{\lambda_i}\right) f_i\right)$$

as claimed.

Applying Lemma 2.2 to  $(R, H, v) = (R, \text{Hess}(f), \nabla f)$ , one obtains

$$\text{tr}\left(\bigwedge^m R|_{\nabla f^\perp} \wedge \bigwedge^{k-1-2m} \text{Hess}(f)|_{\nabla f^\perp}\right) = \frac{P_{n,k,m}(R, \text{Hess}(f), \nabla f)}{\|\nabla f\|^{2(k-1)}}. \quad (2.9)$$

In view of the special cases considered above, and under the nondegeneracy condition for the third equation, one has

$$P_{n,1}(\text{Hess}(f), \nabla f) = 1, \quad (2.10)$$

$$P_{n,2}(\text{Hess}(f), \nabla f) = \|\nabla f\|^2 \Delta f - \text{Hess}(f)(\nabla f, \nabla f), \quad (2.11)$$

$$P_{n,n}(\text{Hess}(f), \nabla f) = \frac{\det(\text{Hess}(f)) \|\nabla f\|^{2n}}{\text{Hess}(f)(\nabla f, (\nabla f)^{\text{Hess}(f)})}. \quad (2.12)$$

We now state the version of the divergence theorem that will be useful to us. The *divergence* of a  $\mathcal{C}^1$ -vectorfield  $X \in \mathfrak{X}(M)$  is defined by  $\mathcal{L}_X \text{vol}_M = (\text{div } X) \text{vol}_M$ , where  $\mathcal{L}$  denotes the Lie derivative. Many generalizations of the standard divergence theorem have been proved, relaxing hypotheses on the regularity and compactness of the manifold or stratified space and on the regularity of the vectorfield, encompassing the present statement. We include a proof for the convenience of the reader.

**THEOREM 2.5 (Divergence theorem).** — *Let  $(M, g)$  be a compact  $n$ -dimensional Riemannian manifold with boundary. Let  $X \in \mathfrak{X}(M)$  be a continuous vectorfield on  $M$  which is of class  $\mathcal{C}^1$  on  $\text{int } M$  and such that  $\text{div } X \in L^1(M)$ . Then,*

$$\int_M (\text{div } X) \text{vol}_M = \int_{\partial M} \langle X, \nu \rangle \text{vol}_{\partial M}. \quad (2.13)$$

where  $\nu$  is the outward unit normal vectorfield on  $\partial M$ .

*Proof.* — If  $X$  is of class  $\mathcal{C}^1$  on  $M$ , then this is the standard divergence theorem. Else, we consider the geodesic flow from the boundary of  $M$  along the outward unit normal vectorfield  $\nu$ . For  $\epsilon > 0$  small enough, set

$\theta_\epsilon: \partial M \rightarrow M, x \mapsto \exp(x, -\epsilon\nu_x)$  and set  $M_\epsilon := M \setminus \bigcup_{s \in [0, \epsilon[} \theta_s(\partial M)$ . For  $\epsilon$  small enough,  $M_\epsilon$  is a compact submanifold with boundary of  $M$ , and  $\theta_\epsilon$  induces a diffeomorphism  $\vartheta_\epsilon: \partial M \xrightarrow{\sim} \partial M_\epsilon$ . Applying the standard divergence theorem on  $M_\epsilon$ , one obtains  $\int_{M_\epsilon} (\operatorname{div} X) \operatorname{vol}_M = \int_{\partial M_\epsilon} \langle X, \nu_\epsilon \rangle \operatorname{vol}_{\partial M_\epsilon}$ . When  $\epsilon \rightarrow 0$ , the left-hand side converges to  $\int_M (\operatorname{div} X) \operatorname{vol}_M$  by Lebesgue's dominated convergence theorem, since  $\operatorname{div} X \in L^1(M)$ . The right-hand side is equal, by change of variable, to  $\int_{\partial M} \langle \vartheta_\epsilon^* X, \nu \rangle (\det T\vartheta_\epsilon) \operatorname{vol}_{\partial M}$ , which converges to  $\int_{\partial M} \langle X, \nu \rangle \operatorname{vol}_{\partial M}$  since the integrand is uniformly convergent and  $\partial M$  is compact.  $\square$

We can now prove a first general result.

**THEOREM 2.6.** — *Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold. Let  $(f, a) \in \operatorname{Reg}_c^3(M, \mathbb{R})$ .*

*For  $1 \leq k \leq n$  and  $0 \leq m \leq \lfloor \frac{k-1}{2} \rfloor$ , let  $F_{k,m} \in \mathcal{C}^1(TM, TM)$  be fiber-preserving such that  $F_{k,m}(u) \sim_\infty \frac{u}{\|u\|}$ . Then, for  $0 \leq k \leq n$ , one has*

$$\begin{aligned} \mathcal{L}_{n-k}(M^a) &= a_k \int_{M^a} \operatorname{tr} \left( \bigwedge^{k/2} R \right) \operatorname{vol}_M \\ &\quad + \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{k,m} \int_{M^a} \operatorname{div} \left( \|\nabla f\|^{2m+3(1-k)} \right. \\ &\quad \left. P_{n,k,m}(R, \operatorname{Hess}(f), \nabla f) \left( F_{k,m} \circ \frac{\nabla f}{a-f} \right) \right) \operatorname{vol}_M \end{aligned} \quad (2.14)$$

*under the condition that the divergence appearing in the integral exists and is integrable.*

*Assume that  $M$  is flat. For  $1 \leq k \leq n$ , let  $F_k \in \mathcal{C}^1(TM, TM)$  be fiber-preserving such that  $F_k(u) \sim_\infty \frac{u}{\|u\|}$ . Then, for  $1 \leq k \leq n$ , one has*

$$\mathcal{L}_{n-k}(M^a) = b_k \int_{M^a} \operatorname{div} \left( \|\nabla f\|^{3(1-k)} P_{n,k}(\operatorname{Hess}(f), \nabla f) \left( F_k \circ \frac{\nabla f}{a-f} \right) \right) \operatorname{vol}_M. \quad (2.15)$$

*under the same conditions.*

**Remark 2.7.** — By " $F(u) \sim_\infty \frac{u}{\|u\|}$ ", we mean that  $\lim_{\|u\| \rightarrow +\infty} \|F(u) - \frac{u}{\|u\|}\| = 0$  locally uniformly in  $\tau_M(u) \in M$ .

*Proof.* — Starting with the definition (1.1), we use the expression of the second fundamental form (2.4) and Equation (2.9) to obtain

$$\begin{aligned} \mathcal{L}_{n-k}(M) &= a_k \int_M \operatorname{tr} \left( \bigwedge^{k/2} R \right) \operatorname{vol}_M \\ &\quad + \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{k,m} \int_{\partial M} \frac{P_{n,k,m}(R, \operatorname{Hess}(f), \nabla f)}{\|\nabla f\|^{3(k-1)-2m}} \operatorname{vol}_{\partial M}. \end{aligned} \quad (2.16)$$

The asymptotic property of  $F_{k,m}$  ensures that the vectorfield whose divergence is considered in the statement is continuous on  $\partial M^a$  and its value there is

$$\|\nabla f\|^{2m+3(1-k)} P_{n,k,m}(R, \operatorname{Hess}(f), \nabla f) \frac{\nabla f}{\|\nabla f\|}.$$

Finally, the hypotheses of the proposition ensure that the divergence theorem applies.  $\square$

*Remark 2.8.* — Since  $P_{n,1} = 1$ , the theorem for  $k = 1$  holds for  $(f, a) \in \operatorname{Reg}_c^2(M, \mathbb{R})$ .

Our next step is to find explicit functions  $F$  (in particular proving that some exist) making the divergence appearing in the theorem integrable. We consider radial maps of the form  $F_{k,m}(u) = \|u\|^{3(k-1)-2m} G_{k,m}(\|u\|)u$  with  $G_{k,m} \in \mathcal{C}^1(\mathbb{R}_{\geq 0}, \mathbb{R})$ . The condition  $G_{k,m}(x) \sim_{+\infty} x^{2(m+1)-3k}$  ensures that  $F_{k,m}(u) \sim_{\infty} \frac{u}{\|u\|}$ . Examples of functions  $G$  satisfying these conditions are given by  $G_{k,m}(x) := (1 + x^{2(3k-2(m+1))})^{-1/2}$ . We set

$$\eta_{f,\ell} := \sqrt{f^{2\ell} + \|\nabla f\|^{2\ell}} \quad (2.17)$$

for  $\ell \geq 0$ . With these choices for  $F$  and  $G$ , one obtains

$$F_{k,m} \circ \left( \frac{\nabla f}{a-f} \right) = \|f\|^{3(k-1)-2m} \frac{\nabla f}{\eta_{f-a,3k-2(m+1)}} \quad (2.18)$$

on  $\operatorname{int} M^a$ , which yields the following theorem.

**THEOREM 2.9.** — *Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold. Let  $(f, a) \in \operatorname{Reg}_c^3(M, \mathbb{R})$ . For  $0 \leq k \leq n$ , one has*

$$\begin{aligned} \mathcal{L}_{n-k}(M^a) &= a_k \int_{M^a} \operatorname{tr} \left( \bigwedge^{k/2} R \right) \operatorname{vol}_M \\ &\quad + \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{k,m} \int_{M^a} \operatorname{div} \left( \frac{P_{n,k,m}(R, \operatorname{Hess}(f), \nabla f)}{\eta_{f-a,3k-2(m+1)}} \nabla f \right) \operatorname{vol}_M. \end{aligned} \quad (2.19)$$

If  $M$  is flat and  $1 \leq k \leq n$ , then

$$\mathcal{L}_{n-k}(M^a) = b_k \int_{M^a} \operatorname{div} \left( \frac{P_{n,k}(\operatorname{Hess}(f), \nabla f)}{\eta_{f-a, 3k-2}} \nabla f \right) \operatorname{vol}_M. \quad (2.20)$$

For  $k = 1, 2$ , this gives

$$\mathcal{L}_{n-1}(M^a) = \frac{1}{2} \int_{M^a} \operatorname{div} \left( \frac{\nabla f}{\eta_{f-a}} \right) \operatorname{vol}_M, \quad (2.21)$$

$$\begin{aligned} \mathcal{L}_{n-2}(M^a) &= -\frac{1}{2\pi} \int_{M^a} \operatorname{scal} \operatorname{vol}_M \\ &\quad + \frac{1}{2\pi} \int_{M^a} \operatorname{div} \left( \frac{\|\nabla f\|^2 \Delta f - \operatorname{Hess}(f)(\nabla f, \nabla f)}{\eta_{f-a, 4}} \nabla f \right) \operatorname{vol}_M. \end{aligned} \quad (2.22)$$

Similarly, when  $M$  is flat and  $\operatorname{Hess}(f)|_{\nabla f^\perp}$  is nondegenerate, if  $n \geq 1$ , one has

$$\chi(M^a) = b_n \int_{M^a} \operatorname{div} \left( \frac{\det(\operatorname{Hess}(f)) \|\nabla f\|^{2n}}{\operatorname{Hess}(f)(\nabla f, (\nabla f)^{\operatorname{Hess}(f)}) \eta_{f-a, 3n-2}} \nabla f \right) \operatorname{vol}_M. \quad (2.23)$$

*Remark 2.10.* — There are obviously many natural choices for the functions  $F$  and  $G$ . For instance, one can take  $F_{k,m} := F_k$ . With the  $F_k$ 's given above, the divergence corresponding to the  $m^{\text{th}}$  summand reads

$$\operatorname{div} \left( \|\nabla f\|^{2m} \frac{P_{n,k,m}(R, \operatorname{Hess}(f), \nabla f)}{\eta_{f-a, 3k-2}} \nabla f \right). \quad (2.24)$$

In the case of nodal volumes, other choices are given in the next subsection.

*Remark 2.11.* — Intrinsic volumes can be defined for Riemannian manifolds with corners, and even Whitney stratified spaces of positive reach in Riemannian manifolds. Since the divergence theorem admits generalizations to these settings, it is possible to extend the above results to sublevel sets of functions defined on Riemannian manifolds with boundary or corners, and to Whitney stratified spaces in Riemannian manifolds, under the assumption that the function is transverse to the boundary or the strata respectively. Boundary terms will appear in the formulas. We do not carry out this generalization in full and only give a formula for nodal volumes in the next subsection (see Remark 2.13).

### 2.3. Nodal volumes

In this subsection, we show how we can compute the intrinsic volumes of the zero sets, or nodal sets, of functions defined on compact Riemannian

manifolds. Let  $(M, g)$  be a compact  $n$ -dimensional Riemannian manifold. Let  $f \in \mathcal{C}_{0\text{-reg}}^2(M, \mathbb{R})$  (class  $\mathcal{C}^2$  is sufficient by Remark 2.8). The *zero set* of  $f$  is  $\mathcal{Z}_f := f^{-1}(0) = \partial M_f^0 = \partial M_{-f}^0$ . By (1.6), one has  $\text{vol}(\mathcal{Z}_f) = \mathcal{L}_{n-1}(M_f^0) + \mathcal{L}_{n-1}(M_{-f}^0)$ . Since  $M_f^0 \cup M_{-f}^0 = M$  and  $M_f^0 \cap M_{-f}^0 = \mathcal{Z}_f$  is negligible in  $M$ , Formula (2.21) gives an integral on  $M$ . Using the general formula of Theorem 2.6 yields

$$\text{vol}(\mathcal{Z}_f) = -\frac{1}{2} \int_M \text{div} \left( F_1 \circ \frac{\nabla f}{|f|} \right) \text{vol}_M \quad (2.25)$$

(where minus the absolute value appears since  $f$  is negative on  $M_f^0$  and positive on  $M_{-f}^0$ ). Of course, this identity could have been obtained directly by applying the divergence theorem to the identity  $\text{vol}(\mathcal{Z}_f) = \int_{\partial M_f^0} \text{vol}_{\partial M_f^0}$ .

Recalling the definition of  $\eta_{f,\ell}$  by Equation (2.17), we set

$$\eta_f := \eta_{f,1} = \sqrt{f^2 + \|\nabla f\|^2}. \quad (2.26)$$

We also write  $\sigma_f: M \rightarrow \{-1, 0, 1\}$  for the sign of  $f$ .

Setting, in Formula (2.25),  $F_1(u) := G_1(\|u\|)u$  with respectively  $G_1(x) := (1+x^2)^{-1/2}$  and  $\frac{2}{\pi} \frac{\arctan x}{x}$  and  $\frac{\tanh x}{x}$ , one obtains

$$\text{vol}(\mathcal{Z}_f) = \frac{1}{2} \int_M \frac{\sigma_f}{\eta_f^3} (f \|\nabla f\|^2 + \text{Hess}(f)(\nabla f, \nabla f) - \eta_f^2 \Delta f) \text{vol}_M \quad (2.27)$$

and

$$\begin{aligned} \text{vol}(\mathcal{Z}_f) = \frac{1}{\pi} \int_M \left( \|\nabla f\|^{-1} \left( \arctan \circ \frac{\|\nabla f\|}{f} \right) \left( \frac{\text{Hess}(f)(\nabla f, \nabla f)}{\|\nabla f\|^2} - \Delta f \right) \right. \\ \left. + \eta_f^{-2} \left( \|\nabla f\|^2 - \frac{f \text{Hess}(f)(\nabla f, \nabla f)}{\|\nabla f\|^2} \right) \right) \text{vol}_M \end{aligned} \quad (2.28)$$

and

$$\begin{aligned} \text{vol}(\mathcal{Z}_f) = \frac{1}{2} \int_M \left( \|\nabla f\|^{-1} \left( \tanh \circ \frac{\|\nabla f\|}{f} \right) \left( \frac{\text{Hess}(f)(\nabla f, \nabla f)}{\|\nabla f\|^2} - \Delta f \right) \right. \\ \left. + \left( \cosh \circ \frac{\|\nabla f\|}{f} \right)^{-2} \left( \frac{\|\nabla f\|^2}{f^2} - \frac{\text{Hess}(f)(\nabla f, \nabla f)}{f \|\nabla f\|^2} \right) \right) \text{vol}_M \end{aligned} \quad (2.29)$$

(see [15] for the computation details).

*Remark 2.12.* — In the last three formulas, all terms of the integrands are bounded on  $M$  and continuous on  $M \setminus \mathcal{Z}_f$ . Indeed, the Hessian expressions are quadratic in  $\|\nabla f\|$ , the arctan and tanh expressions are linear in  $\|\nabla f\|$  when  $\|\nabla f\|$  is small, and the cosh expression is exponentially small in  $|f|$

when  $|f|$  is small. However, not all terms need be continuous on  $M$ . This problem is dealt with below.

*Remark 2.13.* — Fulfilling the promise made in Remark 2.11, let  $M$  be a compact Riemannian manifold with boundary. If  $f$  intersects  $\partial M$  transversely, then Formula (2.25) becomes

$$\text{vol}(\mathcal{Z}_f) = \frac{1}{2} \left( \int_{\partial M} \left\langle F \circ \frac{\nabla f}{f}, v \right\rangle \text{vol}_{\partial M} - \int_M \left( \text{div} \left( F \circ \frac{\nabla f}{f} \right) \right) \text{vol}_M \right). \quad (2.30)$$

Note that by the transversality assumption,  $\mathcal{Z}_f \cap \partial M$  is negligible in  $\partial M$ . This formula reduces in dimension 1 to [2, Prop. 3].

*Remark 2.14.* — The cases considered in [2] correspond to  $M = (\mathbb{R}/\mathbb{Z})^n$  with the standard flat metric. In particular, Formula (2.27) is essentially [2, Prop. 5] (in the case of  $(\mathbb{R}/\mathbb{Z})^n$  with the standard flat metric). In dimension 1, the general formula (2.25) reduces to [2, Prop. 2], and in that case, only the condition  $\lim_{x \rightarrow \pm\infty} F(x) = \pm 1$  is required if one considers the integral as an improper Lebesgue integral. Similarly, Formula (2.27) reduces to [2, Prop. 1] and Formula (2.28) to [2, Cor. 1 of Prop. 2].

More generally, the intrinsic volumes of subsets have the additivity property

$$\mathcal{L}_{n-k}(A) + \mathcal{L}_{n-k}(B) = \mathcal{L}_{n-k}(A \cup B) - \mathcal{L}_{n-k}(A \cap B) \quad (2.31)$$

when  $A, B$  are subsets of a compact  $n$ -dimensional Riemannian manifold  $M$  such that all terms are well-defined (see [9, Thm. 5.16(6)]). Therefore, if  $f \in \mathcal{C}_{0\text{-reg}}^3(M, \mathbb{R})$ , then

$$\mathcal{L}_{n-k}(\mathcal{Z}_f) = \mathcal{L}_{n-k}(M_f^0) + \mathcal{L}_{n-k}(M_{-f}^0) - \mathcal{L}_{n-k}(M). \quad (2.32)$$

The terms corresponding to the first summand in (2.19) cancel out, so that

$$\mathcal{L}_{n-k}(\mathcal{Z}_f) = \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} b_{k,m} \int_M \sigma_f^k \text{div} \left( \frac{P_{n,k,m}(R, \text{Hess}(f), \nabla f)}{\eta_{f,3k-2(m+1)}} \nabla f \right) \text{vol}_M. \quad (2.33)$$

The exponent  $k$  of  $\sigma_f$  is congruent modulo 2 to  $\deg_{\text{Hess}(f)} P_{n,k,m} + \deg_{\nabla f} P_{n,k,m} + 1 = 3k - 2(1+m)$  by Remark 2.4. In particular,  $\mathcal{L}_{n-k}(\mathcal{Z}_f) = 0$  for  $k$  even, as expected. One can also consider  $\mathcal{Z}_f$  as a Riemannian manifold with curvature  $\tilde{R}$  and obtain

$$\mathcal{L}_{n-1-k}(\mathcal{Z}_f) = a_k \int_{\mathcal{Z}_f} \text{tr} \left( \bigwedge^{k/2} \tilde{R} \right) \text{vol}_{\mathcal{Z}_f} \quad (2.34)$$

where  $\tilde{R}$  is given by the Gauss formula for the curvature of submanifolds,

$$\tilde{R}(X, Y, Z, T) = R(X, Y, Z, T) + S(X, Z)S(Y, T) - S(X, T)S(Y, Z) \quad (2.35)$$

for  $X, Y, Z, T \in \mathfrak{X}(\mathcal{Z}_f)$ .

We return to the question raised in Remark 2.12 of having continuous integrands. The only non-continuous terms in the integrands of Equations (2.27), (2.28), (2.29) are of the form

$$\sigma_f h \left( \text{Hess}(f)(\nabla f, \nabla f) - \Delta f \|\nabla f\|^2 \right) \quad (2.36)$$

with  $h \in \mathcal{C}^1(M, \mathbb{R})$ , respectively  $h = \eta_f^{-3}$  and  $h = \|\nabla f\|^{-3} \left( \arctan \circ \frac{\|\nabla f\|}{|f|} \right)$  and  $h = \|\nabla f\|^{-3} \left( \tanh \circ \frac{\|\nabla f\|}{|f|} \right)$ . This is dealt with in [2] (in the case of Equation (2.27) on a flat torus) using an integration by parts. The same method extends to compact Riemannian manifolds as follows. One has

$$\text{Hess}(f)(\nabla f, \nabla f) - (\Delta f)\|\nabla f\|^2 = \langle \nabla f, \nabla^{\text{LC}}_{\nabla f} \nabla f - (\Delta f)\nabla f \rangle.$$

Therefore,

$$\sigma_f h \left( \text{Hess}(f)(\nabla f, \nabla f) - (\Delta f)\|\nabla f\|^2 \right) = \langle \nabla |f|, h(\nabla^{\text{LC}}_{\nabla f} \nabla f - (\Delta f)\nabla f) \rangle.$$

We temporarily assume that  $f$  is of class  $\mathcal{C}^3$  and we use the fact that  $\text{div}(|f|h(\nabla^{\text{LC}}_{\nabla f} \nabla f - (\Delta f)\nabla f))$  has a vanishing integral on  $M$  (by the standard divergence theorem). Therefore,

$$\begin{aligned} \int_M \sigma_f h \left( \text{Hess}(f)(\nabla f, \nabla f) - (\Delta f)\|\nabla f\|^2 \right) \text{vol}_M \\ = \int_M |f| \text{div} \left( h((\Delta f)\nabla f - \nabla^{\text{LC}}_{\nabla f} \nabla f) \right) \text{vol}_M. \end{aligned}$$

One has  $\text{div}((\Delta f)\nabla f) = (\Delta f)^2 + \langle \nabla \Delta f, \nabla f \rangle$ . The Bochner formula yields

$$\begin{aligned} \text{div}(\nabla^{\text{LC}}_{\nabla f} \nabla f) &= \text{div} \left( \frac{1}{2} \nabla \|\nabla f\|^2 \right) \\ &= \frac{1}{2} \Delta \|\nabla f\|^2 \\ &= \langle \nabla \Delta f, \nabla f \rangle + \|\text{Hess } f\|^2 + \text{Ric}(\nabla f, \nabla f) \end{aligned}$$

where the norm of the Hessian is the Hilbert–Schmidt norm. Therefore, the third derivatives cancel out. Since  $\mathcal{C}^2(M, \mathbb{R})$  is dense in  $\mathcal{C}^3(M, \mathbb{R})$  for the (Whitney) strong  $\mathcal{C}^2$ -topology (see for instance [14, Thm. II.2.6] and the next section for function space topologies) and the involved quantities are continuous in this topology, one has, for any  $f$  of class  $\mathcal{C}^2$ ,

$$\begin{aligned} \int_M \sigma_f h \left( \text{Hess}(f)(\nabla f, \nabla f) - (\Delta f)\|\nabla f\|^2 \right) \text{vol}_M \\ = \int_M |f| \left( h \left( (\Delta f)^2 - \|\text{Hess } f\|^2 - \text{Ric}(\nabla f, \nabla f) \right) \right. \\ \left. + \langle \nabla h, (\Delta f)\nabla f - \nabla^{\text{LC}}_{\nabla f} \nabla f \rangle \right) \text{vol}_M. \quad (2.37) \end{aligned}$$

For example, one has  $\nabla\eta_f^{-3} = -3\eta_f^{-5} (f\nabla f + \nabla^{\text{LC}}_{\nabla f} \nabla f)$ , so (2.27) becomes

$$\begin{aligned} & \text{vol}(\mathcal{Z}_f) \\ &= \frac{1}{2} \int_M \left( \frac{|f|}{\eta_f^3} (\|\nabla f\|^2 - |f| \Delta f + (\Delta f)^2 - \|\text{Hess } f\|^2 - \text{Ric}(\nabla f, \nabla f)) \right. \\ & \quad \left. + 3\eta_f^{-5} \left( f \text{Hess}(f)(\nabla f, \nabla f) + \text{Hess}(f)(\nabla f, \nabla^{\text{LC}}_{\nabla f} \nabla f) \right. \right. \\ & \quad \left. \left. - (\Delta f)^2 (f\|\nabla f\|^2 + \text{Hess}(f)(\nabla f, \nabla f)) \right) \right) \text{vol}_M. \quad (2.38) \end{aligned}$$

In the case of a flat torus, this is [2, Prop. 7].

*Remark 2.15.* — These formulas can also be written in terms of the tracefree Hessian. Recall that  $\text{Hess}^0(f) = \text{Hess}(f) - \frac{\Delta f}{n} \text{id}$ . A tracefree linear map is Hilbert–Schmidt-orthogonal to the identity, so  $\|\text{Hess } f\|^2 = \frac{(\Delta f)^2}{n} + \|\text{Hess}^0 f\|^2$ .

In Equation (2.38), the integrand is a Lipschitz continuous functional of  $f \in \mathcal{C}^2_{0\text{-reg}}(M, \mathbb{R})$  (see next section for the precise setting), so one can apply techniques of the Malliavin calculus (see [2]). The only difference between (2.38) and [2, Prop. 7] is the additional term involving the Ricci curvature,  $|f|\eta_f^{-3} \text{Ric}(\nabla f, \nabla f)$ , and this term is in the required domain of the Malliavin calculus by the same proof as [2, Lem. 2 p. 26]. Therefore, [2, Thm. 1] holds on any compact Riemannian manifold. Similarly, Formula (2.30) shows that the extra boundary terms are not problematic, so [2, Thm. 1] holds on any compact Riemannian manifold with corners, a generalization which includes [2, Thm. 2] as a special case.

### 3. Continuity of the intrinsic volumes of sublevel sets

#### 3.1. Review of function space topologies

For this subsection, we refer to [14, Ch. II] for details. Let  $(M, g)$  be a Riemannian manifold and  $p \in \overline{\mathbb{N}}$ . We will use two different topologies on the set  $\mathcal{C}^p(M, \mathbb{R})$ , the *uniform  $\mathcal{C}^p$ -topology*, and the finer (Whitney) *strong  $\mathcal{C}^p$ -topology*. The resulting topological spaces will be denoted with the subscripts  $U$  and  $S$  respectively.

If  $0 \leq i \leq p$  and  $f \in \mathcal{C}^p(M, \mathbb{R})$ , then  $\nabla^{\text{LC}^i} f \in \Gamma^{(p-i)}(\bigotimes^i T^*M \rightarrow M)$ . We denote by  $\|\nabla^{\text{LC}^i} f(x)\|$  the norm of this multilinear form induced by

the norm  $g_x$  on  $T_x^*M$ , and by  $\|\nabla^{\text{LC}^i} f\|_\infty$  the supremum of these norms for  $x \in M$ . Let  $p \in \mathbb{N}$ . The uniform  $\mathcal{C}^p$ -topology is the group topology such that a neighborhood basis of 0 is given by

$$U_p(\epsilon) := \left\{ f \in \mathcal{C}^p(M, \mathbb{R}) \left| \sum_{i=0}^p \|\nabla^{\text{LC}^i} f\|_\infty < \epsilon \right. \right\} \quad (3.1)$$

for  $\epsilon \in \mathbb{R}_{>0}$ . The strong  $\mathcal{C}^p$ -topology is the group topology such that a neighborhood basis of 0 is given by

$$S_p(\epsilon) := \left\{ f \in \mathcal{C}^p(M, \mathbb{R}) \left| \forall x \in M \sum_{i=0}^p \|\nabla^{\text{LC}^i} f(x)\| < \epsilon(x) \right. \right\} \quad (3.2)$$

for  $\epsilon \in \mathcal{C}^0(M, \mathbb{R}_{>0})$ . The uniform and strong  $\mathcal{C}^\infty$ -topologies are obtained as the unions of the corresponding  $\mathcal{C}^p$ -topologies.

Then,  $\mathcal{C}^p(M, \mathbb{R})_S$  is a complete Baire topological ring (but not metrizable nor a topological vector space if  $M$  is not compact), and  $\mathcal{C}^p(M, \mathbb{R})_U$  is a Fréchet space. The strong  $\mathcal{C}^p$ -topology does not depend on the Riemannian metric (it could actually be defined using norms of usual derivatives in charts as in [14, II.1]). These topologies differ in the control of functions at infinity (in particular, they are equal when  $M$  is compact, and equal to the compact-open  $\mathcal{C}^p$ -topology). Furthermore, results involving the strong topology will often remain true for the uniform topology when restricted to proper functions.

The product  $\mathcal{C}^p(M, \mathbb{R}) \times \mathbb{R}$  will be considered with the corresponding product topology, and the sets  $\text{Reg}^p(M, \mathbb{R})$  and  $\text{Reg}_c^p(M, \mathbb{R})$  defined in Equations (2.2) and (2.3) with the corresponding subspace topologies.

Although  $\mathcal{C}^p(M, \mathbb{R})_U$  is metrizable, there is no natural choice of a metric if  $p > 0$  (and  $M$  has strictly positive dimension) or if  $M$  is not compact. On the other hand, the space  $\mathcal{C}^0(M, \mathbb{R})_U$  has a natural extended norm (a norm that can take infinite values) given by  $\|f\|_\infty := \sup_{x \in M} |f(x)|$  for  $f \in \mathcal{C}^0(M, \mathbb{R})$  and giving rise to an extended distance. When we speak of Lipschitz continuity of a map from  $\mathcal{C}^p(M, \mathbb{R})_U$  (and similarly from its product with  $\mathbb{R}$  or a subspace of one of these) to  $\mathbb{R}$ , we mean  $\mathcal{C}^p(M, \mathbb{R})_U$ -local  $\|-\|_\infty$ -Lipschitz continuity. More precisely, the map  $L: \mathcal{C}^p(M, \mathbb{R})_U \rightarrow \mathbb{R}$  is *Lipschitz continuous* if  $\mathcal{C}^p(M, \mathbb{R})_U$  has an open cover such that for all  $V \subseteq \mathcal{C}^p(M, \mathbb{R})$  in this cover, there exists  $C \in \mathbb{R}$  such that for all  $f, g \in V$ , one has  $|L(g) - L(f)| \leq C\|g - f\|_\infty$ . Since  $\|-\|_\infty$  is continuous at 0, Lipschitz continuity implies continuity. Note however that with this definition,

Lipschitz continuity does not imply uniform continuity when the domain is not compact.<sup>(2)</sup>

The strong topology has the disadvantage that the inclusion of constant functions,  $\mathbb{R} \rightarrow \mathcal{C}^p(M, \mathbb{R})_S, a \mapsto (x \mapsto a)$ , is not continuous when  $M$  is not compact. For example, the function

$$\begin{aligned} \tau: \mathcal{C}^p(M, \mathbb{R})_U \times \mathbb{R} &\longrightarrow \mathcal{C}^p(M, \mathbb{R})_U \\ (f, a) &\longmapsto f - a. \end{aligned} \tag{3.3}$$

is Lipschitz continuous, but the analogous result (for mere continuity) with the strong topology does not hold. Therefore, when studying sublevel sets at varying heights, we will use the uniform topology and we will restrict our attention to proper functions, and when studying sublevel sets at a fixed height, we will use the strong topology if we want to allow nonproper functions.

We denote by  $\mathcal{C}_p^p(M, \mathbb{R})$  (resp.  $\mathcal{C}_b^p(M, \mathbb{R})$ ) the set of proper (resp. bounded below) functions in  $\mathcal{C}^p(M, \mathbb{R})$ , and by  $\mathcal{C}_{pb}^p(M, \mathbb{R}) := \mathcal{C}_p^p(M, \mathbb{R}) \cap \mathcal{C}_b^p(M, \mathbb{R})$  the set of proper bounded below functions. Similarly, we set  $\text{Reg}_*^p := \text{Reg}^p(M, \mathbb{R}) \cap (\mathcal{C}_*^p(M, \mathbb{R}) \times \mathbb{R})$  for  $*$  = b, p, pb.

**PROPOSITION 3.1.** — *Let  $p \in \overline{\mathbb{N}}$ . At least one (resp. all) sublevel set(s) of  $f \in \mathcal{C}^p(M, \mathbb{R})$  is/are compact if and only if  $f$  is bounded below (resp. proper bounded below). In particular,  $\text{Reg}_{pb}^p(M, \mathbb{R}) \subseteq \text{Reg}_c^p(M, \mathbb{R})$ . The subsets  $\mathcal{C}_b^p(M, \mathbb{R})$  and  $\mathcal{C}_p^p(M, \mathbb{R})$  and  $\mathcal{C}_{pb}^p(M, \mathbb{R})$  are open and closed in  $\mathcal{C}^p(M, \mathbb{R})_U$ .*

*Proof.* — Obvious. □

**Example 3.2.** — For a given function, the set of real numbers such that the associated sublevel set is compact can be any downset. Indeed, consider the functions on  $\mathbb{R}$  which send  $x$  to respectively  $x$  or  $a$  or  $a + e^x$  or  $x^2$ . The sets of real numbers such that the associated sublevel set is compact are  $\emptyset$  and  $]-\infty, a[$  and  $]-\infty, a]$  and  $\mathbb{R}$  respectively.

Recall that  $\eta_f$  was defined by Equation (2.26) and  $\tau$  by Equation (3.3).

**LEMMA 3.3.** — *The function  $\inf: \mathcal{C}^0(M, \mathbb{R})_U \rightarrow \mathbb{R} \cup \{-\infty\}$  is Lipschitz continuous. The function  $\eta: \mathcal{C}^1(M, \mathbb{R})_U \rightarrow \mathcal{C}^0(M, \mathbb{R})_U$  is Lipschitz continuous. Let  $p \in \overline{\mathbb{N}}$ . The function  $\eta: \mathcal{C}_{0-\text{reg}}^{1+p}(M, \mathbb{R})_X \rightarrow \mathcal{C}^p(M, \mathbb{R}_{>0})_X$  is Lipschitz continuous for  $X = U$  and continuous for  $X = S$ . The function  $m := \inf \circ \eta \circ \tau: \mathcal{C}^{1+p}(M, \mathbb{R})_U \times \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous.*

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<sup>(2)</sup> In a more elementary setting, the square function on real numbers is an example, and for an example where the domain is totally bounded, consider the sign function  $\sigma: [-1, 1] \setminus \{0\} \rightarrow \mathbb{R}$ , which is Lipschitz continuous since locally constant, and not uniformly continuous.

*Proof.* — Obvious. □

PROPOSITION 3.4. — *Let  $p \in \overline{\mathbb{N}}_{\geq 1}$ . The subset  $\text{Reg}_p^p(M, \mathbb{R})$  is open and dense in  $\mathcal{C}_p^p(M, \mathbb{R})_U \times \mathbb{R}$ . The subset  $\mathcal{C}_{0-\text{reg}}^p(M, \mathbb{R})$  (resp.  $\mathcal{C}_{0-\text{reg},c}^p(M, \mathbb{R})$ ) is open and dense in  $\mathcal{C}^p(M, \mathbb{R})_S$  (resp.  $\mathcal{C}_{0-c}^p(M, \mathbb{R})_S$ ). The three openness results actually hold for the (uniform or strong)  $\mathcal{C}^0$ -topology.*

*Proof.* — One has,  $\text{Reg}_p^p(M, \mathbb{R}) = (m|_{\mathcal{C}_p^p(M, \mathbb{R})_U})^{-1}(\mathbb{R}_{>0})$ , which is therefore open in  $\mathcal{C}_p^p(M, \mathbb{R})_U$ . Similarly,  $\mathcal{C}_{0-\text{reg}}^p(M, \mathbb{R}) = \eta^{-1}(\mathcal{C}^{p-1}(M, \mathbb{R}_{>0}))$  is open in  $\mathcal{C}_p^p(M, \mathbb{R})_S$ .

As for density, by the Morse–Sard theorem, if  $f \in \mathcal{C}^n(M, \mathbb{R})$ , then the set of regular values of  $f$  is dense. This implies that  $\text{Reg}_p^p(M, \mathbb{R})$  is dense in  $\mathcal{C}^{\max(n,p)}(M, \mathbb{R})_U \times \mathbb{R}$ , which is dense in  $\mathcal{C}^p(M, \mathbb{R})_U \times \mathbb{R}$  (see [14, Thm. II.2.6]).

For  $\mathcal{C}_{0-\text{reg}}^p(M, \mathbb{R})$ , one can use the transversality theorem as follows. Let  $f \in \mathcal{C}^p(M, \mathbb{R})$  and  $\epsilon \in \mathcal{C}^1(M, \mathbb{R}_{>0})$ . Let  $(\phi_i)_{i \in I}$  be a smooth partition of unity subordinated to some locally finite atlas of  $M$ . Consider the map  $\Phi: M \times \mathbb{R}^{|I|} \rightarrow \mathbb{R}, (x, (\lambda_i)) \mapsto f(x) + \epsilon(x) \sum_i \lambda_i \phi_i(x)$ . Then,  $\Phi$  is submersive, so for almost all tuples  $(\lambda_i)$ , the map  $\Phi(-, (\lambda_i))$  is transverse to 0.

The proofs work similarly with the compactness requirement added. □

### 3.2. Continuity of the intrinsic volumes

We begin with the special cases of the volume and the nodal volume, which will be needed in the proof of the general case. We actually prove a more general statement, where  $\Delta$  denotes the symmetric difference of two sets.

PROPOSITION 3.5. — *Let  $(M, g)$  be a Riemannian manifold. The functions*

$$\begin{aligned} \text{Reg}_{\text{pb}}^1(M, \mathbb{R})_U^2 &\longrightarrow \mathbb{R}_{\geq 0} \\ ((f, a), (g, b)) &\longmapsto \text{vol}(M_f^a \Delta M_g^b) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \text{Reg}_{\text{pb}}^1(M, \mathbb{R})_U &\longrightarrow \mathbb{R}_{\geq 0} \\ (f, a) &\longmapsto \text{vol}(M_f^a) \end{aligned} \tag{3.5}$$

*are continuous (resp. Lipschitz continuous) when the domains are given the uniform  $\mathcal{C}^0$ -topology (resp. are restricted to functions of class  $\mathcal{C}^2$  and given the uniform  $\mathcal{C}^2$ -topology).*

*The functions*

$$\begin{aligned} \mathcal{C}_{0-\text{reg},c}^1(M, \mathbb{R})_S^2 &\longrightarrow \mathbb{R}_{\geq 0} \\ (f, g) &\longmapsto \text{vol}(M_f^0 \triangle M_g^0) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \mathcal{C}_{0-\text{reg},c}^1(M, \mathbb{R})_S &\longrightarrow \mathbb{R}_{\geq 0} \\ f &\longmapsto \text{vol}(M_f^0) \end{aligned} \tag{3.7}$$

are continuous when the domains are given the strong  $\mathcal{C}^0$ -topology.

*The function*

$$\begin{aligned} \text{Reg}_{\text{pb}}^1(M, \mathbb{R})_U &\longrightarrow \mathbb{R}_{\geq 0} \\ (f, a) &\longmapsto \text{vol}(\partial M_f^a) \end{aligned} \tag{3.8}$$

is continuous when the domain is given the uniform  $\mathcal{C}^1$ -topology.

*Remark 3.6.* — The continuity of nodal volumes was proved in the Euclidean case in [3, Thm. 3], with a similar proof.

We first prove a lemma.

LEMMA 3.7. — *Let  $f \in \mathcal{C}_{0-\text{reg},c}^1(M, \mathbb{R})$ . For any neighborhood  $V$  of  $\mathcal{Z}_f$ , there exists an open neighborhood  $W$  of  $f$  in the strong  $\mathcal{C}^0$ -topology (and, if  $f$  is proper, in the uniform  $\mathcal{C}^0$ -topology) such that for any  $h, k \in W$ , one has  $\mathcal{Z}_h \subseteq V$  and  $M_h^0 \triangle M_k^0 \subseteq V$ .*

*Proof.* — Let  $f$  and  $V$  be as in the statement. Then  $M_f^0$  is compact. Let  $H, K$  be compact subsets such that  $M_f^f \setminus V \subseteq H \subset\subset M_f^f \subset\subset K \subseteq M_f^0 \cup V$ . By Proposition 2.1,  $f$  is strictly negative on  $\text{int } M_f^f$ , so  $\epsilon_0 := \inf\{-f(x) \mid x \in H\} > 0$ .

In the proper case, one has  $\epsilon_1 := \inf\{f(x) \mid x \in M \setminus K\} > 0$  and we set  $W := f + U_0(\min(\epsilon_0, \epsilon_1))$ . In the nonproper case, let  $\epsilon_2 := \min(\epsilon_0, \inf\{f(x) \mid x \in \partial K\}) > 0$ , let  $\epsilon \in \mathcal{C}^0(M, \mathbb{R}_{>0})$  be the function equal to  $\epsilon_2$  on  $K$  and  $\min(\epsilon_2, f)$  on  $M \setminus K$ , and set  $W := f + S_0(\epsilon)$ . In both cases, if  $h \in W$ , one has  $\mathcal{Z}_h \subseteq K \setminus H \subseteq V$ , and similarly  $M_h^0 \triangle M_k^0 \subseteq V$  if  $h, k \in W$ .  $\square$

*Proof of the proposition.* —

(i). — Let  $\Phi$  be the first function in the proposition. Since  $M_f^a = M_{\tau(f,a)}^0$  and  $\tau$  is Lipschitz continuous, it suffices to consider 0-sublevel sets. Let  $((f, 0), (g, 0)) \in \text{Reg}_{\text{pb}}^1(M, \mathbb{R})^2$ . If  $((h, 0), (k, 0)) \in \text{Reg}_{\text{pb}}^1(M, \mathbb{R})^2$ , then  $(M_f^0 \triangle M_g^0) \triangle (M_h^0 \triangle M_k^0) \subseteq (M_f^0 \triangle M_h^0) \cup (M_g^0 \triangle M_k^0)$ , so  $|\Phi((f, 0), (g, 0)) - \Phi((h, 0), (k, 0))| \leq \Phi((f, 0), (h, 0)) + \Phi((g, 0), (k, 0))$ . Therefore, it suffices to

prove the continuity (resp. Lipschitz continuity) of  $\Phi$  on the diagonal. Continuity is proved by Lemma 3.7 and we defer the proof of Lipschitz continuity to the end of the proof.

(ii). — In the non-proper case, it is similarly sufficient to prove the continuity of the third function, say  $\Psi$ , on the diagonal, and Lemma 3.7 also proves the result.

(iii). — The second (resp. fourth) function is equal to  $\Phi(-, (1, 0))$  (resp.  $\Psi(-, 1)$ ), so it is continuous.

(iv). — We now prove continuity of nodal volumes. By Lipschitz continuity of  $\tau$ , we can restrict our attention to  $(f, 0) \in \text{Reg}_{\text{pb}}^1(M, \mathbb{R})_U$ . Then,  $\mathcal{Z}_f$  is a compact  $\mathcal{C}^1$ -hypersurface. For any  $x \in \mathcal{Z}_f$ , there exists a smooth chart  $\phi: U \rightarrow \mathbb{R}^n$  such that  $\phi(U) = ]0, 1[^n$  and  $\phi(\mathcal{Z}_f \cap U)$  is the graph of a  $\mathcal{C}^1$ -function  $f^\phi: ]0, 1[^{n-1} \rightarrow ]1/3, 2/3[$  with  $\|df^\phi\|_\infty < 1$ . Since  $\mathcal{Z}_f$  is compact, there exists a finite cover  $(U_i)_{i \in I}$  of  $\mathcal{Z}_f$  by such sets, and we denote by  $\phi_i: U_i \rightarrow \mathbb{R}^n$  the corresponding charts. Let  $(\psi_i)_{i \in I}$  be a smooth partition of unity subordinated to  $(U_i)_{i \in I}$ .

By Lemma 3.7, there exists a neighborhood  $V$  of  $f$  such that the nodal set of any  $h \in V$  is included in  $\bigcup_{i \in I} U_i$ . For any  $i \in I$ , by the implicit function theorem, there exists a neighborhood  $V_i \subseteq V$  of  $f$  in the  $\mathcal{C}^1$ -topology such that for any  $h \in V_i$ , the hypersurface  $\phi_i(U_i \cap \mathcal{Z}_h)$  is the graph of a function  $h_i: ]0, 1[^{n-1} \rightarrow ]0, 1[$  with  $\|dh_i\|_\infty < 2$ . One then has  $\nabla h_i = -\left(\frac{\partial(h \circ \phi_i^{-1})}{\partial x_d}\right)^{-1} \nabla(h \circ \phi_i^{-1})$ . Set  $W := \bigcap_{i \in I} V_i$ . For any  $i \in I$ , let  $j_i: \Gamma(h_i) \hookrightarrow ]0, 1[^n$  be the inclusion of the graph of  $h_i$ . Finally, denote by  $\text{pr}_{n-1}: ]0, 1[^n \rightarrow ]0, 1[^{n-1}$  the projection on the first  $n - 1$  factors.

Let  $h \in W$  and let  $j: \mathcal{Z}_h \hookrightarrow M$  be the inclusion. Then,

$$\begin{aligned} \text{vol}(\mathcal{Z}_h) &= \sum_{i \in I} \int_{U_i \cap \mathcal{Z}_h} j^*(\psi_i \text{vol}_M) \\ &= \sum_{i \in I} \int_{\Gamma(h_i)} j_i^* \phi_{i*}(\psi_i \text{vol}_M) \\ &= \sum_{i \in I} \int_{]0, 1[^{n-1}} \sqrt{1 + \|\nabla h_i\|^2} \text{pr}_{n-1*} \phi_{i*}(\psi_i \text{vol}_M) \\ &= \sum_{i \in I} \int_{]0, 1[^{n-1}} \sqrt{1 + \left(\frac{\partial(h \circ \phi_i^{-1})}{\partial x_d}\right)^{-2} \|\nabla(h \circ \phi_i^{-1})\|^2} (\text{pr}_{n-1} \circ \phi_i)_*(\psi_i \text{vol}_M). \end{aligned}$$

On this expression, the continuity of  $\text{vol}(\mathcal{Z}_-)$  for the uniform  $\mathcal{C}^1$ -topology is clear.

(v). — We now prove the Lipschitz continuity statement. Let  $(f, 0) \in \text{Reg}_{\text{pb}}^2(M, \mathbb{R})$ . Let  $K$  be a compact neighborhood of  $\mathcal{Z}_f$  on which  $\nabla f$  does not vanish. By Lemma 3.7, there exists an open neighborhood  $V_0$  of  $f$  in the uniform  $C^0$ -topology such that for any  $h, k \in V_0$ , one has  $\mathcal{Z}_h \subseteq K$  and  $M_h \Delta M_k \subseteq K$ . Set  $\epsilon := \frac{1}{2} \inf\{\|\nabla f(x)\| \mid x \in K\} > 0$  and  $V := V_0 \cap (f + U_1(\epsilon))$ . Set  $C := \|\nabla f\|_{\infty, K} + \epsilon$ . Let  $h, k \in \mathcal{C}_{0\text{-reg}, c}^2(M, \mathbb{R}) \cap V_0$ . Then,  $M_h^0 \Delta M_k^0 \subseteq K$ . Furthermore,  $M_h^0 \Delta M_k^0$  is included in the union of the tubular neighborhoods of  $\mathcal{Z}_h$  of radius  $\frac{C}{\epsilon^2} \|h - k\|_{\infty, K}$  and of  $\mathcal{Z}_k$  of same radius, as we now prove.

Let  $x \in M_h^0 \Delta M_k^0$ . One can suppose that  $k(x) \leq 0 \leq h(x)$ . Therefore,  $0 \leq h(x) \leq \|h - k\|_{\infty, K}$ . Consider the maximal integral curve  $c$  defined by  $c(0) = x$  and  $c'(t) = -\nabla h(c(t))$ . Let  $T := \inf\{t \geq 0 \mid c(t) \in \mathcal{Z}_h \cup \mathcal{Z}_k\}$ . One has  $c([0, T]) \subseteq M_h^0 \Delta M_k^0 \subseteq K$ . One has  $(h \circ c)' = \langle \nabla h, c'(t) \rangle = \|\nabla h(c(t))\|^2 \geq (\|\nabla f(c(t))\| - \epsilon)^2 \geq \epsilon^2$ . Since  $0 \leq h(c(T)) \leq h(c(0)) = h(x)$ , the mean value theorem gives  $T \leq \frac{1}{\epsilon^2} h(x) \leq \frac{1}{\epsilon^2} \|h - k\|_{\infty, K}$ . On the other hand, one has  $\|c'(t)\| \leq C$  for  $t \in [0, T]$ . Therefore,  $d(c(0), c(T)) \leq \frac{C}{\epsilon^2} \|h - k\|_{\infty, K}$

Now we need to bound the volume of these tubular neighborhoods of  $\mathcal{Z}_h$  (and  $\mathcal{Z}_k$ ). By part (iv) of the proof, nodal volumes are continuous, hence locally bounded. By Gauss's equation (2.35) for the curvature of submanifolds and Equation (2.4) for the second fundamental form, there exists a  $\mathcal{C}_U^2$ -neighborhood of  $f$  such that the curvature of  $\mathcal{Z}_h$  is bounded for  $h$  in this neighborhood. Therefore, by [13, Thm. 8.16(i)] (and the comments preceding it concerning the passage from area to volume of tubes and the adaptation to negative lower bounds on the curvature of the ambient manifold), the volume of tubes around  $\mathcal{Z}_h$  is bounded by a linear function of the radius, uniformly in a  $\mathcal{C}_U^2$ -neighborhood  $W$  of  $f$ . Therefore, there exists  $A \in \mathbb{R}$  such that if  $h, k \in \mathcal{C}_{0\text{-reg}, c}^2(M, \mathbb{R}) \cap V \cap W$ , then  $\text{vol}(M_h^0 \Delta M_k^0) \leq \frac{A}{\epsilon} \|h - k\|_{\infty}$ .  $\square$

*Remark 3.8.* — The volume function is not uniformly continuous, as the pairs of constant functions equal to  $\pm \frac{1}{n}$  on a nonempty compact manifold  $M$  show: the volume of the 0-sublevel set jumps from 0 to  $\text{vol}(M)$  for two arbitrarily close functions. Similar examples can be given for any nonzero intrinsic volume. Recall indeed (see Subsection 3.1) that (local) Lipschitz continuity does not imply uniform continuity when the domain is not complete.

**THEOREM 3.9.** — *Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold. If  $0 \leq k \leq n$ , then the function*

$$\begin{aligned} \mathcal{L}_{n-k}^{\text{sub}} : \text{Reg}_{\text{pb}}^3(M, \mathbb{R})_U &\longrightarrow \mathbb{R} \\ (f, a) &\longmapsto \mathcal{L}_{n-k}(M_f^a) \end{aligned} \tag{3.9}$$

is Lipschitz continuous, and the function

$$\begin{aligned} \mathcal{L}_{n-k}(M_-^0) : \mathcal{C}_{0-\text{reg},c}^3(M, \mathbb{R})_S &\longrightarrow \mathbb{R} \\ f &\longmapsto \mathcal{L}_{n-k}(M_f^0) \end{aligned} \quad (3.10)$$

is continuous.

*Remark 3.10.* — By Remark 2.8, the functions  $\mathcal{L}_{n-1}^{\text{sub}}$  and  $\mathcal{L}_{n-1}(M_-^0)$  are also defined for  $\mathcal{C}^2$ -functions, and the proof below applies. For mere continuity in the proper case, the result even holds for  $\mathcal{C}^1$ -functions, as proved in Proposition 3.5.

*Remark 3.11.* — Since the Euler–Poincaré characteristic is an integer, one obtains that  $\chi(M_-)$  is locally constant on  $\text{Reg}_{\text{pb}}^3(M, \mathbb{R})_U$  and  $\chi(M_-^0)$  is locally constant on  $\mathcal{C}_{0-\text{reg},c}^3(M, \mathbb{R})_S$ . Note that this result also follows from Morse theory.

LEMMA 3.12. — *Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold and  $p \in \bar{\mathbb{N}}$ . Let  $n, k, m \in \mathbb{N}$ . The function*

$$\begin{aligned} Y_{n,k,m} : \text{Reg}^{p+3}(M, \mathbb{R})_U &\longrightarrow \mathcal{C}^p(M, \mathbb{R})_U \\ (f, a) &\longmapsto \text{div} \left( \frac{P_{n,k,m}(R, \text{Hess}(f), \nabla f)}{\eta_{f-a, 3k-2(m+1)}} \nabla f \right) \end{aligned} \quad (3.11)$$

is Lipschitz continuous. The function  $Y_{n,k,m}(-, 0) : \mathcal{C}_{0-\text{reg}}^{p+3}(M, \mathbb{R})_S \rightarrow \mathcal{C}^p(M, \mathbb{R})_S$  is continuous.

*Proof.* — The result follows from the continuity of the four functions

$$\begin{aligned} \tau : \text{Reg}^{p+3}(M, \mathbb{R})_U &\longrightarrow \mathcal{C}_{0-\text{reg}}^{p+3}(M, \mathbb{R})_U, \\ \eta_{-, 3k-2(m+1)} : \mathcal{C}_{0-\text{reg}}^{p+3}(M, \mathbb{R})_X &\longrightarrow \mathcal{C}^{p+2}(M, \mathbb{R}_{>0})_X, \\ P_{n,k,m}(R, \text{Hess}(-), \nabla -) \nabla - : \mathcal{C}^{p+3}(M, \mathbb{R})_X &\longrightarrow \mathfrak{X}^{p+1}(M)_X, \\ \text{div} : \mathfrak{X}^{p+1}(M)_X &\longrightarrow \mathcal{C}^p(M, \mathbb{R})_X \end{aligned}$$

for  $X = U, S$ , with Lipschitz continuity when  $X = U$ .  $\square$

*Proof of the theorem.* — Because of the general identity  $M_f^b = M_{f+a-b}^a$ , it is sufficient to consider  $(f, 0), (g, 0) \in \text{Reg}_{\text{p}}^3(M, \mathbb{R})$ . By Formula (2.19), one has

$$\begin{aligned} &|\mathcal{L}_{n-k}(M_g^0) - \mathcal{L}_{n-k}(M_f^0)| \\ &\leq |a_k| \left| \int_{M_g^0} \text{tr} \left( \bigwedge^{k/2} R \right) \text{vol}_M - \int_{M_f^0} \text{tr} \left( \bigwedge^{k/2} R \right) \text{vol}_M \right| \\ &\quad + \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} |b_{k,m}| \left| \int_{M_g^0} Y_{n,k,m}(g, 0) \text{vol}_M - \int_{M_f^0} Y_{n,k,m}(f, 0) \text{vol}_M \right|. \end{aligned} \quad (3.12)$$

As for the first term, the integrand  $\text{tr} \left( \bigwedge^{k/2} R \right)$  is locally bounded on  $M$ , and  $\text{vol}(M_g^0 \triangle M_f^0)$  is controlled by  $d(f, g)$  by Proposition 3.5.

As for the summands, it suffices to bound

$$\int_{M_g^0 \triangle M_f^0} \max(|Y_{n,k,m}(f, 0)|, |Y_{n,k,m}(g, 0)|) \text{vol}_M$$

and

$$\int_{M_f^0 \cap M_g^0} |Y_{n,k,m}(g, 0) - Y_{n,k,m}(f, 0)| \text{vol}_M.$$

The first term is dealt with by Proposition 3.5 (and the fact that  $Y_{n,k,m}(-, 0)$  is continuous by the lemma, hence locally bounded) and the second by the lemma. □

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