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Persisting entropy structure for nonlocal cross-diffusion systems [(*)]

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Abstract. — For cross-diffusion systems possessing an entropy (i.e. a Lyapunov functional) we study nonlocal versions and exhibit sufficient conditions to ensure that the nonlocal version inherits the entropy structure. These nonlocal systems can be understood as population models *per se* or as approximation of the classical ones. With the preserved entropy, we can rigorously link the approximating nonlocal version to the classical local system. From a modelling perspective, this gives a way to prove a derivation of the model and, from a PDE perspective, this provides a regularisation scheme to prove the existence of solutions. A guiding example is the SKT model [22]. In this context, we answer positively the question raised by [12] for the derivation and thus complete the derivation.

1. Introduction

1.1. Cross-diffusion systems with entropy structure

Our starting points are cross-diffusion systems of n species with densities $u = (u_i)_{1\leqslant i\leqslant n}$ solving a system

$$\partial_t u_i - \operatorname{div}\left(\sum_{j=1}^{n} a_{ij}(u)\nabla u_j\right) = 0, \qquad \text{for } i = 1,\dots,n, \tag{1.1}$$

on a domain Ω supplemented with boundary conditions and initial data u^{init}. Here a_{ij} are given scalar functions ($\mathbb{R}^n_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$) and the unknowns are the

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Article proposé par Isabelle Gallagher.

model densities u_i's, which are therefore expected to be nonnegative. The matrix $A(u) := (a_{ij}(u))$ is called the *diffusion* matrix and is always assumed to be positive definite. As this work focuses on the entropy structure for the diffusion, we do not consider here any reaction terms.

Without any assumptions on the a_{ij}'s, the only estimate that we have on system (1.1) is the conservation of the overall mass, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_i = 0,$$

for $i = 1, \ldots, n$. Due to the severe nonlinearity of the system, this sole control is not sufficient to obtain the existence of global solutions. Searching for a Lyapunov functional of the form

$$H(u) := \int_\Omega \sum_{i=1}^n h_i(u_i), \tag{1.2}$$

where $h_i \in \mathscr{C}^0(\mathbb{R}_{\geqslant 0}) \cap \mathscr{C}^2(\mathbb{R}_{>0})$, we find formally without boundary terms that

$$\frac{\mathrm{d}}{\mathrm{d}t}H(u) = -\int_\Omega \begin{pmatrix} \nabla u_1 \\ \vdots \\ \nabla u_n \end{pmatrix} \cdot M(u) \begin{pmatrix} \nabla u_1 \\ \vdots \\ \nabla u_n \end{pmatrix},$$

with $M : \mathbb{R}^n_{>0} \to \mathbb{R}^{n\times n}$ defined by

$$M(y) = \begin{pmatrix} h_1''(y_1) & 0 & \ldots & 0 \\ 0 & h_2''(y_2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & h_n''(y_n) \end{pmatrix} \begin{pmatrix} a_{11}(y) & a_{12}(y) & \ldots & a_{1n}(y) \\ a_{21}(y) & a_{22}(y) & \ldots & a_{2n}(y) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(y) & a_{n2}(y) & \ldots & a_{nn}(y) \end{pmatrix}. \tag{1.3}$$

Hence we have a positive dissipation $I = -\mathrm{d}H/\mathrm{d}t$ if (the symmetric part of) M is positive semi-definite. This motivates the following definition, where the second part quantifies the dissipation.

Definition 1.1 (Entropy structure). — *We say that the system* (1.1) *has an* entropy structure *if there exist n functions $h_1, \ldots, h_n \in \mathscr{C}^0(\mathbb{R}_{\geqslant 0}) \cap \mathscr{C}^2(\mathbb{R}_{>0})$ such that the corresponding matrix map $M : \mathbb{R}^n_{>0} \to \mathbb{R}^{n\times n}$ defined by* (1.3) *takes its values in the cone of positive definite matrices. We say that this entropy structure is* uniform *when there exist furthermore n functions $\alpha_1, \ldots, \alpha_n : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that for all $(z, v) \in \mathbb{R}^n \times \mathbb{R}^n_{>0}$ it holds*

$$z^T \cdot M(v)z \geqslant \sum_{i=1}^n \alpha_i(v_i)^2 z_i^2. \tag{1.4}$$

For a given entropy structure the functions h_i's are called the entropy densities*, α_i's are the* dissipations *and the functional H defined in* (1.2) *is called the* entropy.

Remark 1.2. — From the assumed positive definiteness of the diffusion matrix A, it directly follows that for every entropy structure all the functions h_i, $i = 1, \dots, n$, are convex.

Remark 1.3. — For typical examples, as the SKT system (1.5) below, the entropy densities h_i have diverging derivative towards the origin so that we define the matrix map M only for positive arguments. In this work, we also take $\mathbb{R}_{\geqslant 0}$ for the range of the densities which is the most common case. In general the entropy structure can also be defined for bounded subsets of $\mathbb{R}$, cf. [15].

Smooth solutions for the system (1.1) are known to exist, at least locally in time, thanks to the work of Amann [1] which gives also a criteria of explosion for such solutions. Apart from the very specific case of triangular system [13], for global solutions the current literature allows only weak solutions and relies crucially on the entropy structure.

For an overview of such cross-diffusion systems we refer to [15], which gives a list of examples in the introduction and also uses the quantified condition (1.4). Note that [15] allows in principle more general entropies but, apart from the volume-filling models, all examples have the additive form (1.2) required in this work.

A guiding example is the SKT system with densities u_1 and u_2

$$\begin{cases} \partial_t u_1 = \Delta\big((d_1 + d_{12}u_2)u_1\big), \\ \partial_t u_2 = \Delta\big((d_2 + d_{21}u_1)u_2\big) \end{cases} \tag{1.5}$$

with parameters $d_1, d_2, d_{12}, d_{21} \geqslant 0$. This system has been introduced by Shigesada, Kawasaki and Teramoto [22]. Writing the system in divergence form (1.1), the matrix $(a_{ij}(u))_{ij}$ reads

$$\begin{pmatrix} d_1 + d_{12}u_2 & d_{12}u_1 \\ d_{21}u_2 & d_2 + d_{21}u_1 \end{pmatrix}.$$

For nonnegative solutions this matrix has nonnegative trace and determinant. As remarked by Chen and Jüngel [6] the following entropy allows to symmetrize the system

$$H(u_1, u_2) := \int_{\mathbb{T}^d} \big(h_1(u_1) + h_2(u_2)\big), \tag{1.6}$$

with

$$h_1(z) = d_{21}\psi(z), \quad h_2(z) := d_{12}\psi(z), \quad \psi(z) = z\log(z) - z + 1. \tag{1.7}$$

Indeed, one checks that

$$M(u_1, u_2) = \begin{pmatrix} h_1''(u_1) & 0 \\ 0 & h_2''(u_2) \end{pmatrix} \begin{pmatrix} d_1 + d_{12}u_2 & d_{12}u_1 \\ d_{21}u_2 & d_2 + d_{21}u_1 \end{pmatrix} = d_{12}d_{21} \begin{pmatrix} \star & 1 \\ 1 & \star \end{pmatrix},$$

so that M is symmetric and still has nonnegative determinant and trace. Thus M is positive semi-definite and forms with H an entropy structure again under the necessary condition that the solution is nonnegative.

It is also known (see [10, 15, 16] for instance) that the previous entropy structure of the additive form (1.2) can be found for substantial generalization of (1.5) in the following general class of cross-diffusion systems

$$\begin{cases} \partial_t u_1 = \Delta\big(\mu_1(u_1,u_2)\, u_1\big), \\ \partial_t u_2 = \Delta\big(\mu_2(u_1,u_2)\, u_2\big), \end{cases} \tag{1.8}$$

where the nonlinear functions μ_1 and μ_2 are assumed $\mathscr{C}^0(\mathbb{R}^2_{\geqslant 0}) \cap \mathscr{C}^1(\mathbb{R}^2_{>0})$ so that (1.8) can be written in divergence form (1.1) in order for the entropy structure to makes sense. For the analysis of the PDE the difficulty comes from the cross-diffusion effect so that we will focus on the case without self-diffusion (imposing that μ_i does not depend on u_i)

$$\begin{cases} \partial_t u_1 = \Delta\big(\mu_1(u_2)\, u_1\big), \\ \partial_t u_2 = \Delta\big(\mu_2(u_1)\, u_2\big). \end{cases} \tag{1.9}$$

The contribution of this paper is a constructive answer to the following question.

Main question. — *For a cross-diffusion system with an entropy H of the form* (1.2)*, does there exist a spatial mollification of the diffusion such that the mollified system still has an entropy?*

These mollified systems are called *nonlocal* because the diffusion rate of one species at a given point x does not depend anymore solely on the population density at this place, but on a *space average* around it.

We provide a family of spatial mollifications of the cross-diffusion keeping the entropy structure, where our intuition takes its origin from the article [8] in which the first author of the current article exhibited an entropy structure for the SKT systems under a spatial discretization. To the best of our knowledge, the current literature does not offer any prior example of persisting entropy structure for a nonlocal cross-diffusion systems.

Approximation results. The usage of a spatial mollification was first proposed by Bendahme et al. [3] and Lepoutre et al. [17] where no rigorous link with the original model was established.

Having the entropy structure at hand, we can rigorously perform the limit from the mollified nonlocal system to the original local system. This gives immediately a new proof of existence of global weak entropy solutions.

Such an existence result is nontrivial because an adequate approximation scheme has to create nonnegative solutions and to keep the entropy structure, as it has been done before by an entropic change of variable [15] or with a semi-discrete scheme [10].

Derivation from particle models. The main motivation comes from the derivation of many particle models as a mean-field limit. The starting point is by Fontbona and Méléard [12] who performed a stochastic derivation of a regularised cross-diffusion system with a nonlocal spatial regularisation. Their aim was not to produce an adequate approximation scheme but to derive the SKT system from a particle model. However, they could not handle the last step of the derivation and they explicitly raised the question, whether it is possible to find the classical local cross-diffusion system in the limit of small regularisation.

A partial answer in this direction is given in [20] in the special case of triangular diffusion coefficients. The recent work by Chen et al. [5] uses the same approximation by spatial mollifiers and manage to prove rigorously the limit of small regularization *and* large population at the same time. In the aspect of taking both asymptotic limits at once, the analysis of [5] goes beyond the program of [12]. However, for proving the uniform stability of the mollified systems, [5] impose the assumption of small cross-diffusion coefficients so that the cross-diffusion terms can be handled perturbatively. Hence the result of [5] does *not* cover the full SKT system (1.5).

In general, our proposed regularisation scheme is different to the one used in [5, 12] but agrees on the important example of a linear rate SKT system (1.5). Hence we provide, to the best of our knowledge, the first complete derivation of this popular cross-diffusion model *via* nonlocal approximation.

For completeness, we note that other approaches for the derivation of cross-diffusion systems are fast reaction asymptotics [9, 14, 24] or spatial discretisations (without convolution). The later method was formally proposed in [8] and recently revisited rigorously in [2].

Plan. In the following Sections 1.2 and 1.3, we introduce our nonlocal mollifications keeping the entropy structure. We then state the existence and convergence results for the mollified systems in the following Section 1.4. In the remainder of the paper these results are then proved.

As we focus on the approximation scheme, we will show existence of solutions of the regularised systems by PDE techniques, where we already see the effectiveness of the regularisation. We expect that the stochastic derivation can be adapted; but leave a general derivation from particle models for future work.

Another future direction is the study of the gradient flow structure. Formally, the original local system often has a gradient flow structure which in most studies is only used in the form of the dissipation inequality (an exception is [25]). Having found a regularisation, we plan for future work to investigate the gradient flow formulation of the nonlocal system and the limit towards the local system. Such limits of gradient flows are an active field and we only mention [4, 18, 21] as starting points.

1.2. Regularisation on the torus

The starting point was [8], where the entropy structure was understood for the linear rates SKT model in a spatial discretisation. In this paper the intuition is to relate the entropy structure to the reversibility of a Markov chain modelling an N-particle system whose mean-field limit converges (formally) to the spatially discrete system.

Briefly, the idea in [8] is that, on a particle model with discrete space variable, the entropy structure is obtained by imposing that a pair of particles is jumping together with a suitable rate. Trying to use this idea for a nonlocal approximation, we intuitively want to make pairs of particle with a given distance jump together. In order to identify the pairs, we therefore take the convolution reflected between the two species.

For $\Omega = \mathbb{T}^d$ and a nonnegative measure ρ on $\mathbb{T}^d$ this motivates the following regularisation of (1.5)

$$\begin{cases} \partial_t u_1 = \Delta\big((d_1 + d_{12}\, u_2 \star \rho) u_1\big), \\ \partial_t u_2 = \Delta\big((d_2 + d_{21}\, u_1 \star \check{\rho}) u_2\big), \end{cases} \tag{1.10}$$

where $\check{\rho}$ is the reflected measure. To understand why such a system may preserve an entropy structure if any, it is instructive to study the particular case of a Dirac measure $\rho = \delta_a$. In that case, using the translation operator $\tau_a f := f(\,\cdot - a)$, and applying this operator to the second equation, the previous system becomes

$$\begin{cases} \partial_t u_1 = \Delta\big((d_1 + d_{12}\, \tau_a u_2) u_1\big), \\ \partial_t \tau_a u_2 = \Delta\big((d_2 + d_{21}\, u_1) \tau_a u_2\big), \end{cases}$$

that is, we recover a standard SKT system in which the second species u_2 is replaced by the shifted one $\tau_a u_2$. In particular, for H of the form (1.2) we recover directly

$$\frac{\mathrm{d}}{\mathrm{d}t} H(u_1, \tau_a u_2) = -\int_{\mathbb{T}^d} \begin{pmatrix} \nabla u_1 \\ \nabla \tau_a u_2 \end{pmatrix} \cdot M\big(u_1, \tau_a u_2\big) \begin{pmatrix} \nabla u_1 \\ \nabla \tau_a u_2 \end{pmatrix},$$

where M is the same matrix associated to H, as in the usual local case. Due to the translation invariance of the Lebesgue measure, we have $H(u_1, \tau_a u_2) = H(u_1, u_2)$ and we therefore recover the entropy structure in the sense that H still defines a Lyapunov functional with a dissipation.

In the case of a general nonnegative measure ρ, the previous computation can be reproduced once noted that the system (1.10) can be rewritten (the Laplacian does not act on the translation variable y)

$$\begin{cases} \partial_t u_1 = \displaystyle\int_{\mathbb{T}^d} \Delta\big((d_1 + d_{12}\, \tau_y u_2) u_1\big)\, \mathrm{d}\rho(y), \\ \partial_t u_2 = \displaystyle\int_{\mathbb{T}^d} \Delta\big((d_2 + d_{21}\, \tau_{-y} u_1) u_2\big)\, \mathrm{d}\rho(y), \end{cases}$$

and using once more the translation invariance of the Lebesgue measure (after integrating in the variable x), one recovers this time

$$\frac{\mathrm{d}}{\mathrm{d}t} H(u_1, u_2) = -\int_{\mathbb{T}^d} \left\{ \int_{\mathbb{T}^d} \begin{pmatrix} \nabla u_1 \\ \nabla \tau_y u_2 \end{pmatrix} \cdot M\big(u_1, \tau_y u_2\big) \begin{pmatrix} \nabla u_1 \\ \nabla \tau_y u_2 \end{pmatrix} \right\} \mathrm{d}\rho(y).$$

This computation can be adapted to the generalised SKT system (1.8) with the following caution: the spatial regularisation has to be applied after the nonlinearity, without affecting the self-diffusion. By rescaling the time, the kernel can be normalised (i.e. ρ defines a probability measure) and we find, more precisely, the following proposition.

Proposition 1.4. — *Consider two smooth functions $\mu_1, \mu_2 : \mathbb{R}^2_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ and the corresponding system* (1.8)*. If this system has an entropy H, then for any nonnegative smooth kernel ρ of integral* 1*, any smooth solution of the following nonlocal system (τ_y is the translation operator)*

$$\begin{cases} \partial_t u_1 - \Delta \left[\displaystyle\int_{\mathbb{T}^d} \rho(y)\, \mu_1\big(u_1, \tau_y u_2\big)\big)\, \mathrm{d}y\, u_1 \right] = 0, \\ \partial_t u_2 - \Delta \left[\displaystyle\int_{\mathbb{T}^d} \check{\rho}(y)\, \mu_2\big(\tau_y u_1, u_2\big)\big)\, \mathrm{d}y\, u_2 \right] = 0 \end{cases} \tag{1.11}$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} H(u_1(t), u_2(t)) \leqslant 0.$$

If the entropy structure of the system (1.8) *is furthermore assumed uniform with dissipation α_1 and α_2, then we have formally*

$$\frac{\mathrm{d}}{\mathrm{d}t} H(u_1(t), u_2(t)) + D(t) \leqslant 0,$$

where

$$D(t) := \int_{\mathbb{T}^d} \alpha_1(u_1(t))^2|\nabla u_1(t)|^2 + \int_{\mathbb{T}^d} \alpha_2(u_2(t))^2|\nabla u_2(t)|^2.$$

Remark 1.5. — We considered here a smooth setting in order to avoid tedious justifications for the computations below, but we will show later how this can be made rigorous in the a weaker setting (see Theorem 1.10).

Proof. — Denoting by h_1 and h_2 the entropy densities, we find by multiplying the first equation of (1.11) by $h_1'(u_1)$ and integrating over $\mathbb{T}^d$ that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} h_1(u_1)\,\mathrm{d}x &= -\int_{\mathbb{T}^d} h_1''(u_1)\nabla u_1 \cdot \left\{\int_{\mathbb{T}^d} \rho(y)\nabla(\mu_1(u_1,\tau_y u_2)u_1)\,\mathrm{d}y\right\}\mathrm{d}x\\ &= -\int_{\mathbb{T}^d} \rho(y)\left\{\int_{\mathbb{T}^d} h_1''(u_1)\nabla u_1\cdot\nabla(\mu_1(u_1,\tau_y u_2)u_1)\,\mathrm{d}x\right\}\mathrm{d}y,\end{aligned}$$

where τ_y is the translation operator and ∇ acts on the x (not noted) variable only. We have a similar formula for the second equation, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} h_2(u_2)\,\mathrm{d}x = -\int_{\mathbb{T}^d}\check{\rho}(y)\left\{\int_{\mathbb{T}^d} h_1''(u_2)\nabla u_2\cdot\nabla(\mu_2(\tau_y u_1,u_2)u_2)\,\mathrm{d}x\right\}\mathrm{d}y.$$

Intuitively speaking, we want to collect the pairs $u_1(x)$ and $u_2(x-y)$ in both expressions. This motivates in the double integral in the variables x, y of the last right hand side the change of variable $(x,y)\mapsto(z-w,-w)$. Using that the translation commutes with differential operators, we find

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} h_2(u_2)\\ &\quad= -\int_{\mathbb{T}^d}\check{\rho}(-w)\left\{\int_{\mathbb{T}^d} h_2''(\tau_w u_2)\nabla\tau_w u_2\cdot\nabla(\mu_1(u_1,\tau_w u_2)\tau_w u_2)\mathrm{d}z\right\}\mathrm{d}w.\end{aligned}$$

Since $\check{\rho}(-w)=\rho(w)$, renaming the variables as before, we collect both contributions as

$$\frac{\mathrm{d}}{\mathrm{d}t}H(u_1(t),u_2(t)) = -\int_{\mathbb{T}^d}\rho(y)\left\{\int_{\mathbb{T}^d}\begin{pmatrix}\nabla u_1\\ \nabla\tau_y u_2\end{pmatrix}\cdot M\Big(u_1,\tau_y u_2\Big)\begin{pmatrix}\nabla u_1\\ \nabla\tau_y u_2\end{pmatrix}\right\}\mathrm{d}y,$$

where M is given by (1.3), the coefficients of the matrix A being the one used to write (1.8) in divergence form (1.1). The fact that H is a Lyapunov functional and the precised dissipation in case of uniform entropy follow (for the latter, use the normalisation of ρ). □

If there is no self diffusion in the generalised system (1.8) like in (1.9), the nonlocal system (1.11) becomes simply

$$\begin{cases}\partial_t u_1 = \Delta\big((\mu_1(u_2)\star\rho)u_1\big),\\ \partial_t u_2 = \Delta\big((\mu_2(u_1)\star\check{\rho})u_2\big).\end{cases}\tag{1.12}$$

Thus, compared to [12], the spatial regularisation is applied after the nonlinearity, while they do it the opposite way in their stochastic derivation. However, in the fundamental case of the (linear) SKT system (1.5), we get the same system.

For these systems with the Laplace structure, another important role is played by the duality estimates, see [10, 16, 20]. An advantage of the previous scheme is that these duality estimates naturally continue to work in the nonlocal versions.

Remark 1.6. — In the regularisation (1.11) the rate $(\mu_i)_{i=1,2}$ is averaged with respect to the cross-diffusion influence but a possible nonlinear self-diffusion is not regularised. However, a nonlinear self-diffusion tends to improve the entropy-dissipation estimates and we thus focus on cases without self-diffusion.

For stochastic derivations it is, nevertheless, interesting to also regularise the self-diffusion. In a general setting this destroys the entropy structure and we need a compatibility with the entropy structure. For such a regularisation consider a symmetric kernel σ, i.e. $\check{\sigma} = \sigma$, and assume that (1.8) can be written as

$$\begin{cases} \partial_t u_1 = \Delta\big((\mu_1(u_2) + \kappa_1(u_1))\, u_1\big), \\ \partial_t u_2 = \Delta\big((\mu_2(u_1) + \kappa_2(u_2))\, u_2\big), \end{cases}$$

where the system without the κ has an entropy structure with an entropy H consisting of h_1 and h_2 and matrix map M. We then propose the regularisation

$$\begin{cases} \partial_t u_1(x) = \Delta_x\Bigg[\bigg(\displaystyle\int_{y\in\mathbb{T}^d} \rho(x-y)\, \mu_1\big(u_2(y)\big)\,\mathrm{d}y \\ \qquad\qquad\qquad + \displaystyle\int_{y\in\mathbb{T}^d} \sigma(x-y)\, \kappa_1\big(u_1(y)\big)\,\mathrm{d}y\bigg) u_1(x)\Bigg], \\ \partial_t u_2(y) = \Delta_y\Bigg[\bigg(\displaystyle\int_{x\in\mathbb{T}^d} \rho(x-y)\, \mu_2\big(u_1(x)\big)\,\mathrm{d}x \\ \qquad\qquad\qquad + \displaystyle\int_{x\in\mathbb{T}^d} \sigma(x-y)\, \kappa_2\big(u_2(x)\big)\,\mathrm{d}x\bigg) u_2(y)\Bigg]. \end{cases}$$

For the dissipation we then find

$$-\frac{\mathrm{d}}{\mathrm{d}t} H(u_1, u_2) = I_\mu + I_1 + I_2,$$

where I_μ is the dissipation with $\kappa_1 = \kappa_2 = 0$ and thus has a good sign. The new terms are after using symmetrisation $\check{\sigma} = \sigma$

$$I_i = \frac{1}{2}\int_{x\in\Omega}\int_{y\in\mathbb{R}^d} \sigma(y) \begin{pmatrix} \nabla u_i(x) \\ \nabla u_i(x-y) \end{pmatrix} \cdot N_i \begin{pmatrix} \nabla u_i(x) \\ \nabla u_i(x-y) \end{pmatrix} \mathrm{d}y\,\mathrm{d}x$$

with

$$N_i = \begin{pmatrix} h_i''(u_i(x)) & 0 \\ 0 & h_i''(u_i(x-y)) \end{pmatrix} \begin{pmatrix} \kappa_1(u_i(x-y)) & u_i(x)\kappa_1'(u_i(x-y)) \\ u_i(x-y)\kappa_1'(u_i(x)) & \kappa_1(u_i(x)) \end{pmatrix}$$

for $i = 1, 2$.

Hence H is still an entropy if $(N_i)_{i=1,2}$ are always positive semi-definite which gives an extra condition on the system. We note, however, that for the studied SKT system (1.5) this condition is always satisfied under the natural assumption that effect on the other species is of the same form as the self-diffusion effect, i.e. that it takes the form

$$\begin{cases} \partial_t u_1 = \Delta\big((d_1 + d_{12}u_2^\alpha + d_{11}u_1^\beta)u_1\big), \\ \partial_t u_2 = \Delta\big((d_2 + d_{21}u_1^\beta + d_{22}u_2^\alpha)u_2\big), \end{cases}$$

for constants $d_1, d_2, d_{11}, d_{12}, d_{21}, d_{22}, \alpha, \beta \in \mathbb{R}_{>0}$ with $\alpha\beta \leqslant 1$ (see, e.g., [10] for the discussion of the local case).

Remark 1.7 (Dirichlet boundary condition by penalisation). — For cross-diffusion systems where the entropy-dissipation is sufficient for controlling the nonlinearity, we can impose Dirichlet boundary conditions by a penalisation method. We sketch the argument and the estimate *formally* for the linear rate SKT system

$$\begin{cases} \partial_t u_1 = \Delta\big((d_1 + d_{11}u_1 + d_{12}u_2)u_1\big), \\ \partial_t u_2 = \Delta\big((d_2 + d_{21}u_1 + d_{22}u_2)u_2\big) \end{cases}$$

with active self-diffusion, i.e. $d_{11}, d_{22} > 0$, which has the same entropy functional (1.6) as (1.5).

For the penalisation problem take N large enough such that $\overline{\Omega} \subset\subset (-N, N)^d$ and identify the hypercube $[-N, N]^d$ with the flat torus $\mathbb{T}_N^d := (\mathbb{R}/2N\mathbb{Z})^d$. Assume constant in time Dirichlet boundary conditions for species u_i given by b_i and suppose the boundary data can be extended to a twice continuously differentiable map $b_i : \mathbb{T}_N^d \to (0, \infty)$ with $\|\nabla^2 b_i\|_\infty \lesssim 1$.

Using the spatial regularisation with parameter $\epsilon > 0$ and a penalisation with parameter $\delta > 0$, we arrive at the system

$$\begin{cases} \partial_t u_{1,\varepsilon,\delta} - \Delta((d_1 + d_{11}u_{1,\varepsilon\,\delta} + d_{12}u_{2,\varepsilon,\delta} \star \rho_\varepsilon)u_{1,\varepsilon,\delta}) = -\dfrac{1}{\delta}(u_{1,\varepsilon,\delta} - b_1)\mathbb{1}_{\mathbb{T}_N^d \setminus \Omega}, \\ \partial_t u_{2,\varepsilon,\delta} - \Delta((d_2 + d_{21}u_{1,\varepsilon,\delta} \star \check{\rho}_\varepsilon + d_{22}u_{2,\varepsilon,\delta})u_{2,\varepsilon,\delta}) = -\dfrac{1}{\delta}(u_{2,\varepsilon,\delta} - b_2)\mathbb{1}_{\mathbb{T}_N^d \setminus \Omega}. \end{cases}$$

As discussed before, the convolution keeps the entropy structure but the penalisation term may break it. The idea is that the entropy structure only depends on the second derivative so that we can shift the entropy functional accordingly, i.e. we consider the new entropy

$$H(u_1,u_2)=\int_{\mathbb{T}^d_N}\Big(h_1(u_1)-h_1'(b_1)u_1-h_1(b_1)+h_1'(b_1)b_1 \\ +h_2(u_2)-h_2'(b_2)u_2-h_2(b_2)+h_2'(b_2)b_2\Big)$$

so that the penalisation term creates a term with a good sign. We then find the dissipation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H=&-\int_{\mathbb{T}^d_N}\nabla u_{1,\varepsilon,\delta}\,h_1''(u_{1,\varepsilon,\delta})\,\nabla\big((d_1+d_{11}u_{1,\varepsilon\,\delta}+d_{12}u_{2,\varepsilon,\delta}\star\rho_\varepsilon)u_{1,\varepsilon,\delta}\big)\\&-\int_{\mathbb{T}^d_N}\nabla u_{2,\varepsilon,\delta}\,h_2''(u_{2,\varepsilon,\delta})\,\nabla\big((d_2+d_{22}u_{2,\varepsilon\,\delta}+d_{21}u_{1,\varepsilon,\delta}\star\check{\rho}_\varepsilon)u_{2,\varepsilon,\delta}\big)\\&-\int_{\mathbb{T}^d_N}\Delta(h_1'(b_1))\,\big((d_1+d_{11}u_{1,\varepsilon\,\delta}+d_{12}u_{2,\varepsilon,\delta}\star\rho_\varepsilon)u_{1,\varepsilon,\delta}\big)\\&-\int_{\mathbb{T}^d_N}\Delta(h_2'(b_2))\,\big((d_2+d_{22}u_{2,\varepsilon\,\delta}+d_{21}u_{1,\varepsilon,\delta}\star\check{\rho}_\varepsilon)u_{2,\varepsilon,\delta}\big)\\&-\frac{1}{\delta}\int_{\mathbb{T}^d_N}\Big[(h_1'(u_{1,\varepsilon,\delta})-h_1'(b_1))(u_{1,\varepsilon,\delta}-b_1)+(h_2'(u_{2,\varepsilon,\delta})-h_2'(b_1))(u_{2,\varepsilon,\delta}-b_2)\Big].\end{aligned}$$

The first two lines are exactly the dissipation we have found for the regularised system. The next two lines are error terms and the last line is a good dissipation term from the penalisation. Hence we find for a constant c that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H\leqslant&-\int_{\mathbb{T}^d_N}(d_{11}d_{21}|\nabla u_{1,\varepsilon,\delta}|^2+d_{22}d_{12}|\nabla u_{2,\varepsilon,\delta}|^2)\\&-\frac{1}{\delta}\int_{\mathbb{T}^d_N}\Big[(h_1'(u_{1,\varepsilon,\delta})-h_1'(b_1))(u_{1,\varepsilon,\delta}-b_1)+(h_2'(u_{2,\varepsilon,\delta})-h_2'(b_1))(u_{2,\varepsilon,\delta}-b_2)\Big]\\&+c\,(1+\|u_{1,\varepsilon,\delta}\|_2^2+\|u_{2,\varepsilon,\delta}\|_2^2).\end{aligned}$$

As the error term can be controlled by the entropy and dissipation, this yields a uniform control. Hence we can pass to the limit. By the dissipation we can control the nonlinearity and by the penalisation term the prescribed boundary data are obtained.

1.3. General regularisation scheme

In the previous subsection we considered the special case of two species on the torus. In this subsection we will generalise the regularisation scheme to several species and general domains Ω. Here the boundary implies that the specific Laplace structure as in (1.8) is not preserved and we have the general divergence structure as in (1.1), see Remark 1.9.

For the two densities case on the torus, we used the convolution in order to define how a pair is interacting in the cross-diffusion. In the general case of n densities on a domain Ω, the suitable generalisation is a kernel $K : \Omega^n \to \mathbb{R}_{\geqslant 0}$ between all densities and the intuitive idea is that the cross-diffusion between the densities $u_1(x_1), u_2(x_2), \dots, u_n(x_n)$ at positions $x_1, x_2, \dots, x_n \in \Omega$ happens with the intensity $K(x_1, x_2, \dots, x_n)$. The idea of using a kernel on a bounded domain has been proposed in [17], where K is the fundamental solution the (Neumann) operator $\mathrm{Id} - \delta\Delta$ with $0 < \delta \ll 1$. However, the authors kept the Laplace structure and applied the regularisation before the nonlinearity so that the entropy structure was lost, see Remark 1.9 below.

At the boundary such a general tuple cannot diffuse freely if we impose no-flux boundary conditions. Hence in order to rule out boundary terms we further assume that

$$K(x_1, \dots, x_n) = 0 \qquad \text{if } x_i \in \partial\Omega \text{ for } i = 1, \dots, n. \tag{1.13}$$

A family of kernel K^ϵ for $\epsilon > 0$ then yields an approximation of the local system if the kernel is concentrating on the diagonal as $\epsilon \to 0$, i.e. for a species $i = 1, \dots, n$, a point $x_i \in \Omega$ and a sufficiently nice test function $\phi : \Omega^n \to \mathbb{R}$ it holds that

$$\prod_{j \neq i} \int_{x_j \in \Omega} \mathrm{d}x_j K^\epsilon(x_1, \dots, x_n)\, \phi(x_1, \dots, x_n) \longrightarrow \phi(x_i, \dots, x_i) \qquad \text{as } \epsilon \longrightarrow 0,$$

where we introduced the notation $\prod_{j \neq i} \int_{x_j \in \Omega} \mathrm{d}x_j$ to denote the repeated integral over all coordinates x_j with $j \neq i$, i.e.

$$\prod_{j \neq i} \int_{x_j \in \Omega} \mathrm{d}x_j := \int_{x_1 \in \Omega} \mathrm{d}x_1 \cdots \int_{x_{i-1} \in \Omega} \mathrm{d}x_{i-1} \int_{x_{i+1} \in \Omega} \mathrm{d}x_{i+1} \cdots \int_{x_n \in \Omega} \mathrm{d}x_n.$$

A natural candidate of such kernels K^ϵ is a smoothing of

$$K^\epsilon(x_1, \dots, x_n) = C_\epsilon 1_{|x_i - x_j| \leqslant \epsilon, i, j = 1, \dots, n}$$

with a cutoff towards the boundary and a suitable constant C_ϵ.

We can now state our proposed general regularisation.

Proposition 1.8. — *Let $n \in \mathbb{N}$ be the number of densities and assume rates $(a_{ij})_{i,j=1,\dots,n}$ such that the local system* (1.1) *has an entropy H.*

For a constant $\epsilon > 0$*, a domain* $\Omega \in \mathbb{R}^d$ *and a kernel* $K : \Omega^n \to \mathbb{R}_{\geqslant 0}$ *satisfying* (1.13)*, suppose of densities* $u_1, \dots, u_n : \Omega \to \mathbb{R}_{\geqslant 0}$ *evolving in time* t *by the nonlocal system* $(i = 1, \dots, n)$

$$\begin{aligned} &\partial_t u_i(x_i) - \epsilon \Delta u_i(x_i) \\ &\quad - \operatorname{div}_{x_i}\left(\prod_{k \neq i} \int_{x_k \in \Omega} \mathrm{d}x_k \, K(x_1, \dots, x_n) \sum_{j=1}^n a_{ij}\big(u_1(x_1), \dots, u_n(x_n)\big) \nabla u_j(x_j) \right) \\ &= 0 \end{aligned} \tag{1.14}$$

supplemented in the case of boundaries with von Neumann boundary conditions

$$n \cdot \nabla u_i(x) = 0 \text{ for } x \in \partial\Omega. \tag{1.15}$$

Then it holds formally that

$$\frac{\mathrm{d}}{\mathrm{d}t} H(u_1(t), \dots, u_n(t)) \leqslant 0.$$

In the case of uniform dissipations $(\alpha_i)_i$ *it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t} H(u_1(t), \dots, u_n(t)) + D(t) \leqslant 0,$$

where

$$D(t) = \sum_{i=1}^n \int_\Omega \left[\epsilon \, h_i''(u_i(x)) + \alpha_i (u_i(x))^2 w_i(x)\right] |\nabla u_i(x)|^2 \, \mathrm{d}x$$

with the weights $(w_i)_{i=1,\dots,n}$ *defined as*

$$\prod_{j \neq i} \int_{x_j \in \Omega} \mathrm{d}x_j K(x_1, \dots, x_n) = w_i(x_i) \geqslant 0, \qquad \forall \, x_i \in \Omega. \tag{1.16}$$

Here we added a small global diffusion with ϵ in order to compensate that the kernel K vanishes at the boundary so that the system becomes uniformly parabolic and we can obtain global regularity estimates for the regularised system. By the assumption (1.13) the imposed von Neumann boundary conditions imply zero-flux boundary conditions.

Proof. — In order to obtain the estimate, the idea is to collect the interaction in a tuple $u_1(x_1), \dots, u_n(x_n)$. We then find for the dissipation

$$\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t} H(u(t)) \\
&= -\sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t} \int_{x_i \in \Omega} h_i(x_i)\, \mathrm{d}x_i \\
&= \sum_{i=1}^{n} \int_{x_i} \nabla u_i(x_i)\, h_i''(u_i(x_i)) \\
&\qquad \cdot \left[\epsilon + \prod_{k \neq i} \int_{x_k \in \Omega} \mathrm{d}x_k\, K(x_1, \dots, x_n) \sum_{j=1}^{n} a_{ij}\big(u_1(x_1), \dots, u_n(x_n)\big) \nabla u_j(x_j) \right] \mathrm{d}x_i \\
&= \int_{x_1} \mathrm{d}x_1 \cdots \int_{x_n} \mathrm{d}x_n K(x_1, \dots, x_n) \begin{pmatrix} \nabla u_1(x_1) \\ \vdots \\ \nabla u_n(x_n) \end{pmatrix} \cdot M\big(u_1(x_1), \dots, u_n(x_n)\big) \begin{pmatrix} \nabla u_1(x_1) \\ \vdots \\ \nabla u_n(x_n) \end{pmatrix} \\
&\qquad + \epsilon \int_{x \in \Omega} \sum_{i=1}^{n} h_i''(u_i(x))\, |\nabla u_i(x)|^2\, \mathrm{d}x,
\end{aligned}$$

where M is the matrix from the entropy structure, Definition 1.1, and the boundary terms vanish due to the von Neumann boundary condition and (1.13).

By the assumed sign of the matrix M and the lower bound by α_i, respectively, the result follows. □

Remark 1.9. — The previous regularisation (1.10) for two species in the simple setting $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$ is exactly recovered by setting $K(x_1, x_2) = \rho(x_1 - x_2)$ and dropping the normal diffusion with ϵ.

This leaves the question whether the Laplace structure of a system of a the form (1.8) can be preserved in the nonlocal version. Applying the regularisation procedure for a general kernel K, we can rewrite the regularised evolution in the Laplace structure if

$$\nabla_x K(x, y) = -\nabla_y K(x, y).$$

This, however, is only true if K has a convolution structure and thus does not work for domains with boundaries. Indeed we find for the two species system (1.8) the regularisation

$$\begin{cases} \partial_t u_1(x) - \epsilon\Delta u_1 - \Delta_x\Big[\displaystyle\int_{y\in\Omega} K(x,y)\,\mu_1\big(u_1(x),u_2(y)\big)\,\mathrm{d}y\,u_1(x)\Big] \\ \qquad = -\nabla_x\Big[\displaystyle\int_{y\in\Omega}\big[(\partial_x K)(x,y)+(\partial_y K)(x,y)\big]\mu_1\big(u_1(x),u_2(y)\big)\,\mathrm{d}y\,u_1(x)\Big], \\ \partial_t u_2(y) - \epsilon\Delta u_2 - \Delta_y\Big[\displaystyle\int_{x\in\Omega} K(x,y)\,\mu_2\big(u_1(x),u_2(y)\big)\,\mathrm{d}x\,u_2(y)\Big] \\ \qquad = -\nabla_y\Big[\displaystyle\int_{x\in\Omega}\big[(\partial_x K)(x,y)+(\partial_y K)(x,y)\big]\mu_2\big(u_1(x),u_2(y)\big)\,\mathrm{d}x\,u_2(y)\Big], \end{cases} \tag{1.17}$$

which contains corrector terms for the defect of the convolution structure on the right hand side.

For the linear rate SKT system, this matches the regularisation following the discrete structure in [8], where we identified the entropy with the reversibility of a corresponding Markov chain, see Appendix A.

1.4. Results

Having introduced the regularisation schemes, we can now state our rigorous existence and approximation results.

Our first result shows that the regularisation (1.12) for (1.9) is sufficient to find solutions satisfying the entropy-dissipation inequality. It will be clear from the proof below that the diffusivity μ_i's could be assumed sublinear, instead of being controlled by the entropy densities. The self-diffusion could be included *via* the more general approximation (1.11) (for which there is a similar existence result) but we have chosen to avoid it to simplify the presentation.

Theorem 1.10. — *Consider the generalised SKT system* (1.9) *with* $\mu_1,\mu_2\in\mathscr{C}^0(\mathbb{R}_{\geqslant 0})\cap\mathscr{C}^1(\mathbb{R}_{>0})$. *Assume that it admits a uniform entropy structure with entropy* H, *entropy densities* $h_1,h_2\in\mathscr{C}^0(\mathbb{R}_{\geqslant 0})\cap\mathscr{C}^2(\mathbb{R}_{>0})$ *and dissipations* $\alpha_1,\alpha_2:\mathbb{R}_{\geqslant 0}\to\mathbb{R}_{\geqslant 0}$. *Assume furthermore two positive constants* δ,A *such that for all* $z\in\mathbb{R}_{\geqslant 0}$

$$\delta\leqslant\mu_1(z)\leqslant \mathrm{A}(1+h_2(z))\quad\textit{and}\quad \delta\leqslant\mu_2(z)\leqslant \mathrm{A}(1+h_1(z)). \tag{1.18}$$

Fix $\rho\in\mathscr{C}^2(\mathbb{T}^d)$ *nonnegative having integral* 1 *over* $\mathbb{T}^d$, *and a bounded initial data* $u_1^{\mathrm{init}},u_2^{\mathrm{init}}$ *satisfying for some positive constant* γ

$$\gamma\leqslant u_i^{\mathrm{init}}\leqslant\gamma^{-1},$$

so that $H^{\mathrm{init}} := H(u_1^{\mathrm{init}}, u_2^{\mathrm{init}}) < +\infty$*. Then, there exist positive functions*

$$u_1, u_2 \in \mathscr{C}^0([0,T];\mathrm{L}^2(\mathbb{T}^d)) \cap \mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^d)) \cap \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\mathbb{T}^d)), \quad (1.19)$$

such that (u_1, u_2) *is a distributional solution to the system* (1.12) *initiated by* $(u_1^{\mathrm{init}}, u_2^{\mathrm{init}})$*. This solution* (u_1, u_2) *satisfies furthermore the following estimates for* $i = 1, 2$*:*

- conservation of the mass: $u_i \in \mathscr{C}^0([0,T];\mathrm{L}^1(\mathbb{T}^d))$ *and for* $t \in [0,T]$

$$\int_{\mathbb{T}^d} u_i(t) = \int_{\mathbb{T}^d} u_i^{\mathrm{init}}. \quad (1.20)$$

- entropy estimate: $h_i(u_i) \in \mathscr{C}^0([0,T];\mathrm{L}^1(\mathbb{T}^d))$ *and for* $t \in [0,T]$

$$H(u_1(t), u_2(t)) + \int_0^t D(s)\,\mathrm{d}s \leqslant H^{\mathrm{init}}, \quad (1.21)$$

where

$$D(t) := \int_{\mathbb{T}^d} \alpha_1(u_1(t))^2 |\nabla u_1(t)|^2 + \alpha_2(u_2(t))^2 |\nabla u_2(t)|^2.$$

- maximum principle:

$$\gamma \exp\big(-\mathrm{AB}_{T,\mathrm{init}} \|\Delta\rho\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\big) \leqslant u_i \leqslant \gamma^{-1} \exp\big(\mathrm{AB}_{T,\mathrm{init}} \|\Delta\rho\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\big), \quad (1.22)$$

where

$$\mathrm{B}_{T,\mathrm{init}} := T(1 + H^{\mathrm{init}}).$$

- duality estimate:

$$\int_{Q_T} \big(\big[\mu_1(u_2) \star \rho\big] u_1 + \big[\mu_2(u_1) \star \check{\rho}\big] u_2\big)\,(u_1 + u_2) \lesssim_d (1 + 2\mathrm{AB}_{T,\mathrm{init}}) \left(\int_{\mathbb{T}^d} (u_1^{\mathrm{init}})^2 + \int_{\mathbb{T}^d} (u_2^{\mathrm{init}})^2\right), \quad (1.23)$$

where the constant behind $\lesssim_d$ *depends only on the dimension* d*.*

Remark 1.11. — The upper bound in assumption (1.18) is natural for many cross-diffusion systems. For instance if μ_1 and μ_2 are given by power-laws (as in [10]), the entropy densities are precisely given by the same exponents (with an exception for the linear case). See also Remark 1.13.

Using the uniform control by the entropy, we can prove the following limit theorem. Together with the previous existence result, this shows, as a by-product, the (known) existence of weak solutions to the generalised SKT system (1.9).

Theorem 1.12. — *Consider the assumptions of Theorem 1.10, for a sequence of nonnegative functions $(\rho_n)_n \in \mathscr{C}^2(\mathbb{T}^d)^{\mathbb{N}}$ which converges weakly towards the Dirac mass, with dissipation rates α_1 and α_2 vanishing on a set of measure* 0. *Assume furthermore that the diffusivities are strictly subquadratic or controlled by the entropy densities, that is*

$$\lim_{z\to+\infty} \frac{\mu_1(z)}{h_2(z)+z^2} + \frac{\mu_2(z)}{h_1(z)+z^2} = 0. \tag{1.24}$$

Then, the corresponding sequence of solutions $(u_{1,n}, u_{2,n})_n$ given by Theorem 1.10 converges (up to a subsequence) in $\mathrm{L}^1(Q_T)$ towards a weak global solution (u_1, u_2) of the SKT system which satisfies for a.e. $t \in [0,T]$ the conservation of the mass (1.20)*, the entropy estimate* (1.21) *and the following duality estimate*

$$\int_{Q_T} \big(\mu_1(u_2)u_1 + \mu_2(u_1)u_2\big)(u_1+u_2) \\ \lesssim_d (1 + 2\mathrm{AB}_{T,\mathrm{init}}) \int_{\mathbb{T}^d} (u_1^{\mathrm{init}})^2 + \int_{\mathbb{T}^d} (u_2^{\mathrm{init}})^2. \tag{1.25}$$

Remark 1.13. — The assumption (1.24) is crucial to avoid any concentration in the nonlinearities of the system. However, in practice (see for instance the power-law case in [10]) the control of gradients of the entropy estimate gives raise (by Sobolev embedding) to another estimate on $\mu_1(u_2)$ and $\mu_2(u_1)$.

In a general setting we described the regularisation scheme (1.14), for which we can state the following existence result.

Theorem 1.14. — *Consider a cross-diffusion system* (1.1) *for n species and rates $a_{ij} \in \mathscr{C}^0(\mathbb{R}^n_{\geqslant 0})$, $i,j = 1,\dots,n$. Assume that it admits a uniform entropy structure with entropy H, entropy densities $h_1,\dots,h_n \in \mathscr{C}^0(\mathbb{R}_{\geqslant 0}) \cap \mathscr{C}^2(\mathbb{R}_{>0})$ and dissipations $\alpha_1,\dots,\alpha_n : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$.*

For $i \neq j$ define $\widetilde{a}_{ij} : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}$ by

$$\partial_j \widetilde{a}_{ij}(v) = a_{ij}(v), \quad v \in \mathbb{R}^n_{\geqslant 0}, \qquad \textit{and} \qquad \widetilde{a}_{ij}(v) = 0, \quad \textit{if } v_j = 0.$$

Suppose that $v \mapsto \widetilde{a}_{ij}(v)$ is continuously differentiable with respect to v_i and that there exists a constant A such that for all $v \in \mathbb{R}^n_{\geqslant 0}$ and $i = 1,\dots,n$

$$a_{ii}(v) \leqslant A\,(1 + h_1(v_1) + \cdots + h_n(v_n)),$$
$$\frac{\widetilde{a}_{ij}(v)}{v_i} \leqslant A\,(1 + h_1(v_1) + \cdots + h_n(v_n)),$$

and

$$\partial_i \widetilde{a}_{ij}(v) \leqslant A\,(1 + h_1(v_1) + \cdots + h_n(v_n)).$$

Let $\Omega \in \mathbb{R}^d$ be a domain with piecewise $\mathscr{C}^1$ boundary and $K \in \mathscr{C}^2_c(\Omega^n)$ be a nonnegative kernel satisfying (1.13). *Further fix bounded initial data $u^{\mathrm{init}} = (u_1^{\mathrm{init}}, \dots, u_n^{\mathrm{init}})$ satisfying for some positive constant γ*

$$\gamma \leqslant u_i^{\mathrm{init}} \leqslant \gamma^{-1},$$

so that $H^{\mathrm{init}} := H(u^{\mathrm{init}}) < +\infty$. Then, there exists positive functions

$$u_1, \dots, u_n \in \mathscr{C}^0([0,T]; \mathrm{L}^2(\Omega)) \cap \mathrm{L}^2(0,T; \mathrm{H}^1(\Omega)) \cap \mathrm{L}^\infty(0,T; \mathrm{L}^\infty(\Omega)), \tag{1.26}$$

such that $(u_1, \dots, u_n)$ is a distributional solution to the system (1.17) *with initial data u^{init} and von Neumann boundary data* (1.15). *Furthermore, the solution $u = (u_1, \dots, u_n)$ satisfies the following estimates for $i = 1, \dots, n$:*

- conservation of the mass: $u_i \in \mathscr{C}^0([0,T]; \mathrm{L}^1(\Omega))$ *and for $t \in [0,T]$*
$$\int_\Omega u_i(t) = \int_\Omega u_i^{\mathrm{init}}. \tag{1.27}$$
- entropy estimate: $h_i(u_i) \in \mathscr{C}^0([0,T]; \mathrm{L}^1(\Omega))$ *and for $t \in [0,T]$*
$$\begin{gathered} H(u(t))) + \int_0^t D(s)\,\mathrm{d}s \leqslant H^{\mathrm{init}}, \\ D(t) = \sum_{i=1}^n \int_\Omega \left[\epsilon\, h_i''(u_i(x)) + \alpha_i(u_i(x))^2 w_i(x)\right] |\nabla u_i(x)|^2 \,\mathrm{d}x \end{gathered} \tag{1.28}$$
with the weights $(w_i)_{i=1,\dots,n}$ defined in (1.16).
- maximum principle:
$$\gamma \exp\left(-MT\right) \leqslant u_i \leqslant \gamma^{-1} \exp\left(MT\right), \tag{1.29}$$
where
$$M = 2A \max(\|K\|_\infty, \|\nabla K\|_\infty, \|\nabla^2 K\|_\infty)\,(|\Omega| + H^{\mathrm{init}}). \tag{1.30}$$
- ϵ regularity:

$$\sup_{t\in[0,T]} \|u_i(t,\cdot)\|^2_{\mathrm{L}^2(\Omega)} + \epsilon \int_0^T \|\nabla u_i(t,\cdot)\|^2_{\mathrm{L}^2(\Omega)} \leqslant \exp\left[TM\left(2 + \frac{1}{\epsilon}\right)\right] \|u_i^{\mathrm{init}}\|^2_{\mathrm{L}^2(\Omega)}.$$

Under the assumption that the dissipation is big enough, one can conclude that the approximations converge to the local version. In the setting of their time-discretisation approximation scheme, [7] discusses possible conditions for such a convergence. Nevertheless, they need to treat the SKT case separately.

As the SKT case is the motivating example, we focus on the SKT case, where we replace the duality estimate with a positive self-diffusion. For

n species with densities $u = (u_1, \ldots, u_n)$ the SKT system corresponds to the evolution

$$\partial_t u_i = \Delta \left(d_i u_i + \sum_{j=1}^{n} d_{ij} u_j u_i \right) \tag{1.31}$$

with constants $d_1, \ldots, d_n \geqslant 0$ and $(d_{ij})_{ij} \geqslant 0$. Furthermore, suppose that there exist weights $\pi_1, \ldots, \pi_n \geqslant 0$ such that the diffusion coefficients satisfy the detailed balance condition

$$\pi_i d_{ij} = \pi_j d_{ji}, \qquad \text{for } i, j = 1, \ldots, n, \tag{1.32}$$

see [8] for a discussion on the condition. Then the evolution (1.31) has an entropy structure with

$$h_i(z) = \pi_i \left(z \log z - z + 1 \right)$$

and dissipation

$$\alpha_i(z) = \pi_i d_{ii}.$$

Theorem 1.15. — *Given a bounded domain $\Omega \subset \mathbb{R}^d$ with $\mathscr{C}^1$ boundary and an increasing sequence of sets $(A_m)_{m\in\mathbb{N}}$ with $A_m \subset\subset \Omega$ and $A_m \uparrow \Omega$ as $m \to \infty$. Suppose that there exists a constant c and extension operators $\mathcal{E}_m : W^{1,2}(A_m) \to W^{1,2}(\mathbb{R}^d)$ such that $\|\mathcal{E}_m\|_{W^{1,2}\to W^{1,2}} \leqslant c$ and $\|\mathcal{E}_m\|_{L^p \to L^p} \leqslant c$ for $p = 2 + 2/d$.*

Assume a corresponding sequence of nonnegative regularisation kernels $(K^m)_{m\in\mathbb{N}}$ in $\mathscr{C}_c^2(\Omega^n)$ for n densities satisfying (1.13) with weights w_i^m, $i = 1, \ldots, n$, as in (1.16). Suppose that the weights always map to $[0, 1]$ and

$$w_i^m(x) = 1, \qquad \text{for } x \in A_m \text{ and } i = 1, \ldots, n.$$

Moreover, suppose that K^m concentrates along the diagonal, i.e.

$$K^m(x_1, \ldots, x_n) = 0, \quad \text{if } |x_i - x_j| \geqslant \frac{1}{m} \text{ for some } i, j = 1, \ldots, n.$$

Consider the SKT system (1.31) for n densities with constants $d_1, \ldots, d_n \geqslant 0$ and $(d_{ij})_{ij} \geqslant 0$ and weights $\pi_1, \ldots, \pi_n \geqslant 0$ satisfying (1.32) and $d_{ii} > 0$ for $i = 1, \ldots, n$ with initial data $u^{\text{init}} = (u_1^{\text{init}}, \ldots, u_n^{\text{init}})$ with

$$\gamma \leqslant u_i^{\text{init}} \leqslant \gamma^{-1}$$

for $i = 1, \ldots, n$ and a constant $\gamma \in \mathbb{R}_{>0}$.

Then there exists a sequence of $(\epsilon_m)_{m\in\mathbb{N}}$ with $\epsilon_m \downarrow 0$ as $m \to \infty$ such that the approximating solutions $(u^m)_m$ as constructed in Theorem 1.14 converge along a subsequence to u in $\mathrm{L}^q([0,T) \times \Omega; \mathbb{R}^n_{\geqslant 0})$ with $q = 2 + (1/2d)$. The limit

u is a nonnegative weak solution to (1.31) *with the no-flux boundary conditions satisfying the entropy-dissipation inequality, i.e. for $\phi \in \mathscr{C}^\infty([0,T]\times\Omega)$ with $\phi(T,\cdot\,) \equiv 0$ and $i = 1,\dots,n$ it holds*

$$-\int_0^T \int_\Omega u_i\,\partial_t\phi + \int_0^T \int_\Omega \left(\sum_{j=1}^n a_{ij}(u)\nabla u_j\right)\cdot\nabla\phi = \int_\Omega u_i^{\mathrm{init}}\phi(0,\cdot\,),$$

where

$$a_{ij}(u) = \begin{cases} d_i + 2d_{ii}u_i + \sum_{j\neq i} d_{ij}u_j & \text{if } i = j,\\ d_{ij}u_i & \text{otherwise.}\end{cases}$$

Remark 1.16. — A sequence of such sets A_m can be constructed for locally Lipschitz domains. For this, locally write the boundary as a graph of $d-1$ variables and locally then such a sequence can be constructed. For the construction of extensions we refer to the treatment of [11, Section 5.4] and [23, Section VI].

2. The convolution scheme on the flat torus

This section is dedicated to the proofs of Theorem 1.10 and Theorem 1.12 on the torus. For the domain, we introduce the notation

$$Q_T := [0,T)\times\mathbb{T}^d,$$

and start by recalling some useful results about the Kolmogorov equation, that is

$$\partial_t z - \Delta(\mu z) = G, \tag{2.1}$$

$$z(0,\cdot\,) = z^{\mathrm{init}}, \tag{2.2}$$

where G, μ and z^{init} are given and z is the unknown. Solutions will be understood in the following sense:

Definition 2.1. — *Given a measurable function $\mu : Q_T \to \mathbb{R}$, and two square-integrable functions $G, z^{\mathrm{init}} : Q_T \to \mathbb{R}$ we say that $z \in \mathrm{L}^1(Q_T)$ is a distributional solution of* (2.1)–(2.2) *if $z\mu$ is integrable on Q_T and for all test function $\varphi \in \mathscr{D}(Q_T)$ it holds that*

$$-\int_{Q_T} z(\partial_t\varphi + \mu\Delta\varphi) = \int_{\mathbb{T}^d} z^{\mathrm{init}}\varphi(0,\cdot\,) + \int_{Q_T} G\varphi.$$

2.1. Reminder on the Kolmogorov equation

The following result is directly extracted from [20], more precisely merging results obtained in Theorem 3, Proposition 2 and Proposition 3 therein.

Theorem 2.2. — *Fix $\mu \in \mathrm{L}^\infty(Q_T)$ such that $\inf_{Q_T}\mu > 0$. For any $z^{\mathrm{init}} \in \mathrm{L}^2(\mathbb{T}^d)$ there exists a unique solution z to (2.1) – (2.2) in the sense of Definition 2.1. This solution belongs to $\mathrm{L}^2(Q_T)$ and satisfies*

- maximum principle*: if G and z^{init} are nonnegative, then so is z;*
- duality estimate*: $\mu^{1/2}z \in \mathrm{L}^2(Q_T)$ and*
$$\int_{Q_T}\mu z^2 \lesssim_d \left(1+\int_{Q_T}\mu\right)\left(\int_{\mathbb{T}^d}(z^{\mathrm{init}})^2 + T\int_{Q_T}G^2\right),$$
where the constant behind $\lesssim_d$ depends only on the dimension;
- sequential stability*: for fixed G and z^{init} as above, the map $\mu \mapsto z$, restricted to those μ who are bounded and positively lower-bounded, is continuous in the $\mathrm{L}^1(Q_T)$ topology for the argument μ and the $\mathrm{L}^2(Q_T)$ topology for the image z.*

We will use two corollaries of the previous theorem.

Corollary 2.3. — *Consider the assumptions of Theorem 2.2, with $G = 0$. If furthermore z^{init} is bounded with $\gamma \leqslant z^{\mathrm{init}} \leqslant \gamma^{-1}$ for some positive constant γ and if $\Delta\mu \in \mathrm{L}^1(0,T;\mathrm{L}^\infty(\mathbb{T}^d))$, then $z \in \mathrm{L}^\infty(Q_T)$ with the estimate*
$$\gamma\exp\left(-\int_0^T \|(\Delta\mu)^-(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\mathrm{d}s\right) \leqslant z \leqslant \gamma^{-1}\exp\left(\int_0^T \|(\Delta\mu)^+(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s\right),$$
where the exponents $^+$ and $^-$ refer to (respectively) the positive and negative parts.

Proof. — Define
$$\Phi(t) := \gamma^{-1}\exp\left(\int_0^t \|(\Delta\mu)^+(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s\right) - z,$$
$$\Psi(t) := z - \gamma\exp\left(-\int_0^t \|(\Delta\mu)^-(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s\right),$$
which satisfy
$$(\partial_t\Phi - \Delta(\mu\Phi))(t,x) = (\Phi+z)(t,x)(\|(\Delta\mu)^+(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)} - \Delta\mu(t,x)) \geqslant 0,$$
$$(\partial_t\Psi - \Delta(\mu\Psi))(t,x) = (z-\Psi)(t,x)(\Delta\mu(t,x) + \|(\Delta\mu)^-(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}) \geqslant 0.$$
The conclusion follows using the maximum principle of Theorem 2.2, since Φ and Ψ are initially nonnegative. ☐

Corollary 2.4. — *Consider the assumptions of Theorem 2.2, with* $G = 0$*. If furthermore* $\Delta\mu \in \mathrm{L}^1(0,T;\mathrm{L}^\infty(Q_T))$*, then* $z \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{T}^d)) \cap \mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^d))$ *with the following estimate for a.e.* $t \in [0,T]$

$$\int_{\mathbb{T}^d} z(t)^2 + \int_0^t \int_{\mathbb{T}^d} \mu|\nabla z|^2 \leqslant \exp\left(\int_0^t \|(\Delta\mu)^+(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s\right)\int_{\mathbb{T}^d} (z^{\mathrm{init}})^2. \quad (2.3)$$

Proof. — We first assume that μ and the initial data are smooth. In that case, we can rewrite the Kolmogorov equation (2.1) as standard parabolic equation, and we get the smoothness of the solution z. In this situation, we can rigorously multiply the equation by z and integrating by parts, to get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} z(t)^2 + \int_{\mathbb{T}^d} \mu(t)|\nabla z(t)|^2 &= -\int_{\mathbb{T}^d} z(t)\nabla z(t)\cdot\nabla\mu(t) \\ &= \frac{1}{2}\int_{\mathbb{T}^d} z(t)^2\Delta\mu(t) \leqslant \frac{1}{2}\|(\Delta\mu)^+(t)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\int_{\mathbb{T}^d} z(t)^2.\end{aligned}$$

We have thus

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\left\{\exp\left(-\int_0^t \|(\Delta\mu)^+(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s\right)\int_{\mathbb{T}^d} z(t)^2\right\} \\ &\qquad + \exp\left(-\int_0^t \|(\Delta\mu)^+(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s\right)\int_{\mathbb{T}^d} \mu(t)|\nabla z(t)|^2 \leqslant 0,\end{aligned}$$

and we infer after time integration the stated estimate. For the moment, we only established the estimate in the case of smooth data. Replacing μ and z^{init} by smooth approximations $(\mu_n)_n$ and $(z_n^{\mathrm{init}})_n$, approaching them in $\mathrm{L}^1(Q_T)$ and $\mathrm{L}^2(\mathbb{T}^d)$ respectively, with furthermore $\|(\Delta\mu_n)^+\|_{\mathrm{L}^\infty(Q_T)} \leqslant \|(\Delta\mu)^+\|_{\mathrm{L}^\infty(Q_T)}$, we get a sequence $(z_n)_n$ which, by the sequential stability of Theorem 2.2, approaches z in $\mathrm{L}^2(Q_T)$. The usual semi-continuity argument for weak convergence allows to obtain that $z \in \mathrm{L}^\infty(0,T;\mathrm{L}^2(\mathbb{T}^d)) \cap \mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^d))$, with the estimate (2.3) being satisfied for a.e. t. ☐

2.2. Proof of Theorem 1.10

Proof. — We start by proving the four *a priori* estimates, under the assumption of positivity and regularity (1.19).

- *conservation of the mass:* since $u_i \in \mathscr{C}^0([0,T];\mathrm{L}^2(\mathbb{T}^d))$, it also belongs to $\mathscr{C}^0([0,T];\mathrm{L}^1(\mathbb{T}^d))$ and this is sufficient (*via* a density argument) to use $\mathbb{1}_{\mathbb{T}^d}$ as test function which allows to recover (1.20).
- *entropy estimate:* the h_i's and μ_i's are locally Lipschitz, so boundedness of the u_i's and their belonging to $\mathscr{C}^0([0,T];\mathrm{L}^2(\mathbb{T}^d))$ imply $h_i(u_i), \mu_i(u_i) \in \mathscr{C}^0([0,T];\mathrm{L}^1(\mathbb{T}^d))$, for $i = 1, 2$. With the same type

of arguments we recover $h_i'(u_i) \in \mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^d))$. This is sufficient to justify the following formula for all $t \in [0,T]$, by density of smooth functions,

$$\int_0^t \int_{\mathbb{T}^d} h_i'(u_i)\partial_t u_i = \int_{\mathbb{T}^d} h_i(u_i(t)) - \int_{\mathbb{T}^d} h_i(u_i^{\mathrm{init}}).$$

Similarly, we have that (with the analogous formula for the other species)

$$-\int_0^t \int_{\mathbb{T}^d} h_1'(u_1)\Delta\big(\big[\mu_1(u_2)\star\rho\big]u_1\big) = \int_0^t \int_{\mathbb{T}^d} h_1''(u_1)\nabla u_1 \cdot \nabla\big(\big[\mu_1(u_2)\star\rho\big]u_1\big),$$

which is sufficient to reproduce rigorously the computation done in the proof of Proposition 1.4 and integrate it time to get (1.21).

- *maximum principle:* we have (using assumption (1.18) and the entropy estimate)

$$\begin{aligned}\int_0^T \|\Delta\left(\mu_1(u_2)\star\rho\right)(s)\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\,\mathrm{d}s &\leqslant \|\mu_1(u_2)\|_{\mathrm{L}^1(Q_T)}\|\Delta\rho\|_{\mathrm{L}^\infty(\mathbb{T}^d)} \\ &\leqslant \mathrm{A}T(1+H^{\mathrm{init}})\|\Delta\rho\|_{\mathrm{L}^\infty(\mathbb{T}^d)}\end{aligned}$$

  and likewise for $\Delta(\mu_2(u_1)\star\check{\rho})$ so that (1.22) follows directly from Corollary 2.3.

- *duality estimate:* the function $z := u_1 + u_2$ solves the following Kolmogorov equation

$$\begin{aligned}\partial_t z - \Delta(\mu z) = 0,& \\ z(0,\cdot\,) = u_1^{\mathrm{init}} + u_2^{\mathrm{init}},&\end{aligned}$$

  with

$$\mu := \frac{(\mu_1(u_2)\star\rho)u_1 + (\mu_2(u_1)\star\check{\rho})u_2}{u_1+u_2},$$

  where μ is well-defined thanks to the positivity of the u_i's and furthermore bounded. The duality estimate of Theorem 2.2 implies

$$\begin{aligned}\int_{Q_T} \big(\big[\mu_1(u_2)\star\rho\big]u_1 + \big[\mu_2(u_1)\star\check{\rho}\big]u_2\big)(u_1+u_2)& \\ \lesssim_d \left(1+\int_{Q_T}\mu\right)\left(\int_{\mathbb{T}^d}(u_1^{\mathrm{init}})^2 + \int_{\mathbb{T}^d}(u_2^{\mathrm{init}})^2\right).&\end{aligned} \tag{2.4}$$

  To recover (1.23), simply notice that $\mu \leqslant \mu_1(u_2)\star\rho + \mu_2(u_1)\star\check{\rho}$ so that using the normalisation of ρ and assumption (1.18),

$$\begin{aligned}\int_{Q_T}\mu \leqslant \int_{Q_T}\mu_1(u_2) + \int_{Q_T}\mu_2(u_1) &\leqslant \mathrm{A}\left(\int_{Q_T}(2+h_1(u_1)+h_2(u_2))\right) \\ &\leqslant 2\mathrm{A}B_{T,\mathrm{init}},\end{aligned}$$

where we used the entropy estimate and the constant $\mathrm{B}_{T,\mathrm{init}} := T(1+H^{\mathrm{init}})$.

These estimates have been proven for a positive solution with regularity (1.19) whose existence has been assumed. We now construct a solution by a fixed-point argument.

On the set $E := \mathrm{L}^1(Q_T) \times \mathrm{L}^1(Q_T)$ we define the map $\Theta : E \to E$ which sends (u_1, u_2) to the solutions $(u_1^\bullet, u_2^\bullet)$ (in the sense of Definition 2.1) of

$$\begin{cases} \partial_t u_1^\bullet = \Delta\big(\big[\mu_1(u_2^+ \wedge M) \star \rho\big] u_1^\bullet\big), \\ \partial_t u_2^\bullet = \Delta\big(\big[\mu_2(u_1^+ \wedge M) \star \check{\rho}\big] u_2^\bullet\big), \end{cases}$$

where the cutoff constant $M > 0$ will be fixed later on. By continuity of μ_1 and μ_2 we have

$$\max(\mu_1(x), \mu_2(x)) \leqslant C, \qquad \forall\, x \in [0, M]. \tag{2.5}$$

In particular, Theorem 2.2 applies and ensures that the previous map is well-defined. Moreover, (2.5) implies for $(u_1, u_2) \in E$ that

$$|\Delta\big(\mu_1(u_2^+ \wedge M) \star \rho\big)| = |\mu_1(u_2^+ \wedge M) \star \Delta\rho| \leqslant C\|\Delta\rho\|_{\mathrm{L}^1(\mathbb{T}^d)},$$

and likewise for $\Delta\big(\mu_2(u_1^+ \wedge M) \star \rho\big)$. Hence we infer from Corollaries 2.3 and 2.4 that the images $u_1^\bullet$ and $u_2^\bullet$ are nonnegative and uniformly bounded in $\mathrm{L}^\infty(Q_T)$ and $\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^d))$. Moreover by the equations, the time derivatives are also uniformly bounded in $\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\mathbb{T}^d))$. That means that there exists a constant c such that

$$\Theta(E) \subset K := \left\{ (v_1, v_2) \in E : \begin{array}{r} \max\big(\|v_i\|_{\mathrm{L}^\infty(Q_T)}, \|v_i\|_{\mathrm{L}^2(0,T;\mathrm{H}^1(\mathbb{T}^d))}, \\ \|\partial_t v_i\|_{\mathrm{L}^2(0,T;\mathrm{H}^{-1}(\mathbb{T}^d))}\big) \leqslant c \text{ for } i = 1, 2 \end{array} \right\}.$$

Then by the Aubin–Lions lemma the convex set K is also compact in E. Hence Schauder's fixed-point theorem applies and ensures that there exists a fixed-point $(u_1, u_2) \in K$, solving therefore (the u_i's are nonnegative)

$$\begin{cases} \partial_t u_1 = \Delta\big(\big[\mu_1(u_2 \wedge M) \star \rho\big] u_1\big), \\ \partial_t u_2 = \Delta\big(\big[\mu_2(u_1 \wedge M) \star \check{\rho}\big] u_2\big). \end{cases} \tag{2.6}$$

The bounds obtained for elements in K also ensure that u_1 and u_2 both belong to $\mathscr{C}^0([0,T];\mathrm{L}^2(\mathbb{T}^d))$ so that we have the required regularity (1.19). In order to conclude we just need to fix a constant M such that the corresponding saturation vanishes. For this purpose, we consider

$$M = 2\gamma^{-1} \exp(\mathrm{A}\mathrm{B}_{T,\mathrm{init}}\|\Delta\rho\|_\infty),$$

where the constants A and $\mathrm{B}_{T,\mathrm{init}}$ are defined in the statement of Theorem 1.10 and we recall that $\gamma > 0$ is such that

$$\gamma \leqslant u_i^{\mathrm{init}} \leqslant \gamma^{-1}.$$

We now define

$$t^\star := \sup\left\{t \in [0,T] : \max_{i=1,2} \|u_i\|_{\mathrm{L}^\infty([0,t]\times\mathbb{T}^d)} \leqslant \frac{3M}{4}\right\}.$$

By Corollary 2.3, we have $t^\star > 0$ and up to any $t \in (0,t^\star)$ the cutoff M has been irrelevant. Thus, for $t \in (0,t^\star)$ all the a priori estimates apply and in particular the entropy estimate which implies

$$\|\mu_1(u_2)\|_{\mathrm{L}^1([0,t]\times\mathbb{T}^d)} \leqslant \mathrm{A}\int_0^t \left(1 + \int_{\mathbb{T}^d} h_2(u_2)\right) \leqslant \mathrm{A}t(1+H^{\mathrm{init}}) \leqslant \mathrm{AB}_{T,\mathrm{init}},$$

with a similar estimate for the other species. This in turn implies by Corollary 2.3 that for $t < t^\star$

$$\max_{i=1,2} \|u_i\|_{\mathrm{L}^\infty([0,t]\times\mathbb{T}^d)} \leqslant \frac{M}{2},$$

which proves that $t^\star = T$ by the usual continuity argument and our fixed-point (u_1,u_2) is the required solution. □

2.3. From nonlocal to local SKT

We start with a compactness tool already used in [16] that we adapt slightly to our setting. The proofs are only included for the reader's convenience.

Lemma 2.5. — *Fix $\alpha : \mathbb{R}_{>0} \to \mathbb{R}_{\geqslant 0}$ having a negligible set of zeros. Consider a sequence of positive functions $(w_n)_n \in \mathrm{W}^{1,1}(Q_T)$ such that*

(i) *$(w_n)_n$ bounded in $\mathrm{L}^2(Q_T)$;*
(ii) *$(\partial_t w_n)_n$ bounded in $\mathrm{L}^1(0,T;\mathrm{H}^{-m}(\mathbb{T}^d))$ for some integer m;*
(iii) *$(\alpha(w_n)\nabla w_n)_n$ bounded in $\mathrm{L}^2(Q_T)$.*

Then $(w_n)_n$ admits an a.e. converging subsequence.

Proof. — By assumption (iii) the sequence $(\nabla F(w_n))_n$ is bounded in $\mathrm{L}^2(Q_T)$, where $F : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is defined by

$$F(z) := \int_0^z 1 \wedge \alpha.$$

Moreover, F is an increasing (because $\alpha > 0$ a.e.) 1-Lipschitz function vanishing at 0. In particular, we infer from (i) the same bound for $(F(w_n))_n$. Up to a subsequence we can thus assume that $(w_n)_n$ and $(F(w_n))_n$ respectively converge weakly to w and $\widetilde{w}$ in $\mathrm{L}^2(Q_T)$. Using (ii) we thus infer from [19, Proposition 3] that (up to a subsequence),

$$\int_{Q_T} w_n F(w_n) \underset{n\to\infty}{\longrightarrow} \int_{Q_T} w\widetilde{w}. \tag{2.7}$$

At this stage we use the Minty–Browder or Leray–Lions trick: one first establishes that

$$\int_{Q_T} \overbrace{(F(w_n) - F(w))(w_n - w)}^{:=h_n}$$
$$= \int_{Q_T} F(w_n)w_n + \int_{Q_T} F(w)w - \int_{Q_T} F(w_n)w - \int_{Q_T} F(w)w_n \underset{n\to\infty}{\longrightarrow} 0$$

by exploiting the $\mathrm{L}^2(Q_T)$ weak convergences $(w_n)_n \rightharpoonup_n w$, $(F(w_n))_n \rightharpoonup_n \widetilde{w}$, together with (2.7). Then, since F is increasing, we have $h_n \geqslant 0$ so that the previous convergence may be seen as the convergence of $(h_n)_n$ to 0 in $\mathrm{L}^1(Q_T)$. In particular, up to some subsequence, we get that $(h_n)_n$ converges a.e. to 0 which in turn implies (increasingness of F) that $(w_n)_n \to w$. $\square$

Proof of Theorem 1.12. — Using the duality estimate of Theorem 1.10 we first have

$$\int_{Q_T}\Big(\big[\mu_1(u_{2,n})\star\rho_n\big]u_{1,n} + \big[\mu_2(u_{1,n})\star\check{\rho}_n\big]u_{2,n}\Big)\,(u_{1,n}+u_{2,n}) \lesssim_{d,\mathrm{init}} 1, \quad (2.8)$$

where the constant depends on the dimension and initial data but is uniform in n. In particular, both species satisfy (since μ_1 and μ_2 are positively lower-bounded) assumptions (i) and (ii) of Lemma 2.5. Using the entropy estimate of Theorem 1.10, we have also for both species that $(\alpha_i(u_{i,n})\nabla u_n)_n$ bounded in $\mathrm{L}^2(Q_T)$, which validates assumption (iii) of the lemma since the dissipation rates α_i are assumed a.e. positive on $\mathbb{R}_{>0}$. We infer therefore from the previous lemma that, up to a subsequence (that we do not label), $(u_{1,n})_n$ and $(u_{2,n})_n$ converge a.e. to some u_1 and u_2, respectively.

We now pass to the limit (in $\mathscr{D}'(Q_T)$) in the products

$$\big[\mu_1(u_{2,n})\star\rho_n\big]u_{1,n} \text{ and } \big[\mu_2(u_{1,n})\star\check{\rho}_n\big]u_{2,n}.$$

W.l.o.g. we can focus on the first one. Since $(u_{2,n})_n$ converges to u_2 a.e., so does $(\mu_1(u_{2,n})_n$ to $\mu_1(u_2)$, by continuity of μ_1.

The assumption (1.18) and the entropy estimate of Theorem 1.10 imply that $(\mu_1(u_{2,n}))_n$ is bounded in $\mathrm{L}^\infty(0,T;\mathrm{L}^1(\mathbb{T}^d))$. As this is not sufficient to prevent possible concentration in the space variable, we use the growth assumption (1.24), to establish the uniform integrability of $(\mu_1(u_{2,n}))_n$. Indeed, since μ_1 is continuous, the sequence $c_R := \inf\{z \geqslant 0 \,:\, \mu_1(z) \geqslant R\}$ diverges to $+\infty$ with R and we have

$$\int_{Q_T} \mu_1(u_{2,n})\mathbb{1}_{\mu_1(u_{2,n})\geqslant R} \leqslant \int_{Q_T} \mu_1(u_{2,n})\mathbb{1}_{u_{2,n}\geqslant c_R}$$
$$\leqslant \sup_{z\geqslant c_R}\Phi(z)\int_{Q_T} h_2(u_{2,n}) + u_{2,n}^2,$$

where $\Phi(z) := \frac{\mu_1(z)}{h_2(z)+z^2}$ goes to 0 as $z \to +\infty$, by assumption (1.24). Since μ_2 is positively lower-bounded (thanks to assumption (1.18)), we infer from (2.8) a bound for $(u_{2,n})_n$ in $\mathrm{L}^2(Q_T)$ (we use here the nonnegativity of all the involved functions). Using the entropy estimate and the previous inequalities, we therefore infer

$$\lim_{R\to+\infty} \sup_n \int_{Q_T} \mu_1(u_{2,n}) \mathbb{1}_{\mu_1(u_{2,n})\geqslant R} = 0,$$

which establishes uniform integrability.

Therefore, Vitali's convergence theorem implies that $(\mu_1(u_{2,n}))_n$ converges to $\mu_1(u_2)$ in $\mathrm{L}^1(Q_T)$. The sequence $(\mu_1(u_{2,n}) \star \rho_n)_n$ shares the same behaviour. In particular, $(\mu_1(u_{2,n}) \star \rho_n)_n$ is also uniformly integrable and adding a subsequence if necessary, we can assume that it converges a.e. towards $\mu_1(u_2)$. Now, to conclude we write

$$w_n := \left[\mu_1(u_{2,n}) \star \rho_n\right] u_{1,n} = \left[\mu_1(u_{2,n}) \star \rho_n\right]^{1/2} \left[\mu_1(u_{2,n}) \star \rho_n\right]^{1/2} u_{1,n}.$$

As already noticed, $(w_n)_n$ converges a.e. to the expected limit $\mu_1(u_2)u_1$. The previous writing together with the duality estimate (2.8) and the Cauchy–Schwarz inequality shows that $(w_n)_n$ is bounded in $\mathrm{L}^1(Q_T)$. Even better, $(w_n)_n$ is the product of a L^2-uniformly integrable sequence with an L^2-bounded one so that $(w_n)_n$ is uniformly integrable and the Vitali convergence theorem applies once more to get the convergence of $(w_n)_n$ towards $\mu_1(u_2)u_1$.

The previous reasoning (which applies to both species) allows to pass to the limit of the equations. The limit satisfies the estimates by Fatou's lemma. ☐

3. General regularised scheme on a domain

In this section, we study the general regularisation scheme introduced in Proposition 1.8 and prove the corresponding results Theorem 1.14 and Theorem 1.15.

3.1. Existence of regularised solutions

We start with proving the existence of solutions for the regularised scheme, i.e. Theorem 1.14.

The advantage of the regularisation is that the cross-diffusion terms are controllable and we thus rewrite the evolution as

$$\partial_t u_i(x_i) - \operatorname{div}_{x_i}\left[\left(\epsilon + \prod_{k\neq i}\int_{x_k\in\Omega} \mathrm{d}x_k\, K(x_1,\dots,x_n)\, a_{ii}(u_1(x_1),\dots,u_n(x_n))\right)\nabla u_i(x_i)\right]$$
$$= \operatorname{div}_{x_i}\left[\prod_{k\neq i}\int_{x_k\in\Omega} \mathrm{d}x_k\, K(x_1,\dots,x_n) \sum_{j\neq i} a_{ij}(u_1(x_1),\dots,u_n(x_n))\nabla u_j(x_j)\right].$$

For the cross-diffusion terms, the $\widetilde{a}_{ij}$ in Theorem 1.14 are defined such that

$$a_{ij}(u_1(x_1),\dots,u_n(x_n))\nabla u_j(x_j) = \nabla_{x_j}\widetilde{a}_{ij}(u_1(x_1),\dots,u_n(x_n))$$

so that the partial derivative can formally be integrated by parts onto the kernel K, where no boundary terms appear due to (1.15). Hence the evolution can be rewritten as

$$\partial_t u_i - \nabla\big((\epsilon + \bar{a}_i[u])\nabla u_i\big) + \bar{b}_i[u]\nabla u_i + \bar{c}_i[u]u_i = 0, \tag{3.1}$$

with von Neumann boundary conditions and

$$\bar{a}_i[u](x_i) = \prod_{k\neq i}\int_{x_k\in\Omega} \mathrm{d}x_k\, K(x_1,\dots,x_n)\, a_{ii}(u_1(x_1),\dots,u_n(x_n)), \tag{3.2}$$

$$\bar{b}_i[u](x_i) = \sum_{j\neq i}\prod_{k\neq i}\int_{x_k\in\Omega} \mathrm{d}x_k\, \partial_j K(x_1,\dots,x_n)\, \partial_i\widetilde{a}_{ij}(u_1(x_1),\dots,u_n(x_n)), \tag{3.3}$$

$$\bar{c}_i[u](x_i) = \sum_{j\neq i}\prod_{k\neq i}\int_{x_k\in\Omega} \mathrm{d}x_k\, \partial_{ij} K(x_1,\dots,x_n)\, \frac{\widetilde{a}_{ij}(u_1(x_1),\dots,u_n(x_n))}{u_i(x_i)}. \tag{3.4}$$

The assumptions of Theorem 1.14 then imply for $x_i \in \Omega$ that

$$\begin{aligned} |\bar{a}_i[u](x_i)| &\leqslant A\,\|K\|_\infty(|\Omega| + H(u)),\\ |\bar{b}_i[u](x_i)| &\leqslant A\,\|\nabla K\|_\infty(|\Omega| + H(u)),\\ |\bar{c}_i[u](x_i)| &\leqslant A\,\|\nabla^2 K\|_\infty(|\Omega| + H(u)). \end{aligned} \tag{3.5}$$

This is enough to prove the existence of solutions by a Galerkin scheme.

Proof of Theorem 1.14. — Let $\sigma \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ a nonnegative mollification kernel with $\operatorname{supp}\sigma \subset B_1$ and $\int \sigma \mathrm{d}x = 1$ and define

$$\sigma^m(x) = m^d\,\sigma(mx).$$

Extending $\bar{a}_i$, $\bar{b}_i$, $\bar{c}_i$ with zero outside Ω, we consider for $m \in \mathbb{N}$ the following system

$$\begin{aligned} \partial_t u_i^m - \nabla\big((\epsilon + ((\bar{a}_i[u^m]\wedge M)\star\sigma^m)\nabla u_i^m\big)&\\ + ((\bar{b}_i[u^m]\wedge M)\star\sigma^m)\nabla u_i^m + ((\bar{c}_i[u^m]\wedge M)\star\sigma^m)u_i^m &= 0, \end{aligned} \tag{3.6}$$

with von Neumann boundary conditions, $i = 1, \dots, n$ and the constant M as in (1.30).

By a standard Galerkin scheme (e.g. taking the von Neumann eigenvectors of the Laplacian on Ω), the system (3.6) has a solution u_i^m with initial data u_i^{init} and has any H^k, $k \in \mathbb{N}$, regularity after an arbitrary short time. Hence we can apply the maximum principle for parabolic equations and find as in Corollary 2.3 that

$$\gamma \exp(-TM) \leqslant u_i^m \leqslant \gamma^{-1} \exp(TM).$$

Furthermore, each u_i^m is preserving the mass. Finally, we can test (3.6) against u_i^m to find the followig estimate independent of m:

$$\sup_{t\in[0,T]} \|u_i^m(t,\cdot)\|^2_{\mathrm{L}^2(\Omega)} + \epsilon \int_0^T \|\nabla u_i^m(t,\cdot)\|^2_{\mathrm{L}^2(\Omega)} \leqslant \exp\left[TM\left(2+\frac{1}{\epsilon}\right)\right] \|u_i^{\mathrm{init}}\|^2_{\mathrm{L}^2(\Omega)},$$

where $i = 1, \dots, n$ and we used the cutoff with M. Note that here the right hand side is bounded by assumption. Hence we find for a constant $C(T)$ independent of m that

$$\|\partial_t u_i^m\|_{\mathrm{L}^2(0,T,\mathrm{H}^{-1}(\Omega))} \leqslant C(T)$$

for $i = 1, \dots, n$.

By Aubin–Lions lemma we can therefore find a subsequence (relabelling with m) and

$$u_i \in \mathscr{C}^0([0,T];\mathrm{L}^2(\Omega)) \cap \mathrm{L}^2(0,T;\mathrm{H}^1(\Omega)) \cap \mathrm{L}^\infty(0,T;\mathrm{L}^\infty(\Omega)),$$

for $i = 1, \dots, n$ such that u_i^m converges almost everywhere to u_i and ∇u_i^m converges L^2 weakly to ∇u_i. Moreover, it holds that

$$\gamma \exp(-TM) \leqslant u_i \leqslant \gamma^{-1} \exp(TM).$$

The convergence implies that for $\phi \in \mathscr{C}^\infty(\Omega_T)$ with $\phi(T,\cdot) \equiv 0$ it holds that

$$-\int_0^T\int_\Omega u_i \partial_t \phi + \int_0^T\int_\Omega (\epsilon + \bar{a}_i[u] \wedge M)\nabla u_i \cdot \nabla \phi + \int_0^T\int_\Omega (\bar{b}_i[u] \wedge M) \cdot \nabla u_i\ \phi$$
$$+ \int_0^T\int_\Omega (\bar{c}_i[u] \wedge M) u_i\ \phi = \int_\Omega u_i^{\mathrm{init}} \phi^{\mathrm{init}},$$

i.e. $u = (u_1, \dots, u_n)$ is a weak solution with von Neumann boundary data. Moreover, by the continuity this implies directly the conservation of mass.

Until a time $T^* \leqslant T$ for which

$$\sup_{t\in[0,T^*]} \sup_{x\in\Omega} \max(\bar{a}_i[u], |\bar{b}_i[u]|, \bar{c}_i[u]) \leqslant M,$$

the cutoff M is not applied and we have a weak solution of (3.1). As in the Laplace case on the torus in Section 2, the proven regularity is sufficient to justify rigorously the formal entropy estimate as in Proposition 1.8.

The assumptions (3.5) then imply that at time T^* it holds that

$$\sup_{x\in\Omega} \max(\bar{a}_i[u], |\bar{b}_i[u]|, \bar{c}_i[u]) \leqslant \frac{M}{2}$$

and thus by continuity $T^* = T$ and we have constructed the claimed solution. □

3.2. Limit for the SKT system

Having constructed the nonlocal approximation, we now prove Theorem 1.15.

The assumption of the extension operator allows to find a uniform Gagliardo–Nirenberg inequality.

Lemma 3.1. — *Assume the setup of Theorem 1.15. Then there exists a uniform c for the Gagliardo–Nirenberg inequality*

$$\|f\|^p_{\mathrm{L}^p(A_m)} \leqslant c\left(\|f\|^{(1-\theta)p}_{\mathrm{L}^1(A_m)}\|\nabla f\|^{\theta p}_{\mathrm{L}^2(A_m)} + \|f\|^p_{\mathrm{L}^1(A_m)}\right) \qquad \forall\, f : A_m \longrightarrow \mathbb{R},$$

holds on all A_m, where $m \in \mathbb{N}$, $\theta = 2/p$ and $p = 2 + 2/d$.

Proof. — By the extension operator $\mathcal{E}_m$ and the Gagliardo–Nirenberg inequality on $\mathbb{R}^d$ we find

$$\begin{aligned}\|f\|^p_{\mathrm{L}^p(A_m)} &\leqslant \|\mathcal{E}_m(f)\|^p_{\mathrm{L}^p(\mathbb{R}^d)}\\ &\lesssim \|\mathcal{E}_m(f)\|^{(1-\theta)p}_{\mathrm{L}^1(\mathbb{R}^d)}\,\|\nabla\mathcal{E}_m(f)\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\\ &\lesssim \|f\|^{(1-\theta)p}_{\mathrm{L}^1(A_m)}\left(\|f\|^2_{\mathrm{L}^2(A_m)} + \|\nabla f\|^2_{\mathrm{L}^2(A_m)}\right).\end{aligned}$$

As $p > 2$, we can interpolate $\|f\|_{\mathrm{L}^2(A_m)}$ between $\|f\|_{\mathrm{L}^1(A_m)}$ and $\|f\|_{\mathrm{L}^p(A_m)}$ and absorb the contribution of $\|f\|_{\mathrm{L}^p(A_m)}$ so that the claimed inequality follows. □

The first lemma ensures the integrability and determines the sequence ϵ.

Lemma 3.2. — *Assume the setup of Theorem 1.15. Then there exists a constant C_T and a decreasing sequence $(\epsilon_m)_m$ with $\epsilon_m \downarrow 0$ such that*

$$\|u_i^m\|_{\mathrm{L}^{\widetilde{p}}([0,T)\times\Omega)} \leqslant C_T$$

for $i = 1, \dots, n$ and $\widetilde{p} = 2 + 1/d$.

Proof. — For the regularisation kernel K^m and $\epsilon_m > 0$, we find by Theorem 1.14 a solution u_m which satisfies

(i) $\|u_i^m\|_{\mathrm{L}^\infty(0,T;\mathrm{L}^1(\Omega))} \leqslant c$ for $i = 1, \dots, n$ (conservation of mass),
(ii) $\|\nabla u_i^m\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} \leqslant \epsilon_m^{-1}\exp(\epsilon_m^{-1})c$ (ϵ-dependent estimate),

(iii) $\|\nabla u_i^m\|^2_{\mathrm{L}^2(0,T;\mathrm{L}^2(A_m))} \leqslant c$ (dissipation estimate in the set A_m on which the weights are $w_i^m \equiv 1$)

for a constant c independent of m.

The parameters θ and p of the Gagliardo–Nirenberg in Lemma 3.1 are chosen such that $\theta p = 2$. Hence we find on A_m that for $i = 1, \dots, n$

$$\int_0^T \|u_i^m\|^p_{\mathrm{L}^p(A_m)} \mathrm{d}t \lesssim \int_0^T \left(\|\nabla u_i^m\|^2_{\mathrm{L}^2(A_m)} \, \|u_i^m\|^{(1-\theta)p}_{\mathrm{L}^1(A_m)} + \|u_i^m\|^p_{\mathrm{L}^1(A_m)} \right) \mathrm{d}t.$$

With the gradient control from the dissipation and the conservation of mass this shows

$$\int_{[0,T)\times A_m} |u_i^m|^p \, \mathrm{d}x \, \mathrm{d}t \leqslant c_d$$

for a constant c_d independent of m.

As the domain Ω is assumed to have $\mathscr{C}^1$ boundary, we can also apply the argument of Lemma 3.1 to find over Ω that for $i = 1, \dots, n$

$$\int_0^T \|u_i^m\|^p_{\mathrm{L}^p(\Omega)} \mathrm{d}t \lesssim \int_0^T \left(\|\nabla u_i^m\|^2_{\mathrm{L}^2(\Omega)} \, \|u_i^m\|^{(1-\theta)p}_{\mathrm{L}^1(\Omega)} + \|u_i^m\|^p_{\mathrm{L}^1(\Omega)} \right) \mathrm{d}t.$$

Hence we find for a constant c_e independent of m that

$$\int_{[0,T)\times\Omega} |u_i^m|^p \, \mathrm{d}x \, \mathrm{d}t \leqslant c_e \left(1 + \frac{c_e}{\epsilon_m} \exp(\epsilon_m^{-1}) \right).$$

As $\widetilde{p} < p$ we can find $q \in (0,1)$ so that the Hölder inequality implies

$$\|f\|_{\mathrm{L}^{\tilde{p}}([0,T)\times B)} \leqslant (T\,|B|)^q \, \|f\|_{\mathrm{L}^p([0,T)\times B)}$$

for $B \subset \Omega$ and $f \in \mathrm{L}^p([0,T) \times B)$.

By splitting Ω into A_m and $\Omega \setminus A_m$ we therefore find (as $|A_m| \leqslant |\Omega|$)

$$\begin{aligned} \|u_i^m\|_{\mathrm{L}^{\tilde{p}}([0,T)\times\Omega)} &\leqslant \|u_i^m\|_{\mathrm{L}^{\tilde{p}}([0,T)\times A_m)} + \|u_i^m\|_{\mathrm{L}^{\tilde{p}}([0,T)\times(\Omega\setminus A_m))} \\ &\leqslant T^q \, |\Omega|^q c_d^{1/p} + T^q \, |\Omega \setminus A_m|^q c_e^{1/p} \left(1 + \frac{\exp(\epsilon_m^{-1})}{\epsilon_m} \right)^{1/p}. \end{aligned}$$

As $|\Omega \setminus A_m| \to 0$ and $q \in (0,1)$, we can therefore find a sequence $\epsilon_m \downarrow 0$ such that $|\Omega \setminus A_m|^q (1 + \epsilon_m^{-1} \exp(\epsilon_m^{-1}))^{1/p}$ is bounded by a constant independent of m. The claim then follows directly from the given estimate. □

We can now proceed with the convergence result.

Proof of Theorem 1.15. — Inside each good set $A_{\overline{m}}$, the dissipation and mass conservation give a uniform estimate for u_i^m in $\mathrm{L}^\infty(0,T;\mathrm{L}^1(A_m))$ and $\mathrm{L}^2(0,T;\mathrm{H}^1(A_m))$ for $i = 1, \dots, n$ and $m \geqslant \overline{m}$. By the equation this also gives a uniform estimate of the time-derivative in $\mathrm{L}^1(0,T;\mathrm{H}^{-k}(A_m))$ for a large enough $k \in \mathbb{N}$ (depending only on dimension d). Hence on $A_{\overline{m}}$ we have compactness for u_i^m. As $A_m \uparrow \Omega$, a diagonal argument shows that along a subsequence (which we relabel with m) that for $i = 1, \ldots, n$ there exist $u_i : [0,T) \times \Omega$ such that $u_i^m \to u_i$ a.e. Moreover, choosing ϵ_m as in Lemma 3.2 we find $u_i \in \mathrm{L}^{\tilde{p}}([0,T) \times \Omega)$.

By the dissipation inequality we find that

$$\|\sqrt{w_i^m} \nabla u_i^m\|_{\mathrm{L}^2([0,T)\times\Omega)}$$

is uniformly bounded. Hence along a subsequence $\sqrt{w_i^m} \nabla u_i^m$ converges weakly in L^2 to a limit ψ_i. As w_i^m is the constant 1 inside the set A_m and $A_m \uparrow \Omega$, it follows that $\sqrt{w_i^m} \nabla u_i^m \rightharpoonup \nabla u$ and $\nabla u \in \mathrm{L}^2$.

As u^m preserves the mass, is nonnegative and satisfies the entropy-dissipation inequality, the same is true for the limit u by using the stated regularity. Moreover, the stated regularity gives the claimed convergence.

It thus remains to check that u is a weak solution. As u^m satisfies von Neumann boundary data and K^m vanishes at the boundary, the constructed solutions satisfy for all $\phi \in \mathscr{C}^\infty([0,T] \times \Omega$ with $\phi(T, \cdot\,) \equiv 0$ and $i = 1, \ldots, n$ that

$$\begin{aligned}
&- \int_0^T \int_\Omega u_i^m \,\partial_t \phi + \epsilon_m \int_0^T \int_\Omega \nabla u_i^m \cdot \phi \\
&+ \int_0^T \int_\Omega \left(\prod_{k \neq i} \int_{x_k \in \Omega} \mathrm{d}x_k K^m(x_1, \ldots, x_n) \sum_{j=1}^n a_{ij}(u_1^m(x_1), \ldots, u_n^m(x_n)) \nabla u_j^m(x_j) \right) \cdot \nabla \phi \\
&= \int_\Omega u_i^{\mathrm{init}} \phi(0, \cdot\,).
\end{aligned}$$

For the diffusion from d_{ij} with $i \neq j$ and $i, j = 1, \ldots, n$, we must therefore show that for all test function ϕ

$$\begin{aligned}
&\int_0^T \int_{\Omega^n} K^m(x_1, \ldots, x_n) u_i^m(t, x_i) \nabla u_j^m(t, x_j)\, \nabla \phi(t, x_i)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n \mathrm{d}t \\
&\qquad\qquad \longrightarrow \int_0^T \int_\Omega u_i(t,x) \nabla u_j(t,x)\, \nabla \phi(t,x)\, \mathrm{d}x\, \mathrm{d}t.
\end{aligned}$$

We rewrite the nonlinear diffusion term as

$$\begin{aligned}
&\int_0^T \int_{\Omega^n} K^m(x_1, \ldots, x_n) u_i^m(t, x_i) \nabla u_j^m(t, x_j)\, \nabla \phi(t, x_i)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n \mathrm{d}t \\
&\qquad\qquad = \int_0^T \int_\Omega \sqrt{w_j^m(x_j)} \nabla u_j^m(t, x_j) \psi^m(t, x_j)\, \mathrm{d}x_i\, \mathrm{d}t,
\end{aligned}$$

where

$$\psi^m(t,x_j) = \frac{1}{\sqrt{w_j^m(x_j)}} \prod_{k\neq j} \int_\Omega \mathrm{d}x_k\, K^m(x_1,\dots,x_n) u_i^m(t,x_i)\, \nabla\phi(t,x_i).$$

By the definition of the weight w_j^m, we can apply Jensen's inequality for $\widetilde{p} \geqslant 2$ to find that

$$\|\psi^m\|_{\mathrm{L}^{\tilde{p}}([0,T)\times\Omega)} \leqslant \|u_i^m \nabla\phi\|_{\mathrm{L}^{\tilde{p}}([0,T)\times\Omega)}.$$

By the proven convergence and regularity of u_i^m we find that $\psi^m \to u_i\nabla\phi$ a.e. in $[0,T)\times\Omega$. The previous inequality gives a uniform bound of ψ^m in $\mathrm{L}^{\tilde{p}}$ with $\widetilde{p} > 2$ so that ψ^m converges strongly in L^2 to $u_i\nabla\phi$. As $\sqrt{w_j^m}\nabla u_j^m$ converges weakly in L^2 to ∇u_j, this proves the claimed convergence.

The other terms in the weak formulation converge more directly in the limit and we thus have found a weak solution. $\square$

Appendix A. Microscopic reversibility

In the linear SKT model (1.5), the entropy was understood as reversiblity in a microscopic model in [8] and this gave us the intuition about the nonlocal entropy structure. In this appendix we discuss in the case of two species how the form in Remark 1.9 in the general regularisation on bounded domains by a kernel $K : \Omega^2 \to \mathbb{R}_{\geqslant 0}$ appears formally from the microscopic entropy structure.

In the microscopic picture of [8] we considered a spatial discretisation in the one-dimensional setting so that we have discrete positions $\{1,\dots,N\}$. On this discrete setting we consider many particles of the two species 1 and 2 and we then obtain a reversible cross-diffusion behaviour if a pair consisting of a particle of species 1 at position i and a particle of species 2 at position j jumps together with rate $R_r(i,j)$ to the positions $i+1$ and $j+1$, respectively. Likewise the pair can jump with a rate $R_l(i,j)$ to $i-1$ and $j-1$, respectively. We then have the reversibility (and thus the entropy structure) if

$$R_r(i,j) = R_l(i+1,j+1). \tag{A.1}$$

In the formal mean-field limit we then find the evolution for the densities u_1 and u_2 the following nonlinear system

$$\begin{cases} \partial_t u_1(i) = \displaystyle\sum_{j=1}^{M} \Big\{ R_r(i-1,j)u_1(i-1)u_2(j) + R_l(i+1,j)u_1(i+1)u_2(j) \\ \qquad\qquad - (R_l(i,j)+R_r(i,j))u_1(i)u_2(j) \Big\} \\ \partial_t u_2(j) = \displaystyle\sum_{i=1}^{M} \Big\{ R_r(i,j-1)u_1(i)u_2(j-1) + R_l(i,j+1)u_1(i)u_2(j+1) \\ \qquad\qquad - (R_l(i,j)+R_r(i,j))u_1(i)u_2(j) \Big\} \end{cases}$$

for which we can indeed verify the entropy

$$H = \sum_{i=1}^{M} \Big[h(u_1(i)) + h(u_2(i)) \Big]$$

where $h'(x) = \log x$ as

$$\frac{\mathrm{d}}{\mathrm{d}t} H = -\sum_{i,j} R_r(i,j) \Big[\big(u_1(i+1)u_2(j+1) - u_1(i)u_2(j) \big) \\ \times \big(\log(u_1(i+1)u_2(j+1)) - \log(u_1(i)u_2(j)) \big) \Big].$$

For the formal limit of the discrete system to a PDE, we denote the centred discrete Laplacian

$$(\Delta_d f)(i) = f(i+1) + f(i-1) - 2f(i).$$

We can then rewrite the evolution as

$$\partial_t u_1(i) = \Delta_d \left(\sum_j \frac{R_l(i,j)+R_r(i,j)}{2} u_1(i)u_2(j) \right) \\ + \frac{1}{2} \sum_j \Big\{ u_1(i{+}1)u_2(j) \big[R_l(i{+}1,j) - R_r(i{+}1,j) \big] \\ + u_1(i{-}1)u_2(j) \big[R_r(i{-}1,j) - R_r(i{+}1,j) \big] \Big\}$$

and likewise for u_2. By the microscopic reversibility (A.1) we note that this is exactly the discrete form of the regularisation found in (1.17).

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