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A direct approach to the analytic Bergman projection [(*)]

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ABSTRACT. — We develop a direct approach to the semiclassical asymptotics for Bergman projections in exponentially weighted spaces of holomorphic functions, with real analytic strictly plurisubharmonic weights. In particular, the approach does not make any direct use of the Kuranishi trick and it allows us to shorten and simplify proofs of a result due to [7] and [23], stating that in the analytic case, the amplitude of the asymptotic Bergman projection is a realization of a classical analytic symbol.

RÉSUMÉ. — Nous développons une approche directe pour l'asymptotique semi-classique du projecteur de Bergman sur des espaces de fonctions holomorphes à poids exponentiel, dont le poids est analytique et strictement pluri-sous-harmonique. En particulier, cette approche n'utilise jamais directement l'astuce de Kuranishi et nous permet de raccourcir et de simplifier les preuves du fait, établi dans [7] et [23], que dans le cas analytique, l'amplitude du projecteur de Bergman asymptotique est la réalisation d'un symbole analytique classique.

1. Introduction

Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and let $\Phi \in C^\infty(\Omega; \mathbb{R})$ be a strictly plurisubharmonic function (i.e. the Hermitian matrix $\partial\bar\partial\Phi$ is positive

definite everywhere in Ω). The study of the exponentially weighted L^2-space of holomorphic functions

$$H_\Phi(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{C} \text{ holomorphic}; \int_\Omega |u|^2 e^{-\frac{2}{h}\Phi} < \infty \right\},$$

with a small parameter $h > 0$, plays a basic role in complex analysis. In particular, it serves as a local model for the space of holomorphic sections of a high power of an ample line bundle over a complex manifold. In this article, we are interested in the asymptotic description, in the semiclassical limit $h \to 0^+$, of the orthogonal projection $\Pi : L^2(\Omega; e^{-2\Phi/h}) \to H_\Phi(\Omega)$ and its integral kernel. The Bergman projection Π can be studied in many different ways, sharing as a common core the spectral gap property for the $\bar{\partial}$-operator on $L^2(\Omega; e^{-2\Phi/h})$, or rather for the corresponding Hodge Laplacian, as established in [14]. The spectral gap implies directly that the Bergman kernel is rapidly decreasing away from the diagonal [5, 8]. The existence of a complete asymptotic expansion in powers of h for the Bergman kernel has been shown in [3, 25], by means of a reduction to the main result of [22] on the asymptotic behavior of the Szegő kernel on the boundary of a strictly pseudoconvex smooth domain. The work [1] has subsequently provided a self-contained proof of the existence of the expansion, by constructing local asymptotic Bergman kernels directly, using some of the ideas of analytic microlocal analysis, developed in [24]; see also [20]. Other self-contained strategies for the study of the Bergman kernel and its generalizations include [17, 18].

The case of a real analytic weight Φ has been the subject of a recent intense activity [4, 7, 10, 11, 23]. In this setting, one shows that the amplitude in the asymptotic Bergman kernel is a realization of a classical analytic symbol, in the sense of [21, 24], and one can describe the Bergman projection Π up to an exponentially small error, $\mathcal{O}(e^{-\frac{1}{Ch}})$, for some $C > 0$. In [23], an essential ingredient in the proof of this result consists of exploiting the Kuranishi trick, when showing that analytic Weyl pseudodifferential operators and certain Bergman quantizations with classical analytic symbols agree, up to exponentially small errors. This ingredient is already present in [1], see the discussion following (2.7) there. An alternative proof strategy, used in [4, 7, 11], consists in a direct verification that the coefficients in the complete expansion of the Bergman kernel amplitude form a classical analytic symbol. Both approaches are highly technical and notably require a generalisation and improvement of the pre-existing tools in analytic semiclassical analysis as found, for instance, in [24]. A direct approach to Bergman projections in the real analytic case is therefore desirable, and it is precisely our purpose here to develop such an approach. We hope furthermore that the new approach will be useful in situations when the Levi form $\partial\bar{\partial}\Phi$ of Φ

becomes degenerate or nearly degenerate at a point or along a submanifold. A natural occurrence of such a behavior appears in the work in progress [12], in the context of second microlocalization. See also [19].

The following is the main result of this work.

THEOREM 1.1. — *Assume that Φ is real analytic in Ω, and let $x_0 \in \Omega$. There exist a unique classical analytic symbol $a(x, \widetilde{y}; h)$, defined in a neighborhood of $(x_0, \overline{x_0})$, solving*

$$(Aa)(x, \widetilde{y}; h) = 1, \tag{1.1}$$

where A is an elliptic analytic Fourier integral operator, and small open neighborhoods $U \Subset V \Subset \Omega$ of x_0, with C^∞-boundaries, such that the operator

$$\widetilde{\Pi}_V u(x) = \frac{1}{h^n} \int_V e^{\frac{2}{h} \Psi(x, \overline{y})} a(x, \overline{y}; h) u(y) e^{-\frac{2}{h} \Phi(y)} \, L(\mathrm{d}y) \tag{1.2}$$

satisfies

$$\widetilde{\Pi}_V - 1 = \mathcal{O}(1) e^{-\frac{1}{Ch}} : H_\Phi(V) \longrightarrow H_\Phi(U), \quad C > 0. \tag{1.3}$$

Here in (1.2), the holomorphic function Ψ is the polarization of Φ and $L(\mathrm{d}y)$ is the Lebesgue measure on $\mathbb{C}^n$.

Let us point out that the general strategy of constructing the amplitude of the asymptotic Bergman projection by inverting an elliptic analytic Fourier integral operator, acting on the space of analytic symbols, was also followed in [23]. That work proceeded by means of the Kuranishi trick, and the Fourier integral operator in question was obtained by composing various integral transforms. In contrast, in Theorem 1.1, we remove much of the heavy use of the Kuranishi trick and construct the operator A in (1.1) directly. This article can therefore be regarded as an alternative to the two approaches to asymptotic Bergman kernels mentioned above, and we plan to generalize it to degenerate situations as well. It seems also that the method for determining the amplitude in the Bergman kernel, consisting of solving the equation (1.1), is quite direct.

Remark. — The statement of Theorem 1.1 is essentially the same as that of the key Lemma 4.12 in [23], but now with a shorter, more direct proof. The lemma is the most technical ingredient in the results in Section 5 and 6 of [23] concerning the exponentially accurate approximations of the Bergman kernel for scalar functions on $\mathbb{C}^n$ and for sections of line bundles.

The plan of the article is as follows. In Section 2 we review a resolution of the identity in the H_Φ-spaces related to the Fourier inversion formula in the complex domain. Section 3 is devoted to the construction of a suitable analytic symbol to be used as the amplitude for the asymptotic Bergman

projection. We introduce a complex phase function, with no fiber variables present, whose canonical transformation maps the zero section to itself. We introduce a Fourier integral operator A corresponding to this phase function, and we define the amplitude a in (1.1) as the unique classical analytic symbol of order 0 such that $Aa = 1$, locally. Then, in Section 4 we show that the operator $\widetilde{\Pi}_V$ given in (1.2) satisfies the reproducing property in H_Φ, locally and in the weak formulation: for $u, v \in H_\Phi(\Omega)$, on a small enough set V we have $(\widetilde{\Pi}_V u, v)_{H_\Phi(V)} = (u, v)_{H_\Phi(V)} + \mathcal{O}(e^{-\frac{1}{Ch}})$, provided that v is small near the boundary of V. The proof consists of a contour deformation argument which depends on the resolution of the identity of Section 2. The contour deformation is first justified for elements of H_Φ sufficiently localised near a point, and the decomposition of Section 2 ensures that, by linearity, the reproducing property is true on the whole of H_Φ. In Section 5, we conclude the proof of Theorem 1.1 using the $\bar{\partial}$-method.

Once the local approximate reproducing property of Theorem 1.1 has been established, a global version (uniformly in any compact subset of Ω, or uniformly on a complex compact manifold without boundary) follows from cut-and-paste arguments and, in particular, the L^2-estimates for the $\bar{\partial}$-operator. Such arguments have already been developed carefully in [23], see also [1, 13]. For completeness and convenience of the reader, let us merely give the following corollary to Theorem 1.1, stating that the distribution kernel of the true orthogonal projection Π is locally approximated by the kernel of the operator $\widetilde{\Pi}_V$ in (1.2), up to an exponentially small error.

COROLLARY 1.2. — *Let $\Omega \subset \mathbb{C}^n$ be open pseudoconvex, let Φ be strictly plurisubharmonic real analytic in Ω, and let $K(x, \bar{y}) e^{-2\Phi(y)/h}$ be the Schwartz kernel of the orthogonal projection $\Pi : L^2(\Omega; e^{-2\Phi/h}) \to H_\Phi(\Omega)$. Let $x_0 \in \Omega$. There exists a small open neighborhood $\widetilde{U} \Subset \Omega$ of x_0, such that*

$$e^{-\Phi(x)/h}\left(K(x, \bar{y}) - \frac{1}{h^n} e^{\frac{2}{h}\Psi(x,\bar{y})} a(x, \bar{y}; h)\right) e^{-\Phi(y)/h} = \mathcal{O}(1)\, e^{-\frac{1}{Ch}}, \quad C > 0,$$
(1.4)

uniformly for $x, y \in \widetilde{U}$. Here Ψ is the polarization of Φ and the classical analytic symbol a has been introduced in (1.1).

The appendix is devoted to the proof of Corollary 1.2.

Remark. — As mentioned above, the original proofs of the existence of a complete asymptotic expansion for the Bergman kernel in [3], [25] depend on a reduction to the main result of [22], showing that the Szegő projection on the boundary of a strictly pseudoconvex smooth domain in $\mathbb{C}^n$ is a Fourier integral operator with complex phase, with a precise description of the singularities of the distribution kernel near the diagonal, modulo C^∞.

Let us state explicitly that the present paper does not address the problem of a description of the singularities of the kernel of the Szegő projection in the case when the boundary of the strictly pseudoconvex domain is real analytic. In particular, we do not prove that the amplitude $s(x, y; t)$ in [22, Theorem 1.5] is a classical analytic symbol under these assumptions. An outline of proof has been given by Kashiwara in the seminar proceedings [16]. It would be most interesting to try to see whether the arguments developed in the present paper can be adapted to give a simple direct proof of this fact, and we hope to return to this question in a future work.

We would finally like to emphasize that the majority of the methods and the ideas in this paper stem from [24].

Acknowledgments

2. A resolution of the identity

Let $\Omega \subset \mathbb{C}^n$ be open, and let $\Phi \in C^\infty(\Omega; \mathbb{R})$ be strictly plurisubharmonic in Ω: there exists $0 < c \in C(\Omega)$ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \Phi}{\partial x_j \partial \overline{x}_k}(x) \xi_j \overline{\xi}_k \geqslant c(x) |\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{C}^n. \tag{2.1}$$

Let us define the space

$$H_\Phi(\Omega) = \text{Hol}(\Omega) \cap L^2(\Omega; e^{-2\Phi/h} L(\mathrm{d}x)), \tag{2.2}$$

equipped with its natural Hilbert space norm

$$\|u\|_{L^2_\Phi(\Omega)} = \left(\int_\Omega |u(x)|^2 e^{-2\Phi(x)/h} L(\mathrm{d}x) \right)^{1/2}. \tag{2.3}$$

Let $x_0 \in \Omega$ and let $V \Subset \Omega$ be an open neighborhood of x_0 with C^∞-boundary. The strict plurisubharmonicity of Φ has the following consequence.

PROPOSITION 2.1. — *There exists a small neighborhood $V \Subset \Omega$ of x_0, with C^∞-boundary, such that the $2n$-dimensional manifold $\Lambda(x) \subset \mathbb{C}^{2n}_{y,\theta}$ given by*

$$\theta = \theta(x, y) = \frac{2}{i} \left(\frac{\partial \Phi}{\partial y}(y) + \frac{1}{2} \Phi''_{yy}(y)(x - y) \right), \quad y \in V, \tag{2.4}$$

is a good contour for the plurisubharmonic function $(y, \theta) \mapsto -\operatorname{Im}((x - y) \cdot \theta) + \Phi(y)$, *for* $x \in V$, *in the sense of* [24, Chapter 3]: *it is maximally totally real and such that there exists* $\delta > 0$ *such that for all* $x, y \in V$, *we have*

$$-\operatorname{Im}((x - y) \cdot \theta) + \Phi(y) \leqslant \Phi(x) - \delta|x - y|^2. \qquad (2.5)$$

Moreover, the contour $\Lambda(x)$ *depends holomorphically on* $x \in V$.

Proof. — The estimate (2.5) is a direct consequence of (2.1) and Taylor's formula. To see that the $2n$-dimensional C^∞-submanifold $\Lambda(x)$ (with C^∞-boundary) is maximally totally real, we use the following general observation: let q be a plurisubharmonic quadratic form on $\mathbb{C}^n$, and let $L \subset \mathbb{C}^n$ be a real linear subspace of dimension n such that $q|_L$ is negative definite. Then L is maximally totally real, see [24, Proposition 3.1]. □

Let $V_1 \Subset V_2 \Subset V$ be open neighborhoods of x_0 and let $\chi \in C_0^\infty(V; [0, 1])$ be such that $\chi = 1$ near $\overline{V_2}$. Following [24, Chapter 3], [1], we have the following result, representing the identity operator on $H_\Phi(V)$ as a pseudo-differential operator in the anti-classical quantization.

PROPOSITION 2.2. — *Let* V_1 *and* $\Lambda(x)$ *be as above. There exists* $\eta > 0$ *such that when* $u \in H_\Phi(V)$, *we have for* $x \in V_1$,

$$u(x) = \frac{1}{(2\pi h)^n} \iint_{\Lambda(x)} e^{\frac{i}{h}(x-y)\cdot\theta} u(y)\chi(y) \, \mathrm{d}y \, \mathrm{d}\theta$$
$$+ \mathcal{O}(1)\|u\|_{L^2_\Phi(V)} e^{\frac{1}{h}(\Phi(x) - \eta)}. \quad (2.6)$$

Here it is assumed that the contour $\Lambda(x)$ *has been equipped with a suitable orientation.*

Proof. — Following [1], the proof proceeds by applying the Stokes formula to the $(2n, 0)$-form

$$\frac{1}{(2\pi h)^n} e^{\frac{i}{h}(x-y)\cdot\theta} u(y)\chi(y) \, \mathrm{d}y \wedge \mathrm{d}\theta,$$

integrated over the (oriented) boundary of the $(2n + 1)$-dimensional chain given by

$$V \times [0, s] \ni (y, \lambda) \longmapsto (y, \theta(x, y) + i\lambda(\overline{x - y})) \in \mathbb{C}^{2n}_{y, \theta},$$

and letting $s \to \infty$. □

Remark. — In particular, the resolution of identity given by (2.6) is valid for $u \in H_\Phi(\Omega)$.

It follows from Proposition 2.2 that, for some $\eta > 0$ and for all $u \in H_\Phi(V)$, we have

$$u(x) = \int_V u_y(x) \, \mathrm{d}y \, \mathrm{d}\bar{y} + \mathcal{O}(1)\|u\|_{L^2_\Phi(V)} e^{\frac{1}{h}(\Phi(x) - \eta)}, \quad x \in V_1, \qquad (2.7)$$

with

$$u_y(x) = \frac{1}{(2\pi h)^n} e^{\frac{i}{h}(x-y)\cdot\theta(x,y)} u(y)\chi(y)\det(\partial_{\bar{y}}\theta(x,y)) \in H_{F_y}(V), \quad (2.8)$$

where F_y is strictly plurisubharmonic such that

$$F_y(x) \leqslant \Phi(x) - \delta|x-y|^2, \quad \delta > 0. \quad (2.9)$$

We conclude this section with a pointwise estimate for elements of $H_\Phi(V)$.

PROPOSITION 2.3. — Let $V_1 \Subset V \Subset \Omega$. Then there exists $C > 0$ such that for all $u \in H_\Phi(V)$ and for all $h \in (0,1]$, we have

$$\sup_{V_1} |ue^{-\Phi/h}| \leqslant Ch^{-n}\|u\|_{H_\Phi(V)}. \quad (2.10)$$

Proof. — A holomorphic function is equal to its mean value over an open ball, so that, for all $x \in V_1$ and all $h > 0$ small enough so that $B(x,h) \subset V$, we have

$$u(x) = \frac{C_n}{h^{2n}} \int_{|y-x|<h} u(y)\, L(\mathrm{d}y).$$

Here $C_n > 0$ depends on n only. It follows that

$$|u(x)|e^{-\Phi(x)/h} \leqslant \frac{C_n}{h^{2n}} \int_{|y-x|<h} |u(y)|e^{-\Phi(x)/h}\, L(\mathrm{d}y)$$

$$\leqslant \sup_{|y-x|<h} e^{(\Phi(y)-\Phi(x))/h} \frac{C_n}{h^{2n}} \int_{|y-x|<h} |u(y)|e^{-\Phi(y)/h}\, L(\mathrm{d}y)$$

$$\leqslant \frac{C'}{h^{2n}}\|u\|_{H_\Phi(V)}\|1\|_{L^2(B(x,h))} \leqslant \frac{C'}{h^n}\|u\|_{H_\Phi(V)}. \qquad \square$$

3. A Fourier integral operator with complex phase

Assume that the strictly plurisubharmonic function Φ is real analytic in Ω, and let $x_0 \in \Omega$. Associated to Φ is the polarization $\Psi(x,y)$, which is the unique holomorphic function of $(x,y) \in \mathrm{neigh}((x_0,\overline{x_0}), \mathbb{C}^{2n})$ such that

$$\Psi(x,\bar{x}) = \Phi(x), \quad x \in \mathrm{neigh}(x_0, \mathbb{C}^n). \quad (3.1)$$

The matrix $\Psi''_{xy}(x_0,\overline{x_0}) = \Phi''_{x\bar{x}}(x_0)$ is non-singular and the following classical estimate,

$$\Phi(x) + \Phi(y) - 2\operatorname{Re}\Psi(x,\bar{y}) \asymp |x-y|^2, \quad x,y \in \mathrm{neigh}(x_0, \mathbb{C}^n), \quad (3.2)$$

is implied by the strict plurisubharmonicity of Φ, see for instance [23].

Let us set

$$\varphi(y,\widetilde{x};x,\widetilde{y}) = \Psi(x,\widetilde{y}) - \Psi(x,\widetilde{x}) - \Psi(y,\widetilde{y}) + \Psi(y,\widetilde{x}). \quad (3.3)$$

We have $\varphi \in \mathrm{Hol}(\mathrm{neigh}((x_0, \overline{x}_0; x_0, \overline{x}_0), \mathbb{C}^{4n}))$. Furthermore, at the point $(x_0, \overline{x_0}; x_0, \overline{x_0})$, the $2n \times 2n$-matrix of second derivatives

$$\varphi''_{(y,\widetilde{x}),(x,\widetilde{y})} = \begin{pmatrix} \varphi''_{yx} & \varphi''_{y\widetilde{y}} \\ \varphi''_{\widetilde{x}x} & \varphi''_{\widetilde{x}\widetilde{y}} \end{pmatrix} = \begin{pmatrix} 0 & -\Psi''_{y\widetilde{y}}(y,\widetilde{y}) \\ -\Psi''_{\widetilde{x}x}(x,\widetilde{x}) & 0 \end{pmatrix} \quad (3.4)$$

is invertible; thus this matrix is non-degenerate in a neighbourhood of $(x_0, \overline{x_0}; x_0, \overline{x_0})$. Therefore, $\varphi(y, \widetilde{x}; x, \widetilde{y})$ is a generating function for the canonical transformation

$$\kappa : \left(x, \widetilde{y}; -\frac{2}{i}\partial_x\varphi, -\frac{2}{i}\partial_{\widetilde{y}}\varphi \right) \longmapsto \left(y, \widetilde{x}; \frac{2}{i}\partial_y\varphi, \frac{2}{i}\partial_{\widetilde{x}}\varphi \right). \quad (3.5)$$

PROPOSITION 3.1. — *The canonical transformation κ maps the zero section to the zero section, and we have*

$$\det \varphi''_{(x,\widetilde{y}),(x,\widetilde{y})}(x_0, \overline{x}_0; x_0, \overline{x}_0) \neq 0. \quad (3.6)$$

Proof. — Using the invertibility of $\Psi''_{x\widetilde{x}}(x_0, \overline{x}_0)$ and (3.3), we see that $\partial_x\varphi = 0 \Longleftrightarrow \widetilde{y} = \widetilde{x}$, as well as $\partial_{\widetilde{y}}\varphi = 0 \Longleftrightarrow x = y$, and therefore the unique critical point of φ with respect to the variables $(x, \widetilde{y})$ is given by $x = y$, $\widetilde{y} = \widetilde{x}$. The corresponding critical value is equal to 0. When proving the proposition, we may therefore simplify the notation by considering a holomorphic function $\varphi(z, w)$ defined near $(0, 0)$ in $\mathbb{C}^{2m}$, such that

$$\det \varphi''_{zw}(0,0) \neq 0, \quad \varphi'_w(z, w) = 0 \Longleftrightarrow w = z, \quad \varphi(z, z) = 0. \quad (3.7)$$

It follows that

$$\varphi'_z(z, z) = \partial_z(\varphi(z, z)) = 0, \quad (3.8)$$

and therefore the canonical transformation

$$\kappa : (w, -\partial_w\varphi(z, w)) \longmapsto (z, \partial_z\varphi(z, w)) \quad (3.9)$$

maps the zero section $\{\eta = 0\}$ to the zero section $\{\xi = 0\}$. It only remains to check that $\det \varphi''_{ww}(0, 0) \neq 0$, and to this end we observe that the differential of κ at $(0, 0)$ is given by

$$(\delta_w, -\varphi''_{wz}\delta_z - \varphi''_{ww}\delta_w) \longmapsto (\delta_z, \varphi''_{zz}\delta_z + \varphi''_{zw}\delta_w), \quad \varphi'' = \varphi''(0,0), \quad (3.10)$$

where δ_z and δ_w are infinitesimal increments. If $\varphi''_{ww}\delta_w = 0$, we get $d\kappa(0, 0) :$ $(\delta_w, 0) \mapsto (0, \varphi''_{zw}\delta_w)$, and it follows that $\delta_w = 0$. $\qquad \square$

We now introduce an elliptic analytic Fourier integral operator A in the complex domain, defined in a neighbourhood of $(x_0, \overline{x_0})$. This Fourier integral operator is associated to the canonical transformation κ in (3.5) and acts on the space of analytic symbols H_0^{loc}, defined in a neighborhood of $(x_0, \overline{x_0})$. Here we recall that the space of analytic symbols H_0^{loc} has been introduced in [24, Chapter 1]. To this end, for $(y, \widetilde{x})$ in a neighbourhood of $(x_0, \overline{x_0})$, we let $\Gamma(y, \widetilde{x}) \subset \mathbb{C}^{2n}_{x,\widetilde{y}}$ be a good contour for the pluriharmonic phase function $(x, \widetilde{y}) \mapsto \mathrm{Re}\,\varphi(y, \widetilde{x}; x, \widetilde{y})$, so that $\Gamma(y, \widetilde{x})$ is a $2n$-dimensional contour passing

through the critical point $(y, \widetilde{x})$ and depending holomorphically on $(y, \widetilde{x})$, such that along $\Gamma(y, \widetilde{x})$ we have

$$\operatorname{Re} \varphi(y, \widetilde{x}; x, \widetilde{y}) \leqslant -\frac{1}{C}|x - y|^2 - \frac{1}{C}|\widetilde{y} - \widetilde{x}|^2. \tag{3.11}$$

Given an analytic symbol $u(x, \widetilde{y}; h)$ defined near $(x_0, \overline{x_0})$, we set

$$(Au)(y, \widetilde{x}; h) = \frac{1}{h^n} \iint_{\Gamma(y, \widetilde{x})} e^{\frac{2}{h} \varphi(y, \widetilde{x}; x, \widetilde{y})} u(x, \widetilde{y}; h) \mathrm{d}x \mathrm{d}\widetilde{y}, \tag{3.12}$$

so that Au is an analytic symbol defined in a neighborhood of $(x_0, \overline{x_0})$.

Before stating the main result of this section, following [24, Chapter 1], let us recall the notion of a classical analytic symbol. Let $V \subset \mathbb{C}^n$ be open, $a_k \in \operatorname{Hol}(V)$, $k = 0, 1, \ldots$, and assume that for every $\widetilde{V} \Subset V$, there exists $C = C_{\widetilde{V}} > 0$ such that

$$|a_k(x)| \leqslant C^{k+1} k^k, \quad x \in \widetilde{V}. \tag{3.13}$$

The series $a(x; h) = \sum_{k=0}^{\infty} a_k(x) h^k$ is called a formal classical analytic symbol of order zero. We have a realization of a on $\widetilde{V}$ given by

$$a_{\widetilde{V}}(x; h) = \sum_{0 \leqslant k \leqslant (C_{\widetilde{V}} eh)^{-1}} a_k(x) h^k, \tag{3.14}$$

so that $a_{\widetilde{V}} \in \operatorname{Hol}(\widetilde{V})$, $|a_{\widetilde{V}}(x; h)| \leqslant C_{\widetilde{V}} e/(e-1)$.

PROPOSITION 3.2. — *There is a unique classical analytic symbol of order zero $a(x, \widetilde{y}; h)$, defined in a neighbourhood of $(x_0, \overline{x_0})$ such that*

$$(Aa)(y, \widetilde{x}; h) = 1 + \mathcal{O}(e^{-\frac{1}{Ch}}), \tag{3.15}$$

near $(x_0, \overline{x_0})$.

Proof. — In view of Proposition 3.1 combined with the method of analytic stationary phase, we know that the Fourier integral operator A in (3.12) maps classical analytic symbols defined near $(x_0, \overline{x_0})$ to classical analytic symbols defined in a neighborhood of the same point, see [24, Chapter 2, Chapter 4]. A similar observation has also been used in [23]. Furthermore, in view of the ellipticity of A, from [24, Theorem 4.5], we know that there exists a microlocal inverse B of A having the form

$$(Bb)(x, \widetilde{y}; h) = \frac{1}{h^n} \iint_{\Gamma_1(x, \widetilde{y})} e^{-\frac{2}{h} \varphi(y, \widetilde{x}; x, \widetilde{y})} d(y, \widetilde{x}, x, \widetilde{y}; h) b(y, \widetilde{x}; h) \mathrm{d}y \, \mathrm{d}\widetilde{x}. \tag{3.16}$$

Here $d(y, \widetilde{x}, x, \widetilde{y}; h)$ is an elliptic classical analytic symbol defined in a neighborhood of the point $(x_0, \overline{x_0}; x_0, \overline{x_0}) \in \mathbb{C}^{4n}$, $b(y, \widetilde{x}; h)$ is a classical analytic

symbol defined near $(x_0, \bar{x}_0)$, and $\Gamma_1(x, \widetilde{y})$ is a good contour for the pluriharmonic function $(y, \widetilde{x}) \mapsto - \operatorname{Re} \varphi(y, \widetilde{x}; x, \widetilde{y})$. (Details on the construction of B are given after the end of this proof.) Setting

$$a(x, \widetilde{y}; h) = (B1)(x, \widetilde{y}; h), \tag{3.17}$$

we obtain the desired classical analytic symbol defined in a neighborhood of $(x_0, \bar{x}_0)$. $\qquad \square$

The rest of this section is devoted to a commentary on Proposition 3.2, by exhibiting its relationship with the main claim, and by comparing it to previously existing work on the topic.

Let us first explain how the equation $(Aa)(y, \widetilde{x}; h) = 1$ is related to the expansion of the Bergman kernel using a formal argument, in the sense that all contours of integration are omitted and all exponentially small remainders are neglected. The equation $(Aa)(y, \widetilde{x}; h) = 1$ can be written as follows,

$$\frac{1}{h^n} \iint e^{\frac{2}{h}(\Psi(x,\tilde{y}) - \Psi(x,\tilde{x}) - \Psi(y,\tilde{y}))} a(x, \widetilde{y}; h) \, dx \, d\widetilde{y} = e^{-\frac{2}{h}\Psi(y,\tilde{x})}. \tag{3.18}$$

Introducing the elliptic Fourier integral operators

$$(\mathcal{A}u)(x) = \frac{1}{h^{n/2}} \int e^{\frac{2}{h}\Psi(x,\tilde{y})} a(x, \widetilde{y}; h) u(\widetilde{y}) \, d\widetilde{y}, \tag{3.19}$$

$$(\mathcal{C}u)(\widetilde{x}) = \frac{1}{h^{n/2}} \int e^{-\frac{2}{h}\Psi(y,\tilde{x})} u(y) \, dy, \tag{3.20}$$

we can rewrite (3.18) in the form,

$$\iint \mathcal{K}_{\mathcal{C}}(\widetilde{x}, x) \mathcal{K}_{\mathcal{A}}(x, \widetilde{y}) \mathcal{K}_{\mathcal{C}}(\widetilde{y}, y) \, dx \, d\widetilde{y} = \mathcal{K}_{\mathcal{C}}(\widetilde{x}, y). \tag{3.21}$$

Here $\mathcal{K}_{\mathcal{A}}, \mathcal{K}_{\mathcal{C}}$ are the integral kernels of $\mathcal{A}, \mathcal{C}$, respectively. The equation (3.15) is therefore equivalent to the operator equation

$$\mathcal{C} \circ \mathcal{A} \circ \mathcal{C} = \mathcal{C} \iff \mathcal{A} \circ \mathcal{C} = 1. \tag{3.22}$$

In Sections 4, 5 below, we shall see that the operator of the form

$$(\widetilde{\Pi}u)(x) = \frac{1}{h^n} \iint e^{\frac{2}{h}(\Psi(x,\tilde{y}) - \Psi(y,\tilde{y}))} a(x, \widetilde{y}; h) u(y) \, dy \, d\widetilde{y}$$

enjoys the (approximate) reproducing property on H_Φ, and since we have $\widetilde{\Pi} = \mathcal{A} \circ \mathcal{C}$, the equation (3.22) can be regarded as a formal factorization of the asymptotic Bergman projection. In a sense, the rest of this article is devoted to a proof that the manipulations above can be performed.

Before comparing Proposition 3.2 to the techniques used in [23], let us briefly recall, following the proof of [24, Theorem 4.5], the principal steps in the construction of the operator B in (3.16). First, letting $\widetilde{B}$ be an operator of the form (3.16), where the corresponding amplitude $\widetilde{d}(y, \widetilde{x}, x, \widetilde{y}; h)$

is an elliptic classical analytic symbol, we observe, by an application of the Kuranishi trick, that the composition $A \circ \widetilde{B}$ is an elliptic h-pseudo-differential operator with a standard phase and a classical analytic symbol of the form $\widetilde{c}(z, w, \theta; h)$. Here $h = \frac{1}{\lambda}$ in the notation of [24], and $z, w \in$ neigh$((x_0, \overline{x_0}), \mathbb{C}^{2n})$, $\theta \in$ neigh$((0,0), \mathbb{C}^{2n})$. In [24, Chapter 4], prior to the statement of Theorem 4.5, it is explained how to replace a classical analytic symbol of the form $\widetilde{c}(z, w, \theta; h)$ by a classical analytic symbol $\sigma_{A \circ \widetilde{B}}(z, \zeta; h)$, no longer depending on w. Letting R be a classical analytic h-pseudo-differential parametrix of $A \circ \widetilde{B}$, whose existence is guaranteed by [24, Theorem 1.5], we may then set $B = \widetilde{B} \circ R$ in (3.16).

We would like to emphasize that this route is much simpler than the one followed in [23], which uses that an operator of the form (1.2), where a can be any classical analytic symbol, can be written as an analytic Weyl pseudodifferential operator,

$$\mathrm{Op}^w(b)u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\cdot\theta} b\left(\frac{x+y}{2}, \theta; h\right) u(y) \, \mathrm{d}y \, \mathrm{d}\theta, \quad (3.23)$$

up to an exponentially small error term. Here $b(x, \theta)$ is a classical analytic symbol of order 0 defined in a neighborhood of $\left(x_0, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x_0)\right)$ and $\Gamma(x)$ is a suitable good contour, cf. Proposition 2.1. This result is established relying on the Kuranishi trick, showing that the map $a \mapsto b$ is an elliptic analytic Fourier integral operator associated to a canonical transformation sending the zero section to itself. As explained in [23, Section 3], the map $a \mapsto b$ is given as a composition of various integral transforms, and showing that it is an analytic Fourier integral operator with a canonical transformation which sends the zero section to itself, is an essential accomplishment of [23], whose proof requires some substantial effort. Thus, even though (when inspecting how the operator B in (3.16) is constructed) both proofs use the Kuranishi trick at some point, its usage in [24, Chapter 4] is much simpler and well established. This discussion makes it clear therefore that the direct approach developed in the present paper improves and shortens the arguments of [23].

To conclude this discussion, we note that equation (3.15) in Proposition 3.2 is reminiscent of the condition (2.10) in the work [1], which reads

$$S\left(B\left(x, z(x, y, \theta); h\right) \Delta_0(x, y, \theta)\right)|_{y=x} = 1, \quad (3.24)$$

where $S = e^{hD_y \cdot hD_\theta / h}$. In [1], the equation (3.24) provides recursive equations for the Bergman kernel coefficients, which are also valid in the case of a C^∞ weight. Here, the point is that the amplitude of the asymptotic Bergman projection in the analytic case is constructed directly, by means of an inversion (up to an exponentially small error) of an elliptic analytic Fourier integral operator associated to the canonical transformation κ in (3.5), rather than using (3.24) and proving that the formal symbol is analytic, as in [4, 7].

See also [13] for an application of this recursion relation in the smooth case, more closely related to the point of view of this work.

4. The reproducing property in the weak formulation

Let us recall from Section 2 that $V \Subset \Omega$ is a small open neighborhood of a point $x_0 \in \Omega$, and shrinking V if necessary, we may assume that the polarization Ψ of the real analytic weight function Φ, introduced in (3.1), as well as the classical analytic symbol a, given in Proposition 3.2, are defined in a neighborhood of the closure of the open set $V \times \rho(V)$. Here $\rho(x) = \bar{x}$ is the complex conjugation map.

We introduce the following operator of Bergman type,

$$\widetilde{\Pi}_V u(x) = \frac{1}{h^n} \iint_{\Gamma_V} e^{\frac{2}{h}(\Psi(x,\tilde{y}) - \Psi(y,\tilde{y}))} a(x, \tilde{y}; h) u(y) \, \mathrm{d}y \, \mathrm{d}\tilde{y}, \quad u \in H_\Phi(V), \quad (4.1)$$

where the contour of integration $\Gamma_V \subset V \times \rho(V)$ is given by

$$\Gamma_V = \{\tilde{y} = \bar{y}, \ y \in V\}. \quad (4.2)$$

Here in (4.1) we have also chosen a realization of a on $V \times \rho(V)$. It follows from (3.2), combined with the Schur test, that

$$\widetilde{\Pi}_V = \mathcal{O}(1) : H_\Phi(V) \longrightarrow H_\Phi(V). \quad (4.3)$$

The purpose of this section is to show that the operator $\widetilde{\Pi}_V$ satisfies a reproducing property, in the weak formulation. Specifically, we shall prove that for a convenient class of $(u,v) \in H_\Phi(V)$, the continuous sesquilinear form

$$H_\Phi(V) \times H_\Phi(V) \ni (u,v) \longmapsto (\widetilde{\Pi}_V u, v)_{H_\Phi(V)} \quad (4.4)$$

agrees, modulo an exponentially small error, with the scalar product $(u,v)_{H_\Phi(V)}$. This result cannot be expected to hold if u,v are general elements of $H_\Phi(V)$, since they might both concentrate near the boundary of V where we have cut off the integral operator $\widetilde{\Pi}_V$.

The following is the main result of this section. It will be instrumental in Section 5, when proving Theorem 1.1.

THEOREM 4.1. — *There exists a small open neighborhood $W \Subset V$ of x_0 with C^∞-boundary such that for each $\Phi_1 \in C(\Omega; \mathbb{R})$, $\Phi_1 \leqslant \Phi$, with $\Phi_1 < \Phi$ on $\Omega \backslash \overline{W}$, there exists $C > 0$ such that for all $u \in H_\Phi(V)$, $v \in H_{\Phi_1}(V)$, we have*

$$(\widetilde{\Pi}_V u, v)_{H_\Phi(V)} = (u,v)_{H_\Phi(V)} + \mathcal{O}(1)e^{-\frac{1}{Ch}} \|u\|_{H_\Phi(V)} \|v\|_{H_{\Phi_1}(V)}. \quad (4.5)$$

When proving Theorem 4.1, using also the notation of Section 2, we let $W \Subset V_1 \Subset V$ be an open neighborhood of x_0 with C^∞-boundary, to be chosen small enough, and let $\Phi_1 \in C(\Omega; \mathbb{R})$ be such that

$$\Phi_1 \leqslant \Phi \text{ in } \Omega, \quad \Phi_1 < \Phi \text{ on } \Omega \backslash \overline{W}. \tag{4.6}$$

We shall study the scalar product

$$(\widetilde{\Pi}_V u, v)_{H_\Phi(V)} = \int_V \widetilde{\Pi}_V u(x) \overline{v(x)} e^{-2\Phi(x)/h} \, L(\mathrm{d}x),$$
$$u \in H_\Phi(V), \ v \in H_{\Phi_1}(V), \quad (4.7)$$

and let us first write, using (4.3), (4.6), and the Cauchy–Schwarz inequality,

$$(\widetilde{\Pi}_V u, v)_{H_\Phi(V)} = \int_{V_1} \widetilde{\Pi}_V u(x) \overline{v(x)} e^{-2\Phi(x)/h} \, L(\mathrm{d}x)$$
$$+ \mathcal{O}(1) e^{-\frac{1}{Ch}} \|u\|_{H_\Phi(V)} \|v\|_{H_{\Phi_1}(V)}. \quad (4.8)$$

Here and in what follows we let $C > 0$ stand for constants which may depend on Φ, Φ_1, but not on u, v. Let next V_2 be an open set such that $V_1 \Subset V_2 \Subset V$ and observe that in view of (3.2), we have

$$\|\widetilde{\Pi}_V (1 - \chi_{V_2}) u\|_{L^2_\Phi(V_1)} \leqslant \mathcal{O}(1) e^{-\frac{1}{Ch}} \|u\|_{H_\Phi(V)}. \tag{4.9}$$

Here χ_{V_2} denotes the characteristic function of V_2. Using (4.8) and (4.9), we may therefore write

$$(\widetilde{\Pi}_V u, v)_{H_\Phi(V)} = \int_{V_1} \widetilde{\Pi}_{V_2} u(x) \overline{v(x)} e^{-2\Phi(x)/h} \, L(\mathrm{d}x)$$
$$+ \mathcal{O}(1) e^{-\frac{1}{Ch}} \|u\|_{H_\Phi(V)} \|v\|_{H_{\Phi_1}(V)}, \quad (4.10)$$

where, similarly to (4.1), we set

$$\widetilde{\Pi}_{V_2} u(x) = \frac{1}{h^n} \iint_{\Gamma_{V_2}} e^{\frac{2}{h}(\Psi(x,\tilde{y}) - \Psi(y,\tilde{y}))} a(x, \tilde{y}; h) u(y) \, \mathrm{d}y \, \mathrm{d}\tilde{y}. \tag{4.11}$$

The advantage of representing the scalar product $(\widetilde{\Pi}_V u, v)_{H_\Phi(V)}$ in the form (4.10) is due to the fact that in the right hand side of (4.10), both the integrations in x and y are confined to suitable relatively compact subsets of the open set V, where good pointwise estimates on the holomorphic functions u and v are available, in view of Proposition 2.3.

We would next like to apply the resolution of the identity (2.7) to the holomorphic function $v \in H_{\Phi_1}(V)$ in the integral in the right hand side of (4.10). To this end, let us first observe that thanks to the exponential decay of v in $H_\Phi(V)$ away from $\overline{W}$, committing an exponentially small error, we may restrict the domain of integration in the right hand side of (2.7) to

an arbitrarily small but fixed neighborhood W_1 of $\overline{W}$, $W_1 \Subset V_1$. In precise terms, we may write

$$v(x) = \int_{W_1} v_z(x)\, dz\, d\bar{z} + \mathcal{O}(1)\|v\|_{H_{\Phi_1}(V)} e^{\frac{1}{h}(\Phi(x) - \frac{1}{C})}, \quad x \in V_1, \quad (4.12)$$

where, similarly to (2.8), we have

$$v_z(x) = \frac{1}{(2\pi h)^n} e^{\frac{i}{h}(x-z)\cdot\theta(x,z)} v(z)\chi(z) \det(\partial_{\bar{z}}\theta(x,z)) \in \mathrm{Hol}(V) \quad (4.13)$$

is well localized at the point $z \in W_1$, see (2.5). Combining (4.10), (4.12), and (4.3), we get

$$(\widetilde{\Pi}_V u, v)_{H_\Phi(V)} = \int_{W_1}\int_{V_1} \widetilde{\Pi}_{V_2}u(x)\overline{v_z(x)}e^{-2\Phi(x)/h}\, L(dx)\, dz\, d\bar{z}$$
$$+ \mathcal{O}(1)e^{-\frac{1}{Ch}}\|u\|_{H_\Phi(V)}\|v\|_{H_{\Phi_1}(V)}. \quad (4.14)$$

Let us rewrite (4.14) as follows,

$$(\widetilde{\Pi}_V u, v)_{H_\Phi(V)} = \int_{W_1} (\widetilde{\Pi}_{V_2}u, v_z)_{H_\Phi(V_1)}\, dz\, d\bar{z}$$
$$+ \mathcal{O}(1)e^{-\frac{1}{Ch}}\|u\|_{H_\Phi(V)}\|v\|_{H_{\Phi_1}(V)}. \quad (4.15)$$

When proving Theorem 4.1, it will be convenient to work with the decomposition (4.15), in view of the good localization properties of the holomorphic functions v_z, for $z \in W_1$.

The crucial role in the proof is played by the following observation.

PROPOSITION 4.2. — *Given $z \in V$, let us set for some $\delta > 0$ small,*

$$F_z(\widetilde{x}) = \Phi(\overline{\widetilde{x}}) - \delta|\widetilde{x} - \bar{z}|^2, \quad \widetilde{x} \in \rho(V). \quad (4.16)$$

Let G_z be the following real analytic plurisubharmonic function:

$$G_z(x, \widetilde{x}, y, \widetilde{y})$$
$$= 2\,\mathrm{Re}\,\Psi(x, \widetilde{y}) - 2\,\mathrm{Re}\,\Psi(y, \widetilde{y}) + \Phi(y) + F_z(\widetilde{x}) - 2\,\mathrm{Re}\,\Psi(x, \widetilde{x}). \quad (4.17)$$

Then G_z has a non-degenerate critical point at $(z, \bar{z}, z, \bar{z})$ of signature $(4n, 4n)$, with the critical value equal to 0. Furthermore, the following two submanifolds of $V \times \rho(V) \times V \times \rho(V) \subset \mathbb{C}^{4n}$ are good contours for G_z in a neighbourhood of $(z, \bar{z}, z, \bar{z})$, in the sense that they are both contours of maximal real dimension $4n$ passing through the critical point, along which the Hessian of G_z is negative definite:

   (1) *The contour*

$$\Gamma_V \times \Gamma_V = \{(x, \widetilde{x}, y, \widetilde{y}); \widetilde{x} = \bar{x}, \widetilde{y} = \bar{y}, x \in V, y \in V\} \quad (4.18)$$

(2) *The composed contour*

$$\{(x, \widetilde{x}, y, \widetilde{y}); (y, \widetilde{x}) \in \Gamma_V, (x, \widetilde{y}) \in \Gamma(y, \widetilde{x})\}. \tag{4.19}$$

Here $\Gamma(y, \widetilde{x}) \subset \mathbb{C}^{2n}_{x,\widetilde{y}}$ is a good contour for the pluriharmonic function

$$(x, \widetilde{y}) \longmapsto \operatorname{Re} \varphi(y, \widetilde{x}; x, \widetilde{y}),$$

described in (3.11), (3.12).

Proof. — Let us observe first that the two contours clearly pass through the point $(z, \overline{z}, z, \overline{z})$ and that $G_z(z, \overline{z}, z, \overline{z}) = 0$, in view of (4.16), (3.1). In order to show that $(z, \overline{z}, z, \overline{z})$ is a non-degenerate critical point of signature $(4n, 4n)$, it suffices, in view of the plurisubharmonicity of $G_z(x, \widetilde{x}, y, \widetilde{y})$, to observe that, using (3.2), (4.16), we have

$$G_z(x, \overline{x}, y, \overline{y}) \leqslant -\frac{1}{C}|y - x|^2 - \delta|x - z|^2 \leqslant -\frac{1}{C}|x - z|^2 - \frac{1}{C}|y - z|^2. \tag{4.20}$$

This establishes at the same time that the contour (4.18) is a good contour for G_z. It only remains to prove that the second submanifold given in (4.19) also defines a good contour. To this end, let us write, using (3.3), (4.17),

$$G_z(x, \widetilde{x}, y, \widetilde{y}) = 2 \operatorname{Re} \varphi(y, \widetilde{x}; x, \widetilde{y}) - 2 \operatorname{Re} \Psi(y, \widetilde{x}) + \Phi(y) + F_z(\widetilde{x}). \tag{4.21}$$

Using (3.11), (4.16), (3.1), we get therefore for $(y, \widetilde{x}) \in \Gamma_V, (x, \widetilde{y}) \in \Gamma(y, \widetilde{x})$,

$$G_z(x, \widetilde{x}, y, \widetilde{y})$$
$$\leqslant -\frac{1}{C}|y - x|^2 - \frac{1}{C}|\widetilde{y} - \widetilde{x}|^2 - 2\operatorname{Re}\Psi(y, \widetilde{x}) + \Phi(y) + \Phi(\overline{\widetilde{x}}) - \delta|\widetilde{x} - \overline{z}|^2$$
$$= -\frac{1}{C}|y - x|^2 - \frac{1}{C}|\widetilde{y} - \widetilde{x}|^2 - \delta|y - z|^2. \tag{4.22}$$

It follows that

$$G_z(x, \widetilde{x}, y, \widetilde{y}) \leqslant -\frac{1}{C}|x - z|^2 - \frac{1}{C}|y - z|^2 - \frac{1}{C}|\widetilde{x} - \overline{z}|^2 - \frac{1}{C}|\widetilde{y} - \widetilde{x}|^2$$
$$\leqslant -\frac{1}{C}|x - z|^2 - \frac{1}{C}|y - z|^2 - \frac{1}{C}|\widetilde{x} - \overline{z}|^2 - \frac{1}{C}|\widetilde{y} - \overline{z}|^2, \tag{4.23}$$

which demonstrates that the composed contour (4.19) is also good and concludes the proof. $\square$

We are now ready to take a closer look at the scalar product $(\widetilde{\Pi}_{V_2} u, v_z)_{H_\Phi(V_1)}$, occuring in the right hand side of (4.15).

PROPOSITION 4.3. — *There exists an open neighborhood $W_1 \Subset V_1$ of x_0 such that, uniformly in $z \in W_1$, we have*

$$(\widetilde{\Pi}_{V_2} u, v_z)_{H_\Phi(V_1)} = (u, v_z)_{H_\Phi(V_1)} + \mathcal{O}(1)e^{-\frac{1}{C\hbar}}\|u\|_{H_\Phi(V)}|v(z)|e^{-\Phi(z)/\hbar}. \tag{4.24}$$

Here v_z is given in (4.13).

Proof. — The scalar product in the Hilbert space of holomorphic functions $H_\Phi(V_1)$ can be expressed as follows,

$$(f, g)_{H_\Phi(V_1)} = \int_{V_1} f(x)\overline{g(x)}e^{-\frac{2\Phi(x)}{h}} L(\mathrm{d}x)$$

$$= C_n \iint_{\Gamma_{V_1}} f(x)g^*(\widetilde{x})e^{-\frac{2}{h}\Psi(x,\tilde{x})} \, \mathrm{d}x \, \mathrm{d}\widetilde{x}. \quad (4.25)$$

Here the contour Γ_{V_1} is defined similarly to (4.2) and C_n is a numerical factor, depending on n only, such that the Lebesgue measure $L(\mathrm{d}x)$ on $\mathbb{C}^n$ satisfies $L(\mathrm{d}x) = C_n \mathrm{d}x \, \mathrm{d}\bar{x}$. In (4.25) we have also set

$$g^*(\widetilde{x}) = \overline{g(\overline{\widetilde{x}})} \in H_{\widehat{\Phi}}(\rho(V_1)), \quad \widehat{\Phi}(\widetilde{x}) = \overline{\Phi(\overline{\widetilde{x}})}. \quad (4.26)$$

Recalling (4.11) and using (4.25), we see that the scalar product $(\widetilde{\Pi}_{V_2}u, v_z)_{H_\Phi(V_1)}$ takes the form

$$\frac{C_n}{h^n} \iint_{\Gamma_{V_1}} \left(\iint_{\Gamma_{V_2}} e^{\frac{2}{h}(\Psi(x,\tilde{y})-\Psi(y,\tilde{y}))}a(x,\widetilde{y};h)u(y) \, \mathrm{d}y \, \mathrm{d}\widetilde{y} \right)$$

$$\times v_z^*(\widetilde{x})e^{-\frac{2}{h}\Psi(x,\tilde{x})} \, \mathrm{d}x \, \mathrm{d}\widetilde{x}. \quad (4.27)$$

Here using (4.13), (2.5), we observe that

$$|v_z^*(\widetilde{x})| \leqslant \frac{\mathcal{O}(1)}{h^n}|v(z)|e^{-\Phi(z)/h}e^{F_z(\tilde{x})/h}, \quad \widetilde{x} \in \rho(V_1), \quad (4.28)$$

where F_z is the strictly plurisubharmonic function in $\rho(V_1)$ given by

$$F_z(\widetilde{x}) = \widehat{\Phi}(\widetilde{x}) - \delta|\widetilde{x} - \bar{z}|^2, \quad (4.29)$$

see also (4.16). Combining (4.28) with Proposition 2.3 we conclude that the absolute value of the holomorphic integrand in (4.27)

$$V_1 \times \rho(V_1) \times V_2 \times \rho(V_2) \ni (x, \widetilde{x}, y, \widetilde{y})$$

$$\longmapsto e^{\frac{2}{h}(\Psi(x,\tilde{y})-\Psi(y,\tilde{y}))}a(x,\widetilde{y};h)u(y)v_z^*(\widetilde{x})e^{-\frac{2}{h}\Psi(x,\tilde{x})} \quad (4.30)$$

does not exceed

$$\frac{\mathcal{O}(1)}{h^{2n}}\|u\|_{H_\Phi(V)}|v(z)|e^{-\Phi(z)/h}e^{G_z(x,\tilde{x},y,\tilde{y})/h}. \quad (4.31)$$

Here the plurisubharmonic function $G_z(x, \widetilde{x}, y, \widetilde{y})$ has been defined in (4.17), and the contour of integration $\Gamma_{V_1} \times \Gamma_{V_2}$ in (4.27) is therefore good for G_z, in view of Proposition 4.2. In particular, only a small neighborhood of the critical point $(z, \bar{z}, z, \bar{z})$ gives a contribution that is not exponentially small to the integral (4.27). In view of (4.16), (4.17), let us also remark that $G_z = G_{x_0} + \mathcal{O}(\delta|x_0 - z|)$.

We shall now carry out a contour deformation in (4.27), making use of Proposition 4.2. When doing so, let us recall from [24, Chapter 3], [9,

Proposition 3.5] that all good contours are homotopic, with the homotopy through good contours. As explained in [24, Chapter 3], a homotopy between two good contours is obtained by working in the Morse coordinates in a neighborhood of the critical point. An application of the Stokes formula and Proposition 4.2 allow us therefore to conclude that there exists a small open neighborhood $W_1 \Subset V_1$ of x_0 such that for all $z \in W_1$, the integral (4.27) is equal to the integral

$$
C_n \iint_{\Gamma_{V_1}} \left(\frac{1}{h^n} \iint_{\Gamma(y,\tilde{x}) \cap (V_1 \times \rho(V_1))} e^{\frac{2}{h}\varphi(y,\tilde{x};x,\tilde{y})} a(x,\tilde{y};h) \, \mathrm{d}x \, \mathrm{d}\tilde{y} \right)
$$
$$
\times u(y) v_z^*(\tilde{x}) e^{-\frac{2}{h}\Psi(y,\tilde{x})} \, \mathrm{d}y \, \mathrm{d}\tilde{x}, \quad (4.32)
$$

modulo an error term of the form

$$
\mathcal{O}(1) \|u\|_{H_\Phi(V)} |v(z)| e^{-\Phi(z)/h} e^{-\frac{1}{Ch}}. \tag{4.33}
$$

Here we have also used (4.31). An application of Proposition 3.2 shows that the integral (4.32) is equal to

$$
(u, v_z)_{H_\Phi(V_1)} + \mathcal{O}(1) e^{-\frac{1}{Ch}} \|u\|_{H_\Phi(V)} |v(z)| e^{-\Phi(z)/h}, \tag{4.34}
$$

which completes the proof. □

Remark. — The advantage of exploiting the resolution of the identity given in Proposition 2.2 is due precisely to the fact that it is thanks to it that we are able to reduce the study of the scalar product $(\widetilde{\Pi}_V u, v)_{H_\Phi(V)}$ to a superposition of integrals over good contours (see (4.15), (4.27)).

It is now easy to finish the proof of Theorem 4.1. To this end, we let $W \Subset W_1$, where W_1 is as in Proposition 4.3. Combining (4.15) with (4.24) we get

$$
(\widetilde{\Pi}_V u, v)_{H_\Phi(V)} = \int_{W_1} (u, v_z)_{H_\Phi(V_1)} \, \mathrm{d}z \, \mathrm{d}\bar{z} + \mathcal{O}(1) e^{-\frac{1}{Ch}} \|u\|_{H_\Phi(V)} \|v\|_{H_{\Phi_1}(V)}. \tag{4.35}
$$

On the other hand, using (4.12), we can write

$$
(u, v)_{H_\Phi(V)} = \int_{W_1} (u, v_z)_{H_\Phi(V_1)} \, \mathrm{d}z \, \mathrm{d}\bar{z} + \mathcal{O}(1) e^{-\frac{1}{Ch}} \|u\|_{H_\Phi(V)} \|v\|_{H_{\Phi_1}(V)}. \tag{4.36}
$$

The proof of Theorem 4.1 is complete.

5. End of the proof of Theorem 1.1

To conclude the proof of Theorem 1.1, we shall first pass from the scalar products in Theorem 4.1 to weighted L^2 norm estimates. To this end, let

$\Phi_1 \in C(\Omega; \mathbb{R})$ be such that

$$\Phi_1 \leqslant \Phi \text{ in } \Omega, \quad \Phi_1 < \Phi \text{ on } \Omega \setminus \overline{W}. \tag{5.1}$$

Let us notice that while the weighted space $H_{\Phi_1}(V)$ is not preserved by the action of the operator $\widetilde{\Pi}_V$ in (4.1), we still have

$$\widetilde{\Pi}_V = \mathcal{O}(1) : H_{\Phi_1}(V) \longrightarrow H_{\Phi_2}(V), \tag{5.2}$$

where similarly to (5.1), the weight function $\Phi_2 \in C(\Omega; \mathbb{R})$ satisfies

$$\Phi_2 \leqslant \Phi \text{ in } \Omega, \quad \Phi_2 < \Phi \text{ on } \Omega \setminus \overline{W}. \tag{5.3}$$

Indeed, let us write $\Phi_1 = \Phi - \psi_1$, $\psi_1 \geqslant 0$, with strict inequality on $\Omega \setminus \overline{W}$. Using (3.2) together with the Schur test, we obtain (5.2) with $\Phi_2 = \Phi - \psi_2$, where $0 \leqslant \psi_2 \in C(\Omega; \mathbb{R})$ is the infimal convolution

$$\psi_2(x) = \inf_{y \in V} \left(\frac{|x - y|^2}{2C} + \psi_1(y) \right). \tag{5.4}$$

Here $C > 0$ is sufficiently large. It is therefore clear that (5.3) holds.

Let $u \in H_{\Phi_1}(V)$, where $\Phi_1 \in C(\Omega; \mathbb{R})$ satisfies (5.1), and let us apply Theorem 4.1, with $v = (\widetilde{\Pi}_V - 1)u \in H_{\max(\Phi_1, \Phi_2)}(V)$, and $\max(\Phi_1, \Phi_2)$ in place of Φ_1. We obtain, using also (5.2),

$$\|(\widetilde{\Pi}_V - 1)u\|_{H_\Phi(V)} \leqslant \mathcal{O}(1)e^{-\frac{1}{Ch}} \|u\|_{H_{\Phi_1}(V)}. \tag{5.5}$$

The estimate (5.5) is very close to the approximate reproducing property for $\widetilde{\Pi}_V$ that we seek but we still need to free ourselves from the auxiliary weight Φ_1. This will be accomplished by $\bar{\partial}$-surgery. Without loss of generality, in what follows we shall assume therefore that the bounded open set V is pseudoconvex, and we may even choose it to be a ball centered at x_0.

Let $U \Subset W \Subset V$ be an open neighborhood of x_0 with C^∞-boundary. Given $u \in H_\Phi(V)$, we shall estimate

$$\|(\widetilde{\Pi}_V - 1)u\|_{H_\Phi(U)}. \tag{5.6}$$

When doing so, let $\Phi_1 \in C^\infty(\Omega; \mathbb{R})$ be such that

$$\Phi_1 = \Phi \text{ in } \overline{W}, \quad \Phi_1 < \Phi \text{ on } \Omega \setminus \overline{W}, \tag{5.7}$$

with $\|\Phi - \Phi_1\|_{C^2(\overline{V})}$ small enough. In particular, Φ_1 is strictly plurisubharmonic in V, see also (2.1), so that

$$\sum_{j,k=1}^n \frac{\partial^2 \Phi_1}{\partial x_j \partial \overline{x}_k}(x) \xi_j \bar{\xi}_k \geqslant \frac{|\xi|^2}{\mathcal{O}(1)}, \quad x \in V, \quad \xi \in \mathbb{C}^n. \tag{5.8}$$

Let $\chi \in C_0^\infty(W; [0, 1])$ be such that $\chi = 1$ in a neighborhood of $\overline{U}$. We shall also need an auxiliary weight $\Phi_3 \in C^\infty(V; \mathbb{R})$ such that

$$\Phi_3(x) \leqslant \Phi_1(x) \leqslant \Phi(x), \quad x \in V, \tag{5.9}$$

which furthermore satisfies

$$\Phi_3 = \Phi \text{ near } \operatorname{supp}(\nabla\chi), \tag{5.10}$$

$$\Phi_3 < \Phi \text{ near } \overline{U}. \tag{5.11}$$

We may also arrange so that Φ_3 is strictly plurisubharmonic in V,

$$\sum_{j,k=1}^n \frac{\partial^2 \Phi_3}{\partial x_j \partial \overline{x}_k}(x)\xi_j\overline{\xi}_k \geqslant \frac{|\xi|^2}{\mathcal{O}(1)}, \quad x \in V, \quad \xi \in \mathbb{C}^n. \tag{5.12}$$

When estimating (5.6), we write

$$u = \chi u + (1-\chi)u, \quad u \in H_\Phi(V).$$

Here

$$\overline{\partial}(\chi u) = u\overline{\partial}\chi$$

satisfies

$$\|\overline{\partial}(\chi u)\|_{L^2_{\Phi_3}(V)} \leqslant \mathcal{O}(1)\|u\|_{H_\Phi(V)}, \tag{5.13}$$

in view of (5.10). By an application of Hörmander's L^2-estimate for the $\overline{\partial}$-equation in the pseudoconvex open set V for the weight Φ_3 ([15, Proposition 4.2.5]), there exists $w \in L^2_{\Phi_3}(V)$ such that

$$\overline{\partial}w = \overline{\partial}(\chi u), \tag{5.14}$$

with

$$\|w\|_{L^2_{\Phi_3}(V)} \leqslant \mathcal{O}(h^{1/2})\|\overline{\partial}(\chi u)\|_{L^2_{\Phi_3}(V)} \leqslant \mathcal{O}(h^{1/2})\|u\|_{H_\Phi(V)}. \tag{5.15}$$

Here we have also used (5.13). Using (5.7), (5.9), and (5.15), we see that the function $\chi u - w \in \operatorname{Hol}(V)$ satisfies

$$\|\chi u - w\|_{H_{\Phi_1}(V)} \leqslant \|\chi u\|_{L^2_{\Phi_1}(V)} + \|w\|_{L^2_{\Phi_1}(V)} = \mathcal{O}(1)\|u\|_{H_\Phi(V)}, \tag{5.16}$$

and therefore by (5.5) we conclude that

$$\|(\widetilde{\Pi}_V - 1)(\chi u - w)\|_{H_\Phi(V)} \leqslant \mathcal{O}(1)e^{-\frac{1}{Ch}}\|u\|_{H_\Phi(V)}. \tag{5.17}$$

Next, similarly to (4.9), using (3.2), we obtain that

$$\|(\widetilde{\Pi}_V - 1)(1-\chi)u\|_{L^2_\Phi(U)} \leqslant \mathcal{O}(1)e^{-\frac{1}{Ch}}\|u\|_{H_\Phi(V)}. \tag{5.18}$$

We finally come to estimate the norm $\|(\widetilde{\Pi}_V - 1)w\|_{L^2_\Phi(U)}$, and we remark first that in view of (5.11), (5.15), we have

$$\|w\|_{L^2_\Phi(U)} \leqslant \mathcal{O}(1)e^{-\frac{1}{Ch}}\|u\|_{H_\Phi(V)}. \tag{5.19}$$

Next, let $U \Subset U_1 \Subset W$ be such that we still have

$$\Phi_3 < \Phi \text{ on } \overline{U_1}, \tag{5.20}$$

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and let χ_{U_1} stand for the characteristic function of U_1. Using (3.2), (5.9), and (5.15), we get

$$
\begin{aligned}
\|\widetilde{\Pi}_V w\|_{L^2_\Phi(U)} &\leqslant \|\widetilde{\Pi}_V(1-\chi_{U_1})w\|_{L^2_\Phi(U)} + \|\widetilde{\Pi}_V \chi_{U_1} w\|_{L^2_\Phi(U)} \\
&\leqslant \mathcal{O}(1)e^{-\frac{1}{Ch}}\|w\|_{L^2_\Phi(V)} + \|\widetilde{\Pi}_V \chi_{U_1} w\|_{L^2_\Phi(U)} \\
&\leqslant \mathcal{O}(1)e^{-\frac{1}{Ch}}\|u\|_{H_\Phi(V)} + \|\widetilde{\Pi}_V \chi_{U_1} w\|_{L^2_\Phi(U)} \\
&\leqslant \mathcal{O}(1)e^{-\frac{1}{Ch}}\|u\|_{H_\Phi(V)} + \mathcal{O}(1)\|\chi_{U_1} w\|_{L^2_\Phi(V)} \\
&\leqslant \mathcal{O}(1)e^{-\frac{1}{Ch}}\|u\|_{H_\Phi(V)}.
\end{aligned}
\tag{5.21}
$$

Here in the final estimate we have also used (5.20) and (5.15).

Combining (5.17), (5.18), (5.19), and (5.21), we get

$$
\|(\widetilde{\Pi}_V - 1)u\|_{H_\Phi(U)} \leqslant \mathcal{O}(1)e^{-\frac{1}{Ch}}\|u\|_{H_\Phi(V)}.
\tag{5.22}
$$

The proof of Theorem 1.1 is complete.

Appendix A. From asymptotic to exact Bergman projections: proof of Corollary 1.2

The purpose of this appendix is to give a proof of Corollary 1.2, showing that the operator $\widetilde{\Pi}_V$ in (1.2), enjoying the local approximate reproducing property (1.3), provides an approximation for the orthogonal projection

$$
\Pi : L^2(\Omega; e^{-2\Phi/h}L(\mathrm{d}x)) \longrightarrow H_\Phi(\Omega),
\tag{A.1}
$$

up to an exponentially small error, locally near $x_0 \in \Omega$. The following arguments are essentially well known and follow [1] closely. See also [13] for the corresponding discussion in the case of C^∞ weights.

Let $u \in H_\Phi(V)$. Theorem 1.1 gives that the holomorphic function $\widetilde{\Pi}_V u - u$ satisfies

$$
\|\widetilde{\Pi}_V u - u\|_{H_\Phi(U)} \leqslant \mathcal{O}(1)e^{-\frac{1}{Ch}}\|u\|_{H_\Phi(V)}, \quad C > 0,
\tag{A.2}
$$

and therefore, letting $\widetilde{U} \Subset U$ be an open neighborhood of x_0, we conclude by Proposition 2.3 that

$$
|\widetilde{\Pi}_V u(x) - u(x)| \leqslant \mathcal{O}(1)e^{-\frac{1}{Ch}}\, e^{\frac{\Phi(x)}{h}}\|u\|_{H_\Phi(V)}, \quad x \in \widetilde{U},
\tag{A.3}
$$

where $C > 0$. We get

$$
u(x) = \frac{1}{h^n}\int_V e^{\frac{2}{h}\Psi(x,\bar{y})}a(x,\bar{y};h)u(y)e^{-\frac{2}{h}\Phi(y)}\,L(\mathrm{d}y)
$$

$$
+ \mathcal{O}(1)e^{-\frac{1}{Ch}}\, e^{\frac{\Phi(x)}{h}}\|u\|_{H_\Phi(V)}, \quad x \in \widetilde{U}.
\tag{A.4}
$$

We shall apply (A.4) to $u \in H_\Phi(\Omega)$. To this end, let us observe that in view of Proposition 2.3, we have

$$|u(x)| \leqslant \mathcal{O}(1)h^{-n}e^{\frac{\Phi(x)}{h}}\|u\|_{H_\Phi(\Omega)}, \quad x \in V. \tag{A.5}$$

Let $\chi \in C_0^\infty(V;[0,1])$ be such that $\chi = 1$ near $\overline{U}$. Using (A.5) and (3.2), we see that for some $C > 0$,

$$\left| \frac{1}{h^n} \int_V e^{\frac{2}{h}\Psi(x,\bar{y})}(1-\chi(y))a(x,\bar{y};h)u(y)e^{-\frac{2}{h}\Phi(y)} L(\mathrm{d}y) \right|$$
$$\leqslant \mathcal{O}(1)e^{-\frac{1}{Ch}} e^{\frac{\Phi(x)}{h}}\|u\|_{H_\Phi(\Omega)}, \quad x \in \widetilde{U}. \tag{A.6}$$

We get, combining (A.4) and (A.6), when $u \in H_\Phi(\Omega)$,

$$u(y) = \int_\Omega \widetilde{K}(y,\bar{z})\chi(z)u(z)e^{-\frac{2}{h}\Phi(z)} L(\mathrm{d}z)$$
$$+ \mathcal{O}(1)e^{-\frac{1}{Ch}} e^{\frac{\Phi(y)}{h}}\|u\|_{H_\Phi(\Omega)}, \quad y \in \widetilde{U}, \tag{A.7}$$

where

$$\widetilde{K}(y,\bar{z}) = 1_V(y)1_V(z)\frac{1}{h^n}e^{\frac{2}{h}\Psi(y,\bar{z})}a(y,\bar{z};h), \quad (y,z) \in \Omega \times \Omega. \tag{A.8}$$

It has been established in [23, Section 5], see also [6, Appendix A], that the Schwartz kernel of the orthogonal projection in (A.1) is of the form $K(x,\bar{y})e^{-2\Phi(y)/h}$, where $K(x,z) \in \mathrm{Hol}(\Omega \times \overline{\Omega})$ satisfies

$$y \longmapsto \overline{K(x,\bar{y})} \in H_\Phi(\Omega), \quad x \longmapsto K(x,\bar{y}) \in H_\Phi(\Omega). \tag{A.9}$$

Following [1] and applying (A.7) to the function $y \mapsto K(y,\bar{x}) \in H_\Phi(\Omega)$, we get

$$K(y,\bar{x}) = \int_\Omega \widetilde{K}(y,\bar{z})\chi(z)K(z,\bar{x})e^{-\frac{2}{h}\Phi(z)} L(\mathrm{d}z)$$
$$+ \mathcal{O}(1)e^{-\frac{1}{Ch}} e^{\frac{\Phi(y)}{h}}\|K(\,\cdot\,,\bar{x})\|_{H_\Phi(\Omega)}, \quad y \in \widetilde{U}. \tag{A.10}$$

Here we have

$$\|K(\,\cdot\,,\bar{x})\|_{H_\Phi(\Omega)} \leqslant \mathcal{O}(1)h^{-n/2}e^{\frac{\Phi(x)}{h}}, \quad x \in \widetilde{U}, \tag{A.11}$$

see [2, Chapter 4], and we get using (A.10) and (A.11),

$$K(y,\bar{x}) = \int_\Omega \widetilde{K}(y,\bar{z})\chi(z)K(z,\bar{x})e^{-\frac{2}{h}\Phi(z)} L(\mathrm{d}z)$$
$$+ \mathcal{O}(1)e^{-\frac{1}{Ch}} e^{\frac{\Phi(x)+\Phi(y)}{h}}, \quad x,y \in \widetilde{U}. \tag{A.12}$$

Taking the complex conjugates in (A.12) and using the Hermitian property $K(x, \bar{y}) = \overline{K(y, \bar{x})}$, we obtain

$$K(x, \bar{y}) = \int_\Omega K(x, \bar{z})\chi(z)\widehat{K}(z, \bar{y})e^{-\frac{2}{\hbar}\Phi(z)} \, L(\mathrm{d}z)$$
$$+ \mathcal{O}(1)e^{-\frac{1}{C\hbar}} \, e^{\frac{\Phi(x)+\Phi(y)}{\hbar}}, \quad x, y \in \widetilde{U}, \quad \text{(A.13)}$$

where, in view of (A.8),

$$\widehat{K}(z, \bar{y}) = \overline{\widetilde{K}(y, \bar{z})} = 1_V(y)1_V(z)\frac{1}{h^n}e^{\frac{2}{\hbar}\Psi(z, \bar{y})}b(z, \bar{y}; h),$$
$$b(z, \bar{y}; h) = \overline{a(y, \bar{z}; h)}. \quad \text{(A.14)}$$

Here we have also used that the polarization Ψ of Φ enjoys the Hermitian property

$$\Psi(x, y) = \overline{\Psi(\bar{y}, \bar{x})}, \quad (x, y) \in \mathrm{neigh}((x_0, \overline{x_0}), \mathbb{C}^{2n}). \quad \text{(A.15)}$$

Recalling that

$$\Pi u(x) = \int_\Omega K(x, \bar{y})u(y)e^{-2\Phi(y)/h} \, L(\mathrm{d}y), \quad u \in L^2(\Omega; e^{-2\Phi/h}L(\mathrm{d}x)), \quad \text{(A.16)}$$

we may rewrite (A.13) as follows,

$$K(x, \bar{y}) = \Pi\big(\widehat{K}(\,\cdot\,, \bar{y})\chi\big)(x) + \mathcal{O}(1)e^{-\frac{1}{C\hbar}} \, e^{\frac{\Phi(x)+\Phi(y)}{\hbar}}, \quad x, y \in \widetilde{U}. \quad \text{(A.17)}$$

Here we would like to show that $\Pi\big(\widehat{K}(\,\cdot\,, \bar{y})\chi\big)(x)$ is exponentially close to $\widehat{K}(x, \bar{y})\chi(x) = \widehat{K}(x, \bar{y})$ for $x \in \widetilde{U}$, and to this end we follow an argument in [1], relying on Hörmander's L^2-estimate for $\bar{\partial}$ in the pseudoconvex domain Ω. The function

$$\Omega \ni x \longmapsto u_y(x) = \widehat{K}(x, \bar{y})\chi(x) - \Pi\big(\widehat{K}(\,\cdot\,, \bar{y})\chi\big)(x) \quad \text{(A.18)}$$

is the solution of the $\bar{\partial}$-problem

$$\bar{\partial}u_y = \bar{\partial}\big(\widehat{K}(\,\cdot\,, \bar{y})\chi\big) = \widehat{K}(\,\cdot\,, \bar{y})\bar{\partial}\chi, \quad \text{(A.19)}$$

in Ω of the minimal $L^2(\Omega; e^{-2\Phi/h}L(\mathrm{d}x))$ norm, and therefore, by Hörmander's L^2-estimate for the $\bar{\partial}$ operator, see [15, Proposition 4.2.5], we get for

any $y \in \widetilde{U}$,

$$
\int_\Omega |u_y(x)|^2 e^{-2\Phi(x)/h} L(\mathrm{d}x)
$$

$$
\begin{aligned}
&\leqslant \mathcal{O}(h) \int_\Omega \frac{1}{c(x)} \big|\bar\partial_x\big(\widehat{K}(x,\bar{y})\chi(x)\big)\big|^2 e^{-2\Phi(x)/h} L(\mathrm{d}x) \\
&\leqslant \mathcal{O}(h) \int_\Omega |\nabla\chi(x)|^2 \big|\widehat{K}(x,\bar{y})\big|^2 e^{-2\Phi(x)/h} L(\mathrm{d}x) \\
&\leqslant \mathcal{O}(h) \int_{V\setminus U} \big|\widehat{K}(x,\bar{y})\big|^2 e^{-2\Phi(x)/h} L(\mathrm{d}x) \\
&= \mathcal{O}(1)\, e^{2\Phi(y)/h} e^{-1/Ch}, \quad y \in \widetilde{U}.
\end{aligned} \tag{A.20}
$$

Here we have also used (A.14) and (3.2). Therefore,

$$
\|u_y\|_{L^2_\Phi(\Omega)} \leqslant \mathcal{O}(1) e^{-\frac{1}{Ch}}\, e^{\Phi(y)/h}, \quad y \in \widetilde{U}, \tag{A.21}
$$

and it only remains for us to pass from the weighted L^2-bound (A.21) on u_y to a pointwise estimate. To this end, using that u_y is holomorphic in U we get, in view of (A.21) and Proposition 2.3,

$$
|u_y(x)| \leqslant \mathcal{O}(1) e^{-\frac{1}{Ch}}\, e^{(\Phi(x)+\Phi(y))/h}, \quad x,y \in \widetilde{U}. \tag{A.22}
$$

We infer, combining (A.17), (A.18), and (A.22), with a new constant $C > 0$,

$$
K(x,\bar{y}) = \widehat{K}(x,\bar{y}) + \mathcal{O}(1) e^{-\frac{1}{Ch}}\, e^{(\Phi(x)+\Phi(y))/h}, \quad x,y \in \widetilde{U}. \tag{A.23}
$$

Recalling also (A.14), we obtain

$$
K(x,\bar{y}) = \overline{\widetilde{K}(y,\bar{x})} + \mathcal{O}(1) e^{-\frac{1}{Ch}}\, e^{(\Phi(x)+\Phi(y))/h}, \quad x,y \in \widetilde{U}, \tag{A.24}
$$

and taking the complex conjugates and using the Hermitian symmetry of K, we get

$$
K(y,\bar{x}) = \widetilde{K}(y,\bar{x}) + \mathcal{O}(1) e^{-\frac{1}{Ch}}\, e^{(\Phi(x)+\Phi(y))/h}, \quad x,y \in \widetilde{U}. \tag{A.25}
$$

Switching the variables x and y in (A.25), we complete the proof of Corollary 1.2.

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