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On the extension of Kähler currents on compact Kähler manifolds: holomorphic retraction case

On the extension of Kähler currents on compact Kähler manifolds: holomorphic retraction case (*)

Jiafu Ning [(1)], Zhiwei Wang [(2)] and Xiangyu Zhou [(3)]

Abstract. — In the present paper, we show that given a compact Kähler manifold (X, ω) with a Kähler metric (not necessarily Hodge metric) ω, and a complex submanifold $V \subset X$ of positive dimension, if V has a holomorphic retraction structure in X, then any quasi-plurisubharmonic function φ on V such that $\omega|_V + \sqrt{-1}\partial\bar{\partial}\varphi \geqslant \varepsilon\omega|_V$ with $\varepsilon > 0$ can be extended to a quasi-plurisubharmonic function Φ on X, such that $\omega + \sqrt{-1}\partial\bar{\partial}\Phi \geqslant \varepsilon'\omega$ for some $\varepsilon' > 0$. This gives a partial answer to a question raised by Coman–Guedj–Zeriahi [4].

Résumé. — Dans le présent article, nous montrons que étant donné une variété compacte de kählérienne (X, ω) avec une métrique de kählérienne (pas nécessairement une métrique de Hodge) ω, et une sous-variété complexe $V \subset X$ de dimension positif, si V a une structure de rétraction holomorphe dans X, alors toute fonction quasi-plurisousharmonique φ sur V telle que $\omega|_V + \sqrt{-1}\partial\bar{\partial}\varphi \geqslant \varepsilon\omega|_V$ avec $\varepsilon > 0$ peut être étendue é une fonction quasi-plurisousharmonique Φ sur X, telle que $\omega + \sqrt{-1}\partial\bar{\partial}\Phi \geqslant \varepsilon'\omega$ pour quelques $\varepsilon' > 0$. Ceci donne une réponse partielle à une question soulevée par Coman–Guedj–Zeriahi [4].

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1. Introduction

In this paper, we study the following important problem raised by Coman–Guedj–Zeriahi.

PROBLEM 1.1 ([4]). — *Let (X, ω) be a compact Kähler manifold of complex dimension n, equipped with a Kähler metric ω. Let $V \subset X$ be a complex submanifold of complex dimension $k > 0$. Does the following hold*

$$\mathrm{Psh}(V, \omega|_V) = \mathrm{Psh}(X, \omega)|_V?$$

Here we list some recent progress about this problem.

- When ω is a Hodge metric and φ is a smooth quasi-psh function on V, such that $\omega|_V + \sqrt{-1}\partial\bar{\partial}\varphi > 0$, then Problem 1.1 has a positive answer by Schumacher [11].
- When ω is a Hodge metric, then Problem 1.1 has a positive answer by Coman–Guedj–Zeriahi [4], and when ω is a Kähler metric and φ is a smooth quasi-psh function on V, such that $\omega|_V + \sqrt{-1}\partial\bar{\partial}\varphi > 0$, then Problem 1.1 has a positive answer in the same paper [4]. Quite recently, this result was strengthened by the same authors [5] as follows: if ω is a Kähler metric and $\{\omega\} \in NS_{\mathbb{R}}(X)$, then Problem 1.1 has a positive answer, where $NS_{\mathbb{R}}(X)$ is the real Neron–Severi space of X, and V is only assumed to be an analytic subvariety of X.
- When ω is a Kähler metric and φ is a quasi-psh function on V, which has analytic singularities, such that $\omega|_V + \sqrt{-1}\partial\bar{\partial}\varphi > \epsilon\omega|_V$ for some $\epsilon > 0$, there is a quasi-psh function Φ on X, such that $\Phi|_V = \varphi$ and $\omega + \sqrt{-1}\partial\bar{\partial}\Phi > \epsilon'\omega$ on X for some $\epsilon' > 0$ by Collins–Tosatti [3].
- When ω is a Kähler metric and φ is a quasi-psh function with arbitrary singularity on V, such that $\omega|_V + \sqrt{-1}\partial\bar{\partial}\varphi > \epsilon\omega|_V$ for some $\epsilon > 0$. Suppose that V has a holomorphic tubular neighborhood in X, then there is a quasi-psh function Φ on X, such that $\Phi|_V = \varphi$ and $\omega + \sqrt{-1}\partial\bar{\partial}\Phi > \epsilon'\omega$ on X for some $\epsilon' > 0$ by Wang–Zhou [13].

The main theorem of this paper is as follows.

THEOREM 1.2. — *Let (X, ω) be a compact Kähler manifold of complex dimension n, equipped with a Kähler metric ω. Let $V \subset X$ be a complex submanifold of complex dimension $k > 0$. Suppose that there is an open neighborhood U of V in X, and a holomorphic retraction $\pi : U \to V$. Let φ be a quasi-psh function with arbitrary singularity on V, such that $\omega|_V + \sqrt{-1}\partial\bar{\partial}\varphi > \epsilon\omega|_V$ for some $\epsilon > 0$. Then there is a quasi-psh function Φ on X, such that $\Phi|_V = \varphi$ and $\omega + \sqrt{-1}\partial\bar{\partial}\Phi > \epsilon'\omega$ on X for some $\epsilon' > 0$.*

Remark 1.3. — The main theorem is slightly stronger than the result in [13], by weakening the assumption that V has a holomorphic tubular neighborhood structure in X to the assumption that V has a holomorphic retraction structure in X. By a holomorphic retraction, we mean that there is an open neighborhood U of V in X, and a holomorphic map $\pi : U \to V$, such that $\pi|_V : V \to V$ is the identity map. Without the holomorphic tubular neighborhood structure, we need to compute the complex Hessian of the square of the distance function to V on X.

Remark 1.4. — Form Siu's work [12], any Stein submanifold in a complex manifold automatically has a holomorphic retraction structure. This plays a key role in Coman–Guedj–Zeriahi's work [4]. Thus, it seems reasonable to assume the existence of holomorphic retraction structure in the compact Kähler manifold setting. Meanwhile, our result does not require the reference metric ω to be a Hodge metric, even in the projective manifold setting.

We also consider the extension of Kähler currents in a big class. Note that if the assumption that ω is a Hodge metric in [4] was weaken as the rational class $\{\omega\}$ contains a Kähler current, there are counterexamples for the extension, cf. [10, Example 4.1]. In [2], the non-Kähler locus of a big class $\alpha \in H^{1,1}(X, \mathbb{R})$, is defined as

$$E_{nK}(\alpha) := \bigcap_{T \in \alpha} E_+(T),$$

where $E_+(T)$ is the set of points of X such that the Kähler current T has positive Lelong numbers, and T varies in all the Kähler currents in α. From Siu's semicontinuity of Lelong number upper level sets and strong Noether property, $E_{nK}(\alpha)$ is an analytic subvariety. Similar with [13], we have the following

THEOREM 1.5. — *Let (X, ω) be a compact Kähler manifold of complex dimension n, and $V \subset X$ be a complex submanifold of positive dimension. Suppose that V has a holomorphic retraction structure in X. Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class such that any of the irreducible components of $E_{nK}(\alpha)$ either does not intersect with V, or is contained in V. Then any Kähler current in $\alpha|_V$ is the restriction of a Kähler current in α.*

The structure of the paper is organized as follows. In Section 2, we compute the complex Hessian of the square of distance function to a complex submanifold. In Section 3, we give the proof of the Theorem 1.2 and Theorem 1.5. In Section 4, we provide some examples of the pair of complex manifolds (V, X), such that V has holomorphic retraction structure in X.

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2. Complex Hessian of square of distance to a complex submanifold

In this section, we compute the complex Hessian of the square of Riemannian distance to a complex submanifold. It is believed that the complex Hessian of the square of Riemannian distance to a complex submanifold is positive along the normal direction and trivial along the tangent direction on the submanifold. However, the explicit computation seems to be lacked in the literature. Here we take the opportunity to give a detailed proof.

We adopt Matsumoto's notations in [9]. Let (M, g) be a C^∞ Rimannian manifold of dimension n. For $x, y \in M$, we denote by $\delta(x, y)$ the distance between x and y induced by the metric g.

It is known that for any $p \in M$, there is an open coordinate neighborhood U of p, and a coordinate $x_1, x_2, \ldots, x_n$ on U, with $x(p) = 0$ and $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, $i, j = 1, 2, \ldots, n$. For $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, we may view $v \in T_x M$ as $\sum_{j=1}^n v_j \frac{\partial}{\partial x_j}|_x$. Up to shrinking, there is an open neighborhood $B \subset \mathbb{R}^n$ of 0, such that $\Phi(x, v) = (x, \exp_x(v))$ is bijection from $U \times B$ to $\Phi(U \times B)$, both Φ and Φ^{-1} are C^∞. As $\Phi(x, 0) = (x, x)$, and from the property of exponential map $y = \exp_x v$, we can get

$$J\Phi(0,0) = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}. \tag{2.1}$$

As $\Phi(U \times B)$ is an open neighborhood of (p, p), we may take an open set $V \subset U$, such that $p \in V$, and $\Phi(U \times B) \supset V \times V$. Write $(x, v(x, y)) = \Phi^{-1}(x, y)$, then

$$y = \exp_x(v(x, y)), \quad \delta(x, y)^2 = \sum_{i,j=1}^n g_{ij}(x) v_i(x, y) v_j(x, y)$$

and from (2.1), we have

$$v(0,0) = 0, \quad \frac{\partial v_i}{\partial y_j}(0,0) = -\frac{\partial v_i}{\partial x_j}(0,0) = \delta_{ij}, \quad 1 \leqslant i, j \leqslant n. \tag{2.2}$$

Let $S \in M$ be a C^∞ submanifold of M with $\dim S = k$, $0 \leqslant r < n$. We define the distance function $\operatorname{dist}(\,\cdot\,, S)$ to be

$$\operatorname{dist}(x, S) := \inf\{\delta(x, y) : y \in S\}, \ x \in M,$$

and set

$$h(x) := \operatorname{dist}^2(x, S), x \in M$$

It is proved in [9] that $h(x)$ is C^∞ in a tube neighborhood of S. As we need to do calculation on h, and for the sake of completeness, we introduce Matsumoto's proof about the smoothness.

Given $p \in S$, we can choose a coordinate $(U, x = (x_1, x_2, \ldots, x_n))$ around p, such that $x(p) = 0$, $S \cap U = \{x_{k+1} = \cdots = x_n = 0\}$, and

$$g_{ij}(0) = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)(0) = \delta_{ij}, \quad i, j = 1, 2, \ldots, n. \tag{2.3}$$

Take a small neighborhood $V \subset U$ of p, such that $\mathrm{dist}(x, S) = \mathrm{dist}(x, S \cap U)$ for any $x \in V$. Let

$$f(x, y) = \delta(x, y)^2 = \sum_{i,j=1}^{n} g_{ij}(x) v_i(x, y) v_j(x, y)$$

and

$$F_\mu(x, y) = \frac{\partial f}{\partial y_\mu}(x, y), \quad \mu = 1, 2, \ldots, k$$

for $x \in V$ and $y = (y_1, \ldots, y_k, 0, \ldots, 0) \in S \cap U$. Notice that $F_\mu(0, 0) = 0, \mu = 1, 2, \ldots, k$.

From (2.2), (2.3), we can get

$$\frac{\partial F_\mu}{\partial y_\nu}(0, 0) = 2 \sum_{i,j=1}^{n} g_{ij}(0) \frac{\partial v_i}{\partial y_\mu}(0, 0) \frac{\partial v_j}{\partial y_\nu}(0, 0) = 2\delta_{\mu\nu}. \tag{2.4}$$

Therefore, from the implicit function theorem, we can find a neighborhood $V_0 \subset V$ of p, so that each $x \in V_0$ has a unique solution $y = y(x) \in S \cap U$ of equations $F_\mu(x, y) = 0, \mu = 1, 2, \ldots, k$, $y(0) = 0$ and $y = y(x)$ is C^∞ on V_0. As for each $x \in V_0$, there is at least one point $y \in S \cap U$, such that $\mathrm{dist}(x, S) = \mathrm{dist}(x, S \cap U) = \delta(x, y)$. Therefore, the point y is uniquely determined by x and it must coincide to $y(x)$ because $f(x, y) = \delta(x, y)^2$ is minimal at $y = y(x)$ for each x.

Hence

$$h(x) = \mathrm{dist}^2(x, S \cap U) = \delta(x, y(x))^2 = f(x, y(x))$$

for $x \in V_0$.

From $F_\mu(x, y(x)) = 0$, taking the partial derivatives of this equation, we can get

$$\frac{\partial F_\mu}{\partial x_j}(0, 0) + \sum_{\nu=1}^{k} \frac{\partial F_\mu}{\partial y_\nu}(0, 0) \frac{\partial y_\nu}{\partial x_j}(0) = 0.$$

Combining with equation (2.4), we can get $\frac{\partial y_\mu}{\partial x_j}(0) = -\frac{1}{2}\frac{\partial F_\mu}{\partial x_j}(0, 0)$ for $\mu = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n$.

From (2.2), (2.3), we have

$$\frac{\partial F_\mu}{\partial x_l}(0,0) = \sum_{i,j=1}^n g_{ij}(0)\frac{\partial v_i}{\partial y_\mu}(0,0)\frac{\partial v_j}{\partial x_l}(0,0) + \sum_{i,j=1}^n g_{ij}(0)\frac{\partial v_i}{\partial x_l}(0,0)\frac{\partial v_j}{\partial y_\mu}(0,0)$$

$$= -2\delta_{\mu l}.$$

Hence,

$$\frac{\partial y_\mu}{\partial x_j}(0) = \delta_{\mu j} \text{ for } \mu = 1,2,\ldots,k; j = 1,2,\ldots,n. \tag{2.5}$$

Let $a_j(x) = v_j(x,y(x)), j = 1,2,\ldots,n$, then $a_j(0) = 0$, and

$$\frac{\partial a_j}{\partial x_l}(0) = \frac{\partial v_j}{\partial x_l}(0,0) + \sum_{\mu=1}^k \frac{\partial v_j}{\partial y_\mu}(0,0)\frac{\partial y_\mu}{\partial x_l}(0)$$

$$= \sum_{\mu=1}^k \delta_{j\mu}\delta_{\mu l} - \delta_{jl}. \tag{2.6}$$

As $h(x) = \sum_{i,j=1}^n g_{ij}(x)v_i(x,y(x))v_j(x,y(x)) = \sum_{i,j=1}^n g_{ij}(x)a_i(x)a_j(x)$, from (2.6), then

$$\frac{\partial^2 h}{\partial x_s \partial x_t}(0) = 2\sum_{i,j=1}^n g_{ij}(0)\frac{\partial a_i}{\partial x_s}(0)\frac{\partial a_j}{\partial x_t}(0)$$

$$= \begin{cases} 0, & s \text{ or } t \leqslant k; \\ 2\delta_{st}, & s,t > k. \end{cases} \tag{2.7}$$

Now let (X,ω) be a compact Hermitian manifold with a Hermitian metric ω. Let $V \subset X$ be a complex submanifold of complex dimension $k > 0$. Fix any $p \in V$. There is a holomorphic coordinate $(U, z = (z_1,\ldots,z_k,z_{k+1},\ldots,z_n))$ centered at p in X, such that $U \cap V = \{z_{k+1} = \cdots = z_n = 0\}$, and $\omega = \sqrt{-1}\sum_{i,j=1}^n g_{i\bar{j}}dz_i \wedge d\bar{z}_j$ with $g_{i\bar{j}}(0) = \delta_{i\bar{j}}$ for $i,j = 1,\ldots,n$. Write $z_i = x_{2i-1} + \sqrt{-1}x_{2i}$. Note that the Riemannian metric induced by ω has the form $\sum_{\alpha=1}^{2n} dx_\alpha \otimes dx_\alpha$ at p. Since

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_{2i-1}} - \sqrt{-1}\frac{\partial}{\partial x_{2i}}\right), \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_{2i-1}} + \sqrt{-1}\frac{\partial}{\partial x_{2i}}\right),$$

we get that

$$\frac{\partial^2 h}{\partial z_i \partial \bar{z}_j} = \frac{1}{4}\left(\frac{\partial^2 h}{\partial x_{2i-1}\partial x_{2j-1}} + \frac{\partial^2 h}{\partial x_{2i}\partial x_{2j}}\right.$$

$$\left. - \sqrt{-1}\frac{\partial^2 h}{\partial x_{2i}\partial x_{2j-1}} + \sqrt{-1}\frac{\partial^2 h}{\partial x_{2i-1}\partial x_{2j}}\right). \tag{2.8}$$

Combining (2.7) and (2.8), we obtain the following

PROPOSITION 2.1. — *Let (X, ω) be a complex n-dimensional Hermitian manifold with a Hermitian metric ω. Let $V \subset X$ be a complex submanifold of complex dimension k, and $h(z) := \operatorname{dist}^2(\,\cdot\,, V)$ be the square of the distance function $\operatorname{dist}(\,\cdot\,, V)$ on X with respect to the Reimannian metric induced by ω. Let $p \in V$ be an arbitrarily fixed point in V, then there is a holomorphic coordinate chart $(U, z = (z_1, \ldots, z_k, z_{k+1}, \ldots, z_n))$ centered at p such that $U \cap V = \{z_{k+1} = \cdots = z_n = 0\}$, $\omega = \sqrt{-1}\sum_{i,j=1}^n g_{i\bar{j}}\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$ with $g_{i\bar{j}}(0) = \delta_{i\bar{j}}$ for $i, j = 1, \ldots, n$, and*

$$\frac{\partial^2 h}{\partial z_i \partial \bar{z}_j}(0) = \begin{cases} 0, & i \text{ or } j \leqslant k; \\ \delta_{i\bar{j}}, & i, j > k. \end{cases}$$

3. Proof of the main theorem

In this section, we give the proof of Theorem 1.2. The idea of the proof is similar to that in [13]. The main difference lies in the construction of the local uniform extension. For the sake of completeness, we give the detailed proof.

We need the following two lemmas.

LEMMA 3.1 ([1, 13]). — *Let φ be a quasi-psh function on a compact Hermitian manifold (X, ω), such that $\omega + \sqrt{-1}\partial\bar{\partial}\varphi \geqslant \varepsilon\omega$ and $\varphi < -C < 0$. Then there is a sequence of smooth functions φ_m and a decreasing sequence $\varepsilon_m > 0$ converging to 0, satisfying the following*

(a) $\varphi_m \searrow \varphi$;
(b) $\omega + \sqrt{-1}\partial\bar{\partial}\varphi_m \geqslant (\varepsilon - \varepsilon_m)\omega$;
(c) $\varphi_m \leqslant -\frac{C}{2}$.

LEMMA 3.2 (cf. [6]). — *There exists a function $F : X \to [-\infty, +\infty)$ which is smooth on $X \setminus V$, with logarithmic singularities along V, and such that $\omega + \sqrt{-1}\partial\bar{\partial}F \geqslant \varepsilon\omega$ is a Kähler current on X. By subtracting a large constant, we can make that $F < 0$ on X.*

Let $T = \omega|_V + \sqrt{-1}\partial\bar{\partial}\varphi \geqslant \varepsilon\omega|_V$ be the given Kähler current in the Kähler class $[\omega|_V]$, where φ is a strictly $\omega|_V$-psh function. By subtracting a large constant, we may assume that $\sup_V \varphi < -C$ for some positive constant C.

By Lemma 3.1, there is a non-increasing sequence of smooth strictly $\omega|_V$-psh functions φ_m on V, and a decreasing sequence of positive numbers ε_m such that as $m \to \infty$

- $\varphi_m \searrow \varphi$;
- $\omega|_V + \sqrt{-1}\partial\bar{\partial}\varphi_m > \frac{\varepsilon}{2}\omega|_V$;
- $\varphi_m \leqslant -\frac{C}{2}$.

We say a smooth strictly $\omega|_V$-psh function ϕ on V satisfies *assumption* $\bigstar_{\varepsilon,C}$, if $\omega|_V + \sqrt{-1}\partial\bar{\partial}\phi > \frac{\varepsilon}{2}\omega|_V$ and $\phi < -\frac{C}{2}$.

Note that for all $m \in \mathbb{N}^+$, φ_m satisfy *assumption* $\bigstar_{\varepsilon,C}$. In the following, we will extend all the φ_m simultaneously to non-increasing strictly ω-psh functions on the ambient manifold X.

Step 1: Local uniform extensions of φ_m for all m.

Let $U \subset X$ be an open neighborhood of V and let $r : U \to V$ be a holomorphic retraction. Let ϕ be a function satisfying *assumption* $\bigstar_{\varepsilon,C}$. Let h be the square of the distance function, which is a smooth function defined in Section 2. We define

$$\bar{\phi} := \phi \circ r + Ah$$

where A is a positive constant to be determined later.

Fix arbitrary $p \in V$, choose a holomorphic coordinate chart $(W_p \subset U, z = (z_1, \ldots, z_n))$ centered at p such that $W_p \cap V = \{z_{k+1} = \cdots = z_n = 0\}$, $g_{i\bar{j}}(0) = \delta_{i\bar{j}}$, and such that Proposition 2.1 holds, where $\omega = \sqrt{-1} \sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\bar{z}_j$. Then on W_p, we have that

$$\bar{\phi}(z) := (\phi \circ r)(z) + Ah(z).$$

Note that on W_p,

$$\omega + \sqrt{-1}\partial\bar{\partial}\bar{\phi}(z) = (\omega - r^*(\omega|_V)) + r^*(\omega|_V + \sqrt{-1}\partial\bar{\partial}\phi) + A\sqrt{-1}\partial\bar{\partial}h \quad (3.1)$$

$$\geqslant (\omega - r^*(\omega|_V)) + \frac{\varepsilon}{2}r^*(\omega|_V) + A\sqrt{-1}\partial\bar{\partial}h$$

The second inequality follows from the fact that $\omega|_V + \sqrt{-1}\partial\bar{\partial}\phi \geqslant \frac{\varepsilon}{2}\omega|_V$ on V and r is a holomorphic retraction map. The key point is that the last term in above inequality is independent of ϕ.

CLAIM 3.3. — *There is an open neighborhood W_p (independent of ϕ), of p in U, and positive constants $A > 0$ and $\varepsilon' > 0$ (independent of ϕ), such that on W_p,*

$$\omega + \sqrt{-1}\partial\bar{\partial}\bar{\phi} \geqslant \frac{\varepsilon'}{2}\omega \quad and \quad \bar{\phi} \leqslant -\frac{C}{4}.$$

Proof. — Under the local coordinate chosen as above, one can see that

$$r(z) = (r_1(z_1, \ldots, z_n), \ldots, r_k(z_1, \ldots, z_n), 0, \ldots, 0);$$
$$r(z_1, \ldots, z_k, 0, \ldots, 0) = (z_1, \ldots, z_k, 0, \ldots, 0);$$
$$\mathrm{d}r_i(z_1, \ldots, z_k, 0, \ldots, 0) = \mathrm{d}z_i + \sum_{k+1 \leqslant j \leqslant n} \frac{\partial r_i}{\partial z_j}(z_1, \ldots, z_k, 0 \ldots, 0)\mathrm{d}z_j.$$

Since $\omega|_V = \sqrt{-1}\sum_{1 \leqslant i,j \leqslant k} g_{i\bar{j}}(z_1, \ldots, z_k, 0, \ldots, 0)\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$, it follows that at $(z_1, \ldots, z_k, 0, \ldots, 0)$,

$$r^*(\omega|_V)$$
$$= \sqrt{-1} \sum_{1 \leqslant i,j \leqslant k} g_{i\bar{j}}\left(\mathrm{d}z_i + \sum_{k+1 \leqslant l \leqslant n} \frac{\partial r_i}{\partial z_l}\mathrm{d}z_l\right) \wedge \left(\mathrm{d}\bar{z}_j + \sum_{k+1 \leqslant m \leqslant n} \frac{\partial \bar{r}_j}{\partial \bar{z}_m}\mathrm{d}\bar{z}_m\right)$$
$$= \sqrt{-1} \sum_{1 \leqslant i,j \leqslant k} g_{i\bar{j}}\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j + \sqrt{-1} \sum_{1 \leqslant i \leqslant k, k+1 \leqslant m \leqslant n} \sum_{1 \leqslant j \leqslant k} g_{i\bar{j}}\frac{\partial \bar{r}_j}{\partial \bar{z}_m}\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_m$$
$$+ \sqrt{-1} \sum_{1 \leqslant j \leqslant k, k+1 \leqslant l \leqslant n} \sum_{1 \leqslant i \leqslant k} g_{i\bar{j}}\frac{\partial r_i}{\partial z_l}\mathrm{d}z_l \wedge \mathrm{d}\bar{z}_j$$
$$+ \sqrt{-1} \sum_{k+1 \leqslant l,m \leqslant n} \sum_{1 \leqslant i,j \leqslant k} g_{i\bar{j}}\frac{\partial r_i}{\partial z_l}\frac{\partial \bar{r}_j}{\partial \bar{z}_m}\mathrm{d}z_l \wedge \mathrm{d}\bar{z}_m.$$

Thus, at $p \in V$, which corresponds to $0 \in W_p$, we get the following

$$(\omega - r^*(\omega|_V)) + \frac{\varepsilon}{2}r^*(\omega|_V) + A\sqrt{-1}\partial\bar{\partial}h$$
$$= \sqrt{-1} \sum_{1 \leqslant i,j \leqslant k} \left(\frac{\varepsilon}{2}g_{i\bar{j}} + Ah_{i\bar{j}}\right)\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$$
$$+ \sqrt{-1} \sum_{1 \leqslant i \leqslant k, k+1 \leqslant m \leqslant n} \left(g_{i\bar{m}} + Ah_{i\bar{m}} + \left(\frac{\varepsilon}{2} - 1\right)\sum_{1 \leqslant j \leqslant k} g_{i\bar{j}}\frac{\partial \bar{r}_j}{\partial \bar{z}_m}\right)\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_m$$
$$+ \sqrt{-1} \sum_{k+1 \leqslant l \leqslant n, 1 \leqslant j \leqslant k} \left(g_{l\bar{j}} + Ah_{l\bar{j}} + \left(\frac{\varepsilon}{2} - 1\right)\sum_{1 \leqslant i \leqslant k} g_{i\bar{j}}\frac{\partial r_i}{\partial z_l}\right)\mathrm{d}z_l \wedge \mathrm{d}\bar{z}_j$$
$$+ \sqrt{-1} \sum_{k+1 \leqslant l,m \leqslant n} \left(g_{l\bar{m}} + Ah_{l\bar{m}} + \left(\frac{\varepsilon}{2} - 1\right)\sum_{1 \leqslant i,j \leqslant k} g_{i\bar{j}}\frac{\partial r_i}{\partial z_l}\frac{\partial \bar{r}_j}{\partial \bar{z}_m}\right)\mathrm{d}z_l \wedge \mathrm{d}\bar{z}_m$$

$$= \sqrt{-1} \sum_{1 \leqslant i,j \leqslant k} \frac{\varepsilon}{2} \delta_{i\bar{j}} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$$

$$+ \sqrt{-1} \sum_{1 \leqslant i \leqslant k, k+1 \leqslant m \leqslant n} \left(\delta_{i\bar{m}} + \left(\frac{\varepsilon}{2} - 1 \right) \sum_{1 \leqslant j \leqslant k} \delta_{i\bar{j}} \frac{\partial \bar{r}_j}{\partial \bar{z}_m} \right) \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_m$$

$$+ \sqrt{-1} \sum_{k+1 \leqslant l \leqslant n, 1 \leqslant j \leqslant k} \left(\delta_{l\bar{j}} + \left(\frac{\varepsilon}{2} - 1 \right) \sum_{1 \leqslant i \leqslant k} \delta_{i\bar{j}} \frac{\partial r_i}{\partial z_l} \right) \mathrm{d}z_l \wedge \mathrm{d}\bar{z}_j$$

$$+ \sqrt{-1} \sum_{k+1 \leqslant l,m \leqslant n} \left(\delta_{l\bar{m}} + A\delta_{l\bar{m}} + \left(\frac{\varepsilon}{2} - 1 \right) \sum_{1 \leqslant i,j \leqslant k} \delta_{i\bar{j}} \frac{\partial r_i}{\partial z_l} \frac{\partial \bar{r}_j}{\partial \bar{z}_m} \right) \mathrm{d}z_l \wedge \mathrm{d}\bar{z}_m,$$

where the second equality follows from Proposition 2.1 and the fact that $g_{i\bar{j}} = \delta_{i\bar{j}}$ at p. Then can see that when $A > 0$ is sufficiently large (independent of ϕ), there is an open neighborhood of p in X, which is still denoted by W_p (independent of ϕ), such that

$$(\omega - r^*(\omega|_V)) + \frac{\varepsilon}{2} r^*(\omega|_V) + A\sqrt{-1}\partial\bar{\partial}h$$

is positive definite and $\bar{\phi} \leqslant -\frac{C}{4}$ on W_p. From (3.1), the proof of Claim 3.3 is complete. $\qquad\square$

To emphasis the uniformity, it is worth to point out again that the chosen of the open set W_p, and the constant ε' is independent of ϕ, as long as ϕ satisfies *assumption* $\bigstar_{\varepsilon,C}$. We call the above data $(W_p, A, \varepsilon', -\frac{C}{4}, \bar{\phi})$, an *admissible local extension* of ϕ.

Since all the φ_m satisfy the same *assumption* $\bigstar_{\varepsilon,C}$, thus near p, we can choose a *uniform admissible local extension* $(W_p, A, \varepsilon', -\frac{C}{4}, \bar{\varphi}_m)$ of φ_m, for all $m \in \mathbb{N}^+$. Since V is compact, one may choose an open neighborhood W of V in X, and universal constants $A > 0$ and $\varepsilon' > 0$, such that the functions $\widetilde{\varphi}_m := \varphi_m \circ r + Ah$ are defined on W, and $\omega + i\partial\bar{\partial}\widetilde{\varphi}_m \geqslant \varepsilon'\omega$ on W for all m. Since $\{\varphi_m\}$ is a non-increasing sequence, one obtains that $\{\widetilde{\varphi}_m\}$ is a non-increasing sequence.

Step 2: Global extensions of φ_m for all m. Up to shrinking, we may assume that $\widetilde{\varphi}_m$ are defined on the closure of W for all $m \in \mathbb{N}^+$. Let F be the quasi-psh function in Lemma 3.2. Near ∂W (the boundary of W), the function F is smooth, and $\sup_{\partial W} \widetilde{\varphi}_1 = -C''$ for some positive constant $C'' > 0$. Now we choose a small positive ν, such that $\inf_{\partial W}(\nu F) > -\frac{C''}{2}$ and $\omega + i\partial\bar{\partial}\nu F \geqslant \varepsilon'\omega$. Thus $\nu F > \widetilde{\varphi}_1 \geqslant \widetilde{\varphi}_m$ in a neighborhood of ∂W for all $m \in \mathbb{N}^+$, since $\widetilde{\varphi}_m$ is non-increasing. Therefore, we can finally define

$$\Phi_m = \begin{cases} \max\{\widetilde{\varphi}_m, \nu F\}, & \text{on } W; \\ \nu F, & \text{on } X \setminus W, \end{cases}$$

which is defined on the whole of X. It is easy to check that Φ_m satisfies the following properties:

- Φ_m is non-increasing in m,
- $\Phi_m \leqslant 0$ for all $m \in \mathbb{N}^+$,
- $\omega + i\partial\bar\partial\Phi_m \geqslant \varepsilon'\omega$ for all $m \in \mathbb{N}^+$,
- $\Phi_m|_V = \varphi_m$ for all $m \in \mathbb{N}^+$.

Step 3: Taking limit to complete the proof of Theorem 1.2. From above steps, we get a non-increasing sequence of non-positive strictly ω-psh functions Φ_m on X. Then either $\Phi_m \to -\infty$ uniformly on X, or $\Phi := \lim_m \Phi_m \in \mathrm{Psh}(X, \omega)$. But $\Phi_m|_V = \varphi_m \searrow \varphi \not\equiv -\infty$, the first case will not appear. Moreover, we can see that $\Phi := \lim_m \Phi_m$ is a strictly ω-psh function on X from the property $\omega + i\partial\bar\partial\Phi_m \geqslant \varepsilon'\omega$ for all $m \in \mathbb{N}^+$, and $\Phi|_V = \lim_m \Phi_m|_V = \lim_m \varphi_m = \varphi$. It follows that $(\omega + i\partial\bar\partial\Phi)|_V = \omega|_V + i\partial\bar\partial\varphi$. Thus we complete the proof of Theorem 1.2.

Remark 3.4. — By similar arguments as in [13], we can get the following extension results for Kähler currents in a big class.

THEOREM 3.5. — *Let (X, ω) be a compact Kähler manifold of complex dimension n, and $V \subset X$ be a complex submanifold of positive dimension. Suppose that V has a holomorphic retraction structure in X. Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class such that any of the irreducible components of $E_{nK}(\alpha)$ either does not intersect with V, or is contained in V. Then any Kähler current in $\alpha|_V$ is the restriction of a Kähler current in α.*

Proof. — Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class, and $\theta \in \alpha$ be a smooth representative. Let φ be a quasi-psh function on V such that $\theta|_V + i\partial\bar\partial\varphi \geqslant \varepsilon\omega|_V$ on V, for some $\varepsilon > 0$. Then by the same technique as in Step 1 and Step 2, we can get an open neighbourhood U of V in X, and a non-increasing sequence of smooth functions $\widetilde\varphi_m$ on U, such that

- $\theta + i\partial\bar\partial\widetilde\varphi_m \geqslant \varepsilon'\omega$ on U,
- $\widetilde\varphi_m|_V = \varphi$.

In [2, Theorem 3.17], it is proved that there is a Kähler currents T with analytic singularities in α, such that $E_+(T) = E_{nK}(\alpha)$. We write $T = \theta + i\partial\bar\partial\Upsilon$. Since any of the irreducible components of $E_{nK}(\alpha)$ either does not intersect with V, or is contained in V, one can choose $\nu > 0$ and $C > 0$ so that $\inf_{\partial U} \Upsilon + \nu F + C \geqslant \sup_{\partial U} \widetilde\varphi_1$, and $\theta + i\partial\bar\partial(\Upsilon + \nu F + C) \geqslant \varepsilon''\omega$, up to shrinking U if necessary. Set

$$\Phi_m = \begin{cases} \max\{\widetilde\varphi_m, \Upsilon + \nu F + C\}, & \text{on } U; \\ \Upsilon + \nu F + C, & \text{on } X \setminus U, \end{cases}$$

where F is the function in Lemma 3.2. Therefore, we get a non-increasing sequence of continuous strictly θ-psh functions Φ_m on X, and by the same argument as in Step 3, we conclude that $\Phi := \lim \Phi_m$ is a desired extension of φ. □

4. Examples

In [7], Hosono–Koike point out that in Nakayama's example and Zariski's example, the submanifolds have holomorphic tubular neighborhood structure (thus holomorphic retraction structure) in the ambient manifold.

In the following, we list more examples.

Product manifold. Let Y_1 and Y_2 be two compact Kähler manifold and set $X := Y_1 \times Y_2$. Fix an arbitrary point $p \in Y_2$, let $V = Y_1 \times \{p\}$, then the natural map $\pi : Y_1 \times Y_2 \to Y_1 \times \{p\}$ serves as a holomorphic retraction map.

An interesting example of non-product manifold admitting a holomorphic retraction, which does not have a holomorphic tubular neighborhood structure, communicated to us by Koike [8], is the following famous example of Serre.

Serre's example. Let $X := \mathbb{P}_{[x;y]} \times \mathbb{C}_z / \sim$, where $\tau \in \mathbb{H}$ with $\mathbb{H}$ being the upper half plane, and

$$([x;y], z) \sim ([x; y+x], z+1) \sim ([x; y+\bar{\tau} \cdot x], z+\tau).$$

Let $V := \{x = 0\} \subset X$ as a submanifold of X which is obviously isomorphic to the elliptic curve $\mathbb{C}/\langle 1, \tau \rangle$. It is easy to check that the projection map $\pi : X \to \mathbb{C}/\langle 1, \tau \rangle =: V$ is a holomorphic retraction.

Remark 4.1. — In [8], Koike gives an interesting proof of Theorem 1.2 for Serre's example, which however seems not to be applicable to the general case treated in this paper.

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