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Renormalization operator for substitutions (*)

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Abstract. — This paper studies properties of *renormalization operators for potentials* in symbolic dynamics. These operators first appeared in [1] and the link with substitutions was done in [4]. They are linear, have a fixed direction and potentials in this fixed direction are natural candidates to have pathologic behavior with respect to Thermodynamical formalism such as phase transitions.

We define the family of marked substitutions, which contains the Thue–Morse substitution, and study how the fixed-direction for the associated renormalization operator $\mathcal{R}$ is attracting (or repelling). Namely, we show that $\mathcal{R}^n(\varphi)$ converges provided that φ has the right germ close to the attractor of the substitution.

Résumé. — Nous étudions ici les propriétés d'un opérateur de renormalisation sur les potentiels en dynamique symbolique. Cet opérateur a été défini la première fois dans [1]. Dans [4] un lien est fait avec les substitutions en dynamique symbolique. C'est un opérateur linéaire qui agît sur les fonctions (avec une certaine régularité) et qui admet une direction invariante. Les fonctions dans cette direction propre sont des candidates naturelles pour exhiber des comportements pathologiques comme des transitions de phases congelantes.

Nous définissons ici une famille de substitutions dites marquées. Cette famille contient la substitution de Thue–Morse. Nous montrons que pour ces substitutions, l'itération de la renormalisation a un comportement hyperbolique sur les potentiels, au sens où elle attire vers la direction fixe les potentiels avec un germe bien choisi proche de l'attracteur de la substitution.

Ce résultat améliore celui de [4] où le caractère hyperbolique n'était établi que pour la moyenne des itérations.

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Article proposé par Marie Claude Arnaud.

1. Introduction

1.1. Background

This paper deals with dynamical properties for some renormalization procedures. More precisely, it studies some *substitutions from the outside.*

The main motivation for this work was to continue investigations between renormalization and some phase transitions (within the ergodic viewpoint) done in [1, 4, 5]. In [1], a renormalization operator $\mathcal{R}$ on potentials was defined for the Manneville–Pomeau map and pushed to its symbolic model. Then, it was proved that fixed points for this renormalization operator were the ones which produce a freezing phase transition (*Hofbauer*-type potentials). In [4] it was proved that this renormalization operator could be defined in the shift space for the Thue–Morse substitution. In [5], the Fibonacci case was studied, showing that the renormalization operator could be extended to a non-constant length substitution. For a potential φ, the main question was the convergence of $\mathcal{R}^n\varphi$. It was proved a pointwise convergence for the Fibonacci substitution, nevertheless it was only proved a Cesaro mean convergence for the Thue–Morse substitution. It is noteworthy that if links between fixed points of the operator and phase transitions were investigated in these works, the precise nature of these links have still not been discovered or understood.

The present paper proves a general statement in the study of fixed point for renormalization for potentials associated to substitutions, instead of the study of one example as in [4]. We define the class of *marked* substitutions. We mention that after the first version of this paper, an independently, the notion of marked substitution has also been used in [3] under the terminology of *permutative*. Roughly speaking it means that if a is a digit and H the substitution, it is sufficient to know the first[(1)] or the last[(2)] letter of $H(a)$ to know what a is. Clearly, the Thue–Morse substitution which is defined by $H(0) = 01$ and $H(1) = 10$ is left and right marked. In this paper, it is proved that for left and right marked substitutions, the renormalization operator $\mathcal{R}$ admits a unique continuous and non-nul fixed point. Conditions on the potentials φ are given to insure that $\mathcal{R}^n(\varphi)$ converges to the fixed point. We point out that after the first version of this work has been announced, J. Emme managed to get a similar result for the k-bonacci case, which are right-marked but not left-marked substitutions (see [8]). Moreover it seems that the notion of accident corresponds to the notion of *minimal forbidden words* introduced in [2]. It could be use to generalize results of this paper to sturmian words for example.

(1) left marked

(2) right marked

We would like to thank the referee for usefull comments who help us to improve the paper.

1.2. Results

Let $\mathcal{A}$ be a finite set called the alphabet with cardinality $D \geqslant 2$. Elements of $\mathcal{A}$ are called *letters* or *digits*. A word is a finite or infinite string of digits. If v is the finite word $v = v_0 \dots v_{n-1}$ then n is called the length of the word v and is denoted by $|v|$. Moreover the number of occurences of the letter i in v is denoted $|v|_i$. The set of all finite words over $\mathcal{A}$ is denoted by $\mathcal{A}^*$.

If $u = u_0 \dots u_{n-1}$ is a finite word and $v = v_0 \dots$ is a word, the concatenation uv is the new word $u_0 \dots u_{n-1} v_0 \dots$. If v is a finite word, v^n denotes the concatenated word

$$v^n = \underbrace{v \dots v}_{n \text{ times}}.$$

If $u = u_0 \dots u_{n-1}$ is a word, a prefix of u is either the empty word ϵ, or any word $u_0 \dots u_j$ with $j \leqslant n-1$. A suffix of u is either the empty word ϵ or any word of the form $u_j \dots u_{n-1}$ with $0 \leqslant j \leqslant n-1$.

The shift map is the map defined on $\mathcal{A}^{\mathbb{N}}$ by $\sigma(u) = v$ with $v_n = u_{n+1}$ for all integers $n \geqslant 0$. We endow $\mathcal{A}$ with the discrete topology and consider the product topology on $\mathcal{A}^{\mathbb{N}}$. This topology is compatible with the distance d on $\mathcal{A}^{\mathbb{N}}$ defined by

$$d(x,y) = \frac{1}{D^n} \quad \text{if} \quad n = \min\{i \geqslant 0, x_i \neq y_i\}. \tag{1.1}$$

Definition 1.1. — *An infinite word u is said to be periodic (for σ) if it is the infinite concatenation of a finite word v, that is $u = vvvv\dots$ In that case we set $u = v^\infty$.*

A substitution H is a map from an alphabet $\mathcal{A}$ to the set $\mathcal{A}^* \setminus \{\epsilon\}$ of nonempty finite words on $\mathcal{A}$. It extends to a morphism of the monoid $\mathcal{A}^*$ by concatenation, that is $H(uv) = H(u)H(v)$.

Several basic notions on substitutions are recalled in Section 2. We also refer to [9]. We recall here the notions we need to state our results.

Definition 1.2. — *If H is a substitution, its* incidence matrix *is the $D \times D$ matrix $\mathcal{M}_H$ with entries a_{ij} where a_{ij} is the number of j's in $H(i)$. Then, H is said to be* primitive *if all entries of $\mathcal{M}_H^k$ are positive for some $k \geqslant 1$.*

A k-periodic point of H is an infinite word u with $H^k(u) = u$ for some $k > 0$. If $k = 1$ the point is said to be fixed.

We point out an equivalent definition for being primitive. The substitution H is primitive if and only if there exists an integer k such that for every couple of letters (i,j), j appears in $H^k(i)$.

Let H be a substitution over the alphabet $\mathcal{A}$, the subshift associated to H is the subset $\mathbb{K}$ of $\mathcal{A}^{\mathbb{N}}$ such that $x \in \mathbb{K}$ if for all integers i,j the word $x_i \dots x_{j+i}$ appears in some $H^n(a)$ for some letter a and some integer n. It is called the *subshift* associated to the substitution. Then, H is said to be *aperiodic* if there is no periodic point for σ inside the subshift. If H is aperiodic and primitive, then $\mathbb{K}$ is uniquely ergodic but not reduced to a σ-periodic orbit. In that case, the unique σ-invariant probability is denoted by $\mu_{\mathbb{K}}$. Moreover in this case $\mathbb{K}$ is also the orbit closure of a fixed point of H under the shift action.

We recall that the *language of a primitive substitution* is the set of finite words which appear in a fixed point of H. It is denoted by $\mathcal{L}_H$.

Definition 1.3. — *A substitution is said to be* 2-full *if any word of length 2 in $\mathcal{A}^*$ belongs to the language of the substitution. A substitution is said to be* marked *if the set of the first letters of the images of the letters by the substitution is in bijection with the alphabet and if the same thing is true with the set of last letters. It is* left marked *if the set of first letters is in bijection with the alphabet, and* right marked *if the set of last letters is in bijection with the alphabet.*

Definition 1.4. — *Let n be a positive integer. For $x \in \mathcal{A}^{\mathbb{N}}$ of the form $x = a\dots$ and for a substitution H, we set $t_n(x) = |H^n(a)|$.*

Let us define $\mathcal{R}$ by:

$$\begin{aligned} \mathcal{R} : \mathcal{C}(\mathcal{A}^{\mathbb{N}}, \mathbb{R}) &\longrightarrow \mathcal{C}(\mathcal{A}^{\mathbb{N}}, \mathbb{R}) \\ \varphi(x) &\longmapsto \mathcal{R}(\varphi)(x) = \sum_{i=0}^{t_1(x)-1} \varphi \circ \sigma^i \circ H(x) \end{aligned} \tag{1.2}$$

Then we have:

Theorem 1.5. — *Let H be a 2-full, marked, aperiodic and primitive substitution. Then there exists a unique $U : \mathcal{A}^{\mathbb{N}} \to \mathbb{R}$ continuous, such that*

- $\mathcal{R}(U) = U$,
- $U_{|\mathbb{K}} \equiv 0$,
- *Let $\alpha \in (0,+\infty)$, consider a map $\varphi : \mathcal{A}^{\mathbb{N}} \to \mathbb{R}$ such that $\varphi_{|\mathbb{K}} \equiv 0$ and $\varphi(x) = \frac{g(x)}{p^{\alpha}}$ if $d(x,\mathbb{K}) = D^{-p}$, where g is a continuous function satisfiyng $g_{|\mathbb{K}} > 0$ and $d(x,\mathbb{K})$ is the distance between x and $\mathbb{K}$*

(see (1.1)*). For every x in $\mathcal{A}^{\mathbb{N}}$ we have*

$$\lim_{m\to+\infty}\mathcal{R}^m\varphi(x)=\begin{cases}0 & \textit{if } \alpha>1,\\ +\infty & \textit{if } \alpha<1,\\ \int g\,\mathrm{d}\mu_{\mathbb{K}}.\,U(x) & \textit{if } \alpha=1.\end{cases}$$

We emphasize some points related to our main theorem:

(1) The map U is non zero by construction.
(2) The expression of U outside $\mathbb{K}$ is explicit as soon as the substitution is given. It will be explained during the proof see Corollary 3.13. There is an explicit example in Section 4.
(3) The map $\mathcal{R}$ is linear. Hence, it makes no sense to inquire for fixed points but rather to study fixed directions, i.e., eigenvectors. The exact spectrum of $\mathcal{R}$ is far from being known, even in $\mathcal{C}^0$. Actually, studying the spectrum is a natural question and would be a natural continuation for this work.
(4) Thue–Morse substitution is 2-full, marked, aperiodic and primitive. Therefore, Theorem 1.5 improves [4] where only the Cesaro-convergence was proved.
(5) The theorem points out that to study chaos close to $\mathbb{K}$ (but *outside*), it is important to fix a germ for the test functions φ. We remind that this has already been pointed out for the Manneville–Pomeau case, as it is mentioned in [1]. We mention a physical approach for intermittent maps in [12].

In the following, we denote by Ξ_α the set of potentials $V=-\varphi$ of the form $\varphi(x)=\frac{g(x)}{p^\alpha}$ as in Theorem 1.5.

1.3. Outline of the paper

First of all in Section 2 we recall some classical definitions and results on substitutions and symbolic dynamics. The last part of this section is devoted to some background on the notion of accidents, defined in [4].

Then in Section 3 we prove Theorem 1.5. The proof is decomposed in several parts. We obtain a formula for $\mathcal{R}^m\varphi$ in Lemma 3.1. To study the convergence of this term we need to get good estimates for $\delta_i^n(x)$ (defined in Section 2.3) for $i<t_n(x)$ and for any $x\notin\mathbb{K}$. This is done in Corollary 3.8. Finally we compute the limit in two steps: one for the simplest case $g\equiv 1$ and one for the general case, see Section 3.4.3.

In Section 4 we give a detailed and explicit computation of the function U of Theorem 1.5 for the example of the Thue–Morse subshift.

2. More definitions and tools

2.1. Words, languages and special words

For this paragraph we refer to [9].

Definition 2.1. — *A word $v = v_0 \dots v_{r-1}$ is said to occur at position m in an infinite word u if there exists an integer m such that for all $i \in [0; r-1]$ we have $u_{m+i} = v_i$. We say that the word v is a* factor *of u.*

For an infinite word u, the language of u (respectively the language of length n) is the set of all words (respectively all words of length n) in $\mathcal{A}^*$ which appear in u. We denote it by $\mathcal{L}(u)$ (respectively $\mathcal{L}_n(u)$). Then, the sequence of finite languages $(\mathcal{L}_n(u))_{n\in\mathbb{N}}$ is said to be the factorial language for $\mathcal{L}(u)$.

Definition 2.2 ([7, §7]). — *The dynamical system associated to an infinite word u is the system $(\mathbb{K}_u, \sigma)$ where σ is the shift map and $\mathbb{K}_u = \overline{\{\sigma^n(u), n \in \mathbb{N}\}}$. An infinite word u is said to be recurrent if every factor of u occurs infinitely often.*

Remark that u being recurrent is equivalent to the fact that σ is onto on $\mathbb{K}_u$. Moreover we have equivalence between $\omega \in \mathbb{K}_u$ and $\mathcal{L}(\omega) \subset \mathcal{L}(u)$. Thus the language of $\mathbb{K}_u$ is equal to the language of u. A language is said to be *factorial* if it is closed under taking factors. It is called *extendable* if every word of length n in the language can be extended to a word of length $n+1$ for every integer n.

Definition 2.3. — *Let $\mathcal{L} = (\mathcal{L}_n)_{n\in\mathbb{N}}$ be a factorial and extendable language. The* complexity function *$p : \mathbb{N} \to \mathbb{N}$ is the function defined by $p(n) := \mathrm{card}(\mathcal{L}_n)$. For $v \in \mathcal{L}_n$ let us define*

$$\begin{aligned}
m_l(v) &= \mathrm{card}\{a \in \mathcal{A}, av \in \mathcal{L}_{n+1}\},\\
m_r(v) &= \mathrm{card}\{b \in \mathcal{A}, vb \in \mathcal{L}_{n+1}\},\\
m_b(v) &= \mathrm{card}\{(a,b) \in \mathcal{A}^2, avb \in \mathcal{L}_{n+2}\},\\
i(v) &= m_b(v) - m_r(v) - m_l(v) + 1.
\end{aligned}$$

- *A word v is called right special if $m_r(v) \geqslant 2$.*
- *A word v is called left special if $m_l(v) \geqslant 2$.*
- *A word v is called bispecial if it is right and left special.*

Definition 2.4. — *A word v such that $i(v) < 0$ is called a weak bispecial. A word such that $i(v) > 0$ is called a strong bispecial. A bispecial word v such that $i(v) = 0$ is called a neutral bispecial.*

2.2. Substitutions

2.2.1. Some more definitions

For a word, we recall that a strict prefix is a prefix different from the entire word. We have a similar definition for a strict suffix. In all the following, we allow eventually empty word for prefix and suffix.

Definition 2.5. — *Let H be a substitution. The set of all strict prefixes and all strict suffixes for $H(a)$, $a \in \mathcal{A}$, are respectively denoted by $\mathcal{P}(a)$ and $\mathcal{S}(a)$. Unions (over a) of the $\mathcal{P}(a)$'s and the $\mathcal{S}(a)$'s are respectively denoted by $\mathcal{P}$ and $\mathcal{S}$.*

For a substitution H, we recall that its language is denoted by $\mathcal{L}_H$.

Definition 2.6. — *Let H be a substitution. We say that the word $u \in \mathcal{L}_H$ is* uniquely desubstituable *if there exists only one way to write $u = \widehat{\tau}H(v)\widehat{\rho}$ with*

(1) *$\tau v \rho$ is a word in $\mathcal{L}_H$,*
(2) *$\widehat{\rho} \in \mathcal{P}(\rho)$,*
(3) *$\widehat{\tau} \in \mathcal{S}(\tau)$.*

We recall the following theorem, which is also true without the hypothesis marked.

Theorem 2.7 ([11, Thm. 2.4]). — *Let H be a marked, primitive, aperiodic substitution. There exists a constant N_H such that for every word $w \in \mathcal{L}_H$ the word w^{N_H} does not belong to this language.*

Remark 2.8. — Remark that N_H can be computed by an algorithm (see [10]).

2.2.2. Length of words in the language of a substitution

If H is a primitive substitution, the Perron Frobenius theorem shows that the incidence matrix $\mathcal{M}_H$ admits a single and simple dominating eigenvalue. We denote it by λ. It is a positive real number. The rest of the spectrum is contained inside a disc $\mathbb{D}(0,\theta)$ with $0 < \theta < \lambda$. Moreover we know that there exists a matrix A (product of the right and left eigenvectors of the incidence matrix for the eigenvalue λ) such that $\mathcal{M}_H^n \sim \lambda^n A$. This yields that there exists $\kappa > 0$ such that for every words $v = v_0 \dots v_{p-1}$ and for every n, if we denote $V = (|v|_i)_{0 \leqslant i \leqslant D-1}$:

$$\big| |H^n(v)| - \lambda^n \|AV\|_1 \big| \leqslant \kappa\theta^n. \tag{2.1}$$

2.3. Accidents

Let $\mathbb{K}$ be the subshift associated to the substitution H. Let x be an element of $\mathcal{A}^{\mathbb{N}}$ which does not belong to $\mathbb{K}$, then we define and denote:

- The word w is the maximal prefix of x such that w belongs to the language of $\mathbb{K}$. Thus we have by (1.1) $d(x,\mathbb{K}) = D^{-d}$ with $x = w\dots$ and $w = x_0\dots x_{d-1}$. Let us denote $\delta(x) := d$, and $\delta_k^n(x) := \delta(\sigma^k \circ H^n(x))$ for all integers k and n. Note that $\delta = \delta_0^0$.

  We emphasize that the word w is non-empty since every letter is in the language of $\mathbb{K}$ if the substitution is primitive. Then, w is the unique word such that
  $$x = wx', \quad w \in \mathcal{L}_H, \quad wx'_0 \notin \mathcal{L}_H.$$
- If there exists an integer b such that
  $$d(\sigma^b(x),\mathbb{K}) \leqslant d(\sigma^{b-1}(x),\mathbb{K}),$$
  then we say that an *accident appears at time b for x.*

By convention, 0 is an accident time for any $x \notin \mathbb{K}$. For a fixed $x \notin \mathbb{K}$, the accident times are ordered which allows to define the notion of j^{th} accident with $j \geqslant 0$. By convention, the 0^{th} accident occurs at time 0.

Accidents will play a crucial role in the study of our problem. For that we need a way to detect them and we need to consider the set of accident times for a point. This later point is done in Definition 2.11. Next lemma explains how to detect accidents. Figure 2.1 illustrates the definition and the next lemma which was stated and proved in [4] Proposition 3, see also [2].

Lemma 2.9. — *Let x be an infinite word not in $\mathbb{K}$. Assume that $\delta(x) = d$ and that the first accident appears at time $0 < b \leqslant d$. Then the word $x_b \dots x_{d-1}$ is a bispecial word of $\mathcal{L}_H$. It is called the first* accident-word.

Remark 2.10. — If $\mathcal{A}$ has cardinality two, then $x_0\dots x_{d-1}$ is not right-special. Moreover, and always if $\mathcal{A}$ has cardinality two, if $x = \sigma(z)$ and there is an accident at time 1 for z, then $x_0\dots x_{d-1}$ is not left-special.

For the next definition, please look at Figure 2.2.

Definition 2.11. — *We define inductiveley*

$$\begin{aligned}
b_1 &= b = \min\{j \geqslant 1, d(\sigma^j x,\mathbb{K}) \leqslant d(\sigma^{j-1}x,\mathbb{K})\}\\
b_2 &= \min\{j \geqslant 1, d(\sigma^{j+b_1} x,\mathbb{K}) \leqslant d(\sigma^{j+b_1-1}x,\mathbb{K})\}\\
b_3 &= \min\{j \geqslant 1, d(\sigma^{j+b_1+b_2} x,\mathbb{K}) \leqslant d(\sigma^{j+b_1+b_2-1}x,\mathbb{K})\}\\
&\vdots
\end{aligned}$$

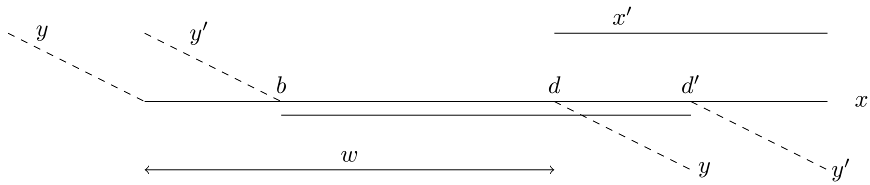

Figure 2.1. Dashed lines indicate infinite words y, y' in $\mathbb{K}$. The accident appears at b, w is the prefix of x of length d. The length of the accident-word is $d - b$ and the depth of the accident is d'.

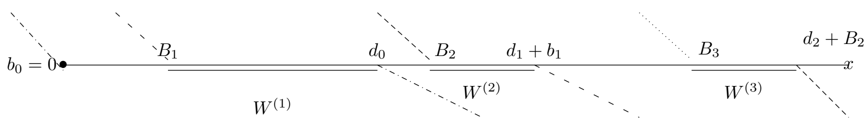

Figure 2.2. Accidents inside a word.

Set $b_0 = 0$, *and inductively* $B_j = b_0 + \cdots + b_j$. *Then,*

(1) *the integer* $B_j, j \geqslant 1$ *is the* j^{th} accident time for x,
(2) *the integer* $d_j := \delta(\sigma^{B_j} x)$ *is its* depth,
(3) *the prefix of* $\sigma^{B_j} x$ *of length* d_j *is its depth-word,*
(4) *the word* $x_{B_j} \dots x_{d_{j-1}-1}$ *is called the* j^{th} accidents-word for x,
(5) *its length is called the* length of the j^{th} accident for x.

Lemma 2.12. — *Let d be a positive integer. Consider x such that $\delta(x) = d$. Denote by B_1, B_2 the times of first and second accidents. Assume the two bispecial words defined by the accidents do not overlap, then we have:*

$$\begin{cases} \delta_i^0(x) = d - i, 0 \leqslant i < B_1 \\ \delta_i^0(x) = d - B_1 - i, B_1 \leqslant i < B_2. \end{cases}$$

Proof. — It is a simple application of the definition of accident. See also Figure 2.1 with $B_1 = b$. □

We recall that for $x \in \mathcal{A}^{\mathbb{N}}$ of the form $x = a \dots$ and for a primitive, 2-full and marked substitution H, we have set $t_n(x) = |H^n(a)|$. Then, we set:

Definition 2.13. — *We denote by $\mathcal{B}_n(x)$ the set of j^{th} accidents-words for $H^n(x)$ with $j \leqslant t_n(x)$.*

3. Proof of Theorem 1.5

3.1. Renormalization operator and accidents

In order to prove Theorem 1.5 we need to compute $\mathcal{R}^n\varphi$. We give here a formula for $\mathcal{R}^n\varphi(x)$ and explain why $\lim_{n\to+\infty}\mathcal{R}^n\varphi(x)$ only depends on the germ of φ close to $\mathbb{K}$. The uniqueness of U and the fact that it is fixed by the operator will be clear consequences.

3.1.1. A formula for $\mathcal{R}^n\varphi$

We emphasize that σ satisfies the following renormalization equation (with respect to H)

$$H\circ\sigma(x)=\sigma^{t_1(x)}\circ H(x).$$

This equality is the key point to prove the formula that gives an expression for R^n:

Lemma 3.1. — *For every integer n and for every $x\in\mathcal{A}^{\mathbb{N}}$ we have*

$$\mathcal{R}^n\varphi(x)=\sum_{i=0}^{t_n(x)-1}\varphi\circ\sigma^i\circ H^n(x).$$

Proof. — We make a proof by induction.

For $n=1$ it is clear. Assume the result is true for n.

By induction hypothesis applied to the new potential $R\varphi$ we deduce

$$\mathcal{R}^{n+1}\varphi(x)=\mathcal{R}^n\circ\mathcal{R}\varphi(x)=\sum_{i=0}^{t_n(x)-1}\sum_{j=0}^{t_1(x)-1}\varphi\circ\sigma^j\circ H\circ\sigma^i\circ H^n(x).$$

For all $i\in[0\dots t_1(x)-1]$ we have:

$$H\circ\sigma^i(x)=\sigma^{s(i,x)}\circ H(x),\quad\text{where}\quad s(i,x)=\sum_{j=1}^{i}t_1(\sigma^{j-1}(x)).$$

We deduce

$$\begin{aligned}\mathcal{R}^{n+1}\varphi(x)&=\sum_{i=0}^{t_n(x)-1}\sum_{j=0}^{t_1(x)-1}\varphi\circ\sigma^{s(i,x)+j}\circ H^{n+1}(x)\\&=\sum_{i=0}^{t_{n+1}(x)-1}\varphi\circ\sigma^i(x)\circ H^{n+1}(x).\end{aligned}$$

We used the fact that $t_{n+1}(x)=|H^{n+1}(a)|=|H(H^n(a))|=\sum_{i=1}^{t_n(x)} t_1(\sigma^i(x))$. The induction hypothesis is proved. $\square$

3.1.2. Distance between $\sigma^j(H^n(x))$ and $\mathbb{K}$

Lemma 3.1 shows why it is so important to know the numbers $\delta_k^n(x) = \delta(\sigma^k(H^n(x)))$ for every x and for $k \leqslant t_n(x) - 1$. We shall see below why accidents perturb the computation of $\mathcal{R}^n(\varphi)(x)$. This explains why we need to control them.

Moreover, $\mathcal{R}^n\varphi(x)$ involves a Birkhoff sum at point $H^n(x)$ which changes when n increases. Clearly, $H^n(x)$ converges to a fixed point of H (up to take a power of H), thus goes to $\mathbb{K}$ when n increases. But this convergence may be faster than what we could expect, just knowing for how many digits x coincides with $\mathbb{K}$. We give here two examples illustrating this point:

Example. — Consider $H : \begin{cases} a \to abbaaa \\ b \to baaaab \end{cases}$. The word bbb does not belong to the language. Nevertheless $H(bbb)$ belongs to $\mathcal{L}$ as seen by the computation of

$$H(aaaa) = abbaaaabbaaaabbaaaabbaaa = abH(bbb)aa.$$

Here, for $x = bbb\dots$ we have $\delta(x) = 2$ and $\delta(H(x)) = \delta_0^1(x) \geqslant 3 * 6 > 2 * 6$.

Consider $H : \begin{cases} a \mapsto aaab \\ b \mapsto abaa \end{cases}$. We have $H(a^3) = a^3ba^3ba^3b = a^2H(bb)ab$, thus bb does not belong to the language, and H is not 2-full. Nevertheless we have $H(bb) = aba^3ba^2$, which is a factor of $H(aaa)$. Now let $x = b\sigma^3H^\infty(a)$, then we obtain $x = bba^3ba^3baba^5ba^3b\dots$ Remark that $\delta(x) = 1$. Moreover $H(x) = aba^3ba^5b\dots$, thus we obtain $\delta_0^1(x) = 7$.

3.1.3. Necessity of 2-full hypothesis and germ of a potential close to $\mathbb{K}$

We can now explain why knowing the germ close to $\mathbb{K}$ is sufficient to determine $\lim_{n\to+\infty} \mathcal{R}^n\varphi(x)$. Note that H is 2-full which means that for every x, $\delta(x) \geqslant 2$. Set $x = ab\dots$, it follows that $\delta_0^n(x)$ is greater than $t_n(a) + t_n(b)$, and then for every $k \leqslant t_n(a) - 1$

$$\delta_n^k(x) \geqslant t_n(b) + t_n(a) - k. \tag{3.1}$$

Remember that $t_n(b)$ is bounded by $c.\lambda^n$ with $c > 0$. This computation shows that among all the points $\sigma^k(H^n(x))$, the farthest from $\mathbb{K}$ is at distance at most $D^{-t_n(b)-1} \sim D^{-\lambda^n}$. It thus makes sense to replace $\varphi(\sigma^k(H^n(x)))$ by $g(\sigma^k(H^n(x)))/(\delta_k^n(x))^\alpha$.

Counter-example. — On the contrary, consider the following substitution

$$H = \begin{cases} a \to abba \\ b \to bab. \end{cases}$$

This substitution is primitive, marked but is not 2-full since aa does not belong to the language.

Then consider $x = aa\ldots$ we have $\delta(x) = 1$. Therefore, $H^n(x) = H^n(a)H^n(a)\ldots$. Note that $H^n(a)$ finishes and starts with a and then $H^n(a)H^n(a)$ contains the word aa in its middle. Furthermore, any suffix of $H^n(a)$ is in the language but no suffix of $H^n(a)a$ belongs to the language. Therefore, for any $i \leqslant n$ $\delta_i^n(x) = |H^n(a)| - i$. We will see at the end of the paper that $\mathcal{R}^n(\varphi)(x)$ does not converge. This shows that knowing the germ close to $\mathbb{K}$ is not sufficient to determine the limit for $\mathcal{R}^n(\varphi)(x)$.

3.2. Bispecial words for marked substitutions

As we have seen above, it is important to detect accidents. We also pointed out that accidents are related to occurrences of bispecial words in the language. It is therefore of prime importance to study these bispecial words. We prove here a strong version of Theorem 2.7 in Theorem 3.4. This allows us to get a complete description of the set of bispecial words (see Proposition 3.6). Remark that after the first version of this paper, a similar result appears in a more general context, see [3]. It could be used to generalize our result.

Lemma 3.2. — *Assume that H is a marked substitution. If $z = H(x) = \widehat{\tau}H(y)$ is an infinite word with $\widehat{\tau} \in \mathcal{S}(\tau) \cup \{\epsilon\}$. Then either $\widehat{\tau}$ is empty and $x = y$ or the word z is eventually periodic.*

Proof. — If $\widehat{\tau}$ is the empty word, then the left marking proves the result. If not, then let us denote by t the length of $\widehat{\tau}$. Denote $x = x_0x_1\ldots$. The infinite word $H(x)$ can be cut by construction into words corresponding to the images of the letters by H, i.e. $H(x) = H(x_0)H(x_1)\ldots$. Let us do the same thing for $H(y)$. Since H is left marked, the first letters of the image are in bijection with the alphabet, thus we can assume that $H(x_i)$ begins with x_i for every integer. We denote by $t' = \big||H(x_0)| - t\big|$, see Fig. 3.1.

First of all assume that $t + |H(y_0)| = |H(x_0)| + |H(x_1)|$. Then we have $\widehat{\tau}H(y_0) = H(x_0x_1)$, the hypothesis of right marking allows us to deduce $y_0 = x_1$ and $\widehat{\tau} = H(x_0)$ which is impossible.

By the same argument, if $t+[H(y_0)| = |H(x_0)|$, the left marking hypothesis proves that t is null and $x = y$. Thus we can assume it never happens, and

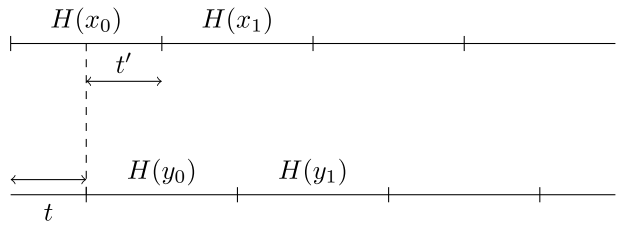

Figure 3.1. $\sigma^t H(x) = H(y)$

define a sequence $(x_n, y_n, t_n)_{n\in\mathbb{N}}$ with $(x_n, y_n, t_n) \in \mathcal{A}^2 \times [0, \dots, \max|H(a)|]$ by induction:

$$\mathcal{A}^2 \times [0, \dots, \max|H(a)|] \longrightarrow \mathcal{A}^2 \times [0, \dots, \max|H(a)|]$$
$$(x_0, y_0, t) \longmapsto \psi(x_0, y_0, t) = \begin{cases} (x_1, y_0, t') & t < |H(x_0)| \\ (y_0, x_1, t') & t > |H(x_0)|. \end{cases}$$

This algorithm is defined on a finite set and can be iterated by the previous argument, thus the sequence is ultimately periodic. This implies that the word z is ultimately periodic by the pigeonhole principle. $\square$

From Lemma 3.2 we deduce a very important result. If x belongs to $\mathcal{A}^{\mathbb{N}} \setminus \mathbb{K}$, then so does $H(x)$:

Corollary 3.3. — *Consider a marked, aperiodic substitution H. For each word $x = wx'$ with $w \in \mathcal{L}_H$ and $wx'_1 \notin \mathcal{L}_H$, for every integer k there exists $m < \infty$ such that $\delta[H^k(x)] = m$.*

Proof. — The proof is by contradiction and by induction. Assume $H(x) \in \mathbb{K}$ thus it can be written $\widehat{\tau}H(y)$ with $y \in \mathbb{K}$. Then we apply Lemma 3.2. If $\widehat{\tau} = \epsilon$ (the empty word) then, $x = y$ and it is a contradiction with our assumption. If $\widehat{\tau} \neq \epsilon$, then y is ultimately periodic which is in contradiction with Theorem 2.7. This shows

$$x \notin \mathbb{K} \Longrightarrow H(x) \notin \mathbb{K}.$$

Then, the result follows by induction. $\square$

Theorem 3.4. — *Consider a primitive, aperiodic and marked substitution. There exists $l(H) > 0$ such that for every $z \in \mathcal{L}_H$ with $|z| > l(H)$ there exists a unique decomposition $z = \widehat{\tau}H(x)\widehat{\rho}$ with $\widehat{\tau} \in \mathcal{S}(\tau)$, $\widehat{\rho} \in \mathcal{P}(\rho)$, $\tau x \rho \in \mathcal{L}_H$.*

Proof. — The existence of the decomposition is clear because $\mathbb{K} = \{\sigma^n(v), n \in \mathbb{N}\}$ where v is any fixed point for H. Now assume we have two decompositions

$$\widehat{\tau}H(x)\widehat{\rho} = \widetilde{\tau}H(y)\widetilde{\rho}.$$

We will apply an effective version of the proof of Lemma 3.2. Let us denote $k = \max_a |H(a)|$. The same proof can be applied, it suffices to remark that the period and the pre-period are bounded by the cardinality D of the finite alphabet $\mathcal{A}$. Recall that N_H is defined in Theorem 2.7. Consider the minimum l_0 of the integers l such that $(D^2k)^l + kD^2 > N_H$. The proof is done with $l(H) = (D^2k)^{l_0} + kD^2$. We deduce $\widehat{\tau} = \widetilde{\tau}$, then the same argument shows that $\widehat{\rho} = \widetilde{\rho}$. □

An immediate corollary of Theorem 3.4 for marked substitution is

Corollary 3.5. — *Let x be such that $l(H) < \delta(x) < +\infty$. Let w be the prefix of x of length $\delta(x)$. Then, $\delta(H(x)) = |H(w)|$.*

Proof. — Set $p := \delta(x)$. The word $w := x_0 \dots x_{p-1}$ belongs to $\mathcal{L}_H$ and $x_0 \dots x_p$ does not belong to $\mathcal{L}_H$. Then, $H(w)$ belongs to $\mathcal{L}_H$, hence $\delta(H(x)) \geqslant |H(w)|$.

If we assume that $\delta(H(x)) > |H(w)|$ holds, then the prefix w' of length $\delta(H(x))$ of $H(x)$ can be written as $\widehat{\tau}H(w'')\widehat{\rho}$, with $\widehat{\tau} \in \mathcal{S}(\tau)$, $\widehat{\rho} \in \mathcal{P}(\rho)$, $\tau x \rho \in \mathcal{L}_H$. Because w' starts as $H(w)$, then $\widehat{\tau} = \epsilon$ and because H is marked, x_{p+1} is a prefix of $\widehat{\rho}$. This contradicts the fact that wx_{p+1} is not in $\mathcal{L}_H$. □

Proposition 3.6. — *Let H be a primitive, aperiodic and marked substitution. Let $\mathcal{W}_b$ be the set of bispecial words of length less than $l(H)$. Then every bispecial word can be written as $H^n(v)$ with $v \in \mathcal{W}_b$ and n some integer.*

Proof. — Consider a bispecial word u. By Theorem 3.4 we can write $u = \widehat{\tau}H(v)\widehat{\rho}$ where v has maximal length, v, $\widehat{\tau}$ and $\widehat{\rho}$ are unique.

We claim that $\widehat{\tau}$ is empty. Indeed, since u is a bispecial word, there exist two letters such that au and bu belong to the language. If $\widehat{\tau}$ is non-empty, then $a\widehat{\tau}, b\widehat{\tau}$ are the suffixes with the same length of $H(c)$ where c is a letter (unique by assumption on H). We deduce $a = b$, which is impossible. The same argument applies for $\widehat{\rho}$.

Now we prove that v is a bispecial word. If $aH(v)$ belongs to the language $\mathcal{L}_H$, the properties of H show that it is the suffix of a unique word $H(c)H(v)$. The same argument works for $bH(v)$ the other left extension of $H(v)$. The two left extensions of v are different by assumption on H. By the same argument v is right special. The proof finishes by an iteration of this process. □

3.3. Crucial Proposition

By Lemma 3.1, we have a formula for $\mathcal{R}^n(\varphi)(x)$. To study the convergence of this term we need to get good estimates for $\delta_i^n(x)$ for $i < t_n(x)$ and for any $x \notin \mathbb{K}$ (see also the discussion after Lemma 3.1). We have an easy bound from above :

$$\delta_i^n(x) \geqslant \delta_0^n(x) - i,$$

but we need a sharper estimate. For that purpose, we need to know the accident words $\mathcal{B}_n(x)$ (recall Definition 2.13).

The following main proposition shows that for sufficiently large $n \geqslant k$, the number of accidents, their depth, their time, and the associated accident word (see Definition 2.11) for $H^n(x)$ are all obtained from the accidents, their depth, their time and the accident word for $H^k(x)$ via the renormalization procedure given by H^{n-k} (see Figure 3.2).

Proposition 3.7. — *Let H be a 2-full, marked, aperiodic and primitive substitution. Let $x \notin \mathbb{K}$ and p be such that $\delta_0^0(x) = p$. Set $x = w_0 . \ldots w_{p-1}$ $x_p \ldots \notin \mathbb{K}$ and let k be such that $|H^k(w_1 \ldots w_{p-1})| \geqslant l(H)$. Then for $n \geqslant k$*

1. *we have $\#\mathcal{B}_n(x) = \#\mathcal{B}_k(x)$.*
2. *The word w is the j^{th} accident word for $H^n(x)$ if and only if $w = H^{n-k}(w')$ and w' is the j^{th} accident word for $H^k(x)$.*
3. *The j^{th}-accident time for $H^n(x)$ denoted by $t_{j,n}(x)$ is equal to $|H^{n-k}(w'')|$, where w'' is the prefix of length $t_{j,k}(x)$ for $H^k(x)$.*
4. *If w''' is the depth-word for the j^{th} accident for $H^k(x)$, then $H^{n-k}(w''')$ is the depth-word for the j^{th} accident for $H^n(x)$.*

Proof. — Note that $x = wx_p \ldots$ and $w \in \mathcal{L}_H$. Let us write $H^k(x) = e_0 \ldots e_{m_k-1}e_{m_k} \ldots$ with $m_k = \delta_0^k(x)$. Corollary 3.3 shows that m_k is finite.

First we prove that

$$\delta_0^n(x) = \delta(H^n(x)) = |H^{n-k}(e_0 \ldots e_{m_k-1})|$$

holds. Note that

$$H^n(x) = H^{n-k}H^k(w_0 \ldots w_{p-1} \ldots) = H^{n-k}(e_0 \ldots e_{m_k-1}e_{m_k} \ldots)$$

holds, which yields $\delta_0^n(x) \geqslant |H^{n-k}(e_0 \ldots e_{m_k-1})|$ because $e_0 \ldots e_{m_k-1}$ belongs to $\mathcal{L}_H$.

Now, we show by induction on $n \geqslant k+1$ that $\delta_0^n(x) \leqslant |H^{n-k}(e_0 \ldots e_{m_k-1})|$ holds.

First we prove the inequality for $n = k+1$. Assume by contradiction that $\delta_0^{k+1}(x)$ is strictly greater than the number $|H(e_0 \ldots e_{m_k-1})|$. This means that there exists a letter a such that $H(e_0 \dots e_{m_k-1})a \in \mathcal{L}_H$. As

$$|H(e_0 \dots e_{m_k-1})| > |H^k(w_1 \dots w_p)| \geqslant l(H),$$

we can apply Theorem 3.4 to the word $H(e_0 \dots e_{m_k-1})a$. By the left marking of H we deduce that $e_0 \dots e_{m_k-1}e \in \mathcal{L}_H$ with the letter e such that $H(e)$ begins with a. Thus we have $e = e_{m_k}$. This is a contradiction with the definition of m_k. We then iterate this argument, noting that $|H^j(e_0 \dots e_{m_k-1})|$ increases in j and is thus greater than $l(H)$. The induction process is done.

Now consider the time of the first accident for $H^k(x)$ and denote it by $j_1 \leqslant t_k(x)$. We argue by contradiction and prove that $H^n(x)$ cannot have an accident for $i < |H^{n-k}(e_0 \dots e_{j_1-1})| =: j'_1$.

By definition of the accident $\delta^k_{j_1}(x) \geqslant m_k + 1 - j_1$ whereas $\delta^k_{j_1-1}(x) = m_k - j_1 + 1$.

Pick $0 < i < j'_1$ and assume that $\delta^n_i(x) > \delta^n_0(x) - i$. We have $H^n(x) = H^{n-k}(e_0)H^{n-k}(e_1)\dots$. Let us introduce l the smallest integer such that $i < |H^{n-k}(e_0 \dots e_{l-1})|$. A prefix of $\sigma^i H^n(x)$ can be written $sH^{n-k}(e_l \dots e_{m_k-1})a \in \mathcal{L}_H$ with s suffix of $H^{n-k}(e_{l-1})$ and $a \in \mathcal{A}$. Note that $l \leqslant j_1 < t_k(x)$, which yields that $H^n(w_1 \dots w_{p-1}) = H^{n-k}(H^k(w_1 \dots w_{p-1}))$ is a factor of $H^{n-k}(e_l \dots e_{m_k-1})$. We can thus apply Theorem 3.4 and by the right marking of H^k, we obtain a word suffix of $e_{l-1} \dots e_{m_k-1}e \in \mathcal{L}_H$. This means that $H^k(x)$ has an accident at time $l - 1 < j_1$ and this is a contradiction with the definition of j_1. Finally we have proved

$$\delta^n_i(x) = \delta^n_0(x) - i, 0 \leqslant i \leqslant |H^{n-k}(e_0 \dots e_{j_1-1})| - 1,$$

or equivalently, that the first accident for $H^n(x)$ cannot occur before time j'_1.

Now, we prove that j'_1 is an accident time for $H^n(x)$. By definition of an accident, we know that $e_{j_1} \dots e_{m_k}e \in \mathcal{L}_H$ for some letter e. Then by application of H^{n-k} we deduce that there exists some letter a such that $H^{n-k}(e_{j_1} \dots e_{m_k})a \in \mathcal{L}_H$. Thus the first accident of H^n appears at time $|H^{n-k}(e_0 \dots e_{j_1})|$. The same reasoning shows that the accident-word is the image by H^{n-k} of the first accident-word of H^k.

At that stage, we have proved that items (2) and (3) hold for $j = 1$ (and they obviously hold for $j = 0$). Figure 3.2 illustrates this renormalization procedure.

Let us denote by $j_2 \leqslant t_k(x)$ the time of the second accident of $H^k(x)$. Note that the key argument is that $H^n(w_2 \dots w_p)$ has length greater than $l(H)$ and is still a factor of $H^{n-k}(e_{j_2} \dots e_{m_k})$ (because $j_2 \leqslant t_k(x)$). Note also that $\sigma^{j_1}(H^k(x))$ coincides with a word of $\mathbb{K}$ for at least $m_k - j_1 + 1$ digits. In other words, $H^{n-k}(e_{j_1} \dots e_{m_k}e_{m_{k+1}})$ is a suffix of the coincidence of $\sigma^{j_1}(H^n(x))$ with $\mathbb{K}$. This suffix contains $H^{n-k}(e_{j_2} \dots e_{m_k})$, thus it also

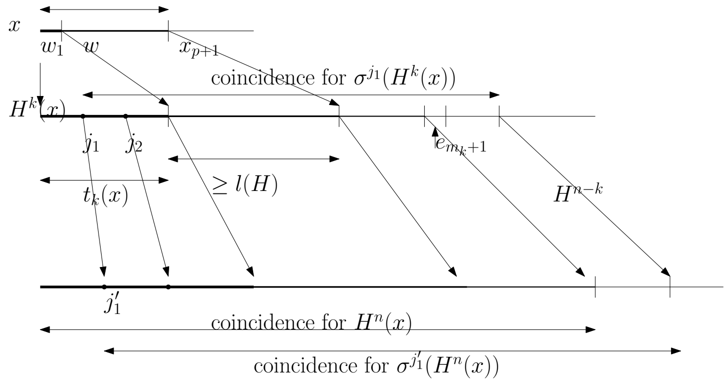

Figure 3.2. Renormalization by marked substitution H^{n-k} forces accidents.

contains $H^n(w_2 \dots w_p)$. We can thus repeat the same process to j_2 and more generally to each accident of $H^k(x)$ occurring before time $t_k(x)$.

Item (4) is a direct consequence of Corollary 3.5. $\square$

Corollary 3.8. — *Assume that $H^k(x)$ admits q accidents before $t_k(x)$. Let $W^1, \dots, W^q$ be the associated accident-words. Let $V^1, \dots, V^q$ be the associated depth-words.*

Then there exists positive constants $C_j = C_j(W^j)$ and $C'_j = C'_j(V^j)$, $1 \leqslant j \leqslant q$, and $\kappa > 0$ such that for every $n \geqslant k$,

- *The j^{th} accident times of $H^n(x)$ before $t_n(x)$, denoted $t_{j,n-k}$, fulfills*
$$|t_{j,n-k} - \lambda^{n-k} C_j| \leqslant \kappa\theta^{n-k}.$$
- *Its depth is denoted $\Delta_{j,n-k}$ and we obtain*
$$|\Delta_{j,n-k} - \lambda^{n-k} C'_j| \leqslant \kappa\theta^{n-k}.$$

Furthermore, the sequence (C_j) is increasing.

Proof. — We remind that these notations imply that the j^{th} accident time is $|W^j|$ and its depth is $|V^j|$. By Proposition 3.7 $H^n(x)$ admits exactly s accidents before $t_n(x)$. Furthermore, their times are $|H^{n-k}(W^j)|$ and their depths are $|H^{n-k}(V^j)|$. Then we use Inequality (2.1) for each W^j and each V^j. Since A is a positive matrix and W^j is a prefix of W^{j+1}, we conclude $C_j < C_{j+1}$. $\square$

Remark 3.9. — Up to a constant independant of i, C_i is equal to B_i and C'_i is equal to $|W^i| + |T^i| + |W^{i+1}|$, where $|T^i|$ is the length between the end of the bispecial word and the beginning of the next one (see Figure 2.2).

Thus $C_{i+1} - C_i$ is equal to b_{i+1}, and thus $C'_i - (C_{i+1} - C_i) = |W^i| + |T^i|$ is positive.

3.4. Proof of Theorem 1.5

3.4.1. Preliminary lemma

LEMMA 3.10. — *Let a, λ be some positive real numbers with $\lambda > 1$. Let f be a Lipschitz function defined on a neighborhood of $[0, a]$. Let $\phi : \mathbb{N} \to \mathbb{R}$ be a real sequence such that $|\phi(n)| \leqslant C\theta^n$ with $C > 0$ and $0 < \theta < \lambda$. We have*

$$\lim_{n\to+\infty} \frac{1}{\lambda^n} \sum_{k=0}^{[a\lambda^n]} f\left(\frac{k+\phi(n)}{\lambda^n}\right) = \int_0^a f(x)\mathrm{d}x.$$

Proof. — Let us denote S_n the sum $\frac{1}{\lambda^n}\sum_{k=0}^{[a\lambda^n]} f\left(\frac{k+\phi(n)}{\lambda^n}\right)$ and K the Lipschitz constant of the function f. We obtain

$$\begin{aligned}\left|S_n - \frac{1}{\lambda^n}\sum_{k=0}^{[a\lambda^n]} f\left(\frac{k}{\lambda^n}\right)\right| &\leqslant \frac{1}{\lambda^n}\sum_{k=0}^{[a\lambda^n]}\left|f\left(\frac{k+\phi(n)}{\lambda^n}\right) - f(\frac{k}{\lambda^n})\right| \\ &\leqslant \frac{1}{\lambda^n} a\lambda^n.K.\frac{|\phi(n)|}{\lambda^n} \leqslant Ka\frac{|\phi(n)|}{\lambda^n}.\end{aligned}$$

The upper bound converges to zero as n goes to infinity. The term $\frac{1}{\lambda^n}\sum_{k=0}^{[a\lambda^n]} f\left(\frac{k}{\lambda^n}\right)$ is a Riemann sum, thus we deduce the result. □

Remark 3.11. — The same type of proof works if f is an uniformly continuous function. It also holds if the sum is done up to $a\lambda^n + o(\lambda^n)$ instead of $a\lambda^n$.

3.4.2. Computation of $\lim_{m\to+\infty} \mathcal{R}^m\varphi$: the case $g \equiv 1$

We want to compute $\lim_{m\to+\infty} \mathcal{R}^m(\varphi)$. By Lemma 3.1 we have

$$\mathcal{R}^m\varphi(x) = \sum_{i=0}^{t_m(x)-1} \varphi\circ\sigma^i\circ H^m(x).$$

The potential φ has the following form $\varphi(x) = \frac{1}{p^\alpha}$ where $d(x, \mathbb{K}) = D^{-p}$.

First of all consider the case $\alpha = 1$. Since $\varphi(x) = \frac{1}{p}$ if $\delta(x) = p$, we obtain

$$\mathcal{R}^m\varphi(x) = \sum_{j=0}^{t_m(x)-1} \frac{1}{\delta_j^m(x)}.$$

We pick some $x \notin \mathbb{K}$ and reemploy notations from Corollary 3.8. Let $p = \delta(x)$ and k be such that $|H^k(x_2 \dots x_p)| > l(H)$.

Moreover, by Lemma 2.12

$$\delta_j^m(x) = \Delta_{i,m-k} - (j - t_{i,m-k}), \quad t_{i,m-k} \leqslant j < t_{i+1,m-k}$$

holds.

Recall that q is the number of accidents of $H^k(x)$, see Corollary 3.8. We split the sum $\sum_{j=0}^{t_m(x)-1}$ into the sums $\sum_{j=t_{i,m-k}}^{t_{i+1,m-k}-1}$ with the convention $t_{0,m-k} = 0$ and $t_{q+1,m-k} = t_m(x)$. To make notations consistent we also set $C_0 = 0$, $\Delta_0 = \delta_0^k(x)$ and $C_{q+1} = t_k(x) - 1$. Then we obtain the following, where q is the number of accidents of $H^k(x)$ before $t_k(x)$.

$$\begin{aligned}\mathcal{R}^m\varphi(x) = & \sum_{l=0}^{t_{1,m-k}-1} \frac{1}{\Delta_{0,m-k} - l} + \sum_{l=t_{1,m-k}}^{t_{2,m-k}-1} \frac{1}{\Delta_{1,m-k} - l + t_{1,m-k}} \\ & + \dots + \sum_{l=t_{q,m-m}}^{t_m(x)-1} \frac{1}{\Delta_{q,m-k} - l + t_{q,m-k}} \\ = & \sum_{i=0}^{q} \sum_{l=0}^{t_{i+1,m-k}-t_{i,m-k}-1} \frac{1}{\Delta_{i,m-k} - l}.\end{aligned}$$

By Corollary 3.8 we obtain

$$\lambda^{m-k}(C_{i+1}-C_i)-2\kappa\theta^{m-k} \leqslant |t_{i+1,m-k}-t_{i,m-k}| \leqslant \lambda^{m-k}(C_{i+1}-C_i)+2\kappa\theta^{m-k}$$
$$\lambda^{m-k}C_i' - \kappa\theta^{m-k} \leqslant \Delta_{i,m-k} \leqslant \lambda^{m-k}C_i' + \kappa\theta^{m-k}$$

The computation of the sums is made with Lemma 3.10 $a = C_{i+1} - C_i$, $f(x) = \frac{1}{C_i' - x}$ and Remark 3.9. We finally obtain

$$U(x) = \lim_{+\infty} \mathcal{R}^m\varphi(x) = \sum_{i=0}^{q} \log\left(\frac{C_i'}{C_i' - (C_{i+1} - C_i)}\right).$$

By the formula we deduce that U is locally constant, thus continuous: Note that this last quantity only depends on the distance between $H^k(x)$ and $\mathbb{K}$. If y coincides with x for a very long time, then $H^k(x)$ and $H^k(y)$ do coincide for a greater time (of order λ^k times the first coincidence time).

This later coincident time can be adjusted such that it is greater than all the accidents and depths for $\mathcal{B}_k(x)$. This means that for such a y, $\mathcal{B}_k(y) = \mathcal{B}_k(x)$.

To finish the proof of the continuity of U it remains to compute it close to $\mathbb{K}$. First, note that if $x \in \mathbb{K}$, then $U(x) = 0$. Moreover, if $\delta_0^0(x) > l(\mathbb{K})+1$, then the k in Proposition 3.7 is equal to 0. We remind that $\mathcal{B}_n(x)$ stands for the set of accident words for $H^n(x)$ that are lower or equal to $t_n(x) := |H^n(x_0)|$ if $x = x_0x_1\dots$. By definition, the set $\mathcal{B}_0(x)$ is empty. Therefore, in our case $\delta_0^0(x) > l(\mathbb{K}) + 1$, $\mathcal{B}_n(x)$ is empty for every n, which yields

$$U(x) = \log\left(\frac{\Delta_0}{\Delta_0 - t_0(x)}\right) = \log\left(\frac{\delta_0^0(x)}{\delta_0^0(x) - 1}\right),$$

because for every a, $|H^0(a)| = |a| = 1$. This shows that $U(x) \to 0$ if $d(x, \mathbb{K})$ goes to 0, and thus U is continuous on $\mathbb{K}$.

It remains to consider the cases $\alpha \neq 1$. The proof is simpler and is based on convergence of Riemann sums. In all the cases, the renormalization term to get a Riemann sum is $\lambda^{-\alpha(m-k)}$ and the sums have λ^{m-k} summands. For $\alpha > 1$, the renormalization term is too heavy and the sum goes to 0. For $\alpha < 1$ the renormalization term is too light and the sum goes to $+\infty$. We left the exact computations to the reader and refer to [4, 5] Section 3.3 and 4 for similar computations.

3.4.3. Limit for $\mathcal{R}^m\varphi(x)$. The general case

We consider φ of the form $\varphi(x) = \frac{g(x)}{p^\alpha}$ if $\delta(x) = p$ and with g a positive and continuous function. First, we emphasize that continuity and positiveness for g imply that g is bounded from above and from below away from zero. Therefore, the proof for $\alpha \neq 1$ is the same as for $g \equiv 1$. We can thus focus on $\alpha = 1$.

In that case we need to compute for $x \notin \mathbb{K}$

$$\mathcal{R}^m\varphi(x) = \sum_{j=0}^{t_m(x)-1} \frac{g \circ \sigma^j(H^m(x))}{\delta_j^m(x)}.$$

There are two main arguments to deal with these extra terms. First, we show that the terms $g\circ\sigma^j(H^n(x))$ can be exchanged by terms $g\circ\sigma^k(H^n(y_{k,j}))$ with $y_{k,j} \in \mathbb{K}$. Then, we use a technical lemma to show the convergence to the desired quantity.

Replacing $g \circ \sigma^j(H^n(x))$. — We reemploy notations from above. Let $j_1, \dots j_q$ the times of accidents for $H^k(x)$, We also set $j_0 = 0$ and $j_{q+1} =$

$t_k(x)-1$. Recall we have defined $t_{i,m-k}$ and $\Delta_{i,m-k}$. There exist points $y^0,\dots,y^q$ in $\mathbb{K}$ such that we have $d(\sigma^{j_i}(H^k(x)),y^i)=d(\sigma^{j_i}(H^k(x)),\mathbb{K})$. In other words, the y^i's are points in $\mathbb{K}$ and coincide with $\sigma^{j_i}(H^k(x))$ for exactly $\delta^k_{j_i}(x)$-digits.

Now, we refer the reader to Figure 3.3 for the next discussion. Note that Corollary 3.5 yields that for every $m\geqslant k$, for every $t_{i,m-k}\leqslant j<t_{i+1,m-k}$,

$$\delta^m_j(x)=d(\sigma^j(H^m(x)),\mathbb{K})=d(\sigma^j(H^m(x)),H^{m-k}(y^i)). \tag{3.2}$$

As H is 2-full, for every i, $\delta^k_{j_i}(x)\geqslant j_{i+1}-j_i+1$ (otherwise $j_{i+1}-1$ would be an accident) and then for $0\leqslant j\leqslant t_{i+1,m-k}-t_{i,m-k}$

$$d(\sigma^{t_{i,m-k}+j}(H^m(x)),\sigma^j(H^{m-k}(y^i)))=D^{-\Delta_{i,m-k}+j}\leqslant D^{-\lambda^{m-k}}, \tag{3.3}$$

where we use that accident $i+1$ arrives before the $\sigma^{t_{i,m-k}+j}(H^m(x))$ and the $\sigma^j(H^{m-k}(y^i))$ split (see overlapping in Figure 3.3).

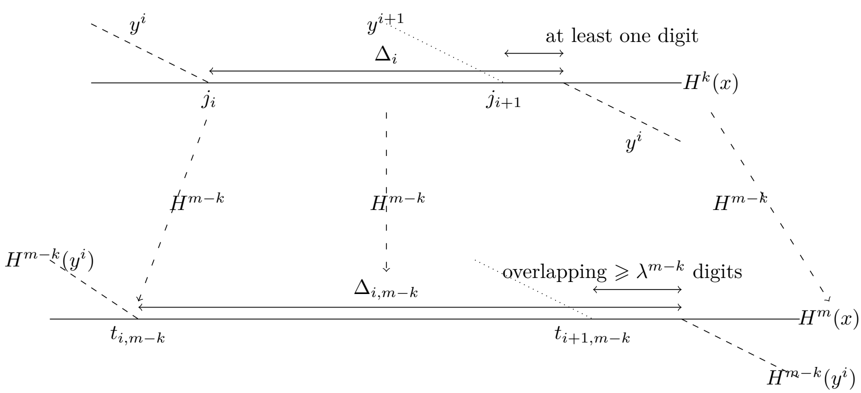

Figure 3.3. H^{m-k} renormalization

In other words, pieces of orbits $\sigma^{t_{i,m-k}+j}(H^m(x))$ and $\sigma^j(H^{m-k}(y^i))$ move away from each other as j goes from 0 to $t_{i+1,m-k}-t_{i,m-k}$, but the largest distance is of order $D^{-\lambda^{m-k}C'_i}$. This quantity goes to 0 if m goes to $+\infty$.

Furthermore, we remind that g is continuous thus uniformly continuous and positive. Hence, considering a modulus of continuity for g, replacing $g(\sigma^j(H^m(x)))$ by $g(\sigma^{j-t_{i,m-k}}(H^{m-k}(y^i)))$ for $t_{i,m-k}\leqslant j<t_{i+1,m-k}$ just add a multiplicative error term of order $(1+\varepsilon(m))$ with $\varepsilon(m)\to 0$ if m goes

to $+\infty$. More precisely we have

$$\begin{aligned}
\mathcal{R}^m\varphi(x) &= \sum_{j=0}^{t_m(x)-1} \frac{g\circ\sigma^j(H^m(x))}{\delta_j^m(x)}\\
&= \sum_{i=0}^{q}\sum_{l=0}^{t_{i+1,m-k}-t_{i,m-k}-1} \frac{g\circ\sigma^l\circ\sigma^{t_{i,m-k}}H^m(x)}{\Delta_{i,m-k}-l}\\
&= \sum_{i=0}^{q}\sum_{l=0}^{t_{i+1,m-k}-t_{i,m-k}-1} \frac{g\circ\sigma^l H^{m-k}(y^i)}{\Delta_{i,m-k}-l}(1+\varepsilon_{i,l}(m)).
\end{aligned}$$

We deduce two inequalities

$$(1-\varepsilon(m))\sum_{i=0}^{q}\sum_{l=0}^{t_{i+1,m-k}-t_{i,m-k}-1} \frac{g\circ\sigma^l H^{m-k}(y^i)}{\Delta_{i,m-k}-l} \leqslant \mathcal{R}^m\varphi(x)$$

$$\mathcal{R}^m\varphi(x) \leqslant (1+\varepsilon(m))\sum_{i=0}^{q}\sum_{l=0}^{t_{i+1,m-k}-t_{i,m-k}-1} \frac{g\circ\sigma^l H^{m-k}(y^i)}{\Delta_{i,m-k}-l}.$$

Therefore, the sandwich theorem shows that if the later term

$$\sum_{i=0}^{q}\sum_{l=0}^{t_{i+1,m-k}-t_{i,m-k}-1} \frac{g\circ\sigma^l H^{m-k}(y^i)}{\Delta_{i,m-k}-l}$$

converges as $m\to+\infty$, then $\mathcal{R}^m\varphi(x)$ does also converge to the same limit.

Now we need a technical lemma:

Lemma 3.12. — *Let (X,σ,μ) be an uniquely ergodic subshift. Let f be a continuous integrable function on $(0,1)$, let $g : X\to\mathbb{R}$ be a continuous function on X. Then we have uniformly in $x\in X$:*

$$\lim_{+\infty}\frac{1}{n}\sum_{k=0}^{n} f\left(\frac{k}{n}\right)g(\sigma^k x) = \int_0^1 f(t)\,\mathrm{d}t\int_X g\,\mathrm{d}\mu.$$

Proof. — Let us define $a_k = f(\frac{k}{n})$ and the Birkhoff sum $S_n(x) = \sum_{k=0}^{n-1} g(\sigma^k x)$ with $S_0=0$. Finally denote $X_n = \frac{1}{n}\sum_{k=0}^{n} f(\frac{k}{n})g(\sigma^k x)$. We have

$$X_n = \frac{1}{n}\sum_{k=0}^{n} a_k(S_{k+1}(x)-S_k(x)) = \frac{1}{n}\left[\sum_{k=1}^{n+1} a_{k-1}S_k(x) - \sum_{k=0}^{n} a_k S_k(x)\right]$$

$$X_n = \frac{1}{n}\sum_{k=1}^{n}(a_{k-1}-a_k)S_k(x) + \frac{a_nS_{n+1}(x)-a_0S_0}{n}.$$

Now by unique ergodicity we have $\lim_{n\to+\infty} \frac{S_n(x)}{n} = \int_X g(x)\mathrm{d}\mu$ uniformly in x. Thus for all $\varepsilon > 0$, there exists N such that for $n \geqslant N$ we have

$$S_n(x) = n \int_X g \,\mathrm{d}\mu + n\varepsilon(n), \quad |\varepsilon(n)| \leqslant \varepsilon. \tag{3.4}$$

First of all assume $f \in \mathcal{C}^1([0,1])$.

$$\begin{aligned}
X_n &= \frac{1}{n}\sum_{k=1}^{n}(a_{k-1}-a_k)S_k(x) + \frac{a_n S_{n+1}(x) - a_0 S_0}{n}, \\
&= \frac{1}{n}\sum_{k=1}^{n}(a_{k-1}-a_k)\left(k\int_X g\,\mathrm{d}\mu + k\varepsilon(k)\right) + \frac{a_n S_{n+1}(x) - a_0 S_0}{n}, \\
&= \frac{1}{n}\sum_{k=1}^{n-1} a_k \int_X g\,\mathrm{d}\mu - \frac{a_0 + n a_n}{n}\int_X g\,\mathrm{d}\mu \\
&\qquad\qquad + \frac{1}{n}\sum_{k=1}^{n}(a_{k-1}-a_k)k\varepsilon(k) + \frac{a_n S_{n+1}(x) - a_0 S_0}{n}, \\
&= \frac{1}{n}\sum_{k=1}^{n-1} a_k \int_X g\,\mathrm{d}\mu + \frac{1}{n}\sum_{k=1}^{n}(a_{k-1}-a_k)k\varepsilon(k) \\
&\qquad\qquad + a_n\left(\frac{S_{n+1}(x)}{n} - \int_X g\,\mathrm{d}\mu\right) - \frac{a_0 S_0}{n} - \frac{a_0}{n}\int_X g\,\mathrm{d}\mu.
\end{aligned}$$

Now by property of f, there exists c_k such that $a_k - a_{k-1} = \frac{f'(c_k)}{n}$

$$\begin{aligned}
X_n = \frac{1}{n}\sum_{k=1}^{n-1} a_k \int_X g\,\mathrm{d}\mu + \frac{1}{n^2}\sum_{k=1}^{n} f'(c_k)k\varepsilon(k) + a_n\left(\frac{S_{n+1}(x)}{n} - \int_X g\,\mathrm{d}\mu\right) \\
- \frac{a_0 S_0}{n} - \frac{a_0}{n}\int_X g\,\mathrm{d}\mu.
\end{aligned}$$

We deduce from (3.4) there exists two constants $C, C' > 0$ such that

$$\left|\frac{1}{n^2}\sum_{k=1}^{n} f'(c_k)k\varepsilon(k)\right| \leqslant \frac{1}{n^2}\sum_{k=1}^{N} Ck|\varepsilon(k)| + \frac{n(n-N)}{n^2}C\varepsilon \leqslant C'\varepsilon$$

Thus X_n converges to $\int_0^1 f(t)\,\mathrm{d}t \int_X g\,\mathrm{d}\mu$ uniformly in x.

Now if f is only a continuous function, it is a uniform limit of $\mathcal{C}^1$ functions. We apply the previous proof. □

With this lemma we can conclude

Corollary 3.13. — *We consider φ of the form $\varphi(x)=\frac{g(x)}{p^\alpha}$ if $\delta(x)=p$ and with g a positive and continuous function. Then we have for all $x\notin\mathbb{K}$*

$$\lim_{+\infty}\mathcal{R}^m\varphi(x)=\int_{\mathbb{K}}g\,\mathrm{d}\mu\times\sum_{i=0}^{q}\log\left(\frac{C_i'}{C_i'-(C_{i+1}-C_i)}\right).$$

Proof. — We apply the previous lemma to $H^n(x)$, which is possible due to the uniform convergence, and use the computation in the case $g\equiv 1$. □

Remark 3.14. — Note that in that case, continuity for U is an immediate consequence of continuity of U for $g\equiv 1$. Remark also that if $x\in\mathbb{K}$, then $U(x)=0$ by definition.

3.4.4. Back to 2-full assumption

We gave an example above (see page 208) where the substitution is not 2-full. We can now complete this example and check that for any m,

$$\mathcal{R}^m\varphi(x)=\sum_{k=1}^{|H^m(a)|-1}\frac{1}{|H^m(a)|-k},$$

which diverges.

We emphasize that the 2-full assumption is important to guaranty some fast convergence to $\mathbb{K}$ iterating H^m and taking the images by σ^j. For instance, we used the assumption in the previous proof to check that $\Delta_{i-j,i+1}$ is positive, which is a crucial point to exchange the $\sigma^j(H^m(x))$ by the $\sigma^j(H^{m-k}(y^i))$.

4. The Thue–Morse substitution: example with explicit computations

Consider the Thue–Morse substitution $H:\left\{\begin{smallmatrix}0\mapsto 01\\ 1\mapsto 10\end{smallmatrix}\right.$.

For this example we rephrase the proof of Theorem 1.5 and give an explicit form for the potential U.

Theorem 4.1. — *For the Thue–Morse substitution, there exists a unique function U such that, for all $x\in\mathcal{A}^{\mathbb{N}}$, we have $U(x)=\lim_m\mathcal{R}^m\varphi(x)$ for all potentials $\varphi:\mathcal{A}^{\mathbb{N}}\to\mathbb{R}$ satisfying $\varphi(x)=\frac{1}{p}+o(\frac{1}{p})$ if $d(x,K)=2^{-p}$. Moreover, if we set $p=\delta(x)$ we obtain*

$$U(x)=\begin{cases}\log(\frac{p}{p-1}) & p\geqslant 3\\ \frac{1}{2}\log\left(\frac{4}{3}\right) & p=2.\end{cases}$$

We will prove that the only accident is at time zero if $p > 2$, thus we can check that we have the same formula as in Corollary 3.13 with $q = 0, C'_i = p, g = 1$.

4.1. Technical lemmas

Lemma 4.2. — *The Thue–Morse substitution and its language $\mathcal{L}$ fulfill:*

- *The fixed point which begins by* 0 *can be written*
$$u = 0110100110010110100101\ldots$$
- *The language contains the words* $\begin{cases} 0, 1 \\ 00, 01, 10, 11 \\ 001, 010, 011, 100, 101, 110 \end{cases}$
- *H is 2-full and marked.*
- *The non uniquely desubstituable words of $\mathcal{L}$ are* 0*,* 1*,* 01*,* 10*,* 010*,* 101*,* 0101*,* 1010*.*
- *Every word of length at least* 5 *in $\mathcal{L}$ is uniquely desubstituable inside the language.*

Proof. — We refer to [9] and [6] for these classical results. □

Let x be an infinite word outside $\mathbb{K}$ which begins by a word w of the language. We can always *assume* that $x = w1\ldots$ (otherwise we exchange 0 and 1 by symmetry). We denote $x = w_0 \ldots w_{p-1}1\ldots$ where $p = \delta(x) \geqslant 2$. We obtain
$$H^n(x) = H^n(w_0)\ldots H^n(w_{p-1})H^n(1)\ldots$$
Let us consider two cases:

First case: $p \geqslant 3$. —

Proposition 4.3. — *For every infinite word x with $\delta(x) \geqslant 3$ we have*
$$\delta(\sigma^k \circ H^n(x)) = p2^n - k,$$
for all $k \in [0, 2^n - 1]$.

Proof. — We begin by the case $k = 0$: The substitution has constant length, thus the length of $H^n(w)$ is equal to $p2^n$, thus we have $\delta_0(x) \geqslant p\, 2^n$. Remark that $H^n(x) = H^{n-1}(H(w))H^n(1)\ldots$, The word $H(w)$ belongs to $\mathcal{L}$ and its length is equal to $2p > 4$. Assume $\delta_0 > p2^n$, then $H(w)1 \in \mathcal{L}$ by Lemma 4.2 since $H(1)$ begins by 1. We deduce $w1 \in \mathcal{L}$: this yields a contradiction. Thus we have $\delta_0^n = p2^n$.

Assume $1 \leqslant k \leqslant 2^{n-1} - 1$. Let us denote $H(w) = u_0 \ldots u_{2p-1}$. We have
$$\sigma^k(H^n(x)) = \sigma^k H^{n-1}(u_0).H^{n-1}(u_1 \ldots u_{2p-1})H^{n-1}(1)\ldots$$

First of all remark that $\sigma^k(H^n(x))$ begins with a strict suffix of $H^{n-1}(u_0)$. We know that $\delta(\sigma^k(H^n(x)) \geqslant p\ 2^n - k$.

Assume that the word $\sigma^k H^{n-1}(u_0).H^{n-1}(u_1 \dots u_{2p-1})1$ belongs to $\mathcal{L}$. Remark that the word $\sigma^k H^{n-1}(u_0)$ is non empty and that $p \geqslant 3$, thus we have $2p-1 \geqslant 5$. By last point of Lemma 4.2 we deduce that $w1$ belongs to the language: contradiction. Thus we obtain $\delta_k^n = p2^n - k$.

Now assume $k = 2^{n-1} + l$ with $0 \leqslant l < 2^{n-1}$, then we have

$$\sigma^k H^n(x) = \sigma^l(H^{n-1}(u_1)).H^{n-1}(u_2 \dots u_{2p-1})H^{n-1}(10)\dots$$

The shift acts at most on the image of u_1. We know $\delta_k^n \geqslant p\ 2^n - k$, and $|u_2 \dots u_{2p-1}| = 2p-2 > 3$. The same argument goes on: If $H^{n-1}(u_1 \dots u_{2p-1})1$ belongs to $\mathcal{L}$, the same is true for $u_1 u_2 \dots u_{2p-1}1$. It is equal to $u_1 H(w_1 \dots w_{p-1})1$, by Lemma 4.2 since $2p-1 \geqslant 3$. Thus it is the unique suffixe of $H(w_0 w_1 \dots w_{p-1})1$: contradiction. We deduce that $\delta_k^n = p2^n - k$. □

Second case: $p < 3$. —

First of all the case $p = 1$ is impossible, because the substitution is 2-full. By Lemma 4.2 the word w is not right special thus it is equal either to 11 or to 00. The word 001 belongs to $\mathcal{L}$, thus the only possibility is $w = 11$ (and $111 \notin \mathcal{L}$).

Proposition 4.4. — *Let x be an infinite word with $\delta(x) \leqslant 2$, we obtain*

$$\delta(\sigma^k \circ H^n(x)) = \begin{cases} 2.2^n - k & k < 2^{n-1} \\ 2^{n+1} - l & k = 2^{n-1} + l, 0 \leqslant l \leqslant 2^{n-1} - 1. \end{cases}$$

Thus there is an accident.

Proof. — The argument before the proof shows that $x = 111\dots$

First assume $k = 0$. We have

$$\begin{aligned} H^n(x) &= H^n(1)H^n(1)H^n(1)\dots \\ &= H^{n-1}(1010)H^{n-1}(10)\dots \end{aligned}$$

Remark that $\delta_0^n \geqslant 2.2^n$. Assume that $H^n(11)1$ belongs to $\mathcal{L}$. The word 1010 has length 4, we apply Lemma 4.2, we deduce that 10101 belongs to $\mathcal{L}$. Since $10101 = H(11)1$ we deduce that 111 belongs also to $\mathcal{L}$: contradiction. We have proved $\delta_0^n = 2.2^n = 2^{n+1}$.

Now assume $1 \leqslant k < 2^{n-1}$, then we have

$$\sigma^k H^n(x) = \sigma^k(H^{n-1}(1010))H^n(1)\dots$$

$$\sigma^k H^n(x) = \sigma^k[H^{n-1}(1)]H^{n-1}(010)H^n(1)\dots$$

We prove by contradiction that $\delta_k^n = 2^{n+1} - k$. Since $k < 2^{n-1}$ the last letter of $H^{n-1}(1)$ is not shifted by σ: we denote it a. The word $aH^{n-1}(010)1$ belongs to the language. Once again we apply Lemma 4.2, we deduce $\bar{a}0101 \in \mathcal{L}$ (with $\bar{a} = 1 - a$): contradiction whatever the value of a is.

Now assume $k = 2^{n-1}$. We obtain

$$\sigma^k H^n(x) = H^{n-1}(010)1\dots$$

The word 0101 belongs to the language, thus we obtain $\delta^n_{2^{n-1}} \geqslant 2^{n+1}$. There is an accident. Assume $\delta^n_{2^{n-1}} > 2^{n+1}$. This implies that $H^{n-1}(0101)0$ also belongs to $\mathcal{L}$, and the same for 01010: contradiction since $01010 = H(00)0 = 0H(11)$. Thus we have $\delta^n_{2^{n-1}} = 2^{n+1}$.

The last case is identical and left to the reader: For $k = 2^{n-1} + l$, we obtain $\delta_k^n = 2^{n+1} - l$. □

4.2. Proof of Theorem 4.1

Consider $\varphi(x) = \frac{1}{p} + o(1/p)$ with $d(x, \mathbb{K}) = 2^{-p}$.

- If $p \leqslant 2$ the last proposition shows:
$$\mathcal{R}^n\varphi(x) = 2\sum_{k=0}^{2^{n-1}-1} \frac{1}{2.2^n - k} = \frac{1}{2^{n-2}} \sum_{k=0}^{2^{n-1}-1} \frac{1}{4 - k/2^{n-1}}.$$
It converges to $\frac{1}{2}\int_0^1 \frac{\mathrm{d}x}{4-x} = \frac{1}{2}\log\left(\frac{4}{3}\right)$.
- If $p \geqslant 3$, then we deduce
$$\mathcal{R}^n\varphi(x) = \sum_{k=0}^{2^n-1} \frac{1}{p.2^n - k} = \frac{1}{2^n}\sum_{k=0}^{2^n-1} \frac{1}{p - k/2^n}.$$
It converges to $\log(\frac{p}{p-1})$.

Finally, with the notation $p = \delta(x)$, the limit is equal to:

$$U(x) = \begin{cases} \log(\frac{p}{p-1}) & p \geqslant 3 \\ \frac{1}{2}\log\left(\frac{4}{3}\right) & p = 2. \end{cases}$$

Bibliography

[1] A. Baraviera, R. Leplaideur & A. Lopes, “The potential point of view for renormalization”, *Stoch. Dyn.* **12** (2012), no. 4, article no. 1250005 (34 pages).

[2] M.-P. Béal, F. Mignosi, A. Restivo & M. Sciortino, “Forbidden words in symbolic dynamics”, *Adv. Appl. Math.* **25** (2000), no. 2, p. 163-193.

[3] V. Berthé, W. Steiner, J. M. Thuswaldner & R. Yassawi, "Recognizability for sequences of morphisms", *Ergodic Theory Dyn. Syst.* **39** (2019), no. 11, p. 2896-2931.

[4] H. Bruin & R. Leplaideur, "Renormalization, thermodynamic formalism and quasicrystals in subshifts", *Commun. Math. Phys.* **321** (2013), no. 1, p. 209-247.

[5] ———, "Renormalization, freezing phase transitions and Fibonacci quasicrystals", *Ann. Sci. Éc. Norm. Supér.* **48** (2015), no. 3, p. 739-763.

[6] J. Cassaigne, "Complexité et facteurs spéciaux", *Bull. Belg. Math. Soc. Simon Stevin* **4** (1997), no. 1, p. 67-88, Journées Montoises (Mons, 1994).

[7] F. Durand, B. Host & C. Skau, "Substitutional dynamical systems, Bratteli diagrams and dimension groups", *Ergodic Theory Dyn. Syst.* **19** (1999), no. 4, p. 953-993.

[8] J. Emme, "Thermodynamic formalism and k-bonacci substitutions", *Discrete Contin. Dyn. Syst.* **37** (2017), no. 7, p. 3701-3719.

[9] N. P. Fogg, V. Berthé, S. Ferenczi, C. Mauduit & A. Siegel (eds.), *Substitutions in dynamics, arithmetics and combinatorics*, Lecture Notes in Mathematics, vol. 1794, Springer, 2002, xviii+402 pages.

[10] K. Klouda & Š. Starosta, "Characterization of circular D0L-systems", *Theor. Comput. Sci.* **790** (2019), p. 131-137.

[11] B. Mossé, "Puissances de mots et reconnaissabilité des points fixes d'une substitution", *Theor. Comput. Sci.* **99** (1992), no. 2, p. 327-334.

[12] H. G. Schuster & W. Just, *Deterministic chaos. An introduction. 4th, revised and enlarged ed*, 4th, revised and enlarged ed. ed., Wiley-VCH, 2005, xxvi+287 pages.