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# Sufficient conditions for rescaling expansivity (*)

Arnoldo Rojas (1), Xiao Wen (2) and Yinong Yang (3)

**Abstract.** — We show that every k*-expansive vector field of a closed manifold is rescaling expansive. This improves the main result in [5]. The proof relies on the new notion of *singular-expansive flow* which will be studied.

**Résumé.** — Nous montrons que tout champ de vecteurs k*-expansif d'une variété fermée est rééchelonné expansif. Cela améliore le résultat principal dans [5]. La preuve repose sur la nouvelle notion de *flux singulier-expansif* qui sera étudiée.

## 1. Introduction

The *expansive flows* were introduced by Bowen and Walters [10] to generalize the corresponding notion for homeomorphisms [16]. The theory behind them though vast excludes important examples as the *geometric Lorenz attractor* [1, 11] and the *Cherry flow* [15]. Attempts to include them into the expansive theory have been given by some authors. For instance, Komuro [12] defined *k*-expansive flows* and proved that the geometric Lorenz attractor

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is k*-expansive. Oka [14] proved that k*-expansivity and expansivity are equivalent for nonsingular flows. Araujo et al. [2] extended Komuro's [12] by proving that every singular-hyperbolic attractor of three-dimensional differentiable flow is k*-expansive. Different authors prove interesting properties of k*-expansive flows, see [3, 4, 8]. More recently, Wen and Wen [17] defined *rescaling expansive flows* and proved that the multisingular hyperbolic flows [7] are rescaling expansive. Artigue [5] gave a sufficient condition for a k*-expansive flow to be rescaling expansive. The condition he found, called efficiency, is satisfied for instance when the fixed points of the flow are hyperbolic. Consequently, $C^1$ generic k*-expansive vector fields on closed manifolds are rescaling expansive.

In this paper we improve Artigue's [5] by showing that every k*-expansive vector field of a closed manifold is rescaling expansive. The proof relies on a new notion of expansivity for flows called *singular-expansivity*. We analyze the dynamics of singular-expansive flows on metric spaces. Indeed, we prove that the set of periodic orbits of a singular-expansive flow is countable and, if the singular set is dynamically isolated, then the set of periodic orbits of a prescribed period is finite. We show that there are singular-expansive flows with the shadowing property which are not expansive. The notion of singular-equicontinuous flows generalizing the classical notion of equicontinuous flow [6] is also considered. We proved that there are flows which are not equicontinuous but both singular-expansive and singular-equicontinuous. We show that the Bowen entropy of the nonsingular points vanishes for the singular-equicontinuous flows. Let us state our main result in a precise way.

Let $M$ denote a *closed manifold* i.e. a compact connected boundaryless manifold endowed with a Riemannian metric $\|\cdot\|$. In this section we let $d$ denote the distance in $M$ induced by $\|\cdot\|$. Denote by $B_a(x)$ the $a$-ball with center at $x$. The exponential map of $M$ will be denoted by exp.

Let $V$ be a vector field of $M$ (all vector fields will be assumed to be $C^1$). Denote by $V_t(x)$ the solution curve of the ODE $\dot{x} = V(x)$ with initial condition $x \in M$. This produces a one-parameter family of diffeomorphisms $\{V_t : M \to M\}_{t\in\mathbb{R}}$. We say that $\Lambda \subset M$ is *invariant* if $V_t(\Lambda) = \Lambda$ for every $t \in \mathbb{R}$.

Definition 1.1 ([17]). — *We say that $V$ is* rescaling expansive *on $\Lambda \subset M$ if for every $\epsilon > 0$ there is $\delta > 0$ such that if $x, y \in \Lambda$ satisfy*

$$d(V_t(x), V_{s(t)}(y)) \leqslant \delta \|V(V_t(x))\|$$

*for every $t \in \mathbb{R}$ and some increasing homeomorphism $s : \mathbb{R} \to \mathbb{R}$, then $V_{s(0)}(y) \in V_{[-\epsilon,\epsilon]}(x)$. If $V$ is rescaling expansive on $M$, we just say that $V$ is a* rescaling expansive vector field.

Actually this is not the original definition [17] but an equivalent one [18]. Next we recall the notion of k*-expansive vector field based on Komuro [12].

Definition 1.2. — *We say that $V$ is* k*-expansive *on $\Lambda$ if for every $\epsilon > 0$ there is $\delta > 0$ such that if $x, y \in \Lambda$ satisfy $d(V_t(x), V_{s(t)}(y)) \leqslant \delta$ for every $t \in \mathbb{R}$ and some increasing homeomorphism $s : \mathbb{R} \to \mathbb{R}$ fixing $0$, then $V_{s(t_0)}(y) \in V_{[t_0-\epsilon, t_0+\epsilon]}(x)$ for some $t_0 \in \mathbb{R}$. If $V$ is k*-expansive on $X$, we just say that $V$ is k*-expansive.*

With these definitions we can state our main result.

Theorem 1.3. — *Let $V$ be a vector field of a closed manifold $M$. If $V$ is k*-expansive on a compact invariant set $\Lambda \subset M$, then $V$ is rescaling expansive on $\Lambda$.*

The proof of this theorem is based on the following definition. Denote by $\mathrm{Sing}(V) = \{x \in X : V(x) = 0\}$ the singular set of a vector field $V$. The distance between $z \in M$ and $A \subset M$ is defined by $\mathrm{dist}(z, A) = \inf_{a \in A} d(z, a)$.

Definition 1.4. — *We say that $V$ is* singular-expansive on $\Lambda \subset M$ *if for every $\epsilon > 0$ there is $\delta > 0$ such that if $x, y \in \Lambda$ satisfy $d(V_t(x), V_{s(t)}(y)) \leqslant \delta\, \mathrm{dist}(V_t(x), \mathrm{Sing}(V))$ for every $t \in \mathbb{R}$ and some increasing homeomorphism $s : \mathbb{R} \to \mathbb{R}$, then $V_{s(t_0)}(y) \in V_{[t_0-\epsilon, t_0+\epsilon]}(x)$ for some $t_0 \in \mathbb{R}$. If $V$ is singular-expansive on $X$, we just say that $V$ is a* singular-expansive flow.

Theorem 1.3 is clearly a direct consequence of the following two propositions.

Proposition 1.5. — *Let $V$ be a vector field of a closed manifold $M$. If $V$ is k*-expansive on $\Lambda \subset M$, then $V$ is singular-expansive on $\Lambda$.*

Proposition 1.6. — *Let $V$ be a vector field of a closed manifold $M$. If $V$ is singular-expansive on a compact invariant set $\Lambda \subset M$, then $V$ is rescaling expansive on $\Lambda$.*

This paper is organized as follows. In Section 2 we will prove Proposition 1.6. In Section 3 we will extend the notion of singular-expansivity from vector fields on closed manifolds to flows on metric spaces and prove Proposition 1.5. In Section 4 we state some topological properties of the singular-expansive flows. These properties will be proved in Section 5.

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## 2. Proof of Proposition 1.6

We first recall a flowbox theorem given in [17]. Let $V$ be a $C^1$ vector field on a closed manifold $M$. Let $x \in M \setminus \mathrm{Sing}(V)$. Let

$$N_x(r) = \{v \in T_xM : v \perp X(x), \|v\| < r\},$$

$$U_x(r) = \{v + tX(x) : v \in N_x(r\|V(x)\|), t \in \mathbb{R}, |t| < r\}.$$

Denote by

$$F_x : U_x \to M, \qquad F_x(v + tX(x)) = \varphi_t(\exp_x(v)).$$

The *conorm* (or mininorm) of a linear operator $L$ is defined by

$$m(L) = \inf_{\|v\|=1} \|L(v)\|.$$

Lemma 2.1 ([17, Proposition 2.2]). — *Let $V$ be a $C^1$ vector field on $M$. There exists a constants $r_0 > 0$ such that for any $x \in M \setminus \mathrm{Sing}(V)$, $F_x : U_x(r_0) \to M$ is an embedding and $\|D_pF_x\| < 3$ and $m(D_pF_x) > 1/3$ for any $p \in U_x(r_0)$.*

Since $M$ is compact, there is $a > 0$ such that

$$\|D_p \exp_x\| < 3/2 \quad \text{and} \quad \|(D_p \exp_x)^{-1}\| < 3/2$$

for all $x \in M$ and $p \in T_xM$ with $\|p\| < a$. Assume $V$ is a $C^1$ vector field on $M$. Also by the compactness of $M$, we can find $L > 0$ such that for any $x \in M$, the vector field

$$\overline{V} = (\exp_x^{-1})_*(V|_{B_a(x)})$$

is Lipschitz with Lipschitz constant $L$. We call $L$ the *local Lipschitz constant* of $V$. The following is an easy lemma.

Lemma 2.2. — *Let $V$ be a $C^1$ vector field on $M$. There is $c > 0$ such that for any $x \in M \setminus \mathrm{Sing}(V)$, if $d(y, x) < c\|V(x)\|$, then*

$$\frac{1}{2}\|V(x)\| \leqslant \|V(y)\| \leqslant 2\|V(x)\|.$$

*Proof.* — Since there is an upper bound of $\|V(x)\|$ by the compactness of $M$, we can find $c > 0$ such that $c\|V(x)\| < a$ for all $x \in M$ at first. Let $L$ be a local Lipschitz constant of $V$, we choose $c > 0$ such that $c\|V(x)\| < a$ and $c \leqslant \frac{1}{4L}$. Then for any $x \in M \setminus \mathrm{Sing}(V)$ and any $y \in M$ with $d(x, y) < c\|V(x)\|$, denoting by $\overline{V} = (\exp_x^{-1})_*(V|_{B_a(x)})$, we have

$$\|\overline{V}(\exp_x^{-1}(y)) - \overline{V}(\exp_x^{-1}(x))\| \leqslant Ld(y, x) \leqslant Lc\|V(x)\| \leqslant \frac{1}{4}\|V(x)\|.$$

Note that $\overline{V}(\exp_x^{-1}(x)) = V(x)$. Thus we have

$$\frac{3}{4}\|V(x)\| \leqslant \|\overline{V}(\exp_x^{-1}(y))\| \leqslant \frac{5}{4}\|V(x)\|.$$

Hence

$$\|V(y)\| = \|(D_{exp_x^{-1}(y)} \exp_x)^{-1}(\overline{V}(\exp_x^{-1}(y)))\| \leqslant \frac{3}{2}\cdot\frac{5}{4}\|V(x)\| < 2\|V(x)\|,$$

$$\|V(y)\| = \|(D_{exp_x^{-1}(y)} \exp_x)^{-1}(\overline{V}(\exp_x^{-1}(y)))\| \geqslant \frac{2}{3}\cdot\frac{3}{4}\|V(x)\| \geqslant \frac{1}{2}\|V(x)\|.$$

This ends proof of the lemma. □

Lemma 2.3 ([17, Lemma 2.3]). — *For every $C^1$ vector field $V$ of a closed manifold $M$ there is $r_0 > 0$ such that*

(1) *if $x \in M \setminus \mathrm{Sing}(V)$, $0 < \delta < r_0/3$ and $t \in \mathbb{R}$ satisfy $V_{[0,t]}(x) \subset B(x, \delta\|V(x)\|)$, then $|t| < 3\delta$.*
(2) *if $x \in M\setminus\mathrm{Sing}(V)$, $0 < \delta < r_0/3$, $|t| < r_0$ and $V_t(x) \in B(x, \delta\|V(x)\|)$, then $|t| < 3\delta$.*

Note that the constants $r_0$ in Lemma 2.1 and Lemma 2.3 are same. The following Lemma follows the idea of Lemma 2.4 of [17].

Lemma 2.4. — *Let $V$ be a $C^1$ vector field on $M$ and $r_0$ be given as in Lemma 2.3. There is $\delta_0 > 0$ such that for any $x \in M \setminus \mathrm{Sing}(V)$, $y \in M$ and any increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$, if*

$$d(V_t(x), V_{h(t)}(y)) < \delta_0\|V(V_t(x))\|$$

*for all $t \in \mathbb{R}$, then $|h(t) - h(0) - t| < r_0/2$ for all $t \in (-r_0, r_0)$.*

*Proof.* — Let $L$ be a local Lipschitz constant of $V$ and $c$ be the constant given in Lemma 2.2. By the continuous dependence of solutions with respect to initial conditions, we know that $d(V_t(x), V_t(y)) \leqslant e^{L|t|}d(x, y)$ for any $x, y \in M$ and $t \in \mathbb{R}$. Then we know that

$$e^{-L|t|} \leqslant \frac{\|V(V_t(x))\|}{\|V(x)\|} \leqslant e^{L|t|}$$

for any $x \in M \setminus \mathrm{Sing}(V)$ and $t \in \mathbb{R}$.

Now we choose $\delta_0$ such that the following properties are satisfied:

$$e^{2Lr_0}\delta_0 < c,\quad 6(1 + 2e^{2Lr_0})\delta_0 < r_0/2,\quad 3(e^{Lr_0} + e^{2Lr_0})\delta_0 < r_0/2,$$
$$9(e^{Lr_0} + e^{2Lr_0})\delta_0 < c,\quad 6[3 + 9(e^{Lr_0} + e^{2Lr_0})]\delta_0 < r_0/2.$$

Assume that $x \in M\setminus\mathrm{Sing}(V)$, $y \in M$ and homeomorphism $h : \mathbb{R} \to \mathbb{R}$ satisfy

$$d(V_t(x), V_{h(t)}(y)) < \delta_0\|V(V_t(x))\|$$

for all $t \in \mathbb{R}$. We show that $|h(t) - h(0) - t| < r_0/2$ for any $t \in (-r_0, r_0)$.

Without loss of generality, we assume that $t \in (0, r_0)$. First we consider the case of $h(t) - h(0) \leqslant t$. In this case we have

$$\begin{aligned} d(V_{h(t)}(y), V_{t+h(0)}(y)) &\leqslant d(V_{h(t)}(y), V_t(x)) + d(V_t(x), V_{t+h(0)}(y)) \\ &< \delta_0 \|V(V_t(x))\| + e^{Lt} d(x, V_{h(0)}(y)) < \delta_0 \|V(V_t(x))\| + \delta_0 e^{Lt} \|V(x)\| \\ &\leqslant \delta_0 \|V(V_t(x))\| + \delta_0 e^{2Lt} \|V(V_t(x))\| < (1 + e^{2Lr_0}) \delta_0 \|V(V_t(x))\|. \end{aligned}$$

In the proof of above inequalities, we have already shown that

$$d(V_t(x), V_{t+h(0)}(y)) < \delta_0 e^{2Lt} \|V(V_t(x))\| < \delta_0 e^{2Lr_0} \|V(V_t(x))\|. \tag{2.1}$$

By the assumption $e^{2Lr_0}\delta_0 < c$ we know that $d(V_t(x), V_{t+h(0)}(y)) < c\|V(V_t(x))\|$, hence $\|V(V_t(x))\| \leqslant 2\|V(V_{t+h(0)}(y))\|$. Then we have

$$\begin{aligned} d(V_{h(t)}(y), V_{t+h(0)}(y)) &< (1 + e^{2Lr_0}) \delta_0 \|V(V_t(x))\| \\ &< 2(1 + e^{2Lr_0}) \delta_0 \|V(V_{t+h(0)}(y))\|. \end{aligned}$$

By the assumption that $6(1+2e^{2Lr_0})\delta_0 < r_0/2$ we know $2(1+e^{2Lr_0})\delta_0 < \frac{r_0}{3}$. Note that

$$-r_0 < -t < h(t) - (t + h(0)) \leqslant 0.$$

We can get that $|h(t) - (t + h(0))| < 6(1 + e^{2Lr_0})\delta_0 < r_0/2$ by the second item of Lemma 2.3.

Now let us consider the case of $h(t) - h(0) > t$. Since $h$ is an increasing homeomorphism, there exists $s \in (0, t)$ such that $h(s) = h(0) + t$. Then

$$\begin{aligned} d(V_s(x), V_t(x)) &\leqslant d(V_s(x), V_{h(0)+t}(y)) + d(V_{h(0)+t}(y), V_t(x)) \\ &= d(V_s(x), V_{h(s)}(y)) + d(V_{t+h(0)}(y), V_t(x)) \\ &\overset{(2.1)}{<} \delta_0 \|V(V_s(x))\| + \delta_0 e^{2Lt} \|V(V_t(x))\| \\ &\leqslant \delta_0 e^{L(t-s)} \|V(V_t(x))\| + \delta_0 e^{2Lt} \|V(V_t(x))\| \\ &< (e^{Lr_0} + e^{2Lr_0}) \delta_0 \|V(V_t(x))\|. \end{aligned}$$

By the assumption that $3(e^{Lr_0} + e^{2Lr_0})\delta_0 < r_0/2$ we know that $(e^{Lr_0} + e^{2Lr_0})\delta_0 < \frac{r_0}{3}$. Noting that $0 < t - s < t < r_0$, we have

$$t - s < 3(e^{Lr_0} + e^{2Lr_0})\delta_0 < r_0/2$$

from the second item of Lemma 2.3. By Lemma 2.1 we know that for any $\tau' \in [s, t]$

$$\begin{aligned} d(V_{\tau'}(x), V_t(x)) &= d(F_{V_t(x)}((\tau' - t)V(V_t(x))), F_{V_t(x)}(0)) \\ &< 3|\tau' - t|\|V(V_t(x))\| = 3(t - \tau')\|V(V_t(x))\| \\ &\leqslant 3(t - s)\|V(V_t(x))\| < 9(e^{Lr_0} + e^{2Lr_0})\delta_0 \|V(V_t(x))\|. \end{aligned}$$

By the assumption that $9(e^{Lr_0}+e^{2Lr_0})\delta_0 < c$ we know

$$\|V(V_{\tau'}(x))\| \leqslant 2\|V(V_t(x))\|.$$

For any $\tau \in [h(0)+t, h(t)] = [h(s), h(t)]$, we can find $\tau' \in [s,t]$ such that $h(\tau') = \tau$, then we have

$$\begin{aligned}
&d(V_\tau(y), V_{h(t)}(y))\\
&\quad\leqslant d(V_\tau(y), V_{\tau'}(x)) + d(V_{\tau'}(x), V_t(x)) + d(V_t(x), V_{h(t)}(y))\\
&\quad< \delta_0\|V(V_{\tau'}(x))\| + 9(e^{Lr_0}+e^{2Lr_0})\delta_0\|V(V_t(x))\| + \delta_0\|V(V_t(x))\|.\\
&\quad\leqslant 2\delta_0\|V(V_t(x))\| + 9(e^{Lr_0}+e^{2Lr_0})\delta_0\|V(V_t(x))\| + \delta_0\|V(V_t(x))\|\\
&\qquad\qquad\qquad = [3+9(e^{Lr_0}+e^{2Lr_0})]\delta_0\|V(V_t(x))\|.
\end{aligned}$$

Noting that $d(V_{h(t)}(y), V_t(x)) < \delta_0\|V(V_t(x))\| < c\|V(V_t(x))\|$, we have $\|V(V_t(x))\| \leqslant 2\|V(V_{h(t)}(y))\|$ from Lemma 2.2. Hence we have

$$d(V_\tau(y), V_{h(t)}(y)) < 2[3+9(e^{Lr_0}+e^{2Lr_0})]\delta_0\|V(V_{h(t)}(y))\|$$

for all $\tau \in [h(0)+t, h(t)]$. This means that

$$V_{[0,h(0)+t-h(t)]}(V_{h(t)}(y)) \subset B(V_{h(t)}(y), 2[3+9(e^{Lr_0}+e^{2Lr_0})]\delta_0\|V(V_{h(t)}(y))\|)$$

By the choice of $\delta_0$ such that $6[3+9(e^{Lr_0}+e^{2Lr_0})]\delta_0 < r_0/2$, we know $2[3+9(e^{Lr_0}+e^{2Lr_0})]\delta_0 < \frac{r_0}{3}$. By the first part of Lemma 2.3 we know $|h(t)-h(0)-t| < 6[3+9(e^{Lr_0}+e^{2Lr_0})]\delta_0 < r_0/2$. This ends the proof of the lemma. □

Lemma 2.5. — *Let $V$ be a $C^1$ vector field on $M$. There exists $\delta_0 < \frac{r_0}{3}$ such that for any $\delta \in (0,\delta_0)$ and any $x \in M \setminus \mathrm{Sing}(V)$ and any $y \in M$ with a increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$, if*

$$d(V_t(x), V_{h(t)}(y)) < \delta\|V(V_t(x))\|$$

*for all $t \in \mathbb{R}$ and $V_{h(0)}(y) \in V_{(-r_0,r_0)}(x)$, then*

$$V_{h(t)}(y) \in V_{(-3\delta,3\delta)}(V_t(x)) \subset V_{(-r_0,r_0)}(V_t(x))$$

*for any $t \in (-r_0, r_0)$.*

*Proof.* — Let $\delta_0$ be chosen as in Lemma 2.4. Without loss of generality, we can assume that $6\delta_0 < r_0$. Assume that there are $\delta \in (0,\delta_0)$ and $x \in M \setminus \mathrm{Sing}(V)$ and $y \in M$ with an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that

$$d(V_t(x), V_{h(t)}(y)) < \delta\|V(V_t(x))\|$$

for all $t \in \mathbb{R}$ and $V_{h(0)}(y) \in V_{(-r_0,r_0)}(x)$. Denote by $V_{t_0}(x) = V_{h(0)}(y)$. Note that $d(V_{h(0)}(y), x) < \delta\|V(x)\|$, hence by Lemma 2.1 we have

$$\|t_0 V(x)\| = \|F_x^{-1}(V_{t_0}(x))\| < 3d(V_{t_0}(x), x) < 3\delta\|V(x)\|,$$

thus $|t_0| < 3\delta$. Let any $t \in (-r_0, r_0)$ be given. It is easy to see that
$V_{h(t)}(y) = V_{h(t)-h(0)}(V_{h(0)}(y)) = V_{h(t)-h(0)}(V_{t_0}(x)) = V_{h(t)-h(0)-t+t_0}(V_t(x))$.
By Lemma 2.4 we know that $|h(t) - h(0) - t| < r_0/2$, hence we have $|h(t) - h(0) - t + t_0| < \frac{r_0}{2} + |t_0| < \frac{r_0}{2} + 3\delta_0 < r_0$. By the fact that

$$d(V_{h(t)}(y), V_t(x)) < \delta\|V(V_t(x))\|$$

we know $d(V_{h(t)-h(0)-t+t_0}(V_t(x)), V_t(x)) < \delta\|V(V_t(x))\|$. From the second part of Lemma 2.3 we know $|h(t) - h(0) - t + t_0| < 3\delta$. This says that

$$V_{h(t)}(y) = V_{h(t)-h(0)-t+t_0}(V_t(x)) \in V_{(-3\delta,3\delta)}(V_t(x))$$

and ends the proof of the lemma. □

Lemma 2.6. — *Let $V$ be a $C^1$ vector field on $M$. There exists $\delta_0$ such that for any $\delta \in (0, \delta_0)$ and any $x \in M \setminus \mathrm{Sing}(V)$ and any $y \in M$ with a increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$, if*

$$d(V_t(x), V_{h(t)}(y)) < \delta\|V(V_t(x))\|$$

*for all $t \in \mathbb{R}$ and $V_{h(0)}(y) \in V_{(-r_0,r_0)}(x)$, then*

$$V_{h(t)}(y) \in V_{(-3\delta,3\delta)}(V_t(x))$$

*for any $t \in \mathbb{R}$.*

*Proof.* — Let $\delta_0$ be the constant chosen in Lemma 2.5 and $\delta \in (0, \delta_0]$ be given. If $x \in M \setminus \mathrm{Sing}(V)$ and $y \in M$ with a increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ satisfy

$$d(V_t(x), V_{h(t)}(y)) < \delta\|V(V_t(x))\|$$

for all $t \in \mathbb{R}$ and $V_{h(0)}(y) \in V_{(-r_0,r_0)}(x)$. Then by Lemma 2.5 we know that

$$V_{h(\frac{r_0}{2})}(y) \in V_{(-r_0,r_0)}(V_{\frac{r_0}{2}}x).$$

Let $\widetilde{h}(t) = h(t + \frac{r_0}{2}) - h(\frac{r_0}{2}), \widetilde{x} = V_{\frac{r_0}{2}}x, \widetilde{y} = V_{h(\frac{r_0}{2})}(y)$, then

$$\begin{aligned} d(V_t(\widetilde{x}), V_{\tilde{h}(t)}(\widetilde{y})) &= d(V_{t+\frac{r_0}{2}}(x), V_{h(t+\frac{r_0}{2})}(y)) \\ &< \delta\|V(V_{t+\frac{r_0}{2}}(x))\| = \delta\|V(V_t(\widetilde{x}))\| \end{aligned}$$

for all $t \in \mathbb{R}$. And we also have $V_{\tilde{h}(0)}(\widetilde{y}) \in V_{(-r_0,r_0)}(\widetilde{x})$. Hence for any $t \in (-r_0, r_0)$ we have

$$V_{\tilde{h}(t)}(\widetilde{y}) \in V_{(-3\delta,3\delta)}(V_t(\widetilde{x})),$$

and then

$$V_{h(t+\frac{r_0}{2})}(y) \in V_{(-3\delta,3\delta)}(V_{t+\frac{r_0}{2}}(x)).$$

This prove that $V_{h(t)}(y) \in V_{(-3\delta,3\delta)}(V_t(x))$ for any $t \in [0, \frac{3}{2}r_0]$. Similarly we prove the same thing holds as $t \in [-\frac{3}{2}r_0, 0]$. By induction we can prove that for any $n \in \mathbb{N}$ we have $V_{h(t)}(y) \in V_{(-3\delta,3\delta)}(V_t(x))$ for any $t \in [-\frac{n}{2}r_0, \frac{n}{2}r_0]$. Hence $V_{h(t)}(y) \in V_{(-3\delta,3\delta)}(V_t(x))$ for all $t \in \mathbb{R}$. □

*Proof of Proposition 1.6.* — Let $V$ be a $C^1$ vector field and $\Lambda$ be a compact invariant set of $V$. Assume that $V$ is singular-expansive on $\Lambda$. Since $V$ is $C^1$ (in particular Lipschitz), there is $B > 0$ such that

$$\|V(z)\| \leqslant B \operatorname{dist}(z, \operatorname{Sing}(V)), \qquad \forall\, z \in M.$$

Fix $\epsilon > 0$. Without loss of generality, we assume that $\epsilon < r_0$. For this $\epsilon$ we let $\delta' > 0$ be given by the singular-expansivity of $V$ on $\Lambda$. Define $\delta = \min\{\frac{\delta'}{B}, \delta_0, \frac{\epsilon}{3}\}$ where $\delta_0$ was given as in Lemma 2.6. Suppose that

$$d(V_t(x), V_{s(t)}(y)) \leqslant \delta \|V(V_t(x))\|$$

for every $t \in \mathbb{R}$ and $x \in \Lambda \backslash \operatorname{Sing}(V)$, $y \in \Lambda$ and an increasing homeomorphism $s : \mathbb{R} \to \mathbb{R}$.

Then, $x, y \in \Lambda$ satisfy

$$\delta \|V(V_t(x))\| \leqslant \frac{\delta'}{B} \|V(V_t(x))\| \leqslant \delta' \operatorname{dist}(V_t(x), \operatorname{Sing}(V)), \qquad \forall\, t \in \mathbb{R}$$

and so

$$d(V_t(x), V_{s(t)}(y)) \leqslant \delta' \operatorname{dist}(V_t(x), \operatorname{Sing}(V)), \qquad \forall\, t \in \mathbb{R}.$$

It follows from singular expansivity that $V_{s(t_0)}(y) = V_\tau(V_{t_0}(x))$ for some $\tau \in [-\epsilon, \epsilon]$ and $t_0 \in \mathbb{R}$. Now let $\widetilde{x} = V_{t_0}(x), \widetilde{y} = V_{s(t_0)}(y), \widetilde{s}(t) = s(t + t_0) - s(t_0)$, then we have

$$d(V_t(\widetilde{x}), V_{\widetilde{s}(t)}(\widetilde{y})) = d(V_{t+t_0}(x), V_{s(t+t_0)}(y)) \leqslant \delta \|V(V_{t+t_0}(x))\| = \delta \|V(V_t(\widetilde{x}))\|,$$

and $V_{\widetilde{s}(0)}(\widetilde{y}) = V_\tau(\widetilde{x}) \in V_{(-r_0, r_0)}(\widetilde{x})$. By Lemma 2.6 we have

$$V_{\widetilde{s}(t)}(\widetilde{y}) \in V_{(-3\delta, 3\delta)}(V_t(\widetilde{x}))$$

for all $t \in \mathbb{R}$. Thus

$$V_{s(t)}(y) \in V_{(-3\delta, 3\delta)}(V_t(x)) \subset V_{[-\epsilon, \epsilon]}(V_t(x))$$

for all $t \in \mathbb{R}$. Therefore, $V$ is rescaling expansive on $\Lambda$ and the proof follows. □

## 3. Singular-expansive flows and Proof of Proposition 1.5

In this section we will transform the notion of singular-expansivity from vector fields on closed manifolds (Definition 1.4) to flows on compact metric spaces. In particular, we will prove Proposition 1.5. Previously, we present some basic definitions and facts.

Let $X$ be a metric space. A *flow* of $X$ is a continuous map $\phi : \mathbb{R} \times X \to X$ satisfying $\phi(0, x) = x$ and $\phi(t + s, x) = \phi(t, \phi(s, x))$ for every $x \in X$ and $t, s \in \mathbb{R}$. Denote by $\phi_t(x) = \phi(t, x)$ the time $t$-map. If $I \subset \mathbb{R}$ we denote

$\phi_I(x) = \{\phi_t(x) : t \in I\}$. We define $\phi_{[a,b]}(x) = \phi_{[b,a]}(x)$ for $a \geqslant b$. Recall that a *singularity* of a flow $\phi$ is a point $\sigma \in X$ such that $\phi_t(\sigma) = \sigma$ for every $t \in \mathbb{R}$. Denote by $\mathrm{Sing}(\phi)$ the set of singularities of $\phi$. Notice that the singularities of a vector field match with the singularities of its corresponding flow.

For the sake of comparison we will present the current notions of expansivity for flows on metric spaces. The first one is the classical notion of expansive flow by Bowen and Walters [10].

Definition 3.1. — *We say that $\phi$ is* expansive *on $\Lambda \subset X$ if for every $\epsilon > 0$ there is $\delta > 0$ such that if $x, y \in \Lambda$ satisfy $d(\phi_t(x), \phi_{s(t)}(y)) \leqslant \delta$ for every $t \in \mathbb{R}$ and some continuous map $s : \mathbb{R} \to \mathbb{R}$ fixing $0$, then $y \in \phi_{[-\epsilon,\epsilon]}(x)$. If $\phi$ is expansive on $X$, we just say that $\phi$ is expansive.*

The second is the notion of k*-expansive flow by Komuro [12].

Definition 3.2. — *We say that $\phi$ is* k*-expansive *on $\Lambda$ if for every $\epsilon > 0$ there is $\delta > 0$ such that if $x, y \in \Lambda$ satisfy $d(\phi_t(x), \phi_{s(t)}(y)) \leqslant \delta$ for every $t \in \mathbb{R}$ and some increasing homeomorphism $s : \mathbb{R} \to \mathbb{R}$ fixing $0$, then $\phi_{s(t_0)}(y) \in \phi_{[t_0-\epsilon, t_0+\epsilon]}(x)$ for some $t_0 \in \mathbb{R}$. If $\phi$ is k*-expansive on $X$, we just say that $\phi$ is k*-expansive.*

Now, by replacing the vector field $V$ by a general flow $\phi$ in Definition 1.4 we obtain the following definition.

Definition 3.3. — *We say that $\phi$ is* singular-expansive on $\Lambda$ *if for every $\epsilon > 0$ there is $\delta > 0$ such that if $x, y \in \Lambda$ satisfy $d(\phi_t(x), \phi_{s(t)}(y)) \leqslant \delta\, \mathrm{dist}(\phi_t(x), \mathrm{Sing}(\phi))$ for every $t \in \mathbb{R}$ and some increasing homeomorphism $s : \mathbb{R} \to \mathbb{R}$, then $\phi_{s(t_0)}(y) \in \phi_{[t_0-\epsilon, t_0+\epsilon]}(x)$ for some $t_0 \in \mathbb{R}$. If $\phi$ is singular-expansive on $X$, we just say that $\phi$ is a* singular-expansive flow.

We can therefore reformulate Proposition 1.6 by saying that if the flow of a vector field $V$ of a closed manifold $M$ is singular-expansive on $\Lambda \subset M$, then such a flow is also rescaling expansive on $\Lambda$. Therefore, the property of being singular-expansive is stronger than rescaling expansive.

On the other hand, there are singular-expansive flows which are not expansive: take for instance the trivial flow namely the one for which every point is a singularity (it is clearly singular-expansive but not expansive unless the space is finite). In particular, singular-expansive flows which are not k*-expansive also exist.

The next result asserts that every k*-expansive flow is singular-expansive.

Theorem 3.4. — *Let $\phi$ be a flow of a compact metric space $X$. If $\phi$ is k*-expansive on $\Lambda \subset X$, then $\phi$ is singular-expansive on $\Lambda$.*

*Proof.* — Fix $\epsilon > 0$ and let $\delta > 0$ be given by the k*-expansivity of $\phi$ on $\Lambda$. Take $x, y \in X$ and an increasing homeomorphism $s : \mathbb{R} \to \mathbb{R}$ such that $d(\phi_t(x), \phi_{s(t)}(y)) \leqslant \frac{\delta}{\operatorname{diam}(X)} \operatorname{dist}(\phi_t(x), \operatorname{Sing}(\phi))$ for every $t \in \mathbb{R}$. Since $\operatorname{dist}(z, \operatorname{Sing}(\phi)) \leqslant \operatorname{diam}(X)$ for every $z \in X$, we get $d(\phi_t(x), \phi_{s(t)}(y) \leqslant \delta$ for every $t \in \mathbb{R}$. Then, by k*-expansivity and the choice of $\delta$ we get $\phi_{s(t_0)}(y) \in \phi_{[t_0-\epsilon, t_0+\epsilon]}(x)$ proving the result. □

*Proof of Proposition 1.5.* — Just apply Theorem 3.4 to the flow generated by $V$. □

We finish this section by proving that both the rescaling and singular-expansivity are equivalent when the vector field is invertible at the singularities. More precisely, the following result holds.

THEOREM 3.5. — *Let $V$ be a $C^1$ vector field of a closed manifold $M$ such that $DV(\sigma)$ is invertible for every $\sigma \in \operatorname{Sing}(V)$. Then, $V$ is singular-expansive on $\Lambda \subset M$ if and only if it is rescaling expansive on $\Lambda$.*

*Proof.* — Clearly the following property implies that $V$ is singular-expansive on $\Lambda$:

(∗) For every $\epsilon > 0$ there is $\delta > 0$ such that if $x, y \in \Lambda$ satisfy
$$d(V_t(x), V_{s(t)}(y)) \leqslant \delta \operatorname{dist}(V_t(x), \operatorname{Sing}(V))$$
for every $t \in \mathbb{R}$ and some increasing homeomorphism $s : \mathbb{R} \to \mathbb{R}$, then $V_{s(0)}(y) \in V_{(-\epsilon,\epsilon)}(x)$.

On the other hand, by Proposition 1.6, the singular-expansivity of $V$ on $\Lambda$ implies the rescaling expansivity of $V$ on $\Lambda$. It remains to prove that the rescaling expansivity of $V$ on $\Lambda$ implies (∗).

First note that by Lemma 3.9 in [5] there is $C > 0$ such that
$$\operatorname{dist}(z, \operatorname{Sing}(V)) \leqslant C\|V(z)\|, \qquad \forall\, z \in M.$$
Fix $\epsilon > 0$ and let $\delta'$ be given by rescaling-expansivity. Let $\delta = \frac{\delta'}{C}$ and suppose that
$$d(V_t(x), V_{s(t)}(y)) \leqslant \delta \operatorname{dist}(V_t(x), \operatorname{Sing}(V))$$
for every $t \in \mathbb{R}$ and some increasing homeomorphism $s : \mathbb{R} \to \mathbb{R}$. Then,
$$\delta \operatorname{dist}(V_t(x), \operatorname{Sing}(V)) \leqslant \frac{\delta'}{C} \operatorname{dist}(V_t(x), \operatorname{Sing}(V)) \leqslant \delta' \|V(V_t(x))\|, \quad \forall\, t \in \mathbb{R}$$
hence $d(V_t(x), V_{s(t)}(y)) \leqslant \delta' \|V(V_t(x))\|$, for every $t \in \mathbb{R}$ and so $V_{s(0)}(y) = V_t(x)$ for some $t \in [-\epsilon, \epsilon]$. This completes the proof. □

These results motivate the study of the singular-expansive flows. In the next section we will present our results in this direction.

## 4. Properties of singular-expansive flows

In the sequel we will study the topological properties of the singular-expansive flows. This requires some notations.

Let $\phi$ be a flow of a compact metric space $X$. We say that $x \in X$ is a *periodic point* of $\phi$ if $x \notin \mathrm{Sing}(\phi)$ but there is $t > 0$ such that $\phi_t(x) = x$. The minimal of such $t's$ is called the *period* of $x$. We say that $x \in X$ is *nonwandering* if $U \cap (\bigcap_{t \geqslant T} \phi_t(U)) \neq \emptyset$ for every $T > 0$ and every neighborhood $U$ of $x$. The set of nonwandering points is denoted by $\Omega(\phi)$.

We say that $K \subset X$ is *invariant* if $\phi_t(K) = K$ for every $t \in \mathbb{R}$. In such a case we denote by $\phi|_K$ the flow restricted to $K$. We say that $K$ is *dynamically isolated* if there is a neighborhood $U$ of $K$ (called *isolated block*) such that

$$K = \bigcap_{t \in \mathbb{R}} \phi_t(U).$$

THEOREM 4.1. — *The following properties hold for every singular-expansive flow $\phi$ of a compact metric space $X$.*

(1) *The set of periodic orbits of $\phi$ is countable.*
(2) *If* $\mathrm{Sing}(\phi) = \emptyset$ *or consists of finitely many isolated points of $X$, then $\phi$ is expansive.*
(3) *If* $\mathrm{Sing}(\phi)$ *is dynamically isolated, the set of periodic orbits with period $\tau \in [0, t]$ is finite ($\forall\, t > 0$).*

Item (3) above is false without the hypothesis that $\mathrm{Sing}(\phi)$ is dynamically isolated. Based on Subsection 3.5 in [5] we obtain the following counterexample.

*Example 4.2.* — There is a compact metric space exhibiting a singular-expansive flow with infinitely many periodic orbits of period $2\pi$.

*Proof.* — Define

$$X = \{(0,0)\} \cup \bigcup_{n \in \mathbb{N}} \{z \in \mathbb{R}^2 : \|z\| = e^{-n}\}.$$

We have that $X$ is a compact metric space if equipped with the Euclidean metric. Consider the flow $\phi$ on $X$ obtained by restricting (to $X$) the flow of the vector field in $\mathbb{R}^2$ defined by $V(x, y) = (-y, x)$. Proposition 3.21 in [5] implies that $V$ is rescaling expansive on $X$. Since $DV(0, 0)$ is invertible, we conclude from Theorem 3.5 that $\phi$ singular-expansive. On the other hand, by direct integration we obtain

$$\phi_t(z) = (-y \sin t + x \cos t, x \sin t + y \cos y), \qquad \forall\, z = (x, y) \in X, \quad t \in \mathbb{R}.$$

Then, $X \setminus \mathrm{Sing}(\phi) = X \setminus \{(0,0)\}$ consists of infinitely many periodic orbits of period $2\pi$. □

On the other hand, we say that $\phi$ is *equicontinuous* when for every $\epsilon > 0$ there is $\delta > 0$ such that if $x, y \in X$ and $d(x,y) \leqslant \delta$, then $d(\phi_t(x), \phi_t(y)) \leqslant \epsilon$ for every $t \in \mathbb{R}$. The equicontinuous flows have been widely studied in the literature [6]. Their singular version are given below.

Definition 4.3. — *A flow $\phi$ of a metric space $X$ is* singular-equicontinuous *if for every $\epsilon > 0$ there is $\delta > 0$ such that if $x, y \in X$ and $d(x,y) \leqslant \delta \operatorname{dist}(x, \mathrm{Sing}(\phi))$, then $d(\phi_t(x), \phi_t(y)) \leqslant \epsilon$ for every $t \in \mathbb{R}$.*

Every equicontinuous flow $\phi$ on a compact metric space is singular-equicontinuous. Indeed, take $\epsilon > 0$ and let $\delta > 0$ be given by the equicontinuity of $\phi$. If $\delta' = \frac{\delta}{\mathrm{diam}(X)}$ and $d(x,y) \leqslant \delta' \operatorname{dist}(x, \mathrm{Sing}(\phi))$, then $d(x,y) \leqslant \delta$ and so $d(\phi_t(x), \phi_t(y)) \leqslant \epsilon$ for every $t \in \mathbb{R}$. Therefore, $\phi$ is singular-equicontinuous.

The converse is true if there are no singularities. More precisely, every singular-equicontinuous flow without singularities on a compact metric space is equicontinuous. It follows that flows which are both singular-expansive and singular-equicontinuous do exist (e.g. the trivial flow). A different example will be reported below.

First recall that if $\delta, T > 0$ a $(\delta, T)$-pseudo orbit of a flow $\phi$ is a sequence $(x_i, t_i)_{i\in\mathbb{Z}}$ formed by points $x_i \in X$ and times $t_i \in \mathbb{R}$ such that $t_i \geqslant T$ and $d(\phi_{t_i}(x_i), x_{i+1}) \leqslant \delta$ for every $i \in \mathbb{Z}$. Given $\epsilon > 0$ we define $Rep(\epsilon)$ as the set of increasing homeomorphism $s : \mathbb{R} \to \mathbb{R}$ such that

$$\left| \frac{s(t) - s(r)}{t - r} - 1 \right| \leqslant \epsilon \quad (\forall\, t \neq r).$$

We say that $(x_i, t_i)_{i\in\mathbb{Z}}$ can be *$\epsilon$-shadowed* if there are $x \in X$ and $s \in Rep(\epsilon)$ such that

$$d(\phi_{s(t)}(x), \phi_{t-S(i)}(x_i)) \leqslant \epsilon, \qquad \forall\, i \in \mathbb{Z}, \quad \forall\, S(i) \leqslant t < S(i+1)$$

where

$$S(i) = \begin{cases} t_0 + t_1 + \cdots + t_{i-1} & \text{if } i > 0, \\ 0 & \text{if } i = 0, \\ -t_1 - t_2 - \cdots - t_i & \text{if } i < 0. \end{cases}$$

Given $\Lambda \subset X$ we say that $\phi$ *has the shadowing property on* $\Lambda$ if for every $\epsilon > 0$ there is $\delta > 0$ such that every $(\delta, 1)$-pseudo orbit $(x_i, t_i)_{i\in\mathbb{Z}}$ *on* $\Lambda$ (i.e. $x_i \in \Lambda$ for $i \in \mathbb{Z}$) can be $\epsilon$-shadowed. If $\phi$ has the shadowing property on $X$ we just say that $\phi$ has the shadowing property. This definition was abbreviated to SPOTP (strong pseudo-orbit tracing property) in Komuro [13]. Examples of flows with this property are the Anosov ones [15].

Theorem 4.4. — *There is a compact metric space exhibiting a flow with the shadowing property which is singular-expansive, singular-equicontinuous but not equicontinuous.*

An interesting property of the equicontinuous flows is that they have zero topological entropy. Recall the topological entropy of a flow $\phi$ of a compact metric space $X$ defined by $h(\phi) = h(\phi, X)$ where

$$h(\phi, K) = \lim_{\epsilon\to 0} \limsup_{t\to\infty} \frac{1}{t} \ln r(t, \epsilon)$$

for $K \subset X$ compact, and $r(t, \epsilon)$ is the minimal cardinality of $F \subset K$ satisfying

$$K \subset \bigcup_{x\in F} \{y \in X : d(\phi_s(x), \phi_s(y)) \leqslant \epsilon,\ \forall\, 0 \leqslant s \leqslant t\}$$

($F$ is then called $(t, \epsilon)$-spanning set). We would like to prove the same property for the singular-equicontinuous flows. Instead, we will consider the *Bowen entropy* [9] of the nonsingular points namely

$$h^*(\phi) = \sup\{h(\phi, K) : K \subset X \setminus \mathrm{Sing}(\phi) \text{ is compact}\}$$

In general $h^*(\phi) \leqslant h(\phi)$ hence $h^*(\phi) = 0$ if $\phi$ is equicontinuous. Since $\phi$ restricted to $\mathrm{Sing}(\phi)$ is equicontinuous, $h(\phi, \mathrm{Sing}(\phi)) = 0$ we expect $h^*(\phi) = h(\phi)$. These facts motivate the result below.

Theorem 4.5. — *Let $\phi$ be a flow of compact metric spaces.*

(1) *If $\Omega(\phi) \setminus \mathrm{Sing}(\phi)$ is closed, then $h^*(\phi) = h(\phi)$.*
(2) *If $\phi$ is singular-equicontinuous, then $h^*(\phi) = 0$.*

## 5. Proof of Theorems 4.1 to 4.5

The next lemma is closely related to Lemma 2.2 in [5].

Lemma 5.1. — *Let $\phi$ be a flow of a compact metric space $X$. If $\mathrm{Sing}(\phi)$ is dynamically isolated, then there is $\beta_0 > 0$ such that $\mathrm{diam}(\phi_{\mathbb{R}}(x)) \geqslant \beta_0$ for every $x \in X \setminus \mathrm{Sing}(\phi)$.*

*Proof.* — Otherwise, there is a sequence $x_n \in X \setminus \mathrm{Sing}(\phi)$ such that $\mathrm{diam}(\phi_{\mathbb{R}}(x_n)) \to 0$ as $n \to \infty$. Let $U$ be an isolating block of $\mathrm{Sing}(\phi)$. Since $\mathrm{diam}(\phi_{\mathbb{R}}(x_n)) \to 0$ and $X$ is compact, we can assume that $x_n \to \sigma$ for some $\sigma \in \mathrm{Sing}(\phi)$. Then, $\phi_{\mathbb{R}}(x_n) \subset U$ for $n$ large so $x_n \in \bigcap_{t\in\mathbb{R}} \phi_t(U) = \mathrm{Sing}(\phi)$ for $n$ large which is absurd. This completes the proof. □

The next lemma proves that the set of singularities of a singular-expansive flow is dynamically isolated.

Lemma 5.2. — *Let $\phi$ be a flow of a compact metric space $X$. If $\phi$ is expansive on $X \setminus \mathrm{Sing}(\phi)$, then $\mathrm{Sing}(\phi)$ is dynamically isolated.*

*Proof.* — Suppose that $\mathrm{Sing}(\phi)$ is not dynamically isolated. We have three cases to be considered:

(1) There is a sequence of periodic orbits $\{\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots\}$ accumulating on $\mathrm{Sing}(\phi)$ whose periods are bounded away from 0.
(2) There is a sequence of periodic orbits $\{\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots\}$ accumulating on $\mathrm{Sing}(\phi)$ whose periods tent to 0 as $n \to \infty$.
(3) There is a neighborhood $U_0$ of $\mathrm{Sing}(\phi)$ such that no periodic orbit of $\phi$ is entirely contained in $U_0$.

In Case (1) we take $\epsilon_0 = \min\{\frac{\tau}{4}, 1\}$ where $\tau$ is a positive lower bound of the periods of the periodic orbits $\gamma_n$ $(n = 1, 2, \cdots)$. Fix $\delta > 0$. For this $\delta$ there exists a neighborhood $U_\delta$ of $\mathrm{Sing}(\phi)$ such that $d(\phi_t(z), z) < \delta$ for every $z \in U_\delta$ and every $t \in [0, 2]$. Since $\gamma_n$ accumulates on $\mathrm{Sing}(\phi)$, one has $\gamma_n \subset U_\delta$ for some $n \in \mathbb{N}$. Take $z \in \gamma_n$ and $y = \phi_{2\epsilon_0}(z)$. It follows from the choice of $\epsilon_0$ that $2\epsilon_0 \in [0, 2]$. Then, $d(\phi_t(z), \phi_t(y)) = d(\phi_t(z), \phi_{2\epsilon_0}(\phi_t(z))) < \delta$ for every $t \in \mathbb{R}$. Again the choice of $\epsilon_0$ implies $2\epsilon_0 < \tau$ hence $y \notin \phi_{[-\epsilon_0, \epsilon_0]}(z)$.

In Case (2) we take $\epsilon_0 = 1$. In this case have that $\gamma_n \to \sigma$ for some $\sigma \in \mathrm{Sing}(\phi)$. Given $\delta > 0$ we can choose two different periodic orbits $\gamma_n$ and $\gamma_m$ with $d_H(\gamma_n, \gamma_m) < \delta$ ($d_H$ here is the Hausdorff distance). Taking $z \in \gamma_n$ and $y \in \gamma_m$ we have $d(\phi_t(z), \phi_t(y)) < \delta$ for every $t \in \mathbb{R}$. Since $\gamma_n$ and $\gamma_m$ are different orbits, we also have $y \notin \phi_{[-\epsilon_0, \epsilon_0]}(x)$.

In Case (3) we take $\epsilon_0 = 1$ once more. Take $\delta > 0$. As before there is a neighborhood $U_\delta$ of $\mathrm{Sing}(\phi)$ such that $d(\phi_t(z), z) < \delta$ for every $z \in U_\delta$ and every $t \in [0, 2]$. Since $\mathrm{Sing}(\phi)$ is not dynamically isolated, there is $x \in U \setminus \mathrm{Sing}(\phi)$ such that the full orbit of $x$ contained in $U_\delta$. The assumption in Case (3) implies that the orbit of $x$ is not periodic. Take $y = \phi_2(x)$. Then, $d(\phi_t(x), \phi_t(y)) < \delta$ for every $t \in \mathbb{R}$. Since $x$ is not periodic, we have $y \notin \phi_{[-1,1]}(x)$.

Taking $h$ as the identity of $\mathbb{R}$ in all these cases we obtain that $\phi$ is not expansive on $X \setminus \mathrm{Sing}(\phi)$. This completes the proof. □

We also need the following lemma.

Lemma 5.3. — *The properties below hold for every flow $\phi$ of a compact metric space $X$:*

(1) *If $\phi$ is expansive on $X \setminus \mathrm{Sing}(\phi)$, then $\phi$ is singular-expansive.*
(2) *If $\phi$ is singular-expansive and $\mathrm{Sing}(\phi)$ is open, then $\phi$ is expansive on $X \setminus \mathrm{Sing}(\phi)$.*

*Proof.* — First we prove Item (1). By Lemma 5.2 we have that $\mathrm{Sing}(\phi)$ is dynamically isolated. Take an isolating block $U$ of $\mathrm{Sing}(\phi)$. Fix $\Delta > 0$ such that

$$\{z \in X : \mathrm{dist}(z, \mathrm{Sing}(\phi)) \leqslant \Delta\} \subset U. \tag{5.1}$$

Let $\epsilon > 0$ and set $\epsilon' = \min\{\epsilon, \Delta\}$. For this $\epsilon'$ we let $\delta' > 0$ be given by the expansivity of $\phi$ on $X \setminus \mathrm{Sing}(\phi)$. Define $\delta = \frac{\min\{\delta',\Delta\}}{\mathrm{diam}(X)}$ and let $x, y \in X$ such that

$$d(\phi_t(x), \phi_{s(t)}(y)) \leqslant \delta\, \mathrm{dist}(\phi_t(x), \mathrm{Sing}(\phi))$$

for every $t \in \mathbb{R}$ and some increasing homeomorphism $s : \mathbb{R} \to \mathbb{R}$. Define $\widehat{s}(t) = s(t) - s(0)$ for $t \in \mathbb{R}$ and $\widehat{y} = \phi_{s(0)}(y)$. Then, $\widehat{s} : \mathbb{R} \to \mathbb{R}$ is a continuous map fixing 0 such that

$$d(\phi_t(x), \phi_{\hat{s}(t)}(\widehat{y})) \leqslant \delta\, \mathrm{dist}(\phi_t(x), \mathrm{Sing}(\phi)), \qquad \forall\, t \in \mathbb{R}.$$

If $x \in \mathrm{Sing}(\phi)$, $\phi_t(x) = x \in \mathrm{Sing}(\phi)$ for $t \in \mathbb{R}$ hence $\widehat{y} = x$ and so $\phi_{s(0)}(y) = \phi_t(x)$ with $t = 0 \in [-\epsilon, \epsilon]$.

On the other hand,

$$\delta\, \mathrm{dist}(z, \mathrm{Sing}(\phi)) = \min\{\delta', \Delta\} \frac{\mathrm{dist}(z, \mathrm{Sing}(\phi))}{\mathrm{diam}(X)} \leqslant \min\{\delta', \Delta\}, \quad \forall\, z \in X.$$

Therefore,

$$d(\phi_t(x), \phi_{\hat{s}(t)}(\widehat{y})) \leqslant \min\{\delta', \Delta\}, \qquad \forall\, t \in \mathbb{R}.$$

If $y \in \mathrm{Sing}(\phi)$, then $d(\phi_t(x), y) \leqslant \Delta$ for every $t \in \mathbb{R}$. This and the inclusion (5.1) imply $x \in \bigcap_{t\in\mathbb{R}} \phi_t(U) = \mathrm{Sing}(\phi)$ and then $\phi_{s(0)}(y) = \phi_t(x)$ with $t \in [-\epsilon, \epsilon]$ as before. Therefore, we can assume $x, y \in X \setminus \mathrm{Sing}(\phi)$. Since

$$d(\phi_t(x), \phi_{\hat{s}(t)}(\widehat{y})) \leqslant \min\{\delta', \Delta\} \leqslant \delta', \qquad \forall\, t \in \mathbb{R},$$

we conclude from the expansivity on $X \setminus \mathrm{Sing}(\phi)$ that $\phi_{s(0)}(y) = \widehat{y} = \phi_t(x)$ for some $t \in [-\epsilon, \epsilon]$ proving Item (1).

To prove Item (2) we assume that $\phi$ is singular-expansive and that $\mathrm{Sing}(\phi)$ is open. Then, $X \setminus \mathrm{Sing}(\phi)$ is closed and $\phi$ has no singularities there so by Item (ii) of Theorem 1 in [10] we just need to consider increasing homeomorphisms fixing 0 to prove the expansivity of $\phi$ on $X \setminus \mathrm{Sing}(\phi)$. Since $\mathrm{Sing}(\phi)$ and $X \setminus \mathrm{Sing}(\phi)$ are closed disjoint, there is $c > 0$ such that $\inf\{\mathrm{dist}(z, \mathrm{Sing}(\phi)) : z \in X \setminus \mathrm{Sing}(\phi)\} \geqslant c$. Now, let $\epsilon > 0$ and consider $\delta'$ from the singular-expansivity of $\phi$ for this $\epsilon$. Define $\delta = \delta' c$ and take $x, y \in X \setminus \mathrm{Sing}(\phi)$ such that $d(\phi_t(x), \phi_{s(t)}(y)) \leqslant \delta$, for every $t \in \mathbb{R}$ and some increasing homeomorphism $s : \mathbb{R} \to \mathbb{R}$ fixing 0. Since $x \in X \setminus \mathrm{Sing}(\phi)$ which is invariant, $\phi_t(x) \in X \setminus \mathrm{Sing}(\phi)$ for every $t \in \mathbb{R}$. Then, $c \leqslant \mathrm{dist}(\phi_t(x), \mathrm{Sing}(\phi))$ for every $t \in \mathbb{R}$ and then

$$d(\phi_t(x), \phi_{s(t)}(y)) \leqslant \delta = \delta' c \leqslant \delta'\, \mathrm{dist}(\phi_t(x), \mathrm{Sing}(\phi)), \qquad \forall\, t \in \mathbb{R}.$$

Therefore, $y = \phi_{s(0)}(y) = \phi_t(x)$ for some $t \in [-\epsilon, \epsilon]$ proving that $\phi$ is expansive on $X \setminus \mathrm{Sing}(\phi)$. This completes the proof. □

*Example 5.4.* — It is natural to ask if we can remove the hypothesis that $\mathrm{Sing}(\phi)$ is open in Item (2) of Lemma 5.3. However, this is false and a counterexample is given by the geometric Lorenz attractor.

We also need the result below.

Lemma 5.5. — *Let $\phi$ be a singular-expansive flow of a compact metric space $X$. Then, $\phi$ is expansive on every nonsingular compact invariant set of $\phi$.*

*Proof.* — Let $\Lambda$ be a nonsingular compact invariant set of $\phi$. We assert that $\phi|_\Lambda$ is singular-expansive.

Fix $\epsilon > 0$ and let $\delta'$ be given by the singular-expansivity of $\phi$ for this $\epsilon$. Since $\Lambda$ is compact and nonsingular, there exists $\delta' > 0$ such that if $a, b \in \Lambda$ and $d(a, b) \leqslant \delta' \,\mathrm{diam}(X)$, then $d(a, b) \leqslant \delta \,\mathrm{dist}(a, \mathrm{Sing}(\phi))$.

Now suppose that $x, y \in \Lambda$ and $d(\phi_t(x), \phi_{s(t)}(y)) \leqslant \delta' \mathrm{dist}(\phi_t(x), \mathrm{Sing}(\phi|_\Lambda))$ for every $t \in \mathbb{R}$ and some increasing homeomorphism $s : \mathbb{R} \to \mathbb{R}$. Since $\phi$ is nonsingular, $\mathrm{Sing}(\phi|_\Lambda) = \emptyset$ hence $\mathrm{dist}(\phi_t(x), \mathrm{Sing}(\phi|_\Lambda)) = \mathrm{dist}(\phi_t(x), \emptyset) = \mathrm{diam}(X)$ for every $t \in \mathbb{R}$. Then, $d(\phi_t(x), \phi_{s(t)}(y)) \leqslant \delta' \,\mathrm{diam}(X)$ for every $t \in \mathbb{R}$. Taking $a = \phi_t(x)$ and $b = \phi_{s(t)}(y)$ we get $d(\phi_t(x), \phi_{s(t)}(y)) \leqslant \delta \,\mathrm{dist}(\phi_t(x), \mathrm{Sing}(\phi))$ for all $t \in \mathbb{R}$. and so $\phi_{s(t_0)}(y) \in \phi_{[t_0-\epsilon, t_0+\epsilon]}(x)$ for some $t_0 \in \mathbb{R}$ proving the assertion.

On the other hand, since $\Lambda$ is nonsingular, $\mathrm{Sing}(\phi|_\Lambda) = \emptyset$ and so $\mathrm{Sing}(\phi|_\Lambda)$ is open. Then, $\phi|_\Lambda$ is expansive by Lemma 5.3 proving the result. □

This corollary motivates the question if conversely every flow which is expansive on every nonsingular compact invariant set is singular-expansive. But the answer is negative by the following example.

*Example 5.6.* — There is a compact metric space exhibiting a flow which not singular-expansive but expansive on every nonsingular compact invariant set.

*Proof.* — Following the ideas of Example 4.2 we define

$$X = \{(0, 0)\} \cup \bigcup_{n \in \mathbb{N}} \{z \in \mathbb{R}^2 : \|z\| = n^{-1}\}.$$

Again $X$ is a compact metric space if equipped with the Euclidean metric. Once more we consider the flow $\phi$ on $X$ obtained by restricting that of the vector field in $\mathbb{R}^2$ defined by $V(x, y) = (-y, x)$ on $X$. As in Example 4.2 we have that

$$\phi_t(z) = (-y \sin t + x \cos t, x \sin t + y \cos y), \qquad \forall\, z = (x, y) \in X, \quad t \in \mathbb{R}.$$

Then, $\mathrm{Sing}(\phi) = \{(0,0)\}$ and $\phi_t$ is a linear isometry for every $t \in \mathbb{R}$. Now take $\epsilon = 1$. Define the sequences $z_n, z'_n \in X$ by $z_n = (\frac{1}{n}, 0)$ and $z'_n = (\frac{1}{n+1}, 0)$ for $n \in \mathbb{N}$. Notice that $z_n$ and $z'_n$ belong to different circles of $X$ and so their orbits are different. From this we obtain $z'_n \notin \phi_{[-1,1]}(z_n)$. On the other hand,

$$d(\phi_t(z_n), \phi_t(z'_n)) = \|z_n - z'_n\| = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \frac{1}{n^2},$$

and $\mathrm{dist}(\phi_t(z_n), \mathrm{Sing}(\phi)) = \|z_n - (0,0)\| = \frac{1}{n}$ so

$$d(\phi_t(z_n), \phi_t(z'_n)) < \frac{1}{n}\,\mathrm{dist}(\phi_t(z_n), \mathrm{Sing}(\phi)), \qquad \forall\, t \in \mathbb{R}.$$

Since $z_n$ and $z'_n$ belong to different circles, $\phi$ is not singular-expansive. Finally, since every nonsingular compact invariant set consists of finitely many periodic orbits, one has that $\phi$ is expansive on all such sets. This completes the proof. □

The lemma below will be used to prove Theorem 4.4.

Lemma 5.7. — *Let $\phi$ be a flow of a compact metric space $X$. If for every $\epsilon > 0$ there is $\delta > 0$ such that $B[x, \delta\,\mathrm{dist}(x, \mathrm{Sing}(\phi))] \subset \phi_{[-\epsilon,\epsilon]}(x)$ for every $x \in X$, then $\phi$ is both singular-expansive and singular-equicontinuous.*

*Proof.* — Let $\epsilon > 0$ and $\delta > 0$ be given by the condition. If $s : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism satisfying

$$d(\phi_t(x), \phi_{s(t)}(y)) \leqslant \delta\,\mathrm{dist}(\phi_t(x), \mathrm{Sing}(\phi))$$

for all $t \in \mathbb{R}$, then $\phi_{s(0)}(y) \in B(x, \delta\,\mathrm{dist}(x, \mathrm{Sing}(\phi)))$ and so $\phi_{s(0)}(y) \in \phi_{[-\epsilon,\epsilon]}(x)$ by the condition. Therefore, $\phi$ is singular-expansive.

To prove that $\phi$ is singular-equicontinuous, take $\epsilon > 0$ and $\eta > 0$ such that if $y = \phi_s(x)$ with $|s| \leqslant \eta$, then $d(\phi_t(x), \phi_t(y)) \leqslant \epsilon$ for every $t \in \mathbb{R}$ (cf. [10, p. 181]). For this $\eta$ we take $\delta > 0$ given by the condition. Therefore, if $d(x,y) \leqslant \delta\,\mathrm{dist}(x, \mathrm{Sing}(\phi))$, that is $y \in B[x, \delta\,\mathrm{dist}(x, \mathrm{Sing}(\phi))]$, then $y = \phi_s(x)$ for some $|s| \leqslant \eta$ thus $d(\phi_t(x), \phi_t(y)) \leqslant \epsilon$ for every $t \in \mathbb{R}$ proving the result. □

*Proof of Theorem 4.1.* — Let $\phi$ be a singular-expansive flow of a compact metric space $X$. Given $\delta > 0$ we denote by $U_\delta(\mathrm{Sing}(\phi))$ the open $\delta$-ball around $\mathrm{Sing}(\phi)$. Define

$$X_\delta = \bigcap_{t \in \mathbb{R}} \phi_t(X \setminus U_\delta(\mathrm{Sing}(\phi))). \tag{5.2}$$

It follows that $X_\delta$ is a compact invariant set without singularities of $\phi$. Since $\phi$ is singular-expansive, $\phi$ is expansive on $X_\delta$ by Lemma 5.5. On the other hand, as is well known [10], the set of periodic orbits of an expansive flow is countable. Since the periodic orbits of $\phi$ are contained in $\bigcup_{n\in\mathbb{N}} X_{\frac{1}{n}}$, we conclude that the set of periodic orbits of $X$ is countable. This proves Item (1).

To prove Item (2) we see that if $\mathrm{Sing}(\phi) = \emptyset$ or consists of finitely many isolated points of $X$, then $\mathrm{Sing}(\phi)$ is open. Therefore, $\phi$ is expansive on $X \setminus \mathrm{Sing}(\phi)$ by Item (2) of Lemma 5.3. Since $X \setminus \mathrm{Sing}(\phi)$ and $\mathrm{Sing}(\phi)$ are closed disjoint, we conclude that $\phi$ is expansive. This proves Item (2).

To prove Item (3) we further assume that $\mathrm{Sing}(\phi)$ is isolated. Fix $t > 0$ and suppose by contradiction that $\phi$ has infinitely many distinct periodic orbits $O_n$ with period $t_n \leqslant t$. If $inf_{n\in\mathbb{N}} \mathrm{dist}(O_n, \mathrm{Sing}(\phi)) > 0$, then $\bigcup_{n\in\mathbb{N}} O_n \subset X_\delta$ for some $\delta > 0$ which is a contradiction since $\phi$ is expansive on $X_\delta$ (see [10]). Then, we can assume that there is a sequence $x_n \in O_n$ and $\sigma \in \mathrm{Sing}(\phi)$ such that $x_n \to \sigma$. Since $\sigma \in \mathrm{Sing}(\phi)$ and the period of $O_n$ is bounded by $t$, we have that the whole $O_n \to \sigma$ with respect to the Hausdorff metric of compact subsets of $X$. In particular, $\mathrm{diam}(O_n) \to 0$ contradicting Lemma 5.1. This completes the proof. □

*Proof of Theorem 4.4.* — Let $X = [0, 1]$ be the unit interval endowed with the Euclidean metric. For every $\lambda \in \mathbb{R}$ we define $\phi : \mathbb{R} \times X \to X$ by

$$\phi_t(x) = \frac{xe^{\lambda t}}{1 + x(e^{\lambda t} - 1)}, \qquad \forall\, 0 \leqslant x \leqslant 1, t \in \mathbb{R}.$$

It is not difficult to see that $\phi$ is a flow of $X$. If $\lambda = 0$, then $\phi$ is the trivial flow. If $\lambda \neq 0$, then $\mathrm{Sing}(\phi) = \{0, 1\}$ and the remainder orbits go from 0 to 1 or viceversa depending on whether $\lambda > 0$ or $\lambda < 0$. Then, $\phi$ is Morse–Smale and so it has the shadowing property but is not equicontinuous. We shall prove that if $\lambda = 1$, then $\phi$ satisfies the condition in Lemma 5.7.

First note that $\mathrm{dist}(z, \mathrm{Sing}(\phi)) = \min\{z, 1-z\}$ for every $z \in X$. It follows that $\mathrm{dist}(z, \mathrm{Sing}(\phi)) = z$ (if $z \leqslant \frac{1}{2}$) or $1 - z$ (otherwise).

Take $\epsilon > 0$ and $0 < \delta < \frac{1}{2}$ such that

$$\ln\left(\frac{1+\delta}{1-\delta}\right) \leqslant \epsilon.$$

We will show that

$$|x - y| \leqslant \delta \min\{x, 1-x\} \quad \Rightarrow \quad y \in \phi_{[-\epsilon,\epsilon]}(x).$$

Notice that the left-hand side of the above implication is equivalent to

$$x - \delta \min\{x, 1-x\} \leqslant y \leqslant x + \delta \min\{x, 1-x\}.$$

Since $\delta < \frac{1}{2}$, one has $y \in X \setminus \mathrm{Sing}(\phi)$.

Assume $x \leqslant \frac{1}{2}$. It follows that

$$(1-\delta) \leqslant \frac{y}{x} \leqslant (1+\delta) \quad \text{and} \quad \frac{1-x}{1-x(1-\delta)} \leqslant \frac{1-x}{1-y} \leqslant \frac{1-x}{1-x(1+\delta)}.$$

Since $0 < x \leqslant \frac{1}{2}$, $0 < \frac{x}{1-x} \leqslant 1$ and then

$$\frac{1-x}{1-x(1-\delta)} = \frac{(1-x)}{(1-x)+x\delta} = \frac{1}{1+\left(\frac{x}{1-x}\right)\delta} \geqslant \frac{1}{1+\delta}.$$

Likewise,

$$\frac{1-x}{1-x(1+\delta)} \leqslant \frac{1}{1-\delta}$$

so

$$\frac{1}{1+\delta} \leqslant \frac{1-x}{1-y} \leqslant \frac{1}{1-\delta}$$

thus

$$\ln\left(\frac{1-\delta}{1+\delta}\right) \leqslant \ln\left(\frac{y}{x}\cdot\frac{1-x}{1-y}\right) \leqslant \ln\left(\frac{1+\delta}{1-\delta}\right).$$

On the other hand, it follows from the definition of $\phi$ that the equation $\phi_t(x) = y$ is solved by

$$t = \ln\left(\frac{y}{x}\cdot\frac{1-x}{1-y}\right).$$

Then, the choice of $\delta$ implies

$$-\epsilon \leqslant t \leqslant \epsilon$$

yielding $y \in \phi_{[-\epsilon,\epsilon]}(x)$ for $x \leqslant \frac{1}{2}$. Interchanging the roles of $x$ and $y$ above by $1-x$ and $1-y$ respectively we get $y \in \phi_{[-\epsilon,\epsilon]}(x)$ when $x > \frac{1}{2}$ too. Therefore, $\phi$ satisfies the condition in Lemma 5.7, and so, it is both singular-expansive and singular-equicontinuous. This completes the proof. ☐

*Proof of Theorem 4.5.* — First suppose that $\Omega(\phi) \setminus \mathrm{Sing}(\phi)$ is closed. Then, $K = \Omega(\phi) \setminus \mathrm{Sing}(\phi)$ is compact contained in $X \setminus \mathrm{Sing}(\phi)$ hence $h(\phi, K) \leqslant h^*(\phi)$. On the other hand, by well-known properties of the topological entropy ([9, p. 403]), $h(\phi) = h(\phi, X) = h(\phi, K \cup \mathrm{Sing}(\phi)) \leqslant \max\{h(\phi, K), h(\phi, \mathrm{Sing}(\phi))\} = h(\phi, K) \leqslant h^*(\phi)$. Since $h^*(\phi) \leqslant h(\phi)$, we are done.

Now suppose that $\phi$ is singular-equicontinuous. Take $K \subset X \setminus \mathrm{Sing}(\phi)$ compact and $\epsilon > 0$. For this $\epsilon$ let $\delta$ be given by the singular-equicontinuity of $\phi$. Since $K \cap \mathrm{Sing}(\phi) = \emptyset$, $\delta\,\mathrm{dist}(x, \mathrm{Sing}(\phi)) > 0$ for every $x \in K$. Then, since $K$ is compact, there is $F \subset K$ finite such that

$$K \subset \bigcup_{x\in F} B[x, \delta\,\mathrm{dist}(x, \mathrm{Sing}(\phi))].$$

Now take $y \in K$ and $t > 0$. Then,

$$d(x,y) \leqslant \delta \operatorname{dist}(x, \operatorname{Sing}(\phi))$$

for some $x \in F$, and so, by singular-equicontinuity,

$$d(\phi_l(x), \phi_l(y)) \leqslant \epsilon \qquad \forall\, 0 \leqslant l \leqslant t.$$

It follows that $F$ is $(t,\epsilon)$-spanning for every $t > 0$ thus $r(t,\epsilon) \leqslant \operatorname{car}(F)$ the cardinality of $F$ for every $t > 0$. Therefore,

$$\limsup_{t\to\infty} \frac{1}{t} \ln r(t,\epsilon) \leqslant \limsup_{t\to\infty} \frac{\ln \operatorname{car}(F)}{t} = 0, \qquad \forall\, \epsilon > 0.$$

Then, $h(\phi, K) = 0$ for every compact subset $K \subset X \setminus \operatorname{Sing}(\phi)$ so $h^*(\phi) = 0$. □

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