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Currents relative to a malnormal subgroup system (*)

Yassine Guerch [1]

Abstract. — This paper introduces a new topological space associated with a nonabelian free group F_n of rank n and a malnormal subgroup system $\mathcal{A}$ of F_n, called the space of currents relative to $\mathcal{A}$, which are F_n-invariant measures on an appropriate subspace of the double boundary of F_n. The extension from free factor systems as considered by Gupta to malnormal subgroup systems is necessary in order to fully study the growth under iteration of outer automorphisms of F_n, and requires the introduction of new techniques on cylinders. We in particular prove that currents associated with elements of F_n which are not contained in a conjugate of a subgroup of $\mathcal{A}$ are dense in the space of currents relative to $\mathcal{A}$.

Résumé. — Dans cet article, nous introduisons un nouvel espace topologique associé à un groupe libre non abélien F_n de rang n et à un système de sous-groupes malnormal $\mathcal{A}$ de F_n. Appelé espace des courants relatifs à $\mathcal{A}$, cet espace est constitué de mesures F_n-invariantes à support dans un sous-espace approprié du double bord de F_n. L'extension du cas des systèmes de facteurs libres considéré par Gupta au cas des systèmes de sous-groupes malnormaux est nécessaire afin d'étudier la croissance sous itération d'automorphismes extérieurs de F_n, et requiert l'introduction de nouvelles techniques sur les cylindres. Nous démontrons en particulier que l'ensemble des courants associés aux éléments de F_n qui ne sont contenus dans aucun conjugué de sous-groupes de $\mathcal{A}$ est dense dans l'espace des courants relatifs à $\mathcal{A}$.

1. Introduction

Let $n \geqslant 2$. This paper is the first of a sequence of papers where we study the exponential growth of elements of $\mathrm{Out}(F_n)$, the outer automorphism group of a nonabelian free group F_n of rank n. Let $[g]$ be the conjugacy class

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of a nontrivial element g of F_n, let $\phi \in \mathrm{Out}(F_n)$ and let $\Phi \in \mathrm{Aut}(F_n)$ be a representative of ϕ. We say that $[g]$ has *exponential growth under iterates of* ϕ if there exists a basis $\mathcal{B}$ of F_n such that the length of $[\Phi^n(g)]$ with respect to the word metric relative to $\mathcal{B}$ grows exponentially fast with n. It is known, using for instance the technology of relative train tracks (see [2]) that, otherwise, $[g]$ has polynomial growth under iterates of ϕ. Let $\mathrm{Poly}(\phi)$ be the set of conjugacy classes of elements of F_n whose growth under iteration of ϕ is polynomial. For a subgroup H of $\mathrm{Out}(F_n)$, let $\mathrm{Poly}(H) = \bigcap_{\phi \in H} \mathrm{Poly}(\phi)$. The aim of these three papers is to prove the following result.

Theorem 1.1 ([12]). — *Let $n \geqslant 3$ and let H be a subgroup of* $\mathrm{Out}(F_n)$*. There exists $\phi \in H$ such that* $\mathrm{Poly}(\phi) = \mathrm{Poly}(H)$.

Theorem 1.1 is proved using dynamical methods developed mainly in [13]. In the present article, we introduce the topological space associated with the dynamics. Informally, Theorem 1.1 shows that the exponential growth of a subgroup H of $\mathrm{Out}(F_n)$ is encaptured by the exponential growth of a single element of H. In this paper, we construct a space which is well-adapted for our considerations, the *space of currents relative to a malnormal subgroup system*. These relative currents are nonnegative F_n-invariant Radon measures on an appropriate subspace of the double boundary at infinity of F_n. Let $\phi \in \mathrm{Out}(F_n)$. When the malnormal subgroup system is appropriately chosen, this space has the property that its points corresponding to conjugacy classes of elements in $F_n - \mathrm{Poly}(\phi)$ are dense in it (see Theorem 1.2).

The space of currents that we construct in this paper builds on objects introduced for similar purposes. For instance, the study of the mapping class group $\mathrm{Mod}(S)$ of a connected, compact, oriented surface S has benefited from the study of the action of $\mathrm{Mod}(S)$ on the *space of geodesic currents* $\mathrm{Curr}(S)$, introduced by Ruelle and Sullivan in [22] (see also the work of Bonahon [4]). It is defined as the space of $\pi_1(S)$-invariant nonnegative Radon measures on the double boundary $\partial^2\widetilde{S}$ of a universal cover $\widetilde{S}$ of S, equipped with the weak-star topology. Considering the space of projective geodesic currents $\mathbb{P}\mathrm{Curr}(S)$, one can show that $\mathbb{P}\mathrm{Curr}(S)$ can be viewed as a completion of the currents associated with weighted nontrivial homotopy classes of closed curves on S. The space $\mathbb{P}\mathrm{Curr}(S)$ is well-adapted to the study of $\mathrm{Mod}(S)$. For instance, it can be used for counting closed geodesics whose length is bounded by a given constant when the surface S is equipped with a hyperbolic metric (see [10] for a survey). Concerning dynamical properties, a result of Thurston ([24], see also [25]) implies that pseudo-Anosov diffeomorphisms act with North-South dynamics on the space $\mathbb{P}\mathrm{Curr}(S)$: every pseudo-Anosov element $f \in \mathrm{Mod}(S)$ has exactly two fixed points in $\mathbb{P}\mathrm{Curr}(S)$ and any other nonfixed point in $\mathbb{P}\mathrm{Curr}(S)$ converges to one of the fixed points under positive or negative iterates of f. Moreover, this convergence can be made uniform on compact subsets of $\mathbb{P}\mathrm{Curr}(S)$ which do not contain the fixed points.

In the specific context of free groups, building on [3] for general hyperbolic groups, the space of currents $\mathrm{Curr}(F_n)$ was first studied by Martin [21]. It is defined as the space of F_n-invariant nonnegative Radon measure on the double boundary $\partial^2 F_n$ of F_n equipped with the weak-star topology. Martin showed that the set of currents associated with conjugacy classes of nontrivial elements of F_n is dense in the space $\mathbb{P}\mathrm{Curr}(F_n)$ of projective currents. Currents for free groups have also been studied in [8, 17, 18]. Similarly to pseudo-Anosov elements of $\mathrm{Mod}(S)$ on $\mathbb{P}\mathrm{Curr}(S)$, fully irreducible automorphisms of F_n and atoroidal automorphisms of F_n act with North-South type dynamics on $\mathbb{P}\mathrm{Curr}(F_n)$ (see [25, 26]).

Currents on free groups have also been studied in a relative context, more precisely, in the context of *free factor systems*. A free factor system $\mathcal{F}$ is a finite set of conjugacy classes $\mathcal{F} = \{[A_1], \dots, [A_k]\}$ of nontrivial subgroups $A_1, \dots, A_k$ of F_n such that there exists a subgroup B of F_n with $F_n = A_1 * \dots A_k * B$. Gupta [15] (see also Guirardel–Horbez [14]) introduced the space $\mathrm{Curr}(F_n, \mathcal{F})$ of currents relative to the free factor system $\mathcal{F}$. Relative currents are then F_n-invariant nonnegative Radon measures on a subspace of the double boundary of F_n which does not intersect the double boundary of any conjugate of A_i, equipped with the weak-star topology. Gupta [15] then showed that the set of currents associated with conjugacy classes of *nonperipheral elements of* F_n, that is, elements of F_n that do not belong to any conjugate of some A_i, is dense in $\mathbb{P}\mathrm{Curr}(F_n, \mathcal{F})$. She then showed that fully irreducible outer automorphisms relative to $\mathcal{F}$ act with a North-South type dynamics on $\mathbb{P}\mathrm{Curr}(F_n, \mathcal{F})$.

In order to study the purely exponential growth part of an outer automorphism of F_n, we need to consider currents relative to a class of subgroup systems which is larger than the class of free factor systems. Indeed, if $\phi \in \mathrm{Out}(F_n)$, the set of all maximal conjugacy classes of subgroups of F_n consisting of elements with polynomial growth under iterates of ϕ is not necessarily a free factor system. However, Levitt [20, Proposition 1.4] proved that this set is a *malnormal subgroup system*. A malnormal subgroup system $\mathcal{A}$ is a finite set of conjugacy classes $\mathcal{A} = \{[A_1], \dots, [A_k]\}$ of nontrivial subgroups of F_n such that, for every $i \in \{1, \dots, k\}$, the group A_i is malnormal and, for all subgroups B_1, B_2 of F_n with $[B_1], [B_2] \in \mathcal{A}$, if the intersection $B_1 \cap B_2$ is nontrivial, then $B_1 = B_2$. A free factor system is, in particular, a malnormal subgroup system but the converse does not hold (see Section 2).

Let $\mathcal{A} = \{[A_1], \ldots, [A_k]\}$ be a malnormal subgroup system. We define the space $\mathrm{Curr}(F_n, \mathcal{A})$ of *currents relative to* $\mathcal{A}$ as the space of F_n-invariant non-negative Radon measures on a natural space $\partial^2(F_n, \mathcal{A})$, the double boundary of F_n relative to $\mathcal{A}$, equipped with the weak-star topology. The space $\partial^2(F_n, \mathcal{A})$ is a subspace of $\partial^2 F_n$ which does not intersect the double boundary of any conjugate of A_i (see Section 2.4 for precise definitions). In this article, we prove the following result. An element of F_n is *non-$\mathcal{A}$-peripheral* if it is not contained in any conjugate of any A_i with $i \in \{1, \ldots, k\}$.

Theorem 1.2. — *Let $n \geqslant 3$ and let $\mathcal{A}$ be a malnormal subgroup system. The set of positive linear combinations of currents associated with conjugacy classes of non-$\mathcal{A}$-peripheral elements of F_n is dense in the space $\mathbb{P}\mathrm{Curr}(F_n, \mathcal{A})$ of projective currents relative to $\mathcal{A}$.*

Let $\phi \in \mathrm{Out}(F_n)$. If $\mathcal{A}$ is the set of conjugacy classes of maximal polynomial subgroups of ϕ, then Theorem 1.2 shows that the set of projective currents associated with exponentially growing elements of F_n under iterates of ϕ is dense in $\mathbb{P}\mathrm{Curr}(F_n, \mathcal{A})$. Therefore, the space $\mathbb{P}\mathrm{Curr}(F_n, \mathcal{A})$ is a natural topological space for the study of the action of ϕ on elements of F_n with exponential growth under iterates of ϕ. A subsequent paper [13] will then show that ϕ acts with North-South type dynamics on $\mathbb{P}\mathrm{Curr}(F_n, \mathcal{A})$. This North-South dynamics will be a central argument in the proof of Theorem 1.1.

We now give an outline of the proof of Theorem 1.2. The proof follows the one of a similar result in the context of currents relative to free factor systems due to Gupta [15]. However, in the case of free factor systems, the proof relies on the existence of an adapted free basis of F_n associated with the free factor system, which does not necessarily exist in the case of malnormal subgroup systems. Our new argument in order to overcome this difficulty is the description of a finite set of elements of F_n associated with a malnormal subgroup system and a free basis of F_n which completely determines whether an element of F_n is contained in a conjugate of a subgroup of the malnormal subgroup system or not (see Lemma 2.3).

Let $\mathcal{A}$ be a malnormal subgroup system and let $\mu \in \mathbb{P}\mathrm{Curr}(F_n, \mathcal{A})$. We first show that μ can be extended into a *signed measured current* $\widetilde{\mu}$ *on* F_n, that is an F_n-invariant Radon measure on $\partial^2 F_n$. Even though $\widetilde{\mu}$ might have negative values, we show that $\widetilde{\mu}$ can be chosen so that $\widetilde{\mu}$ gives positive value to sufficiently many Borel subsets of $\partial^2 F_n$. One can then use the density of currents associated with conjugacy classes of nontrivial elements of F_n in the space $\mathrm{Curr}(F_n)$ in order to conclude the proof.

To our knowledge, the objects we construct in this paper have not been studied or constructed for larger classes of groups, such as relatively hyperbolic groups and quasi-convex almost malnormal subgroups of hyperbolic groups. Nevertheless, the extension of our definitions to this context seems natural since a result of Bowditch [6, Theorem 7.11] shows that the group F_n is always hyperbolic relative to a malnormal subgroup system $\mathcal{A}$. But as we explain in Remark 2.8, the natural double boundary associated with a relative hyperbolic group will have less information than the boundary $\partial^2(F_n, \mathcal{A})$. Therefore, it would require new techniques to develop the notion of currents for relative hyperbolic groups or quasi-convex almost malnormal subgroups of hyperbolic groups.

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2. Malnormal subgroup systems

2.1. Malnormal subgroup systems

Let n be an integer greater than 1 and let F_n be a free group of rank n. In this section, we define, following Handel and Mosher [16, Section I.1.1.2], malnormal subgroup systems and study some of their properties.

A *subgroup system of* F_n is a finite (possibly empty) set $\mathcal{A}$ whose elements are conjugacy classes of nontrivial (that is distinct from $\{1\}$ and F_n) finite rank subgroups of F_n. Note that a subgroup system $\mathcal{A}$ is completely determined by the set of subgroups A of F_n such that $[A] \in \mathcal{A}$. There exists a preorder on the set of subgroup systems of F_n, where $\mathcal{A}_1 \leqslant \mathcal{A}_2$ if for every subgroup A_1 of F_n such that $[A_1] \in \mathcal{A}_1$, there exists a subgroup A_2 of F_n such that $[A_2] \in \mathcal{A}_2$ and A_1 is a subgroup of A_2. The *stabilizer in* $\mathrm{Out}(F_n)$ *of a subgroup system* $\mathcal{A}$, denoted by $\mathrm{Out}(F_n, \mathcal{A})$, is the set of all elements $\phi \in \mathrm{Out}(F_n)$ such that $\phi(\mathcal{A}) = \mathcal{A}$.

Recall that a subgroup A of F_n is *malnormal* if for every element $x \in F_n - A$, we have $xAx^{-1} \cap A = \{e\}$. A subgroup system $\mathcal{A}$ is said to be *malnormal* if every subgroup A of F_n such that $[A] \in \mathcal{A}$ is malnormal and, for any subgroups A_1, A_2 of F_n such that $[A_1], [A_2] \in \mathcal{A}$, if $A_1 \cap A_2$ is nontrivial then $A_1 = A_2$.

There are equivalent formulations of malnormality which we present now (see [16, Section I.1.1.2]). Let T be the Cayley graph of F_n with respect to some given free basis of F_n. For every subgroup A of F_n, let T_A be the minimal A-invariant subtree of T. Then a subgroup system $\mathcal{A}$ consisting in conjugacy classes of malnormal subgroups is malnormal if and only if there exists a finite constant $L > 0$ such that for any distinct subgroups A_1, A_2 of F_n such that $[A_1], [A_2] \in \mathcal{A}$, the diameter of the intersection $T_{A_1} \cap T_{A_2}$ is at most equal to L. Malnormality of a subgroup system $\mathcal{A}$ consisting in conjugacy classes of malnormal subgroups is also equivalent to the fact that, for any distinct subgroups A_1 and A_2 of F_n such that $[A_1], [A_2] \in \mathcal{A}$, we have $\partial_\infty T_{A_1} \cap \partial_\infty T_{A_2} = \varnothing$.

2.2. Properness at infinity

Let $\partial_\infty F_n$ be the Gromov boundary of F_n. Let $\mathcal{B}$ be a free basis of F_n and let T be the Cayley graph of F_n with respect to $\mathcal{B}$. For convenience, we suppose that $\mathcal{B}^{-1} = \mathcal{B}$. The boundary of T is naturally homeomorphic to $\partial_\infty F_n$. For an element $w \in F_n$, we denote by γ_w the path in T starting from e corresponding to the word w. We denote by $w^{+\infty}$ the element in $\partial_\infty F_n$ corresponding to the quasi-geodesic starting at e obtained by concatenating paths in T labeled by w.

Let A be a subgroup of F_n of finite rank. The inclusion $A \subseteq F_n$ induces an A-equivariant inclusion $\partial_\infty A \hookrightarrow \partial_\infty F_n$. Note that the F_n-orbit of the image of this map only depends on the conjugacy class of A in F_n.

Let $\mathcal{A}$ be a subgroup system of F_n. The subgroup system $\mathcal{A}$ is said to be *proper at infinity* if, for every element g of F_n and every subgroup A of F_n with $[A] \in \mathcal{A}$, we have $g^{+\infty} \in \partial_\infty A$ if and only if $g \in A$.

For the proof of Lemma 2.2 below, we need the following result. This is a particular case of a result, due to Swenson [23], valid for all quasi-convex subgroups A_1, A_2 of any word hyperbolic group.

Lemma 2.1 ([23, Theorem 13]). — *For any finitely generated subgroups A_1 and A_2 of F_n, we have*

$$\partial_\infty(A_1 \cap A_2) = \partial_\infty A_1 \cap \partial_\infty A_2.$$

A subgroup A of F_n is *root-closed* if for every $g \in F_n$ and every $k \in \mathbb{N}^*$ such that $g^k \in A$, we have $g \in A$.

Lemma 2.2. — *Let $\mathcal{A}$ be a subgroup system. The following are equivalent:*

(1) *the subgroup system $\mathcal{A}$ is proper at infinity;*
(2) *every subgroup A of F_n such that $[A] \in \mathcal{A}$ is root-closed.*

In particular, a malnormal subgroup system is proper at infinity.

Proof. — Suppose that $\mathcal{A}$ is proper at infinity and let A be a subgroup of F_n such that $[A] \in \mathcal{A}$. Let $g \in F_n$ and $k \in \mathbb{N}^*$ be such that $g^k \in A$. Let us prove that $g \in A$. Since $g^k \in A$, we see that $g^{+\infty} \in \partial_\infty A$. Since $\mathcal{A}$ is proper at infinity, we have $g \in A$. Hence A is root-closed.

Suppose now that every subgroup A of F_n such that $[A] \in \mathcal{A}$ is root-closed. Let $g \in F_n$ and let A be a subgroup of F_n such that $[A] \in \mathcal{A}$ and $g^{+\infty} \in \partial_\infty A$. By Lemma 2.1 applied to $\langle g \rangle$ and A, there exists $k \in \mathbb{N}^*$ such that $g^k \in A$. Since A is root-closed, we see that $g \in A$. Hence $\mathcal{A}$ is proper at infinity. This shows the equivalence.

Let $\mathcal{A}$ be a malnormal subgroup system and let A be a subgroup of F_n such that $[A] \in \mathcal{A}$. We prove that A is root-closed. Let $g \in F_n$ and let $k \in \mathbb{N}^*$ be such that $g^k \in A$. We claim that $g \in A$. Indeed, suppose towards a contradiction that $g \notin A$. Then $g^k = gg^kg^{-1}$ belongs to $A \cap gAg^{-1}$ which is equal to $\{e\}$, a contradiction. □

Let $\mathcal{A}$ be a malnormal subgroup system. An element $g \in F_n$ is $\mathcal{A}$-*peripheral* (or simply *peripheral* if there is no ambiguity) if it is trivial or conjugate into one of the subgroups of $\mathcal{A}$, and *non-$\mathcal{A}$-peripheral* otherwise. Note that, since $\mathcal{A} \neq \{[F_n]\}$, there always exists a non-$\mathcal{A}$-peripheral element. Since $\mathcal{A}$ is proper at infinity by Lemma 2.2, we see that an element g of F_n is $\mathcal{A}$-peripheral if and only if there exists a subgroup A of F_n such that $[A] \in \mathcal{A}$ and $g^{+\infty} \in \partial_\infty A$.

Let $\mathcal{A} = \{[A_1], \ldots, [A_r]\}$ be a malnormal subgroup system of F_n. For every element $i \in \{1, \ldots, r\}$, let T_{A_i} be the minimal A_i-invariant subtree of T. Suppose that for every $i \in \{1, \ldots, r\}$, the representative A_i of $[A_i]$ is chosen so that the tree T_{A_i} contains the base point e of T.

By malnormality of $\mathcal{A}$, there exists $L \in \mathbb{N}^*$ such that for any distinct subgroups A, B of F_n such that $[A], [B] \in \mathcal{A}$, the diameter of the intersection $T_A \cap T_B$ is at most L. Let $i \in \{1, \ldots, r\}$. Let Γ_i be the set of subgroups B of F_n such that there exists $g_B \in F_n$ such that $B = g_B A_i g_B^{-1}$ and the tree T_B contains the base point e of T. Note that, by malnormality of $\mathcal{A}$, for every $i \in \{1, \ldots, r\}$, the set Γ_i is finite. Let C_i be the set of elements w of F_n such that the length of γ_w is equal to $L+2$ and, for every $B \in \Gamma_i$, the path γ_w is not contained in T_B. Let $\mathscr{C} = \bigcap_{i=1}^r C_i$. Since we are looking at geodesic paths of length equal to $L+2$, the set $\mathscr{C}$ is finite. If γ is a path in T, the *element of* F_n *corresponding to* γ is the element $h \in F_n$ such that the path γ is labeled by h.

LEMMA 2.3. — *Let* $\mathcal{B}$, T, $\mathcal{A} = \{[A_1], \dots, [A_r]\}$, $L \in \mathbb{N}^*$, $\Gamma_1, \dots, \Gamma_r$, $\mathscr{C}$ *be as above. The finite set* $\mathscr{C} = \mathscr{C}(A_1, \dots, A_r)$ *is nonempty. Moreover, it satisfies the following:*

(1) *every element* $g \in F_n$ *such that the length of* γ_g *is at least equal to* $L+2$ *and such that* γ_g *is not contained in a tree* T_B *with* $B \in \bigcup_{i=1}^r \Gamma_i$ *contains an element of* $\mathscr{C}$ *as a subword. In particular, every non-*$\mathcal{A}$*-peripheral cyclically reduced element* $g \in F_n$ *has a power which contains an element of* $\mathscr{C}$ *as a subword;*

(2) *for every non-*$\mathcal{A}$*-peripheral cyclically reduced element* $g \in F_n$*, if* c_g *is the geodesic ray in* T *starting from* e *obtained by concatenating edge paths labeled by* g*, there exists an edge path in* c_g *labeled by a word in* $\mathscr{C}$ *at distance at most* $L+2$ *from* $\bigcup_{i=1}^r \bigcup_{B \in \Gamma_i} T_B$*;*

(3) *if an element* $w \in F_n$ *contains an element of* $\mathscr{C}$ *as a subword, then for every* $i \in \{1, \dots, r\}$*, the element* w *is not contained in* A_i*.*

Proof. — We first prove that (1) and (2) hold and that $\mathscr{C}$ is nonempty. Let g be as in the first claim of Assertion (1). First note that, by the choice of L, for every $i, j \in \{1, \dots, r\}$ and every distinct $A \in \Gamma_i$ and $B \in \Gamma_j$, the intersection $T_A \cap T_B$ is contained in the closed ball of radius L centered at e. We consider the geodesic path $c_g \colon [0,1] \to T$ such that $c(0) = e$ and such that $c_g(1)$ is the terminal endpoint of γ_g. Let

$$t_0 = \max \left\{ t \in [0,1] \mid c_g(t) \in \bigcup_{j=1}^r \bigcup_{A \in \Gamma_j} T_A \right\}.$$

The point $c_g(t_0)$ is a vertex and is distinct from $c_g(1)$ by assumption. We denote by $c_{\mathcal{A}}$ the geodesic segment $c_g \cap \bigcup_{j=1}^r \bigcup_{A \in \Gamma_j} T_A$. Observe that $c_{\mathcal{A}}$ is connected.

Suppose first that the length of $c_{\mathcal{A}}$ is at most equal to $L+1$. Let c_0 be the geodesic segment contained in c_g which originates at $c_g(t_0)$ and such that the length of $c_{\mathcal{A}}c_0$ is equal to $L+2$. Such a path $c_{\mathcal{A}}c_0$ exists since the length of γ_g is at least equal to $L+2$. Then the element h of F_n corresponding to $c_{\mathcal{A}}c_0$ is in $\mathscr{C}$ and is a subword of g. This concludes the proof in this case.

Suppose now that the length of $c_{\mathcal{A}}$ is greater than $L+1$. Let $c_{\mathcal{A}}(t_0 - L - 1)$ be the vertex in $c_{\mathcal{A}}$ at distance $L+1$ from $c_g(t_0)$, and let g_0 be the corresponding element of F_n. Let s_0 be the geodesic path between $c_{\mathcal{A}}(t_0 - L - 1)$ and $c_g(t_0)$. Since the geodesic path s_0 has length equal to $L+1$, there exist a unique $i \in \{1, \dots, r\}$ and a unique $A \in \Gamma_i$ such that s_0 is contained in T_A. Let e_0 be the edge in c_g which originates at $c_g(t_0)$.

Let $h \in F_n$ be the element corresponding to the edge path s_1 between $c_g(t_0 - L - 1)$ and the terminal point of e_0. We claim that $h \in \mathscr{C}$. Indeed, suppose towards a contradiction that $h \notin \mathscr{C}$. Then there exists $j \in \{1, \dots, r\}$ and $B \in \Gamma_j$ such that the edge path γ_h is contained in T_B. Since γ_h has length equal to $L+2$, the integer j and the subgroup B are unique. Remark that g_0^{-1} sends the geodesic path s_0 to the initial segment of length $L+1$ of γ_h. Since $g_0^{-1}s_0$ has length equal to $L+1$, the subgroup B is the unique element of $\bigcup_{\ell=1}^{r} \Gamma_\ell$ such that the tree T_B contains $g_0^{-1}s_0$. But s_0 is contained in T_A and the tree T_A is sent by g_0^{-1} to the tree $T_{g_0^{-1}Ag_0}$. Therefore, we see that $B = g_0^{-1}Ag_0$. But g_0^{-1} induces an isometry between T_A and $T_{g_0^{-1}Ag_0}$. Therefore, since s_1 is not contained in T_A, we see that $\gamma_h = g_0^{-1}s_1$ is not contained in $T_{g_0^{-1}Ag_0}$. This leads to a contradiction. Hence $h \in \mathscr{C}$ and h is a subword of g. This proves the first claim of Assertion (1).

We now prove the second claim of Assertion (1). Let g be a non-$\mathcal{A}$-peripheral cyclically reduced element of F_n. Let $c'_g \colon \mathbb{R}_+ \to T$ be the geodesic ray in T starting from e obtained by concatenating edge paths labeled by g. Recall that, for every $i \in \{1, \dots, r\}$, the set Γ_i is finite. Therefore, since g is nonperipheral and since $\mathcal{A}$ is proper at infinity by Lemma 2.2, the intersection of c'_g with $\bigcup_{i=1}^{r} \bigcup_{A \in \Gamma_i} T_A$ is compact. Hence there exists a power of g which satisfies the first claim of Assertion (1). This proves (1).

Moreover, the terminal endpoint of the path in c_g labeled by h which we have constructed is either at distance $L+2$ from e or is at distance at most 1 from $\bigcup_{i=1}^{r} \bigcup_{B \in \Gamma_i} T_B$. This proves (2). This also proves that $\mathscr{C}$ is nonempty as there exists a non-$\mathcal{A}$-peripheral element.

We now prove (3). Suppose towards a contradiction that there exist $i \in \{1, \dots, r\}$ and $a \in A_i$ such that a contains a word of $\mathscr{C}$ as a subword. Thus there exist $x \in \mathscr{C}$, $b, c \in F_n$ such that $a = bxc$ and the word bxc is reduced. Then since e is contained in T_{A_i}, the path γ_a is contained in T_{A_i}. But the element b^{-1} sends the tree T_{A_i} to the tree $T_{b^{-1}A_ib}$. Moreover, since T_{A_i} contains the vertex labeled by b, the tree $T_{b^{-1}A_ib}$ contains the base point e of T. But then $T_{b^{-1}A_ib}$ contains the geodesic segment γ_x. This contradicts the fact that $x \in \mathscr{C} \subseteq C_i$. This concludes the proof. $\square$

2.3. Examples of malnormal subgroup systems

Let n be an integer greater than 1 and let F_n be a free group of rank n. In this section, we give some examples of malnormal subgroup systems. The first one that we describe, following Handel and Mosher [16], is an $\mathbb{R}$-*vertex group system*. Let T be an $\mathbb{R}$-tree equipped with a minimal, isometric action of F_n for which no point or end of T is fixed by the whole group and with trivial arc stabilizers. A proper, nontrivial subgroup A of F_n is an $\mathbb{R}$-*vertex group of* T if there exists a point $x \in T$ such that $A = \mathrm{Stab}(x)$. Note that every free factor of F_n is an $\mathbb{R}$-vertex group of some simplicial tree. Every $\mathbb{R}$-vertex group has rank at most equal to n (see [11]).

The $\mathbb{R}$-*vertex group system of* T, denoted by $\mathcal{A}_T$, is the set consisting of all conjugacy classes of nontrivial point stabilizers in T. The set $\mathcal{A}_T$ is finite and its cardinality is bounded from above by a finite constant depending only on n (see [11]). Therefore the set $\mathcal{A}_T$ is a subgroup system. Note that every free factor system of F_n is an $\mathbb{R}$-vertex group system of some simplicial tree. However, there exist $\mathbb{R}$-vertex group systems which are not free factor systems. For example, let S be a compact connected oriented hyperbolic surface with one totally geodesic boundary component such that $\pi_1(S)$ is isomorphic to F_n. Let T be the $\mathbb{R}$-tree dual to the lift $\widetilde{\Lambda}$ to $\mathbb{H}_2$ of a measured geodesic lamination Λ without compact leaves on S. An identification of $\pi_1(S)$ with F_n induces an action of F_n on T which has trivial arc stabilizers. Moreover, the fundamental group of the connected component containing the boundary curve of S is the stabilizer of a point in T. Since the fundamental group of this connected component is not a free factor of F_n, this shows that $\mathcal{A}_T$ is not a free factor system. More generally, Handel and Mosher [16, Proposition 3.3] give general constructions of $\mathbb{R}$-vertex group systems which are not free factor systems.

Lemma 2.4 ([16, Lemma 3.1]). — *The subgroup system $\mathcal{A}_T$ is a malnormal subgroup system.*

Another example of malnormal subgroup systems is the following. An outer automorphism $\phi \in \mathrm{Out}(F_n)$ is *exponentially growing* if there exists $g \in F_n$ such that the length of the conjugacy class $[g]$ of g in F_n with respect to some basis of F_n grows exponentially fast under iteration of ϕ. If $\phi \in \mathrm{Out}(F_n)$ is not exponentially growing, then the length of the conjugacy class of every element of F_n is polynomially growing under iteration of ϕ and ϕ is said to be *polynomially growing*. One similarly says that an automorphism $\alpha \in \mathrm{Aut}(F_n)$ is exponentially growing or polynomially growing. Let $\phi \in \mathrm{Out}(F_n)$ be exponentially growing. A subgroup P of F_n is a *polynomial subgroup* of ϕ if there exist $k \in \mathbb{N}^*$ and a representative α of ϕ^k such that $\alpha(P) = P$ and $\alpha|_P$ is polynomially growing. By [20, Proposition 1.4], there exist finitely many conjugacy classes $[H_1], \dots, [H_k]$ of maximal polynomial subgroups of ϕ and the set $\mathcal{H} = \{[H_1], \dots, [H_k]\}$ is a malnormal subgroup system.

2.4. Double boundary of F_n relative to a malnormal subgroup system

In this section, we construct a boundary of F_n relative to a malnormal subgroup system. We follow a similar construction made by Gupta in [15, Section 3.1] in the case of the boundary relative to a free factor system.

The *double boundary of* F_n is the quotient topological space

$$\partial^2 F_n = (\partial_\infty F_n \times \partial_\infty F_n \setminus \Delta) / \sim,$$

where $\sim$ is the equivalence relation generated by the flip relation $(x, y) \sim (y, x)$ and Δ is the diagonal, endowed with the diagonal action of F_n. We denote by $\{x, y\}$ the equivalence class of (x, y).

Let $\mathcal{A} = \{[A_1], \ldots, [A_r]\}$ be a malnormal subgroup system of F_n. Let $\mathcal{B}$, T, $L \in \mathbb{N}^*$, $\Gamma_1, \ldots, \Gamma_r$, $\mathscr{C}$ be as above Lemma 2.3. The boundary of T is naturally homeomorphic to $\partial_\infty F_n$ and the set $\partial^2 F_n$ is then identified with the set of unoriented bi-infinite geodesics in T. Let γ be a finite geodesic path in T. The path γ determines a subset in $\partial^2 F_n$ called the *cylinder set of* γ, denoted by $C(\gamma)$, which consists in all unoriented bi-infinite geodesics in T that contain γ. Such cylinder sets form a basis for a topology on $\partial^2 F_n$, and in this topology, the cylinder sets are both open and compact, hence closed since $\partial^2 F_n$ is Hausdorff. The action of F_n on $\partial^2 F_n$ has a dense orbit.

Let A be a nontrivial subgroup of F_n of finite rank. The induced A-equivariant inclusion $\partial_\infty A \hookrightarrow \partial_\infty F_n$ induces an inclusion $\partial^2 A \hookrightarrow \partial^2 F_n$. Let

$$\partial^2 \mathcal{A} = \bigcup_{i=1}^{r} \bigcup_{g \in F_n} \partial^2 g A_i g^{-1}.$$

Let $\partial^2(F_n, \mathcal{A}) = \partial^2 F_n - \partial^2 \mathcal{A}$ be the *double boundary of* F_n *relative to* $\mathcal{A}$. This subset is invariant under the action of F_n on $\partial^2 F_n$ and inherits the subspace topology of $\partial^2 F_n$, denoted by τ.

LEMMA 2.5. — *Let* $\mathrm{Cyl}(\mathscr{C})$ *be the set of cylinder sets of the form* $C(\gamma)$*, where the element of* F_n *determined by the geodesic edge path* γ *contains an element of* $\mathscr{C}$ *as a subword. We have*

$$\partial^2(F_n, \mathcal{A}) = \bigcup_{C(\gamma) \in \mathrm{Cyl}(\mathscr{C})} C(\gamma).$$

In particular, the space $\partial^2(F_n, \mathcal{A})$ *is an open subset of* $\partial^2 F_n$.

Proof. — Let $y \in \partial^2(F_n, \mathcal{A})$. Let c be an oriented geodesic line c in T which belongs to the equivalence class y. Let v be a vertex of T contained in c and let g_0 be the corresponding element of F_n. Note that $\bigcup_{i=1}^{r} \bigcup_{B \in \Gamma_i} T_B$

is connected as every tree in the union contains the base point. Thus, the intersection of c with any translate of $\bigcup_{i=1}^{r}\bigcup_{B\in\Gamma_i} T_B$ is a subpath of c.

Suppose first that the intersection $c\cap g_0\left(\bigcup_{i=1}^{r}\bigcup_{B\in\Gamma_i} T_B\right)$ is either compact or a half-line. In particular, there exists a geodesic ray c' contained in c whose intersection with $g_0\left(\bigcup_{i=1}^{r}\bigcup_{B\in\Gamma_i} T_B\right)$ is a point v' (the starting point of c'). Let x be a vertex in c' at distance at least equal to $L+2$ from v' and such that the element $g\in F_n$ corresponding to the geodesic edge path between v and x pointing towards x is cyclically reduced. The vertex x exists by construction of v' and by the fact that c is a geodesic line. Note that x is not contained in $g_0\left(\bigcup_{i=1}^{r}\bigcup_{B\in\Gamma_i} T_B\right)$. Thus, the edge path γ_g is not contained in $\bigcup_{i=1}^{r}\bigcup_{B\in\Gamma_i} T_B$. By Lemma 2.3 (2), the word g contains a word of $\mathscr{C}$ as a subword. Then $y\in g_0C(\gamma_g)$, and $g_0C(\gamma_g)\in \mathrm{Cyl}(\mathscr{C})$.

Suppose now that the intersection $c\cap g_0\left(\bigcup_{i=1}^{r}\bigcup_{B\in\Gamma_i} T_B\right)$ is not compact or a half-line. Thus, c is contained in $g_0\left(\bigcup_{i=1}^{r}\bigcup_{B\in\Gamma_i} T_B\right)$. Since $y\in \partial^2(F_n,\mathcal{A})$, the path c cannot be contained in a single tree g_0T_B with $B\in \bigcup_{i=1}^{r}\Gamma_i$. Since, for every $B\in\bigcup_{i=1}^{r}\Gamma_i$, the tree g_0T_B contains v, by convexity of the trees, there exist two subgroups $A,B\in\bigcup_{i=1}^{r}\Gamma_i$ such that c is contained in $g_0T_A\cup g_0T_B$. By the definition of the constant L, the subgroups A and B are unique and the intersection $g_0T_A\cap g_0T_B$ has diameter at most equal to L. Let c_0 be the subpath of c of length $2L+2$ whose middle point is v and whose starting point is in g_0T_A and let g be the element of F_n corresponding to c_0. Let v' be the initial vertex of c_0 and let g' be the element of F_n associated with v'. Note that the intersection of c_0 with g_0T_A and g_0T_B has length at least equal to $L+1$. Up to considering a larger path c_0, we may suppose that g is cyclically reduced. We claim that g contains an element of $\mathscr{C}$ as a subword. Indeed, suppose towards a contradiction that g does not contain an element of $\mathscr{C}$ as a subword. By Lemma 2.3 (1), there exist $i\in\{1,\dots,r\}$ and $H\in\Gamma_i$ such that $\gamma_g\subseteq T_H$. But then $g'\gamma_g=c_0$ and is contained in $g'T_H$. Thus the diameter of the intersection $g'T_H$ with g_0T_A and g_0T_B is at least equal to $L+1$. By definition of L, this means that $g'T_H=g_0T_A=g_0T_B$. This means that $A=B$, a contradiction. Hence g contains an element of $\mathscr{C}$ as a subword. Thus we have $y\in g_0C(\gamma_g)$, with $g_0C(\gamma_g)\in\mathrm{Cyl}(\mathscr{C})$. Therefore, we see that

$$\partial^2(F_n,\mathcal{A})\subseteq \bigcup_{C(\gamma)\in\mathrm{Cyl}(\mathscr{C})} C(\gamma).$$

Conversely, let γ be a geodesic path in T such that $C(\gamma)\in\mathrm{Cyl}(\mathscr{C})$. Suppose towards a contradiction that there exists $y\in\partial^2\mathcal{A}$ such that $y\in C(\gamma)$. Thus, there exist elements $i\in\{1,\dots,r\}$, $g\in F_n$ and $a\in gA_ig^{-1}$ such that $\{a^{+\infty},a^{-\infty}\}\in C(\gamma)$. Therefore, we see that γ is a subpath of $T_{gA_ig^{-1}}$. Decompose γ as $\gamma=\tau_1\delta\tau_2$ where δ is labeled by a word w in $\mathscr{C}$.

Let v be the origin of δ and let h be the element of F_n corresponding to v. Then $h^{-1}T_{gA_ig^{-1}} = T_{h^{-1}gA_ig^{-1}h} \in \Gamma_i$ and contains γ_w with $w \in \mathscr{C}$, a contradiction. □

Note that Lemma 2.5 implies that we can define a topology on $\partial^2(F_n, \mathcal{A})$, denoted by τ', where cylinder sets in $\mathrm{Cyl}(\mathscr{C})$ generate the topology. Lemma 2.5 also implies that the two topologies τ and τ' are equal.

Since $\partial^2 F_n$ is locally compact and since $\partial^2(F_n, \mathcal{A})$ is an open subset of $\partial^2 F_n$ by Lemma 2.5, we have the following results.

Lemma 2.6. — *The space $\partial^2(F_n, \mathcal{A})$ is locally compact.*

Lemma 2.7. — *The action of F_n on $\partial^2(F_n, \mathcal{A})$ has a dense orbit.*

Proof. — Recall that there exists $g \in F_n$ such that the action of g on $\partial^2 F_n$ has a dense orbit. Let $x \in \partial^2 F_n$ be such that the orbit of x under iteration of g is dense in $\partial^2 F_n$. Since $\partial^2(F_n, \mathcal{A})$ is an open subset of $\partial^2 F_n$, there exists $m \in \mathbb{N}$ such that $g^m x \in \partial^2(F_n, \mathcal{A})$. Since $\partial^2 \mathcal{A}$ is invariant under the action of F_n, we see that $x \in \partial^2(F_n, \mathcal{A})$. Thus, the element g also acts on $\partial^2(F_n, \mathcal{A})$ with a dense orbit. □

Remark 2.8. — We now compare our definition with other natural constructions of double boundaries. The first one is to see the double boundary of F_n relative to a malnormal subgroup system as the double boundary of a Gromov hyperbolic space. Indeed, if $\mathcal{A} = \{[A_1], \dots, [A_r]\}$ is a malnormal subgroup system, by a result of Bowditch (see [6, Theorem 7.11]), the group F_n is hyperbolic relative to $\mathcal{A}$. In particular, there is a natural (that is well-defined up to quasi-isometry) proper geodesic Gromov hyperbolic space X on which F_n acts by isometries and such that the subgroups of F_n whose conjugacy classes are in $\mathcal{A}$ are precisely the maximal parabolic subgroups of the action of F_n on the Gromov-boundary of X (see [6] for a precise description of X). Thus a natural construction for another type of double boundary of F_n relative to $\mathcal{A}$ is to define it as the double boundary of X. This definition seems to extend to the more general case of relatively hyperbolic groups. However, the relative double boundary $\partial^2(F_n, \mathcal{A})$ has the advantage of being an open subset of $\partial^2 F_n$, so that one can use the cylinder sets of $\partial^2 F_n$ as a basis for the topology of $\partial^2(F_n, \mathcal{A})$. Moreover, the natural application from ∂F_n to ∂X sends the boundary of a parabolic subgroup to a point. Therefore, the relative double boundary $\partial^2(F_n, \mathcal{A})$ seems to contain more information about the geodesic lines whose endpoints are in the Gromov boundary of distinct parabolic subgroups.

Another candidate for the double boundary of the pair $(F_n, \mathcal{A})$ is the following. Let $\widehat{T}$ be the graph obtained from T by adding one vertex $v(gA)$ for every coset gA with A a subgroup of F_n such that $[A] \in \mathcal{A}$ and by adding an edge from $v(gA)$ to every vertex of T labeled by an element in gA. The graph $\widehat{T}$ is Gromov hyperbolic (see for instance [19, Proposition 2.6] or [6]) and the Gromov boundary of $\widehat{T}$ is homeomorphic to the space $\partial_\infty F_n - \bigcup_{i=1}^{r} \bigcup_{g \in F_n} \partial_\infty gA_i$ (see for instance [1, Theorem 1.6] or [6, 9]). However, the double boundary $\partial^2 \widehat{T}$ does not contain any geodesic line whose endpoints are in distinct parabolic subgroups, which makes it a proper subspace of $\partial^2(F_n, \mathcal{A})$ which does not seem to be a union of cylinder sets.

3. Currents relative to a malnormal subgroup system

In this section, we define currents of F_n relative to a malnormal subgroup system. We follow the construction of Gupta [15, Section 3.2] of currents relative to a free factor system.

Let $\mathcal{A} = \{[A_1], \dots, [A_r]\}$ be a malnormal subgroup system of F_n. Let $\mathcal{B}$, T, $L \in \mathbb{N}^*$, $\Gamma_1, \dots, \Gamma_r$, $\mathscr{C}$ be as above Lemma 2.3.

A *relative current on* $(F_n, \mathcal{A})$ is an F_n-invariant nonnegative Radon measure μ on the locally compact space (by Lemma 2.6) $\partial^2(F_n, \mathcal{A})$ (that is μ gives finite measure to compact subsets of $\partial^2(F_n, \mathcal{A})$, is inner and outer regular). The set $\mathrm{Curr}(F_n, \mathcal{A})$ of all relative currents on $\partial^2(F_n, \mathcal{A})$ is equipped with the weak-star topology: a sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\mathrm{Curr}(F_n, \mathcal{A})^{\mathbb{N}}$ converges to a current $\mu \in \mathrm{Curr}(F_n, \mathcal{A})$ if and only if for every Borel subset $B \subseteq \partial^2(F_N, \mathcal{A})$ such that $\mu(\partial B) = 0$ (where ∂B is the topological boundary of B), the sequence $(\mu_n(B))_{n \in \mathbb{N}}$ converges to $\mu(B)$. The space $\mathrm{Curr}(F_n, \mathcal{A})$ is naturally identified with the space of nonnegative, F_n-invariant, continuous linear functionals on the space $C_c(\partial^2(F_n, \mathcal{A}))$ (equipped with the uniform norm) of continuous compactly supported functions of $\partial^2(F_n, \mathcal{A})$ (see [7, Theorem 7.5.5]). Therefore, the space $\mathrm{Curr}(F_n, \mathcal{A})$ is homeomorphic to a subspace of $C_c(\partial^2(F_n, \mathcal{A}))^*$ equipped with the weak-star topology. Equipped with the uniform structure induced by the weak-star topology on $C_c(\partial^2(F_n, \mathcal{A}))^*$, we see that the space $\mathrm{Curr}(F_n, \mathcal{A})$ is metrisable and complete (see [5, Chapter 3, Section 1, Proposition 14]).

The group $\mathrm{Out}(F_n, \mathcal{A})$ acts on $\mathrm{Curr}(F_n, \mathcal{A})$ as follows. Let $\phi \in \mathrm{Out}(F_n, \mathcal{A})$, let Φ be a representative of ϕ, let $\mu \in \mathrm{Curr}(F_n, \mathcal{A})$ and let C be a Borel subset of $\partial^2(F_n, \mathcal{A})$. Then, since ϕ preserves $\mathcal{A}$, we see that $\Phi^{-1}(C)$ is a Borel subset of $\partial^2(F_n, \mathcal{A})$. Then we set

$$\phi(\mu)(C) = \mu(\Phi^{-1}(C)),$$

which is independent of the choice of the representative Φ since μ is F_n-invariant and the extension to the boundary of the action by conjugation and by left translation of F_n on itself coincide.

We now describe some coordinates for $\mathrm{Curr}(F_n,\mathcal{A})$. Recall that $\mathrm{Cyl}(\mathscr{C})$ is the set of cylinder sets of the form $C(\gamma)$, where the element of F_n determined by the geodesic path γ contains an element of $\mathscr{C}$ as a subword. Recall that

$$\partial^2(F_n,\mathcal{A}) = \bigcup_{C(\gamma)\in\mathrm{Cyl}(\mathscr{C})} C(\gamma).$$

Let $\eta \in \mathrm{Curr}(F_n,\mathcal{A})$. Let $w \in F_n$ be such that $C(\gamma_w) \in \mathrm{Cyl}(\mathscr{C})$ and let $w = w_1 \dots w_k$ be the reduced word associated with w written in the basis $\mathcal{B}$. Then $C(\gamma_w) = \coprod C(\gamma_{wb})$, where the union is taken over all elements b of $\mathcal{B} = \mathcal{B}^{-1}$ except $b = w_k^{-1}$. The σ-additivity of a relative current η implies that:

$$\eta(C(\gamma_w)) = \sum_{b\neq w_k^{-1}} \eta(C(\gamma_{wb})).$$

Finally, we note that, for every element $w \in F_n$ such that $C(\gamma_w) \in \mathrm{Cyl}(\mathscr{C})$, we have $\eta(C(\gamma_w)) = \eta(C(\gamma_{w^{-1}}))$. Indeed, this follows from the fact that $C(\gamma_w) = wC(\gamma_{w^{-1}})$ and from the F_n-invariance of η.

Lemma 3.1. — *Let $n\geqslant 3$ and let C be a compact open subset of $\partial^2(F_n,\mathcal{A})$. There exist finite geodesic edge paths $\gamma_1,\dots,\gamma_k$ such that:*

(1) *For every $i \in \{1,\dots,k\}$, we have $C(\gamma_i) \in \mathrm{Cyl}(\mathscr{C})$;*
(2) *for all distinct $i,j \in \{1,\dots,k\}$ we have $C(\gamma_i)\cap C(\gamma_j) = \varnothing$;*
(3) *we have $C = \bigcup_{i=1}^k C(\gamma_i)$.*

Proof. — Since C is a compact open subset of $\partial^2 F_n$, using the topology τ', the set C can be written as a union of cylinder sets $C(\gamma_1),\dots,C(\gamma_\ell)$, where, for every $i \in \{1,\dots,\ell\}$, we have $C(\gamma_i) \in \mathrm{Cyl}(\mathscr{C})$. We may suppose that for all distinct $i,j \in \{1,\dots,\ell\}$, we have $C(\gamma_i) \nsubseteq C(\gamma_j)$. In particular, there do not exist $i,j \in \{1,\dots,\ell\}$ such that $\gamma_i \subseteq \gamma_j$. Let m be the number of pairs of distinct elements $i,j \in \{1,\dots,\ell\}$ such that $C(\gamma_i)\cap C(\gamma_j) \neq \varnothing$. We prove Lemma 3.1 by induction on m. If for every distinct $i,j \in \{1,\dots,\ell\}$, we have $C(\gamma_i)\cap C(\gamma_j) = \varnothing$, then the set $\{\gamma_1,\dots,\gamma_\ell\}$ satisfies the conclusion of the lemma. Suppose that there exists m pairs of distinct elements $i,j \in \{1,\dots,\ell\}$ such that $C(\gamma_i)\cap C(\gamma_j) \neq \varnothing$, with $m \geqslant 1$.

Claim. — Let i,j be as above. There exist finite geodesic paths $\gamma_1^{(i)},\dots,\gamma_{k_i}^{(i)},\gamma_1^{(j)},\dots,\gamma_{k_j}^{(j)}$ in T which satisfy the following:

(a) for every $s \in \{1,\dots,k_i\}$ and every $t \in \{1,\dots,k_j\}$, we have $\gamma_i \subseteq \gamma_s^{(i)}$ and $\gamma_j \subseteq \gamma_t^{(j)}$;
(b) for every $p \in \{i,j\}$, for all distinct $s,t \in \{1,\dots,k_p\}$, we have $C(\gamma_s^{(p)})\cap C(\gamma_t^{(p)}) = \varnothing$;

(c) for every $s \in \{1,\dots,k_i\}$ and every $t \in \{1,\dots,k_j\}$, either $C(\gamma_s^{(i)}) = C(\gamma_t^{(j)})$ or $C(\gamma_s^{(i)}) \cap C(\gamma_t^{(j)}) = \varnothing$;

(d) for every $p \in \{i,j\}$, we have

$$C(\gamma_p) = \bigcup_{s=1}^{k_p} C(\gamma_s^{(p)}).$$

Proof. — See Figure 3.1 to follow the construction. Notice that we either have $\gamma_i \cap \gamma_j = \varnothing$ or $\gamma_i \cap \gamma_j \neq \varnothing$. In both cases, we construct a path τ and vertices v_i, v_i', v_j, v_j' that we will use in the rest of the proof. First suppose that $\gamma_i \cap \gamma_j = \varnothing$. Let τ be the unoriented geodesic path in T which realizes the distance between γ_i and γ_j. Since, by assumption, $C(\gamma_i) \cap C(\gamma_j) \neq \varnothing$, the endpoints of τ are endpoints of γ_i and γ_j. For every $p \in \{i,j\}$, let v_p be the common endpoint of γ_p and τ and let v_p' be the other endpoint of γ_p. Suppose now that $\gamma_i \cap \gamma_j \neq \varnothing$. Then, since $C(\gamma_i) \cap C(\gamma_j) \neq \varnothing$ there exist three paths τ, a_i and a_j such that, up to changing the orientation of γ_i and γ_j, we have: $\gamma_i = a_i\tau$ and $\gamma_j = \tau a_j$. For every $p \in \{i,j\}$, let v_p be the common endpoint of a_p and τ and let v_p' be the other endpoint of a_p.

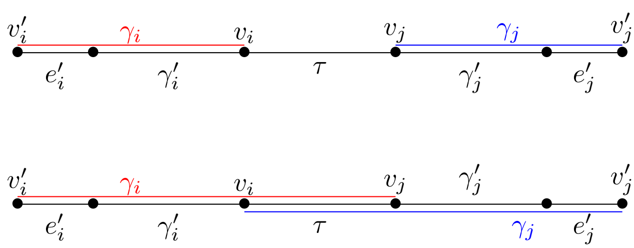

Figure 3.1. The paths constructed in the proof of Lemma 3.1.

For every $p \in \{i,j\}$, let e_p' be the edge of γ_p adjacent to v_p', which exists since γ_p is not reduced to a vertex. For every $p \in \{i,j\}$, let γ_p' be the edge path such that either $\gamma_p = \gamma_p' e_p'$ or $\gamma_p = e_p'\gamma_p'$. For all $p \in \{i,j\}$ and $\ell \in \{i,j\} - \{p\}$, let $\gamma_1^{(p)}, \dots, \gamma_{k_p}^{(p)}$ be the edge paths of T which start at v_p', which properly contain γ_p and such that for every $s \in \{1,\dots,k_p\}$, the endpoint of $\gamma_s^{(p)}$ distinct from v_p' is at distance exactly 1 from the minimal edge path of T which contains τ and γ_ℓ'. Note that for every $p \in \{i,j\}$ and $\ell \in \{i,j\} - \{p\}$, there exists a unique $s_p \in \{1,\dots,k_p\}$ such that $\gamma_{s_p}^{(p)}$ contains e_ℓ'. Note that for every $p \in \{i,j\}$, the integer s_p is the unique integer $s \in \{1,\dots,k_p\}$ such that $\gamma_s^{(p)}$ contains both γ_i and γ_j. Note also that $\gamma_{s_i}^{(i)} = (\gamma_{s_j}^{(j)})^{-1}$.

We claim that the paths $\gamma_1^{(i)},\dots,\gamma_{k_i}^{(i)},\gamma_1^{(j)},\dots,\gamma_{k_j}^{(j)}$ satisfy the conclusion of the claim. Indeed, (a) is satisfied by construction.

We prove (b). Let $p\in\{i,j\}$. Let $s,t\in\{1,\dots,k_p\}$ be distinct. Then $\gamma_s^{(p)}$ and $\gamma_t^{(p)}$ share the path γ_p as an initial segment. But, by construction of the paths $\gamma_s^{(p)}$ and $\gamma_t^{(p)}$, the endpoints of $\gamma_s^{(p)}$ and $\gamma_t^{(p)}$ distinct from v_p' are at distance exactly 1 from the minimal edge path of T which contains τ and γ_ℓ'. Therefore, the endpoint of $\gamma_s^{(p)}$ distinct from v_p' is not contained in $\gamma_t^{(p)}$. Hence the subtree of T generated by $\gamma_s^{(p)}$ and $\gamma_t^{(p)}$ is a tripod. This shows that $C(\gamma_s^{(p)})\cap C(\gamma_t^{(p)})=\varnothing$ and this proves (b).

We now prove (c). Let $s\in\{1,\dots,k_i\}$ and let $t\in\{1,\dots,k_j\}$. Suppose that we have $C(\gamma_s^{(i)})\cap C(\gamma_t^{(i)})\neq\varnothing$. Then there exists a path γ' of T such that γ' contains both $\gamma_s^{(i)}$ and $\gamma_t^{(j)}$. Thus γ' contains both γ_i and γ_j. This implies that $\gamma_s^{(i)}=\gamma_{s_i}^{(i)}=(\gamma_{s_j}^{(j)})^{-1}=(\gamma_t^{(j)})^{-1}$ and that $C(\gamma_s^{(i)})=C(\gamma_t^{(j)})$. This proves (c).

Finally, the fact that (d) holds follows from the fact that $C(\gamma)=\bigcup_{b\in ET,\gamma b\not\subseteq\gamma}C(\gamma b)$. This proves the claim. □

For every $p\in\{i,j\}$, replace γ_p by the paths $\gamma_1^{(p)},\dots,\gamma_{k_i}^{(p)}$. Then we obtain a new set $\{\gamma_1',\dots,\gamma_{\ell_1}'\}$ such that, by the point (d) of the claim, $C=\bigcup_{i=1}^{\ell_1}C(\gamma_i')$. Recall that for every $p\in\{i,j\}$, we have $C(\gamma_p)\in\mathrm{Cyl}(\mathscr{C})$. By the point (a) of the claim, for every $p\in\{i,j\}$ and every $s\in\{1,\dots,k_i\}$, we have $\gamma_p\subseteq\gamma_s^{(p)}$. Therefore, we see that for every $p\in\{i,j\}$ and every $s\in\{1,\dots,k_i\}$, we have $C(\gamma_s^{(p)})\in\mathrm{Cyl}(\mathscr{C})$. Hence the set $\{\gamma_1',\dots,\gamma_{\ell_1}'\}$ satisfies (1). Point (a) of the claim also implies that, for every $m'\in\{1,\dots,\ell\}$, and every $p\in\{i,j\}$, if $C(\gamma_{m'})\cap C(\gamma_p)=\varnothing$ then for every $s\in\{1,\dots,k_p\}$, we have $C(\gamma_{m'})\cap C(\gamma_s^{(p)})=\varnothing$. Combined with points (b) and (c) of the claim, we see that the number of distinct elements $m_1,m_2\in\{1,\dots,\ell_1\}$ such that $C(\gamma_{m_1})\cap C(\gamma_{m_2})\neq\varnothing$ is strictly less than m. An inductive argument then concludes the proof. □

We denote by $F_n-\mathcal{A}$ the subset of F_n consisting in every element $w\in F_n$ such that $C(\gamma_w)\in\mathrm{Cyl}(\mathscr{C})$. Note that $F_n-\mathcal{A}$ is closed under inversion since $\mathscr{C}$ is closed under inversion by Lemma 2.3. The next lemma gives a criterion to extend some functions defined on $F_n-\mathcal{A}$ to a relative current in $\mathrm{Curr}(F_n,\mathcal{A})$ (see [15, Lemma 3.9] for the free factor system case). First we need some definitions.

Let $w\in F_n$, and let $k\in\mathbb{N}^*$. A *length k extension of w* is a word $w'=wx_1\dots x_k$ where for every $i\in\{1,\dots,k-1\}$, we have $x_i\neq x_{i+1}^{-1}$ and x_1 is not the inverse of the last letter of w. An *extension of* w is a word w' such that there exists $k \in \mathbb{N}^*$ such that w' is a length k extension of w.

Lemma 3.2. — *Let* $\eta\colon F_n - \mathcal{A} \to \mathbb{R}_+$ *be a function invariant under inversion and which satisfies, for every* $w \in F_n - \mathcal{A}$*:*

$$\eta(w) = \sum_{v \text{ is a length one extension of } w} \eta(v). \tag{3.1}$$

There exists a unique element $\widetilde{\eta} \in \mathrm{Curr}(F_n, \mathcal{A})$ *such that for every element* $w \in F_n - \mathcal{A}$*, we have*

$$\eta(w) = \widetilde{\eta}(C(\gamma_w)).$$

Proof. — Since $\partial^2(F_n, \mathcal{A})$ is totally disconnected and locally compact by Lemma 2.6, and since a relative current is a Radon measure, a relative current is uniquely determined by its values on compact open subsets of $\partial^2(F_n, \mathcal{A})$. Let C be a compact open subset of $\partial^2(F_n, \mathcal{A})$. By Lemma 3.1, the subset C is a disjoint union of cylinders of finitely many geodesic edge paths $\gamma_1, \ldots, \gamma_k$ such that for every $i \in \{1, \ldots, k\}$, we have $C(\gamma_i) \in \mathrm{Cyl}(\mathscr{C})$. For every $i \in \{1, \ldots, k\}$, let g_i be the element of F_n which is the label of γ_i. For every $i \in \{1, \ldots, r\}$, since g_i contains an element of $\mathscr{C}$ as a subword, we have $g_i \in F_n - \mathcal{A}$. Hence we can set $\widetilde{\eta}(C) = \sum_{i=1}^{k} \eta(g_i)$. We claim that the value $\widetilde{\eta}(C)$ does not depend on the choice of the paths γ_i. Indeed, let $\alpha_1, \ldots, \alpha_\ell$ be another set of geodesic edge paths given by Lemma 3.1 and let $h_1, \ldots, h_\ell$ be the corresponding elements in F_n. Note that for every $i \in \{1, \ldots, k\}$ and every $j \in \{1, \ldots, \ell\}$ such that $C(\gamma_i) \cap C(\alpha_j) \neq \varnothing$, we have $C(\gamma_i) \cap C(\alpha_j) = C(\beta_{i,j})$, where $\beta_{i,j}$ is a minimal edge path in T that contains both γ_i and α_j.

We claim that for every $i \in \{1, \ldots, k\}$, there do not exist distinct $j_1, j_2 \in \{1, \ldots, \ell\}$ and paths a_1 and a_2 such that $\beta_{i,j_1} = a_1\gamma_i$ and $\beta_{i,j_2} = \gamma_i a_2$. Indeed, otherwise the path $a_1\gamma_i a_2$ is a finite path that contains both α_{j_1} and α_{j_2}. Hence $C(\alpha_{j_1}) \cap C(\alpha_{j_2}) \neq \varnothing$, a contradiction. The claim follows.

For every $i \in \{1, \ldots, k\}$ and every $j \in \{1, \ldots, \ell\}$ such that $C(\gamma_i) \cap C(\alpha_j) \neq \varnothing$, let $g_{i,j}$ be an element in F_n corresponding to $\beta_{i,j}$. By the above claim, for every $i \in \{1, \ldots, k\}$, one of the following holds:

(a) for every $j \in \{1, \ldots, \ell\}$ such that $C(\gamma_i) \cap C(\alpha_j) \neq \varnothing$, the element $g_{i,j}$ is an extension of g_i;
(b) for every $j \in \{1, \ldots, \ell\}$ such that $C(\gamma_i) \cap C(\alpha_j) \neq \varnothing$, the element $g_{i,j}^{-1}$ is an extension of g_i^{-1}.

Since η is invariant under inversion, we may suppose that for every $i \in \{1, \ldots, k\}$, and for every $j \in \{1, \ldots, \ell\}$ such that $C(\gamma_i) \cap C(\alpha_j) \neq \varnothing$, the element $g_{i,j}$ is an extension of g_i. Thus for every $j \in \{1, \ldots, \ell\}$, and for every

$i \in \{1, \dots, k\}$ such that $C(\gamma_i) \cap C(\alpha_j) \neq \varnothing$, the element $g_{i,j}^{-1}$ is an extension of h_j^{-1}.

Note that, since $C = \bigcup_{i=1}^{k} C(\gamma_i) = \bigcup_{j=1}^{\ell} C(\alpha_j)$, for every $i \in \{1, \dots, k\}$, the subset $C(\gamma_i)$ is covered by a disjoint union of finitely many $C(\alpha_j)$. Hence, for every $i \in \{1, \dots, k\}$, Equation (3.1) implies that:

$$\eta(g_i) = \sum_{j \,|\, C(\gamma_i) \cap C(\alpha_j) \neq \varnothing} \eta(g_{i,j}).$$

Similarly, for every $j \in \{1, \dots, \ell\}$, we have:

$$\eta(h_j^{-1}) = \sum_{i \,|\, C(\gamma_i) \cap C(\alpha_j) \neq \varnothing} \eta(g_{i,j}^{-1}).$$

Thus, since η is invariant under inversion, we have:

$$\begin{aligned}\sum_{j=1}^{\ell} \eta(h_j) = \sum_{j=1}^{\ell} \eta(h_j^{-1}) &= \sum_{j=1}^{\ell} \sum_{i \,|\, C(\gamma_i) \cap C(\alpha_j) \neq \varnothing} \eta(g_{i,j}^{-1}) \\ &= \sum_{i=1}^{k} \sum_{j \,|\, C(\gamma_i) \cap C(\alpha_j) \neq \varnothing} \eta(g_{i,j}) = \sum_{i=1}^{k} \eta(g_i).\end{aligned}$$

Hence the value of $\widetilde{\eta}(C)$ does not depend on the choice of the paths γ_i.

Therefore $\widetilde{\eta}$ is an additive, F_n-invariant and nonnegative function on the set of compact open subsets of $\partial^2(F_n, \mathcal{A})$. We claim that $\widetilde{\eta}$ is in fact σ-additive. Indeed, by [7, Proposition 1.2.6], it suffices to prove that for every decreasing sequence $(C_n)_{n \in \mathbb{N}}$ of compact open subsets of $\partial^2(F_n, \mathcal{A})$ such that $\bigcap_{n \in \mathbb{N}} C_n = \varnothing$, we have $\lim_{n \to \infty} \widetilde{\eta}(C_n) = 0$. But since a decreasing sequence of nonempty compact subsets is a nonempty compact subset, there exists $n \in \mathbb{N}$ such that $C_n = \varnothing$. This proves the claim. By Carathéodory extension theorem (see [7, Proposition 1.2.6, Theorem 1.3.6]), the function $\widetilde{\eta}$ has a unique extension as a Radon measure on the σ-algebra of Borel sets of $\partial^2(F_n, \mathcal{A})$. □

Let

$$\mathbb{P}\mathrm{Curr}(F_n, \mathcal{A}) = \left(\mathrm{Curr}(F_n, \mathcal{A}) - \{0\}\right) / \mathbb{R}_+^*$$

be the set of projectivized relative currents (where $\mathbb{R}_+^*$ acts on $\mathrm{Curr}(F_n, \mathcal{A})$ by homothety), equipped with the quotient topology which is metrizable. The next result is a generalization of [15, Lemma 3.11].

LEMMA 3.3. — *The metrisable space* $\mathbb{P}\mathrm{Curr}(F_n, \mathcal{A})$ *is compact.*

Proof. — Let $([\eta_n])_{n\in\mathbb{N}}$ be a sequence of projective currents relative to $\mathcal{A}$. We prove that it has a convergent subsequence. Let $\mathscr{C}$ be the finite set given by Lemma 2.3. For every $n \in \mathbb{N}$, let η_n be a representative of $[\eta_n]$ such that, for every $w \in \mathscr{C}$, we have $\eta(C(\gamma_w)) \leqslant 1$, with equality for some $w \in \mathscr{C}$, independent of n up to extraction. The set $\mathscr{C}$ being finite, there exists a subsequence $(\eta_{n_k})_{k\in\mathbb{N}}$ such that for every $u \in \mathscr{C}$, the sequence $(\eta_{n_k}(C(\gamma_u)))_{k\in\mathbb{N}}$ converges. Moreover, there exists $u_0 \in \mathscr{C}$ such that the limit $\lim_{k\to\infty}(\eta_{n_k}(C(\gamma_{u_0})))_{k\in\mathbb{N}}$ is not equal to zero. Let $w \in F_n$ be such that $C(\gamma_w) \in \mathrm{Cyl}(\mathscr{C})$. There exists $u_w \in \mathscr{C}$ such that u_w is a subword of w. Therefore, for every $k \in \mathbb{N}$, we have

$$\eta_{n_k}(C(\gamma_w)) \leqslant \eta_{n_k}(C(\gamma_{u_w})) \leqslant 1.$$

Therefore, for every element $w \in F_n - \mathcal{A}$, the sequence $(\eta_{n_k}C((\gamma_w)))_{k\in\mathbb{N}}$ has a convergent subsequence. By a diagonal argument, up to extraction, for every $C(\gamma_w) \in \mathrm{Cyl}(\mathscr{C})$, the sequence $(\eta_{n_k}(C(\gamma_w)))_{k\in\mathbb{N}}$ converges. Moreover, there exists $C(\gamma_w) \in \mathrm{Cyl}(\mathscr{C})$ such that $(\eta_{n_k}(C(\gamma_w)))_{k\in\mathbb{N}}$ converges to a nonzero element.

Let $\eta\colon F_n - \mathcal{A} \to \mathbb{R}_+$ be the function defined by, for every $w \in F_n - \mathcal{A}$:

$$\eta(w) = \lim_{k\to\infty} \eta_{n_k}(C(\gamma_w)).$$

Since for every $k \in \mathbb{N}$, the function η_{n_k} is a relative current, the function η satisfies the assumptions of Lemma 3.2. Therefore, by Lemma 3.2, there exists a unique relative current $\widetilde{\eta} \in \mathrm{Curr}(F_n, \mathcal{A})$ such that for every element $w \in F_n - \mathcal{A}$, we have

$$\eta(w) = \widetilde{\eta}(C(\gamma_w)).$$

Hence $([\eta_{n_k}])_{k\in\mathbb{N}}$ converges to $[\widetilde{\eta}]$. □

4. Density of rational currents

In this section, let $n \geqslant 3$. Let $r \in \mathbb{N}$ and let $\mathcal{A} = \{[A_1], \ldots, [A_r]\}$ be a malnormal subgroup system of F_n. Let $\mathcal{B}$, T, $L \in \mathbb{N}^*$, $\Gamma_1, \ldots, \Gamma_r$, $\mathscr{C}$ be as above Lemma 2.3. Let $\ell\colon F_n \to \mathbb{N}$ be the length function corresponding to $\mathcal{B}$.

Every conjugacy class of nonperipheral element $g \in F_n$ determines a relative current η_g as follows. Suppose first that g is *root-free*, that is, g is not a proper power of any element in F_n. Let γ be a finite geodesic path in the Cayley graph T such that $C(\gamma) \in \mathrm{Cyl}(\mathscr{C})$. Then $\eta_g(C(\gamma))$ is the number of unoriented translation axes in T of conjugates of g that contain the path γ. If $g = h^k$ with $k \geqslant 2$ and h root-free, we set $\eta_g = k\,\eta_h$. Such currents are called *rational currents*. Note that for every nonperipheral element $g \in F_n$, the current η_g only depends on the conjugacy class of g. Therefore, we can talk about rational currents induced by conjugacy classes of nonperipheral elements of F_n and write $\eta_{[g]}$ for the rational current associated with the conjugacy class of a nonperipheral element $g \in F_n$. We prove the following proposition.

Proposition 4.1. — *Let $n \geqslant 3$ and let $\mathcal{A}$ be a malnormal subgroup system of F_n. The set of positive linear combinations of projectivized rational currents induced by conjugacy classes of nonperipheral elements of F_n is dense in $\mathbb{P}\mathrm{Curr}(F_n, \mathcal{A})$.*

We follow Gupta's proof ([15, Proposition 3.12]) in the special case of free factor systems. The proof consists in approximating currents in $\mathbb{P}\mathrm{Curr}(F_n, \mathcal{A})$ with *signed measured currents* on $\partial^2 F_n$, which are F_n-invariant and σ-additive real-valued functions on the set of Borel subsets of $\partial^2 F_n$. We will then conclude using the following lemma, due to Martin (see also [15, Lemma 3.15]).

Lemma 4.2 ([21, Lemma 15]). — *Let $n \geqslant 3$. Suppose that $\mathcal{A} = \varnothing$. Let $k' \geqslant 1$, let $k \geqslant 2$ with $k' \leqslant k$ and let η be a signed measured current such that, for every $w \in F_n$ with $k' \leqslant \ell(w) \leqslant k$, we have $\eta(C(\gamma_w)) \geqslant 0$. Let $P = 2n(2n-1)^{2n(2n-1)^{k-2}}$. If there exists $w_0 \in F_n$ such that $\ell(w_0) = k$ and $\eta(C(\gamma_{w_0})) \geqslant P$, then there exists $\alpha \in F_n - \{e\}$ such that, for every $w \in F_n$ with $k' \leqslant \ell(w) \leqslant k$, we have $\eta(C(\gamma_w)) \geqslant \eta_{[\alpha]}(C(\gamma_w))$.*

Remark 4.3. —

(1) The hypotheses in [21, Lemma 15] requires that $k' = 1$. However, the proof of Martin works by studying words of length exactly k and then extend the result to words of length at most k by additivity of the measures. Thus the proof with $k' > 1$ is identical.
(2) For the rational current $\eta_{[\alpha]}$ constructed in Lemma 4.2, there exists $w \in F_n$ with $k' \leqslant \ell(w) \leqslant k$ such that $\eta_{[\alpha]}(C(\gamma_w)) > 0$.

Recall that $\mathrm{Cyl}(\mathscr{C})$ is the set of cylinder sets of the form $C(\gamma_w)$, where w is a word of F_n containing a word of $\mathscr{C}$ as a subword. Let $\eta_0 \in \mathrm{Curr}(F_n, \mathcal{A})$ and let $k \geqslant L+2$. Let η be a signed measured current such that, for every element $w \in F_n$ with $C(\gamma_w) \in \mathrm{Cyl}(\mathscr{C})$, we have $\eta(C(\gamma_w)) = \eta_0(C(\gamma_w))$ and for every element $w \in F_n$ of length between $L+2$ and k, we have $\eta(C(\gamma_w)) \geqslant 0$. Then η is called a *k-extension of* η_0. The key lemma in order to prove Proposition 4.1 is the following result (see [15, Lemma 3.15] for the same statement in the particular case of free factor systems):

Lemma 4.4. — *Let η_0 be a relative current and let $k \geqslant L+2$. There exists a signed measured current $\eta \colon \partial^2 F_n \to \mathbb{R}$ which is a k-extension of η_0.*

Let η_0 be a relative current. In order to prove Lemma 4.4, we need some preliminary results. We follow [15, Section 8.1]. For $k \in \mathbb{N}^*$, let S_k be the set of elements of F_n of length k which do not contain an element of $\mathscr{C}$ as a subword. Note that, since $\mathscr{C}$ is closed under inversion by Lemma 2.3, we see that, for every $k \in \mathbb{N}^*$, the set S_k is closed under inversion. For $k = 0$, we set $S_0 = \{e\}$. Note also that, if $k < L+2$, then S_k contains all words of length k since every element of $\mathscr{C}$ has length equal to $L+2$.

Lemma 4.5. —

(1) *If $\mathcal{A} \neq \varnothing$, for every $k \in \mathbb{N}^*$, the set S_k is not empty.*
(2) *For every $k \geqslant L+2$ and every $w \in S_k$, there exist $w' \in S_{k+1}$, $i \in \{1, \dots, r\}$, $g \in F_n$ and $a \in gA_ig^{-1}$ such that w' is a length 1 extension of w and a is an extension of w'.*

Proof. —

(1). — Since the group A_1 is infinite, the corresponding minimal subtree T_{A_1} is infinite. Recall that the tree T_{A_1} is supposed to contain the origin e of T. Let γ be a geodesic path contained in T_{A_1}, starting from e and of length equal to k, and let $h \in F_n$ be the corresponding element of F_n. Then there exists $a \in A_1$ such that a is an extension of h. We have $h \in S_k$ as otherwise a would contradict Lemma 2.3(3). This proves (1).

(2). — Let $k \geqslant L+2$ and let $w \in S_k$. By Lemma 2.3(1), there exist $i \in \{1, \dots, r\}$ and $g \in F_n$ such that γ_w is contained in $T_{gA_ig^{-1}}$. As $T_{gA_ig^{-1}}$ does not contain any univalent vertex, there exists a geodesic ray c in $T_{gA_ig^{-1}}$ starting from e which contains the path γ_w. Let γ' be the geodesic path in c of length $k+1$ containing γ_w, and let w' be the corresponding element in F_n. Then $w' \in S_{k+1}$ and w' is a length 1 extension of w. This proves (2) and this concludes the proof. □

Let $k \geqslant L+2$. Let S_k^0 be a subset of S_k (chosen once and for all) such that for every $w \in S_k$ exactly one of w or w^{-1} appears in S_k^0. In what follows, we adopt the convention that whenever an extension of a word w by a letter $b \in \mathcal{B}$ is written as wb (resp. bw), we assume that b is not the inverse of the last letter (resp. first letter) of the word w.

In order to construct the signed measured current which satisfies the conclusion of Lemma 4.4, we will define a signed measured current on cylinders of words in S_{k-1} and use those values together with the additivity laws in order to define η on cylinders of words of length k. First we set $\eta(C(\gamma_b)) = 1$ for every letter b of $\mathcal{B}$ not contained in $\mathscr{C}$. By induction, assume that for every element $v \in S_{k-1}$, the value $\eta(C(\gamma_v))$ is defined. By additivity of a signed measured current, for every $v \in S_{k-1}^0$, we want to have:

$$\eta(C(\gamma_v)) = \sum_{b\in\mathcal{B}, vb\in S_k} \eta(C(\gamma_{vb})) + \sum_{b\in\mathcal{B}, vb\notin S_k} \eta_0(C(\gamma_{vb}))$$
$$\eta(C(\gamma_{v^{-1}})) = \sum_{b\in\mathcal{B}, v^{-1}b\in S_k} \eta(C(\gamma_{v^{-1}b})) + \sum_{b\in\mathcal{B}, v^{-1}b\notin S_k} \eta_0(C(\gamma_{v^{-1}b}))$$

Since η is invariant under taking inverses, the equation obtained by using forward extensions of v^{-1} is the same one as the equation obtained by using backward extensions of v. After rearranging the equations in order to have the unknown terms on the left hand side, we obtain:

$$\begin{aligned} \sum_{b\in\mathcal{B}, vb\in S_k} \eta(C(\gamma_{vb})) &= \sum_{b\in\mathcal{B}, vb\notin S_k} \eta_0(C(\gamma_{vb})) - \eta(C(\gamma_v)) = c_v \\ \sum_{b\in\mathcal{B}, v^{-1}b\in S_k} \eta(C(\gamma_{v^{-1}b})) &= \sum_{b\in\mathcal{B}, v^{-1}b\notin S_k} \eta_0(C(\gamma_{v^{-1}b})) - \eta(C(\gamma_{v^{-1}})) = c_{v^{-1}}. \end{aligned} \tag{4.1}$$

Since η is invariant under taking inverse, this shows that there are $|S_{k-1}|$ equations in $|S_k|/2 = |S_k^0|$ variables.

Denote the system of equations (4.1) by E_{k-1}^1. These are equations obtained from length 1 extensions of words in S_{k-1}. Similarly, for every $i \in \{1, \ldots, k-1\}$, we define E_{k-i}^i as the system of equations obtained from length i extensions of words in S_{k-i}.

Let $[M|c]$ be the augmented matrix for the system of equations E_{k-1}^1 with rows labeled by words in S_{k-1}, columns by words in S_k^0 and such that for every $w \in S_k^0$ and every $v \in S_{k-1}$, we have $M_{v,w} = 1$ if there exists $b \in \mathcal{B}$ such that $w = vb$ or $w^{-1} = vb$; and $M_{v,w} = 0$ otherwise. Let c be the column vector indexed by words in S_{k-1} such that for every $v \in S_{k-1}$, the coordinate of c at v is equal to c_v. If $v \in S_{k-1}$, we will denote by r_v the corresponding row vector of M. Observe that each column has exactly two entries which are equal to 1. Indeed, $M_{v,w}$ is equal to 1 exactly when w or w^{-1} is a length 1 extension of v. Observe also that any two distinct row vectors r_{v_1} and r_{v_2} can have at most one common coordinate which is equal to 1. Indeed, let $w \in S_k^0$ be such that $M_{v_1,w} = M_{v_2,w} = 1$. Then there exist $b_1, b_2 \in \mathcal{B}$ such that $w = v_1b_1$ or $w = b_1^{-1}v_1^{-1}$ and $w = v_2b_2$ or $w = b_2^{-1}v_2^{-1}$. Therefore, the word v_1 starts with b_2^{-1} and v_2 starts with b_1^{-1}. This shows that w is uniquely determined.

The next lemma is the same one as [15, Lemma 8.2] in the special case of free factor systems.

Lemma 4.6. —

(1) *For every* $i \geqslant 1$*, an equation in the system* E^{i+1}_{k-i-1} *is a linear combination of equations in the system* E^i_{k-i}*. Thus it is sufficient to look at the system* E^1_{k-1} *in order to obtain every constraint satisfied by* $\eta(C(\gamma_w))$ *for every* $w \in S^0_k$*.*

(2) *Let* $u \in S_{k-2}$*. Then the following two linear combinations of rows of* M *are equal:*

$$\sum_{b\in\mathcal{B}, bu\in S_{k-1}} r_{bu} = \sum_{b\in\mathcal{B}, bu^{-1}\in S_{k-1}} r_{bu^{-1}}. \tag{4.2}$$

(3) *Every relation among the rows of* M *is a linear combination of relations in the set of relations* (4.2) *where* u *varies in* S_{k-2}*.*

(4) *We have*

$$\sum_{b\in\mathcal{B}, bu\in S_{k-1}} c_{bu} = \sum_{b\in\mathcal{B}, bu^{-1}\in S_{k-1}} c_{bu^{-1}},$$

where for every $v \in S_{k-1}$*,* c_v *is given by Equation* (4.1)*.*

(5) *The system of equations* E^1_{k-1} *is consistent and hence has a solution. Thus we can define* η *on words of length* k*.*

Proof. —

(1). — Let $i \geqslant 1$ and $u \in S_{k-i-1}$. Then by the system E^1_{k-i-1}

$$\eta(C(\gamma_u)) = \sum_{b\in\mathcal{B}} \eta(C(\gamma_{ub})).$$

By the equations in E^i_{k-i}, we have, for every $b \in \mathcal{B}$:

$$\eta(C(\gamma_{ub})) = \sum_{y\in F_n, \ell(y)=i} \eta(C(\gamma_{uby})).$$

Adding all these equations over $b \in \mathcal{B}$, we have:

$$\eta(C(\gamma_u)) = \sum_{b,y\in F_n, \ell(b)=1, \ell(y)=i} \eta(C(\gamma_{uby})) = \sum_{z\in F_n, \ell(z)=i+1} \eta(C(\gamma_{uz})).$$

Thus we have recovered an equation in E^{i+1}_{k-i-1} as a linear combination of equations in E^i_{k-i}.

(2). — Let $u \in S_{k-2}$ and let $w \in S^0_k$. For every $b \in \mathcal{B}$ such that $bu \in S_{k-1}$, we have $M_{bu,w} \neq 0$ exactly when there exists $y \in \mathcal{B}$ such that $w = buy^{-1}$ or $w = yu^{-1}b^{-1}$ (recall that the basis $\mathcal{B}$ is supposed to be symmetric). Therefore, if $M_{bu,w} \neq 0$, there exists a unique $y \in \mathcal{B}$ such that $M_{yu^{-1},w} \neq 0$.

(3). — Let R be a relation given by $\sum_{v\in S_{k-1}} d_v r_v = 0$, where $d_v \in \mathbb{R}$. Suppose that the number of terms in the sum associated with R is minimal. Such an assumption is possible as every relation is a linear combination of relations whose number of terms is minimal. We can rescale the equation so that there exist $b \in \mathcal{B}$ and $u \in S_{k-2}$ such that $d_{bu} = 1$. For every $y \in \mathcal{B}$ such that $buy^{-1} \in S_k^0$, we have

$$M_{bu,buy^{-1}} = M_{yu^{-1},buy^{-1}} = 1.$$

This implies, as explained above the lemma, that the rows r_{bu} and $r_{yu^{-1}}$ share exactly one common nonzero coordinate, which is buy^{-1}. Moreover, the rows r_{bu} and $r_{yu^{-1}}$ are the only rows which have a nonzero coordinate in buy^{-1}. This shows that $d_{yu^{-1}} = -1$.

Let $y \in \mathcal{B}$ be such that $yu^{-1} \in S_{k-1}$. For every $z \in \mathcal{B}$ such that $yu^{-1}z \in S_k^0$, we have $M_{yu^{-1},yu^{-1}z} = M_{z^{-1}u,yu^{-1}z} = 1$. Thus we have $d_{z^{-1}u} = 1$. Therefore we see that

$$\begin{aligned}\sum_{b\in\mathcal{B},bu\in S_{k-1}} d_{bu}r_{bu} + \sum_{y\in\mathcal{B},yu^{-1}\in S_{k-1}} d_{yu^{-1}}r_{yu^{-1}}\\ = \sum_{b\in\mathcal{B},bu\in S_{k-1}} r_{bu} - \sum_{y\in\mathcal{B},yu^{-1}\in S_{k-1}} r_{yu^{-1}} = 0.\end{aligned}$$

Hence the minimal relation R is just

$$\sum_{b\in\mathcal{B},bu\in S_{k-1}} r_{bu} - \sum_{y\in\mathcal{B},yu^{-1}\in S_{k-1}} r_{yu^{-1}} = 0.$$

(4). — Let $u \in S_{k-2}$. We have, by the definition of c_v:

$$\begin{aligned}&-\sum_{b\in\mathcal{B},bu\in S_{k-1}} c_{bu}\\ &\quad= \sum_{b\in\mathcal{B},bu\in S_{k-1}} \eta(C(\gamma_{bu})) - \sum_{\substack{b,y\in\mathcal{B},bu\in S_{k-1},\\ buy\notin S_k}} \eta(C(\gamma_{buy}))\\ &\quad= \eta(C(\gamma_u)) - \sum_{b\in\mathcal{B},bu\notin S_{k-1}} \eta(C(\gamma_{bu})) - \sum_{\substack{b,y\in\mathcal{B},bu\in S_{k-1},\\ buy\notin S_k}} \eta(C(\gamma_{buy}))\\ &\quad= \eta(C(\gamma_u)) - \sum_{b,y\in\mathcal{B},bu\notin S_{k-1}} \eta(C(\gamma_{buy})) - \sum_{\substack{b,y\in\mathcal{B},bu\in S_{k-1},\\ buy\notin S_k}} \eta(C(\gamma_{buy})).\end{aligned}$$

Note that we have:

$$\begin{aligned}\sum_{b,y\in\mathcal{B},bu\notin S_{k-1}} &\eta(C(\gamma_{buy}))\\ &= \sum_{\substack{b,y\in\mathcal{B},bu\notin S_{k-1},\\ uy\in S_{k-1}}} \eta(C(\gamma_{buy})) + \sum_{\substack{b,y\in\mathcal{B},bu\notin S_{k-1},\\ uy\notin S_{k-1}}} \eta(C(\gamma_{buy})) \quad (4.3)\end{aligned}$$

Similarly, we have:

$$\begin{aligned}-\sum_{b\in\mathcal{B},bu^{-1}\in S_{k-1}} c_{bu^{-1}} = \eta(C(\gamma_u)) - &\sum_{\substack{b,y\in\mathcal{B},\\ bu^{-1}\notin S_{k-1}}} \eta(C(\gamma_{bu^{-1}y}))\\ &- \sum_{\substack{b,y\in\mathcal{B},bu^{-1}\in S_{k-1},\\ bu^{-1}y\notin S_k}} \eta(C(\gamma_{bu^{-1}y})).\end{aligned}$$

The right hand side is also equal to:

$$\eta(C(\gamma_u)) - \sum_{\substack{b,y\in\mathcal{B},\\ ub^{-1}\notin S_{k-1}}} \eta(C(\gamma_{y^{-1}ub^{-1}})) - \sum_{\substack{b,y\in\mathcal{B},ub^{-1}\in S_{k-1},\\ y^{-1}ub^{-1}\notin S_k}} \eta(C(\gamma_{y^{-1}ub^{-1}})).$$

Observe that the sum $\sum_{b,y\in\mathcal{B},ub^{-1}\notin S_{k-1}} \eta(C(\gamma_{y^{-1}ub^{-1}}))$ equals:

$$\sum_{\substack{b,y\in\mathcal{B},ub^{-1}\notin S_{k-1},\\ y^{-1}u\in S_{k-1}}} \eta(C(\gamma_{y^{-1}ub^{-1}})) + \sum_{\substack{b,y\in\mathcal{B},ub^{-1}\notin S_{k-1},\\ y^{-1}u\notin S_{k-1}}} \eta(C(\gamma_{y^{-1}ub^{-1}})). \quad (4.4)$$

Suppose first that $k \leqslant L+2$. Then S_{k-1} contains all words of length $k-1$. Hence we have

$$-\sum_{b\in\mathcal{B},bu\in S_{k-1}} c_{bu} = \eta(C(\gamma_u)) - \sum_{b,y\in\mathcal{B},buy\notin S_k} \eta(C(\gamma_{buy}))$$

and

$$-\sum_{b\in\mathcal{B},bu^{-1}\in S_{k-1}} c_{bu^{-1}} = \eta(C(\gamma_u)) - \sum_{b,y\in\mathcal{B},y^{-1}ub^{-1}\notin S_k} \eta(C(\gamma_{y^{-1}ub^{-1}})),$$

so that Assertion (4) holds in this case with $y = b^{-1}$.

Suppose now that $k > L+2$. Then since every element of $\mathscr{C}$ has length equal to $L+2$, an element of $\mathscr{C}$ contained in a word x of length k is properly contained in x. Hence if $b, y \in \mathcal{B}$ are such that $bu \in S_{k-1}$ and $buy \notin S_k$, then $uy \notin S_{k-1}$. Thus, we see that:

$$\sum_{b,y\in\mathcal{B},bu\in S_{k-1},buy\notin S_k} \eta(C(\gamma_{buy})) = \sum_{b,y\in\mathcal{B},bu\in S_{k-1},uy\notin S_{k-1}} \eta(C(\gamma_{buy})). \quad (4.5)$$

Similarly, we have:

$$\sum_{\substack{b,y\in\mathcal{B},ub^{-1}\in S_{k-1},\\ y^{-1}ub^{-1}\notin S_k}} \eta(C(\gamma_{y^{-1}ub^{-1}})) = \sum_{\substack{b,y\in\mathcal{B},ub^{-1}\in S_{k-1},\\ y^{-1}u\notin S_{k-1}}} \eta(C(\gamma_{y^{-1}ub^{-1}})). \tag{4.6}$$

Using Equations (4.3), (4.4), (4.5) and (4.6) with $y = b^{-1}$, we see that

$$\sum_{\substack{b,y\in\mathcal{B},\\ bu\notin S_{k-1}}} \eta(C(\gamma_{buy})) + \sum_{\substack{b,y\in\mathcal{B},\\ bu\in S_{k-1},buy\notin S_k}} \eta(C(\gamma_{buy}))$$
$$= \sum_{\substack{b,y\in\mathcal{B},\\ ub^{-1}\notin S_{k-1}}} \eta(C(\gamma_{y^{-1}ub^{-1}})) + \sum_{\substack{b,y\in\mathcal{B},ub^{-1}\in S_{k-1},\\ y^{-1}ub^{-1}\notin S_k}} \eta(C(\gamma_{y^{-1}ub^{-1}})).$$

This shows that

$$\sum_{b\in\mathcal{B},bu\in S_{k-1}} c_{bu} = \sum_{b\in\mathcal{B},bu^{-1}\in S_{k-1}} c_{bu^{-1}}.$$

(5). — By Assertions (3) and (4) if R is a linear combination of relations among the rows of M equal to zero, then the corresponding linear combination among coordinates of the vector c is also equal to zero. Therefore, the system $[M|c]$ has a solution. □

Proof of Lemma 4.4. — Let η_0 be a relative current. By Lemma 4.6, there exists a signed measured current η such that, for every element w of F_n which satisfies $C(\gamma_w) \in \mathrm{Cyl}(\mathscr{C})$, we have $\eta_0(C(\gamma_w)) = \eta(C(\gamma_w))$. This extension is not necessarily nonnegative on every element of length between $L+2$ and k. Let

$$-M = \min_{w\in F_n,\ L+2\leqslant \ell(w)\leqslant k} \eta(C(\gamma_w)).$$

Let S be a finite set of elements of $\bigcup_{i=1}^{r} A_i$ such that for every element $w \in S_k$, there exists $g_w \in S$ such that g_w is an extension of w. The set exists by Lemma 4.5 (2). Let

$$\eta_{\mathcal{A}} = \sum_{g\in S} \eta_{[g]}.$$

By Lemma 2.3 (3), for every $w \in F_n$ such that $C(\gamma_w) \in \mathrm{Cyl}(\mathscr{C})$, we have $\eta_{\mathcal{A}}(C(\gamma_w)) = 0$. Moreover for every $w \in \bigcup_{i=L+2}^{k} S_i$, Lemma 4.5 (2) implies that there exists $w' \in S_k$ such that w' is an extension of w. In particular, for every $w \in \bigcup_{i=L+2}^{k} S_i$, we have $\eta_{\mathcal{A}}(C(\gamma_w)) > 0$. By finiteness of $\bigcup_{i=L+2}^{k} S_i$, there exists a constant $R > 0$ such that for every element w in $\bigcup_{i=L+2}^{k} S_i$, we have $R\,\eta_{\mathcal{A}}(C(\gamma_w)) \geqslant M$.

Then $\eta + R\,\eta_{\mathcal{A}}$ is nonnegative on words of length between $L+2$ and k and coincides with η_0 on elements $w \in F_n$ such that $C(\gamma_w) \in \mathrm{Cyl}(\mathscr{C})$. This concludes the proof. □

Proof of Proposition 4.1. — The proof follows [15, Lemma 3.15] (see also [21]). Let $\mathscr{C}$ be the set defined above Lemma 2.3. Let η_0 be a relative current and let $k \geqslant L+2$. Note that every word in $\mathscr{C}$ has length at most equal to k. Let P be the constant given by Lemma 4.2. Note that there exists an element w' in $\mathscr{C}$ such that $\eta_0(C(\gamma_{w'})) > 0$. By additivity of η_0, there exists an element $w_0 \in F_n$ with $\ell(w_0) = k$ and $C(\gamma_{w_0}) \in \mathrm{Cyl}(\mathscr{C})$ and such that $\eta_0(C(\gamma_{w_0})) > 0$. Let $R > 0$ be such that $R\,\eta_0(C(\gamma_{w_0})) > P$. By Lemma 4.4, there exists a signed measured current η which is a k-extension of η_0. By Lemma 4.2 applied to $R\,\eta$ and $k' = L+2$, there exists $\alpha_1 \in F_n - \{e\}$ such that for every $w \in F_n - \{e\}$ of length between $L+2$ and k, we have

$$R\,\eta(C(\gamma_w)) \geqslant \eta_{[\alpha_1]}(C(\gamma_w)).$$

Suppose first that for every $w \in F_n$ of length between $L+2$ and k, we have

$$R\,\eta(C(\gamma_w)) \leqslant \eta_{[\alpha_1]}(C(\gamma_w)) + P.$$

Then we stop the process and choose α_1. Otherwise, we apply Lemma 4.2 to $R\eta - \eta_{[\alpha_1]}$ and $k' = L+2$. This shows that there exists $\alpha_2 \in F_n - \{e\}$ such that for every $w \in F_n - \{e\}$ of length between $L+2$ and k, we have

$$R\,\eta(C(\gamma_w)) - \eta_{[\alpha_1]}(C(\gamma_w)) \geqslant \eta_{[\alpha_2]}(C(\gamma_w)).$$

Applying these arguments iteratively (the process stops by Remark 4.3 (2)), we see that there exist $\alpha_1, \ldots, \alpha_p \in F_n - \{e\}$ such that for every element $w \in F_n - \{e\}$ of length between $L+2$ and k, we have:

$$\sum_{i=1}^{p} \eta_{[\alpha_i]}(C(\gamma_w)) \leqslant R\,\eta(C(\gamma_w)) \leqslant \sum_{i=1}^{p} \eta_{[\alpha_i]}(C(\gamma_w)) + P.$$

We claim that there exists $i \in \{1, \ldots, p\}$ such that α_i is nonperipheral. Indeed, suppose towards a contradiction that for every $i \in \{1, \ldots, p\}$, the element α_i is peripheral. By Lemma 2.3 (3), we have

$$\sum_{i=1}^{p} \eta_{[\alpha_i]}(C(\gamma_{w_0})) = 0.$$

This implies that $R\,\eta(C(\gamma_{w_0})) \leqslant P$. This contradicts the construction of η. Therefore there exists $i \in \{1, \ldots, p\}$ such that α_i is nonperipheral. Let S be the subset of $\{\alpha_1, \ldots, \alpha_p\}$ containing every nonperipheral element. Then, for every element $w \in F_n$ of length k such that $C(\gamma_w) \in \mathrm{Cyl}(\mathscr{C})$ we have:

$$\left| \eta(C(\gamma_w)) - \frac{\sum_{\alpha \in S} \eta_{[\alpha]}(C(\gamma_w))}{R} \right| \leqslant \frac{P}{R}.$$

For $\alpha \in S$, let $\overline{\eta}_{[\alpha]}$ be the restriction of $\eta_{[\alpha]}$ to the Borel subsets of $\partial^2(F_n, \mathcal{A})$. By construction of η, for every element $w \in F_n$ of length at most k such that $C(\gamma_w) \in \mathrm{Cyl}(\mathscr{C})$, we have:

$$\left|\eta_0(C(\gamma_w)) - \frac{\sum_{\alpha \in S} \overline{\eta}_{[\alpha]}(C(\gamma_w))}{R}\right| \leqslant \frac{P}{R}.$$

Since R can be chosen arbitrarily large, we can approximate relative currents by sum of rational relative currents. For $m \in \mathbb{N}^*$, let $\beta^m = \prod_{\alpha \in S} \alpha^m$ (for any total order on S). Then there exists $m \in \mathbb{N}^*$ such that $\sum_{\alpha \in S} \eta_{[\alpha_i]}$ can be approximated by $\frac{1}{m}\eta_{[\beta^m]}$. This concludes the proof. □

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