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On the invariance of the total Monge–Ampère volume of Hermitian metrics (*)

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Abstract. — In this note, we describe the Hermitian metrics that leave the total Monge–Ampère volume invariant. In particular, we give several characterizations of the Hermitian metrics which satisfy the comparison principle for the complex Monge–Ampère operator.

Résumé. — Dans cette note, nous décrivons les métriques hermitiennes qui laissent le volume total de Monge–Ampère invariant, et nous donnons plusieurs caractérisations des métriques hermitiennes qui satisfont le principe de comparaison pour l'opérateur complexe de Monge–Ampère.

Let g be a Hermitian metric on a compact complex manifold X of dimension n. Suppose that it satisfies the equations

$$i\partial\bar{\partial}g^k = 0, \quad \forall\, k = \overline{1, n-1} \tag{1}$$

(we denote by the same letter g the fundamental form of the Hermitian metric g). These metrics were introduced by Guan and Li [7] because these metrics leave the total Monge–Ampère volume unchanged, i.e., they satisfy

$$\int_X (g + i\partial\bar{\partial}u)^n = \int_X g^n, \quad \forall\, u \in \mathrm{PSH}(X, g) \tag{2}$$

where $\mathrm{PSH}(X, g)$ denotes the space of g-plurisubharminic functions on X, that is the $\mathcal{C}^\infty$ real functions u on X such that

$$g + i\partial\bar{\partial}u \geqslant 0.$$

Indeed, the equality (2) follows from (1) from Stokes' theorem applied twice.

In this note, we prove the converse of this statement, namely we show that a Hermitian metric which satisfies (2) has to satisfy (1). This answers Question 29 in [5] (see also Problem 11.1 in [4]).

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Article proposé par Vincent Guedj.

Theorem 1. — *Let (X,g) be a compact complex Hermitian manifold of dimension n such that*

$$\int_X (g+i\partial\bar\partial u)^n = \int_X g^n \tag{3}$$

for any smooth g-plurisubharmonic function u on X. Then

$$i\partial\bar\partial g^k = 0, \quad \forall\, k=\overline{1,n-1}. \tag{4}$$

Proof. — First note that if u is an arbitrary $\mathcal{C}^\infty$ real function on X, then there exists $\varepsilon_0=\varepsilon_0(u)>0$ such that εu is g-plurisubharmonic for $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$. Therefore, for any $u\in\mathcal{C}^\infty(X,\mathbb{R})$, we have

$$\int_X (g+\varepsilon i\partial\bar\partial u)^n = \int_X g^n, \quad \forall\,\varepsilon\in(-\varepsilon_0,\varepsilon_0) \tag{5}$$

If we expand the expression on the left and write it as a polynomial in ε, we obtain that

$$\int_X g^k\wedge(i\partial\bar\partial u)^{n-k}=0, \quad \forall\, k=\overline{1,n-1}. \tag{6}$$

Now fix $k\in\{1,2,\dots,n-1\}$ and if u_1, $u_2,\dots,u_{n-k}$ are arbitrary $\mathcal{C}^\infty$ functions on X and t_1, $t_2,\dots,t_{n-k}$ are arbitrary real numbers, then, from

$$\int_X g^k\wedge\left(i\partial\bar\partial\sum_{i=1}^{n-k}t_iu_i\right)^{n-k}=0 \tag{7}$$

it follows that

$$\int_X g^k\wedge i\partial\bar\partial u_1\wedge i\partial\bar\partial u_2\wedge\cdots\wedge i\partial\bar\partial u_{n-k}=0. \tag{8}$$

Now use Stokes' theorem twice to obtain that, for arbitrary $\mathcal{C}^\infty$ functions u_1, $u_2,\dots,u_{n-k}$ on X we have

$$\int_X u_1 i\partial\bar\partial g^k\wedge i\partial\bar\partial u_2\wedge\cdots\wedge i\partial\bar\partial u_{n-k}=0. \tag{9}$$

Since the function u_1 is arbitrary on X, we obtain that

$$i\partial\bar\partial g^k\wedge i\partial\bar\partial u_2\wedge\cdots\wedge i\partial\bar\partial u_{n-k}=0 \tag{10}$$

on X for any $u_2,\dots,u_{n-k}\in\mathcal{C}^\infty(X,\mathbb{R})$. It is clear that we can let the functions $u_2,\dots,u_{n-k}$ be complex valued. Fix $x\in X$ and let $z_1,z_2,\dots,z_n$ be local coordinates around x. If

$$i\partial\bar\partial g^k=\sum_{\substack{i_1<\cdots<i_{k+1}\\ j_1<\dots j_{k+1}}} g_{i_1\dots i_{k+1}\bar j_1\dots\bar j_{k+1}}\mathrm{d}z_{i_1}\wedge\cdots\wedge\mathrm{d}z_{i_{k+1}}\wedge\mathrm{d}\bar z_{j_1}\wedge\cdots\wedge\mathrm{d}\bar z_{j_{k+1}} \tag{11}$$

then let

$$\{l_1,\dots,l_{n-k-1}\}=\{1,\dots,n\}\setminus\{i_1,\dots,i_{k+1}\} \tag{12}$$

and

$$\{m_1, \ldots, m_{n-k-1}\} = \{1, \ldots, n\} \setminus \{j_1, \ldots, j_{k+1}\} \tag{13}$$

and take $u_2 = z_{l_1}\bar{z}_{m_1}, \ldots, u_{n-k} = z_{l_{n-k-1}}\bar{z}_{m_{n-k-1}}$ near x to obtain

$$g_{i_1 \ldots i_{k+1}\bar{j}_1 \ldots \bar{j}_{k+1}}(x) = 0 \tag{14}$$

hence $i\partial\bar{\partial}g^k = 0$. □

Combining the above result with [6], we obtain the following

Theorem 2. — *Let X be a compact complex manifold of dimension n with a Hermitian metric g. Then the following are equivalent:*

(i) $i\partial\bar{\partial}g = i\partial g \wedge \bar{\partial}g = 0$;
(ii) $i\partial\bar{\partial}g = i\partial\bar{\partial}g^2 = 0$;
(iii) $i\partial\bar{\partial}g^k = 0, \ \forall\, k = \overline{1, n-1}$;
(iv) $i\partial\bar{\partial}g \geqslant 0,\ i\partial g \wedge \bar{\partial}g \geqslant 0$;
(v) $\displaystyle\int_{\{u<v\}} (g + i\partial\bar{\partial}v)^n \leqslant \int_{\{u<v\}} (g + i\partial\bar{\partial}u)^n, \ \forall\, u, v \in \mathrm{PSH}(X, g)$;
(vi) $\displaystyle\int_X (g + i\partial\bar{\partial}u)^n = \int_X g^n, \ \forall\, u \in \mathrm{PSH}(X, g)$.

Proof. — It is clear that (i), (ii) and (iii) are equivalent and that they imply (iv). Proposition 3.2 in [6] shows that (iv) implies (v). If $u \in \mathrm{PSH}(X, g)$, then $u - C < 0 < u + C$ for a constant C large enough. This shows that (v) implies (vi) since $u - C, 0, u + C \in \mathrm{PSH}(X, g)$. Finally, (vi) implies (iii) from the above Theorem 1. □

Remark 3. — (v) is called the comparison principle and it plays a crucial role in the pluripotential study of the complex Monge–Ampère equation.

Remark 4. — The above result implies that a Hermitian metric g on a compact complex manifold which satisfies $i\partial\bar{\partial}g \geqslant 0$ and $i\partial g \wedge \bar{\partial}g \geqslant 0$ actually satisfies $i\partial\bar{\partial}g = i\partial g \wedge \bar{\partial}g = 0$, and the proof goes through Proposition 3.2 in [6]. However, there is a more direct proof of this fact. Indeed, from

$$d(ig^{n-2} \wedge \bar{\partial}g) = (n-2)g^{n-3} \wedge i\partial g \wedge \bar{\partial}g + g^{n-2} \wedge i\partial\bar{\partial}g \tag{15}$$

and Stokes' theorem, it follows that

$$(n-2)\int_X g^{n-3} \wedge i\partial g \wedge \bar{\partial}g + \int_X g^{n-2} \wedge i\partial\bar{\partial}g = 0 \tag{16}$$

and, since both integrals are positive, it follows that they have to be zero. Therefore $g^{n-3} \wedge i\partial g \wedge \bar{\partial}g = g^{n-2} \wedge i\partial\bar{\partial}g = 0$, that is the trace measures of the positive currents $i\partial g \wedge \bar{\partial}g$ and $i\partial\bar{\partial}g$ with respect to g are zero, hence they have to be zero.

Remark 5. — If a compact complex manifold admits a Hermitian metric g as above, i.e., it satisfies $i\partial\bar{\partial}g = i\partial\bar{\partial}g^2 = 0$, and if $p : Y \to X$ is a blowup with a smooth center, then Y supports a Hermitian metric g' with the same property. Indeed, if E is the exceptional divisor, then the line bundle $\mathcal{O}(-[E])$ has a metric with curvature β such that $g' = Np^*g + \beta$ is positive for N a large enough constant. Then it is trivial to show that g' satisfies the same equations as g. However, it is not true that this property is a bimeromorphic invariant, and this follows from the fact that a manifold bimeromorphic to a Kähler manifold need not be Kähler [9], combined with the fact that a Fujiki manifold which supports a SKT metric is Kähler [2].

Remark 6. — It is fairly easy to construct metrics as in Theorem 2. Take (X, g) to be a manifold as in Theorem 2 and (Y, h) a Kähler manifold. If p_X and p_Y denote the two projections, then the metric $p_X^* g + p_Y^* h$ also satisfies the above equations on $X \times Y$. Now take g to be a Gauduchon metric on a non-Kähler surface X to obtain examples of non-Kähler manifolds that admit metrics as above.

Remark 7. — A similar question is the following:

Question 8. — *Characterize the Hermitian metrics g on a compact complex manifold X such that there exists a constant C such that*

$$\int_X (g + i\partial\bar{\partial}u)^n \leqslant C, \quad \forall\, u \in \mathrm{PSH}(X, g).$$

This question is related to the proof of Theorem 4.1 in [3]. For instance, on surfaces, any Hermitian metric has the above property, while on 3-folds, this class includes the Hermitian metrics g that satisfy $i\partial\bar{\partial}g \geqslant 0$. Indeed, if $u \in \mathrm{PSH}(X, g)$, then

$$\int_X (g + i\partial\bar{\partial}u)^3 = \int_X g^3 + 3\int_X g^2 \wedge i\partial\bar{\partial}u + 3\int_X g \wedge (i\partial\bar{\partial}u)^2$$

and the second integral on the right is known to be bounded, while the third integral is negative if $i\partial\bar{\partial}g \geqslant 0$ since

$$\int_X g \wedge (i\partial\bar{\partial}u)^2 = -\int_X i\partial\bar{\partial}g \wedge i\partial u \wedge \bar{\partial}u \leqslant 0$$

from Stokes' theorem. In particular, as in [3], we obtain that if a Hermitian 3-fold (X, g) such that $i\partial\bar{\partial}g \geqslant 0$ admits a nef class α of non-negative self-intersection, then X is Kähler (see Theorem 2.3 in [2]).

Remark 9. — Partial results in this direction have been obtained recently by Guedj and Lu in [8] and by Angella, Guedj and Lu in [1]. In [8], the authors associated to a Hermitian metric g on a compact complex manifold

X of dimension n two numbers,

$$v_+(g) = \sup \left\{ \int_X (g + i\partial\bar{\partial}u)^n \;\middle|\; u \in \mathrm{PSH}(X,g) \cap \mathcal{C}^\infty(X,\mathbb{R}) \right\}$$

and

$$v_-(g) = \inf \left\{ \int_X (g + i\partial\bar{\partial}u)^n \;\middle|\; u \in \mathrm{PSH}(X,g) \cap \mathcal{C}^\infty(X,\mathbb{R}) \right\}.$$

Note that Question 8 is equivalent to finding the metrics g for which $v_+(g) < \infty$. In [8], the authors showed that the conditions $v_+(g) < \infty$ and $v_-(g) < \infty$ are independent of the choice of g, therefore they are an intrinsic property of the manifold X. In [1] the authors further study these conditions.

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