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On some elliptic fractional $s(\,\cdot\,)$ problems with singular potential and general datum [(*)]

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Abstract. — The purpose this work is to address the question of existence and regularity of solutions to a class of nonlocal elliptic problems with variable-order fractional Laplace operator and whose behaviors are complicated by the presence of singular nonlinearities. First, we prove the existence of weak solutions for a large class of data, including measures in some cases. We also obtain additional regularity properties under suitable extra assumptions. Second, we show that, in the case of measures datum, existence analysis is strongly related to the fractional capacity associated to the fractional Sobolev spaces. As a consequence, we get *the natural form* of the adequate "fractional gradient" when dealing with the Hamilton–Jacobi fractional equation with nonlocal gradient term in the sense of Boccardo–Gallouët–Orsina decomposition Problem.

Résumé. — Le but de ce travail est d'étudier la question de l'existence et la régularité des solutions d'une classe de problèmes elliptiques non locaux gouvernés par l'opérateur de Laplace fractionnaire d'ordre variable, et dont le second membre est non linéaire et comporte des singularités. En premier lieu, nous prouvons l'existence de solutions faibles pour une grande classe de données, y compris pour des données mesures. Ensuite, nous montrons que lorsque les données sont régulières, les solutions le sont aussi. Enfin, nous montrons que, dans le cas de données de mesures, l'existence de solutions est fortement liée à la capacité fractionnaire associée aux espaces de Sobolev fractionnaires. Ce qui nous a permis d'obtenir la forme naturelle du « gradient fractionnaire » adéquat lorsque nous traitons l'équation fractionnaire de Hamilton–Jacobi avec un gradient non local dans le sens de décomposition de Boccardo–Gallouët–Orsina.

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1. Introduction

Let us consider the following problem

$$\begin{cases} (-\Delta)^{s(\,\cdot\,)}u = \dfrac{g(x)}{u^{\sigma(x)}} + f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P}$$

where $s(\,\cdot\,) : \mathbb{R}^N \times \mathbb{R}^N \to (0,1)$ is a continuous function, Ω is a bounded regular domain of $\mathbb{R}^N$ ($\mathcal{C}^{1,1}$ regularity is sufficient) with $N > 2s(x,y)$ for all $(x,y) \in \Omega \times \Omega$, f and g are nonnegative measurable functions or Radon measures with suitable assumptions, σ is a positive continuous function on $\overline{\Omega}$.

The variable-order fractional Laplace operator is defined as follows: for each $x \in \mathbb{R}^N$ and for all $u \in C_0^\infty(\Omega)$

$$(-\Delta)^{s(\,\cdot\,)}u(x) := 2\,\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s(x,y)}}\,\mathrm{d}y, \tag{1.1}$$

where P.V. denotes the Cauchy principal value.

If $s(\,\cdot\,) \equiv \text{constant} \in (0,1)$, $(-\Delta)^{s(\,\cdot\,)}$ is nothing but the so-called regional fractional Laplacian. See, for instance, [13, 19, 21, 24, 29, 33, 47, 53] and the references therein for more details about this operator.

In the case where $s(\,\cdot\,) \neq \text{constant}$, the operator $(-\Delta)^{s(\,\cdot\,)}$ arises in a quite way in many applications such as continuum mechanics, phase transition phenomena, …; see [48, 49] for more details. We refer also to [41, 46, 55, 60] and the references therein for the analytic theory of variable-order fractional Laplace operators and their properties.

It is a natural question to ask for which data g, f and σ, Problem (P) admits a solution and to study the regularity of the solution according to the regularity of the data. This is the main goal of this work. As far as we know, this paper is the first time to study the Problem (P) driven by the variable-order fractional Laplace operator $(-\Delta)^{s(\,\cdot\,)}$.

To put our work in context, let us review some well known results about Problem (P) in some particular cases.

(1) The case $s(\,\cdot\,) \equiv 1$, $\sigma(\,\cdot\,) \equiv \textit{const}$ and $f \equiv 0$. The problem was studied by Crandall, Rabinowitz and Tartar in [26], where they proved existence of solution. Additional properties of the solution are also obtained. In [44], the regularity of the solution is analyzed up to the boundary of the domain in the case where g is a continuous function. In [17], according to the summability of the datum g and

the value of σ, the authors studied the summability of the solution. Other properties of the solution are also obtained in [22, 23] such as symmetric properties if Ω is symmetric. The case $f(x,u) = \lambda f(x)$ is considered in [39] where existence of bounded solution is proved if $f, g \in L^p(\Omega)$ with $p > \frac{N}{2}$.

(2) The case $0 < s(\cdot) = const < 1$, $\sigma(\cdot) \equiv const \equiv \sigma > 0$ and $f \equiv 0$. The problem has been recently considered in [15]. The authors have shown the existence of a positive solution under suitable assumptions on g. The behavior of the solution near the boundary is also studied for bounded data g. See also [1] and [40].

(3) The case $0 < s(\cdot) = const < 1$, $\sigma(\cdot) \equiv const \equiv \sigma$ and $f \neq 0$, has been studied using monotony and variational arguments. The existence of a positive solution is obtained when f has some subcritical or critical behavior in u. See for instance [15] and [40] and the references therein.

Before ending this short review, let us mention two related works:

- the first one considers a similar problem where the variable-order fractional Laplacian in the left-hand side is replaced by the local anisotropic operator $\sum_{1\leqslant i\leqslant N} \partial_i \left(|\partial_i u_i|^{p_i-2}\partial_i u_i\right)$, see [52] for more details. In that paper, existence and regularity of the solution is established under some conditions on the behavior of the function $\sigma(x)$ near the boundary of Ω;
- the second one deals with the problem (P_λ) : $(-\Delta)^{s(\cdot)}u = \frac{\lambda+h(x)}{|x|^{2s(\cdot)}}u(x) + k(x)u^{2^*_{s(\cdot)}-1}$ and $u > 0$ set in $\mathbb{R}^N$ where $0 < s(\cdot) = const < 1$, $\lambda > 0$, h and k are nonnegative functions. We refer the interested reader to [5] where the second author and his coworkers have shown that (P_λ) has multiple positive solutions and have determined their precise behavior near the extremal points under some suitable assumptions on the data.

In this work, we will consider the general case $s(\cdot) \neq$ constant and $f, g \in L^1(\Omega)$. We treat also some particular cases where f and g are nonnegative Radon measures. We will analyze the impact of the singular term on the existence and nonexistence of a positive solution according to the regularity of the data.

One of the major difficulties is to estimate the singular term on the set where $u = 0$. When $s(\cdot) =$ constant, this is done by using the classical strong maximum principle, which is a consequence of the weak Harnack inequality for weak solutions. To the best of our knowledge, there is no similar result in the case where $s(\cdot)$ is variable. Thus, the first part of this work will be dedicated to prove a suitable weak Harnack inequality for our operator. The proof is inspired from [31, 43], see also [9], taking into consideration the lack of homogeneity caused by the new form of the function $s(\cdot)$.

As a consequence, we get a strong maximum principle for nonnegative solution that allows us to obtain suitable estimates on the singular term.

The other major difficulty is that, unlike the case where $s(\cdot) =$ constant, the behavior near the boundary $\partial\Omega$ of the solution to the Poisson equation

$$\begin{cases} (-\Delta)^{s(\cdot)} w = h & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ w > 0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

is unknown. If $s(\cdot) =$ constant, then for bounded datum h, we know that w behaves like $\operatorname{dist}^s(., \partial\Omega)$. This estimate turns out to be useful in order to control the singular term.

To circumvent this difficulty in our case, we adopt a new approach using suitable test functions. As a first step towards understanding the full Problem (P), we begin by analyzing the Problem (1.2), where the function $s(\cdot)$ satisfies two basic assumptions (H_1) and (H_2) (see Section 2), and h is a measurable nonnegative function satisfying suitable summability conditions that will be specified later. We prove the existence, uniqueness and summability of the *weak solution* in the corresponding fractional Sobolev space. To establish the corresponding regularity result in our case, we will follow the nonlinear approach of [4] and [7].

Then, we will focus our attention on the main Problem (P). In order to take in consideration the difficulties due to the singular term, one is naturally led to distinguish between the case where $\sigma(\cdot)$ is constant or not.

Case where $\sigma(\cdot)$ is constant. —

- First, we begin with the sub-case where f depends only on x. We show the existence of a solution for all $(f, g) \in L^1(\Omega) \times L^1(\Omega)$.
- Second, we treat a model case where f depends also on u.
- Third, we study the case where g is a bounded Radon measure and $f = 0$. Here we will show that the solution u lives in a suitable fractional Sobolev space.

It is interesting to note that, one of our main contributions is to relate the existence of a solution to the regularity of the measure g with respect to the corresponding fractional capacity. A strong nonexistence result is also proved if the measure g is singular respect to a suitable fractional capacity. Partial uniqueness results are also proved according to additional hypothesis on σ. As we will show, a relation between the measure g and the fractional capacity is needed in order to prove the existence of a weak solution (or a solution obtained as a limit of approximations). Therefore, when $g \equiv \mu_s$, a singular measure with respect to the relative capacity $\mathrm{Cap}^{\overline{\Omega}}_{s(\,\cdot\,),2}$ (see Definition 5.7), we are able to prove a strong nonexistence result.

However, if $g = g_0 + \mu_s$ with $g_0 \in L^1(\Omega)$ and μ_s is as above, then the sequence of solutions to the corresponding approximated problem converges to the unique solution of the singular problem with datum g_0.

Case where $\sigma(\,\cdot\,)$ is nonconstant. — In this case, the situation is more complicated as we will see later. However, we are also able to prove the existence of a solution but with less regularity compared to the case $\sigma(\,\cdot\,) =$ constant.

The paper is organized as follows. In Section 2, we give some auxiliary results on the variable-order fractional Sobolev spaces and some functional inequalities, as well as and some useful tools that will be used systematically throughout the paper. The existence and the regularity of the Poisson equation 1.2 is dealt with in Section 3. In Section 4, we prove a weak Harnack inequality for our operator. This result will be used to obtain some useful a priori estimates. In Section 5, we deal with the main Problem (P) in the case where $\sigma(\,\cdot\,)$ is constant (i.e. does not depend of x). In Subsection 5.1, we consider the case where $g, f \in L^1(\Omega)$. Subsection 5.2 is devoted to a model case where f depends on u. The case where g is a general nonnegative Radon measure is treated in Subsection 5.3. Finally, we study the main Problem (P) in the case where $\sigma(\,\cdot\,)$ is nonconstant in Section 6.

To carry out our study, particularly in the case where g is a Radon measure, we shall need a number of technical results form capacity theory. We have felt it necessary to expound them in order of make the paper self-contained. However, as this theme is not central in the paper, we have gathered them in a quite long appendix, which can be skipped in a first reading.

Nevertheless, the appendix may be useful, as a first introduction, to readers not already familiar with capacity theory. More precisely, it deals with two main issues:

(1) A theory of capacities linked to fractional Sobolev spaces of variable order, where two types of capacities are treated: variable fractional $(s(\,\cdot\,), p)$-capacity (see Definition 7.2) and variable fractional relative $(s(\,\cdot\,), p)$-capacity with respect to Ω (see Definition 7.4).
(2) A decomposition Theorem for regular signed measures with respect to (s, p)-capacity, inspired by the paper [16]. More precisely:
   (a) any such measure can be written as the sum of an element of $W^{-s,p'}(\Omega)$ (dual space of $W^{s,p}_0(\Omega)$) and a function in $L^1(\Omega)$;

(b) any bounded Radon measure μ, with $\mu \in L^1(\Omega) + W^{-s,p'}(\Omega)$, is regular with respect to (s,p)-capacity.

As an interesting application of the above decomposition, we extend the famous result of [16] about a nonlinear problem involving a usual Laplacian, with gradient term and measure data. More precisely, let $D_s^2(u)$ be a version of a nonlocal gradient term given by

$$(D_s^2(u))(x) = \frac{a_{N,s}}{2}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}y$$

where

$$a(N,s) := \frac{s4^s}{\pi^{\frac{N}{2}}}\frac{\Gamma(s+\frac{N}{2})}{\Gamma(1-s)}. \tag{1.3}$$

Notice that this operator can be at least tracked to [58]. Moreover, it naturally appears as the nonlocal equivalent to the gradient when considering the minimization of fractional harmonic maps into the sphere. See for instance the recent papers [20, 54, 56]. Then, by using the above decomposition result, we prove that if the problem

$$(-\Delta)^s u + uD_s^2(u) = \mu \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \tag{1.4}$$

has a nonnegative solution u with $uD_s^2(u) \in L^1(\Omega)$, then μ must be a regular measure.

It should be noted that several “fractional gradients” can be defined. The two examples often found in the literature are:

- the so-called “half s-Laplacien”

$$(-\Delta)^{s/2}(u)(x) = \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+s}}\mathrm{d}y, \quad x \in \mathbb{R}^N;$$

- the so-called “Riesz s-Laplacien”

$$\nabla^s u(x) := \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^s}\frac{x-y}{|x-y|}\frac{\mathrm{d}y}{|x-y|^N}, \quad x \in \mathbb{R}^N.$$

In this regard, let us mention that the following nonlocal Kardar–Parisi–Zhang Problem

$$\begin{cases} \partial_t w + (-\Delta)^s w = |(-\Delta)^{s/2} w|^p + f & \text{in } \Omega \times (0,T) \\ w(x,t) = 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ w(x,0) = w_0(x) & \text{in } \Omega \end{cases}$$

is considered in [2]. For other existence and nonexistence results involving the fractional Laplacian and local gradient, see [12] and the references therein.

Now, setting

$$\partial^s_{x_i} u(x) := \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^s} \frac{x_i - y_i}{|x-y|} \frac{\mathrm{d}y}{|x-y|^N}, \quad x \in \mathbb{R}^N, i = 1, \ldots, N,$$

then it is interesting to note that for $u \in \mathcal{C}^\infty_0(\mathbb{R}^N)$, we have

$$(-\Delta)^s u = -\operatorname{div}^s(\nabla^s u(x)) := \sum_{i=1}^N \partial^s_{x_i} u(x).$$

We refer to [20, 50, 57] for additional properties of these nonlocal gradients and some relations between them.

In our opinion, our existence result for Problem (1.4) can be seen as a justification of the fact that $D_s(u)$ is the natural fractional gradient best suited to fractional capacity.

2. The functional setting and tools

In this section, we present some useful tools related to our operator. Inspired by the works [7, 45, 60], *we assume in the whole paper* that $s(\,\cdot\,) : \mathbb{R}^N \times \mathbb{R}^N \to (0,1)$ is a bounded continuous function and satisfying these two following assumptions:

(H_1) $0 < s_0 := \inf_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x,y) \leqslant s_1 := \sup_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x,y) < 1$;

(H_2) $s(\,\cdot\,)$ is symmetric, that is $s(x,y) = s(y,x)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

2.1. The functional setting

In this subsection, Ω denotes an arbitrary (unless otherwise specified) open subset of $\mathbb{R}^N$ and $p \in [1,+\infty)$.

As in the case where $s(\,\cdot\,)$ is constant, we start this paragraph by introducing the definition of Fractional Sobolev Spaces $W^{s(\,\cdot\,),p}(\Omega)$. Then, we define the spaces $\mathbb{W}^{s(\,\cdot\,),p}_0(\Omega)$ which are the appropriate space functional framework to study our Problem (P).

Fractional Sobolev Spaces $W^{s(\,\cdot\,),p}(\Omega)$, $W_0^{s(\,\cdot\,),p}(\Omega)$, $\widetilde{W}^{s(\,\cdot\,),p}(\Omega)$.

We define

(1) the space $W^{s(\,\cdot\,),p}(\Omega)$ by

$$W^{s(\,\cdot\,),p}(\Omega) := \left\{ u \in L^p(\Omega)\ ;\ \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps(x,y)}}\mathrm{d}x\mathrm{d}y < +\infty \right\}.$$

Endowed with the norm

$$\|u\|_{W^{s(\,\cdot\,),p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps(x,y)}}\mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}},$$

$W^{s(\,\cdot\,),p}(\Omega)$ is a Banach space.

(2) $W_0^{s(\,\cdot\,),p}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{s(\,\cdot\,),p}(\Omega)}$.

(3) $\widetilde{W}^{s(\,\cdot\,),p}(\Omega) := \overline{W^{s(\,\cdot\,),p}(\Omega) \cap \mathcal{C}_0(\overline{\Omega})}^{W^{s(\,\cdot\,),p}(\Omega)}$.

Let us observe that $W_0^{s(\,\cdot\,),p}(\Omega)$ is the smaller closed subspace of $W^{s(\,\cdot\,),p}(\Omega)$ containing $C_0^\infty(\Omega)$. In addition, $W_0^{s(\,\cdot\,),p}(\Omega) \subset \widetilde{W}^{s(\,\cdot\,),p}(\Omega)$; in the case where $s(\,\cdot\,)$ is constant, see [59] for additional details about this inclusion.

Other type of fractional Sobolev Spaces $\mathbb{W}_0^{s(\,\cdot\,),p}(\Omega)$.

In order to take in consideration the interaction between Ω and $\mathbb{R}^N \setminus \Omega$, we need appropriate fractional Sobolev spaces. For this purpose, we introduce the following space

$$\mathbb{W}_0^{s(\,\cdot\,),p}(\Omega) := \left\{ u \in W^{s(\,\cdot\,),p}(\mathbb{R}^N)\ ;\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}$$

endowed with the norm induced by $\|u\|_{W^{s(\,\cdot\,),p}(\mathbb{R}^N)}$.

It is clear that $\mathbb{W}_0^{s(\,\cdot\,),p}(\Omega)$ is a Banach space. Moreover, if Ω is a bounded domain, then by using Poincaré's inequality, we can endowed the space $\mathbb{W}_0^{s(\,\cdot\,),p}(\Omega)$ with the norm

$$\|u\|_{\mathbb{W}_0^{s(\,\cdot\,),p}(\Omega)} := \left(\iint_{D_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps(x,y)}}\mathrm{d}x\mathrm{d}y \right)^{1/p},$$

where $D_\Omega := \mathbb{R}^{2N}\backslash(\Omega^c \times \Omega^c)$.

Particular case $p = 2$.

In that case, we denote by $\mathbb{H}_0^{s(\,\cdot\,)}(\Omega) = \mathbb{W}_0^{s(\,\cdot\,),2}(\Omega)$. It is worth to mention that, as in the case where $s(\,\cdot\,)$ is a constant, $(\mathbb{H}_0^{s(\,\cdot\,)}(\Omega), \|\cdot\|_{\mathbb{H}_0^{s(\,\cdot\,)}(\Omega)})$ is a Hilbert space.

Under the assumptions (H_1) and (H_2), we can show that the space $(\mathbb{H}^{s(\,\cdot\,)}(\Omega), \|\cdot\|_{s(\,\cdot\,)})$ is separable. The proof closely follows the arguments of [53] and [60]. For the sake of completeness, let us provide some details. Assume that (H_1), (H_2) hold.

Define the operator $T : \mathbb{H}^{s(\cdot)}(\Omega) \to L^2(\Omega) \times L^2(\Omega \times \Omega)$ with

$$T(u) = \left(u, \frac{u(x) - u(y)}{|x-y|^{\frac{N}{2}+s(x,y)}} \right)$$

Notice that $L^2(\Omega) \times L^2(\Omega \times \Omega)$ is a separable space. Using the fact that $T(\mathbb{H}^{s(\cdot)}(\Omega))$ is a closed subspace in $L^2(\Omega) \times L^2(\Omega \times \Omega)$ and since T is an isometry between $\mathbb{H}^{s(\cdot)}(\Omega)$ and $T(\mathbb{H}^{s(\cdot)}(\Omega))$, we deduce that $\mathbb{H}^{s(\cdot)}(\Omega)$ is a separable space.

Remark 2.1. — As in [60], under the assumptions (H_1) and (H_2), we can prove that the embedding $\mathbb{W}_0^{s_1,p}(\Omega) \hookrightarrow \mathbb{W}_0^{s(\cdot),p}(\Omega) \hookrightarrow \mathbb{W}_0^{s_0,p}(\Omega)$ are continuous. Moreover, if $N > ps_0$ for any fixed constant exponent $q \in [1, \frac{2N}{N-s_0p}]$, and the space $\mathbb{W}_0^{s(\cdot),p}(\Omega)$ is continuously embedded into $L^q(\Omega)$.

Let us recall the next Hardy inequality proved in [35] or [38].

THEOREM 2.2. — *Let $\Omega \subset \mathbb{R}^N$ be a bounded regular domain. Then, there exists a positive constant $C = C(s_0, \Omega)$ such that for all $\phi \in \mathbb{W}_0^{s_0,p}(\Omega)$, we have*

$$C \int_\Omega \frac{|\phi(x)|^p}{\rho^{ps_0}(x)} \, dx \leqslant \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+ps_0}} \, dx dy, \tag{2.1}$$

where $\rho(x) = \mathrm{dist}(x, \partial\Omega)$.

Since $\mathbb{W}_0^{s(\cdot),p}(\Omega) \hookrightarrow \mathbb{W}_0^{s_0,p}(\Omega)$, under the same hypotheses as in Theorem 2.2, we obtain the next result.

THEOREM 2.3. — *Let $\Omega \subset \mathbb{R}^N$ be a bounded regular domain. Then, there exists a positive constant $C = C(s_0, \Omega)$ such that for all $\phi \in \mathbb{W}_0^{s(\cdot),p}(\Omega)$, we have*

$$C \int_\Omega \frac{|\phi(x)|^p}{\rho^{ps_0}(x)} \, dx \leqslant \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+ps(x,y)}} \, dx dy. \tag{2.2}$$

As we will consider problems with general data, we need the concept of the truncation. For $k > 0$, we define $T_k(t)$ by

$$T_k(t) = \begin{cases} t & \text{if } |t| \leqslant k \\ k\frac{t}{|t|} & \text{if } |t| > k\,; \end{cases} \tag{2.3}$$

and

$$G_k(t) := t - T_k(t).$$

The following lemma will be very useful in this paper.

Lemma 2.4. — *Let $\phi \in Lip(\mathbb{R})$ (the space of global lipschitz functions) with $\phi(0)=0$. Then, $\phi(u) \in \mathbb{H}_0^{s(\cdot)}(\Omega)$. Moreover, for any $k \geqslant 0$, $T_k(u)$ and $G_k(u) \in \mathbb{H}_0^{s(\cdot)}(\Omega)$; and if $u \geqslant 0$, we have*

$$\|G_k(u)\|^2_{\mathbb{H}_0^{s(\cdot)}(\Omega)} \leqslant \int_\Omega G_k(u)(-\Delta)^{s(\cdot)} u \mathrm{d}x,$$
$$\|T_k(u)\|^2_{\mathbb{H}_0^{s(\cdot)}(\Omega)} \leqslant \int_\Omega T_k(u)(-\Delta)^{s(\cdot)} u \mathrm{d}x.$$

Proof. — The proof is similar to the proof of [45, Proposition 3]. □

2.2. Some useful inequalities

First, we recall the next Kato type inequality, whose proof can be found in [45] in the special case of the classical fractional Laplacian.

Theorem 2.5. — *Let $\Phi \in \mathcal{C}^2(\mathbb{R}^N)$ be a convex function. Let $u \in \mathbb{W}_0^{s(\cdot),1}(\Omega)$ such that $\Phi(u) \in \mathbb{W}_0^{s(\cdot),1}(\Omega)$, $(-\Delta)^{s(\cdot)} u \in L^1(\Omega)$, $(-\Delta)^{s(\cdot)}\Phi(u) \in L^1(\Omega)$ and $\Phi'(u)(-\Delta)^{s(\cdot)} u \in L^1(\Omega)$. Then*

$$(-\Delta)^{s(\cdot)}\Phi(u) \leqslant \Phi'(u)(-\Delta)^{s(\cdot)} u \text{ in the weak sense,} \tag{2.4}$$

namely, for all $\psi \in C_0^\infty(\Omega)$ with $\psi \geqslant 0$, we have

$$\int_\Omega \Phi(u)(-\Delta)^{s(\cdot)}\psi \,\mathrm{d}x \leqslant \int_\Omega \psi\Phi'(u)(-\Delta)^{s(\cdot)} u \,\mathrm{d}x. \tag{2.5}$$

In particular, we have the next corollary.

Corollary 2.6. — *Let $u \in \mathbb{H}_0^{s(\cdot)}(\Omega)$ be a nonnegative function. Suppose that $(-\Delta)^{s(\cdot)} u \in L^1(\Omega)$ is a nonnegative function. Then for any $k > 0$, we have*

$$(-\Delta)^{s(\cdot)} T_k(u) \geqslant \chi_{\{x\in\Omega\,;\,u(x)\leqslant k\}}(-\Delta)^{s(\cdot)} u \quad \text{weakly in } \Omega.$$

Namely, for all $\psi \in C_0^\infty(\Omega)$ with $\psi \geqslant 0$, we have

$$\int_\Omega T_k(u)(-\Delta)^{s(\cdot)}\psi \,\mathrm{d}x \geqslant \int_\Omega \psi\chi_{\{x\in\Omega\,;\,u(x)\leqslant k\}}(-\Delta)^{s(\cdot)} u \,\mathrm{d}x. \tag{2.6}$$

The next Picone type inequality will be useful in order to prove comparison principles. The proof is a simple variation of the arguments used in [4] and [45].

Proposition 2.7. — *Let $u \in \mathbb{H}_0^{s(\cdot)}(\Omega)$ be a nonnegative function such that $(-\Delta)^{s(\cdot)}u \in L^1(\Omega)$ with $-\Delta)^{s(\cdot)}u \geqslant 0$ a.e. in Ω. Suppose in addition that $\frac{(-\Delta)^{s(\cdot)}u}{u} \in L^1_{\mathrm{loc}}(\Omega)$. Then, for all $\phi \in \mathcal{C}_0^\infty(\Omega)$, we have*

$$\int_\Omega \frac{(-\Delta)^{s(\cdot)}u}{u}\phi^2\,\mathrm{d}x \leqslant \frac{1}{2}\iint_{D_\Omega} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s(x,y)}}\,\mathrm{d}x\mathrm{d}y. \tag{2.7}$$

As a consequence, we have the next comparison principle that extends to the fractional framework the classical one obtained by Brezis and Kamin in [18]. See [4] and [45] for the case where $0 < s(\cdot) \equiv$ *constant* < 1.

Theorem 2.8. — *Let f be a nonnegative continuous function with $f(x,r) > 0$ if $r > 0$ and $r \mapsto \frac{f(x,r)}{r}$ is decreasing for $r > 0$. Let $u, v \in \mathbb{H}_0^{s(\cdot)}(\Omega)$ be such that $u, v > 0$ a.e. in Ω and*

$$\begin{aligned}(-\Delta)^{s(\cdot)}u &\geqslant f(x,u) \quad \text{in } \Omega,\\ (-\Delta)^{s(\cdot)}v &\leqslant f(x,v) \quad \text{in } \Omega.\end{aligned}$$

Then $u \geqslant v$ in Ω.

Proof. — The proof follows from suitable rescaling arguments. □

Before closing this section, let us recall some useful algebraic inequalities that will be used throughout this paper.

Lemma 2.9. — *Let $(a,b) \in [0,+\infty) \times [0,+\infty)$ and $(\alpha,k) \in (0,+\infty)^2$. Then, there exist positive constants $c_i, i = 1,\dots,5$ such that*

$$(a+b)^\alpha \leqslant c_1 a^\alpha + c_2 b^\alpha\ ; \tag{2.8}$$

$$(a-b)(a^\alpha - b^\alpha) \geqslant c_3 |a^{\frac{\alpha+1}{2}} - b^{\frac{\alpha+1}{2}}|^2\ ; \tag{2.9}$$

$$(a-b)(T_k(a^\alpha) - T_k(b^\alpha)) \geqslant c_4 \left(T_k(a^{\frac{\alpha+1}{2}}) - T_k(b^{\frac{\alpha+1}{2}})\right)^2\ ; \tag{2.10}$$

moreover, if $\alpha \geqslant 1$ we have

$$|a+b|^{\alpha-1}|a-b|^2 \leqslant c_5 |a^{\frac{\alpha+1}{2}} - b^{\frac{\alpha+1}{2}}|^2. \tag{2.11}$$

Proof. — The proof is elementary and is left to the reader. □

Finally, we need also the following well-know iteration lemma.

Lemma 2.10. — *Let $\beta > 0$ and let $\{A_j\}_{j\leqslant 0}$ be a sequence of real positive numbers such that*

$$A_{j+1} \leqslant c_0 b^j A_j^{\beta+1},$$

with $c_0 > 0$ and $\beta > 1$. If $A_0 \leqslant c_0^{-\frac{1}{\beta}} b^{-\frac{1}{\beta^2}}$, then

$$A_j \leqslant b^{-\frac{j}{\beta}} A_0.$$

In particular, $\lim_{j\to\infty} A_j = 0$.

3. Existence and regularity results for the Poisson problem

In this section, we prove the main existence results for Problem (P) in the case where $g \equiv 0$. More precisely, we consider the following problem

$$\begin{cases} (-\Delta)^{s(\,\cdot\,)} u = h & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega \end{cases} \tag{3.1}$$

where $h \in L^m(\Omega)$ with $m \geqslant 1$.

We begin by analyzing the Poisson equation with the variable-order fractional Laplace operator. According to the regularity of the datum, we will prove that the solution lives in a suitable fractional Sobolev space. Our approach is based on the choice of suitable test functions as in [4]. To make the paper self-contained as possible, we will include details of the proofs.

To carry out this study, we have to distinguish two cases depending on whether h is only integrable or not.

The case where $h \in L^m(\Omega)$ with $m > \frac{2N}{N+2s_0}$

We will start by specifying the sense of solution to (3.1) in this case.

Definition 3.1. — *Let $\Omega \subset \mathbb{R}^N$ be bounded regular domain and $h \in \mathbb{H}^{-s(\,\cdot\,)}(\Omega)$ where $\mathbb{H}^{-s(\,\cdot\,)}(\Omega) \equiv (\mathbb{H}_0^{s(\,\cdot\,)}(\Omega))'$. We say that $u \in \mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$ is a finite energy solution to* (3.1) *if*

$$a(u,v) = \langle h, v \rangle \quad \forall\, v \in \mathbb{H}_0^{s(\,\cdot\,)}(\Omega),$$

where

$$a(u,v) := \iint_{D_\Omega} \frac{(v(x)-v(y))(u(x)-u(y))}{|x-y|^{N+2s(x,y)}} \mathrm{d}x\mathrm{d}y.$$

Since $h \in \mathbb{H}^{-s(\,\cdot\,)}(\Omega)$, then using Lax–Milgram Theorem we get the existence and the uniqueness of u. Notice that if $h \in L^a(\Omega)$ with $a > \frac{2N}{N+2s_0}$, then $h \in \mathbb{H}^{-s(\,\cdot\,)}(\Omega)$.

Following closely the argument used in [45], we are able to prove the next regularity result.

THEOREM 3.2. — *Assume that* (H_1), (H_2) *hold and* $h \in L^m(\Omega)$ *where* $m > \frac{2N}{N+2s_0}$. *Then, we have:*

(1) *if* $m > \frac{N}{2s_0}$ *with* s_0 *is defined in* (H_1), *there exists a constant* $C \equiv C(N, \Omega, \|h\|_{L^m(\Omega)}) > 0$ *such that the unique energy solution of* (3.1) *satisfies,*
$$\|u\|_{L^\infty(\Omega)} \leqslant C\|h\|_{L^m(\Omega)} \; ; \tag{3.2}$$
(2) *if* $m = \frac{N}{2s_0}$, *then, there exists* $\alpha > 0$ *depending only on the data and it is independent of* h *and* u *such the if* u *is the unique solution to* (3.1) *(in the sense of Definition 3.1), then,*
$$\int_\Omega e^{\alpha u} < \infty \; ;$$
(3) *if* $\frac{2N}{N+2s_0} < m < \frac{N}{2s_0}$, *then* $u \in L^{m_{s_0^{**}}}(\Omega)$ *where* $m_{s_0^{**}} = \frac{mN}{N-2ms_0}$ *and there exists a constant* C *such that*
$$\|u\|_{L^{m_{s_0^{**}}}(\Omega)} \leqslant C\|h\|_{L^m(\Omega)}. \tag{3.3}$$

Remark 3.3. — Notice that if (H_1), (H_2) hold and $h \geqslant 0$, then $u \geqslant 0$. Indeed, by using u^- as a test function in (3.1), we get,
$$0 \leqslant C\|u^-\|^2_{\mathbb{H}_0^{s(\cdot)}(\Omega)} \leqslant \int_\Omega hu^- \leqslant 0.$$
Hence $u^- = 0$ and the result follows.

The case where $h \in L^m(\Omega)$ with $1 \leqslant m \leqslant \frac{2N}{N+2s_0}$

Unfortunately, we cannot expect the existence of a solution of finite energy, unlike the local case. To address this difficulty, we extend the meaning of solutions, and we prove the existence of solution, in this weaker sense, to Problem (3.1).

DEFINITION 3.4. — *First of all, let us define the class of test functions,*
$$\mathcal{T}(\Omega) = \{\phi \in \mathbb{H}_0^{s(\cdot)}(\Omega) \; ; \; (-\Delta)^{s(\cdot)}\phi = \psi \text{ in } \Omega \text{ where } \psi \in C_0^\infty(\Omega)\}. \tag{3.4}$$
For $h \in L^1(\Omega)$, *we say that* $u \in L^1(\Omega)$ *is a weak solution to* (3.1) *if*
$$\int_\Omega u\,(-\Delta)^{s(\cdot)}\phi \,\mathrm{d}x = \int_\Omega h\phi \,\mathrm{d}x, \tag{3.5}$$
for any $\phi \in \mathcal{T}(\Omega)$.

Remark 3.5. — Observe that, if $\phi \in \mathcal{T}(\Omega)$, then under assumptions (H_1) and (H_2) it follows that $\phi \in \mathbb{H}_0^{s(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Assume now that (H_1) and (H_2) hold. Suppose in addition that $h \in (L^m(\Omega))^+$ with $1 \leqslant m < \frac{2N}{N+2s_0}$. If u is a solution to Problem (3.1) in the sense of Definition 3.4, then $u \geqslant 0$.
In fact, let $\psi \in \mathcal{C}_0^\infty(\Omega)$ be a nonnegative function. Define $\phi \in \mathcal{T}(\Omega)$ to be the unique solution to the problem $(-\Delta)^{s(\cdot)}\phi = \psi$ in Ω, then $\phi \geqslant 0$. Using ϕ as test function in Problem (3.1), we get,

$$\int_\Omega u\psi \,\mathrm{d}x = \int_\Omega h\phi \,\mathrm{d}x \geqslant 0.$$

Hence $\int_\Omega u\psi \,\mathrm{d}x \geqslant 0$ for all $\psi \in \mathcal{C}_0^\infty(\Omega)$ with $\psi \geqslant 0$. Thus $u \geqslant 0$ a.e. in $\mathbb{R}^N$.

Now, we are ready to state our existence result for L^1-data.

Theorem 3.6. — *Let $\Omega \subset \mathbb{R}^N$ be bounded regular domain. Then, for any $h \in L^1(\Omega)$, there exits a unique weak solution u in the sense of Definition 3.4 to Problem* (3.1) *such that*

$$u \in L^q(\Omega), \quad \forall\, q \in \left[1, \frac{N}{N-2s_0}\right), \tag{3.6}$$

$$\forall\, k > 0, \quad T_k(u) \in \mathbb{H}_0^{s(\cdot)}(\Omega), \tag{3.7}$$

and

$$\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+p\hat{s}}} \,\mathrm{d}x\mathrm{d}y < \infty, \quad \forall\, p \in \left[1, \frac{N}{N-s_0}\right) \text{ and } \forall\, \widehat{s} < s_0. \tag{3.8}$$

Proof. — Without loss of generality, we can assume that $h \gneqq 0$. We follow closely the argument used in [4, 6, 45]. The proof will be split into several steps.

Step 1. Uniqueness. — Let u be the weak positive solution in sense of definition 3.4 with $h \equiv 0$, then

$$\int_\Omega u\psi \,\mathrm{d}x = 0, \quad \forall\, \psi \in C_0^\infty(\Omega).$$

Thus, $u \equiv 0$ and then the uniqueness follows.

Step 2. Existence. — Let u_n be the solution to the following problem:

$$\begin{aligned} (-\Delta)^{s(\cdot)} u_n &= h_n \quad \text{in } \Omega,\\ u_n &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,\\ u_n &> 0 \quad \text{in } \Omega, \end{aligned} \tag{3.9}$$

where $h_n = \min\{h, n\}$. Therefore, $h_n \in L^\infty(\Omega)$ and $h_n \nearrow h$ in $L^1(\Omega)$. Moreover, we claim that, there exists a positive constant $C \equiv C(N, s_0, \Omega)$ such that

$$\|u_n\|_{L^q(\Omega)} \leqslant C\|h\|_{L^1(\Omega)}, \quad \forall\, q \in \left[1, \frac{N}{N-2s_0}\right). \tag{3.10}$$

To establish (3.10), we take $T_k(u_n)$ as test function in (3.9) where $k > 0$. Then

$$\|T_k(u_n)\|^2_{\mathbb{H}_0^{s(\cdot)}(\Omega)} \leqslant \int_\Omega T_k(u_n)(-\Delta)^{s(\cdot)} u_n \mathrm{d}x \leqslant k\|h_n\|_{L^1(\Omega)}. \tag{3.11}$$

Therefore, using Remark 2.1 and the Sobolev inequality, we get the existence of $C_0 > 0$ such that,

$$\|T_k(u_n)\|^2_{L^{2^*_{s_0}}(\Omega)} \leqslant \frac{k}{C_0}\|h_n\|_{L^1(\Omega)}. \tag{3.12}$$

Let $A_{n,k}(u_n) := \{x \in \Omega\,;\, u_n \geqslant k\}$, then

$$k^2|A_{n,k}(u_n)|^{\frac{N-2s_0}{N}} \leqslant \|T_k(u_n)\|^2_{L^{2^*_{s_0}}(\Omega)} \leqslant \frac{k}{C_0}\|h_n\|_{L^1(\Omega)}.$$

Thus

$$|A_{n,k}(u_n)| \leqslant (C_0)^{-\frac{N}{N-2s_0}} \left(\frac{\|h\|_{L^1(\Omega)}}{k}\right)^{\frac{N}{N-2s_0}}. \tag{3.13}$$

This means that $\{u_n\}$ is bounded in the Marcinkiewicz space $\mathcal{M}^{\frac{N}{N-2s_0}}(\Omega)$ and estimation (3.10) holds true.

Step 3. — Now, we will establish an estimate of u_n in a suitable fractional Sobolev space. For this purpose, we will choose a suitable test function taking into consideration that the datum h_n is only bounded in $L^1(\Omega)$.

Let $\widehat{s} < s_0$ be fixed and assume that $p < \frac{N}{N-\hat{s}}$. We claim that

$$\iint_{\Omega\times\Omega} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+p\hat{s}}}\,\mathrm{d}x\mathrm{d}y \leqslant C \text{ for all } n.$$

To prove the claim, we follow closely the arguments used in [4].

Let $\alpha > 0$ to be chosen later. Define $z_n(x) = 1 - \frac{1}{(u_n(x)+1)^\alpha}$, then $z_n \in \mathbb{H}_0^{s(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $\|z_n\|_{L^\infty(\Omega)} \leqslant 1$.

Using z_n as a test function in (3.9), it follows that

$$\frac{1}{2}\iint_{D_\Omega} \frac{(u_n(x) - u_n(y))\big((u_n(x)+1)^\alpha - (u_n(y)+1)^\alpha\big)\mathrm{d}y\mathrm{d}x}{(u_n(x)+1)^\alpha(u_n(y)+1)^\alpha|x-y|^{N+2s(x,y)}} \leqslant \int_\Omega h_n(x)\,\mathrm{d}x \leqslant C.$$

Let $w_n(x) = u_n(x) + 1$, then

$$(u_n(x) - u_n(y))\big((u_n(x)+1)^\alpha - (u_n(y)+1)^\alpha\big) \\ = (w_n(x) - w_n(y))\big(w_n^\alpha(x) - w_n^\alpha(y)\big).$$

Hence

$$\iint_{D_\Omega} \frac{(w_n(x) - w_n(y))\big(w_n^\alpha(x) - w_n^\alpha(y)\big)}{w_n^\alpha(x) w_n^\alpha(y)|x-y|^{N+2s(x,y)}} \leqslant C_1.$$

Since $w_n \geqslant 0$, by Lemma 2.9 there exists $C_2 > 0$ such that

$$\iint_{D_\Omega} \frac{|w_n^{\frac{1+\alpha}{2}}(x) - w_n^{\frac{1+\alpha}{2}}(y)|^2}{w_n^\alpha(x) w_n^\alpha(y)|x-y|^{N+2s(x,y)}} \leqslant C_2. \tag{3.14}$$

Now, since $p < 2$, using Hölder's inequality, we obtain

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|w_n(x) - w_n(y)|^p}{|x-y|^{N+p\hat{s}}} \mathrm{d}x\mathrm{d}y \\
&= \int_\Omega\int_\Omega \frac{|w_n(x) - w_n(y)|^p}{|x-y|^{ps(x,y)}} \frac{(w_n(x)+w_n(y))^{\alpha-1}}{(w_n(x)w_n(y))^\alpha} \frac{(w_n(x)w_n(y))^\alpha}{(w_n(x)+w_n(y))^{\alpha-1}} \\
&\qquad \times \frac{|x-y|^{ps(x,y)-p\hat{s}}\mathrm{d}y\mathrm{d}x}{|x-y|^N} \\
&\leqslant \left(\int_\Omega\int_\Omega \frac{|w_n(x) - w_n(y)|^2 (w_n(x)+w_n(y))^{\alpha-1}}{|x-y|^{N+2s(x,y)}(w_n(x)w_n(y))^\alpha} \mathrm{d}y\mathrm{d}x\right)^{\frac{p}{2}} \\
&\qquad \times \left(\int_\Omega\int_\Omega \frac{(w_n(x)+w_n(y))^{\alpha-1}}{(w_n(x)w_n(y))^\alpha} \frac{(w_n(x)w_n(y))^{\alpha\frac{2}{2-p}}}{(w_n(x)+w_n(y))^{(\alpha-1)\frac{2}{2-p}}}\right. \\
&\qquad\qquad \left. \times |x-y|^{\frac{2p(s(x,y)-\hat{s})}{2-p}} \frac{\mathrm{d}y\mathrm{d}x}{|x-y|^N}\right)^{\frac{2-p}{2}}.
\end{aligned} \tag{3.15}$$

But

$$|w_n(x) - w_n(y)|^2 (w_n(x)+w_n(y))^{\alpha-1} \leqslant C|w_n(x)^{\frac{1+\alpha}{2}} - w_n(y)^{\frac{1+\alpha}{2}}|^2.$$

Hence, taking in consideration that $\Omega \times \Omega \subset D_\Omega$ and by (3.14), we get

$$\begin{aligned}
&\left(\int_\Omega\int_\Omega \frac{|w_n(x) - w_n(y)|^2 (w_n(x)+w_n(y))^{\alpha-1}}{|x-y|^{N+2s(x,y)}(w(x)w(y))^\alpha} \mathrm{d}y\mathrm{d}x\right)^{\frac{p}{2}} \\
&\qquad \leqslant C\left(\iint_{D_\Omega} \frac{|w_n(x)^{\frac{1+\alpha}{2}} - w_n(y)^{\frac{1+\alpha}{2}}|^2}{|x-y|^{N+2s(x,y)}(w_n(x)w_n(y))^\alpha} \mathrm{d}y\mathrm{d}x\right)^{\frac{p}{2}} \leqslant C_3.
\end{aligned} \tag{3.16}$$

Therefore we conclude that

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|w_n(x)-w_n(y)|^p}{|x-y|^{N+p\hat{s}}}\mathrm{d}y\mathrm{d}x \leqslant C_4\Bigg(\int_\Omega\int_\Omega\Bigg(\frac{(w_n(x)w_n(y))^\alpha}{(w_n(x)+w_n(y))^\alpha}\Bigg)^{\frac{p}{2-p}} \\ \times (w_n(x)+w_n(y))^{\frac{p}{2-p}}\frac{\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\frac{2p(s(x,y)-\hat{s})}{2-p}}}\Bigg)^{\frac{2-p}{2}}. \quad (3.17)\end{aligned}$$

On the other hand

$$\begin{aligned}(w_n(x)+w_n(y))\Bigg(\frac{w_n(x)w_n(y)}{w_n(x)+w_n(y)}\Bigg)^\alpha &\leqslant (w_n(x)+w_n(y))^{\alpha+1}\\ &\leqslant C_5\big(w_n^{\alpha+1}(x)+w_n^{\alpha+1}(y)\big),\end{aligned}$$

then

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|w_n(x)-w_n(y)|^p}{|x-y|^{N+p\hat{s}}}\mathrm{d}y\mathrm{d}x \leqslant C\Bigg(\int_\Omega\int_\Omega \frac{w_n^{\frac{(\alpha+1)p}{2-p}}(x)\mathrm{d}x\mathrm{d}y}{|x-y|^{N-\frac{2p(s(x,y)-\hat{s})}{2-p}}}\Bigg)^{\frac{2-p}{2}} \\ + C\Bigg(\int_\Omega\int_\Omega \frac{w_n^{\frac{(\alpha+1)p}{2-p}}(y)\mathrm{d}x\mathrm{d}y}{|x-y|^{N-\frac{2p(s(x,y)-\hat{s})}{2-p}}}\Bigg)^{\frac{2-p}{2}}. \quad (3.18)\end{aligned}$$

Notice that

$$\int_\Omega\int_\Omega \frac{w_n^{\frac{(\alpha+1)p}{2-p}}(x)\mathrm{d}x\mathrm{d}y}{|x-y|^{N-\frac{2p(s(x,y)-\hat{s})}{2-p}}} = \int_\Omega\int_\Omega \frac{w_n^{\frac{(\alpha+1)p}{2-p}}(y)\mathrm{d}x\mathrm{d}y}{|x-y|^{N-\frac{2p(s(x,y)-\hat{s})}{2-p}}}.$$

Now, since $s(x,y)-\widehat{s}\geqslant s_0-\widehat{s}>0$ for all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$, then using the fact that Ω is a bounded domain,

$$\begin{aligned}\int_\Omega\int_\Omega \frac{w_n^{\frac{(\alpha+1)p}{2-p}}(x)\mathrm{d}x\mathrm{d}y}{|x-y|^{N-\frac{2p(s(x,y)-\hat{s})}{2-p}}} &= \int_\Omega w_n^{\frac{(\alpha+1)p}{2-p}}(x)\mathrm{d}x\int_\Omega \frac{\mathrm{d}y}{|x-y|^{N-\frac{2p(s(x,y)-\hat{s})}{2-p}}}\\ &\leqslant C\int_\Omega w_n^{\frac{(\alpha+1)p}{2-p}}(x)\mathrm{d}x.\end{aligned}$$

Since $p<\frac{N}{N-s_0}$, by choosing α small enough we deduce that $\frac{(\alpha+1)p}{2-p}<\frac{N}{N-2s_0}$.

Using the fact that $u_n(x)-u_n(y)=w_n(x)-w_n(y)$ and since

$$\int_\Omega w_n^{\frac{(\alpha+1)p}{2-p}}(x)\mathrm{d}x \leqslant C+\int_\Omega u_n^{\frac{(\alpha+1)p}{2-p}}(x)\mathrm{d}x\leqslant C \text{ for all n},$$

we reach that for all

$$\iint_{\Omega\times\Omega} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+p\hat{s}}}\,\mathrm{d}x\mathrm{d}y \leqslant C \text{ for all } n$$

with $\widehat{s} < s_0$ and $p < \frac{N}{N-\hat{s}}$. Hence the claim follows.

Step 4. — Taking into consideration the previous estimates and since $\{u_n\}_n$ is an increasing sequence, we get the existence of a measurable function u such that $u_n \uparrow u$ a.e. in Ω, $u_n \uparrow u$ strongly in $L^a(\Omega)$ for all $a < \frac{N}{N-2s_0}$ and $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $\mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$. Moreover, using Fatou's lemma we deduce that

$$\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+p\hat{s}}}\,\mathrm{d}x\mathrm{d}y \leqslant C$$

for all $\widehat{s} < s_0$ and $p < \frac{N}{N-\hat{s}}$.

To end this proof, we show that u is a weak solution to Problem (3.1) in the sense of definition 3.4.

Let $\phi \in \mathcal{T}(\Omega)$. Let us recall that $\phi \in \mathbb{H}_0^{s(\,\cdot\,)}(\Omega) \cap L^\infty(\Omega)$ and $(-\Delta)^{s(\,\cdot\,)}\phi = \psi$ where $\psi \in C_0^\infty(\Omega)$.

Using ϕ as a test function in (3.9), it holds that

$$\int_\Omega u_n(-\Delta)^{s(\,\cdot\,)}\phi\,\mathrm{d}x = \int_\Omega h_n\phi\,\mathrm{d}x.$$

It is clear that

$$\int_\Omega h_n\phi\,\mathrm{d}x \to \int_\Omega h\phi\,\mathrm{d}x.$$

Now, since $\int_\Omega u_n(-\Delta)^{s(\,\cdot\,)}\phi\,\mathrm{d}x = \int_\Omega u_n\psi\,\mathrm{d}x$, then using the strong convergence of the sequence $\{u_n\}_n$ in $L^a(\Omega)$ for all $a < \frac{N}{N-2s_0}$, we get, as $n \to \infty$,

$$\int_\Omega u_n\psi\,\mathrm{d}x \to \int_\Omega u\psi\,\mathrm{d}x = \int_\Omega u(-\Delta)^{s(\,\cdot\,)}\phi\,\mathrm{d}x.$$

Hence u solves Problem (3.1) in the sense of Definition 3.4. □

In the next theorem, we show more regularity result on the solution u if h is more regular. More precisely, we have:

Theorem 3.7. — *Assume that* $1 \leqslant m \leqslant \frac{2N}{N+2s_0}$ *(where* s_0 *is defined in* (H$_1$)*) and let* u *be the unique weak solution to Problem* (3.1) *in the sense of Definition 3.4. Then*

$$\iint_{D_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+p\hat{s}}}\,\mathrm{d}x\mathrm{d}y < \infty \textit{ for all } p < \bar{p} := \frac{mN}{N-ms_0} \textit{ and for all } \widehat{s} < s_0.$$

Proof. — Without loss of generality we can assume that $h \gneqq 0$. We follow closely the argument used in [4]. Notice that, since $m \leqslant \frac{2N}{N+2s_0}$, then $\bar{p} \leqslant 2$.

Let $\alpha = \frac{N}{(N-2s_0)m'-N} = \frac{N(m-1)}{N-2s_0 m}$, then $\alpha m' = \frac{2^*_{s_0}}{2}(\alpha+1)$. Using a suitable approximating argument, we can take u^α as a test function in (3.1) to obtain

$$\frac{1}{2}\iint_{D_\Omega} \frac{(u(x)-u(y))(u^\alpha(x)-u^\alpha(y))}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y = \int_\Omega h(x)u^\alpha(x)\,\mathrm{d}x.$$

Hence

$$C\iint_{D_\Omega} \frac{(u(x)-u(y))(u^\alpha(x)-u^\alpha(y))}{|x-y|^{N+2s_0}}\mathrm{d}x\mathrm{d}y \leqslant \int_\Omega h(x)u^\alpha(x)\mathrm{d}x.$$

From the algebraic inequality (2.9), we deduce that

$$C\iint_{D_\Omega} \frac{(u^{\frac{\alpha+1}{2}}(x)-u^{\frac{\alpha+1}{2}}(y))^2}{|x-y|^{N+2s_0}}\mathrm{d}x\mathrm{d}y \leqslant \int_\Omega h(x)u^\alpha(x)\mathrm{d}x.$$

Hence, by using Hölder's inequality, we obtain

$$C\frac{1}{2}\iint_{D_\Omega} \frac{(u^{\frac{\alpha+1}{2}}(x)-u^{\frac{\alpha+1}{2}}(y))^2}{|x-y|^{N+2s_0}}\mathrm{d}x\mathrm{d}y \leqslant \left(\int_\Omega h^m(x)\mathrm{d}x\right)^{\frac{1}{m}}\left(\int_\Omega u^{\alpha m'}(x)\mathrm{d}x\right)^{\frac{1}{m'}}. \tag{3.19}$$

Using the definition of α, we obtain $\alpha m' = \frac{mN}{N-2s_0 m}$. Hence, estimate (3.3) in Theorem 3.2 implies

$$\left(\int_\Omega u^{\alpha m'}(x)\mathrm{d}x\right)^{\frac{1}{\alpha m'}} \leqslant C\|h\|_{L^m(\Omega)}.$$

Going back to (3.19), we obtain

$$\iint_{D_\Omega} \frac{(u^{\frac{\alpha+1}{2}}(x)-u^{\frac{\alpha+1}{2}}(y))^2}{|x-y|^{N+2s_0}}\mathrm{d}x\mathrm{d}y \leqslant C\|h\|^{\alpha+1}_{L^m(\Omega)}. \tag{3.20}$$

By using Hardy's inequality stated in Theorem 2.3, it follows that

$$C\int_\Omega \frac{u^{1+\alpha}(x)}{\delta^{2s_0}(x)}\,\mathrm{d}x \leqslant C\iint_{D_\Omega} \frac{(u^{\frac{\alpha+1}{2}}(x)-u^{\frac{\alpha+1}{2}}(y))^2}{|x-y|^{N+2s_0}}\mathrm{d}x\mathrm{d}y \leqslant C\|h\|^{\alpha+1}_{L^m(\Omega)}. \tag{3.21}$$

Now, let $\widehat{s} < s_0$ be fixed and define $p = \frac{\hat{s}}{s_0}\bar{p} < \bar{p}$. Thanks to Hölder's inequality, we obtain

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+\hat{s}p}} \\
&\quad= \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{ps_0}}\frac{(u(x)+u(y))^{\alpha-1}}{(u(x)+u(y))^{\alpha-1}}|x-y|^{p(s_0-\hat{s})}\frac{\mathrm{d}y\mathrm{d}x}{|x-y|^N} \\
&\quad\leqslant \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2(u(x)+u(y))^{\alpha-1}}{|x-y|^{N+2s_0}}\mathrm{d}y\mathrm{d}x\right)^{\frac{p}{2}} \\
&\qquad\qquad\times\left(\int_\Omega\int_\Omega \frac{(u(x)+u(y))^{\alpha-1}}{(u(x)+u(y))^{(\alpha-1)\frac{2}{2-p}}}\frac{\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\frac{2p(s_0-\hat{s})}{2-p}}}\right)^{\frac{2-p}{2}}. \qquad (3.22)
\end{aligned}$$

Recall that, from (2.11), we have

$$|u(x)-u(y)|^2(u(x)+u(y))^{\alpha-1} \leqslant C(u^{\frac{\alpha+1}{2}}(x)-u^{\frac{\alpha+1}{2}}(y))^2.$$

Hence, using (3.20), we get

$$\begin{aligned}
&\left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2(u(x)+u(y))^{\alpha-1}}{|x-y|^{N+2s_0}}\mathrm{d}y\mathrm{d}x\right)^{\frac{p}{2}} \\
&\qquad\qquad\qquad\leqslant C\left(\iint_{D_\Omega}\frac{(u^{\frac{\alpha+1}{2}}(x)-u^{\frac{\alpha+1}{2}}(y))^2}{|x-y|^{N+2s_0}}\mathrm{d}x\mathrm{d}y\right)^{\frac{p}{2}} \\
&\qquad\qquad\qquad\leqslant C\|h\|_{L^m(\Omega)}^{\frac{(\alpha+1)p}{2}}. \qquad (3.23)
\end{aligned}$$

Now, we deal with the term

$$\left(\int_\Omega\int_\Omega \frac{(u(x)+u(y))^{\alpha-1}}{(u(x)+u(y))^{(\alpha-1)\frac{2}{2-p}}}\frac{\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\frac{2p(s_0-\hat{s})}{2-p}}}\right)^{\frac{2-p}{2}}.$$

As $\alpha < 1$, we get

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{(u(x)+u(y))^{\alpha-1}}{(u(x)+u(y))^{(\alpha-1)\frac{2}{2-p}}}\frac{\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\frac{2p(s_0-\hat{s})}{2-p}}} \\
&\quad= \int_\Omega\int_\Omega (u(x)+u(y))^{\frac{p(1-\alpha)}{2-p}}\frac{\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\frac{2p(s_0-\hat{s})}{2-p}}} \\
&\quad\leqslant C_1\int_\Omega\int_\Omega \frac{(u(x))^{\frac{p(1-\alpha)}{2-p}}\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\frac{2p(s_0-\hat{s})}{2-p}}} + C_2\int_\Omega\int_\Omega \frac{(u(y))^{\frac{p(1-\alpha)}{2-p}}\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\frac{2p(s_0-\hat{s})}{2-p}}}. \qquad (3.24)
\end{aligned}$$

Using a symmetric argument, we deduce that

$$\int_\Omega\int_\Omega \frac{(u(x))^{\frac{p(1-\alpha)}{2-p}}\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\frac{2p(s_0-\hat{s})}{2-p}}} = \int_\Omega\int_\Omega \frac{(u(y))^{\frac{p(1-\alpha)}{2-p}}\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\frac{2p(s_0-\hat{s})}{2-p}}}.$$

Hence we have just to estimate the first term, indeed

$$\int_\Omega\int_\Omega \frac{(u(x))^{\frac{p(1-\alpha)}{2-p}}\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\frac{2p(s_0-\hat{s})}{2-p}}} = \int_\Omega (u(x))^{\frac{p(1-\alpha)}{2-p}}\mathrm{d}x\int_\Omega \frac{1}{|x-y|^{N-\frac{2p(s_0-\hat{s})}{2-p}}}\mathrm{d}y \leqslant C(\Omega)\int_\Omega (u(x))^{\frac{p(1-\alpha)}{2-p}}\mathrm{d}x. \tag{3.25}$$

Since $\frac{\bar{p}(1-\alpha)}{2-\bar{p}} = \frac{mN}{N-2s_0m}$ and $\frac{p(1-\alpha)}{2-p} < \frac{\bar{p}(1-\alpha)}{2-\bar{p}}$, then by using Hölder's inequality and estimate (3.3), it follows that,

$$\int_\Omega\int_\Omega \frac{(u(x))^{\frac{p(1-\alpha)}{2-p}}\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\frac{2p(s_0-\hat{s})}{2-p}}} \leqslant C(\Omega)\int_\Omega (u(x))^{\frac{p(1-\alpha)}{2-p}}\mathrm{d}x \leqslant C\|h\|_{L^m(\Omega)}^{\frac{p(1-\alpha)}{2-p}}.$$

Thus

$$\left(\int_\Omega\int_\Omega \frac{(u(x)+u(y))^{\alpha-1}}{(u(x)+u(y))^{(\alpha-1)\frac{2}{2-p}}}\frac{\mathrm{d}y\mathrm{d}x}{|x-y|^{N-\frac{2p(s_0-\hat{s})}{2-p}}}\right)^{\frac{2-p}{2}} \leqslant C\|h\|_{L^m(\Omega)}^{\frac{(1-\alpha)p}{2}}. \tag{3.26}$$

And then

$$\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+\hat{s}p}} \leqslant C\|h\|_{L^m(\Omega)}^p. \tag{3.27}$$

As a conclusion, we obtain

$$\begin{aligned}\iint_{D_\Omega} &\frac{|u(x)-u(y)|^p\mathrm{d}y\mathrm{d}x}{|x-y|^{N+\hat{s}p}}\\ &= \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p\mathrm{d}y\mathrm{d}x}{|x-y|^{N+\hat{s}p}} + 2\int_\Omega\int_{\mathbb{R}^N\setminus\Omega}\frac{|u(x)|^p}{|x-y|^{N+\hat{s}p}}\mathrm{d}y\mathrm{d}x\\ &\leqslant C\|h\|_{L^m(\Omega)}^p + C(\Omega)\int_\Omega \frac{|u(x)|^p}{(\delta(x))^{\hat{s}p}}\mathrm{d}x,\end{aligned} \tag{3.28}$$

where we have used the fact that

$$\int_{\mathbb{R}^N\setminus\Omega}\frac{1}{|x-y|^{N+\hat{s}p}}\mathrm{d}y \leqslant \frac{C(\Omega)}{(\delta(x))^{\hat{s}p}}.$$

Since $p < 2$, by using estimate (3.21) and Hölder's inequality, we deduce that

$$\begin{aligned}\int_\Omega \frac{|u(x)|^p}{(\delta(x))^{\hat{s}p}}\mathrm{d}x &= \int_\Omega \frac{u^{\frac{p}{2}(1+\alpha)}}{(\delta(x))^{\hat{s}p}} u^{\frac{p}{2}(1-\alpha)}\mathrm{d}x \\ &\leqslant \left(\int_\Omega \frac{|u(x)|^{1+\alpha}}{(\delta(x))^{2\hat{s}}}\mathrm{d}x\right)^{\frac{p}{2}} \left(\int_\Omega |u(x)|^{\frac{p(1-\alpha)}{2-p}}\mathrm{d}x\right)^{\frac{2-p}{2}} \\ &\leqslant C\|h\|_{L^m(\Omega)}^{\frac{p}{2}(1+\alpha)} \left(\int_\Omega |u(x)|^{\frac{p(1-\alpha)}{2-p}}\mathrm{d}x\right)^{\frac{2-p}{2}} \\ &\leqslant C\|h\|_{L^m(\Omega)}^{\frac{p}{2}(1+\alpha)} \|u\|_{L^{\frac{\bar{p}(1-\alpha)}{2-\bar{p}}}(\Omega)}^{\frac{p}{2}(1-\alpha)} \end{aligned} \tag{3.29}$$

Recall that $\alpha = \frac{N(m-1)}{N-2s_0m}$, then $\frac{\bar{p}(1-\alpha)}{2-\bar{p}} = \frac{mN}{N-2s_0m}$. Thus

$$\int_\Omega \frac{|u(x)|^p}{(\delta(x))^{\hat{s}p}}\mathrm{d}x \leqslant C\|h\|_{L^m(\Omega)}^p.$$

Hence

$$\iint_{D_\Omega} \frac{|u(x)-u(y)|^p \mathrm{d}y\mathrm{d}x}{|x-y|^{N+\hat{s}p}} \leqslant C\|h\|_{L^m(\Omega)}^p \tag{3.30}$$

and the result follows. □

As a consequence of the previous theorem, we get the next compactness result.

Theorem 3.8. — *Let $\widehat{s} < s_0$ and $p < \frac{N}{N-\hat{s}}$ be fixed. Consider the operator $T : L^1(\Omega) \to \mathbb{W}_0^{\hat{s},p}(\Omega)$ defined by $T(h) = u$ where u is the unique weak solution to Problem* (3.1). *Then, T is continuous and compact.*

Proof. — We begin by proving that the operator T is compact.
Let $\widehat{s} < s_0$ and $p < \frac{N}{N-\hat{s}}$ be fixed and consider $\{h_n\}_n$ to be a bounded sequence in $L^1(\Omega)$. Define $\{u_n\}_n$ to be the unique solution to Problem (3.9).

From the previous results, we deduce that the sequence $\{T_k(u_n)\}_n$ is bounded in $\mathbb{H}_0^{s(\cdot)}(\Omega)$ for all $k > 0$ fixed and

$$\iint_{D_\Omega} \frac{|u_n(x)-u_n(y)|^{\check{p}}}{|x-y|^{N+\check{p}\check{s}}}\,\mathrm{d}x\mathrm{d}y < C \text{ for all } \check{p} < \frac{mN}{N-ms_0} \text{ and for all } \check{s} < s_0,$$

where $C > 0$ does not depend on n.

Thus we conclude that the sequence $\{T_k(u_n)\}_n$ is bounded in $\mathbb{H}_0^{s_0}(\Omega)$ for all $k > 0$. Hence we get the existence of a measurable function u such that, up to a subsequence, $T_k(u) \in \mathbb{H}_0^{s(\cdot)}(\Omega), u \in L^\sigma(\Omega)$ for all $\sigma < \frac{N}{N-2s_0}$ and

$T_k(u_n) \rightharpoonup T_k(u)$ in $\mathbb{H}_0^{s(\cdot)}(\Omega)$. Thus, using the Rellich–Kondrachov compactness result, then up to a subsequence, we deduce that $u_n \to u$ a.e. in Ω. Moreover $u_n = 0$ in $\mathbb{R}^N \setminus \Omega$, it follows that $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$.

Now, let us introduce

$$U_n(x,y) := \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+p\hat{s}}} \quad \text{and} \quad U(x,y) := \frac{|u(x) - u(y)|^p}{|x-y|^{N+p\hat{s}}},$$

with $p < \frac{N}{N-\hat{s}}$.

Thanks to Theorem 3.7, $\{U_n\}$ is bounded in $L^1(\Omega \times \Omega)$. Hence, by using Vitali's lemma, we conclude that, $U_n \to U$, strongly in $L^1(\Omega \times \Omega)$, and compactness of T follows.

Finally, to show that T is continuous, we will use the same argument as above showing the strong convergence of the whole sequence. □

4. Weak Harnack inequality

In order to address the difficulties caused by the singular term, we need to know the precise location of the set where the solution is zero. When $s(\cdot) =$ constant, this information is given by the strong maximum principe, which is a direct consequence of the weak Harnack inequality. Thus, for our purpose, we begin by proving a similar inequality in the more general case where $s(\cdot)$ is not constant. To this end, we closely follow the argument used in [31, 43] (see also [9]), where a weak Harnack inequality is proved for weighted fractional operators.

Let $u \in \mathbb{H}_0^{s(\cdot)}(\Omega)$ be a nonnegative weak solution of

$$\begin{aligned} (-\Delta)^{s(\cdot)} u &= h \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \backslash \Omega \end{aligned} \tag{4.1}$$

where $h \in L^2(\Omega)$ and $h \geqslant 0$. Then, we have the next weak Harnack inequality.

Theorem 4.1 (Weak Harnack inequality). — *Let $x_0 \in \Omega$ and $r > 0$ be such that $B_{2r}(x_0) \subset \Omega$. Let $v \in \mathbb{H}_0^{s(\cdot)}(\Omega)$ be a supersolution (see Definition 4.2 below) to* (4.1) *with $v \gneqq 0$ in $\mathbb{R}^N$. Then, for every $q < \frac{N}{N-2s_0}$ there exists a positive constant $C = C(N, s_0, r)$ such that*

$$\left(\int_{B_r(x_0)} v^q \mathrm{d}x \right)^{\frac{1}{q}} \leqslant C \operatorname{essinf}_{B_{\frac{3}{2}r}(x_0)} v. \tag{4.2}$$

Notation. — For the sake of legibility, and since x_0 is a generic point in Ω, we will denote $B_r(x_0)$ by B_r.

Before starting the demonstration of the previous theorem, let us make precise what we mean by sub and surpersolution to Problem (4.1) in the following definition.

Definition 4.2. — *Let $h \in \mathbb{H}_0^{s(\cdot)}(\Omega)_+$. We say that, $u, v \in (\mathbb{H}_0^{s(\cdot)}(\Omega))_+$ sub and supersolution respectively to* (4.1) *if*

$$\frac{1}{2}\iint_{D_\Omega} \frac{(\phi(x)-\phi(y))(u(x)-u(y))}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y \leqslant \int_\Omega h\phi\,\mathrm{d}x, \quad \forall\, \phi \in \mathbb{H}_0^{s(\cdot)}(\Omega)_+$$

and

$$\frac{1}{2}\iint_{D_\Omega} \frac{(\psi(x)-\psi(y))(u(x)-u(y))}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y \geqslant \int_\Omega h\psi\,\mathrm{d}x, \quad \forall\, \psi \in \mathbb{H}_0^{s(\cdot)}(\Omega)_+$$

are satisfied

Let us now come back to the proof of Theorem 4.1. This proof is quite long and will be decomposed into six lemmas.

First of all, we start by the following nonlocal Caccioppoli-type inequality.

Lemma 4.3 (Caccioppoli's inequality). — *Let $u \in \mathbb{H}_0^{s(\cdot)}(\Omega)$ be the weak solution to* (4.1) *with $u \gneqq 0$ in $\mathbb{R}^N$. Then, for any $B_r \subset \Omega$ and any nonnegative function $\phi \in C_0^\infty(B_r)$, the following estimate holds true*

$$\begin{aligned}\int_{B_r}\int_{B_r} &\frac{|w_\pm(x)\phi(x)-w_\pm(y)\phi(y)|^2}{|x-y|^{N+2s(x,y)}}\,\mathrm{d}x\mathrm{d}y \\ &\leqslant c\int_{B_r}\int_{B_r} \frac{(\max\{w_\pm(x),w_\pm(y)\})^2|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s(x,y)}}\,\mathrm{d}x\mathrm{d}y \\ &\quad+\int_{B_r} w_\pm(x)\phi^2(x)\mathrm{d}x\left(\int_{\mathbb{R}^N\setminus B_r} \operatorname{esssup}_{y\in \operatorname{Supp}\phi} \frac{w_\pm(x)}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\right), \quad (4.3)\end{aligned}$$

where $w_\pm := (u-k)_\pm$ and c is a positive constant.

Proof. — The proof follows the same ideas as the proof of [32, Theorem 1.4], (see also [31, Theorem 2.2]). □

Let us begin by proving the next useful lemma.

Lemma 4.4. — *Let $R > 0$ such that $B_R \subset \Omega$ and assume that $u \in \mathbb{H}_0^{s(\cdot)}(\Omega)$ with $u \gneqq 0$ is a supersolution to* (4.1). *Let $k > 0$ and suppose that for some $\sigma \in (0,1]$ we have*

$$|B_r \cap \{u \geqslant k\}| \geqslant \sigma |B_r| \tag{4.4}$$

with $0<r<\frac{R}{16}$. Then, there exists a positive constant $C=C(N,s_0,s_1)$ such that

$$|B_{6r}\cap\{u\leqslant 2\delta k\}| \leqslant \frac{C}{\sigma r^{s_1-s_0}\log(\frac{1}{2\delta})}|B_{6r}| \tag{4.5}$$

for all $\delta \in (0,\frac{1}{4})$.

Proof. — Without loss of generality, we can assume that $u > 0$ in B_R, (if not, we can consider $u + \varepsilon$ and we let $\varepsilon \to 0$ at the end.) Let $\psi \in \mathcal{C}_0^\infty(B_R)$ be such that $0 \leqslant \psi \leqslant 1$, $\operatorname{supp}\psi \subset B_{7r}$, $\psi = 1$ in B_{6r} and $|\nabla\psi| \leqslant \frac{C}{r}$. Taking $\psi^2 u^{-1}$ as a test function in (4.1), it follows that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\psi^2(x)u^{-1}(x)-\psi^2(y)u^{-1}(y))}{|x-y|^{N+2s(x,y)}}\,\mathrm{d}y\mathrm{d}x \geqslant 0.$$

Then

$$0 \leqslant \int_{B_{8r}}\int_{B_{8r}} \frac{(u(x)-u(y))\left(\frac{\psi^2(x)}{u(x)}-\frac{\psi^2(y)}{u(y)}\right)}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y \\ + 2\int_{\mathbb{R}^N\setminus B_{8r}}\int_{B_{8r}} \frac{(u(x)-u(y))\frac{\psi^2(x)}{u(x)}}{|x-y|^{N+2s(x,y)}}\mathrm{d}y\mathrm{d}x. \quad (4.6)$$

Denote $x = |x|x'$ and $y = \rho y'$ where $|x'| = |y'| = 1$ and $\rho := |y|$ as in [9]. Then, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_{8r}}\int_{B_{8r}} &\frac{(u(x)-u(y))\frac{\psi^2(x)}{u(x)}}{|x-y|^{N+2s(x,y)}}\mathrm{d}y\mathrm{d}x \\ &\leqslant \int_{\mathbb{R}^N\setminus B_{8r}}\int_{B_{8r}} \frac{\psi^2(x)}{|x-y|^{N+2s(x,y)}}\,\mathrm{d}x\mathrm{d}y \\ &\leqslant C\int_{B_{7r}}\psi^2(x)\int_{8r}^{\infty}\frac{\rho^{N-1}}{|x|^{N+2s_1}}\Big(\int_{\mathbb{S}^{N-1}}\frac{\mathrm{d}y'}{|\frac{\rho}{|x|}y'-x'|^{N+2s_1}}\Big)\mathrm{d}\rho\,\mathrm{d}x. \quad (4.7)\end{aligned}$$

We set $\tau := \frac{\rho}{|x|}$, then

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_{8r}}\int_{B_{8r}} &\frac{\psi^2(x)}{|x-y|^{N+2s(x,y)}}\,\mathrm{d}x\mathrm{d}y \\ &\leqslant C\int_{B_{7r}}\frac{\psi^2(x)}{|x|^{2s_1}}\int_{\frac{8}{7}}^{\infty}\tau^{N-1}\bigg(\int_{\mathbb{S}^{N-1}}\frac{\mathrm{d}y'}{|\tau y'-x'|^{N+2s_1}}\bigg)\mathrm{d}\tau\mathrm{d}x, \quad (4.8)\end{aligned}$$

Now, introduce

$$D(\tau) =: \int_{\mathbb{S}^{N-1}}\frac{\mathrm{d}y'}{|\tau y'-x'|^{N+2s_1}}.$$

Then using the fact that $|x'| = |y'| = 1$, taking into consideration that we are integrating in $\mathbb{S}^{N-1}$ and using an orthogonal transformation, we can show that $D(\tau)$ does not depend on x'. Now as in the proof of Theorem 1.1, p. 5–6 in [37], using the spherical coordinates, it holds

$$D(\tau) = 2\frac{\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})}\int_0^{\pi}\frac{\sin^{N-2}(\theta)}{(1-2\tau\cos(\theta)+\tau^2)^{\frac{N+2s_1}{2}}}\mathrm{d}\theta.$$

Hence,

$$\int_{\mathbb{R}^N\setminus B_{8r}}\int_{B_{8r}}\frac{(u(x)-u(y))\frac{\psi^2(x)}{u(x)}}{|x-y|^{N+2s(x,y)}}\mathrm{d}y\mathrm{d}x \leqslant C\int_{B_{7r}}\frac{\psi^2(x)}{|x|^{2s_1}}\int_{\frac{8}{7}}^{\infty}\tau^{N-1}D(\tau)\mathrm{d}\tau\mathrm{d}x \quad (4.9)$$

Therefore, we have that, $\tau^{N-1}D(\tau)\sim\tau^{-1-2s_1}$ as $\tau\to+\infty$. Thus

$$L:=\int_{\frac{8}{7}}^{\infty}\tau^{N-1}D(\tau)\mathrm{d}\tau<C.$$

Hence, we obtain

$$\int_{\mathbb{R}^N\setminus B_{8r}}\int_{B_{8r}}\frac{(u(x)-u(y))\frac{\psi^2(x)}{u(x)}}{|x-y|^{N+2s(x,y)}}\mathrm{d}y\mathrm{d}x\leqslant Cr^{N-2s_1}. \quad (4.10)$$

Now, from [32, Proof of Lemma 1.3], we get

$$(u(x)-u(y))\left(\frac{\psi^2(x)}{u(x)}-\frac{\psi^2(y)}{u(y)}\right) \leqslant -C_1(\log(u(x))-\log(u(y)))^2\psi^2(y)+C_2(\psi(x)-\psi(y))^2, \quad (4.11)$$

where C_1 and C_2 are two universal positive constants. Hence

$$\int_{B_{8r}}\int_{B_{8r}}\frac{(u(x)-u(y))\left(\frac{\psi^2(x)}{u(x)}-\frac{\psi^2(y)}{u(y)}\right)}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y \leqslant -C_1\int_{B_{8r}}\int_{B_{8r}}\frac{(\log(u(x))-\log(u(y)))^2\psi^2(y)}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y + C_2\int_{B_{8r}}\int_{B_{8r}}\frac{(\psi(x)-\psi(y))^2}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y. \quad (4.12)$$

We claim that

$$\int_{B_{8r}}\int_{B_{8r}}\frac{(\psi(x)-\psi(y))^2}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y\leqslant Cr^{N-2s_1}.$$

Indeed, since $|\psi(x)-\psi(y)|\leqslant\frac{C|x-y|}{r}$, it follows that

$$\int_{B_{8r}}\int_{B_{8r}}\frac{(\psi(x)-\psi(y))^2}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y\leqslant\frac{C}{r^2}\int_{B_{8r}}\int_{B_{8r}}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+2s(x,y)-2}}.$$

Then setting $z=y-x$, it follows that

$$\int_{B_{8r}}\int_{B_{8r}}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+2s(x,y)-2}}\leqslant\int_{B_{8r}}\mathrm{d}x\int_{B_{16r}}\frac{\mathrm{d}z}{|z|^{N+2s_1-2}}\mathrm{d}z\leqslant Cr^{N-2s_1+2}.$$

Hence the claim follows.

Now going back to (4.12), we conclude that

$$\int_{B_{8r}}\int_{B_{8r}} \frac{(u(x)-u(y))\left(\frac{\psi^2(x)}{u(x)}-\frac{\psi^2(y)}{u(y)}\right)}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y$$

$$\leqslant -C_1\int_{B_{8r}}\int_{B_{8r}} \frac{(\log(u(x))-\log(u(y)))^2\psi^2(y)}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y + Cr^{N-2s_1}. \quad (4.13)$$

Combining estimates (4.6), (4.10) and (4.13) and using the fact that $\psi=1$ in B_{6r}, it holds that

$$\int_{B_{6r}}\int_{B_{6r}} \frac{(\log(u(x))-\log(u(y)))^2}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y \leqslant Cr^{N-2s_1}. \quad (4.14)$$

Let $\delta\in(0,1/4)$ and define $w(x):=\min\left\{\log\left(\frac{1}{2\delta}\right),\log\left(\frac{k}{u}\right)\right\}_+$. Thanks to (4.14), we have

$$\int_{B_{6r}}\int_{B_{6r}} \frac{(w(x)-w(y))^2}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y \leqslant Cr^{N-2s_1}.$$

Also, let us denote

$$\langle w\rangle_{B_{6r}} := \frac{1}{|B_{6r}|}\int_{B_{6r}} w(x)\,\mathrm{d}x.$$

Thus, using Remark 2.1, Hölder and fractional Poincaré inequalities (see [51, formula (4.2), p. 297]), we get

$$\begin{aligned}\int_{B_{6r}} |w(x)-\langle w\rangle_{B_{6r}}|\mathrm{d}x &\leqslant Cr^{\frac{N}{2}}\left(\int_{B_{6r}}|w(x)-\langle w\rangle_{B_{6r}}|^2\right)^{\frac12}\\ &\leqslant Cr^{s_0+\frac{N}{2}}\left(\int_{B_{6r}}\int_{B_{6r}}\frac{(w(x)-w(y))^2}{|x-y|^{N+2s_0}}\mathrm{d}x\mathrm{d}y\right)^{\frac12}\\ &\leqslant Cr^{s_0+\frac{N}{2}}\left(\int_{B_{6r}}\int_{B_{6r}}\frac{(w(x)-w(y))^2}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y\right)^{\frac12}\\ &\leqslant Cr^{N+s_0-s_1}. \end{aligned} \quad (4.15)$$

On the other hand, it is clear that $\{x\in\Omega\,;\,w(x)=0\}=\{x\in\Omega\,;\,u(x)\geqslant k\}$, and then from (4.4) we have

$$|B_{6r}\cap\{w=0\}| \geqslant \frac{\sigma}{6^N}|B_{6r}|.$$

Moreover

$$\left|B_{6r}\cap\left\{w=\log\left(\frac{1}{2\delta}\right)\right\}\right| \leqslant \frac{6^N}{\sigma\log(\frac{1}{2\delta})}\int_{B_{6r}}|w(x)-\langle w\rangle_{B_{6r}}|.$$

Hence we get the result by applying (4.15) and using the fact that

$$B_{6r} \cap \{u \leqslant 2\delta k\} = B_{6r} \cap \Big\{w = \log\Big(\frac{1}{2\delta}\Big)\Big\}. \qquad \square$$

The key lemma in our proof is the following.

Lemma 4.5. — *Assume that the assumptions of Lemma 4.4 hold. Then, there exists $\delta \in (0, \frac{1}{2})$ depending only on N, s_0, s_1, σ and r and it is independent of u such that*

$$\inf_{B_{4r}} u \geqslant \delta k. \tag{4.16}$$

Proof. — We follows closely the same arguments used in [31]. For the reader's convenience we include here some details.

Let $w := (\ell - u)_+$ where $\ell \in (\delta k, 2\delta k)$ and consider $\psi \in \mathcal{C}_0^\infty(B_\rho)$, with $r \leqslant \rho < 6r$. Thus, using $w\psi^2$ as a test function in (4.1) and following the computations as in the proof of Lemma 3.2 in [31], it holds that

$$\begin{aligned}\int_{B_\rho}\int_{B_\rho} &\frac{(w(x)\psi(x) - w(y)\psi(y))^2}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y \\ &\leqslant C_1 \int_{B_\rho}\int_{B_\rho} \frac{\max\{w(x), w(y)\}^2(\psi(x)-\psi(y))^2}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y \\ &\quad + \ell^2 |B_\rho \cap \{v < \ell\}|_{d\mu} \sup_{\{x \in \mathrm{supp}(\psi)\}} \int_{\mathbb{R}^N \setminus B_\rho} \frac{\mathrm{d}y}{|x-y|^{N+2s(x,y)}}. \end{aligned} \tag{4.17}$$

To estimate the terms of (4.17) and in order to apply Lemma 2.10, we define the sequences, $\{\ell_j\}_{j\in\mathbb{N}}$, $\{\rho_j\}_{j\in\mathbb{N}}$ and $\{\overline{\rho_j}\}_{j\in\mathbb{N}}$ such that

$$\ell_j := \delta k + 2^{-j-1}\delta k,\ \rho_j := 4r + 2^{1-j}r,\ \bar{\rho}_j := \frac{\rho_j + \rho_{j+1}}{2}.$$

Let us define

$$w_j := (\ell_j - v)_+, \qquad B_j := B_{\rho_j}.$$

Consider $\psi_j \in \mathcal{C}_0^\infty(B_{\overline{\rho_j}})$ such that $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ in B_{j+1} with $|\nabla \psi_j| \leqslant 2^{j+3}/r$.

Using Remark 2.1, it follows that

$$C\bigg(\int_{B_j} |w_j\psi_j|^{2^*_{s_0}}\,\mathrm{d}x\bigg)^{\frac{2}{2^*_{s_0}}} \leqslant \int_{B_j}\int_{B_j} \frac{(w(x)\psi(x) - w(y)\psi(y))^2}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y.$$

Therefore, taking into consideration that

$$w_j\psi_j \geqslant (\ell_j - \ell j + 1) \text{ in } B_{j+1} \cap \{v < \ell_{j+1}\},$$

with $\overline{C}$ independent of j, we reach that,

$$\left(\int_{B_j}|w_j\psi_j|^{2^*_{s_0}}\,\mathrm{d}x\right)^{\frac{2}{2^*_{s_0}}} \geqslant (\ell_j-\ell_{j+1})^2|B_{j+1}\cap\{v<\ell_{j+1}\}|^{\frac{2}{2^*_{s_0}}}.$$

Hence we conclude that

$$\begin{aligned}(\ell_{j-l}-\ell_{j+1})^2&\left(\frac{|B_{j+1}\cap\{v<\ell_{j+1}\}|_{d\mu}}{|B_{j+1}|_{d\mu}}\right)^{\frac{2}{2^*_{s_0}}}\\ &\leqslant Cr^{-(N-2s_0)}\int_{B_j}\int_{B_j}\frac{(w(x)\psi(x)-w(y)\psi(y))^2}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Applying (4.17) to w_j, it holds that

$$\begin{aligned}(\ell_j-\ell_{j+1})^2&\left(\frac{|B_{j+1}\cap\{v<\ell_{j+1}\}|}{|B_{j+1}|}\right)^{\frac{2}{2^*_{s_0}}}\\ \leqslant&\frac{C}{r^{(N-2s_0)}}\left(C_1\int_{B_\rho}\int_{B_\rho}\frac{\max\{w(x),w(y)\}^2(\psi(x)-\psi(y))^2}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y\right.\\ &\left.+\ell_j^2|B_j\cap\{v<\ell_j\}|\sup_{\{x\in\mathrm{supp}(\psi_j)\}}\int_{\mathbb{R}^N\setminus B_j}\frac{\mathrm{d}y}{|x-y|^{N+2s(x,y)}}\right).\end{aligned}\tag{4.18}$$

Using again Remark 2.1, we get

$$\begin{aligned}\int_{B_\rho}\int_{B_\rho}&\frac{\max\{w(x),w(y)\}^2(\psi(x)-\psi(y))^2}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y\\ &\leqslant C\int_{B_\rho}\int_{B_\rho}\frac{\max\{w(x),w(y)\}^2(\psi(x)-\psi(y))^2}{|x-y|^{N+2s_1}}\mathrm{d}x\mathrm{d}y\\ &\leqslant C\ell_j^2\|\nabla\psi_j\|^2_{L^\infty(B_j)}\int_{B_j\cap\{v<\ell_j\}}\mathrm{d}x\int_{B_j}\frac{|x-y|^{2-2s_1}}{|x-y|^N}\mathrm{d}y\\ &\leqslant C2^{2j}\ell_j^2r^{-2s_1}\int_{B_j\cap\{v<\ell_j\}}\mathrm{d}x=C2^{2j}\ell_j^2r^{-2s_1}|B_j\cap\{v<\ell_j\}|.\end{aligned}\tag{4.19}$$

Now, we estimate the term

$$\sup_{\{x\in\mathrm{supp}(\psi_j)\}}\int_{\mathbb{R}^N\setminus B_j}\frac{\mathrm{d}y}{|x-y|^{N+2s(x,y)}}.$$

Since $s(x,y)<s_1$, we get

$$\sup_{\{x\in\mathrm{supp}(\psi_j)\}}\int_{\mathbb{R}^N\setminus B_j}\frac{\mathrm{d}y}{|x-y|^{N+2s(x,y)}}\leqslant C\sup_{\{x\in\mathrm{supp}(\psi_j)\}}\int_{\mathbb{R}^N\setminus B_j}\frac{\mathrm{d}y}{|x-y|^{N+2s_1}}.$$

Thus, as in [31, Lemma 3.2], and by taking into consideration (4.18) and (4.19), it follows that

$$(\ell_j-\ell_{j+1})^2\left(\frac{|B_{j+1}\cap\{v<j+1\}|}{|B_{j+1}|}\right)^{\frac{2}{2^*_{s_0}}}$$
$$\leqslant 2^{j(2+2s_1+N)}\ell_j^2\frac{C}{r^{(N-2s_0)}}r^{-2s_1}|B_j\cap\{v<\ell_j\}|$$
$$\leqslant \frac{\widetilde{C}2^{j(2+2s_1+N)}\ell_j^2}{r^{2(s_1-s_0)}}\frac{|B_j\cap\{v<j\}|}{|B_j|}$$

where $\widetilde{C}$ is a positive constant which is independent of j.

Setting

$$A_j:=\frac{|B_j\cap\{v<\ell_j\}|}{|B_j|}. \tag{4.20}$$

Notice that,

$$\frac{|B_{6r}\cap\{u<\ell_0\}|}{|B_{6r}|}\leqslant\frac{\overline{C}}{\sigma r^{s_1-s_0}}\frac{1}{\log\left(\frac{1}{2\delta}\right)}$$

Then, the previous estimate can be written as follows

$$A_{j+1}^{\frac{2}{2^*_{s_0}}}\leqslant\frac{\widetilde{C}2^{j(2+2s_1+N)}\ell_j^2}{r^{2(s_1-s_0)}(\ell_j-\ell_{j+1})^2}A_j\leqslant\frac{C_2 2^{j(4+2s_1+N)}}{r^{2(s_1-s_0)}}A_j.$$

Hence, we get that,

$$A_{j+1}\leqslant\frac{C_3 2^{j\left(\frac{2^*_{s_0}N}{2}+22^*_{s_0}+s_12^*_{s_0}\right)}}{r^{2^*_{s_0}(s_1-s_0)}}A_j^{1+\frac{2s_0}{N-2s_0}}\equiv C_4 2^{j\left(\frac{2^*_{s_0}N}{2}+22^*_{s_0}+s_12^*_{s_0}\right)}A_j^{1+\frac{2s_0}{N-2s_0}}$$

where $C_4\equiv\frac{C_3}{r^{2^*_{s_0}(s_1-s_0)}}$ is a positive constant independent of j. Now, we are able to use Lemma 2.10 with

$$c_0=C_4,\ b=2^{\left(\frac{2^*_{s_0}N}{2}+22^*_{s_0}+s_12^*_{s_0}\right)}>1\quad\text{and}\quad\beta=\frac{2s_0}{N-2s_0}>0.$$

Then, if

$$0<\delta:=\frac{1}{2}\exp\left\{-\frac{\overline{C}C_3^{\frac{N-2s_0}{2s_0}}2^{(\frac{N}{2}+s_1+2)\frac{N(N-2s_0)}{2s_0^2}}}{\sigma r^{(\frac{N}{s_0}+1)(s_1-s_0)}}\right\}<\frac{1}{2}$$

and it depends only on N,s_0,s_1,r, we deduce from (4.20) and (4.5)

$$A_0\leqslant C_3^{-(\frac{N-2s_0}{2s_0})}2^{-(\frac{N}{2}+s_1+2)\frac{N(N-2s_0)}{2s_0^2}}r^{\frac{N}{s_0}(s_1-s_0)}.$$

Hence, Lemma 2.10 implies

$$\lim_{j\to\infty}A_j=0.$$

Taking into consideration that $\ell_j \searrow k\delta$ and $\rho_j \searrow 4r$ as $j \to \infty$, it follows that $|B_{4r} \cap \{v < \delta k\}| = 0$. Hence $\inf_{B_4 r} u \geqslant \delta k$ and the result follows. $\square$

Now, we prove the following reverse Hölder's inequality.

Lemma 4.6. — *Let $r > 0$ such that $B_{3r/2} \subset \Omega$ and suppose that u is a supersolution to* (4.1) *with $u \gneqq 0$. Then, for every $0 < \beta_1 < \beta_2 < \frac{N}{N-2s_0}$, we have*

$$\left(\frac{1}{|B_r|}\int_{B_r} u^{\beta_2}(x)\,\mathrm{d}x\right)^{\frac{1}{\beta_2}} \leqslant C\left(\frac{1}{|B_{3r/2}|}\int_{B_{3r/2}} u^{\beta_1}(x)\,\mathrm{d}x\right)^{\frac{1}{\beta_1}} \tag{4.21}$$

with $C = C(N, s_0, s_1, \beta_1, \beta_2) > 0$.

Proof. — We follow closely [9, 32]. Let $q \in (1,2)$ and $d > 0$. Set $\widetilde{u} := (u + d)$, and assume that $\psi \in \mathcal{C}_0^\infty(\Omega)$ such that $\operatorname{supp}(\psi) \subset B_{\tau r}$, $\psi = 1$ in $B_{\tau' r}$ and $|\nabla\psi| \leqslant \frac{C}{(\tau-\tau')r}$ where $\frac{1}{2} \leqslant \tau' < \tau < \frac{3}{2}$.

Using $\widetilde{u}^{1-q}\psi^2$ as a test function in (4.1), it follows that,

$$0 \leqslant \int_{B_{\tau r}}\int_{B_{\tau r}} \frac{(\widetilde{u}(x) - \widetilde{u}(y))\left(\frac{\psi^2(x)}{\widetilde{u}^{q-1}(x)} - \frac{\psi^2(y)}{\widetilde{u}^{q-1}(y)}\right)}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y + 2\int_{\mathbb{R}^N\setminus B_{\tau r}}\int_{B_{\tau r}} \frac{(\widetilde{u}(x) - \widetilde{u}(y))\frac{\psi^2(x)}{\widetilde{u}^{q-1}(x)}}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y.$$

By using the fact, $|x| < |y|$ in $B_{\tau r} \times (\mathbb{R}^N \setminus B_{\tau r})$, and the positivity of $\widetilde{u}$ we obtain,

$$\int_{\mathbb{R}^N\setminus B_{\tau r}}\int_{B_{\tau r}} \frac{(\widetilde{u}(x) - \widetilde{u}(y))\frac{\psi^2(x)}{\widetilde{u}^{q-1}(x)}}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y \leqslant \left(\int_{B_{\tau r}} \widetilde{u}^{2-q}\psi^2\,\mathrm{d}x\right)\left(\sup_{\{x\in\operatorname{supp}(\psi)\}}\int_{\mathbb{R}^N\setminus B_{\tau r}} \frac{\mathrm{d}y}{|x-y|^{N+2s(x,y)}}\right).$$

Now, by the pointwise inequality in [36, Lemma 3.3 (i)], there exist two positive constants C_1 and C_2, depending on q such that

$$\begin{aligned}\int_{B_{\tau r}}\int_{B_{\tau r}} &(\widetilde{u}(x) - \widetilde{u}(y))\left(\frac{\psi^2(x)}{\widetilde{u}^{q-1}(x)} - \frac{\psi^2(y)}{\widetilde{u}^{q-1}(y)}\right)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+2s(x,y)}} \\ &\leqslant -C_1\int_{B_{\tau r}}\int_{B_{\tau r}} (\widetilde{u}^{\frac{2-q}{2}}(x)\psi(x) - \widetilde{u}^{\frac{2-q}{2}}(y)\psi(y))^2 d\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+2s(x,y)}} \\ &\quad + C_2\int_{B_{\tau r}}\int_{B_{\tau r}} ((\widetilde{u}^{2-q}(x) + \widetilde{u}^{2-q}(y))(\psi(x) - \psi(y))^2\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+2s(x,y)}}.\end{aligned} \tag{4.22}$$

By symmetry, we get

$$\int_{B_{\tau r}}\int_{B_{\tau r}}\frac{((\widetilde{u}^{2-q}(x)+\widetilde{u}^{2-q}(y))(\psi(x)-\psi(y))^2\mathrm{d}x\mathrm{d}y}{|x-y|^{N+2s(x,y)}} = 2\int_{B_{\tau r}}\int_{B_{\tau r}}\frac{(\widetilde{u}^{2-q}(x)(\psi(x)-\psi(y))^2\mathrm{d}x\mathrm{d}y}{|x-y|^{N+2s(x,y)}} \tag{4.23}$$

and using the same reasoning as in [9, Proof of Lemma 3.7], it follows that

$$\int_{B_{\tau r}}\int_{B_{\tau r}}\frac{((\widetilde{u}^{2-q}(x)+\widetilde{u}^{2-q}(y))(\psi(x)-\psi(y))^2\mathrm{d}x\mathrm{d}y}{|x-y|^{N+2s(x,y)}} \leqslant \frac{Cr^{-2s_0}}{(\tau-\tau')^{2s_0}}\int_{B_{\tau r}}\widetilde{u}^{2-q}\,\mathrm{d}x.$$

On the other hand, we have

$$\sup_{\{x\in \mathrm{Supp}(\psi)\}}\int_{\mathbb{R}^N\setminus B_{\tau r}}\frac{\mathrm{d}y}{|x-y|^{N+2s(x,y)}}\leqslant Cr^{-2s_0}.$$

Then, by combining the estimates above we obtain

$$\int_{B_{\tau r}}\int_{B_{\tau r}}\frac{(\widetilde{u}^{\frac{2-q}{2}}(x)\psi(x)-\widetilde{u}^{\frac{2-q}{2}}(y)\psi(y))^2}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y\leqslant \frac{Cr^{-2s_0}}{(\tau-\tau')^{2s_0}}\int_{B_{\tau r}}\widetilde{u}^{2-q}\,\mathrm{d}x.$$

Hence, we deduce from the previous inequality, Remark 2.1 and Sobolev's inequality:

$$\left(\frac{1}{|B_{\tau' r}|}\int_{B_{\tau' r}}\widetilde{u}^{\frac{(2-q)N}{N-2s_0}}\,\mathrm{d}x\right)^{\frac{N-2s_0}{N}} \leqslant \left(\frac{1}{|B_{\tau' r}|}\int_{B_{\tau r}}(\widetilde{u}^{\frac{2-q}{2}}\psi)^{2^*_{s_0}}\,\mathrm{d}x\right)^{\frac{N-2s_0}{N}} \leqslant \frac{C}{|B_{\tau r}|(\tau-\tau')^{2s_0}}\int_{B_{\tau r}}\widetilde{u}^{2-q}\,\mathrm{d}x. \tag{4.24}$$

Morover $q\in(1,2)$ is arbitrary and $\frac{N}{N-2s_0}>1$, then, by using Hölder's inequality we obtain the estimate (4.21) for $\widetilde{u}=u+d$ and $0<\beta_1<\beta_2<\frac{N}{N-2s_0}$. Finally, letting $d\to 0$ and applying *Monotone Convergence Theorem* to conclude. □

Lemma 4.7. — *Let $r>0$ such that $B_r\subset\Omega$. Assume that u is a nonnegative supersolution to* (4.1). *Then, there exists a constant $\eta\in(0,1)$ depending only on N and s_0 such that*

$$\left(\frac{1}{|B_r|}\int_{B_r}u^{\eta}\mathrm{d}x\right)^{\frac{1}{\eta}}\leqslant C\inf_{B_r}u \tag{4.25}$$

where $C>0$ depends on N, s_0, s_0, r and it is independent of u.

To prove Lemma 4.7 (see [31], [42] and the pioneering work [30]), we will use the next covering lemma in the spirit of Krylov–Safonov theory. This Lemma is taken from [42, Lemma 7.2].

Lemma 4.8. — *Let $x_0 \in \mathbb{R}^N$ and $r > 0$. Let $E \subset B_r(x_0)$ be a measurable set. For $\bar{\delta} \in (0,1)$, we consider the set of balls $B_{3\rho}(x) \subset \mathbb{R}^N$ with $x \in B_{x_0}(r)$ and $|E \cap B_{3\rho}(x)| > \bar{\delta}|B_\rho(x)|$. Now, define covering*

$$[E]_{\bar{\delta}} := \bigcup_{\rho>0} \big\{ B_{3\rho}(x) \cap B_r(x_0), x \in B_r(x_0) : |E \cap B_{3\rho}(x)| > \bar{\delta}|B_\rho(x)| \big\}.$$

Then, either

(1) $|[E]_{\bar{\delta}}| \geqslant \frac{\tilde{C}}{\bar{\delta}}|E|$, *or*
(2) $[E]_{\bar{\delta}} = B_r(x_0)$.

where $\widetilde{C}$ depends only on N.

Proof of Lemma 4.7. — Let us recall that for every $\eta > 0$, we have

$$\frac{1}{|B_r|}\int_{B_r} u^\eta \mathrm{d}x = \eta \int_0^\infty t^{\eta-1}\frac{|B_r \cap \{u > t\}|}{|B_r|}\mathrm{d}t. \tag{4.26}$$

For $t > 0$ and $i \in \mathbb{N}$, we set $A_t^i := \{x \in B_r\ ;\ u(x) > t\delta^i\}$ where δ is given by Lemma 4.5. It is clear that $A_t^{i-1} \subset A_t^i$. Let $\rho > 0$ and $x \in B_r$ such that $B_{3\rho}(x) \cap B_r \subset [A_t^{i-1}]_{\bar{\delta}}$. Hence, we get

$$|A_t^{i-1} \cap B_{3\rho}(x)| > \bar{\delta}|B_\rho| = \frac{\bar{\delta}}{3^N}|B_{3\rho}|.$$

Thus, thanks to Lemma 4.5, we obtain

$$u(x) > \delta(t\delta^{i-1}) = t\delta^i \text{ for all } x \in B_r,$$

and therefore $[A_t^{i-1}]_{\bar{\delta}} \subset A_t^i$. Hence by Lemma 4.8, it follows that,

$$A_t^i = B_r \text{ or } |A_t^i| \geqslant \frac{\widetilde{C}}{\bar{\delta}}|A_t^{i-1}|. \tag{4.27}$$

So, if for some $m \in \mathbb{N}$ we have

$$|A_t^0| > \left(\frac{\bar{\delta}}{\widetilde{C}}\right)^m |B_r|, \tag{4.28}$$

then $A_t^m = B_r$. If not, it follows from (4.27) that

$$|A_t^m| \geqslant \frac{\widetilde{C}}{\bar{\delta}}|A_t^{m-1}|.$$

On the other hand, we have $A_t^{i-1} \subset A_t^m \subsetneq B_r$ for all $i \leqslant m$. Thus, by the second point of the alternative (4.27) we reach A_t^{i-1} and then

$$|A_t^{m-1}| \geqslant \frac{\widetilde{C}}{\bar{\delta}}|A_t^{m-2}| \geqslant \cdots \geqslant \left(\frac{\widetilde{C}}{\bar{\delta}}\right)^{m-1} |A_t^0| > \left(\frac{\widetilde{C}}{\bar{\delta}}\right)^{-1} |B_r|.$$

Therefore $|A_t^m| > |B_r|$. Or this contradicts the fact that $A_t^m \subsetneq B_r$. Hence $A_t^m = B_r$.

Now let us observe that (4.28) holds if

$$m > \frac{1}{\log(\frac{\bar{\delta}}{\widetilde{C}})} \log\left(\frac{|A_t^0|}{|B_r|}\right), \tag{4.29}$$

and fixing m to be the smallest integer such that (4.29). Then $m \geqslant 1$ and

$$0 \leqslant m-1 \leqslant \frac{1}{\log(\frac{\bar{\delta}}{\widetilde{C}})} \log\left(\frac{|A_t^0|}{|B_r|}\right).$$

Thus, using the fact that $\delta \in (0, \frac{1}{2})$, it can be checked that

$$\inf_{B_r} u > t\delta^m = t\delta\delta^{m-1} \geqslant t\delta \left(\frac{|A_t^0|}{|B_r|}\right)^{\frac{1}{\beta}},$$

with $\beta := \frac{\log(\frac{\bar{\delta}}{\widetilde{C}})}{\log(\delta)}$.

Setting $\xi := \inf_{B_r} u$. Then, we have

$$\frac{|B_r \cap \{u > t\}|}{|B_r|} = \frac{|A_t^0|}{|B_r|} \leqslant \widetilde{C}\delta^{-\beta}t^{-\beta}\xi^\beta.$$

Moreover, we deduce from (4.26)

$$\frac{1}{|B_r|}\int_{B_r} u^\eta \mathrm{d}x \leqslant \eta \int_0^a t^{\eta-1}\mathrm{d}t + \eta\widetilde{C}\int_a^\infty t^{\eta-1}\delta^{-\beta}t^{-\beta}\xi^\beta \mathrm{d}t$$
$$= a^\eta - \eta\widetilde{C}\delta^{-\beta}\xi^\beta \frac{a^{\eta-\beta}}{\eta-\beta}. \tag{4.30}$$

By choosing $a := \xi$ and $\eta := \frac{\beta}{2}$, we can conclude. □

Now, we are ready to prove the weak Harnack inequality stated in Theorem 4.1.

Proof of Theorem 4.1. — Thanks to Lemma 4.7, we obtain

$$\left(\frac{1}{|B_r|}\int_{B_r} u^\eta \mathrm{d}x\right)^{\frac{1}{\eta}} \leqslant C \inf_{B_r} u$$

for some $\eta \in (0,1)$. Fixing $1 \leqslant q < \frac{N}{N-2s_0}$. By Lemma 4.6 with $\beta_1 = \eta$ and $\beta_2 = q$, we get,

$$\left(\frac{1}{|B_r|}\int_{B_r} u^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leqslant C\left(\frac{1}{|B_{\frac{3}{2}r}|}\int_{B_{\frac{3}{2}r}} u^\eta \,\mathrm{d}x\right)^{\frac{1}{\eta}}.$$

Hence

$$\left(\frac{1}{|B_r|}\int_{B_r} u^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leqslant C \inf_{B_{\frac{3}{2}r}} u$$

and we conclude. □

As a consequence, we get the next strong maximum principle.

Theorem 4.9 (Strong maximum principle). — *Let $u \in \mathbb{H}_0^{s(\cdot)}(\Omega)$ be a nonnegative function such that $(-\Delta)^{s(\cdot)}u \geqslant 0$ in the weak sense. Then, either $u = 0$ or $u > 0$ a.e. in Ω. Moreover, if $u \gneqq 0$, then for all $x_0 \in \Omega$ and $r_0 > 0$ such that $B_{2r_0}(x_0) \subset \Omega$ and for all $q < \frac{N}{N-s_0}$, we have*

$$\operatorname{essinf}_{B_{\frac{3}{2}r_0}(x_0)} u \geqslant C(r,N,s_0,s_1)\left(\int_{B_{r_0}(x_0)} u^q \,\mathrm{d}x\right)^{\frac{1}{q}}$$

where $C(r,N,s_0,s_1) > 0$.

5. Existence results for the singular problem in the case where σ is constant

In this section, we come back to the main Problem:

$$\begin{cases} (-\Delta)^{s(\cdot)}u = \dfrac{g(x)}{u^\sigma} + f(x,u) & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.1}$$

in the case where σ is *constant.*

As mentioned in the Introduction, we will concentrate on the following sub-cases:

(i) $f = f(x)$ and $g \in L^1(\Omega)$ are nonnegative measurable functions;
(ii) $f(x,u) = u^\alpha$ with $0 < \alpha < 1$ and
(iii) $f = 0$ and g is a nonnegative Radon measure satisfying additional assumptions that will be specified below.

Let us begin by defining the concept of distributional solution in the case where $f, g \in L^1_{\mathrm{loc}}(\Omega)$.

Definition 5.1. — *Let u be a nonnegative function such that $u \in L^1(\Omega)$. We say that u is a distributional solution to (5.1) if for any compact set $K \subset\subset \Omega$ there exists a positive constant $C(K)$ such that* $\operatorname{essinf}_K u \geqslant C(K) > 0$ *and for all $\phi \in \mathcal{C}_0^\infty(\Omega)$, we have*

$$\int_\Omega u(x)(-\Delta)^{s(\,\cdot\,)}\phi(x)\mathrm{d}x = \int_\Omega \frac{g(x)}{u(x)^{\sigma(x)}}\phi(x)\mathrm{d}x + \int_\Omega f(x)\phi(x)\mathrm{d}x \tag{5.2}$$

provided that every term of (5.2) exists.

In the case where f depends also on u, we suppose that f is a Carathéodory function such that $f(\,\cdot\,, u(\,\cdot\,)) \in L^1_{\mathrm{loc}}(\Omega)$.

5.1. Case $\sigma =$ constant, $f = f(x)$ and $g \in L^1(\Omega)$

Let us begin by the following useful comparison principle, whose proof follows from the comparison principle of Theorem 2.8.

Proposition 5.2. — *Assume that $f, g \in L^\infty(\Omega)$ are nonnegative functions with $g \gneqq 0$ and $a > 0$. Let $u, v \in \mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$ are nonnegative sub and supersolution, in the sense of Definition 4.2, to the following Problem*

$$\begin{cases} (-\Delta)^{s(\,\cdot\,)} w = \dfrac{g}{(w+a)^{\sigma(\,\cdot\,)}} + f & \text{in } \Omega, \\ \qquad w > 0 & \text{in } \Omega, \\ \qquad w = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{5.3}$$

Then $u \leqslant v$ in Ω.

Proof. — Let u and v are nonnegative sub and supersolution to (5.3) in the sense of Definition 4.2, then

$$(-\Delta)^{s(\,\cdot\,)}(u-v)(x) \leqslant g\left[\frac{1}{(u+a)^{\sigma(x)}} - \frac{1}{(v+a)^{\sigma(x)}}\right].$$

Using $(u-v)^+$ as test function in the last inequality, we get

$$C\|(u-v)^+\|^2_{H_0^{s(\,\cdot\,)}(\Omega)} \leqslant \int_\Omega g\left[\frac{1}{(u+a)^{\sigma(x)}} - \frac{1}{(v+a)^{\sigma(x)}}\right](u-v)^+\mathrm{d}x.$$

Notice that,

$$\left[\frac{1}{(u+a)^{\sigma(x)}} - \frac{1}{(v+a)^{\sigma(x)}}\right](u-v)^+ \leqslant 0,$$

since $g \geqslant 0$, then

$$0 \leqslant \|(u-v)^+\|^2_{\mathbb{H}_0^{s(\,\cdot\,)}(\Omega)} \leqslant 0.$$

Thus $(u-v)^+(x) = C$ a.e. in $\mathbb{R}^N$ with $C \in \mathbb{R}^N$. Since $u = v = 0$ in $\mathbb{R}^N \setminus \Omega$, then $(u-v)^+ = 0$. Therefore the result follows. ☐

As a consequence, Problem (5.3) has a unique positive solution $u \in \mathbb{H}_0^{s(\cdot)}(\Omega)$. Moreover, if $g_1 \leqslant g_2, f_1 \leqslant f_2$ and $a_1 \geqslant a_2$, then $u_1 \leqslant u_2$ where u_i is the solution corresponding to the data f_i, g_i and a_i.

Before stating the main existence result of this subsection, we need the following auxiliary existence result.

Theorem 5.3. — *Let g be a nonnegative measurable function such that $g \in L^m(\Omega)$ with $m \geqslant 1$ and $\sigma = const$. Then, Problem*

$$\begin{cases} (-\Delta)^{s(\cdot)} u = \dfrac{g}{u^\sigma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.4}$$

has a distributional solution u such that $u^{\frac{\sigma+1}{2}} \in \mathbb{H}_0^{s(\cdot)}(\Omega)$. Moreover, u is the unique solution to Problem (5.4) *in the sense of Definition 5.1 such that $u^{\frac{\sigma+1}{2}} \in \mathbb{H}_0^{s(\cdot)}(\Omega)$.*

Proof. — Let $n \geqslant 1$ and define u_n to be the unique positive solution to the approximating problem

$$\begin{cases} (-\Delta)^{s(\cdot)} u_n = \dfrac{g_n}{(u_n + \frac{1}{n})^\sigma} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.5}$$

where $g_n := \min(n, g)$. The existence in this case follows from a classical minimization argument. We set

$$L(x,a) = \frac{g_n}{(a + \frac{1}{n})^\sigma} \text{ where } a \geqslant 0.$$

Using the fact that $\frac{L(x,a)}{a}$ is decreasing for $a > 0$, then by the comparison principle of Theorem 2.8, we deduce that $\{u_n\}_n$ is increasing with respect to n. Moreover, from the weak Harnack inequality, it holds that for any compact set $K \subset\subset \Omega$ and for all $n \geqslant 1$

$$\operatorname{essinf}_K u_n \geqslant \operatorname{essinf}_K u_1 \geqslant C(K). \tag{5.6}$$

Using u_n^σ as a test function in (5.5), it follows that

$$\frac{1}{2} \iint_{D_\Omega} \frac{(u_n(x) - u_n(y))(u_n^\sigma(x) - u_n^\sigma(y))}{|x-y|^{N+2s(x,y)}} \mathrm{d}x\mathrm{d}y \leqslant \int_\Omega g(x)\mathrm{d}x.$$

From the algebraic inequality (2.9), we get

$$C \iint_{D_\Omega} \frac{|u_n^{\frac{\sigma+1}{2}}(x) - u_n^{\frac{\sigma+1}{2}}(y)|^2}{|x-y|^{N+2s(x,y)}} \mathrm{d}x\mathrm{d}y \leqslant \int_\Omega g(x)\mathrm{d}x.$$

Thus the sequence $\{u_n^{\frac{\sigma+1}{2}}\}_n$ is bounded in $\mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$. In addition, the sequence $\{u_n\}_n$ is monotone on n. Then, there exists a measurable function u such that $u_n \uparrow u$ a.e. in $\mathbb{R}^N$, $u_n^{\frac{\sigma+1}{2}} \rightharpoonup u^{\frac{\sigma+1}{2}}$ weakly in $\mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$. Hence we conclude that $u_n \uparrow u$ strongly in $L^a(\Omega)$ for all $a < \frac{(\sigma+1)N}{N-2s_0}$. Since $u \geqslant u_n$ for all n, we deduce from (5.6), that $\operatorname{essinf}_K u \geqslant C(K)$ for any compact set $K \subset\subset \Omega$. Hence, by the above estimate and passing to the limit in Problem 5.5, we can prove that u is a distributional solution to Problem (5.4) with $u^{\frac{\sigma+1}{2}} \in \mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$.
To prove the uniqueness, we suppose that Problem (5.4) has two positive solutions v_1, v_2 with $v_1^{\frac{\sigma+1}{2}}, v_2^{\frac{\sigma+1}{2}} \in \mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$. Denote v_1 the solution obtained as a limit of the monotone sequence $\{u_n\}_n$. Then, by the comparison principle in Proposition 5.2 applied to Problem (5.5), it holds that $u_n \leqslant v_2$ for all $n \geqslant 1$. Thus $v_1 \leqslant v_2$. Now, by substraction, we obtain that

$$(-\Delta)^{s(\,\cdot\,)}(v_2 - v_1) \leqslant 0. \tag{5.7}$$

Notice that $(v_2^{\frac{\sigma+1}{2}} - v_1^{\frac{\sigma+1}{2}})$ is a nonnegative function with $v_2^{\frac{\sigma+1}{2}}, v_1^{\frac{\sigma+1}{2}} \in \mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$. Define ψ to be the unique positive solution to the problem

$$\begin{cases} (-\Delta)^{s(\,\cdot\,)}\psi = \theta & \text{in } \Omega, \\ \psi = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.8}$$

where $\theta \in \mathcal{C}_0^\infty(\Omega)$ and $\theta \gneqq 0$. By Theorem 3.2, we know that $\psi \in \mathbb{H}_0^{s(\,\cdot\,)}(\Omega) \cap L^\infty(\Omega)$. Using ψ as a test function in (5.7), we deduce that

$$\int_\Omega (v_2 - v_1)\theta \,\mathrm{d}x \leqslant 0.$$

Hence we conclude that $v_2 - v_1 = 0$ and then the result follows. □

Now, we are in position to state the main existence result of this subsection.

Theorem 5.4. — *Assume $\sigma(\,\cdot\,) = \sigma$. Let f and g be nonnegative measurable functions such that $f, g \in L^1(\Omega)$. Then, Problem (5.1) has a unique positive distributional solution u such that $u \in L^a(\Omega)$ for all $a < \frac{N}{N-2s_0}$ and $T_k(u^{\frac{\sigma+1}{2}}) \in \mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$ for all $k > 0$.*

Proof. — The proof is based on sub and supersolution method and iteration arguments.

For this purpose, let us build a supersolution to (5.1). In fact, let v be the unique solution to Problem (5.4) obtained in Theorem 5.3 and let w be the unique solution to Problem (3.1) with $h = f$.

Then, by Theorem 5.3, we know that $v^{\frac{\sigma+1}{2}} \in \mathbb{H}_0^{s(\cdot)}(\Omega)$, thus $v^{\frac{\sigma+1}{2}} \in \mathbb{H}_0^{s_0}(\Omega)$. Hence using Sobolev inequality, we obtain that $v^{\frac{\sigma+1}{2}} \in L^{a_1}$ for all $a_1 \leqslant \frac{2N}{N-2s_0}$. Now, since $f \in L^1(\Omega)$, then $w \in L^{a_2}(\Omega)$ for all $a_2 < \frac{N}{N-2s_0}$. Setting $\bar{u} = v + w$, then, we deduce that $\bar{u} \in L^{\bar{a}}(\Omega)$ for ever $\bar{a} < \min\{\frac{(\sigma+1)N}{N-2s_0}, \frac{N}{N-2s_0}\} = \frac{N}{N-2s_0}$ and

$$(-\Delta)^{s(\cdot)}\bar{u}(x) = \frac{g}{w^\sigma} + f \geqslant \frac{g}{\bar{u}^\sigma} + f.$$

Thus $\bar{u}$ is a positive supersolution to (5.1) with $\bar{u} \in L^a(\Omega)$ for all $a < \frac{N}{N-2s_0}$.

Now, let u_n be the unique solution to the approximating Problem

$$\begin{cases} (-\Delta)^{s(\cdot)}u_n = \dfrac{g_n}{(u_n + \frac{1}{n})^\sigma} + f_n & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.9}$$

By the comparison principle in Proposition 5.2, $u_n \leqslant \bar{u}$ for all n. Moreover, $\{u_n\}_n$ is increasing. Using now $T_k(u_n^\sigma)$ as a test function in (5.9), it follows that

$$\frac{1}{2}\iint_{D_\Omega} \frac{(u_n(x) - u_n(y))(T_k(u_n^\sigma(x)) - T_k(u_n^\sigma(y)))}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y \leqslant \int_\Omega g + k\int_\Omega f.$$

Hence by the algebraic inequality (2.10), we deduce that the sequence $\{T_k(u_n^{\frac{\sigma+1}{2}})\}_n$ is bounded in $\mathbb{H}_0^{s(\cdot)}(\Omega)$. Moreover, the sequence $\{u_n\}_n$ is monotone on n, we get the existence of positive measurable function u such that $u_n \uparrow u$ a.e. in $\mathbb{R}^N$. Thus $u \geqslant C(K)$ a.e. in K, for any compact set of Ω and $T_k(u_n^{\frac{\sigma+1}{2}}) \rightharpoonup T_k(u^{\frac{\sigma+1}{2}})$ weakly in $\mathbb{H}_0^{s(\cdot)}(\Omega)$. Since $u \leqslant \bar{u}$, then $\bar{u} \in L^a(\Omega)$ for all $a < \frac{N}{N-2s_0}$.

Let $\phi \in C_0^\infty(\Omega)$ with $\operatorname{Supp}\phi \subset K$, where K is a compact set of Ω. We have

$$\frac{g_n}{(u_n + \frac{1}{n})^\sigma}|\phi| + f_n|\phi| \leqslant \frac{g}{u_1^\sigma}\phi + f|\phi| \leqslant \frac{g}{C(K)^\sigma}\phi + f|\phi|,$$

where we have used the fact that $u_1 \geqslant C(K)$ a.e. in K. Then, by the Dominated Convergence Theorem, we obtain

$$\int_\Omega \frac{g_n}{(u_n + \frac{1}{n})^\sigma}\phi\,\mathrm{d}x + \int_\Omega f_n\phi \to \int_\Omega \frac{g}{u^\sigma}\phi\,\mathrm{d}x + \int_\Omega f\phi\,\mathrm{d}x \quad \text{as } n \to \infty.$$

On the other hand

$$\begin{aligned} \lim_{n\to+\infty}\int_\Omega \phi(-\Delta)^{s(\cdot)}u_n\mathrm{d}x &= \lim_{n\to+\infty}\int_\Omega u_n(-\Delta)^{s(\cdot)}\phi\,\mathrm{d}x \\ &= \lim_{n\to+\infty}\int_\Omega u(-\Delta)^{s(\cdot)}\phi\,\mathrm{d}x. \end{aligned}$$

Hence, u is a weak solution to Problem (5.1) with $u \in L^a(\Omega)$ for all $a < \frac{N}{N-2s_0}$ and $T_k(u^{\frac{\sigma+1}{2}}) \in \mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$ for all $k > 0$.

Concerning the uniqueness, we proceed in the same way as in the proof of the uniqueness part in Theorem 5.3. □

5.2. Case $\sigma =$ constant, $f(x,u) = u^\alpha$, and $g \in L^1(\Omega)$

Now, we deal with the case where f depends on the unknown function u. To simplify the presentation, we will only consider the potential case i.e. $f(x,u) = u^\alpha$ with $\alpha > 0$. Namely, we study the following Problem:

$$\begin{cases} (-\Delta)^{s(\,\cdot\,)} u = \dfrac{g(x)}{u^{\sigma(\,\cdot\,)}} + u^\alpha & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.10}$$

where $g \in L^1(\Omega)$ and $g \gneqq 0$.

Theorem 5.5. — *Let $0 \lneqq g \in L^1(\Omega)$. Then, for every $\sigma > 0$ and for every $\alpha \in (0,1)$, Problem* (5.10) *has a distributional solution $u \geqslant 0$ such that $u^{\frac{\sigma+1}{2}} \in \mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$.*

Proof. — We will use monotonicity argument. Let $u_0 = 0$ and for $n \geqslant 1$, we define $u_n \in \mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$ to be the unique solution to the Problem

$$\begin{cases} (-\Delta)^{s(\,\cdot\,)} u_n = \dfrac{g_n}{(u_n + \frac{1}{n})^\sigma} + u_{n-1}^\alpha & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.11}$$

where $g_n := \min(n, g)$. The existence of $\{u_n\}_n$ follows by using an induction argument. For the convenience of reader, we will include some details. Recall that $u_0 = 0$, we define u_1 to be the unique solution to the Problem

$$\begin{cases} (-\Delta)^{s(\,\cdot\,)} u_1 = \dfrac{g_1}{(u_1 + 1)^\sigma} & \text{in } \Omega, \\ u_1 > 0 & \text{in } \Omega, \\ u_1 = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Notice that the existence of u_1 follows using minimizing argument. Since $\frac{g_1}{(u_1+1)^\sigma} \leqslant g_1$, then we can show that $u_1 \in L^\infty(\Omega)$. Thus, again using minimizing argument, we can define u_2 as the unique solution to the Problem

$$\begin{cases} (-\Delta)^{s(\cdot)} u_2 = \dfrac{g_2}{(u_2+\frac{1}{2})^\sigma} + u_1^\alpha & \text{in } \Omega, \\ u_2 > 0 & \text{in } \Omega, \\ u_2 = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Notice that $\frac{g_1}{(u_2+\frac{1}{2})^\sigma} + u_1^\alpha \leqslant 2^\sigma g_2 + u_1^\alpha \in L^\infty(\Omega)$. Hence by an induction argument, we get the existence of u_n that solves Problem (5.11) with $u_n \in L^\infty(\Omega) \cap \mathbb{H}_0^{s(\cdot)}(\Omega)$. In addition, $\{u_n\}$ is an increasing sequence.

Now, using u_n^σ as test function in (5.11), we get,

$$C\|u_n^{\frac{\sigma+1}{2}}\|^2_{\mathbb{H}_0^{s(\cdot)}(\Omega)} \leqslant \int_\Omega \frac{g_n u_n^\sigma}{(u_n+\frac{1}{n})^\sigma} \mathrm{d}x + \int_\Omega u_n^{\alpha+\sigma} \mathrm{d}x.$$

Since $\alpha < 1$, it follows that $\alpha + \sigma < \frac{\sigma+1}{2} 2^*_{s_0}$. Hence, applying Hölder and Sobolev inequalities imply that $\{u_n^{\frac{\sigma+1}{2}}\}_n$ is bounded in $\mathbb{H}_0^{s(\cdot)}(\Omega)$. Therefore, there exists a measurable function u such that $u_n^{\frac{\sigma+1}{2}} \rightharpoonup u_n^{\frac{\sigma+1}{2}}$ weakly in $\mathbb{H}_0^{s(\cdot)}(\Omega)$, $u_n \uparrow u$ strongly in $L^{\frac{(\sigma+1)2^*_{s_0}}{2}}(\Omega)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$. Hence, we easily conclude that u is a distributional solution to (5.10). □

Remark 5.6. — The above existence result still holds if we replace the term u^α by the linear term λu with λ small enough. The case where $\alpha > 1$ is more complicated and left as an open problem.

5.3. Case $\sigma =$ constant, $f = 0$, g is a nonnegative Radon measure

In this paragraph, we study Problem (5.1) when $g = \mu$ is a nonnegative Radon measure. To this end, we shall need of definition and some properties of *fractional relative* $(s(\cdot), 2)$-*capacity* with respect to Ω. Readers who are not familiar with the concept of relative capacity might want to start by reading the first part of the Appendix.

Before stating the main results of this section, we need the following three definitions.

Definition 5.7. — *Let $U \subset \overline{\Omega}$ be a relatively open set, that is, open with the relative topology of $\overline{\Omega}$. The variable order fractional relative $(s(\cdot), 2)$-capacity of U with respect to Ω is defined by*

$$\mathrm{Cap}^{\overline{\Omega}}_{(s(\cdot),2)}(U) := \inf\left\{ \|u\|^2_{H^{s(\cdot)}(\Omega)} \,;\, u \in \widetilde{H^{s(\cdot)}}(\Omega) \text{ and } u \geqslant \chi_U \text{ a.e. in } \Omega \right\}.$$

More generally, for any subset $B \subset \overline{\Omega}$,

$$\mathrm{Cap}^{\overline{\Omega}}_{(s(\,\cdot\,),2)}(B) := \inf\Big\{\mathrm{Cap}^{\overline{\Omega}}_{(s(\,\cdot\,),2)}(U)\,;\, U \textit{ relatively open in } \overline{\Omega} \textit{ and } B \subset U\Big\}.$$

In the previous definition, we have denoted $H^{s(\,\cdot\,)}(\Omega) := W^{s(\,\cdot\,),2}(\Omega)$ and $\widetilde{H^{s(\,\cdot\,)}}(\Omega) = \widetilde{W}^{s(\,\cdot\,),2}(\Omega)$.

Definition 5.8. — *Let μ be a Radon measure on Ω. We say that μ is concentrated on a Borel subset E of Ω, if $\mu(B) = \mu(B \cap E)$ for every Borel subset B of $\mathbb{R}^N$. See, for example,* [28, p. 746].

Definition 5.9. — *Let μ be a nonnegative Radon measure in Ω. We say that μ is singular if it is concentrated on a subset $E \subset \Omega$ such that $\mathrm{Cap}^{\overline{\Omega}}_{(s(\,\cdot\,),2)}(E) = 0$. We denote the set of singular positive measures by $\mathcal{M}_s$.*

Now we are ready to state and to prove the principal theorems of this section.

Theorem 5.10. — *Let μ be a nonnegative Radon measure concentrated on a Borel set E such that $\mathrm{Cap}^{\overline{\Omega}}_{(s(\,\cdot\,),2)}(E) = 0$. Let $\{g_n\}_n$ be a sequence of nonnegative bounded functions such that $g_n \rightharpoonup \mu$ in the narrow topology of measures. Let u_n be the unique solution to the approximated Problem* (5.5)*. Then, up to a subsequence,*

$$u_n^{\frac{\sigma+1}{2}} \rightharpoonup 0 \textit{ weakly in } \mathbb{H}^{s(\,\cdot\,)}_0(\Omega) \textit{ as } n \to \infty.$$

Proof. — We follow closely the argument used in [3]. Since μ is a singular measure with respect to the capacity $\mathrm{Cap}^{\overline{\Omega}}_{s(\,\cdot\,),2}$, then for all $\varepsilon > 0$, we get the existence of an open set $U_\varepsilon \subset\subset \Omega$ such that $E \subset U_\varepsilon$ and $\mathrm{Cap}^{\overline{\Omega}}_{s(\,\cdot\,),2}(U_\varepsilon) < \varepsilon$. Thus there exists a regular function ϕ_ε in $\mathcal{C}^\infty_0(\Omega)$ such that $\phi_\varepsilon \geqslant 0$, $\phi_\varepsilon \equiv 1$ in U_ε and

$$\iint_{D_\Omega} \frac{|\phi_\varepsilon(x) - \phi_\varepsilon(y)|^2}{|x-y|^{N+2s(x,y)}} \mathrm{d}x\mathrm{d}y \leqslant \varepsilon.$$

Using Picone's inequality stated in Proposition 2.7 to u_n and ϕ_ε, we obtain

$$\int_\Omega \frac{(-\Delta)^{s(\,\cdot\,)}(u_n + \frac{1}{n})}{(u_n + \frac{1}{n})} \phi^2_\varepsilon(x)\mathrm{d}x \leqslant \iint_{D_\Omega} \frac{|\phi_\varepsilon(x) - \phi_\varepsilon(y)|^2}{|x-y|^{N+2s(x,y)}} \mathrm{d}x\mathrm{d}y \leqslant \varepsilon.$$

Thus

$$\int_{U_\varepsilon} \frac{g_n}{(u_n + \frac{1}{n})^{\sigma+1}}\,\mathrm{d}x \leqslant \varepsilon.$$

Applying this time Hölder's inequality, we obtain

$$\begin{aligned}\int_{U_\varepsilon} \frac{g_n}{(u_n+\frac{1}{n})^\sigma}\,\mathrm{d}x &\leqslant \Big(\int_{U_\varepsilon} \frac{g_n}{(u_n+\frac{1}{n})^{\sigma+1}}\,\mathrm{d}x\Big)^{\frac{\sigma}{\sigma+1}} \Big(\int_{U_\varepsilon} g_n\,\mathrm{d}x\Big)^{\frac{1}{\sigma+1}}\\ &\leqslant C\Big(\int_{U_\varepsilon} \frac{g_n}{(u_n+\frac{1}{n})^{\sigma+1}}\,\mathrm{d}x\Big)^{\frac{\sigma}{\sigma+1}} \leqslant C\varepsilon^{\frac{\sigma}{\sigma+1}}.\end{aligned}$$

We claim that for all $\phi \in \mathcal{C}_0^\infty(\Omega)$,

$$\lim_{n\to\infty} \int_\Omega \frac{g_n}{(u_n+\frac{1}{n})^\sigma}\phi\,\mathrm{d}x = 0.$$

First, we have

$$\begin{aligned}\int_\Omega \frac{g_n}{(u_n+\frac{1}{n})^\sigma}|\phi|\,\mathrm{d}x &\leqslant \|\phi\|_\infty \int_{U_\varepsilon} \frac{g_n}{(u_n+\frac{1}{n})^\sigma}\,\mathrm{d}x + \int_{\Omega\setminus U_\varepsilon} \frac{g_n}{(u_n+\frac{1}{n})^\sigma}|\phi|\,\mathrm{d}x\\ &\leqslant \|\phi\|_\infty\varepsilon + \int_{\Omega\setminus U_\varepsilon} \frac{g_n}{(u_n+\frac{1}{n})^\sigma}|\phi|\,\mathrm{d}x.\end{aligned}$$

Second, since $\mu(\Omega\setminus U_\varepsilon) = 0$, we can choose a sequence $\{g_n\}_n$ such that $\mathrm{Supp}(g_n) \subset U_\varepsilon$ if $n \geqslant n_0$. Thus

$$\int_{\Omega\setminus U_\varepsilon} \frac{g_n}{(u_n+\frac{1}{n})^\sigma}|\phi|\,\mathrm{d}x = 0 \quad \text{if } n \geqslant n_0.$$

Hence

$$\lim_{n\to\infty} \int_\Omega \frac{g_n}{(u_n+\frac{1}{n})^\sigma}\,\phi\,\mathrm{d}x = 0 \tag{5.12}$$

and the claim follows.

By using the same computation as above and again applying Hölder's inequality, we easily get

$$\lim_{n\to\infty} \int_\Omega \frac{g_n}{(u_n+\frac{1}{n})^\alpha}\phi\,\mathrm{d}x = 0 \text{ for all } \alpha \leqslant \sigma \text{ and for all } \phi \in \mathcal{C}_0^\infty(\Omega). \tag{5.13}$$

Now, using u_n^σ as test function in (5.5) as in the proof of Theorem 5.3. Then, we get the sequence $\{u_n^{\frac{\sigma+1}{2}}\}_n$ is bounded in $\mathbb{H}_0^{s(\cdot)}(\Omega)$. Thus, there exists a subsequence still denoted by $\{u_n\}_n$ such that $u_n^{\frac{\sigma+1}{2}} \rightharpoonup u^{\frac{\sigma+1}{2}}$ weakly in $\mathbb{H}_0^{s(\cdot)}(\Omega)$. It is clear that u solves $(-\Delta)^{s(\cdot)}u = 0$. Moreover, by using an approximation argument, we can take u^σ as a test function in 5.5 to obtain that

$$C(\sigma)\|u^{\frac{\sigma+1}{2}}\|^2_{\mathbb{H}_0^{s(\cdot)}(\Omega)} \leqslant \int_\Omega u^\sigma(-\Delta)^{s(\cdot)}u = 0.$$

Since $u = 0$ in $\mathbb{R}^N \setminus \Omega$, then $u \equiv 0$ and the result follows. □

For the more general case $g = \mu_s + h$ where $h \in L^1(\Omega)$ and μ_s is a singular measure, we can prove the next improvement.

Theorem 5.11. — *Let $\{\ell_n\}_n$ and $\{h_n\}_n$ be two sequences of nonnegative bounded functions such that $\ell_n \rightharpoonup \mu$ in the narrow topology of measures and $h_n \to h$ strongly in $L^1(\Omega)$. Let u_n be the unique solution to the approximated Problem* (5.5) *with $g_n = \ell_n + h_n$.*

Let w be the unique distributional solution to the Problem

$$(-\Delta)^{s(\,\cdot\,)} w = \frac{h}{w^\sigma} \text{ in } \mathcal{D}'(\Omega), w^{\frac{\sigma+1}{2}} \in \mathbb{H}_0^{s(\,\cdot\,)}(\Omega), \tag{5.14}$$

obtained by approximation. Then, up to a subsequence,

$$u_n^{\frac{\sigma+1}{2}} \rightharpoonup w^{\frac{\sigma+1}{2}} \text{ weakly in } \mathbb{H}_0^{s(\,\cdot\,)}(\Omega).$$

Proof. — Using the same arguments as in the proof of Theorem 5.3, we deduce that the sequence $\{u_n^{\frac{\sigma+1}{2}}\}$ is bounded in $\mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$. Hence, up to a subsequence, we get the existence of w such that $w^{\frac{\sigma+1}{2}} \in \mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$ and $u_n^{\frac{\sigma+1}{2}} \rightharpoonup w^{\frac{\sigma+1}{2}}$ weakly in $\mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$.

By the same techniques used in the proof of Theorem 5.10, we obtain

$$\lim_{n\to\infty} \int_\Omega \frac{\ell_n}{(u_n + \frac{1}{n})^\sigma} \phi \, \mathrm{d}x = 0 \quad \text{for all } \phi \in \mathcal{C}_0^\infty(\Omega).$$

Let v_n be the unique positive solution to Problem

$$\begin{cases} (-\Delta)^{s(\,\cdot\,)} v_n = \dfrac{h_n}{(v_n + \frac{1}{n})^\sigma} & \text{in } \Omega, \\ v_n > 0 & \text{in } \Omega, \\ v_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then according to the comparison Principle in Proposition 5.2, we obtain $u_n \geqslant v_n \geqslant v_1$ for all $n \geqslant 1$. Thus for any compact set K of Ω, we have $u_n \geqslant C(K) > 0$ for all n.

Finally, we can easily show that w solves (5.14) at least in the sense of distributions. □

6. Existence results for the singular problem where the function $\sigma(\,\cdot\,)$ is not constant

Throughout this section, we assume that $\sigma : x \mapsto \sigma(x)$ is a positive continuous function and nonconstant.

As mentioned in the introduction, the situation is more complicated in this case. In fact, several estimates, valid in the case σ is constant, no longer hold.

In the beginning, assume that $f = 0$. Then, Problem (5.1) can be written as

$$\begin{cases} (-\Delta)^{s(\cdot)} u = \dfrac{g}{u^{\sigma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{6.1}$$

Then we have the next existence result.

THEOREM 6.1. — *Let g be a nonnegative function such that $g \in L^m(\Omega)$ with $m \geqslant 1$. Then, Problem* (6.1) *has a unique distributional solution u such that $u \in L^1(\Omega)$ and $T_k(u^{\frac{\sigma^*+1}{2}}) \in \mathbb{H}_0^{s(\cdot)}(\Omega)$ where $\sigma^* = \|\sigma\|_{L^\infty(\Omega)}$.*

Proof. — Let $n \geqslant 1$, define u_n to be the unique solution to the approximating problem

$$\begin{cases} (-\Delta)^{s(\cdot)} u_n = \dfrac{g_n}{(u_n + \frac{1}{n})^{\sigma(x)}} & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6.2}$$

where $g_n := \min(n, g)$.

First of all, the existence in this case follows from Schauder fixed point Theorem.

To make the paper self-contained as possible, we include here some details.

Fix $n \geqslant 1$ and consider the operator $\mathfrak{R} : L^2(\Omega) \to L^2(\Omega)$ defined by $\mathfrak{R}(v) = u$ with u being the unique solution to the problem

$$\begin{cases} (-\Delta)^{s(\cdot)} u = \dfrac{g_n}{(v_+ + \frac{1}{n})^{\sigma(x)}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6.3}$$

Since $\frac{g_n}{(v_+ + \frac{1}{n})^{\sigma(x)}} \leqslant C(n, \|\sigma\|_\infty)$, then $\mathfrak{R}$ is well defined. Hence we get the existence of $R > 0$ depending only on $n, \|\sigma\|_\infty$ such that if $\mathfrak{R}(\overline{B}_R(0)) \subset \overline{B}_R(0)$, where $\overline{B}_R(0)$ is the closed ball of $L^2(\Omega)$ centered at the origin with radius R. It is not difficult to show that $\mathfrak{R}$ is continuous and compact. Hence using the Schauder fixed point Theorem, we get the existence of $u_n \in L^2(\Omega)$ such that $\mathfrak{R}(u_n) = u_n$. It is clear that u_n is a positive solution of (6.2). The uniqueness of u_n follows using Theorem 2.8, where in the same way we deduce that $\{u_n\}_n$ is increasing in n.

Let $\sigma^* = \max_{x\in\overline{\Omega}} \sigma(x)$. Using $T_k(u_n^{\sigma^*})$, where T_k is defined in (2.3), as a test function in (6.2). We obtain

$$\frac{1}{2}\iint_{D_\Omega} \frac{(u_n(x)-u_n(y))(T_k(u_n^{\sigma^*}(x))-T_k(u_n^{\sigma^*}(y)))}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y \leqslant \int_\Omega \frac{g_n T_k(u_n^{\sigma^*}(x))}{(u_n+\frac{1}{n})^{\sigma(x)}}\mathrm{d}x.$$

Notice that

$$\begin{aligned}\int_\Omega \frac{g_n T_k(u_n^{\sigma^*}(x))}{(u_n+\frac{1}{n})^{\sigma(x)}}\mathrm{d}x &= \int_{\{u_n^{\sigma^*}<k\}} \frac{g_n u_n^{\sigma^*}(x)}{(u_n+\frac{1}{n})^{\sigma(x)}}\mathrm{d}x + k\int_{\{u_n^{\sigma^*}\geqslant k\}} \frac{g_n}{(u_n+\frac{1}{n})^{\sigma(x)}}\mathrm{d}x \\ &\leqslant \int_{\{u_n^{\sigma^*}<k\}} g_n(u_n^{\sigma^*-\sigma(x)}(x))\mathrm{d}x + k\int_{\{u_n^{\sigma^*}\geqslant k\}} \frac{g_n}{(u_n+\frac{1}{n})^{\sigma(x)}}\mathrm{d}x \\ &\leqslant C(k)\int_\Omega g\mathrm{d}x.\end{aligned}$$

Thus we deduce that

$$\frac{1}{2}\iint_{D_\Omega} \frac{(u_n(x)-u_n(y))(T_k(u_n^{\sigma^*}(x))-T_k(u_n^{\sigma^*}(y)))}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y \leqslant C(k)\|g\|_{L^1(\Omega)}.$$

In addition, we have

$$\frac{1}{2}\iint_{D_\Omega} \frac{(u_n(x)-u_n(y))(T_k(u_n^{\sigma^*}(x))-T_k(u_n^{\sigma^*}(y)))}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y \geqslant C\|T_k(u_n^{\frac{\sigma^*+1}{2}})\|^2_{\mathbb{H}_0^{s(\,\cdot\,)}(\Omega)}.$$

Hence as in the past subsection and taking into consideration that $\{u_n\}_n$ is monotone, we get the existence of a measurable function u such that $u_n \uparrow u$ a.e. in $\mathbb{R}^N$, $T_k(u_n^{\frac{\sigma^*+1}{2}}) \rightharpoonup T_k(u^{\frac{\sigma^*+1}{2}})$ weakly in $\mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$. It is clear that $\operatorname{essinf}_K u \geqslant C(K)$ for any compact set $K\subset\Omega$ and that $u_n \to u$ strongly in $L^1(\Omega)$. Hence by the above estimate and passing to the limit in the problem of u_n, we can show that u is a distributional solution to problem (6.1) with $T_k(u^{\frac{\sigma^*+1}{2}}) \in \mathbb{H}_0^{s(\,\cdot\,)}(\Omega)$ and $u \in L^1(\Omega)$. The uniqueness follows thanks to comparison principle in Proposition 5.2 and the monotonicity of the singular term. □

In order to show more regularity result for the solution to Problem (6.1), in the case where $\sigma(\,\cdot\,)$ is not a constant, we will consider the set

$$\Omega_\delta := \{x\in\Omega\,;\ \mathrm{dist}(x,\partial\Omega)\geqslant\delta\} \quad \text{with } \delta>0,$$

and we will distinguish two cases, depending on whether $\|\sigma\|_{L^\infty(\Omega)} > 1$ or not.

More precisely, we have the next regularity result.

Theorem 6.2. — *Assume that σ is positive continuous function in $\overline{\Omega}$. Let u be the unique distributional solution to Problem* (6.1) *obtained in Theorem 6.1 and define $\sigma^* := \|\sigma\|_{L^\infty(\Omega)}$. Then*

(1) *If $\sigma(x) \leqslant 1$ in Ω_δ for some $\delta > 0$ and $g \in L^{(2^*_{s_0})'}(\Omega)$, then $u \in \mathbb{H}_0^{s(\cdot)}(\Omega)$.*
(2) *If $\sigma(x) \geqslant 1$ in Ω_δ, $\sigma^* > 1$ and $g \in L^{m_1}(\Omega)$ where $m_1 = \frac{N(\sigma^*+1)}{N+2s_0\sigma^*}$, then $u^{\frac{\sigma^*+1}{2}} \in \mathbb{H}_0^{s(\cdot)}(\Omega)$.*

Proof. — Let u_n be the unique solution to the approximating problem (6.2) and consider the set $\omega_\delta := \Omega \setminus \Omega_\delta$. Thanks to the Harnack inequality and the monotonicity of $\{u_n\}_n$, we get the existence C_{ω_δ} independent of n such that $u_n \geqslant C_{\omega_\delta}$.

In the first case we have just to show that $\{u_n\}$ is bounded in $\mathbb{H}_0^{s(\cdot)}(\Omega)$. More precisely, using u_n as test function in (6.2), it holds that

$$\begin{aligned}
\|u_n\|^2_{\mathbb{H}_0^{s(\cdot)}(\Omega)} &= \int_{\overline{\Omega}_\delta} \frac{g_n}{(u_n+\frac{1}{n})^{\sigma(x)}} u_n \mathrm{d}x + \int_{\omega_\delta} \frac{g_n}{(u_n+\frac{1}{n})^{\sigma(x)}} u_n \mathrm{d}x \\
&\leqslant \int_{\overline{\Omega}_\delta \cap \{u_n \leqslant 1\}} g_n \mathrm{d}x + \int_{\overline{\Omega}_\delta \cap \{u_n \geqslant 1\}} g_n u_n \mathrm{d}x + \int_{\omega_\delta} \frac{g_n}{C_{\omega_\delta}^{\sigma(x)}} u_n \mathrm{d}x \\
&\leqslant \|g\|_{L^1(\Omega)} + \left(1 + \|C_{\omega_\delta}^{-\sigma(\cdot)}\|_{L^\infty(\Omega)}\right) \int_\Omega g_n u_n \mathrm{d}x.
\end{aligned}$$

Therefore, by applying Hölder and Sobolev inequalities, we get

$$\int_\Omega g_n u_n \mathrm{d}x \leqslant \|g\|_{L^{(2^*_{s_0})'}(\Omega)} \left(\int_\Omega u_n^{2^*_{s_0}}\right)^{\frac{1}{2^*_{s_0}}} \leqslant C\left(\iint_{D_\Omega} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s(x,y)}} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}$$

Thus, we obtain $\|u_n\|_{H_0^{s(\cdot)}(\Omega)} \leqslant C$. Thus we conclude. For the second case, we use $u_n^{\sigma^*}$ as test function in (6.1) and proceed as in the first case. □

Now, going back to Problem (5.1), with $\sigma(\cdot) \neq$ constant. By following the same arguments as in the proof of Theorem 5.4, we get the next existence result.

Theorem 6.3. — *Suppose that $\sigma(\cdot)$ is a nonnegative continuous function in $\overline{\Omega}$. Then for all $f, g \in L^1(\Omega)$ with $f \geqslant 0$ and $g \gneqq 0$, Problem* (5.1) *has a unique positive distributional solution u such that $u \in L^1(\Omega)$ and $T_k(u^{\frac{\sigma^*+1}{2}}) \in \mathbb{H}_0^{s(\cdot)}(\Omega)$ for all $k > 0$.*

In the case where f and g are more regular, we can improve the regularity of the solution. In particular, we have the next improvement.

Theorem 6.4. — *Let σ be a positive continuous function in $\overline{\Omega}$ and consider u to be the unique distributional solution to Problem* (5.1) *obtained in Theorem 6.3. Then*

(1) *If $\sigma(x) \geqslant 1$ in Ω_δ and $f, g \in L^{m_1}(\Omega)$ with $m_1 = \frac{N(\sigma^*+1)}{N+2s_0\sigma^*}$, then $u^{\frac{\sigma^*+1}{2}} \in \mathbb{H}_0^{s(\cdot)}(\Omega)$.*

(2) *If $\sigma(x) \leqslant 1$ in Ω_δ and $f, g \in L^{(2^*_{s_0})'}(\Omega)$, then $u \in \mathbb{H}_0^{s(\cdot)}(\Omega)$.*

Proof. — We use the same reasoning as in the proof of Theorem 6.2. □

In this regard, we can prove the existence of a solution for all $g \in L^1(\Omega)$ when $\sigma(x) \geqslant 1$ in $\overline{\Omega}$. This is the purpose of the following theorem:

Theorem 6.5. — *Assume that hypotheses of Theorem 6.4 hold. Suppose that $g \in L^1(\Omega)$ and $f \in L^{(2^*_{s_0})'}(\Omega)$. Let u be the solution to Problem* (5.1) *obtained in Theorem 6.3. Then, $G_k(u) \in \mathbb{H}_0^{s(\cdot)}(\Omega)$ and $T_k^{\frac{\sigma^*+1}{2}}(u) \in \mathbb{H}_0^{s(\cdot)}(\Omega)$ for all $k > 0$.*

Proof. — Recall that u_n is the solution to the approximating Problem (5.9). Then, using $G_k(u_n)$ as test function in (5.9), Hölder and Sobolev inequalities, we get

$$\begin{aligned} C\|G_k(u_n)\|^2_{\mathbb{H}_0^{s(\cdot)}(\Omega)} &\leqslant \int_\Omega \frac{g_n G_k(u_n)}{(u_n+\frac{1}{n})^{\sigma(x)}}\mathrm{d}x + \int_\Omega f_n G_k(u_n)\mathrm{d}x \\ &\leqslant \int_\Omega \frac{g_n}{k^{\sigma(x)-1}}\mathrm{d}x + \|f\|_{L^{(2^*_{s_0})'}(\Omega)}\|G_k(u_n)\|_{L^{2^*_{s_0}}(\Omega)} \\ &\leqslant \|k^{1-\sigma(\cdot)}\|_{L^\infty(\Omega)}\|g\|_{L^1(\Omega)} + C\|G_k(u_n)\|_{\mathbb{H}_0^{s(\cdot)}}(\Omega). \end{aligned}$$

Hence $\{G_k(u_n)\}_n$ is bounded in $\mathbb{H}_0^{s(\cdot)}(\Omega)$. We prove now that $\{T_k^{\frac{\sigma^*+1}{2}}(u_n)\}_n$ is bounded in $\mathbb{H}_0^{s(\cdot)}(\Omega)$. Testing with $T_k^{\sigma^*}(u_n)$ in (5.9), it follows that,

$$\begin{aligned} C\|T_k^{\frac{\sigma^*+1}{2}}(u_n)\|^2_{\mathbb{H}_0^{s(\cdot)}(\Omega)} &\leqslant \int_\Omega \frac{g_n T_k^{\sigma^*}(u_n)}{(u_n+\frac{1}{n})^{\sigma(x)}}\mathrm{d}x + \int_\Omega f_n T_k^{\sigma^*}(u_n)\mathrm{d}x \\ &\leqslant \int_\Omega g_n T_k^{\sigma^*-\sigma(x)}(u_n)\mathrm{d}x + C(\sigma^*, k, \|f\|_{L^1(\Omega)}) \\ &\leqslant \|k^{\sigma^*-\sigma(\cdot)}\|_{L^\infty(\Omega)}\|g\|_{L^1(\Omega)} + C(\sigma^*, k, \|f\|_{L^1(\Omega)}) \\ &\leqslant C(k, \sigma^*, \|f\|_{L^1(\Omega)}, \|g\|_{L^1(\Omega)}), \end{aligned}$$

Taking into consideration the monotony of the sequence $\{u_n\}_n$, it follows that $G_k(u) \in \mathbb{H}_0^{s(\cdot)}(\Omega)$ and $T_k^{\frac{\sigma^*+1}{2}}(u) \in \mathbb{H}_0^{s(\cdot)}(\Omega)$. Hence we conclude. □

Remark 6.6. — The existence results in Theorems 5.4 and 6.3 hold if we replace f with a nonnegative bounded Radon measure μ. This follows by using the fact that we can approximate μ by a sequence of bounded functions $\{f_n\}_n$ with $\|f_n\|_{L^1(\Omega)} \leqslant C$.

7. Appendix

Inspired by the papers [14, 59], we will introduce two notions of $(s(\cdot), p)$-capacities connected with the variable order fractional Sobolev spaces $W^{s(\cdot),p}(\Omega)$. Then, we will give a decomposition Theorem for regular signed measures with respect to (s, p)-capacity. As an interesting application, we will extend the famous result of [16] about a nonlinear problem involving the usual Laplacian to a nonlinear problem driven by the regional fractional Laplacian.

In order to make this paper self-contained, we include some details for the reader's convenience.

7.1. Different notions of capacity

In this paragraph, we will introduce three notions of capacity and gather some of their properties that we have already used in Section 5 or will use in the last part of this appendix.

Throughout this paragraph, we will assume that $p \in [1, +\infty)$ and the function $s(\cdot)$ satisfies the hypothesis (H_1) and (H_2).

Choquet capacity.

First, let us recall the definition of the so-called Choquet capacity.

Definition 7.1. — *Let $\mathcal{T}$ be a topological space and let $\mathcal{P}(\mathcal{T})$ be the power set of $\mathcal{T}$. A mapping $\mathcal{C} : \mathcal{P}(\mathcal{T}) \to [0, \infty]$ is called a Choquet capacity on $\mathcal{T}$ if the following properties are satisfied:*

(1) $\mathcal{C}(\emptyset) = 0$;
(2) *Let* $(A, B) \in \mathcal{P}(\mathcal{T}) \times \mathcal{P}(\mathcal{T})$ *such that* $A \subset B$, *then* $\mathcal{C}(A) \leqslant \mathcal{C}(B)$,
(3) *Let* $\{A_n\}_n$ *be an increasing sequence of subsets of* $\mathcal{T}$. *Then*

$$\lim_{n\to+\infty} \mathcal{C}(A_n) = \mathcal{C}\left(\bigcup_{n=0}^{\infty} A_n\right);$$

(4) *Let $\{K_n\}_n$ be a decreasing sequence where K_n is a compact subset of $\mathcal{T}$. Then*

$$\lim_{n\to+\infty}\mathcal{C}(K_n)=\mathcal{C}\left(\bigcap_{n=0}^{\infty}K_n\right).$$

For more details on the Choquet capacity, we refer the interested reader to [25, 34].

Variable order fractional $(s(\,\cdot\,),p)$-capacity.

Definition 7.2. — *Let O be an open set of $\mathbb{R}^N$. The variable order fractional $(s(\,\cdot\,),p)$-capacity of O is defined by*

$$\begin{aligned}&\mathrm{Cap}_{(s(\,\cdot\,),p)}(O)\\&\qquad:=\inf\Big\{\|u\|^p_{W^{s(\,\cdot\,),p}(\mathbb{R}^N)}\,;\ u\in W^{s(\,\cdot\,),p}(\mathbb{R}^N)\text{ and } u\geqslant 1\text{ a.e. in } O\Big\}.\end{aligned}$$

Now, we will gather some properties of the variable order fractional $\mathrm{Cap}_{(s(\,\cdot\,),p)}$ that can be proved by using the same approach as for the "classical" Sobolev-capacity, see [10, 33, 59].

Theorem 7.3. — *Assume that* (H_1) *and* (H_2) *hold. Then, the variable order fractional capacity* $\mathrm{Cap}_{(s(\,\cdot\,),p)}$ *has the following properties:*

(i) $\mathrm{Cap}_{(s(\,\cdot\,),p)}$ *is a Choquet capacity.*

(ii) *For any subset M of $\mathbb{R}^N$, we have*

$$\begin{aligned}&\mathrm{Cap}_{(s(\,\cdot\,),p)}(M)\\&\qquad=\inf\big\{\mathrm{Cap}_{(s(\,\cdot\,),p)}(O)\,;\ O\text{ open subset of }\mathbb{R}^N\text{and } M\subset O\big\}\\&\qquad=\sup\big\{\mathrm{Cap}_{(s(\,\cdot\,),p)}(K)\,;\ K\text{ compact subset of }\mathbb{R}^N\text{and } K\subset M\big\}.\end{aligned}$$

(iii) *For any compact subset K of $\mathbb{R}^N$, we have*

$$\mathrm{Cap}_{s(\,\cdot\,),2}(K)=\inf\Big\{\|u\|^p_{W^{s(\,\cdot\,),p}(\mathbb{R}^N)}\,;\ u\in\mathcal{C}_0^\infty(\mathbb{R}^N)\text{ and } u\geqslant 1\text{ a.e. on } K\Big\}.$$

Variable order fractional relative $(s(\,\cdot\,),p)$-capacity.

Before going further, one should note that the fractional Sobolev version of the relative (s,p)-capacitie has been introduced in the case where $s(\,\cdot\,)$ is constant in [59] (see also [14]). Here, we will extend this notion to the case where the function $s(\,\cdot\,)$ is not a constant function.

In what follows, Ω is an arbitrary (unless otherwise specified) open subset of $\mathbb{R}^N$ with boundary $\partial\Omega$.

Definition 7.4. — *Let $U \subset \overline{\Omega}$ be a relatively open set, that is, open with the relative topology of $\overline{\Omega}$. The variable order fractional relative $(s(\cdot), p)$-capacity of U with respect to Ω is defined by*

$$\mathrm{Cap}^{\overline{\Omega}}_{(s(\cdot),p)}(U) := \inf\Big\{ \|u\|^p_{W^{s,p}(\Omega)} \,;\, u \in \widetilde{W^{s,p}}(\Omega) \text{ s.t. } u \geqslant 1 \text{ a.e. on } U \Big\}.$$

Remark 7.5. — If $\Omega = \mathbb{R}^N$, $\mathrm{Cap}^{\overline{\Omega}}_{(s(\cdot),p)} = \mathrm{Cap}_{(s(\cdot),p)}$.

Theorem 7.6. — *Assume that* (H_1) *and* (H_2) *hold. Then, the variable fractional relative capacity* $\mathrm{Cap}^{\overline{\Omega}}_{(s(\cdot),p)}$ *has the following properties:*

(i) $\mathrm{Cap}^{\overline{\Omega}}_{(s(\cdot),p)}$ *is a Choquet capacity on* $\overline{\Omega}$.

(ii) *For any subset $M \subset \overline{\Omega}$, we have*

$$\begin{aligned}
&\mathrm{Cap}^{\overline{\Omega}}_{(s(\cdot),p)}(M)\\
&\quad = \inf\Big\{ \mathrm{Cap}^{\overline{\Omega}}_{s(\cdot),2}(U) \,;\, U \text{ relatively open in } \overline{\Omega} \text{ and } B \subset U \Big\}\\
&\quad = \sup\Big\{ \mathrm{Cap}^{\overline{\Omega}}_{(s(\cdot),p)}(K) \,;\, K \text{ compact subset of } \mathbb{R}^N \text{ and } K \subset M \subset \overline{\Omega} \Big\}.
\end{aligned}$$

(iii) *For any compact K subset of $\overline{\Omega}$, we have*

$$\begin{aligned}
&\mathrm{Cap}^{\overline{\Omega}}_{s(\cdot),p}(K)\\
&\quad = \inf\Big\{ \|u\|^p_{W^{s(\cdot),p}(\Omega)} \,;\, u \in W^{s(\cdot),p}(\Omega) \cap \mathcal{C}_c(\overline{\Omega}) \text{ and } u \geqslant 1 \text{ a.e. on } K \Big\}.
\end{aligned}$$

Decomposition of a signed measure in $\mathcal{M}^p_b(\Omega)$.

Before stating the main result of this paragraph, we will introduce two notations. We denote by:

- $\mathcal{M}_b(\Omega)$ the set of all bounded signed measures on Ω;
- $\mathcal{M}^p_b(\Omega)$ the space of all measures in $\mathcal{M}_b(\Omega)$ such that $\mu(E) = 0$ for every set E fulfilling $\mathrm{Cap}^{\overline{\Omega}}_{s,p}(E) = 0$.

Inspired by the papers [16, 61], we have the following decomposition of a measure of $\mathcal{M}_b(\Omega)$.

Theorem 7.7. — *Let Ω be bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$, μ be an element of $\mathcal{M}_b(\Omega)$, $s \in (0,1)$ and $1 < p < +\infty$. Assume that $\mathrm{Cap}^{\overline{\Omega}}_{s,p}(\partial\Omega) = 0$. Then, $\mu \in L^1(\Omega) + W^{-s,p'}(\Omega)$ if and only if $\mu \in \mathcal{M}^p_b(\Omega)$.*

The main idea in the proof of Theorem 7.7 is similar to the proof of [61, Proposition 2.6], see also [16, 27, 28].

7.2. Application

In this subsection, we deal with the following Problem,

$$\begin{cases} (-\Delta)^s u + u D_s^2(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{7.1}$$

where Ω is bounded regular domain of $\mathbb{R}^N$ and $s \in (\frac{1}{2}, 1)$. Here $D_s^2(u)$ is a nonlocal term that plays the role of the gradient square in the nonlocal case and it is given by

$$D_s^2(x) = \frac{a_{N,s}}{2} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}y$$

with $a_{N,s}$ is the normalization constant given by (1.3).

It is worth to mention that the case, where the nonlocal gradient term $D_s^2(u)$ appears as a reaction term, was studied recently in [8]. Existence of a solution is proved under restrictive condition on the datum μ.

Since we are considering our problem with general datum, we need to specify the concept of solution. We begin by the following definition, see [11] and [4].

Definition 7.8. — *Let u be a measurable function. We say that $u \in \mathcal{T}_0^{s,2}(\Omega)$ if $T_k(u) \in W_0^{s,2}(\Omega)$ for all $k > 0$. Let $u \in \mathcal{T}_0^{s,2}(\Omega)$. We say that u is a renormalized solution to* (7.1) *if $u D_s^2(u) \in L^1(\Omega)$ and the following conditions are satisfied:*

(1) $\displaystyle \lim_{h \to +\infty} \iint_{A(h)} \frac{|u(x) - u(y)|}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y = 0$, *where*

$$A(h) = \left\{ (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \, ; \begin{array}{l} h+1 \leqslant \max\{|u(x)|, |u(y)|\} \\ \textit{and } \min\{|u(x)|, |u(y)|\} \leqslant h \textit{ or } u(x)u(y) < 0 \end{array} \right\}$$

(2) *for any $\phi \in C_0^\infty(\Omega)$ and $S \in W^{1,\infty}(\mathbb{R}^N)$ with compact support, it holds that*

$$\iint_{D_\Omega} \frac{(u(x) - u(y))((S(u))\phi(x) - (S(u))\phi(y)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \int_\Omega S(u)\phi \, u D_s^2(u) = \int_\Omega S(u) \phi d\mu.$$

The main result of this subsection is the following:

Theorem 7.9. — *Let Ω be bounded regular domain in $\mathbb{R}^N$ such that $\mathrm{Cap}^{\overline{\Omega}}_{s,p}(\partial\Omega) = 0$, $\mu \in \mathcal{M}_b(\Omega)$ and $s \in (\frac{1}{2}, 1)$. Assume that u is a solution to* (7.1) *in the sense of the definition 7.8. Then $\mu \in \mathcal{M}^p_b(\Omega)$.*

Proof. — Let u be a renormalized solution to (7.1). To prove that $\mu \in \mathcal{M}^p_b(\Omega)$, it suffices to prove that $u \in W^{s,2}_0(\Omega)$. Moreover $T_k(u) \in W^{s,2}_0(\Omega)$, then we have just to prove that $G_k(u) \in W^{s,2}_0(\Omega)$.

Define the following subsets

$$
\begin{aligned}
A_1 &= \big\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N \ ; \ u(x) \leqslant k \text{ and } u(y) \leqslant k\big\},\\
A_2 &= \big\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N \ ; \ u(x) > k \text{ and } u(y) > k\big\},\\
A_3 &= \big\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N \ ; \ u(x) \leqslant k \text{ and } u(y) > k\big\},\\
\text{and} \quad A_4 &= \big\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N \ ; \ u(x) > k \text{ and } u(y) \leqslant k\big\}.
\end{aligned}
$$

Thus, if $G_k(u) \in W^{s,2}_0(\Omega)$ for every k, then, by using the previous sets, we obtain

$$
\begin{aligned}
\iint_{D_\Omega} &\frac{(G_k(u(x)) - G_k(u(y)))^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y\\
&= \iint_{A_2} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \iint_{A_3} \frac{(u(y) - k)^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y\\
&\qquad + \iint_{A_4} \frac{(u(x) - k)^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y. \qquad (7.2)
\end{aligned}
$$

In addition, $uD^2_s(u) \in L^1(\Omega)$. Then, we obtain

$$
\begin{aligned}
\int_\Omega uD^2_s(x)\mathrm{d}x &= \int_\Omega u(x) \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y\\
&\geqslant k \int_{\{u(x)>k\}} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y\\
&\geqslant k \iint_{A_2} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y. \qquad (7.3)
\end{aligned}
$$

Hence

$$
\iint_{A_2} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \leqslant C(k). \qquad (7.4)
$$

On the other hand, we have

$$\int_{\Omega} uD_s^2(x)\mathrm{d}x = \int_{\mathbb{R}^N} u(x)\int_{\mathbb{R}^N}\frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$
$$= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))^2(u(x)+u(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$
$$\geqslant C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u^{\frac{3}{2}}(x)-u^{\frac{3}{2}}(y))^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y, \tag{7.5}$$

which implies that $u^{\frac{3}{2}} \in W_0^{s,2}(\Omega)$ and $G_k(u^{\frac{3}{2}}) \in W_0^{s,2}(\Omega)$ for all $k > 0$. Thus, by using the same computation as in (7.2), we get,

$$\iint_{D_\Omega}\frac{(G_k(u^{\frac{3}{2}}(x))-G_k(u^{\frac{3}{2}}(y)))^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$
$$= \iint_{\{u^{\frac{3}{2}}(x)>k,u^{\frac{3}{2}}(y)>k\}}\frac{(u^{\frac{3}{2}}(x)-u^{\frac{3}{2}}(y))^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$
$$+\iint_{\{u^{\frac{3}{2}}(x)\leqslant k,u^{\frac{3}{2}}(y)>k\}}\frac{(u^{\frac{3}{2}}(y)-k)^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$
$$+\iint_{\{u^{\frac{3}{2}}(x)>k,u^{\frac{3}{2}}(y)\leqslant k\}}\frac{(u^{\frac{3}{2}}(x)-k)^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$
$$\geqslant \iint_{\{u^{\frac{3}{2}}(x)\leqslant k,u^{\frac{3}{2}}(y)>k\}}\frac{(u^{\frac{3}{2}}(y)-k)^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y$$
$$+\iint_{\{u^{\frac{3}{2}}(x)>k,u^{\frac{3}{2}}(y)\leqslant k\}}\frac{(u^{\frac{3}{2}}(x)-k)^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y. \tag{7.6}$$

Now we claim that, by choosing $k_1 > 0$ such that $u^{\frac{3}{2}}(y) > k_1$, we obtain

$$(u^{\frac{3}{2}}(y)-k_1)^2 \geqslant C(k_1)(u(y)-k_1^{\frac{2}{3}})^2 \tag{7.7}$$

where $C(k_1)$ is a positive constant. Indeed, since $u^{\frac{3}{2}}(y) > k_1$, we have

$$(u^{\frac{3}{2}}(y)-k_1)^2 = u^3(y)\left(1-\frac{k_1}{u^{\frac{3}{2}}(y)}\right)^2 \geqslant k_1u^2(y)\left(1-\frac{k_1^{\frac{2}{3}}}{u(y)}\right)^2.$$

By setting $\tau = \frac{k_1}{u^{\frac{3}{2}}(y)}$, we have $\tau \in (0,1)$ and

$$u(y)\left(1-\tau\right)^2 \geqslant k_1^{\frac{2}{3}}\left(1-\tau\right)^2 \geqslant C(k_1)\left(1-\tau^{\frac{2}{3}}\right)^2,$$

where we have used the fact that $(1-\tau)^2 \geqslant C(1-\tau^{\frac{2}{3}})^2$ for all $\tau \in (0,1)$. Thus (7.7) follows. So, inequality (7.6) implies

$$\iint_{\{u^{\frac{3}{2}}(x)\leqslant k, u^{\frac{3}{2}}(y)>k\}} \frac{(u^{\frac{3}{2}}(y)-k)^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y < \infty \quad \text{for every } k > k_1. \tag{7.8}$$

Hence, by using the same computations as above, we obtain

$$\iint_{\{u^{\frac{3}{2}}(y)\leqslant k, u^{\frac{3}{2}}(x)>k\}} \frac{(u^{\frac{3}{2}}(x)-k)^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y < \infty \quad \text{for every } k > k_1. \tag{7.9}$$

Combining (7.2), (7.4), (7.8) and (7.9), we get $G_k(u) \in W_0^{s,2}(\Omega)$. Or $u = T_k(u) + G_k(u)$, then $u \in W_0^{s,2}(\Omega)$. Hence $(-\Delta)^s u \in W^{-s,2}(\Omega)$ (the dual space of $W_0^{s,2}(\Omega)$) and we conclude that $\mu \in \mathcal{M}_b^p(\Omega)$. □

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