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Stability of equivariant logarithmic tangent sheaves on toric varieties of Picard rank two [(*)]

Achim Napame [(1)]

Abstract. — For an equivariant log pair (X, D) where X is a normal toric variety and D a reduced Weil divisor, we study slope-stability of the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$. We give a complete description of divisors D and polarizations L such that $\mathcal{T}_X(-\log D)$ is (semi)stable with respect to L when X has a Picard rank one or two.

Résumé. — Pour une paire logarithmique équivariante (X, D) où X est une variété torique normale et D un diviseur de Weil réduit, nous étudions la stabilité au sens de la pente du faisceau tangent logarithmique $\mathcal{T}_X(-\log D)$. Nous donnons une description complète des diviseurs réduits D et polarisations L sur X tels que le faisceau tangent logarithmique $\mathcal{T}_X(-\log D)$ est (semi)stable par rapport à L lorsque X est une variété torique lisse de rang de Picard un ou deux.

1. Introduction

The notion of slope-stability was first introduced by Mumford [17] in his construction of moduli spaces of vector bundles over a curve. This notion was generalized in higher dimension by Takemoto [21]. A vector bundle, or more generally a torsion-free sheaf $\mathcal{E}$ on a complex projective variety X is said to be slope-stable (resp. semistable) with respect to a polarization L, if for any proper coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$, one has $\mu_L(\mathcal{F}) < \mu_L(\mathcal{E})$ (resp. $\mu_L(\mathcal{F}) \leqslant \mu_L(\mathcal{E})$) where the *slope* of $\mathcal{E}$ with respect to L is given by

$$\mu_L(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot L^{\dim(X)-1}}{\mathrm{rk}(\mathcal{E})}.$$

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Article proposé par Vincent Guedj.

As the study of stability of reflexive sheaves is a difficult problem, we are interested by the category of torus equivariant reflexive sheaves over normal toric varieties. Using the description of equivariant reflexive sheaves over toric varieties in terms of families of filtrations given by Klyachko [13] and Perling [19], Kool in [14, Proposition 4.13] showed that it is enough to compare slopes for equivariant and reflexive saturated subsheaves.

Tangent sheaves are natural examples of equivariant reflexive sheaves on normal toric varieties. Using its equivariant structure, Hering–Nill–Süss [8] and Dasgupta–Dey–Khan [4] studied slope-stability of the tangent bundle of smooth projective toric varieties of Picard rank one or two. Inspired by Iitaka's philosophy, in this paper, we extend the results of [4, 8] to the case of equivariant logarithmic pairs (X, D). More precisely, if X is a normal toric variety and D a reduced snc (simple normal crossing) divisor such that the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$ is equivariant, we are interested by the set of polarizations L on X such that $\mathcal{T}_X(-\log D)$ is (semi)stable with respect to L.

We first note that the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$ is equivariant if and only if D is a torus invariant divisor of X. For a toric variety X with fan Σ in $N \otimes_{\mathbb{Z}} \mathbb{R}$, we denote by D_ρ the torus invariant divisor of X corresponding to the ray $\rho \in \Sigma(1)$ (see Section 2 for precise definitions). Then we have:

Theorem 1.1. — *Let $\Delta \subseteq \Sigma(1)$ and $D = \sum_{\rho \in \Delta} D_\rho$. The family of filtrations $\left(E, \{E^\rho(j)\}_{\rho \in \Sigma(1),\, j \in \mathbb{Z}}\right)$ of the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$ is given by*

$$E^\rho(j) = \begin{cases} 0 & \text{if } j \leqslant -1 \\ N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if } j \geqslant 0 \end{cases} \qquad \text{if } \rho \in \Delta$$

and by

$$E^\rho(j) = \begin{cases} 0 & \text{if } j \leqslant -2 \\ \operatorname{Span}(u_\rho) & \text{if } j = -1 \\ N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if } j \geqslant 0 \end{cases} \qquad \text{if } \rho \notin \Delta$$

where $u_\rho \in N$ is the minimal generator of the ray ρ.

Remark 1.2. — We will see in Section 3.2 that if $\Delta = \Sigma(1)$, then $\mathcal{T}_X(-\log D)$ is isomorphic to the trivial sheaf of rank $\dim(X)$ and if $\Delta = \varnothing$, then $\mathcal{T}_X(-\log D)$ is the tangent sheaf $\mathcal{T}_X$.

By Theorem 1.1 and the fact that $|\Sigma(1)| = \dim(X) + \operatorname{rk}(\operatorname{Cl}(X))$ on complete normal toric varieties X, we show that:

Proposition 1.3. — *If* $1+\mathrm{rk}(\mathrm{Cl}(X)) \leqslant |\Delta| \leqslant |\Sigma(1)|-1$*, then for any polarization* L*, the logarithmic tangent sheaf* $\mathcal{T}_X(-\log D)$ *is unstable with respect to* L*.*

According to this proposition, it is therefore sufficient to study the stability of $\mathcal{T}_X(-\log D)$ when $|\Delta| \leqslant \mathrm{rk}(\mathrm{Cl}(X))$. Thus, in this paper we study the case where X is smooth, $\mathrm{rk}\,\mathrm{Cl}(X) \in \{1,2\}$ and $1 \leqslant |\Delta| \leqslant \mathrm{rk}\,\mathrm{Cl}(X)$. Note that the only smooth projective toric variety with Picard number one is the projective space $\mathbb{P}^n$.

Proposition 1.4. — *Let* D *be an invariant hyperplane section of* $\mathbb{P}^n$*. Then, the logarithmic tangent sheaf* $\mathcal{T}_{\mathbb{P}^n}(-\log D)$ *is polystable with respect to* $\mathcal{O}_{\mathbb{P}^n}(1)$*.*

By [12, Theorem 1], every smooth toric variety of Picard rank two is of the form $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^s}(a_i))$ with $r,s \in \mathbb{N}^*$ and $a_1,\dots,a_r \in \mathbb{N}$ such that $a_1 \leqslant \dots \leqslant a_r$. Moreover, X blown down to $\mathbb{P}^{r+s}$ if and only if $(a_1,\dots,a_r) = (0,\dots,0,1)$. We denote by $\pi : X \to \mathbb{P}^s$ the projection map. Let $\mathcal{V}$ be a vector bundle associated to the locally free sheaf

$$\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(-a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^s}(-a_r).$$

Then the irreducible invariant divisors of X are given by

$$\begin{cases} D_{w_j} = \pi^{-1}(\{(z_0 : \ldots : z_s) \in \mathbb{P}^s : z_j = 0\}) & \text{for } 0 \leqslant j \leqslant s \\ D_{v_i} = \{s_i = 0\} & \text{for } 0 \leqslant i \leqslant r \end{cases}$$

where the $\{s_i = 0\}$ are the relative hyperplane sections associated to the line subbundles of $\mathcal{V}^\vee$. If $L = \pi^*\mathcal{O}_{\mathbb{P}^s}(\alpha) \otimes \mathcal{O}_X(\beta)$ is a polarization of X, according to the value of $\nu := \alpha/\beta$, in Tables 4.1, 5.1, 5.2 and 5.3, we give a complete classification of reduced divisors D and polarizations L on X such that the equivariant logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$ is (semi)stable with respect to L. In particular:

Proposition 1.5. — *Let* $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^s}(a_i))$ *with* $a_1 = \dots = a_r = 0$*. Then for any*

$$D \in \{D_{v_i} : 0 \leqslant i \leqslant r\} \cup \{D_{w_j} : 0 \leqslant j \leqslant s\} \cup \{D_{v_i} + D_{w_j} : 0 \leqslant i \leqslant r, 0 \leqslant j \leqslant s\}$$

$\mathcal{T}_X(-\log D)$ *is polystable with respect to* L *if and only if* L *is a power of the polarization corresponding to* $-(K_X + D)$*.*

For $a_r \geqslant 1$, we show that:

Theorem 1.6. — *Let* $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^s}(a_i))$ *with* $a_r \geqslant 1$. *There are* $\alpha, \beta \in \mathbb{N}^*$ *such that the logarithmic tangent sheaf* $\mathcal{T}_X(-\log D_{v_r})$ *is (semi)stable with respect to* $\pi^*\mathcal{O}_{\mathbb{P}^s}(\alpha) \otimes \mathcal{O}_X(\beta)$ *if and only if* $a_r = 1$ *and* $a_{r-1} = 0$. *Moreover, if* $a_r = 1$ *and* $a_{r-1} = 0$, *then* $\mathcal{T}_X(-\log D_{v_r})$ *is stable (resp. semistable) with respect to* $\pi^*\mathcal{O}_{\mathbb{P}^s}(\alpha)\otimes\mathcal{O}_X(\beta)$ *if and only if* $0 < \frac{\alpha}{\beta} < \nu_0$ *(resp.* $0 < \frac{\alpha}{\beta} \leqslant \nu_0$*) where* ν_0 *is the unique positive root of*

$$\mathrm{P}_0(x) = \sum_{k=0}^{s-1} \binom{s+r-1}{k} x^k - s\binom{s+r-1}{s} x^s.$$

Theorem 1.6 can be seen as an extension of [8, Theorem 1.4] to the case of the logarithmic pair (X, D_{v_r}). Indeed, in [8, Theorem 1.4] it is shown that for $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^s}(a_i))$ with $a_r \geqslant 1$, there are $\alpha, \beta \in \mathbb{N}^*$ such that the tangent sheaf $\mathcal{T}_X$ is (semi)stable with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\alpha) \otimes \mathcal{O}_X(\beta)$ if and only if $a_r = 1$ and $a_{r-1} = 0$.

Remark 1.7. — In the logarithmic case, it is not just varieties X which blown down to $\mathbb{P}^{r+s}$ which admit divisors D such that $\mathcal{T}_X(-\log D)$ is stable. If $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^s}(a_i))$ with $a_r \geqslant 1$ and $(a_1, \dots, a_r)$ not necessarily equal to $(0, \dots, 0, 1)$, in Theorems 5.8 and 5.10, we will show that there are polarizations L on X such that $\mathcal{T}_X(-\log D_{v_0})$ is (semi)stable with respect to L if and only if $a_1 = \cdots = a_r$ and $(r-1)a_r < (s+1)$.

Remark 1.8. — For these studies of stability when $\operatorname{rk}\operatorname{Cl}(X) = 2$, we use the calculations made in [8] but we simplify their arguments using Lemma 4.9.

Organization

In Section 2 we recall the necessary background on toric varieties, equivariant reflexive sheaves and their families of filtrations. We also recall the notions of slope-stability. In Section 3, we study the logarithmic tangent sheaf. We prove Theorem 1.1 and Proposition 1.3. Sections 4 and 5 deal with the study of the stability of $\mathcal{T}_X(-\log D)$ when $\operatorname{rk}\operatorname{Cl}(X) = 2$. In Section 6, we apply the results of the paper on Hirzebruch surfaces.

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2. Toric varieties, equivariant sheaves and stability notions

In this section, we present the different notions that will be discussed in this paper: toric varieties [2], equivariant sheaves [19] and stability of sheaves [21].

2.1. Normal toric varieties

A n-dimensional *toric variety* is an irreducible variety X containing a torus $T \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T on itself extends to an algebraic action of T on X.

Let N be a rank n lattice and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual with pairing $\langle \cdot , \cdot \rangle : M \times N \to \mathbb{Z}$. Then N is the *lattice of one-parameter subgroups* of the n-dimensional complex torus $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$. We call M the *lattice of characters* of T_N. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we define $N_{\mathbb{K}} = N \otimes_{\mathbb{Z}} \mathbb{K}$ and $M_{\mathbb{K}} = M \otimes_{\mathbb{Z}} \mathbb{K}$. We denote by $\chi^m : T_N \to \mathbb{C}^*$ the character corresponding to $m \in M$ and by $\lambda^u : \mathbb{C}^* \to T_N$ the one-parameter subgroup corresponding to $u \in N$.

A *fan* Σ in $N_{\mathbb{R}}$ is a set of rational strongly convex polyhedral cones in $N_{\mathbb{R}}$ such that:

- Each face of a cone in Σ is also a cone in Σ;
- The intersection of two cones in Σ is a face of each.

We will denote $\tau \preceq \sigma$ the inclusion of a face τ in $\sigma \in \Sigma$. A cone σ in $N_{\mathbb{R}}$ is *smooth* if its minimal generators form part of a $\mathbb{Z}$-basis of N. We say that σ is *simplicial* if its minimal generators are linearly independent over $\mathbb{R}$. A fan Σ is *smooth* (resp. *simplicial*) if every cone σ in Σ is smooth (resp. *simplicial*). The *support* of Σ is given by $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$ and we say that Σ is *complete* if $|\Sigma| = N_{\mathbb{R}}$.

Notation 2.1. — For a finite subset $S \subseteq N$, we denote by $\mathrm{Cone}(S)$ the cone generated by S. For a fan Σ, we denote by

- $\Sigma(r)$ the set of r-dimensional cones of Σ;
- $u_\rho \in N$ the minimal generator of $\rho \in \Sigma(1)$.

Elements of $\Sigma(1)$ will be called *rays.*

For $\sigma \in \Sigma$, let $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ where $\mathbb{C}[S_\sigma]$ is the semi-group algebra of

$$S_\sigma = \sigma^\vee \cap M = \{m \in M \ : \ \langle m,\, u\rangle \geqslant 0 \text{ for all } u \in \sigma\}.$$

If σ, $\sigma' \in \Sigma$, we have $U_\sigma \cap U_{\sigma'} = U_{\sigma\cap\sigma'}$. We denote by X_Σ the toric variety associated to a fan Σ; X_Σ is obtained by gluing the affine charts $(U_\sigma)_{\sigma\in\Sigma}$. The variety X_Σ is normal and its torus is T_N. As every separated normal toric variety comes from a fan, from now on, a normal toric variety will be defined by a fan.

Let X be the toric variety associated to a fan Σ in $N_\mathbb{R}$. For any $\sigma \in \Sigma$, there is a point $\gamma_\sigma \in U_\sigma$ called the *distinguished point* of σ such that the torus orbit $O(\sigma)$ corresponding to σ is given by $O(\sigma) = T \cdot \gamma_\sigma$. We will use the following result:

Theorem 2.2 (Orbit-Cone Correspondence [2, Theorem 3.2.6]). — *Let X be the toric variety associated to a fan Σ with torus T. Then*

(1) *There is a bijective correspondence*
$$\begin{aligned}\{\textit{Cone } \sigma \textit{ in } \Sigma\} &\longleftrightarrow \{T\textit{-orbits in } X\}\\ \sigma &\longleftrightarrow O(\sigma)\end{aligned}$$
with $\dim O(\sigma) = \dim N_\mathbb{R} - \dim \sigma$.

(2) *The affine open subset U_σ is the union of orbits*
$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau).$$

(3) *$\tau \preceq \sigma$ if and only if $O(\sigma) \subseteq \overline{O(\tau)}$, and*
$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma)$$
where $\overline{O(\tau)}$ denotes the closure in both the classical and Zariski topologies.

Notation 2.3. — For any $\rho \in \Sigma(1)$, we set $D_\rho = \overline{O(\rho)}$.

For any $\rho \in \Sigma(1)$, D_ρ defines a T-invariant Weil divisor of X. Divisors of the form $\sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ are precisely the invariant divisors under the torus action on X. Thus,
$$\mathrm{WDiv}_T(X) := \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z} D_\rho$$
is the group of invariant Weil divisors on X. In particular,

Theorem 2.4 ([2, Theorem 8.2.3]). — *The canonical divisor of a toric variety X_Σ is the torus invariant Weil divisor*
$$K_{X_\Sigma} = -\sum_{\rho\in\Sigma(1)} D_\rho.$$

By [2, Proposition 4.1.2], for any $m \in M$, the character χ^m is a rational function on X_Σ, and its divisor is given by

$$\operatorname{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho, \tag{2.1}$$

so $\operatorname{div}(\chi^m)$ defines an invariant principal divisor of X_Σ. A normal toric variety X_Σ has a *torus factor* if and only if the set $\{u_\rho : \rho \in \Sigma(1)\}$ do not span $N_\mathbb{R}$. If X_Σ has no torus factor, then by [2, Theorem 4.1.3] we have the exact sequence

$$0 \longrightarrow M \longrightarrow \mathrm{WDiv}_T(X_\Sigma) \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0 \tag{2.2}$$

where the map $M \to \mathrm{WDiv}_T(X_\Sigma)$ is given by Equation (2.1). Therefore,

Corollary 2.5. — *If X_Σ has no torus factor, then*

$$|\Sigma(1)| = \dim(X_\Sigma) + \operatorname{rk}\mathrm{Cl}(X).$$

We recall that a *lattice polytope* $\mathrm{Conv}(S)$ in $M_\mathbb{R}$ is the convex hull of a finite set $S \subseteq M$. A Cartier divisor $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ on a complete toric variety X_Σ gives the lattice polytope

$$P_D = \{m \in M_\mathbb{R} : \langle m, u_\rho \rangle \geqslant -a_\rho\} \subseteq M_\mathbb{R}.$$

If P is a full dimensional lattice polytope in $M_\mathbb{R}$ given by

$$P = \{m \in M_\mathbb{R} : \langle m, u_F \rangle \geqslant -a_F \text{ for all facets } F \text{ of } P\} \tag{2.3}$$

where $u_F \in N$ is an *inward-pointing normal* of the facet F and $a_F \in \mathbb{Z}$, we define the fan Σ_P of P by

$$\Sigma_P = \{\mathrm{Cone}(u_F : F \text{ contains } Q) : Q \text{ is a face of } P\}.$$

For any facet F of P, we denote by D_F the invariant divisor of the toric variety X_{Σ_P} corresponding to the ray $\mathrm{Cone}(u_F)$ and we set

$$D_P = \sum_{F \preceq P} a_F D_F.$$

So we have:

Theorem 2.6 ([2, Theorem 6.2.1]). — *Let X be a toric variety given by a complete fan Σ. Then, the map*

$$\begin{Bmatrix} \textit{Torus invariant ample} \\ \textit{divisor on } X \end{Bmatrix} \longrightarrow \begin{Bmatrix} \textit{Full dimensional lattice polytope} \\ P \textit{ in } M_\mathbb{R} \textit{ such that } \Sigma_P = \Sigma \end{Bmatrix}$$
$$D \longmapsto P_D$$

is a bijective correspondence.

Let P be the polytope corresponding to an invariant ample divisor D on X_Σ. For each $\rho \in \Sigma(1)$ we denote by P^ρ the facet of P corresponding to the ray $\rho \in \Sigma(1)$. We recall that a lattice M defines a measure ν on $\mathrm{M}_\mathbb{R}$ as the pullback of the Haar measure on $\mathrm{M}_\mathbb{R}/\mathrm{M}$. It is determined by the properties

(i) ν is translation invariant,
(ii) $\nu(\mathrm{M}_\mathbb{R}/\mathrm{M}) = 1$.

For all $\rho \in \Sigma(1)$, we denote by $\mathrm{vol}(P^\rho)$ the volume of P^ρ with respect to the measure determined by the affine span of $P^\rho \cap M$.

Proposition 2.7 ([3, Section 11]). — *Let (X_Σ, D) be a polarized toric variety corresponding to a lattice polytope P. For all $\rho \in \Sigma(1)$, $\mathrm{vol}(P^\rho) = D_\rho \cdot D^{n-1}$.*

2.2. Smooth toric varieties of Picard rank two

Let X be a smooth toric variety of dimension n with fan Σ in $\mathbb{R}^n$ such that $\mathrm{rk}\,\mathrm{Pic}(X) = 2$. By [2, Theorem 7.3.7] due to Kleinschmidt [12], there are $r, s \in \mathbb{N}^*$ with $r + s = n$ and $a_1, \dots, a_r \in \mathbb{N}$ with $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r$ such that

$$X = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^s}(a_i)\right). \tag{2.4}$$

We denote by $\pi : X \to \mathbb{P}^s$ the projection to the base $\mathbb{P}^s$. By [2, Section 7.3], the rays of Σ are given by the half-lines generated by $w_0, w_1, \dots, w_s$, $v_0, v_1, \dots, v_r$ where $(w_1, \dots, w_s)$ is the standard basis of $\mathbb{Z}^s \times 0_{\mathbb{Z}^r}$, $(v_1, \dots, v_r)$ the standard basis of $0_{\mathbb{Z}^s} \times \mathbb{Z}^r$,

$$v_0 = -(v_1 + \cdots + v_r) \quad \text{and} \quad w_0 = a_1 v_1 + \cdots + a_r v_r - (w_1 + \cdots + w_s).$$

We denote by D_{v_i} the divisor corresponding to the ray $\mathrm{Cone}(v_i)$ and D_{w_j} the divisor corresponding to the ray $\mathrm{Cone}(w_j)$. We have the following linear equivalence,

$$\begin{cases} D_{v_i} \sim_{\mathrm{lin}} D_{v_0} - a_i D_{w_0} & \text{for } i \in \{1, \dots, r\} \\ D_{w_j} \sim_{\mathrm{lin}} D_{w_0} & \text{for } j \in \{1, \dots, s\}. \end{cases} \tag{2.5}$$

By (2.5), we deduce that $\mathrm{Pic}(X)$ is generated by D_{v_0} and D_{w_0}.

Proposition 2.8 ([4, Proposition 4.2.1]). — *Let $D = \alpha D_{w_0} + \beta D_{v_0}$ be an invariant divisor of X with $\alpha, \beta \in \mathbb{Z}$. Then, D is ample if and only if $\alpha > 0$ and $\beta > 0$.*

By Theorem 2.4, the anti-canonical divisor of X is given by

$$-K_X = \sum_{i=0}^{r} D_{v_i} + \sum_{j=0}^{s} D_{w_j} \sim_{\text{lin}} (s+1-a_1-\cdots-a_r)D_{w_0} + (r+1)D_{v_0}. \quad (2.6)$$

Thus, X is a Fano variety if and only if $a_1 + \cdots + a_r \leqslant s$.

Remark 2.9. — For $\alpha, \beta \in \mathbb{N}^*$ and $L = \mathcal{O}_X(\alpha D_{w_0} + \beta D_{v_0})$, we have an isomorphism $L \cong \pi^*\mathcal{O}_{\mathbb{P}^s}(\alpha) \otimes \mathcal{O}_X(\beta)$.

Let $L = \pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ be an ample $\mathbb{Q}$-divisor of X with $\nu \in \mathbb{Q}_{>0}$. For $k \in \{1, \ldots, s\}$, we set $\Delta_k = \text{Conv}(0, w_1, \ldots, w_k)$. By [8, Section 4], the polytope corresponding to the $\mathbb{Q}$-polarized toric variety (X, L) is given by

$$P = \text{Conv}\left(\nu\Delta_s \times \{0\} \cup (a_1+\nu)\Delta_s \times \{v_1\} \cup \cdots \cup (a_r+\nu)\Delta_s \times \{v_r\}\right).$$

We denote by P^{v_i} (resp. P^{w_j}) the facet of P corresponding to the ray $\text{Cone}(v_i)$ (resp. $\text{Cone}(w_j)$). The facet P^{v_i} is the convex hull of

$$\nu\Delta_s \times \{0\} \cup \cdots \cup (a_{i-1}+\nu)\Delta_s \times \{v_{i-1}\} \\ \cup (a_{i+1}+\nu)\Delta_s \times \{v_{i+1}\} \cup \cdots \cup (a_r+\nu)\Delta_s \times \{v_r\}$$

and P^{w_i} is isomorphic to

$$\nu\Delta_{s-1} \times \{0\} \cup (a_1+\nu)\Delta_{s-1} \times \{v_1\} \cup \cdots \cup (a_r+\nu)\Delta_{s-1} \times \{v_r\}.$$

By [8, Proposition 4.3], for any $j \in \{0, \ldots, s\}$,

$$\text{vol}(P^{w_j}) = \sum_{k=0}^{s-1} \binom{s+r-1}{k} \left(\sum_{d_1+\cdots+d_r=s-k-1} a_1^{d_1} \ldots a_r^{d_r} \right) \nu^k$$

and

$$\text{vol}(P^{v_0}) = \sum_{k=0}^{s} \binom{s+r-1}{k} \left(\sum_{d_1+\cdots+d_r=s-k} a_1^{d_1} \ldots a_r^{d_r} \right) \nu^k.$$

If $i \in \{1, \ldots, r\}$, we have

$$\text{vol}(P^{v_i}) = \sum_{k=0}^{s} \binom{s+r-1}{k} \left(\sum_{\substack{d_1+\cdots+d_{i-1} \\ +d_{i+1}+\cdots+d_r=s-k}} a_1^{d_1} \ldots a_{i-1}^{d_{i-1}} a_{i+1}^{d_{i+1}} \ldots a_r^{d_r} \right) \nu^k.$$

All these formulas will be used from Section 4.2 when we study the stability of logarithmic tangent sheaves on toric varieties of Picard rank two.

2.3. Equivariant reflexive sheaves and families of filtrations

Let X be a toric variety associated to a fan Σ in $N_{\mathbb{R}}$. Recall that a reflexive sheaf on X is a coherent sheaf $\mathcal{E}$ that is canonically isomorphic to its double dual $\mathcal{E}^{\vee\vee}$.

Let $\theta : T \times X \to X$ be the action of T on X, $\mu : T \times T \to T$ the group multiplication, $p_2 : T \times X \to X$ the projection onto the second factor and $p_{23} : T \times T \times X \to T \times X$ the projection onto the second and the third factor. We call a sheaf $\mathcal{E}$ on X *equivariant* if there exists an isomorphism $\Phi : \theta^*\mathcal{E} \to p_2^*\mathcal{E}$ such that

$$(\mu \times \mathrm{Id}_X)^*\Phi = p_{23}^*\Phi \circ (\mathrm{Id}_T \times \theta)^*\Phi. \tag{2.7}$$

Klyachko gave a description of torus equivariant reflexive sheaves over toric varieties in terms of combinatorial data [13]:

Definition 2.10. — *A family of filtrations $\mathbb{E}$ is the data of a finite dimensional vector space E and for each ray $\rho \in \Sigma(1)$, an increasing filtration $(E^\rho(i))_{i\in\mathbb{Z}}$ of E such that $E^\rho(i) = \{0\}$ for $i \ll 0$ and $E^\rho(i) = E$ for some i.*

Remark 2.11. — Note that we are using increasing filtrations here, as in [19], rather than decreasing as in [13].

To a family of filtrations $\mathbb{E} := \big(E, \{E^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big)$, we can assign an equivariant reflexive sheaf $\mathcal{E} := \mathfrak{K}(\mathbb{E})$ defined by

$$\Gamma(U_\sigma, \mathcal{E}) := \bigoplus_{m\in M} \bigcap_{\rho\in\sigma(1)} E^\rho(\langle m, u_\rho\rangle) \otimes \chi^m \tag{2.8}$$

for all positive dimensional cones $\sigma \in \Sigma$, while $\Gamma(U_{\{0\}}, \mathcal{E}) = E \otimes \mathbb{C}[M]$. The morphisms between families of filtrations are linear maps preserving the filtrations. Then, by [19, Theorem 5.19], the functor $\mathfrak{K}$ induces an equivalence of categories between the families of filtrations and equivariant reflexive sheaves over X.

Notation 2.12. — Let $\mathcal{E}$ be an equivariant reflexive sheaf given by the family of filtrations $\big(E, \{E^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big)$. For any $\rho \in \Sigma$, we denote the space $\Gamma(U_\rho, \mathcal{E})$ by E^ρ and we write

$$E^\rho = \bigoplus_{m\in M} E^\rho_m \otimes \chi^m$$

where for any $m \in M$, $E^\rho_m := E^\rho(\langle m, u_\rho\rangle)$.

Example 2.13 (Tangent sheaf [4, Corollary 2.2.17]*).* — The family of filtrations of the tangent sheaf $\mathcal{T}_X$ of X is given by

$$E^\rho(j) = \begin{cases} 0 & \text{if } j < -1 \\ \mathrm{Span}(u_\rho) & \text{if } j = -1 \\ N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if } j > -1. \end{cases}$$

2.4. Some stability notions

We denote by $\mathrm{Amp}(X) \subset N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ the *ample cone* of X. Let $\mathcal{E}$ be a torsion-free coherent sheaf on X. The *degree* of $\mathcal{E}$ with respect to an ample class $L \in \mathrm{Amp}(X)$ is the real number obtained by intersection

$$\deg_L(\mathcal{E}) = c_1(\mathcal{E}) \cdot L^{n-1}$$

and its *slope* with respect to L is given by

$$\mu_L(\mathcal{E}) = \frac{\deg_L(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}.$$

Definition 2.14. — *A torsion-free coherent sheaf $\mathcal{E}$ is said to be* slope semistable *(or* semistable *for short) with respect to $L \in \mathrm{Amp}(X)$ if for any proper coherent subsheaf of lower rank $\mathcal{F}$ of $\mathcal{E}$, one has*

$$\mu_L(\mathcal{F}) \leqslant \mu_L(\mathcal{E}).$$

When strict inequality always holds, we say that $\mathcal{E}$ is stable*. Finally, $\mathcal{E}$ is said to be* polystable *if it is the direct sum of stable subsheaves of the same slope.*

Proposition 2.15 ([14, Claim 2 of Proposition 4.13]). — *A reflexive polystable sheaf on X is a semistable sheaf on X isomorphic to a (finite, nontrivial) direct sum of reflexive stable sheaves. Let $\mathcal{E}$ be a semistable reflexive sheaf on X, then $\mathcal{E}$ contains a unique maximal reflexive polystable subsheaf of the same slope as $\mathcal{E}$.*

If $\mathcal{E}$ is an equivariant reflexive sheaf on a normal toric variety X given by the family of filtrations $(E, \{E^\rho(j)\})$, according to [14, Proposition 4.13], it is enough to test slope inequalities for equivariant and reflexive saturated subsheaves. By [8, Proposition 2.3], if $\mathcal{F}$ is an equivariant reflexive subsheaf of $\mathcal{E}$ given by the family of filtrations $(F, \{F^\rho(i)\})$ with F a vector subspace of E and $F^\rho(i) \subseteq E^\rho(i)$, then $\mathcal{F}$ is saturated in $\mathcal{E}$ if and only if for all $\rho \in \Sigma(1), i \in \mathbb{Z}$, $F^\rho(i) = E^\rho(i) \cap F$.

Notation 2.16. — Let F be a vector subspace of E. We denote by $\mathcal{E}_F$ the saturated subsheaf of $\mathcal{E}$ defined by the family of filtrations $(F, \{F^\rho(j)\})$ where $F^\rho(j) = F \cap E^\rho(j)$.

By [14, Corollary 3.18], the first Chern class of $\mathcal{E}$ is given by

$$c_1(\mathcal{E}) = -\sum_{\rho\in\Sigma(1)} e^{\rho}(\mathcal{E})\, D_\rho \quad \text{where} \quad e^{\rho}(\mathcal{E}) = \sum_{i\in\mathbb{Z}} i\, e^{\rho}(i) \tag{2.9}$$

with $e^{\rho}(i) = \dim E^{\rho}(i) - \dim E^{\rho}(i-1)$. Therefore, for any $L \in \mathrm{Amp}(X)$,

$$\mu_L(\mathcal{E}) = -\frac{1}{\mathrm{rk}(\mathcal{E})} \sum_{\rho\in\Sigma(1)} e^{\rho}(\mathcal{E}) \deg_L(D_\rho). \tag{2.10}$$

For a reflexive sheaf $\mathcal{E}$ on X, we set

$$\begin{aligned} &\mathrm{Stab}(\mathcal{E}) = \{L \in \mathrm{Amp}(X) : \mathcal{E} \text{ is stable with respect to } L\} \\ \text{and} \quad &\mathrm{sStab}(\mathcal{E}) = \{L \in \mathrm{Amp}(X) : \mathcal{E} \text{ is semistable with respect to } L\}. \end{aligned}$$

3. Description of equivariant logarithmic tangent sheaves

3.1. Logarithmic tangent sheaves

We recall here the definition of the logarithmic tangent sheaf of a pair (X, D) where X is a normal projective variety of dimension n and D a reduced Weil divisor on X.

Definition 3.1. — *We say that a pair (X, D) is* log-smooth *if X is smooth and D is a reduced snc divisor. We denote by $(X, D)_{\mathrm{reg}}$ the snc locus of the pair (X, D), that is, the locus of points $x \in X$ where (X, D) is log-smooth in a neighborhood of x.*

If the pair (X, D) is log-smooth, we define the logarithmic tangent bundle $T_X(-\log D)$ as the dual of the bundle of logarithmic differential form $\Omega^1_X(\log D)$ where $\Omega^1_X(\log D)$ is defined in [9, Section 1]. By [11, Definition 4] and [20, Section 1], we can see the space of sections of $T_X(-\log D)$ as the set of vector fields on X which are tangent to D at its smooth points. If D is locally given by $(z_1 \dots z_k = 0)$, then $T_X(-\log D)$ as a sheaf is the locally free $\mathcal{O}_X$-module generated by

$$z_1\frac{\partial}{\partial z_1}, \dots, z_k\frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_{k+1}}, \dots, \frac{\partial}{\partial z_n}.$$

Definition 3.2 ([6, Definition 3.4]). — *Let (X, D) be a log pair and $X_0 = (X, D)_{\mathrm{reg}}$. The logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$ of (X, D) is defined as $j_*T_{X_0}(-\log D_{|X_0})$ where $j : X_0 \to X$ is the open immersion.*

The sheaf $\mathcal{T}_X(-\log D)$ (as well as its dual) is coherent; by [7, Proposition 1.6], this sheaf is reflexive. We now consider the case where X is a toric variety with torus T. Let Σ be the fan of X and X_0 the toric variety corresponding to the fan $\Sigma^1 = \Sigma(0) \cup \Sigma(1)$. We denote by $j : X_0 \to X$ the open immersion.

Proposition 3.3. — *Let D be a reduced Weil divisor on X. The sheaf $\Omega^1_X(\log D)$ is equivariant if and only if D is an invariant divisor under the torus action.*

Proof. — We assume that D is an invariant divisor under the torus action. Let D_0 be the restriction of D on X_0. For $t \in T$, let $\phi_t : X \to X$ be the map defined by $\phi_t(x) = t \cdot x$. We set $\Phi_t = (d\phi_t)^{-1}$ where $d\phi_t$ is the differential of ϕ_t. If $\mathcal{E} = T_{X_0}$, we get the following diagram.

$$\begin{array}{ccc} (\phi_{t'\cdot t})^*\mathcal{E} & \xrightarrow{\Phi_{t'\cdot t}} & \mathcal{E} \\ & \searrow^{\phi_t^*\Phi_{t'}} \quad \nearrow_{\Phi_t} & \\ & \phi_t^*\mathcal{E} & \end{array} \tag{3.1}$$

If $\mathcal{E} = T_{X_0}(-\log D_0)$, the diagram (3.1) remains true; thus, $T_{X_0}(-\log D_0)$ is equivariant. Therefore $\Omega^1_{X_0}(\log D_0)$ is equivariant. As

$$\Omega^1_X(\log D) \cong j_*\Omega^1_{X_0}(\log D_0) \ , \tag{3.2}$$

we deduce that $\Omega^1_X(\log D)$ is equivariant.

We now assume that $\Omega^1_X(\log D)$ is equivariant. We write $D = \sum_{k=1}^s D_k$ where the D_k are irreducible Weil divisors of X.

First case. — We assume that X is smooth. By [5, Properties 2.3] we have an exact sequence

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log D) \longrightarrow \bigoplus_{k=1}^s \mathcal{O}_{D_k} \longrightarrow 0$$

where $\mathcal{O}_{D_k}$ is viewing as a sheaf on X via extension by zero. The first part of the proof is to show that: for any $t \in T$, $t \cdot Z = Z$ where $Z = X \setminus D$. Let $x \in Z$ and assume that there is $t \in T$ such that $y = t \cdot x \in D$. We have two exact sequences

$$0 \longrightarrow \Omega^1_{X,x} \longrightarrow \Omega^1_X(\log D)_x \longrightarrow \bigoplus_{k=1}^s \mathcal{O}_{D_k,x} \longrightarrow 0$$

$$0 \longrightarrow \Omega^1_{X,y} \longrightarrow \Omega^1_X(\log D)_y \longrightarrow \bigoplus_{k=1}^s \mathcal{O}_{D_k,y} \longrightarrow 0$$

As Ω_X^1 and $\Omega_X^1(\log D)$ are equivariant, we have an isomorphism

$$\bigoplus_{k=1}^{s} \mathcal{O}_{D_k,x} \cong \bigoplus_{k=1}^{s} \mathcal{O}_{D_k,y} \ ;$$

this is absurd. Therefore, for any $t \in T$, we have $t \cdot Z \subseteq Z$, that is $t \cdot Z = Z$. As $\Omega_X^1(\log D)$ is equivariant, by using the fact that $D = X \setminus Z$, for any $t \in T$, we have $t \cdot D = D$; thus, D is a T-invariant divisor.

Second case. — We assume that X is a normal variety. By (3.2), as $\Omega_X^1(\log D)$ is equivariant, we also have the same property for $\Omega_{X_0}^1(\log D_0)$. By the first case, D_0 is an invariant divisor under the action of T on X_0. As $\operatorname{codim}(X \setminus X_0) \geqslant 2$, we deduce that D is the Zariski closure of D_0 on X. Thus, D is an invariant divisor under the action of T on X. ☐

3.2. Families of filtrations of logarithmic tangent sheaves

We give here the proof of Theorem 1.1. Let X be a toric variety of dimension n associated to the fan Σ and D a reduced Weil divisor of X. According to Proposition 3.3, $\mathcal{T}_X(-\log D)$ is equivariant if and only if

$$D = \sum_{\rho \in \Delta} D_\rho$$

where $\Delta \subseteq \Sigma(1)$. In that case, $\mathcal{E}$ is given by a family of filtrations.

Remark 3.4. — If G is an algebraic group acting on the affine toric variety $Y = \operatorname{Spec}(R)$, we define an action of G on R by setting: for any $g \in G$ and $\varphi \in R$, $g \cdot \varphi = \left(\phi_{g^{-1}}\right)^* \varphi$ where $\phi_g(x) = g \cdot x$.

Proof of Theorem 1.1. — For $\rho \in \Sigma(1)$, we set

$$E^\rho = \Gamma(U_\rho, \mathcal{T}_X(-\log D)).$$

By the orbit-cone correspondence (cf. Theorem 2.2), if $\rho \in \Delta$, we have $U_\rho \cap D = U_\rho \cap D_\rho$ and for $\rho \notin \Delta$, $U_\rho \cap D = \varnothing$. We can reduce the problem to the case where Δ contains one ray. For the rest of the proof, we assume that $\Delta = \{\rho_0\}$. Let $\rho \in \Sigma(1)$ and $(u_1, \dots, u_n)$ a basis of N such that $u_1 = u_\rho$. We denote by $(e_1, \dots, e_n)$ the dual basis of $(u_1, \dots, u_n)$ and we set $x_i = \chi^{e_i}$. We have $\mathbb{C}[S_\rho] = \mathbb{C}[x_1, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$.

First case: We assume that $\rho = \rho_0$. — As on U_ρ the divisor D is defined by the equation $x_1 = 0$, we have

$$E^\rho = \left(\mathbb{C}[S_\rho] \cdot x_1 \frac{\partial}{\partial x_1}\right) \oplus \left(\bigoplus_{i=2}^{n} \mathbb{C}[S_\rho] \cdot \frac{\partial}{\partial x_i}\right).$$

We set

$$L_1^\rho = \bigoplus_{m\in S_\rho} \mathbb{C}\cdot\chi^{m+e_1}\frac{\partial}{\partial x_1} \quad \text{and for } i\in\{2,\dots,n\},\quad L_i^\rho = \bigoplus_{m\in S_\rho} \mathbb{C}\cdot\chi^{m}\frac{\partial}{\partial x_i}.$$

According to Remark 3.4, for any $t\in T$ and $m\in M$, $t\cdot\chi^m = \chi^{-m}(t)\chi^m$. Hence, $t\cdot \mathrm{d}x_i = \chi^{-e_i}(t)\mathrm{d}x_i$ and $t\cdot\frac{\partial}{\partial x_i} = \chi^{e_i}(t)\frac{\partial}{\partial x_i}$. For $i\in\{1,\dots,n\}$, we write

$$L_i^\rho = \bigoplus_{m\in M} (L_i^\rho)_m \quad \text{where} \quad (L_i^\rho)_m = \{f\in L_i^\rho : t\cdot f = \chi^{-m}(t)f\}.$$

We have

$$(L_1^\rho)_m = \begin{cases} \mathbb{C}\cdot\chi^{m+e_1}\frac{\partial}{\partial x_1} & \text{if } 0\preceq_\rho m\\ 0 & \text{otherwise}\end{cases}$$

and for $i\in\{2,\dots,n\}$,

$$(L_i^\rho)_m = \begin{cases} \mathbb{C}\cdot\chi^{m+e_i}\frac{\partial}{\partial x_i} & \text{if } -e_i\preceq_\rho m\\ 0 & \text{otherwise.}\end{cases}$$

As the torus T is a Lie group, the tangent space of T at the identity element generated by $\left(\frac{\partial}{\partial x_i}\right)_{1\leqslant i\leqslant n}$ is isomorphic to $N_\mathbb{C}$. Thus, for all $i\in\{1,\dots,n\}$, we can identify $\frac{\partial}{\partial x_i}$ with u_i.

For $i\in\{1,\dots,n\}$, we set $\mathbb{L}_i^\rho = \mathrm{Span}(u_i)$. Let $m\in M$.

- If $i=1$ and $0\preceq_\rho m$, then $(L_i^\rho)_m$ is isomorphic to $\mathbb{L}_1^\rho\otimes\chi^m$.
- If $i\geqslant 2$ and $-e_i\preceq_\rho m$, then $(L_i^\rho)_m$ is isomorphic to $\mathbb{L}_i^\rho\otimes\chi^m$.

We set $j=\langle m,u_1\rangle$. The condition $0\preceq_\rho m$ is equivalent to $j\geqslant 0$ and for $i\in\{2,\dots,n\}$, $-e_i\preceq_\rho m$ is equivalent to $j\geqslant 0$. Thus, for any $i\in\{1,\dots,n\}$, we set

$$L_i^\rho(j) = \begin{cases} 0 & \text{if } j\leqslant -1\\ \mathbb{L}_i^\rho & \text{if } j\geqslant 0.\end{cases}$$

By construction, $\{L_i^\rho(j)\}$ is the family of filtrations of L_i^ρ. As

$$E^\rho = \bigoplus_{m\in M} E^\rho(\langle m,u_1\rangle)\otimes\chi^m$$

where $E^\rho(\langle m,u_1\rangle)\cong\bigoplus_{i=1}^n L_i^\rho(\langle m,u_\rho\rangle)$, we get

$$E^\rho(j)\cong\begin{cases} 0 & \text{if } j\leqslant -1\\ N_\mathbb{C} & \text{if } j\geqslant 0.\end{cases}$$

Second case: We assume that $\rho \neq \rho_0$. — As $U_\rho \cap D = \varnothing$, we have

$$E^\rho = \bigoplus_{i=1}^{n} \mathbb{C}[S_\rho] \cdot \frac{\partial}{\partial x_i} = \bigoplus_{i=1}^{n} \left(\bigoplus_{m \in S_\rho} \mathbb{C} \cdot \chi^m \frac{\partial}{\partial x_i} \right).$$

For all $i \in \{1, \dots, n\}$, we set $L_i^\rho = \mathbb{C}[S_\rho] \cdot \frac{\partial}{\partial x_i}$. We have

$$L_i^\rho = \bigoplus_{m \in M} (L_i^\rho)_m \quad \text{where} \quad (L_i^\rho)_m = \begin{cases} \mathbb{C} \cdot \chi^{m+e_i} \frac{\partial}{\partial x_i} & \text{if } -e_i \preceq_\rho m \\ 0 & \text{otherwise.} \end{cases}$$

For $m \in M$, we set $j = \langle m, u_1 \rangle$. The condition $-e_i \preceq_\rho m$ is equivalent to $j \geqslant -\langle e_i, u_1 \rangle$. Thus, for any $i \in \{2, \dots, n\}$, the family of filtrations of L_i^ρ is given by

$$L_i^\rho(j) = \begin{cases} 0 & \text{if } j \leqslant -1 \\ \mathbb{L}_i^\rho & \text{if } j \geqslant 0 \end{cases}$$

and the family of filtrations of L_1^ρ is given by

$$L_1^\rho(j) = \begin{cases} 0 & \text{if } j \leqslant -2 \\ \mathbb{L}_i^\rho & \text{if } j \geqslant -1. \end{cases}$$

As in the first case, we get

$$E^\rho(j) \cong \begin{cases} 0 & \text{if } j \leqslant -2 \\ \operatorname{Span}(u_\rho) & \text{if } j = -1 \\ N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if } j \geqslant 0 \end{cases}$$

which ends the proof. □

The sheaf of regular sections of the trivial vector bundle $X \times \mathbb{C} \to X$ of rank 1 is $\mathcal{O}_X$. We denote by $\left(F, \{F^\rho(j)\}_{\rho \in \Sigma(1),\, j \in \mathbb{Z}}\right)$ the family of filtration of $\mathcal{O}_X$. For $\rho \in \Sigma(1)$, we set $F^\rho = \mathcal{O}_X(U_\rho)$ and $F_m^\rho = \{f \in F^\rho : t \cdot f = \chi^{-m}(t) f\}$. As

$$F^\rho = \bigoplus_{m \in M} F_m^\rho = \mathbb{C}[S_\rho] = \bigoplus_{m \in S_\rho} \mathbb{C} \cdot \chi^m,$$

we deduce that $F_m^\rho = \mathbb{C} \cdot \chi^m$ if $m \in S_\rho$ and $F_m^\rho = 0$ if $m \notin S_\rho$. Hence,

$$F^\rho(j) = \begin{cases} 0 & \text{if } j \leqslant -1 \\ \mathbb{C} & \text{if } j \geqslant 0. \end{cases}$$

Corollary 3.5. — *Let $\Delta \subseteq \Sigma(1)$ and $D = \sum_{\rho \in \Delta} D_\rho$.*

(1) *If $\Delta = \varnothing$, then $\mathcal{T}_X(-\log D)$ is the tangent sheaf $\mathcal{T}_X$.*
(2) *If $\Delta = \Sigma(1)$, then $\mathcal{T}_X(-\log D)$ is isomorphic to the trivial sheaf of rank n.*

Proof. — If $\Delta = \varnothing$, the family of filtrations of $\mathcal{T}_X(-\log D)$ is identical to the family of filtrations given in Example 2.13. If $\Delta = \Sigma(1)$, for all $\rho \in \Sigma(1)$, we have

$$E^\rho(j) = \begin{cases} 0 & \text{if } j \leqslant -1 \\ N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if } j \geqslant 0. \end{cases}$$

Hence, $\mathcal{T}_X(-\log D)$ is isomorphic to the trivial sheaf of rank n. □

Notation 3.6. — Let G be a vector subspace of $N_{\mathbb{C}}$. We denote by $\mathcal{E}_G$ the subsheaf of $\mathcal{E} = \mathcal{T}_X(-\log D)$ defined by the family of filtrations $\big(E_G, \{G^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big)$ where $E_G = G$ and $G^\rho(j) = E^\rho(j) \cap G$. If $\rho \in \Delta$ or $u_\rho \notin G$, then

$$G^\rho(j) = \begin{cases} 0 & \text{if } j \leqslant -1 \\ G & \text{if } j \geqslant 0. \end{cases}$$

If $\rho \notin \Delta$ and $u_\rho \in G$, then

$$G^\rho(j) = \begin{cases} 0 & \text{if } j \leqslant -2 \\ \operatorname{Span}(u_\rho) & \text{if } j = -1 \\ G & \text{if } j \geqslant 0. \end{cases}$$

3.3. Decomposition of equivariant logarithmic tangent sheaves

In this part, we give some conditions on Σ and Δ which ensure that the logarithmic tangent sheaf is decomposable. We first recall the family of filtrations of a direct sum of equivariant reflexive sheaves.

Proposition 3.7 ([10, Section 6.3]). — *Let $\mathcal{F}$ and $\mathcal{G}$ be two equivariant reflexive sheaves with $\big(F, \{F^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big)$ and $\big(G, \{G^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big)$ for family of filtrations. The family of filtrations of $\mathcal{F} \oplus \mathcal{G}$ is given by*

$$\big(F \oplus G, \{(F \oplus G)^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big) \quad \textit{where} \quad (F \oplus G)^\rho(j) = F^\rho(j) \oplus G^\rho(j). \tag{3.3}$$

We assume that X is a toric variety without torus factor. We denote by p the rank of the class group $\operatorname{Cl}(X)$ of X. By Corollary 2.5, we have $\operatorname{card}(\Sigma(1)) = n + p$.

Proposition 3.8. — *Let $D = \sum_{\rho\in\Delta} D_\rho$ with $\operatorname{card}(\Delta) = p$. We set $\Sigma(1) \setminus \Delta = \{\rho_1, \ldots, \rho_n\}$ where $\rho_k = \operatorname{Cone}(u_k)$ and $u_k \in N$. If $N_{\mathbb{R}} = \operatorname{Span}(u_1, \ldots, u_n)$, then $\mathcal{E} = \mathcal{T}_X(-\log D)$ is decomposable and*

$$\mathcal{E} = \bigoplus_{k=1}^{n} \mathcal{E}_{F_k}$$

where $\mathcal{E}_{F_k}$ *is the subsheaf of* $\mathcal{E}$ *corresponding to the vector space* $F_k = \mathrm{Span}(u_k)$.

Proof. — For all $k \in \{1, \dots, n\}$, the family of filtrations $(F_k, \{F_k^\rho(j)\})$ of $\mathcal{E}_{F_k}$ is given by

$$F_k^\rho(j) = \begin{cases} 0 & \text{if } j \leqslant -1 \\ F_k & \text{if } j \geqslant 0 \end{cases} \quad \text{if } \rho \neq \mathrm{Cone}(u_k)$$

and

$$F_k^\rho(j) = \begin{cases} 0 & \text{if } j \leqslant -2 \\ \mathrm{Span}(u_\rho) & \text{if } j = -1 \\ F_k & \text{if } j \geqslant 0 \end{cases} \quad \text{if } \rho = \mathrm{Cone}(u_k).$$

For all $\rho \in \Sigma(1)$ and $j \in \mathbb{Z}$, we have

$$\bigoplus_{k=1}^{n} F_k^\rho(j) = \begin{cases} 0 & \text{if } j \leqslant -1 \\ N_\mathbb{C} & \text{if } j \geqslant 0 \end{cases} \quad \text{if } \rho \in \Delta$$

and

$$\bigoplus_{k=1}^{n} F_k^\rho(j) = \begin{cases} 0 & \text{if } j \leqslant -2 \\ \mathrm{Span}(u_\rho) & \text{if } j = -1 \\ N_\mathbb{C} & \text{if } j \geqslant 0 \end{cases} \quad \text{if } \rho \notin \Delta.$$

Hence, by (3.3) and Theorem 1.1 we get $\mathcal{E} = \bigoplus_{k=1}^{n} \mathcal{E}_{F_k}$. □

A similar proof gives the following result.

Proposition 3.9. — *We assume that* Δ *satisfies* $1 + p \leqslant \mathrm{card}(\Delta) \leqslant n + p - 1$. *Then the sheaf* $\mathcal{E} = \mathcal{T}_X(-\log D)$ *is decomposable and* $\mathcal{E} = \mathcal{E}_G \oplus \mathcal{E}_F$ *where* $G = \mathrm{Span}(u_\rho : \rho \in \Sigma(1) \setminus \Delta)$ *and* F *a vector subspace of* $N_\mathbb{C}$ *such that* $N_\mathbb{C} = G \oplus F$.

3.4. An instability condition for logarithmic tangent sheaves

Let $\Delta \subseteq \Sigma(1)$ and $D = \sum_{\rho \in \Delta} D_\rho$. Let $\left(E_G, \{G^\rho(j)\}_{\rho \in \Sigma(1),\, j \in \mathbb{Z}}\right)$ be the family of filtrations corresponding to the subsheaf $\mathcal{E}_G$ of $\mathcal{E} = \mathcal{T}_X(-\log D)$ where $G \subseteq N_\mathbb{C}$ is a vector subspace. By Equation (2.10), if L is a polarization of X, we have

$$\mu_L(\mathcal{E}) = \frac{1}{n} \sum_{\rho \notin \Delta} \deg_L(D_\rho) \tag{3.4}$$

and

$$\mu_L(\mathcal{E}_G) = \frac{1}{\dim G} \sum_{\rho \notin \Delta \text{ and } u_\rho \in G} \deg_L(D_\rho). \tag{3.5}$$

Therefore,

$$\mu_L(\mathcal{E}) - \mu_L(\mathcal{E}_G) = \left(\frac{1}{n} - \frac{1}{\dim G}\right) \sum_{\substack{\rho \notin \Delta \\ u_\rho \in G}} \deg_L(D_\rho) + \frac{1}{n} \sum_{\substack{\rho \notin \Delta \\ u_\rho \notin G}} \deg_L(D_\rho). \quad (3.6)$$

To study the stability of $\mathcal{E}$ with respect to $L \in \mathrm{Amp}(X)$, it suffices to compare $\mu_L(\mathcal{E})$ with $\mu_L(\mathcal{E}_G)$ where $G \subseteq \mathrm{Span}(u_\rho : \rho \notin \Delta)$ and $1 \leqslant \dim G \leqslant n-1$.

Proposition 3.10. — *If $1 \leqslant \mathrm{card}(\Sigma(1) \setminus \Delta) \leqslant n-1$, then for any $L \in \mathrm{Amp}(X)$, the logarithmic tangent sheaf $\mathcal{E} = \mathcal{T}_X(-\log D)$ is not semistable with respect to L.*

Proof. — We assume that $\Sigma(1) \setminus \Delta = \{\rho_1, \ldots, \rho_k\}$ where $1 \leqslant k \leqslant n-1$ and we denote by D_j the divisor corresponding to $\rho_j = \mathrm{Cone}(u_j)$. For $G = \mathrm{Span}(u_1, \ldots, u_k)$, we have

$$\mu_L(\mathcal{E}) - \mu_L(\mathcal{E}_G) = \left(\frac{1}{n} - \frac{1}{\dim G}\right) \sum_{j=1}^{k} \deg_L(D_j) < 0.$$

Thus, $\mathcal{E}$ is not semistable with respect to L. □

Hence, by Corollary 2.5, we get:

Corollary 3.11. — *Let $p = \mathrm{rk}\,\mathrm{Cl}(X)$. If $1 + p \leqslant \mathrm{card}(\Delta) \leqslant n + p - 1$, then for any $L \in \mathrm{Amp}(X)$, the logarithmic tangent sheaf $\mathcal{T}_X(-\log D)$ is unstable with respect to L.*

Remark 3.12. — By Corollary 3.5, if $\mathrm{card}(\Delta) = n + p$, $\mathcal{T}_X(-\log D)$ is semistable with respect to any polarizations.

From now on, we will study the (semi)stability of $\mathcal{T}_X(-\log D)$ only on the case where $1 \leqslant \mathrm{card}(\Delta) \leqslant p = \mathrm{rk}\,\mathrm{Cl}(X)$ and $p \in \{1, 2\}$.

4. Stability of equivariant logarithmic tangent sheaves

4.1. Stability on weighted projective spaces

Let $q_0, q_1, \ldots, q_n \in \mathbb{N}^*$ such that

$$\gcd(q_0, \ldots, q_n) = 1.$$

We set $N = \mathbb{Z}^{n+1}/\mathbb{Z} \cdot (q_0, \ldots, q_n)$. The dual lattice of N is

$$M = \{(a_0, \ldots, a_n) \in \mathbb{Z}^{n+1} : a_0\, q_0 + \cdots + a_n\, q_n = 0\}.$$

We denote by $\{u_i : 0 \leqslant i \leqslant n\}$ the images in N of the standard basis vectors in $\mathbb{Z}^{n+1}$. So the relation $q_0\, u_0 + q_1\, u_1 + \cdots + q_n\, u_n = 0$ holds in N. The toric variety X associated to the simplicial fan $\Sigma = \{\mathrm{Cone}(A) : A \subsetneq \{u_0, \ldots, u_n\}\}$ is the weighted projective space $\mathbb{P}(q_0, q_1, \ldots, q_n)$. We denote by D_i the divisor of X corresponding to the ray $\mathrm{Cone}(u_i)$. For $i \in \{0, \ldots, n\}$, we set $\mathcal{E} = \mathcal{T}_X(-\log D_i)$ and $A_i = \{0, \ldots, n\} \setminus \{i\}$.

Proposition 4.1. — *Let $L \in \mathrm{Amp}(X)$. The sheaf $\mathcal{E}$ is polystable with respect to L if and only if there is $q \in \mathbb{N}^*$ such that for all $j \in A_i$, $q_j = q$.*

Proof. — We first show that $q_i D_j \sim_{\mathrm{lin}} q_j D_i$. Let $m = (a_0, \ldots, a_n) \in M$ defined by $a_i = q_j$, $a_j = -q_i$ and $a_k = 0$ if $k \in A_i \setminus \{j\}$. By Equation (2.1), we get $\mathrm{div}(\chi^m) = q_j D_i - q_i D_j$. Hence, $q_i D_j \sim_{\mathrm{lin}} q_j D_i$. Therefore, for any $L \in \mathrm{Amp}(X)$, $q_i \deg_L(D_j) = q_j \deg_L(D_i)$.

The assumptions of Proposition 3.8 are verified. Hence, $\mathcal{E} = \bigoplus_{j \in A_i} \mathcal{E}_{F_j}$ where $F_j = \mathrm{Span}(u_j)$. By Equation (3.5), we get

$$\mu_L(\mathcal{E}_{F_j}) = \deg_L(D_j) = \frac{q_j}{q_i} \deg_L(D_i).$$

If $\mathcal{E}$ is polystable with respect to L, there is $r \in \mathbb{Q}$ such that for all $j \in A_i$, $q_j = r\, q_i$. Hence, we have the existence of $q \in \mathbb{N}^*$ such that for all $j \in A_i$, $q_j = q$. For the converse, if for all $j \in A_i$, we have $q_j = q$, then $\mathcal{E}$ is polystable. □

According to Proposition 2.15, we get:

Corollary 4.2. — *For all $i \in \{0, \ldots, n\}$, $\mathrm{sStab}(\mathcal{T}_X(-\log D_i)) \neq \varnothing$ if and only if there is $q \in \mathbb{N}^*$ such that for all $j \in A_i$, $q_j = q$. Moreover, if for all $j \in A_i$, $q_j = q$, then*

$$\varnothing = \mathrm{Stab}(\mathcal{T}_X(-\log D_i)) \subsetneq \mathrm{sStab}(\mathcal{T}_X(-\log D_i)) = \mathrm{Amp}(X).$$

4.2. Condition of stability on toric varieties of Picard rank two

In this part, we adapt some results of [8, Section 4] for the study of the stability of $T_X(-\log D)$ when $X = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^s}(a_i)\right)$ with $0 \leqslant a_1 \leqslant \cdots \leqslant a_r$. We use notation of Section 2.2. The following lemma will be useful in the proof of Proposition 4.5 which is the main result of this part. Let $z \in \{0, \ldots, r-1\}$ such that $a_z = 0$ and $a_{z+1} > 0$, we have:

Lemma 4.3 ([8, Lemma 4.2]). — *Let $I' \subseteq \{0, 1 \ldots, r\}$ and $G = \mathrm{Span}(v_i : i \in I')$. The vector $a_1 v_1 + \cdots + a_r v_r$ belongs to G if and only if*

(i) *$\{z+1, \ldots, r\} \subseteq I'$ or*

(ii) $\{0,\ldots,z\} \subseteq I'$, $\mathrm{card}(\{z+1,\ldots,r\}\setminus I') \geqslant 1$ *and* $a_i = a_j$ *for all* $i,j \in \{z+1,\ldots,r\}\setminus I'$.

Since passing to multiples of polarizations has no effect on stability, instead of studying the stability of $T_X(-\log D)$ with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\alpha)\otimes\mathcal{O}_X(\beta)$, we will study the stability of $T_X(-\log D)$ with respect to the $\mathbb{Q}$-divisor $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu)\otimes\mathcal{O}_X(1)$ where $\nu = \alpha/\beta$. Let P be the polytope corresponding to the $\mathbb{Q}$-polarized toric variety (X,L) where $L = \pi^*\mathcal{O}_{\mathbb{P}^s}(\nu)\otimes\mathcal{O}_X(1)$ with $\nu \in \mathbb{Q}_{>0}$.

Notation 4.4. — For all $i \in \{0,1,\ldots,r\}$, we set $\mathrm{V}_i = \mathrm{vol}(P^{v_i})$. As for all $j \in \{1,\ldots,s\}$, $\mathrm{vol}(P^{w_j}) = \mathrm{vol}(P^{w_0})$, we set $\mathrm{W} = \mathrm{vol}(P^{w_0})$.

Let $\Delta \subseteq \Sigma(1)$ and D a reduced Weil divisor on X given by $D = \sum_{\rho\in\Delta} D_\rho$. We set

$$\begin{aligned} I_\Sigma &= \{\mathrm{Cone}(v_0),\ldots,\mathrm{Cone}(v_r)\}\ ,\\ J_\Sigma &= \{\mathrm{Cone}(w_0),\ldots,\mathrm{Cone}(w_s)\}\ ,\\ I &= \{i \in \{0,1,\ldots,r\} : \mathrm{Cone}(v_i) \in I_\Sigma \setminus (I_\Sigma\cap\Delta)\} \quad \text{and}\\ J &= \{j \in \{0,1,\ldots,s\} : \mathrm{Cone}(w_j) \in J_\Sigma \setminus (J_\Sigma\cap\Delta)\}. \end{aligned}$$

To study the stability of $\mathcal{E} = T_X(-\log D)$ with respect to L, it suffices to compare $\mu_L(\mathcal{E})$ and $\mu_L(\mathcal{E}_G)$ when $G = \mathrm{Span}(v_i, w_j : i \in I', j \in J')$ with $I' \subseteq I$, $J' \subseteq J$ and $1 \leqslant \dim G < (r+s)$. By Proposition 2.7, (3.4) and (3.5), we get

$$\mu_L(\mathcal{E}) = \frac{1}{r+s}\left(\sum_{i\in I}\mathrm{V}_i + \mathrm{card}(J)\cdot\mathrm{W}\right)$$

and

$$\mu_L(\mathcal{E}_G) = \frac{1}{\dim G}\left(\sum_{i\in I'}\mathrm{V}_i + \mathrm{card}(J')\cdot\mathrm{W}\right).$$

Here is a version of [8, Proposition 4.1] for logarithmic tangent bundle.

Proposition 4.5. — *The logarithmic tangent bundle $\mathcal{E} = T_X(-\log D)$ is stable (resp. semistable) with respect to $L = \pi^*\mathcal{O}_{\mathbb{P}^s}(\nu)\otimes\mathcal{O}_X(1)$ if and only if $\mu_L(\mathcal{E})$ is greater than (resp. greater than or equal to) the maximum of*

(1) V_{i_0} *where* $i_0 = \min I$ *if* $I \neq \varnothing$*;*
(2) $\frac{1}{r'}\left(\sum_{i\in I}\mathrm{V}_i\right)$*, if* $r' = \dim\mathrm{Span}(v_i : i \in I) \neq 0$*;*
(3) $\frac{\mathrm{card}(J)\cdot\mathrm{W}}{s'}$*, if* $0 < s' = \dim\mathrm{Span}(w_j : j \in J) < r+s$*;*
(4) $\frac{1}{s+k}\left(\sum_{i\in I'}\mathrm{V}_i + (s+1)\mathrm{W}\right)$*, if* $\mathrm{card}(J') = s+1$*,* $k = \mathrm{card}(I') < r$ *and* $\{z+1,\ldots,r\} \subseteq I' \subseteq I$*;*
(5) $\frac{1}{s+k}\left(\sum_{i\in I'}\mathrm{V}_i + (s+1)\mathrm{W}\right)$*, if* $\mathrm{card}(J') = s+1$*,* $k = \mathrm{card}(I') < r$ *and* $I' \subseteq I$ *such that the condition* (ii) *of Lemma 4.3 is verified.*

Proof. — Let $G = \mathrm{Span}(v_i, w_j : i \in I', j \in J')$ where $I' \subseteq I$ and $J' \subseteq J$. In Proposition 4.5, each point corresponds to a value of $\mu_L(\mathcal{E}_G)$ for some G. In particular, (1) corresponds to $G = \mathrm{Span}(v_{i_0})$, (2) corresponds to $G = \mathrm{Span}(v_i : i \in I)$ and (3) corresponds to $G = \mathrm{Span}(w_j : j \in J)$.

If $\mathrm{card}(J') = 0$, then for $\varnothing \subsetneq I' \subseteq I$, we have $\dim G \leqslant r$ and

$$\mu_L(\mathcal{E}_G) = \frac{1}{\dim G} \sum_{i \in I'} \mathrm{V}_i \ ;$$

this number is less than or equal to the maximum of the numbers given in (1) and (2).

If $\mathrm{card}(I') = 0$, then for $\varnothing \subsetneq J' \subseteq J$ such that $\dim G < r + s$, we have

$$\mu_L(\mathcal{E}_G) = \frac{\mathrm{card}(J') \cdot \mathrm{W}}{\dim G} \ ;$$

this number is less than or equal to that given in (3).

If $\mathrm{card}(I') = r + 1$, then $\dim G < r + s$ if and only if $s' = \mathrm{card}(J') < s$. If $1 \leqslant s' < s$, then

$$\mu_L(\mathcal{E}_G) = \frac{1}{r + s'} \left(\sum_{i \in I'} \mathrm{V}_i + s'\mathrm{W} \right) \leqslant \max \left(\frac{1}{r} \sum_{i \in I'} \mathrm{V}_i, \mathrm{W} \right).$$

If $1 \leqslant \mathrm{card}(I') \leqslant r$, $1 \leqslant \mathrm{card}(J') \leqslant s$ and $\dim G < r + s$, then $\mu_L(\mathcal{E}_G)$ is less than or equal to the maximum of numbers given in (1), (2) and (3).

It remains to study the case where $\mathrm{card}(J') = s + 1$ and $1 \leqslant \mathrm{card}(I') < r$ (because if $\mathrm{card}(I') \geqslant r$, then $\dim G = r + s$). We will treat it in two cases.

First case: $a_r = 0$. — For all $i \in \{1, \dots, r\}$, $\mathrm{V}_i = \mathrm{V}_0$. If $r' = \mathrm{card}(I')$ and $1 \leqslant r' < r$, then

$$\mu_L(\mathcal{E}_G) = \frac{1}{r' + s} \left(\sum_{i \in I'} \mathrm{V}_i + (s+1)\mathrm{W} \right) \leqslant \max \left(\mathrm{V}_0, \frac{(s+1)\mathrm{W}}{s} \right).$$

Second case: $a_r > 0$. — We set $r' = \mathrm{card}(I')$. If I' satisfies the first (resp. second) condition of Lemma 4.3, then the value of $\mu_L(\mathcal{E}_G)$ is given in the point (4) (resp. (5)). If I' doesn't satisfy the conditions of Lemma 4.3, then $\dim G = r' + (s+1)$. Moreover, if $r' + (s+1) < r + s$, then the number $\mu_L(\mathcal{E}_G)$ is less than or equal to the maximum of the numbers given in (1) and (3). ☐

Remark 4.6. — If $a_1 = \cdots = a_r = 0$, to check the stability of $\mathcal{E}$ with respect to L, it is enough to compare $\mu_L(\mathcal{E})$ with the numbers given by the points 1, 2 and 3 of Proposition 4.5. In that case, we have

$$\mathrm{W} = \binom{s+r-1}{s-1}\nu^{s-1} \quad \text{and} \quad \mathrm{V}_i = \binom{s+r-1}{s}\nu^{s}. \tag{4.1}$$

If $(a_1, \ldots, a_r) \neq (0, \ldots, 0)$, the results below will help us to determine if $\mathcal{E}$ is unstable with respect to L without having to check each point of Proposition 4.5. Let $z \in \{0, 1, \ldots, r-1\}$ such that $a_z = 0$ and $a_{z+1} > 0$ where $a_0 = 0$. Let $k \in \{0, \ldots, s\}$. We set

$$\begin{aligned} \mathrm{V}_{0k} &= \sum_{d_{z+1}+\cdots+d_r = s-k} a_{z+1}^{d_{z+1}} \ldots a_r^{d_r} \\ \text{and} \quad \mathrm{W}_k &= \sum_{d_{z+1}+\cdots+d_r = s-1-k} a_{z+1}^{d_{z+1}} \ldots a_r^{d_r} \end{aligned}$$

where $\mathrm{W}_s = 0$. For $i \in \{z+1, \ldots, r\}$, we set

$$\mathrm{V}_{ik} = \sum_{\substack{d_{z+1}+\cdots+d_{i-1} \\ +d_{i+1}+\cdots+d_r = s-k}} a_{z+1}^{d_{z+1}} \ldots a_{i-1}^{d_{i-1}} a_{i+1}^{d_{i+1}} \ldots a_r^{d_r}$$

and for $i \in \{1, \ldots, z\}$, we set $\mathrm{V}_{ik} = \mathrm{V}_{0k}$.

Remark 4.7. — If $r = 1$, we set $\mathrm{V}_{1s} = 1$ and for $k \in \{0, \ldots, s-1\}$, $\mathrm{V}_{1k} = 0$. We have $\mathrm{W}_{s-1} = 1$ and $\mathrm{V}_{is} = 1$ for any $i \in \{0, \ldots, r\}$.

Lemma 4.8. — *For all $i \in \{1, \ldots, r\}$, $\mathrm{V}_0 = a_i \mathrm{W} + \mathrm{V}_i$.*

Proof. — To show the lemma, it suffices to show that: for any $k \in \{0, \ldots, s-1\}$, $a_i \mathrm{W}_k + \mathrm{V}_{ik} = \mathrm{V}_{0k}$. If $i \in \{1, \ldots, z\}$, the equality is true because $a_i = 0$. We assume that $i \in \{z+1, \ldots, r\}$, we have

$$\begin{aligned} \mathrm{V}_{0k} &= \sum_{d_{z+1}+\cdots+d_r = s-k} a_{z+1}^{d_{z+1}} \ldots a_r^{d_r} \\ &= \sum_{\substack{d_{z+1}+\cdots+d_r = s-k \\ d_i = 0}} a_{z+1}^{d_{z+1}} \ldots a_r^{d_r} + \sum_{\substack{d_{z+1}+\cdots+d_r = s-k \\ d_i \geqslant 1}} a_{z+1}^{d_{z+1}} \ldots a_r^{d_r} \end{aligned}$$

The first term of the second line corresponds to the number V_{ik} and the second to $a_i \mathrm{W}_k$ (it suffices to replace d_i by $d_i' + 1$). Hence, $\mathrm{V}_{0k} = \mathrm{V}_{ik} + a_i \mathrm{W}_k$. □

Lemma 4.9. — *Let $(a_1, \ldots, a_r) \neq (0, \ldots, 0)$.*

(1) *If $a_r \geqslant 2$, then $s\mathrm{V}_0 - (s+1)\mathrm{W} \geqslant s\mathrm{V}_r$.*
(2) *If $r \geqslant 2$ and $i \in \{1, \ldots, r-1\}$ with $a_i < a_r$, then $\mathrm{V}_i - \mathrm{W} \geqslant \mathrm{V}_r$.*

Proof. — If $a_r \geqslant 2$, then $\left(s - \frac{s+1}{a_r}\right) = \frac{a_r s-(s+1)}{a_r} \geqslant \frac{2s-(s+1)}{a_r} \geqslant 0$ because $s \geqslant 1$. Hence,

$$\begin{aligned} s\mathrm{V}_0 - (s+1)\mathrm{W} &= s\mathrm{V}_0 - \frac{s+1}{a_r}(\mathrm{V}_0 - \mathrm{V}_r) \\ &= \left(s - \frac{s+1}{a_r}\right)\mathrm{V}_0 + \frac{s+1}{a_r}\mathrm{V}_r \\ &\geqslant \left(s - \frac{s+1}{a_r}\right)\mathrm{V}_r + \frac{s+1}{a_r}\mathrm{V}_r = s\mathrm{V}_r. \end{aligned}$$

As $\mathrm{V}_0 = a_i\mathrm{W} + \mathrm{V}_i = a_r\mathrm{W} + \mathrm{V}_r$, we get $\mathrm{V}_i = (a_r - a_i)\mathrm{W} + \mathrm{V}_r$. If $a_r > a_i$, then $a_r - a_i \geqslant 1$; therefore $\mathrm{V}_i \geqslant \mathrm{W} + \mathrm{V}_r$. □

4.3. Stability of logarithmic tangent bundles on a product of projective spaces

We assume that $a_1 = \cdots = a_r = 0$. We have $X \cong \mathbb{P}^s \times \mathbb{P}^r$. We denote by $\pi_1 : X \to \mathbb{P}^s$ and $\pi_2 : X \to \mathbb{P}^r$ the projection maps. If $i \in \{0, \dots, r\}$ and $j \in \{0, \dots, s\}$, by Proposition 3.8 the vector bundle $T_X(-\log(D_{v_i} + D_{w_j}))$ is decomposable. Reasoning as in the proof of Proposition 3.8, it is easy to show that:

Lemma 4.10. — *Let $i, i' \in \{0, \dots, r\}$ and $j, j' \in \{0, \dots, s\}$ such that $i \neq i'$ and $j \neq j'$. Then*

(1) $T_X(-\log(D_{v_i} + D_{v_{i'}})) \cong T_{\mathbb{P}^s} \oplus T_{\mathbb{P}^r}(-\log(\pi_2(D_{v_i}) + \pi_2(D_{v_{i'}})))$
(2) $T_X(-\log(D_{w_j} + D_{w_{j'}})) \cong T_{\mathbb{P}^s}(-\log(\pi_1(D_{w_j}) + \pi_1(D_{w_{j'}}))) \oplus T_{\mathbb{P}^r}$
(3) $\mathcal{E} = T_X(-\log D_{v_i})$ *satisfies*

$$\mathcal{E} \cong T_{\mathbb{P}^s} \oplus \left(\bigoplus_{k=0,\, k\neq i}^{r} \mathcal{E}_{F_k}\right)$$

where $F_k = \mathrm{Span}(v_k)$.
(4) $\mathcal{E} = T_X(-\log D_{w_j})$ *satisfies*

$$\mathcal{E} \cong \left(\bigoplus_{k=0,\, k\neq j}^{s} \mathcal{E}_{G_k}\right) \oplus T_{\mathbb{P}^r}$$

where $G_k = \mathrm{Span}(w_k)$.

Let $D = \sum_{\rho\in\Delta} D_\rho$ with $\Delta \subsetneq \Sigma(1)$. As for any Δ such that $\mathrm{card}(\Delta) \in \{1, 2\}$, the vector bundle $\mathcal{E} = T_X(-\log D)$ is decomposable, we deduce that

$$\mathrm{Stab}(\mathcal{E}) = \varnothing.$$

Table 4.1. Stability of $T_X(-\log D)$ when $a_1 = \cdots = a_r = 0$

Divisor D	sStab($\mathcal{E}$)	References
D_{v_i}, $0 \leqslant i \leqslant r$	$\nu = \frac{s+1}{r}$	Proposition 4.11
D_{w_j}, $0 \leqslant j \leqslant s$	$\nu = \frac{s}{r+1}$	Proposition 4.11
$D_{v_j} + D_{w_j}$	$\nu = \frac{s}{r}$	Proposition 4.13
$D_{v_i} + D_{v_j}$, $0 \leqslant i < j \leqslant r$	$\varnothing$	Proposition 4.12
$D_{w_i} + D_{w_j}$, $0 \leqslant i < j \leqslant s$	$\varnothing$	Proposition 4.12

In Table 4.1, we give the values of ν for which $\mathcal{E}$ is semistable with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$. We recall that by Equation (4.1), $\mathrm{V} = \frac{r\nu}{s}\mathrm{W}$.

Proposition 4.11. — *Let $i \in \{0, 1, \dots, r\}$ and $j \in \{0, 1, \dots, s\}$, then*

(1) *$T_X(-\log D_{v_i})$ is polystable with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ if and only if $\nu = \frac{s+1}{r}$;*
(2) *$T_X(-\log D_{w_j})$ is polystable with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ if and only if $\nu = \frac{s}{r+1}$.*

Proof. — We start with $\mathcal{E} = T_X(-\log D_{v_i})$. We have

$$\mu_L(\mathcal{E}) = \frac{1}{r+s}(r\mathrm{V} + (s+1)\mathrm{W}) = \frac{(r^2\nu + s^2 + s)\mathrm{W}}{s(r+s)} = \frac{(r^2\nu + s^2 + s)\mathrm{V}}{r\nu(r+s)}.$$

By Proposition 4.5, to have the semistability, it is enough to compare $\mu_L(\mathcal{E})$ with

$$\max\left(\frac{(s+1)\mathrm{W}}{s}, \mathrm{V}\right).$$

If $\mu_L(\mathcal{E}) \geqslant \mathrm{V}$, then $\frac{r^2\nu+s^2+s}{r\nu(r+s)} \geqslant 1$, i.e. $(r^2\nu + s^2 + s) \geqslant (r^2\nu + rs\nu)$; hence, $\nu \leqslant \frac{s+1}{r}$.

If $\mu_L(\mathcal{E}) \geqslant \frac{(s+1)\mathrm{W}}{s}$, then $\frac{r^2\nu+s^2+s}{s(r+s)} \geqslant \frac{s+1}{s}$, i.e. $\nu \geqslant \frac{s+1}{r}$. Therefore, $T_X(-\log D_{v_i})$ is semistable with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ if and only if $\nu = \frac{s+1}{r}$.

If we regard the case where $\mathcal{E} = T_X(-\log D_{w_j})$, it is enough to compare $\mu_L(\mathcal{E})$ with $\max\left(\frac{(r+1)\mathrm{V}}{r}, \mathrm{W}\right)$. A similar computation gives the result. □

Proposition 4.12. — *Let $i, i' \in \{0, \dots, r\}$ and $j, j' \in \{0, \dots, s\}$ such that $i \neq i'$ and $j \neq j'$. For any $L \in \mathrm{Amp}(X)$, the logarithmic tangent bundles $T_X(-\log(D_{v_i} + D_{v_{i'}}))$ and $T_X(-\log(D_{w_j} + D_{w_{j'}}))$ are unstables with respect to L.*

Proof. — Let $\mathcal{E} = T_X(-\log(D_{v_i} + D_{v_{i'}})$. We have

$$\begin{aligned}\mu_L(\mathcal{E}) &= \frac{(r-1)\mathrm{V} + (s+1)\mathrm{W}}{r+s} \\ &= \frac{(r(r-1)\nu + s^2 + s)\mathrm{W}}{s(r+s)} \\ &= \frac{(r(r-1)\nu + s^2 + s)\mathrm{V}}{r\nu(r+s)}.\end{aligned}$$

To check the semistability, it is enough to compare $\mu_L(\mathcal{E})$ with

$$\max\left(\frac{(s+1)\mathrm{W}}{s}, \mathrm{V}\right).$$

If $r = 1$, then $\mu_L(\mathcal{E}) = \mathrm{W}$. Hence, $T_X(-\log(D_{v_i} + D_{v_{i'}}))$ is not semistable with respect to L. We now consider the case $r \geqslant 2$.

- If $\mu_L(\mathcal{E}) \geqslant \mathrm{V}$, then $\frac{r(r-1)\nu+s^2+s}{r\nu(r+s)} \geqslant 1$, i.e. $\nu \leqslant \frac{s}{r}$.
- If $\mu_L(\mathcal{E}) \geqslant \frac{(s+1)\mathrm{W}}{s}$, then $\frac{r(r-1)\nu+s^2+s}{s(r+s)} \geqslant \frac{s+1}{s}$, i.e. $\nu \geqslant \frac{s+1}{r-1} > \frac{s}{r}$.

As ν cannot satisfy this two conditions, we deduce that $T_X(-\log(D_{v_i}+D_{v_{i'}}))$ is not semistable with respect to L. ☐

Proposition 4.13. — *Let $i \in \{0, \dots, r\}$, $j \in \{0, \dots, s\}$ and $D = D_{v_i} + D_{w_j}$. Then $T_X(-\log D)$ is polystable with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ if and only if $\nu = \frac{s}{r}$.*

Proof. — We have

$$\mu_L(\mathcal{E}) = \frac{1}{r+s}(r\mathrm{V} + s\mathrm{W}) = \frac{(r^2\nu + s^2)\mathrm{W}}{s(r+s)} = \frac{(r^2\nu + s^2)\mathrm{V}}{r\nu(r+s)}.$$

To check the semi-stability, it is enough to compare $\mu_L(\mathcal{E})$ with $\max(\mathrm{V}, \mathrm{W})$.

- If $\mu_L(E) \geqslant \mathrm{V}$, then $\frac{r^2\nu+s^2}{r\nu(r+s)} \geqslant 1$, i.e. $\nu \leqslant \frac{s}{r}$.
- If $\mu_L(E) \geqslant \mathrm{W}$, then $\frac{r^2\nu+s^2}{s(r+s)} \geqslant 1$, i.e. $\nu \geqslant \frac{s}{r}$.

Hence, $T_X(-\log D)$ is semistable with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ if and only if $\nu = \frac{s}{r}$. ☐

Remark 4.14. — According to (2.5) and (2.6), when $a_1 = \cdots = a_r = 0$, we get $D_{v_i} \sim_{\mathrm{lin}} D_{v_0}$, $D_{w_j} \sim_{\mathrm{lin}} D_{w_0}$ and $-K_X \sim_{\mathrm{lin}} (s+1)D_{w_0} + (r+1)D_{v_0}$. By the above study, we see that: if $\mathrm{sStab}(T_X(-\log D)) \neq \varnothing$, then $T_X(-\log D)$ is semistable with respect to L if and only if $L \cong \mathcal{O}_X(-\alpha\,(K_X + D))$ with $\alpha \in \mathbb{Q}_{>0}$.

5. Stability on smooth toric varieties of Picard rank two

In this section, we study the stability of $T_X(-\log D)$ when

$$X = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^s}(a_i)\right)$$

with $a_r \geqslant 1$. Let $\Delta \subseteq \Sigma(1)$ and $D = \sum_{\rho\in\Delta} D_\rho$. By Corollary 3.11, we will only study the case where $\mathrm{card}(\Delta) \in \{1,2\}$. The case $\mathrm{card}(\Delta) = 0$ was treated by Hering–Nill–Süss in [8]. We recall that $a_0 = 0$. We have a version of Lemma 4.10 when $a_r \geqslant 1$.

Lemma 5.1. — *We assume that $a_r \geqslant 1$.*

(1) *If $i \in \{0,\dots,r\}$ and $j \in \{0,\dots,s\}$, then $T_X(-\log(D_{v_i}+D_{w_j}))$ is decomposable and*

$$\mathcal{E} = \left(\bigoplus_{l=0,\, l\neq j}^{s} \mathcal{E}_{G_l}\right) \oplus \left(\bigoplus_{k=0,\, k\neq i}^{r} \mathcal{E}_{F_k}\right)$$

where $G_l = \mathrm{Span}(w_l)$ and $F_k = \mathrm{Span}(v_k)$.

(2) *If $D = D_{w_i}+D_{w_j}$ for $0 \leqslant i < j \leqslant s$, then the sheaf $\mathcal{E} = T_X(-\log D)$ is decomposable and*

$$\mathcal{E} = \left(\bigoplus_{k=0,\, k\neq i}^{s} \mathcal{E}_{G_k}\right) \oplus \mathcal{E}_F$$

where $G_k = \mathrm{Span}(w_k)$ and $F = \mathrm{Span}(v_0,\dots,v_r)$.

(3) *If $D = D_{v_i}+D_{v_j}$ for $0 \leqslant i < j \leqslant r$, then the sheaf $\mathcal{E} = T_X(-\log D)$ is decomposable. If $a_i < a_j$, then*

$$\mathcal{E} = \left(\bigoplus_{l=0}^{s} \mathcal{E}_{G_l}\right) \oplus \left(\bigoplus_{k=0,\, k\notin\{i,j\}}^{r} \mathcal{E}_{F_k}\right)$$

where $G_l = \mathrm{Span}(w_l)$ and $F_k = \mathrm{Span}(v_k)$. If $a_i = a_j$, then

$$\mathcal{E} = \mathcal{E}_G \oplus \mathcal{E}_F$$

where $G = \mathrm{Span}(w_l, v_k : l \in \{0,\dots,s\}, k \in \{0,\dots,r\}\setminus\{i,j\})$ and $F = \mathrm{Span}(v_j)$.

If $D \in \{D_{v_i} : 0 \leqslant i \leqslant r\}$, we will not search to know if $\mathcal{E} = T_X(-\log D)$ is decomposable because it depends on the numbers r, s and $a_1,\dots,a_r$. In particular, if we assume that $r = 2$, $s = 1$ and $(a_1,a_2) = (0,1)$, then $\mathcal{E} = T_X(-\log D_{v_1})$ is decomposable with $\mathcal{E} = \mathcal{E}_F \oplus \mathcal{E}_G$ where $F = \mathrm{Span}(v_2,w_1)$ and $G = \mathrm{Span}(v_0)$ while $\mathcal{F} = T_X(-\log D_{v_2})$ is not decomposable.

For $D = D_{w_i}$ with $0 \leqslant i \leqslant s$, the vector bundle $\mathcal{E} = T_X(-\log D)$ is decomposable and its decomposition is identical to that given in the point (2) of Lemma 5.1. According to Lemma 5.1, if $D = \sum_{\rho\in\Delta} D_\rho$ with $\mathrm{card}(\Delta) = 2$, then $\mathcal{E} = T_X(-\log D)$ is not stable with respect to any polarizations.

Let $L = \pi^*\mathcal{O}_{\mathbb{P}^s}(\nu)\otimes\mathcal{O}_X(1)$ be an element of $\mathrm{Amp}(X) \subseteq N^1(X)\otimes_{\mathbb{Z}}\mathbb{R}$. We recall that the numbers $\mathrm{V}_0,\dots,\mathrm{V}_r$ defined on Section 2.2 are polynomials of ν of degree s and W is a polynomial of degree $s-1$. If $\mathcal{E} = T_X(-\log D)$, the number $\mu_L(\mathcal{E})$ is a polynomial of degree at most s. Let P_0, P_1, P_2 and Q be the polynomials of ν defined by

$$\mathrm{P}_0 = \mu_L(\mathcal{E})-\mathrm{V}_0,\ \mathrm{P}_1 = \mu_L(\mathcal{E})-\mathrm{V}_1,\ \mathrm{P}_2 = \mu_L(\mathcal{E})-\mathrm{V}_2 \ \text{ and } \ \mathrm{Q} = \mu_L(\mathcal{E})-\mathrm{W}.$$

Under certain conditions on a_i, r and s, these polynomials (P_0, P_1, P_2 and Q) have respectively one or no positive root. If the positive root exists, we denote by

- ν_i the unique positive root of P_i where $i \in \{0,1,2\}$
- ν_3 the unique positive root of Q.

In Tables 5.1, 5.2 and 5.3, we give the values of ν for which $\mathcal{E} = T_X(-\log D)$ is (semi)stable with respect to $L = \pi^*\mathcal{O}_{\mathbb{P}^s}(\nu)\otimes\mathcal{O}_X(1)$.

5.1. Case of divisors coming from the base

Proposition 5.2. — *Let $(a_1,\dots,a_r) \neq (0,\dots,0)$. Let $i, j \in \{0,\dots,s\}$ distinct, $\mathcal{E} = T_X(-\log D_{w_i})$ and $\mathcal{F} = T_X(-\log(D_{w_i} + D_{w_j}))$. For any $L \in \mathrm{Amp}(X)$, the vector bundles $\mathcal{E}$ and $\mathcal{F}$ are not semistable with respect to L.*

Proof. — Let $L = \pi^*\mathcal{O}_{\mathbb{P}^s}(\nu)\otimes\mathcal{O}_X(1)$, we have

$$\mu_L(\mathcal{F}) = \frac{(s-1)\mathrm{W} + (\mathrm{V}_0+\cdots+\mathrm{V}_r)}{r+s} < \frac{s\mathrm{W} + (\mathrm{V}_0+\cdots+\mathrm{V}_r)}{r+s} = \mu_L(\mathcal{E}).$$

By Lemma 5.1 and Proposition 4.5, to check the stability of $\mathcal{E}$ (resp. $\mathcal{F}$) with respect to L, it is enough to compare $\mu_L(\mathcal{E})$ (resp. $\mu_L(\mathcal{F})$) with

$$\max\left(\mathrm{V}_0, \frac{\mathrm{V}_0+\mathrm{V}_1+\cdots+\mathrm{V}_r}{r}\right).$$

By Lemma 4.8, we have $\mathrm{V}_0 = a_r\mathrm{W} + \mathrm{V}_r$. As $a_r \geqslant 1$, we get $\mathrm{V}_0 \geqslant \mathrm{W} + \mathrm{V}_r$, i.e. $\mathrm{V}_0 - \mathrm{W} \geqslant \mathrm{V}_r$. Thus,

$$\begin{aligned}(r+s)\left(\mathrm{V}_0 - \mu_L(\mathcal{E})\right) &= s(\mathrm{V}_0-\mathrm{W}) + (r\mathrm{V}_0 - (\mathrm{V}_0+\cdots+\mathrm{V}_{r-1})) - \mathrm{V}_r \\ &\geqslant s(\mathrm{V}_0-\mathrm{W}) - \mathrm{V}_r \qquad \text{because } \mathrm{V}_i \leqslant \mathrm{V}_0 \\ &\geqslant (s-1)\mathrm{V}_r.\end{aligned}$$

Table 5.1. Stability of $T_X(-\log D)$ when $a_r \geqslant 1$

Divisor D	Condition on r and a_i	Condition on s	$\mathrm{Stab}(\mathcal{E})$	$\mathrm{sStab}(\mathcal{E})$
D_{w_j} $0 \leqslant j \leqslant s$ Prop. 5.2	$r \geqslant 1$ and $a_r \geqslant 1$	$s \geqslant 1$	$\varnothing$	$\varnothing$
D_{v_i} $1 \leqslant i \leqslant r-1$ Prop. 5.3	$r \geqslant 2$ and $a_r \geqslant 1$	$s \geqslant 1$	$\varnothing$	$\varnothing$
D_{v_r}	$r \geqslant 1, a_r = 1$ and $a_{r-1} = 0$	$s \geqslant 1$	$0 < \nu < \nu_0$	$0 < \nu \leqslant \nu_0$
Theorem 5.5	$r \geqslant 1$ and ($a_r \geqslant 2$ or $a_{r-1} \neq 0$)	$s \geqslant 1$	$\varnothing$	$\varnothing$
	$r = 1$	$s \geqslant 1$	$0 < \nu < \nu_1$	$0 < \nu \leqslant \nu_1$
D_{v_0}	$r \geqslant 2$ and $a_1 < a_r$	$s \geqslant 1$	$\varnothing$	$\varnothing$
Theorem 5.8	$r \geqslant 2$ and	$a \geqslant \frac{s+1}{r-1}$	$\varnothing$	$\varnothing$
Lemma 5.7	$a_1 = a_r = a$	$\frac{s}{r} \leqslant a < \frac{s+1}{r-1}$	$0 < \nu < \nu_1$	$0 < \nu \leqslant \nu_1$
Theorem 5.10		$a\,r < s$	$\nu_3 < \nu < \nu_1$	$\nu_3 \leqslant \nu \leqslant \nu_1$

If $s \geqslant 2$, then $\mathrm{V}_0 - \mu_L(\mathcal{E}) > 0$ and $\mathrm{V}_0 - \mu_L(\mathcal{F}) > 0$. Thus, $\mathcal{E}$ and $\mathcal{F}$ are not semistables with respect to L. We now assume that $s = 1$. Using the expressions of V_i and W given in Section 2.2, we have

$$\mathrm{W} = 1\ , \quad \mathrm{V}_0 = (a_1 + \cdots + a_r) + r\nu \quad \text{and} \quad \mathrm{V}_i = \mathrm{V}_0 - a_i \quad \text{for } i \in \{1, \ldots, r\}.$$

As

$$\mu_L(\mathcal{E}) = \frac{1 + (r+1)\mathrm{V}_0 - (a_1 + \cdots + a_r)}{r+1}$$

and

$$\frac{\mathrm{V}_0 + \cdots + \mathrm{V}_r}{r} = \frac{(r+1)\mathrm{V}_0 - (a_1 + \cdots + a_r)}{r}\ ,$$

we get

$$\frac{\mathrm{V}_0 + \cdots + \mathrm{V}_r}{r} - \mu_L(\mathcal{E}) = \frac{(r+1)\mathrm{V}_0 - (a_1 + \cdots + a_r) - r}{r(r+1)}.$$

Table 5.2. Stability of $T_X(-\log D)$ when $a_r \geqslant 1$

Divisor D	Condition on r and a_i	Condition on s	sStab($\mathcal{E}$)
$D_{w_i} + D_{w_j}$ $0 \leqslant i < j \leqslant s$ Proposition 5.2	$r \geqslant 1$ and $a_r \geqslant 1$	$s \geqslant 1$	$\varnothing$
$D_{v_i} + D_{v_j}$ $1 \leqslant i < j \leqslant r$ Corollary 5.4	$r \geqslant 2$ and $a_r \geqslant 1$	$s \geqslant 1$	$\varnothing$
$D_{v_i} + D_{w_j}$, $j \geqslant 0$ and $1 \leqslant i \leqslant r-1$ Proposition 5.3	$r \geqslant 2$ and $a_r \geqslant 1$	$s \geqslant 1$	$\varnothing$
$D_{v_r} + D_{w_j}$, $j \geqslant 0$ Corollary 5.6	$r \geqslant 1$ and $a_r \geqslant 1$	$s \geqslant 1$	$\varnothing$
$D_{v_0} + D_{w_j}$, $0 \leqslant j \leqslant s$	$r = 1$	$s \geqslant 1$	$\nu = \nu_3$
Theorem 5.8	$r \geqslant 2$ and $a_1 < a_r$	$s \geqslant 1$	$\varnothing$
Lemma 5.7	$r \geqslant 2$ and	$s \leqslant a(r-1)$	$\varnothing$
Proposition 5.11	$a_1 = a_r = a$	$s > a(r-1)$	$\nu = \nu_3$
$D_{v_0} + D_{v_i}$, $2 \leqslant i \leqslant r$	$r \geqslant 2$ and $a_1 < a_r$	$s \geqslant 1$	$\varnothing$
Lemma 5.7	$r \geqslant 2$ and	$s \leqslant a(r-1)$	$\varnothing$
Theorem 5.11	$a_1 = a_r = a$	$s > a(r-1)$	$\nu = \nu_3$

As $(a_1 + \cdots + a_r - 1) \geqslant 0$ and $\nu > 0$, we have

$$\begin{aligned}(r+1)\mathrm{V}_0 - (a_1 + \cdots + a_r) - r \\ &= (r+1)(a_1 + \cdots + a_r) + (r+1)r\nu - (a_1 + \cdots + a_r) - r \\ &= r(a_1 + \cdots + a_r - 1) + (r+1)r\nu > 0\end{aligned}$$

Thus, $\mathcal{E}$ and $\mathcal{F}$ are not semistables with respect to L. $\square$

Table 5.3. Stability of $T_X(-\log(D_{v_0} + D_{v_1}))$ when $a_r \geqslant 1$

Divisor D	Condition on r and a_i	Condition on s	sStab($\mathcal{E}$)
	$r = 1$	$s \geqslant 1$	$\nu > 0$
$D_{v_0} + D_{v_1}$	$r \geqslant 2$ and $0 = a_1 < a_r$	$s \geqslant 1$	$\varnothing$
	$r \geqslant 2$ and	$s \leqslant a(r-1)$	$\varnothing$
Theorem 5.8	$a_1 = a_r = a$	$s > a(r-1)$	$\nu = \nu_3$
Proposition 5.12	$r = 2$ and	$s \leqslant \delta_2$	$\varnothing$
Proposition 5.11	$0 < a_1 < a_2$	$s > \delta_2$	$\nu = \nu_3$
Proposition 5.13	$r \geqslant 3$ and $a_2 < a_r$	$s \geqslant 1$	$\varnothing$
Proposition 5.15	$r \geqslant 3$ and	$s \leqslant \delta_r$	$\varnothing$
	$0 < a_1 < a_2 = \cdots = a_r$	$s > \delta_r$	$\nu = \nu_3$

5.2. Sum of divisors coming from the base and the bundle: first part

We first study the stability of $T_X(-\log D)$ when $r \geqslant 2$ and

$$D \in \{D_{v_i} : 1 \leqslant i \leqslant r-1\} \cup \{D_{v_i} + D_{w_j} : 1 \leqslant i \leqslant r-1 \text{ and } 0 \leqslant j \leqslant s\} \cup \{D_{v_i} + D_{v_j} : 1 \leqslant i < j \leqslant r\}.$$

Proposition 5.3. — *Let $r \geqslant 2$, $(a_1, \dots, a_r) \neq (0, \dots, 0)$, $i \in \{1, \dots, r-1\}$ and $j \in \{0, \dots, s\}$. For any $L \in \mathrm{Amp}(X)$, the logarithmic tangent bundles $T_X(-\log D_{v_i})$ and $T_X(-\log(D_{v_i} + D_{w_j}))$ are not semistables with respect to L.*

Proof. — We set $\mathcal{E} = T_X(-\log D_{v_i})$ and $\mathcal{F} = T_X(-\log(D_{v_i} + D_{w_j}))$. Let $L \in \mathrm{Amp}(X)$, we have

$$\mu_L(\mathcal{E}) = \frac{(s+1)\mathrm{W} + (\mathrm{V}_0 + \cdots + \mathrm{V}_{i-1} + \mathrm{V}_{i+1} + \cdots + \mathrm{V}_r)}{r+s}$$

and $\mu_L(\mathcal{F}) < \mu_L(\mathcal{E})$. By Lemma 4.9,

$$\begin{aligned}(r+s)(\mathrm{V}_0 - \mu_L(\mathcal{E})) &= (s+1)(\mathrm{V}_0 - \mathrm{W}) - \mathrm{V}_r + (r-1)\mathrm{V}_0 \\ &\qquad - (\mathrm{V}_0 + \cdots + \mathrm{V}_{i-1} + \mathrm{V}_{i+1} + \cdots + \mathrm{V}_{r-1}) \\ &\geqslant (s+1)(\mathrm{V}_0 - \mathrm{W}) - \mathrm{V}_r \\ &\geqslant (s+1)\mathrm{V}_r - \mathrm{V}_r = s\mathrm{V}_r\end{aligned}$$

Hence, by Proposition 4.5, we deduce that $\mathcal{E}$ and $\mathcal{F}$ are not semistables with respect to L. ☐

Corollary 5.4. — *Let $r \geqslant 2$ and $(a_1, \dots, a_r) \neq (0, \dots, 0)$. Let $L \in \operatorname{Amp}(X)$, for any $i, j \in \{1, \dots, r\}$ with $i \neq j$, the logarithmic tangent bundle $T_X(-\log(D_{v_i} + D_{v_j}))$ is not semistable with respect to L.*

Proof. — If we set $\mathcal{G} = T_X(-\log(D_{v_i} + D_{v_j}))$, by using the proof of Proposition 5.3, we have $\mu_L(\mathcal{G}) < \mu_L(\mathcal{E}) < \mathrm{V}_0$. Thus, $\mathcal{G}$ is not semistable with respect to L. ☐

We now study the stability of $T_X(-\log D)$ when $D \in \{D_{v_r}\} \cup \{D_{v_r} + D_{w_j} : 0 \leqslant j \leqslant s\}$.

Theorem 5.5. — *Let $r \geqslant 1$ and $a_r \geqslant 1$. We have $\operatorname{Stab}(T_X(-\log D_{v_r})) \neq \varnothing$ if and only if $\operatorname{sStab}(T_X(-\log D_{v_r})) \neq \varnothing$ if and only if $a_r = 1$ and $a_{r-1} = 0$. If $a_r = 1$ and $a_{r-1} = 0$, then the logarithmic tangent bundle $T_X(-\log D_{v_r})$ is stable (resp. semi-stable) with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ if and only if $0 < \nu < \nu_0$ (resp. $0 < \nu \leqslant \nu_0$) where ν_0 is the positive root of*

$$\mathrm{P}_0(x) = \sum_{k=0}^{s-1} \binom{s+r-1}{k} x^k - s\binom{s+r-1}{s} x^s.$$

Proof. — Let $\mathcal{E} = T_X(-\log D_{v_r})$ and $L = \pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$. We have

$$(r+s)\mu_L(\mathcal{E}) = (s+1)\mathrm{W} + \mathrm{V}_0 + \mathrm{V}_1 + \cdots + \mathrm{V}_{r-1}.$$

If $a_r \geqslant 2$, using the first point of Lemma 4.9 and the fact that $\mathrm{V}_i \leqslant \mathrm{V}_0$, we get:

$$(r+s)[\mathrm{V}_0 - \mu_L(\mathcal{E})] = (s\mathrm{V}_0 - (s+1)\mathrm{W}) + r\mathrm{V}_0 - (\mathrm{V}_0 + \cdots + \mathrm{V}_{r-1}) \geqslant s\mathrm{V}_r.$$

By Proposition 4.5, $T_X(-\log D_{v_r})$ is not semistable with respect to L.

We assume that $r \geqslant 2$ and $a_{r-1} = a_r = 1$. As $\mathrm{V}_{r-1} = \mathrm{V}_r$, we have

$$\begin{aligned}
&(r+s)[\mathrm{V}_0 - \mu_L(\mathcal{E})] \\
&\qquad = (s+1)[\mathrm{V}_0 - \mathrm{W}] - \mathrm{V}_{r-1} + [(r-1)\mathrm{V}_0 - (\mathrm{V}_0 + \cdots + \mathrm{V}_{r-2})] \\
&\qquad \geqslant (s+1)\mathrm{V}_r - \mathrm{V}_{r-1} \quad \text{because } \mathrm{V}_0 - \mathrm{W} \geqslant \mathrm{V}_r \\
&\qquad \geqslant s\mathrm{V}_r
\end{aligned}$$

By Proposition 4.5, $T_X(-\log D_{v_r})$ is not semistable with respect to L.

Let $r \geqslant 1$. We now assume that $a_{r-1} = 0$ and $a_r = 1$. By using the expressions of Section 2.2, we have $\mathrm{V}_0 = \cdots = \mathrm{V}_{r-1} = \mathrm{V}$ where

$$\mathrm{V} = \sum_{k=0}^{s} \binom{s+r-1}{k} \nu^k \quad \text{and} \quad \mathrm{W} = \sum_{k=0}^{s-1} \binom{s+r-1}{k} \nu^k.$$

The points (4) and (5) of Proposition 4.5 are not verified in this case. To check the stability of $\mathcal{E}$ it is enough to compare

$$\mu_L(\mathcal{E}) = \frac{r\mathrm{V} + (s+1)\mathrm{W}}{r+s}$$

with $\max(\mathrm{V}, \mathrm{W})$. We have $(r+s)(\mu_L(\mathcal{E}) - \mathrm{W}) = r\mathrm{V} - (r-1)\mathrm{W} > 0$ because $\mathrm{W} < \mathrm{V}$ and

$$\begin{aligned}(r+s)(\mu_L(\mathcal{E}) - \mathrm{V}) &= (s+1)\mathrm{W} - s\mathrm{V} \\ &= \sum_{k=0}^{s-1} \binom{s+r-1}{k} \nu^k - s\binom{s+r-1}{s}\nu^s = \mathrm{P}_0(\nu).\end{aligned}$$

By the sign rule of Descartes, the polynomial P_0 have a unique positive root ν_0. If $\nu > 0$, then $\mathrm{P}_0(\nu) > 0$ (resp. $\mathrm{P}_0(\nu) \geqslant 0$) if and only if $\nu < \nu_0$ (resp. $\nu \leqslant \nu_0$). Thus, $T_X(-\log D_{v_r})$ is stable (resp. semistable) with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ if and only if $0 < \nu < \nu_0$ (resp. $0 < \nu \leqslant \nu_0$). □

Corollary 5.6. — *We assume that $r \geqslant 1$ and $(a_1, \dots, a_r) \neq (0, \dots, 0)$. Let $j \in \{0, \dots, s\}$ and $D = D_{v_r} + D_{w_j}$. For any $L \in \mathrm{Amp}(X)$, the logarithmic tangent bundle $T_X(-\log D)$ is not semistable with respect to L.*

Proof. — If $\mathcal{E} = T_X(-\log D)$, we have $\mathrm{V}_0 > \mu_L(\mathcal{E})$. By Proposition 4.5, $T_X(-\log D)$ is not semistable with respect to L. □

5.3. Sum of divisors coming from the base and the bundle: second part

In this part we study the stability of the logarithmic tangent bundle $T_X(-\log D)$ when $r \geqslant 2$ and

$$D \in \{D_{v_0}\} \cup \{D_{v_0} + D_{w_j} : 0 \leqslant j \leqslant s\} \cup \{D_{v_0} + D_{v_i} : 2 \leqslant i \leqslant r\}.$$

The last case $D = D_{v_0} + D_{v_1}$ will be studied in Section 5.4.

Lemma 5.7. — *Let $r \geqslant 2$, $(a_1, \dots, a_r) \neq (0, \dots, 0)$ such that $a_1 < a_r$, $i \in \{2, \dots, r\}$ and $j \in \{0, \dots, s\}$. We set $\mathcal{E} = T_X(-\log D_{v_0})$, $\mathcal{F} = T_X(-\log(D_{v_0} + D_{v_i}))$ and $\mathcal{G} = T_X(-\log(D_{v_0} + D_{w_j}))$. For any $L \in \mathrm{Amp}(X)$, the vector bundles $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ are not semistables with respect to L.*

Proof. — We have $\mu_L(\mathcal{E}) > \mu_L(\mathcal{F})$ and $\mu_L(\mathcal{E}) > \mu_L(\mathcal{G})$. We will show that $\mathrm{V}_1 > \mu_L(\mathcal{E})$. By Lemma 4.9, we have $\mathrm{V}_1 - \mathrm{W} \geqslant \mathrm{V}_r$. Therefore

$$\begin{aligned}(r+s)(\mathrm{V}_1 - \mu_L(\mathcal{E})) &= (r+s)\mathrm{V}_1 - (\mathrm{V}_1 + \cdots + \mathrm{V}_r) - (s+1)\mathrm{W} \\ &= (s+1)(\mathrm{V}_1 - \mathrm{W}) - \mathrm{V}_r \\ &\qquad + (r-1)\mathrm{V}_1 - (\mathrm{V}_1 + \cdots + \mathrm{V}_{r-1}) \\ &\geqslant (s+1)(\mathrm{V}_1 - \mathrm{W}) - \mathrm{V}_r \\ &\geqslant s\mathrm{V}_r\end{aligned}$$

By Proposition 4.5, $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ are not semistables with respect to L. ☐

Let $a \in \mathbb{N}^*$. We now study what happen in Lemma 5.7 when $a_1 = \cdots = a_r = a$. We first consider the case $r = 1$.

Theorem 5.8. — *We assume that $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \mathcal{O}_{\mathbb{P}^s}(a))$. Let* P_1 *and* Q *be the polynomials defined by*

$$\mathrm{P}_1(x) = (s+1)\sum_{k=0}^{s-1}\binom{s}{k}a^{s-k-1}x^k - s\,x^s \text{ and } \mathrm{Q}(x) = x^s - \sum_{k=0}^{s-1}\binom{s}{k}a^{s-k-1}x^k.$$

Then:

(1) *$T_X(-\log D_{v_0})$ is stable (resp. semistable) with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ if and only if $0 < \nu < \nu_1$ (resp. $0 < \nu \leqslant \nu_1$) where ν_1 is the unique positive root of* P_1.
(2) *If $j \in \{0, \ldots, s\}$, then $T_X(-\log(D_{v_0} + D_{w_j}))$ is semistable with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ if and only if $\nu = \nu_3$ where ν_3 is the unique positive root of* Q.
(3) $\varnothing = \mathrm{Stab}(T_X(-\log(D_{v_0} + D_{v_1}))) \subsetneq \mathrm{sStab}(T_X(-\log(D_{v_0} + D_{v_1}))) = \mathrm{Amp}(X)$.

Proof. — By the sign rule of Descartes, P_1 and Q have respectively one positive root. Let $L = \pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$, by using the expressions of Section 2.2, we have:

$$\mathrm{V}_1 = \nu^s \quad \text{and} \quad \mathrm{W} = \sum_{k=0}^{s-1}\binom{s}{k}a^{s-k-1}\nu^k.$$

By Proposition 4.5, to check the stability of $\mathcal{E} = T_X(-\log D_{v_0})$, it is enough to compare

$$\mu_L(\mathcal{E}) = \frac{\mathrm{V}_1 + (1+s)\mathrm{W}}{1+s}$$

with $\max(\mathrm{V}_1, \mathrm{W})$. We have $\mu_L(\mathcal{E}) > \mathrm{W}$ and $(1+s)(\mu_L(\mathcal{E}) - \mathrm{V}_1) = \mathrm{P}_1(\nu)$. Thus, $\mathcal{E}$ is stable (resp. semi-stable) with respect to L if and only if $0 < \nu < \nu_1$ (resp. $0 < \nu \leqslant \nu_1$).

Let $\mathcal{F} = T_X(-\log(D_{v_0} + D_{w_j}))$. By Proposition 4.5, it is enough to compare

$$\mu_L(\mathcal{F}) = \frac{\mathrm{V}_1 + s\mathrm{W}}{1+s}$$

with $\max(\mathrm{V}_1, \mathrm{W})$. As $(1+s)(\mu_L(\mathcal{F}) - \mathrm{W}) = \mathrm{Q}(\nu)$ and $(1+s)(\mu_L(\mathcal{F}) - \mathrm{V}_1) = -s\mathrm{Q}(\nu)$, we deduce that $\mathcal{F}$ is semistable with respect to L if and only if $\nu = \nu_3$.

Let $\mathcal{G} = T_X(-\log(D_{v_0} + D_{v_1}))$. We have $\mu_L(\mathcal{G}) = \mathrm{W}$. By Proposition 4.5, $\mathcal{G}$ is semistable with respect to L. □

We now consider the case $r \geqslant 2$ and $a_1 = \cdots = a_r = a$ with $a \in \mathbb{N}^*$.

Lemma 5.9. — *We have*

$$\mathrm{card}\{(\alpha_1, \ldots, \alpha_p) \in \mathbb{N}^p : \alpha_1 + \cdots + \alpha_p = m\} = \binom{m+p-1}{m}.$$

We recall that $\mathrm{V}_{1s} = 1$. By Lemma 5.9, for all $k \in \{0, \ldots, s-1\}$,

$$\mathrm{W}_k = \sum_{d_1 + \cdots + d_r = s-k-1} a_1^{d_1} \ldots a_r^{d_r} = \binom{s-k+r-2}{s-k-1} a^{s-k-1}$$

and

$$\mathrm{V}_{1k} = \sum_{d_2 + \cdots + d_r = s-k} a_2^{d_2} \ldots a_r^{d_r} = \binom{s-k+r-2}{s-k} a^{s-k}.$$

Using the equality $\binom{n}{p-1} = \frac{p}{n-p+1}\binom{n}{p}$, for any $k \in \{0, \ldots, s-1\}$,

$$\mathrm{W}_k = \frac{s-k}{r-1}\binom{s-k+r-2}{s-k} a^{s-k-1} = \frac{s-k}{a(r-1)} \mathrm{V}_{1k}.$$

Theorem 5.10. — *Let $r \geqslant 2$ and $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^s}(a_i))$ with $a_1 = \cdots = a_r = a$ where $a \in \mathbb{N}^*$. We set $\mathcal{E} = T_X(-\log D_{v_0})$. Let P_1 and Q be the polynomials defined by:*

$$\mathrm{P}_1(x) = \sum_{k=0}^{s-1} \left[\left(\frac{(s-k)(s+1)}{a(r-1)} - s \right) \binom{s+r-1}{k} \mathrm{V}_{1k} \right] x^k - s\binom{s+r-1}{s} x^s$$

$$\mathrm{Q}(x) = \sum_{k=0}^{s-1} \left[\left(r - \frac{s-k}{a} \right) \binom{s+r-1}{k} \mathrm{V}_{1k} \right] x^k + r\binom{s+r-1}{s} x^s$$

Then:

(1) *If $a < \frac{s}{r}$, then $\mathcal{E}$ is stable (resp. semistable) with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ if and only if $\nu_3 < \nu < \nu_1$ (resp. $\nu_3 \leqslant \nu \leqslant \nu_1$) where ν_1 and ν_3 are respectively the positive roots of P_1 and Q.*

(2) *If* $\frac{s}{r} \leqslant a < \frac{s+1}{r-1}$, *then* $\mathcal{E}$ *is stable (resp. semistable) with respect to* $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ *if and only if* $0 < \nu < \nu_1$ *(resp.* $0 < \nu \leqslant \nu_1$*) where* ν_1 *is the positive root of* P_1.

(3) *If* $a \geqslant \frac{s+1}{r-1}$, *then for any* $L \in \mathrm{Amp}(X)$, $\mathcal{E}$ *is not semistable with respect to* L.

Proof. — We first explain the condition which ensure the existence of positive roots on P_1 and Q. We write

$$\mathrm{P}_1(x) = \sum_{k=0}^{s} \alpha_k\, x^k \quad \text{and} \quad \mathrm{Q}(x) = \sum_{k=0}^{s} \beta_k\, x^k.$$

For $k \in \{0, \dots, s-1\}$, $\alpha_k > 0$ if and only if $k < \left(1 - \frac{a(r-1)}{s+1}\right) s$. Therefore,

- If $\frac{a(r-1)}{s+1} \geqslant 1$, then for any $x \geqslant 0$, $\mathrm{P}_1(x) < 0$.
- If $\frac{a(r-1)}{s+1} < 1$, then P_1 has only one positive root ν_1.

For $k \in \{0, \dots, s-1\}$, $\beta_k < 0$ if and only if $k < s - ra$. Therefore,

- If $ra \geqslant s$, then for any $x \geqslant 0$, $\mathrm{Q}(x) > 0$.
- If $ra < s$, then Q has only one positive root ν_3.

We now show that: If $a < \frac{s}{r}$, then $\nu_3 < \nu_1$. As

$$\begin{aligned}\frac{\mathrm{P}_1(x)}{-s} - \frac{\mathrm{Q}(x)}{r} &= \sum_{k=0}^{s-1} \left[\left(-\frac{(s-k)(s+1)}{a\,s(r-1)} + \frac{s-k}{r\,a}\right)\binom{s+r-1}{k} \mathrm{V}_{1k}\right] x^k \\ &= \frac{-(r+s)}{a\,s\,r(r-1)} \sum_{k=0}^{s-1} (s-k)\binom{s+r-1}{k} \mathrm{V}_{1k}\, x^k = \mathrm{P}(x)\end{aligned}$$

and $\frac{\mathrm{P}_1(\nu_3)}{-s} - \frac{\mathrm{Q}(\nu_3)}{r} = \mathrm{P}(\nu_3) < 0$, we deduce that $\mathrm{P}_1(\nu_3) > 0$. By using the fact that, for $x \geqslant 0$, $\mathrm{P}_1(x) > 0$ if and only if $0 \leqslant x < \nu_1$, we deduce that $\nu_3 < \nu_1$.

We can now study the stability of $\mathcal{E}$. As $a_1 = \cdots = a_r$, we have $\mathrm{V}_1 = \cdots = \mathrm{V}_r$. Therefore

$$\mu_L(\mathcal{E}) = \frac{r\mathrm{V}_1 + (s+1)\mathrm{W}}{r+s}.$$

By Proposition 4.5, to check the stability of $\mathcal{E}$, it is enough to compare $\mu_L(\mathcal{E})$ with $\max(\mathrm{V}_1, \mathrm{W})$. We have

$$(r+s)(\mu_L(\mathcal{E}) - \mathrm{V}_1) = -s\mathrm{V}_1 + (s+1)\mathrm{W} = \mathrm{P}_1(\nu)$$

and

$$(r+s)(\mu_L(\mathcal{E}) - \mathrm{W}) = r\mathrm{V}_1 - (r-1)\mathrm{W} = \mathrm{Q}(\nu)$$

Therefore,

(i) If $a \geqslant \frac{s+1}{r-1}$, then for any $\nu > 0$, $\mathrm{P}_1(\nu) < 0$.
(ii) If $a < \frac{s+1}{r-1}$, then $\mathrm{P}_1(\nu) > 0$ (resp. $\mathrm{P}_1(\nu) \geqslant 0$) if and only if $0 < \nu < \nu_1$ (resp. $0 < \nu \leqslant \nu_1$).
(iii) If $a \geqslant \frac{s}{r}$, then for any $\nu > 0$, $\mathrm{Q}(\nu) > 0$.
(iv) If $a < \frac{s}{r}$, then $\mathrm{Q}(\nu) > 0$ (resp. $\mathrm{Q}(\nu) \geqslant 0$) if and only if $\nu > \nu_3$ (resp. $\nu \geqslant \nu_3$).

The point (i) shows the third point of the theorem. By using the points (ii) and (iv), we get the first point of theorem. Finally, the points (ii) and (iii) give the second point of the theorem. □

Proposition 5.11. — *Let $r \geqslant 2$ and $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^s}(a_i))$ with $a_1 = \cdots = a_r = a$ where $a \in \mathbb{N}^*$. Let $i \in \{1, \ldots, r\}$ and $j \in \{0, \ldots, s\}$. We set $\mathcal{F}_j = T_X(-\log(D_{v_0} + D_{w_j}))$, $\mathcal{G}_i = T_X(-\log(D_{v_0} + D_{v_i}))$ and*

$$\mathrm{Q}(x) = \sum_{k=0}^{s-1} \left[\left(1 - \frac{s-k}{a(r-1)}\right) \binom{s+r-1}{k} \mathrm{V}_{1k} \right] x^k + \binom{s+r-1}{s} x^s.$$

(1) *If $a \geqslant \frac{s}{r-1}$, then for any $L \in \mathrm{Amp}(X)$, $\mathcal{F}_j$ and $\mathcal{G}_i$ are not semistables with respect to L.*
(2) *If $a < \frac{s}{r-1}$, then $\mathcal{F}_j$ and $\mathcal{G}_i$ are semistables with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ if and only if $\nu = \nu_3$ where ν_3 is the unique root of* Q.

Proof. — We first study the polynomial Q. We write $\mathrm{Q}(x) = \sum_{k=0}^{s} \alpha_k x^k$. For $k \in \{0, \ldots, s-1\}$, $\alpha_k > 0$ if and only if $k < s - a(r-1)$.

- If $a \geqslant \frac{s}{r-1}$, then for any $x \geqslant 0$, $\mathrm{Q}(x) > 0$.
- If $a < \frac{s}{r-1}$, then Q has a unique positive root ν_3.

As $a_1 = \cdots = a_r$, we have $\mathrm{V}_1 = \cdots = \mathrm{V}_r$. Thus,

$$\mu_L(\mathcal{F}_j) = \frac{r\mathrm{V}_1 + s\mathrm{W}}{r+s} \quad \text{and} \quad \mu_L(\mathcal{G}_i) = \frac{(r-1)\mathrm{V}_1 + (s+1)\mathrm{W}}{r+s}.$$

By Proposition 4.5, to check the stability of $\mathcal{F}_j$ (resp. $\mathcal{G}_i$), it is enough to compare $\mu_L(\mathcal{F}_j)$ (resp. $\mu_L(\mathcal{G}_i)$) with $\max(\mathrm{V}_1, \mathrm{W})$. We have

$$\begin{cases} (r+s)(\mu_L(\mathcal{F}_j) - \mathrm{V}_1) = s(\mathrm{W} - \mathrm{V}_1) = -s\mathrm{Q}(\nu) \\ (r+s)(\mu_L(\mathcal{F}_j) - \mathrm{W}) = r(\mathrm{V}_1 - \mathrm{W}) = r\mathrm{Q}(\nu) \end{cases}$$

and

$$\begin{cases} (r+s)(\mu_L(\mathcal{G}_i) - \mathrm{V}_1) = (s+1)(\mathrm{W} - \mathrm{V}_1) = -(s+1)\mathrm{Q}(\nu) \\ (r+s)(\mu_L(\mathcal{G}_i) - \mathrm{W}) = (r-1)(\mathrm{V}_1 - \mathrm{W}) = (r-1)\mathrm{Q}(\nu) \end{cases}$$

If $a \geqslant \frac{s}{r-1}$, then for any $\nu > 0$, $Q(\nu) > 0$; thus, $\mu_L(\mathcal{F}_j) < \mathrm{V}_1$ and $\mu_L(\mathcal{G}_i) < \mathrm{V}_1$. Hence, for any $\nu > 0$, $\mathcal{F}_j$ and $\mathcal{G}_i$ are not semistables with respect to L.

If $a < \frac{s}{r-1}$, then by the above equalities, $\mathcal{F}_j$ and $\mathcal{G}_i$ are semistables with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ if and only if $\nu = \nu_3$ where ν_3 is the positive root of Q. $\square$

5.4. Sum of divisors coming from the bundle

In this part, we assume that $\mathcal{E} = T_X(-\log(D_{v_0} + D_{v_1}))$. We study the stability of $\mathcal{E}$ when $r \geqslant 2$ and $a_1 < a_r$. The stability of $\mathcal{E}$ when $r = 1$ was treated in Theorem 5.8. When $r \geqslant 2$, in Proposition 5.11, we studied the stability of $\mathcal{E}$ when $a_1 = \cdots = a_r$.

PROPOSITION 5.12. — *Let* $(a_1, \ldots, a_r) \neq (0, \ldots, 0)$ *and*

$$\mathcal{E} = T_X(-\log(D_{v_0} + D_{v_1})).$$

(1) *If* $a_1 = 0$, *then for any* $L \in \mathrm{Amp}(X)$, $\mathcal{E}$ *is not semistable with respect to* L.
(2) *If* $r \geqslant 3$ *and* $a_2 < a_r$, *then for any* $L \in \mathrm{Amp}(X)$, $\mathcal{E}$ *is not semistable with respect to* L.

Proof. — We have

$$\mu_L(\mathcal{E}) = \frac{(s+1)\mathrm{W} + \mathrm{V}_2 + \cdots + \mathrm{V}_r}{r+s}.$$

First point. — As $\mathrm{card}\{2, \ldots, r\} = r - 1$, by using the point 4 of Proposition 4.5 with $I' = \{2, \ldots, r\}$, we get

$$\frac{1}{r+s-1}\left(\sum_{i \in I'} \mathrm{V}_i + (s+1)\mathrm{W}\right) = \frac{1}{r+s-1}\left(\mathrm{V}_2 + \cdots + \mathrm{V}_r + (s+1)\mathrm{W}\right).$$

Thus, $\mathcal{E}$ is not semistable with respect to L.

Second point. — By Lemma 4.9, we have $\mathrm{V}_2 - \mathrm{W} \geqslant \mathrm{V}_r$. Therefore,

$$\begin{aligned}(r+s)(\mathrm{V}_2 - \mu_L(\mathcal{E}) &= (r+s)\mathrm{V}_2 - (\mathrm{V}_2 + \cdots + \mathrm{V}_r) - (s+1)\mathrm{W} \\ &= (s+1)(\mathrm{V}_2 - \mathrm{W}) + ((r-1)\mathrm{V}_2 - (\mathrm{V}_2 + \cdots + \mathrm{V}_r)) \\ &\geqslant (s+1)(\mathrm{V}_2 - \mathrm{W}) \\ &\geqslant (s+1)\mathrm{V}_r\end{aligned}$$

Hence, by Proposition 4.5, $\mathcal{E}$ is not semistable with respect to L. $\square$

We now assume that $0 < a_1 < a_2 = \cdots = a_r$. By Proposition 4.5, to check the stability of $\mathcal{E}$, it is enough to compare $\mu_L(\mathcal{E})$ with $\max(\mathrm{V}_2, \mathrm{W})$. We have

$$\mu_L(\mathcal{E}) = \frac{(r-1)\mathrm{V}_2 + (s+1)\mathrm{W}}{r+s}$$

and

$$\begin{aligned} (r+s)[\mu_L(\mathcal{E}) - \mathrm{V}_2] &= (s+1)(\mathrm{W} - \mathrm{V}_2) \\ (r+s)[\mu_L(\mathcal{E}) - \mathrm{W}] &= -(r-1)(\mathrm{W} - \mathrm{V}_2) \end{aligned} \tag{5.1}$$

The vector bundle $\mathcal{E}$ is semistable with respect to $L = \pi^*\mathcal{O}_{\mathbb{P}^s}(\nu_3) \otimes \mathcal{O}_X(1)$ if and only if ν_3 is a positive root of the polynomial $\mathrm{Q}(\nu) = \mathrm{W} - \mathrm{V}_2$ (W and V_2 depend on ν). We first consider the case $r = 2$.

Proposition 5.13. — *Let $r = 2$ and $0 < a_1 < a_2$. We define $\delta = \frac{\ln(1+a_2-a_1)}{\ln(a_2)-\ln(a_1)}$ and the polynomial* Q *by*

$$\mathrm{Q}(x) = \sum_{k=0}^{s-1} \left[\frac{a_1^{s-k}}{a_2 - a_1} \left(\left(\frac{a_2}{a_1}\right)^{s-k} - 1 - a_2 + a_1 \right) \binom{s+1}{k} \right] x^k - (s+1)x^s.$$

Then:

(1) *If $s \leqslant \delta$, then* $\mathrm{sStab}(\,T_X(-\log(D_{v_0} + D_{v_1}))\,) = \varnothing$*;*
(2) *If $s > \delta$, then $T_X(-\log(D_{v_0} + D_{v_1}))$ is semistable with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu) \otimes \mathcal{O}_X(1)$ if and only if $\nu = \nu_3$ where ν_3 is the positive root of* Q.

Proof. — We have

$$\mathrm{V}_2 = \sum_{k=0}^{s} \binom{s+1}{k} a_1^{s-k}\, \nu^k$$

$$\mathrm{W} = \sum_{k=0}^{s-1} \binom{s+1}{k} \left(\sum_{d_1+d_2=s-k-1} a_1^{d_1} a_2^{d_2} \right) \nu^k = \sum_{k=0}^{s-1} \frac{a_2^{s-k} - a_1^{s-k}}{a_2 - a_1} \binom{s+1}{k} \nu^k$$

hence,

$$\mathrm{W} - \mathrm{V}_2 = \sum_{k=0}^{s-1} \left[\frac{a_1^{s-k}}{a_2 - a_1} \left(\left(\frac{a_2}{a_1}\right)^{s-k} - 1 - a_2 + a_1 \right) \binom{s+1}{k} \right] \nu^k - (s+1)\nu^s = \mathrm{Q}(\nu).$$

We write $\mathrm{Q}(x) = \sum_{k=0}^{s} \alpha_k\, x^k$. The inequality $\left(\frac{a_2}{a_1}\right)^{s-k} - 1 - a_2 + a_1 > 0$ gives

$$k < s - \frac{\ln(1 + a_2 - a_1)}{\ln(a_2) - \ln(a_1)} = s - \delta.$$

Hence, by the Descartes rule, Q has a unique positive root ν_3 if and only if $s > \delta$. □

We now consider the case where $r \geqslant 3$ and $a_1 < a_r$. Let $a, b \in \mathbb{N}^*$ such that $a < b$. We assume that $a_1 = a$ and $a_2 = \cdots = a_r = b$. By Lemma 5.9, for any $k \in \{0, \dots, s-1\}$, we have

$$\begin{aligned} \mathrm{W}_k = \sum_{\substack{d_1+\cdots+d_r\\=s-k-1}} a_1^{d_1}\dots a_r^{d_r} &= \sum_{j=0}^{s-k-1} a^{s-k-1-j}\left(\sum_{d_2+\cdots+d_r=j} b^j\right)\\ &= \sum_{j=0}^{s-k-1}\binom{j+r-2}{j} b^j\, a^{s-k-1-j} \end{aligned}$$

and

$$\begin{aligned} \mathrm{V}_{2k} = \mathrm{V}_{rk} = \sum_{\substack{d_1+\cdots+d_{r-1}\\=s-k}} a_1^{d_1}\dots a_{r-1}^{d_{r-1}} &= \sum_{j=0}^{s-k} a^{s-k-j}\left(\sum_{d_2+\cdots+d_{r-1}=j} b^j\right)\\ &= \sum_{j=0}^{s-k}\binom{j+r-3}{j} b^j\, a^{s-k-j}. \end{aligned}$$

For $p \in \{1, \dots, s\}$, we set $\alpha_p = \mathrm{W}_{s-p} - \mathrm{V}_{2,s-p}$. We have

$$\alpha_p = \sum_{j=0}^{p-1}\binom{j+r-2}{j} b^j\, a^{p-1-j} - \sum_{j=0}^{p}\binom{j+r-3}{j} b^j\, a^{p-j}.$$

Let Q_s be the polynomial defined by

$$\mathrm{Q}_s(x) = \sum_{k=0}^{s-1}\binom{s+r-1}{k}\alpha_{s-k}\, x^k - \binom{s+r-1}{s} x^s.$$

We have $\mathrm{W} - \mathrm{V}_2 = \mathrm{Q}_s(\nu)$. We now search a condition on s which ensure the existence of positive root on Q_s. By using the identity $\binom{n}{p-1} = \frac{p}{n-p+1}\binom{n}{p}$, we have

$$\begin{aligned} \alpha_p &= \sum_{j=0}^{p-1}\frac{j+1}{r-2}\binom{j+r-2}{j+1} b^j\, a^{p-1-j} - \sum_{j=1}^{p}\binom{j+r-3}{j} b^j\, a^{p-j} - a^p\\ &= \sum_{j=0}^{p-1}\left[\left(\frac{j+1}{r-2} - b\right)\binom{j+r-2}{j+1} b^j\, a^{p-1-j}\right] - a^p. \end{aligned}$$

If $1 \leqslant p \leqslant b(r-2)$, then for all $j \in \{0, \dots, p-1\}$, we have

$$\frac{j+1}{r-2} - b \leqslant \frac{p}{r-2} - b = \frac{p-b(r-2)}{r-2} \leqslant 0\ ;$$

thus $\alpha_p < 0$. Hence, if $\alpha_p > 0$, then we must have $p > b(r-2)$. If there is $p > b(r-2)$ such that $\alpha_p > 0$, then for any $q \geqslant p$, we have $\alpha_q > 0$; this follows from these equalities.

$$\begin{aligned}\alpha_{p+1} &= \sum_{j=0}^{p-1}\left[\left(\frac{j+1}{r-2}-b\right)\binom{j+r-2}{j+1}b^j\, a^{p-j}\right]-a^{p+1}\\ &\qquad+\left(\frac{p+1}{r-2}-b\right)\binom{p+r-2}{p+1}b^p\\ &= a\,\alpha_p+\left(\frac{p+1}{r-2}-b\right)\binom{p+r-2}{p+1}b^p\end{aligned}$$

We denote by $\lfloor x\rfloor$ the floor of $x\in\mathbb{R}$.

Lemma 5.14. — *Let $m=b(r-2)$. There is a unique integer $\delta_r\in[m+1\,;\,\lfloor 3.2m\rfloor+1]$ such that: if $p\leqslant\delta_r$, then $\alpha_p\leqslant 0$ and if $p>\delta_r$, then $\alpha_p>0$.*

Let δ_r be the integer given in Lemma 5.14. If $s\leqslant\delta_r$, then all coefficients of Q_s are negative; thus, for any $x>0$, $\mathrm{Q}_s(x)<0$. If $s>\delta_r$, then by the Descartes rule, Q_s has a only one positive root ν_3. We deduce:

Proposition 5.15. — *Let $r\geqslant 3$ and $a,b\in\mathbb{N}^*$ such that $a<b$ and $a_1=a\,,\ a_2=\cdots=a_r=b\,.$*

(1) *If $s\leqslant\delta_r$, then $\mathrm{sStab}(\,T_X(-\log(D_{v_0}+D_{v_1}))\,)=\varnothing\,;$*
(2) *If $s>\delta_r$, then $T_X(-\log(D_{v_0}+D_{v_1}))$ is semistable with respect to $\pi^*\mathcal{O}_{\mathbb{P}^s}(\nu)\otimes\mathcal{O}_X(1)$ if and only if $\nu=\nu_3$.*

We now give the proof of Lemma 5.14.

Proof. — We have

$$\alpha_p=\sum_{j=0}^{m-1}\left[\left(\frac{j+1}{r-2}-b\right)\binom{j+r-2}{j+1}b^j\, a^{p-1-j}\right]-a^p+\beta_p$$

where

$$\begin{aligned}\beta_p &= \sum_{l=m}^{p-1}\left[\left(\frac{l+1-(r-2)b}{r-2}\right)\binom{l+r-2}{l+1}b^l\, a^{p-1-l}\right]\\ &= \sum_{l=0}^{p-1-m}\left[\frac{l+1}{r-2}\binom{l+m+r-2}{l+m+1}b^{l+m}\, a^{p-1-(l+m)}\right]\end{aligned}$$

The goal of this proof is to find an integer p such that $\alpha_p>0$. We will search an integer p such that

$$\beta_p\geqslant a^p+\sum_{j=0}^{m-1}b\binom{j+r-2}{j+1}b^j\, a^{p-1-j}. \tag{5.2}$$

We have $\binom{l+m+r-2}{l+m+1} = \binom{l+m+r-2}{r-3} = \frac{r-2}{l+m+1}\binom{l+m+r-2}{r-2}$. From the equality

$$\sum_{j=n}^{r}\binom{j}{n} = \binom{r+1}{n+1} \quad \text{we have} \quad \binom{r+1}{n+1} = 1 + \sum_{j=0}^{r-n-1}\binom{j+n+1}{n}\ ;$$

Hence,

$$\begin{aligned}\binom{l+m+r-2}{r-2} = 1 + \sum_{j=0}^{l+m-1}\binom{j+r-2}{r-3} &= 1 + \sum_{j=0}^{l+m-1}\binom{j+r-2}{j+1} \\ &\geqslant 1 + \sum_{j=0}^{m-1}\binom{j+r-2}{j+1}.\end{aligned}$$

Thus,

$$\begin{aligned}\beta_p &\geqslant \left(1 + \sum_{j=0}^{m-1}\binom{j+r-2}{j+1}\right)\sum_{l=0}^{p-1-m}\frac{l+1}{l+1+m}\, b^{l+m}\, a^{p-1-(l+m)} \\ &\geqslant b^m\, a^{p-1-m}\left(1 + \sum_{j=0}^{m-1}\binom{j+r-2}{j+1}\right)\sum_{l=0}^{p-1-m}\frac{l+1}{l+1+m}\left(\frac{b}{a}\right)^l.\end{aligned}$$

We have

$$\begin{aligned}\sum_{l=0}^{p-1-m}\frac{l+1}{l+1+m} &\geqslant \sum_{l=0}^{p-1-m}\int_l^{l+1}\frac{x}{x+m}\mathrm{d}x \\ &\geqslant \int_0^{p-m}\frac{x}{x+m}\mathrm{d}x \\ &\geqslant p - m - m\ln\left(\frac{p}{m}\right).\end{aligned}$$

If $k = \frac{p}{m} \geqslant 3.2$, then $(k - 1 - \ln(k)) > 1$. If we set $p = \lfloor 3.2\, m\rfloor + 1$, we get

$$\sum_{l=0}^{p-1-m}\frac{l+1}{l+1+m}\left(\frac{b}{a}\right)^l \geqslant \sum_{l=0}^{p-1-m}\frac{l+1}{l+1+m} \geqslant m \geqslant b.$$

For $j \in \{0, \ldots, m-1\}$, we have $b^m\, a^{p-1-m} \geqslant b^j\, a^{p-1-j}$. If $p = \lfloor 3.2\, m\rfloor + 1$, we get

$$\begin{aligned}\beta_p &\geqslant b\, b^m\, a^{p-1-m}\left(1 + \sum_{j=0}^{m-1}\binom{j+r-2}{j+1}\right) \\ &\geqslant b^{m+1}\, a^{p-1-m} + \sum_{j=0}^{m-1} b\binom{j+r-2}{j+1} b^j\, a^{p-1-j}\ ;\end{aligned}$$

this proves the inequality (5.2). Thus, $\alpha_p > 0$ for $p = \lfloor 3.2\,m \rfloor + 1$. Hence, we deduce the existence of the integer δ_r in the interval $[m+1\,;\,\lfloor 3.2\,m\rfloor+1]$. □

6. Application on log smooth toric del Pezzo pairs

The goal of this part is to study the stability of the equivariant logarithmic tangent bundle $T_X(-\log D)$ with respect to $-(K_X+D)$ when the pair (X, D) is log del Pezzo. We assume that $N = M = \mathbb{Z}^2$ and the pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$ is the usual dot product.

Example 6.1. — Let $r \in \mathbb{N}$ and Σ the fan of the Hirzebruch surface $\mathbb{F}_r = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r))$. The rays of Σ are the half lines generated by the vectors $u_1 = e_1$, $u_2 = e_2$, $u_3 = -e_1 + r\,e_2$ and $u_0 = -e_2$. Hence,

$$\Sigma = \{0\} \cup \{\mathrm{Cone}(u_i) : 0 \leqslant i \leqslant 3\} \cup \{\mathrm{Cone}(u_i, u_{i+1}) : 0 \leqslant i \leqslant 3\}$$

where $u_4 = u_0$. For any $i \in \{0, \dots, 3\}$, we denote by D_i the divisor corresponding to the ray $\mathrm{Cone}(u_i)$. By [2, Proposition 6.4.4], we have

$$\begin{cases} D_i \cdot D_i = -\gamma_i & \\ D_k \cdot D_i = 1 & \text{if } k \in \{i-1, i+1\} \\ D_k \cdot D_i = 0 & \text{if } k \notin \{i-1, i, i+1\} \end{cases} \tag{6.1}$$

where $\gamma_i = \det(u_{i-1}, u_{i+1})$. So, $\gamma_0 = -r$, $\gamma_1 = 0$, $\gamma_2 = r$ and $\gamma_3 = 0$. If $\pi : \mathbb{F}_r \to \mathbb{P}^1$ is the projection map, then the invariant divisors D_1, D_3 are the fibers of π and the invariant divisors D_0, D_2 can be seen as sections.

By using the classification of log Del Pezzo surfaces given by Maeda [16, Section 3.4] (see e.g. [18] for a proof in a toric setting), we get the following description of equivariant log Del Pezzo pairs.

Proposition 6.2. — *Let X be a smooth complete toric surface and D a reduced torus-invariant divisor on X. Then, the pair (X, D) is log Del Pezzo if:*

1. *$X = \mathbb{P}^2$ and $D = D'$ where D' is a line;*
2. *$X = \mathbb{P}^2$ and $D = D' + D''$ where D' and D'' are two lines;*
3. *$X = \mathbb{F}_r$ and $D = D'$ where D' is a section with $(D')^2 = -r$;*
4. *$X = \mathbb{F}_r$ and $D = D' + D''$ where D' is a section with $(D')^2 = -r$ and D'' is a fiber;*
5. *$X = \mathbb{F}_1$ and $D = D'$ where D' is a section such that $(D')^2 = 1$;*
6. *$X = \mathbb{F}_0$ and $D = D''$ where D'' is a fiber.*

Remark 6.3. — If D, D' are two invariant lines of $\mathbb{P}^2$, then according to Corollary 4.2 and Corollary 3.11, $T_{\mathbb{P}^2}(-\log D)$ is polystable with respect to $-(K_{\mathbb{P}^2} + D)$ and $T_{\mathbb{P}^2}(-\log(D + D'))$ is unstable with respect to $-(K_{\mathbb{P}^2} + D + D')$.

Proposition 6.4. — *Let $X = \mathbb{F}_r$ and D_0, D_1, D_2, D_3 the divisors defined in Example 6.1. Then:*

(1) *If $r = 0$ and $D \in \{D_i : 0 \leqslant i \leqslant 3\} \cup \{D_0 + D_1, D_0 + D_3\} \cup \{D_2 + D_1, D_2 + D_3\}$, $T_X(-\log D)$ is polystable with respect to $-(K_X + D)$;*
(2) *If $r = 1$, $T_X(-\log D_0)$ is stable with respect to $-(K_X + D_0)$;*
(3) *If $r \geqslant 1$ and $D \in \{D_2, D_2 + D_1, D_2 + D_3\}$, $T_X(-\log D)$ is unstable with respect to $-(K_X + D)$.*

Proof. — We first note that, the divisors D_0, D_1, D_2, D_3 of $\mathbb{F}_r$ defined in Example 6.1 are given in Section 2.2 by $D_0 = D_{v_0}$, $D_1 = D_{w_1}$, $D_2 = D_{v_1}$ and $D_3 = D_{w_0}$ where $v_1 = e_2$ and $w_1 = e_1$. Thus, by Equation (2.5),

$$D_1 \sim_{\mathrm{lin}} D_3 \quad \text{and} \quad D_2 \sim_{\mathrm{lin}} D_0 - rD_3.$$

If $\alpha D_3 + \beta D_0$ is an ample divisor of $\mathbb{F}_r$, then the number ν used in the results of Sections 4.3 and 5 is defined by $\nu = \frac{\alpha}{\beta}$. Using Remark 4.14 and Propositions 4.11 and 4.13, we get the first point.

Let $r = 1$. We have $-(K_X + D_0) \sim_{\mathrm{lin}} D_0 + D_3$ and $\nu = 1$. As the polynomial P_1 defined in Theorem 5.8 is $\mathrm{P}_1 = 2 - x$ and $0 < \nu < 2$, we deduce that $T_X(-\log D_0)$ is stable with respect to $-(K_X + D_0)$.

The polynomial P_0 of Theorem 5.5 is given by $\mathrm{P}_0 = 1 - x$. As $-(K_X + D_2) \sim_{\mathrm{lin}} 2D_3 + D_0$ and $\nu = 2$, we deduce that $T_X(-\log D_2)$ is unstable with respect to $-(K_X + D_2)$.

If $r \geqslant 2$, then according to Theorem 5.5, $T_X(-\log D_2)$ is unstable with respect to $-(K_X + D)$. Finally, if $r \geqslant 1$ and $D \in \{D_2 + D_1, D_2 + D_3\}$, then $T_X(-\log D)$ is unstable with respect to $-(K_X + D)$ (cf. Corollary 5.6). □

Remark 6.5. — If X is a smooth toric variety and D an invariant divisor on X such that $-(K_X + D)$ is ample, by [1, Theorem 1.2], (X, D) admits a toric log Kähler–Einstein metric if and only if 0 is the barycenter of the polytope $P_{(X,D)}$ corresponding to $-(K_X + D)$. In this case, according to [15, Theorem 1.4], the orbifold tangent sheaf $T_X(-\log D)$ is polystable with respect to $-(K_X + D)$. In this paper we studied the stability of $T_X(-\log D)$ when 0 is not the barycenter of $P_{(X,D)}$. Therefore, we do not have the existence of Kähler–Einstein metrics on these logarithmic pairs (X, D).

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