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A counterexample to strong local monomialization in a tower of two independent defect Artin–Schreier extensions (*)

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Abstract. — We give an example of an extension of two dimensional regular local rings in a tower of two independent defect Artin–Schreier extensions for which strong local monomialization does not hold.

Résumé. — Nous donnons un exemple d'extension d'anneaux locaux réguliers à deux dimensions dans une tour de deux extensions d'Artin–Schreier de défauts indépendants pour lesquelles la monomialisation locale forte ne tient pas.

1. Introduction

In characteristic zero, there is a very nice local form for morphisms, called local monomialization. This result is a little stronger than what comes immediately from the assumption that toroidalization is possible. If $R \to S$ is an extension of local rings such that the maximal ideal of S contracts to the maximal ideal of R then we say that S dominates R. If S is dominated by the valuation ring $\mathcal{O}_\omega$ of a valuation ω we say that ω dominates S.

Theorem 1.1 (local monomialization)([2, 3]). — *Suppose that k is a field of characteristic zero and $R \to S$ is an extension of regular local rings such that R and S are essentially of finite type over k and ω is a valuation of the quotient field of S which dominates S and S dominates R. Then there*

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is a commutative diagram

$$\begin{array}{ccc} R_1 & \longrightarrow & S_1 \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

such that ω dominates S_1, S_1 dominates R_1 and the vertical arrows are products of monoidal transforms; that is, these arrows are factored by the local rings of blowups of prime ideals whose quotients are regular local rings. In particular, R_1 and S_1 are regular local rings. Further, $R_1 \to S_1$ has a locally monomial form; that is, there exist regular parameters $u_1, \ldots, u_m$ in R_1 and $x_1, \ldots, x_n$ in S_1, an $m \times n$ matrix $A = (a_{ij})$ with integral coefficients such that $\operatorname{rank}(A) = m$ *and units $\delta_i \in S_1$ such that*

$$u_i = \delta_i \prod_{j=1}^{n} x_j^{a_{ij}}$$

for $1 \leqslant i \leqslant m$.

The difficulty in the proof is to obtain the condition that $\operatorname{rank}(A) = m$. To do this, it is necessary to blow up above both R and S.

In the case when the extension of quotient fields $K \to L$ of the extension $R \to S$ is a finite extension and k has characteristic zero, it is possible to find a local monomialization such that the structure of the matrix of coefficients recovers classical invariants of the extension of valuations in $K \to L$, and this form holds stably along suitable sequences of birational morphisms which generate the respective valuation rings. This form is called strong local monomialization. It is established for rank 1 valuations in [2] and for general valuations in [8]. The case which has the simplest form and will be of interest to us in this paper is when the valuation has rational rank 1. In this case, if $R_1 \to S_1$ is a strong local monomialization, then there exist regular parameters $u_1, \ldots, u_m$ in R_1 and $v_1, \ldots, v_m$ in S_1, a positive integer a and a unit $\delta \in S_1$ such that

$$u_1 = \delta v_1^a, u_2 = v_2, \ldots, u_m = v_m. \tag{1.1}$$

The stable forms of mappings in positive characteristic and dimension $\geqslant 2$ are much more complicated. For instance, local monomialization does not always hold. An example is given in [4] where $R \to S$ are local rings of points on nonsingular algebraic surfaces over an algebraically closed field k of positive characteristic p and $k(X) \to k(Y)$ is finite and separable.

The obstruction to local monomialization is the defect. The defect $\delta(\omega/\nu)$, which is a power of the residue characteristic p of $\mathcal{O}_\omega$, is defined and its basic properties developed in [21, Chapter VI, Section 11], [12], [8, Section 7.1]. The defect is discussed in Subsection 2.1. We have the following theorem, showing that the defect is the only obstruction to strong local monomialization for maps of surfaces.

THEOREM 1.2 ([8, Theorem 7.35]). — *Suppose that $K \to L$ is a finite, separable extension of algebraic function fields over an algebraically closed field k of characteristic $p > 0$, $R \to S$ is an extension of local domains such that R and S are essentially of finite type over k and the quotient fields of R and S are K and L respectively such that S dominates R. Suppose that ω is valuation of L which dominates S. Let ν be the restriction of ω to K. Suppose that the extension is defectless ($\delta(\omega/\nu) = 1$). Then the conclusions of Theorem 1.1 hold. In particular, $R \to S$ has a local monomialization (and a strong local monomialization) along ω.*

Suppose that $K \to L$ is a Galois extension of fields of characteristic $p > 0$ and ω is a valuation of L, ν is the restriction of ω to K. Then there is a classical tower of fields ([11, p. 171])

$$K \to K^s \to K^i \to K^v \to L.$$

where K^s is the splitting field, K^i is the inertia field, K^v is the ramification field and the extension $K \to K^v$ has no defect. Thus the essential difficulty comes from the extension from K^v to L which could have defect. The extension $K^v \to L$ is a tower of Artin–Schreier extensions, so the Artin–Schreier extension is of fundamental importance in this theory.

Kuhlmann has extensively studied defect in Artin–Schreier extensions in [13]. He separated these extensions into dependent and independent defect Artin–Schreier extensions. This definition is reproduced in Subsection 2.4. Kuhlmann also defined an invariant called the distance to distinguish the natures of Artin–Schreier extensions. This definition is given in Subsections 2.3 and 2.4.

We now specialize to the case of a finite separable extension $K \to L$ of two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$, and suppose that ω is a valuation of L which is trivial on k and ν is the restriction of ω to K. If L/K has defect then ω must have rational rank 1 and be nondiscrete. We will assume that ω has rational rank 1 and is nondiscrete for the remainder of the introduction.

With these restrictions, the distance δ of an Artin–Schreier extension is $\leqslant 0^-$ when the extension has defect. The notation 0^- is defined in Subsection 2.2. If it is a defect extension with $\delta = 0^-$ then it is an independent defect extension. If it is a defect extension and the distance is less than 0^- then the extension is a dependent defect extension.

A quadratic transform along a valuation is the center of the valuation at the blow up of a maximal ideal of a regular local ring. There is the sequence of quadratic transforms along ν and ω

$$R \to R_1 \to R_2 \to \cdots \text{ and } S \to S_1 \to S_2 \to \cdots . \tag{1.2}$$

We have that $\bigcup_{i=1}^{\infty} R_i = \mathcal{O}_\nu$, the valuation ring of ν, and $\bigcup_{i=1}^{\infty} S_i = \mathcal{O}_\omega$, the valuation ring of ω. These sequences can be factored by standard quadratic transform sequences (defined in Section 3). It is shown in [8] that given positive integers r_0 and s_0, there exists $r \geqslant r_0$ and $s \geqslant s_0$ such that $R_r \to S_s$ has the following form:

$$u = \tau x^a, v = x^b(y^d\gamma + x\Omega) \tag{1.3}$$

where u, v are regular parameters in R_r, x, y are regular parameters in S_s, γ and τ are units in S_s, $\Omega \in S_s$, a and d are positive integers and b is a non negative integer. If we choose r_0 sufficiently large, then we have that the complexity ad of the extension $R_r \to S_s$ is a constant which depends on the extension of valuations, which we call the stable complexity of (1.2). When $R_r \to S_s$ has this stable complexity, we call the forms (1.3) stable forms.

The strongly monomial form is the case when $b = 0$ and $d = 1$; that is, after making a change of variables in y,

$$u = \tau x^a, v = y.$$

As we observed earlier (Theorem 1.2) if the extension $K \to L$ has no defect, then the stable form is the strongly monomial form. If there is defect, then it is possible for the a and d in stable forms along a valuation to vary wildly, even though their product ad is fixed by the extension, as shown in [6, Theorem 5.4].

An example is constructed in [8], showing failure of strong local monomialization. It is a tower of two defect Artin–Schreier extensions, each of the type of [6, Theorem 5.4] referred to above. The first extension is of type 1 for even integers and of type 2 for odd integers. The second extension is of type 2 for even integers and of type 1 for odd integers. Extensions of types 1 and 2 are defined in Section 3 before the statement of Theorem 3.2. The composite gives a sequence of extensions of regular local rings $R_i \to S_i$, where R_i has regular parameters u_i, v_i and S_i has regular parameters x_i, y_i such that the stable form is

$$u_i = \gamma x_i^p, v_i = y_i^p\tau + x_i\Omega \tag{1.4}$$

for all i. Both of these Artin–Schreier extensions are dependent. This is calculated in [10] and in [6, Section 6]. In keeping with the philosophy that independent Artin–Schreier extensions are better behaved than dependent ones, this leads to the question of if strong monomialization holds in towers of independent Artin–Schreier extensions. However, this is not true as is shown in Theorem 4.1 of this paper. In this theorem, we construct an example in a tower of two independent defect extensions such that strong local monomialization does not hold.

Suppose that $K \to L$ is a finite extension of fields of positive characteristic and ω is a valuation of L with restriction ν to K. It is known that there is no defect in the extension if and only if there is a finite generating sequence in L for the valuation ω over K ([16, 19]). The calculation of generating sequences for extensions of Noetherian local rings which are dominated by a valuation is extremely difficult. This has been accomplished for two dimensional regular local rings in [9] and [18] and for many hypersurface singularities above a regular local ring of arbitrary dimension in [7].

The nature of a generating sequence in an extension of S over R determines the nature of the mappings in the stable forms. It is shown in [5, Theorem 1] that if $R \to S$ is an extension of two dimensional excellent regular local rings whose quotient fields give a finite extension $K \to L$ and ω is a valuation of L which dominates S then the extension is without defect if and only if there exist sequences of quadratic transform $R \to R_1$ and $S \to S_1$ along ν such that ω has a finite generating sequence in S_1 over R_1. This shows us that we can expect good stable forms (as do hold by Theorem 1.2) if there is no defect, but not otherwise.

2. Preliminaries

2.1. Some notation

Let K be a field with a valuation ν. The valuation ring of ν will be donoted by $\mathcal{O}_\nu$, νK will denote the value group of ν and $K\nu$ will denote the residue field of $\mathcal{O}_\nu$.

The maximal ideal of a local ring A will be denoted by m_A. If $A \to B$ is an extension (inclusion) of local rings such that $m_B \cap A = m_A$ we will say that B dominates A. If a valuation ring $\mathcal{O}_\nu$ dominates A we will say that the valuation ν dominates A.

Suppose that K is an algebraic function field over a field k. An algebraic local ring A of K is a local domain which is a localization of a finite type k-algebra whose quotient field is K. A k-valuation of K is a valuation of K which is trivial on k.

Suppose that $K \to L$ is a finite algebraic extension of fields, ν is a valuation of K and ω is an extension of ν to L. Then the reduced ramification index of the extension is $e(\omega/\nu) = [\omega L : \nu K]$ and the residue degree of the extension is $f(\omega/\nu) = [L\omega : K\nu]$.

The defect $\delta(\omega/\nu)$, which is a power of the residue characteristic p of $\mathcal{O}_\omega$, is defined and its basic properties developed in [21, Chapter VI, Section 11], [12] and [8, Section 7.1]. In the case that L is Galois over K, we have the formula

$$[L : K] = e(\omega/\nu)f(\omega/\nu)\delta(\omega/\nu)g \tag{2.1}$$

where g is the number of extensions of ν to L. In fact, we have the equation (cf. [13] or [8, Section 7.1])

$$|G^s(\omega/\nu)| = e(\omega/\nu)f(\omega/\nu)\delta(\omega/\nu),$$

where $G^s(\omega/\nu)$ is the decomposition group of L/K.

If $K \to L$ is a finite Galois extension, then we will denote the Galois group of L/K by $\mathrm{Gal}(L/K)$.

2.2. Initial and final segments and cuts

We review some basic material about cuts in totally ordered sets from [13]. Let $(S, <)$ be a totally ordered set. An initial segment of S is a subset Λ of S such that if $\alpha \in \Lambda$ and $\beta < \alpha$ then $\beta \in \Lambda$. A final segment of S is a subset Λ of S such that if $\alpha \in \Lambda$ and $\beta > \alpha$ then $\beta \in \Lambda$. A cut in S is a pair of sets (Λ^L, Λ^R) such that Λ^L is an initial segment of S and Λ^R is a final segment of S satisfying $\Lambda^L \cup \Lambda^R = S$ and $\Lambda^L \cap \Lambda^R = \emptyset$. If Λ_1 and Λ_2 are two cuts in S, write $\Lambda_1 < \Lambda_2$ if $\Lambda_1^L \subsetneq \Lambda_2^L$. Suppose that $S \subset T$ is an order preserving inclusion of ordered sets and $\Lambda = (\Lambda^L, \Lambda^R)$ is a cut in S. Then define the cut induced by $\Lambda = (\Lambda^L, \Lambda^R)$ in T to be the cut $\Lambda \uparrow T = (\Lambda^L \uparrow T, T \setminus (\Lambda^L \uparrow T))$ where $\Lambda^L \uparrow T$ is the least initial segment of T in which Λ^L forms a cofinal subset.

We embed S in the set of all cuts of S by sending $s \in S$ to

$$s^+ = (\{t \in S \mid t \leqslant s\}, \{t \in S \mid t > s\}).$$

we may identify s with the cut s^+. Define

$$s^- = (\{t \in S \mid t < s\}, \{t \in S \mid t \geqslant s\}).$$

Given a cut $\Lambda = (\Lambda^L, \Lambda^R)$ of an ordered Abelian group S, we define $-\Lambda = (-\Lambda^R, -\Lambda^L)$ where $-\Lambda^L = \{-s \mid s \in \Lambda^L\}$ and $-\Lambda^R = \{-s \mid s \in \Lambda^R\}$. We have that if Λ_1 and Λ_2 are cuts, then $\Lambda_1 < \Lambda_2$ if and only if $-\Lambda_2 < -\Lambda_1$.

2.3. Distances

Let $K \to L$ be an extension of fields and ω be a valuation of L with restriction ν to K. Let $\widetilde{\nu K}$ be the divisible hull of νK. Suppose that $z \in L$. Then the distance of z from K is defined in [13, Section 2.3] to be the cut $\mathrm{dist}(z, K)$ of $\widetilde{\nu K}$ in which the initial segment of $\mathrm{dist}(z, K)$ is the least initial segment of $\widetilde{\nu K}$ in which $\omega(z - K)$ is cofinal. That is,

$$\mathrm{dist}(z, K) = (\Lambda^L(z, K), \Lambda^R(z, K)) \uparrow \widetilde{\nu K}$$

where

$$\Lambda^L(z, K) = \{\omega(z - c) \mid c \in K \text{ and } \omega(z - c) \in \nu K\}.$$

The following notion of equivalence is defined in [13, Section 2.3]. If $y, z \in L$, then $z \sim_K y$ if $\omega(z - y) > \mathrm{dist}(z, K)$.

2.4. Artin–Schreier extensions

Let $K \to L$ be an Artin–Schreier extension of fields of characteristic $p > 0$ and ω be a valuation of L with restriction ν to K. The field L is Galois over K with Galois group $G \cong \mathbb{Z}_p$, where p is the characteristic of K.

Let $\Theta \in L$ be an Artin–Schreier generator of K; that is, there is an expression

$$\Theta^p - \Theta = a$$

for some $a \in K$. We have that

$$\mathrm{Gal}(L/K) \cong \mathbb{Z}_p = \{\mathrm{id}, \sigma_1, \ldots, \sigma_{p-1}\},$$

where $\sigma_i(\Theta) = \Theta + i$.

Since L is Galois over K, we have that $ge(\omega/\nu)f(\omega/\nu)\delta(\omega/\nu) = p$ where g is the number of extensions of ν to L. So we either have that $g = 1$ or $g = p$. If $g = 1$, then ω is the unique extension of ν to L and either $e(\omega/\nu) = p$ and $\delta(\omega/\nu) = 1$ or $e(\omega/\nu) = 1$ and $\delta(\omega/\nu) = p$. In particular, the extension is defect if and only if is an immediate extension ($e = f = 1$) and ω is the unique extension of ν to L.

From now on in this subsection, suppose that L is a defect extension of K. By [13, Lemma 4.1], the distance $\delta = \mathrm{dist}(\Theta, K)$ does not depend on the choice of Artin–Schreier generator Θ, so δ can be called the distance of the Artin–Schreier extension. Since L/K is an immediate extension, the set $\omega(\Theta - K)$ is an initial segment in νK which has no maximal element by [13, Theorem 2.19].

We have, since the extension is defect, that

$$\delta = \mathrm{dist}(\Theta, K) \leqslant 0^- \tag{2.2}$$

by [13, Corollary 2.30].

A defect Artin–Schreier extension L is defined in [13, Section 4] to be a dependent defect Artin–Schreier extension if there exists an immediate purely inseparable extension $K(\eta)$ of K of degree p such that $\eta \sim_K \Theta$. Otherwise, L/K is defined to be an independent defect Artin–Schreier defect extension. We have by [13, Proposition 4.2] that for a defect Artin- Schreier extension,

$$L/K \text{ is independent} \quad \text{if and only if the distance } \delta = \mathrm{dist}(\Theta, K) \text{ satisfies } \delta = p\delta. \tag{2.3}$$

2.5. Extensions of rank 1 valuations in an Artin–Schreier extension

In this subsection, we suppose that L is an Artin–Schreier extension of a field K of characteristic p, ω is a rank 1 valuation of L and ν is the restriction of ω to K. We suppose that L is a defect extension of K. To simplify notation, we suppose that we have an embedding of νL in $\mathbb{R}$. Since L has defect over K and L is separable over K, νL is nondiscrete by the corollary on page 287 of [20], so that νL is dense in $\mathbb{R}$.

We define a cut in $\mathbb{R}$ by extending the cut $\mathrm{dist}(\Theta, K)$ in $\widetilde{\nu K}$ to a cut of $\mathbb{R}$ by taking the initial segment of the extended cut to be the least initial segment of $\mathbb{R}$ in which the cut $\mathrm{dist}(\Theta, K)$ is confinal. This cut is then $\mathrm{dist}(\Theta, K) \uparrow \mathbb{R}$. This cut is either s or s^- for some $s \in \mathbb{R}$. If L is a defect extension of K then $\mathrm{dist}(\Theta, K) \uparrow \mathbb{R} = s^-$ where s is a non positive real number by [13, Theorem 2.19] and [13, Corollary 2.30]. We will set $\mathrm{dist}(\omega/\nu)$ to be this real number s, so that

$$\mathrm{dist}(\Theta, K) \uparrow \mathbb{R} = s^- = (\mathrm{dist}(\omega/\nu))^-.$$

The real number $\mathrm{dist}(\omega/\nu)$ is well defined since it is independent of choice of Artin–Schreier generator of L/K by Lemma 4.1 [13].

With the assumptions of this subsection, by (2.2) and (2.3), the distance $\delta = \mathrm{dist}(\Theta, K)$ of an Artin–Schreier extension is $\leqslant 0^-$ when the extension has defect. If it is a defect extension with distance equal to 0^- then it is an independent defect extension. If it is a defect extension and the distance is less than 0^- then the extension is a dependent defect extension. Thus if L/K is a defect extension, we have that $\mathrm{dist}(\omega/\nu) \leqslant 0$ and the defect extension L/K is independent if and only if $\mathrm{dist}(\omega/\nu) = 0$.

3. Calculations in two dimensional Artin–Schreier Extensions

Suppose that M is a two dimensional algebraic function field over an algebraically closed field k of characteristic $p > 0$ and μ is a nondiscrete rational rank 1 valuation of M. Suppose that A is an algebraic regular local ring of M such that μ dominates A. A quadratic transform of A is an extension $A \to A_1$ where A_1 is a local ring of the blowup of the maximal ideal of A such that A_1 dominates A and A_1 has dimension two. A quadratic transform $A \to A_1$ is said to be along the valuation μ if μ dominates A_1.

Let

$$A = A_0 \to A_1 \to A_2 \to \cdots$$

be the sequence of quadratic transforms along μ. Then the valuation ring $\mathcal{O}_\mu = \bigcup A_i$ (by [1, Lemma 12]).

Suppose that $K \to L$ is a finite extension of two dimensional algebraic function fields, R is an algebraic regular local ring of K which is dominated by a regular algebraic local ring S of L such that $\dim R = \dim S = 2$. Let x, y be regular parameters in S and u, v be regular parameters in R. Then we can form the Jacobian ideal

$$J(S/R) = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right).$$

This ideal is independent of choice of regular parameters.

The following proposition is proven in [17].

Proposition 3.1. — *Suppose that $K \to L$ is an Artin–Schreier extension of two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$, ω is a rational rank 1 nondiscrete valuation of L with restriction $\nu = \omega|K$. Further suppose that A is an algebraic local ring of K and B is an algebraic local ring of L which is dominated by ω such that B dominates A. Then there exists a commutative diagram of homomorphisms*

$$\begin{array}{ccc} R & \longrightarrow & A \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

such that R is a regular algebraic local ring of K with regular parameters u, v, S is a regular algebraic local ring of L with regular parameters x, y such that S is dominated by ω, S dominates R, $R \to S$ is quasi finite, $J(S/R) = (x^{\bar{c}})$ for some non negative integer $\bar{c}$ and one of the following three cases holds:

(0) $u = x$, $v = y$ *($R \to S$ is unramified).*
(1) $u = x$, $v = y^p\gamma + x\Sigma$ *where γ is a unit in S and $\Sigma \in S$.*
(2) $u = \gamma x^p$, $v = y$ *where γ is a unit in S.*

Let $K \to L$ be an Artin–Schreier extension of two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$. Let $R \to S$ be an extension from a regular algebraic local ring of K to a regular algebraic local ring of L such that S dominates R.

Let u, v be regular parameters in R and x, y be regular parameters in S. We will say that $R \to S$ is of type 0 with respect to these parameters if

$$\text{Type 0:}\quad u = \gamma x, v = y\tau + x\Omega$$

where γ, τ are units in S and $\Omega \in S$, so that $R \to S$ is unramified. We will say that $R \to S$ is of type 1 with respect to these parameters if

$$\text{Type 1:}\quad u = \gamma x, v = y^p\tau + x\Omega$$

where γ, τ are units in S and $\Omega \in S$. We will say that $R \to S$ is of type 2 with respect to these parameters if

$$\text{Type 2:}\quad u = \gamma x^p, v = y\tau + x\Omega$$

where γ, τ are units in S and $\Omega \in S$.

These definitions are such that if one of these types hold, and $\bar{u}, \bar{v}$ are regular parameters in R, $\bar{x}, \bar{y}$ are regular parameters in S such that $\bar{u}$ is a unit in R times u and $\bar{x}$ is a unit in S times x then $R \to S$ is of the same type for the new parameters $\bar{u}, \bar{v}$ and $\bar{x}, \bar{y}$.

In the construction of our example (Theorem 4.1), we will make use of some results from [6].

Theorem 3.2 ([6, Theorem 4.1]). — *Suppose that $R \to S$ is of type 1 with respect to regular parameters x, y in S and u, v in R and that $J(S/R) = (x^{\bar{c}})$. Let $\bar{x} = u$, $\bar{y} = y - g(\bar{x})$ where $g(\bar{x}) \in k[\bar{x}]$ is a polynomial with zero constant term, so that $\bar{x}, \bar{y}$ are regular parameters in S. Suppose that m, q are positive integers with $m > 1$ and $\gcd(m, q) = 1$. Let α be a nonzero element of k. Consider the sequence of quadratic transforms $S \to S_1$ so that S_1 has regular parameters x_1, y_1 defined by*

$$\bar{x} = x_1^m (y_1 + \alpha)^{a'}, \bar{y} = x_1^q (y_1 + \alpha)^{b'}$$

where $a', b' \in \mathbb{N}$ are such that $mb' - qa' = 1$.

Computing the Jacobian determinate $J(S/R)$, we see that

$$u = \bar{x}, v = \bar{y}^p\gamma + \bar{x}^{\bar{c}}\bar{y}\tau + f(\bar{x})$$

where γ, τ are unit series in $\widehat{S}$ and $f(\bar{x}) = \sum e_i \bar{x}^i \in k[[\bar{x}]]$. Make the change of variables $\bar{v} = v - \sum e_i u^i$ where the sum is over i such that $i \leqslant \frac{pq}{m}$ so that $u, \bar{v}$ are regular parameters in R.

We have that $R \to S$ is of type 1 with respect to the regular parameters $\overline{x}, \overline{y}$ and u, v. Let $\sigma = \gcd(m, pq)$ which is 1 or p.

There exists a unique sequence of quadratic transforms $R \to R_1$ such that R_1 has regular parameters u_1, v_1 defined by

$$u = u_1^{\overline{m}}(v_1+\beta)^{c'}, \overline{v} = u_1^{\overline{q}}(v_1+\beta)^{d'}$$

with $0 \neq \beta \in k$ giving a commutative diagram of homomorphisms

$$\begin{array}{ccc} R_1 & \longrightarrow & S_1 \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

such that $R_1 \to S_1$ is quasi finite. We have that $J(S_1/R_1) = (x_1^{c_1})$ for some positive integer c_1 and $R_1 \to S_1$ is quasi finite. Further:

(0) *If $\frac{q}{m} \geqslant \frac{\bar{c}}{p-1}$ then $R_1 \to S_1$ is of type 0.*
(1) *If $\frac{q}{m} < \frac{\bar{c}}{p-1}$ and $\sigma = 1$ then $R_1 \to S_1$ is of type 1 and*

$$\left(\frac{c_1}{p-1}\right) = \left(\frac{\bar{c}}{p-1}\right)m - q.$$

(2) *If $\frac{q}{m} < \frac{\bar{c}}{p-1}$ and $\sigma = p$ then $R_1 \to S_1$ is of type 2 and*

$$\left(\frac{c_1}{p-1}\right) = \left(\frac{\bar{c}}{p-1}\right)m - q + 1.$$

In cases (1) and (2), $m = \sigma\overline{m}$, $pq = \sigma\overline{q}$ and $\overline{m}c' - \overline{q}d' = 1$.

Theorem 3.3 ([6, Theorem 4.3]). — *Suppose that $R \to S$ is of type 2 with respect to regular parameters x, y in S and u, v in R and that $J(S/R) = (x^{\bar{c}})$. Let $g(u) \in k[u]$ be a polynomial with no constant term. Make the change of variables, letting $\overline{v} = v - g(u)$ and $\overline{y} = \overline{v}$, so that $x, \overline{y}$ are regular parameters in S and $u, \overline{v}$ are regular parameters in R.*

Suppose that m, q are positive integers with $\gcd(m, q) = 1$. Let α be a nonzero element of k. Consider the sequence of quadratic transforms $S \to S_1$ so that S_1 has regular parameters x_1, y_1 defined by

$$x = x_1^{m}(y_1+\alpha)^{a'}, \overline{y} = x_1^{q}(y_1+\alpha)^{b'}$$

where $a', b' \in \mathbb{N}$ are such that $mb' - qa' = 1$.

Let $\sigma = \gcd(pm, q)$ which is 1 or p. There exists a unique sequence of quadratic transforms $R \to R_1$ such that R_1 has regular parameters u_1, v_1 defined by

$$u = u_1^{\overline{m}}(v_1+\beta)^{c'}, \overline{v} = u_1^{\overline{q}}(v_1+\beta)^{d'}$$

where $pm = \sigma\overline{m}$, $q = \sigma\overline{q}$, $\overline{m}d' - c'\overline{q} = 1$ *and* $0 \neq \beta \in k$, *giving a commutative diagram of homomorphisms*

$$\begin{array}{ccc} R_1 & \longrightarrow & S_1 \\ \uparrow & & \uparrow \\ R & \longrightarrow & S \end{array}$$

such that $R_1 \to S_1$ *is quasi finite. We have that* $J(S_1/R_1) = (x_1^{c_1})$ *for some positive integer* c_1. *Further:*

(1) *If* $\sigma = 1$ *then* $R_1 \to S_1$ *is of type 1 and*

$$\left(\frac{c_1}{p-1}\right) = \left(\frac{\bar{c}}{p-1}\right) m - m.$$

(2) *If* $\sigma = p$ *then* $R_1 \to S_1$ *is of type 2 and*

$$\left(\frac{c_1}{p-1}\right) = \left(\frac{\bar{c}}{p-1}\right) m - m + 1.$$

A proof of the following proposition is given in [6, Proposition 7.9]. More general results are proven in [15].

Proposition 3.4 (Kuhlmann and Piltant, [14]). — *Suppose that* K *and* L *are two dimensional algebraic function fields over an algebraically closed field* k *of characteristic* $p > 0$ *and* $K \to L$ *is an Artin–Schreier extension. Let* ω *be a rational rank one nondiscrete valuation of* L *and let* ν *be the restriction of* ω *to* K. *Suppose that* L *is a defect extension of* K.

Suppose that R *is a regular algebraic local ring of* K *and* S *is a regular algebraic local ring of* L *such that* ω *dominates* S, S *dominates* R *and* $R \to S$ *is of type 1 or 2. Inductively applying Theorems 3.2 and 3.3, we construct a diagram where the horizontal sequences are birational extensions of regular local rings*

$$\begin{array}{ccccccc} S = S_0 & \longrightarrow & S_1 & \longrightarrow & S_2 & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ R = R_0 & \longrightarrow & R_1 & \longrightarrow & R_2 & \longrightarrow & \cdots \end{array} \tag{3.1}$$

with $\bigcup_{i=1}^{\infty} S_i = \mathcal{O}_\omega$. *Further assume that for each map* $R_i \to S_i$, *there are regular parameters* u, v *in* R_i *and* x, y *in* S_i *such that one of the following forms hold:*

$$u = x, v = f \tag{3.2}$$

where $\dim_k S_i/(x, f) = p$, *or*

$$u = \delta x^p, v = y \tag{3.3}$$

where δ is a unit in S_i and in both cases that $x = 0$ is a local equation of the critical locus of $\operatorname{Spec}(S_i) \to \operatorname{Spec}(R_i)$. *Let*

$$J_i = J(S_i/R_i) = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)$$

be the Jacobian ideal of the map $R_i \to S_i$.

Then the distance $\operatorname{dist}(\omega/\nu)$ *is computed by the formula*

$$-\operatorname{dist}(\omega/\nu) = \frac{1}{p-1}\inf_i\{\omega(J(S_i/R_i))\}$$

where the infimum is over the $R_i \to S_i$ in the sequence (3.1).

4. An example of a tower of independent defect extensions in which strong local monomialization doesn't hold

THEOREM 4.1. — *There exists a tower $(K,\nu) \to (L,\omega) \to (M,\mu)$ of independent defect Artin–Schreier extensions of valued two dimensional algebraic function fields over an algebraically closed field k of characteristic $p > 0$ such that there exist algebraic regular local rings A of K and C of M such that μ dominates C and C dominates A but strong local monomialization along μ does not hold above $A \to C$.*

Remark 4.2. — Let $\delta \in \mathbb{R}_{\geqslant 0}$ be a fixed ratio. Suppose that $R \to S$ is of type 1. By taking m and q sufficiently large in Theorem 3.2 such that $R_1 \to S_1$ is of type 2, we can achieve that $v_1 = \lambda y_1 + g(x_1)$ where λ is a unit in S_1 and the order of $g(x_1)$ is arbitrarily large. Suppose that $R \to S$ is of type 2. By taking m and q sufficiently large in Theorem 3.3 such that $R_1 \to S_1$ is of type 1 we can achieve that $v_1 = y_1^p\gamma + x_1^{c_1}y_1\tau + f(x_1)$ where γ and τ are unit series in S_1 and the order of $f(x_1)$ is arbitrarily large. In both cases, we can choose m and q so that $\frac{q}{m}$ is arbitrarily close to δ.

Remark 4.3. — In Theorem 3.3, we have an expression $\overline{v} = \tau y + f(x)$ where τ is a unit in S. Suppose that m and q are positive integers with $\gcd(m,q) = 1$ and such that $\operatorname{ord} f(x) > \frac{q}{m}$. Then the proof of Theorem 3.3 extends to show that the conclusions of Theorem 3.3 hold with $\overline{y}$ replaced with y.

We now give the proof of Theorem 4.1.

Proof. — Let K be a two dimensional algebraic function field over an algebraically closed field, and let R_{-2} be a two dimensional algebraic regular local ring of K. Let u_{-2}, v_{-2} be regular parameters in R_{-2}.

Let e be a positive integer. Let $c_{-2}=(p-1)e$. Let Θ be a root of the Artin–Schreier polynomial $X^p-X-v_{-2}u_{-2}^{-pe}$. Let $L=K(\Theta)$. Set $x_{-2}=u_{-2}$, $y_{-2}=u_{-2}^e\Theta$. Let $S_{-2}=R_{-2}[y_{-2}]_{(x_{-2},y_{-2})}$, which is an algebraic regular local ring of L which dominates R_{-2}. The regular parameters x_{-2},y_{-2} in S_{-2} satisfy $u_{-2}=x_{-2},v_{-2}=y_{-2}^p-x_{-2}^{e(p-1)}y_{-2}$, so that the extension $R_{-2}\to S_{-2}$ is of type 1. We have that $J(S_{-2}/R_{-2})=(x_{-2}^{c_{-2}})$, with $\frac{c_{-2}}{p-1}>0$.

We first construct a commutative diagram

$$\begin{array}{ccc} S_{-2} & \longrightarrow & S_{-1} \\ \uparrow & & \uparrow \\ R_{-2} & \longrightarrow & R_{-1} \end{array}$$

using Theorem 3.2 so that $R_{-1}\to S_{-1}$ is of type 2. Let Σ be a root of the Artin–Schreier polynomial $X^p-X-y_{-1}x_{-1}^{-pe}$. Let $M=L(\Sigma)$. Set $z_{-1}=x_{-1}$, $w_{-1}=x_{-1}^e\Sigma$. Let $T_{-1}=S_{-1}[w_{-1}]_{(z_{-1},w_{-1})}$, which is an algebraic regular local ring of M which dominates S_{-1}. The regular parameters z_{-1},w_{-1} in T_{-1} satisfy $x_{-1}=z_{-1},y_{-1}=w_{-1}^p-z_{-1}^{e(p-1)}w_{-1}$, so that the extension $S_{-1}\to T_{-1}$ is of type 1. We have that $J(T_{-1}/S_{-1})=(z_{-1}^{c'_{-1}})$, with $\frac{c'_{-1}}{p-1}>0$.

From Theorems 3.2 and 3.3, we construct

$$\begin{array}{ccc} T_{-1} & \longrightarrow & T_0 \\ \uparrow & & \uparrow \\ S_{-1} & \longrightarrow & S_0 \\ \uparrow & & \uparrow \\ R_{-1} & \longrightarrow & R_0 \end{array}$$

such that $R_0\to S_0$ is of type 1 and $S_0\to T_0$ is of type 2. Explicitly, $R_{-1},R_0,S_{-1},S_0,T_{-1},T_0$ have respective regular parameters (u_{-1},v_{-1}), (u_0,v_0), (x_{-1},y_{-1}), (x_0,y_0) and (z_{-1},w_{-1}), (z_0,w_0) which are related by equations

$$\begin{aligned} u_{-1}&=u_0^{pm_0}(v_0+\beta_0)^{d'_0}, \quad & v_{-1}&=u_0^{q_0}(v_0+\beta_0)^{e'_0}\\ x_{-1}&=x_0^{m_0}(y_0+\alpha_0)^{a'_0}, \quad & y_{-1}&=x_0^{q_0}(w_0+\alpha_0)^{g'_0}\\ z_{-1}&=z_0^{pm_0}(v_0+\gamma_0)^{f'_0}, \quad & w_{-1}&=z_0^{q_0}(w_0+\gamma_0)^{g'_0} \end{aligned}$$

where $p\not|q_0$ and $\frac{q_0}{pm_0}<\frac{c'_{-1}}{p-1}$ where $J(T_{-1}/S_{-1})=(z_{-1}^{c'_{-1}})$.

By Remarks 4.2 and 4.3, we can construct $R_0\to S_0\to T_0$ so that we have expressions $y_0=\lambda_0w_0+g_0(z_0)$ where λ_0 is a unit in T_0 and $\operatorname{ord}g_0(z_0)$ is arbitrarily large and

$$v_0=\sigma_0y_0^p+\tau_0x_0^{c_0}y_0+f_0(x_0)$$

where σ_0, τ_0 are units in S_0 and $\operatorname{ord} f_0(x_0)$ is arbitrarily large.

We will inductively construct a commutative diagram within $K \to L \to M$ of two dimensional regular algebraic local rings

$$\begin{array}{ccccccc} T_0 & \longrightarrow & T_1 & \longrightarrow & T_2 & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ S_0 & \longrightarrow & S_1 & \longrightarrow & S_2 & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ R_0 & \longrightarrow & R_1 & \longrightarrow & R_2 & \longrightarrow & \cdots \end{array} \tag{4.1}$$

such that $R_i \to S_i$ is of type 1 if i is even and is of type 2 if i is odd, $S_i \to T_i$ is of type 2 if i is even and is of type 1 if i is odd. Further, valuations ν, ω and μ of the respective function fields K, L and M determined by these sequences are such that $K \to L$ and $L \to M$ are independent defect extensions. We will have that R_i has regular parameters (u_i, v_i), S_i has regular parameters (x_i, y_i) and T_i has regular parameters (z_i, w_i) such that

$$\begin{aligned} u_i &= u_{i+1}^{\overline{m}_{i+1}}(v_{i+1}+\beta_{i+1})^{d'_{i+1}}, & v_i &= u_{i+1}^{\bar{q}_{i+1}}(v_{i+1}+\beta_{i+1})^{e'_{i+1}}, \\ x_i &= x_{i+1}^{m_{i+1}}(y_{i+1}+\alpha_{i+1})^{a'_{i+1}}, & y_i &= x_{i+1}^{q_{i+1}}(y_{i+1}+\alpha_{i+1})^{b'_{i+1}}, \\ z_i &= z_{i+1}^{m'_{i+1}}(w_{i+1}+\gamma_{i+1})^{f'_{i+1}}, & w_i &= z_{i+1}^{q'_{i+1}}(w_i+\gamma_{i+1})^{g'_{i+1}} \end{aligned}$$

with $\overline{m}_i, m_i$ and m'_i larger than 1 for all i.

Let $J(S_i/R_i) = (x_i^{c_i})$ and $J(T_i/S_i) = (z_i^{c'_i})$.

If i is even, then $m_{i+1} = p\overline{m}_{i+1}$, $m'_{i+1} = \overline{m}_{i+1}$, $q_{i+1} = \bar{q}_{i+1}$, $q'_{i+1} = q_{i+1}$ and

$$\frac{q_{i+1}}{m_{i+1}} < \frac{c_i}{p-1}.$$

If i is odd, then $\overline{m}_{i+1} = pm_{i+1}$, $m'_{i+1} = \overline{m}_{i+1}$, $q_{i+1} = \bar{q}_{i+1}$, $q'_{i+1} = q_{i+1}$ and

$$\frac{q'_{i+1}}{m'_{i+1}} < \frac{c'_i}{p-1}.$$

In our construction, if r is even, we will have that

$$y_r = \lambda_r w_r + g_r(z_r) \tag{4.2}$$

where λ_r is a unit in T_r and $\operatorname{ord} g_r(z_r)$ is arbitrarily large and

$$v_r = \sigma_r y_r^p + \tau_r x_r^{c_r} y_r + f_r(x_r) \tag{4.3}$$

where σ_r, τ_r are units in S_r and $\operatorname{ord} f_r(x_r)$ is arbitrarily large. If r is even, we will have

$$y_r = \sigma_r w_r^p + \tau_r z_r^{c'_r} w_r + f(z_r) \tag{4.4}$$

where σ_r, τ_r are units in T_r and $\operatorname{ord} f(z_r)$ is arbitrarily large and

$$v_r = \lambda_r y_r + g_r(x_r) \tag{4.5}$$

where λ_r is a unit in S_r and $\operatorname{ord} g_r(x_r)$ is arbitrarily large.

Suppose that r is even, and we have constructed $R_r \to S_r \to T_r$. We will construct

$$\begin{CD}
T_r @>>> T_{r+1} @>>> T_{r+2}\\
@AAA @AAA @AAA\\
S_r @>>> S_{r+1} @>>> S_{r+2}\\
@AAA @AAA @AAA\\
R_r @>>> R_{r+1} @>>> R_{r+2}
\end{CD}$$

There exists an integer $\lambda(r+1) > 1$ and $q_{r+1} \in \mathbb{Z}_+$ such that $\gcd(q_{r+1}, p) = 1$ and

$$\frac{c_r}{p-1} > \frac{q_{r+1}}{p^{\lambda(r+1)}} > \frac{c_r}{p-1} - \frac{1}{2^{r+1}} m_1 \cdots m_r. \tag{4.6}$$

In fact, we can find $\lambda(r+1)$ arbitrarily large satisfying the inequality. Set $m_{r+1} = p^{\lambda(r+1)}$. We have that $\frac{q_{r+1}}{m_{r+1}} < \frac{c_r}{p-1}$ with $\gcd(m_{r+1}, pq_{r+1}) = p$. This choice of m_{r+1} and q_{r+1} (along with a choice of $0 \neq \alpha_{r+1} \in k$) determines $S_r \to S_{r+1}$. We have an expression $v_r = \sigma_r y_r^p + \tau_r x_r^{c_r} y_r + f_r(x_r)$ where $\operatorname{ord} f_r(x_r)$ is arbitrarily large. In particular, we can assume that $\operatorname{ord} f_r(x_r) > \frac{pq_{r+1}}{m_{r+1}}$. Then $R_r \to R_{r+1}$ is defined as desired by Theorem 3.2. By Remark 4.2, since we can take $\lambda(r+1)$ to be arbitrarily large, we can assume that $v_{r+1} = \lambda_{r+1} y_{r+1} + g_{r+1}(x_{r+1})$ where $\operatorname{ord} g_{r+1}(x_{r+1})$ is arbitrarily large.

By Remark 4.3 and Theorem 3.3, $T_r \to T_{r+1}$ is defined as desired, with $m'_{r+1} = \frac{m_{r+1}}{p}$, $q'_{r+1} = q_{r+1}$. Since we can take $\lambda(r+1)$ to be arbitrarily large, we can assume that $y_{r+1} = \sigma_{r+1} w_{r+1}^p + \tau_{r+1} z_{r+1}^{c'_{r+1}} w_r + f_{r+1}(z_{r+1})$ where $\operatorname{ord} f_{r+1}(z_{r+1})$ is arbitrarily large.

We have defined a commutative diagram

$$\begin{CD}
T_r @>>> T_{r+1}\\
@AAA @AAA\\
S_r @>>> S_{r+1}\\
@AAA @AAA\\
R_r @>>> R_{r+1}
\end{CD} \tag{4.7}$$

with the desired properties; in particular, $R_{r+1} \to S_{r+1}$ is of type 2 with

$$\frac{c_{r+1}}{p-1} = \left(\frac{c_r}{p-1}\right) m_{r+1} - q_{r+1} + 1$$

and $S_{r+1} \to T_{r+1}$ is of type 1, with

$$\frac{c'_{r+1}}{p-1} = \frac{c'_r}{p-1} m'_{r+1} - m'_{r+1}.$$

Now choose q'_{r+2}, $m'_{r+2} = p^{\lambda(r+2)}$ such that $p \nmid q'_{r+2}$ and

$$\frac{c'_{r+1}}{p-1} > \frac{q'_{r+2}}{m'_{r+2}} > \frac{c'_{r+1}}{p-1} - \frac{1}{2^{r+2}} m'_1 \cdots m'_{r+1}. \tag{4.8}$$

We can take $\lambda(r+2)$ arbitrarily large. Set $m_{r+2} = \frac{m'_{r+2}}{p} = p^{\lambda(r+2)-1}$, $q_{r+2} = q'_{r+2}$. By (4.8), $\frac{q'_{r+2}}{m'_{r+2}} < \frac{c'_{r+1}}{p-1}$.

Now construct, as in the construction of (4.7), using Theorems 3.2 and 3.3 and Remark 4.3 and these values of m_{r+2} and q_{r+2},

$$\begin{array}{ccc} T_{r+1} & \longrightarrow & T_{r+2} \\ \uparrow & & \uparrow \\ S_{r+1} & \longrightarrow & S_{r+2} \\ \uparrow & & \uparrow \\ R_{r+1} & \longrightarrow & R_{r+2} \end{array}$$

so that $R_{r+2} \to S_{r+2}$ is of type 1 and $S_{r+2} \to T_{r+2}$ is of type 2. By Remark 4.2, we obtain expressions (4.2) and (4.3) for $r+2$.

By induction, we construct the diagram (4.1).

Let $A = R_0$ and $C = T_0$. We will show that strong local monomialization doesn't hold above $A \to C$ along μ. Suppose that $R' \to T'$ has a strongly monomial form above $A \to C$. Then R' has regular parameters u', v' and T' has regular parameters z', w' such that $u' = \lambda(z')^m$ and $v' = w'$ where $m \in \mathbb{Z}_{>0}$ and λ is a unit in T'. We will show that this cannot occur. There exists a commutative diagram

$$\begin{array}{ccccc} T_s & \longrightarrow & T' & \longrightarrow & T_{s+1} \\ \uparrow & & \uparrow & & \uparrow \\ R_s & \longrightarrow & R' & \longrightarrow & R_{s+1} \end{array}$$

for some s. The ring T' has regular parameters $\overline{z}, \overline{w}$ such that

$$z_s = \overline{z}^a \overline{w}^b, w_s = \overline{z}^c \overline{w}^d \tag{4.9}$$

for some $a, b, c, d \in \mathbb{N}$ with $ad - bc = \pm 1$, and R' has regular parameters $\overline{u}, \overline{v}$ such that $u_s = \overline{u}^{\overline{a}} \overline{v}^{\overline{b}}$, $v_s = \overline{u}^{\overline{c}} \overline{v}^{\overline{d}}$, where $\overline{a}\overline{d} - \overline{b}\overline{c} = \pm 1$. We have an expression

$$u_s = \alpha z_s^p, v_s = \beta w_s^p + \Omega \tag{4.10}$$

where α, β are units in T_s and where

$$\Omega = \varepsilon z_s^{pc_s} w_s + M \tag{4.11}$$

or

$$\Omega = \varepsilon z_s^{c'_s} w_s + M \tag{4.12}$$

where $\varepsilon \in T_s$ is a unit and M is a sum of monomials in z_s, w_s of high order in z_s. Further, $\mu(w_s^p) < \mu(z_s^{pc_s} w_s)$ in (4.11) and $\mu(w_s^p) < \mu(z_s^{c'_s} w_s)$ in (4.12).

In particular, $R_s \to T_s$ is not a strongly monomial form.

Substituting (4.9) into u_s and v_s in (4.10), we have

$$u_s = \alpha \overline{z}^{ap} \overline{w}^{bp}, v_s = \beta \overline{z}^{cp} \overline{w}^{dp} + \Omega. \tag{4.13}$$

We necessarily have that $u_s | v_s$ or $v_s | u_s$ in T'.

First suppose that $c \geqslant a$ and $d \geqslant b$. Then we have that

$$u_s = \alpha \overline{z}^{ap} \overline{w}^{bp}, \frac{v_s}{u_s} = \beta \overline{z}^{(c-a)p} \overline{w}^{(d-b)p} + \frac{\Omega}{\alpha \overline{z}^{ap} \overline{w}^{bp}}$$

giving an expression of the form (4.13). We will show that this is not a strongly monomial form. If it is, then we must have that $a = 0$ or $b = 0$ so that either

$$z_s = \overline{w}, w_s = \overline{z} \overline{w}^d \tag{4.14}$$

or

$$z_s = \overline{z}, w_s = \overline{z}^c \overline{w} \tag{4.15}$$

and we must have that $\frac{\Omega}{u_s}$ is part of a regular system of parameters in T'. Substituting into (4.11) or (4.12), we see that this cannot occur except possibly in the case that (4.12) holds and $\frac{z_s^{c'_s} w_s}{u_s}$ is part of a regular system of parameters in T'.

Suppose that (4.12) and (4.14) hold with

$$\frac{z_s^{c'_s} w_s}{u_s} = \frac{\overline{w}^{c'_s + d} \overline{z}}{\alpha \overline{w}^p}$$

being part of a regular system of parameters in T'. Now in this case, $\mu(w_s) > \mu(z_s)$ and $\mu(w_s^p) < \mu(z_s^{c'_s} w_s)$ so $p \leqslant c'_s$. Thus $\frac{\overline{w}^{c'_s + d} \overline{z}}{\alpha \overline{w}^p}$ cannot be part of a regular system of parameters in T'. A similar argument shows that we do not obtain a strongly monomial form when (4.12) and (4.15) hold.

Suppose that $c < a$ and $d < b$. Then we have expressions

$$v_s = \gamma \overline{z}^{cp} \overline{w}^{dp}, \frac{u_s}{v_s} = \alpha \gamma^{-1} \overline{z}^{(a-c)p} \overline{w}^{(b-d)p}$$

where $\gamma \in T'$ is a unit, giving an expression of the form of (4.13), which is not strongly monomial. Thus we reduce to the case where $(c-a)(d-b) < 0$.

We then have that $u_s \not| v_s$ since $u_s \not| \overline{z}^{cp}\overline{w}^{dp}$. Suppose that $v_s|u_s$. Then $v_s = \lambda\overline{z}^{cp}\overline{w}^{dp}$ where λ is a unit in T'. But this is impossible since $(c-a)(d-b) < 0$. Thus $R' \to T'$ has a form (4.13) with $a, b, c, d > 0$ and so cannot be a strongly monomial form. We have established that strong local monomialization along μ does not hold above $A \to C$.

It remains to show that L/K and M/L have independent defect.

From Theorem 3.2, we have that

$$\left(\frac{c_{r+1}}{p-1}\right)\frac{1}{m_1\cdots m_{r+1}} = \left(\frac{c_r}{p-1}\right)\frac{1}{m_1\cdots m_r} - \frac{q_{r+1}}{m_{r+1}}\left(\frac{1}{m_1\cdots m_r}\right) + \frac{1}{m_1\cdots m_{r+1}}. \tag{4.16}$$

Then from Theorem 3.3, we have that

$$\frac{c_{r+2}}{p-1} = \left(\frac{c_{r+1}}{p-1}\right)m_{r+2} - m_{r+2},$$

and so

$$\left(\frac{c_{r+2}}{p-1}\right)\frac{1}{m_1\cdots m_{r+2}} = \left(\frac{c_r}{p-1}\right)\frac{1}{m_1\cdots m_r} - \frac{q_{r+1}}{m_1\cdots m_{r+1}}. \tag{4.17}$$

By equation (4.6) we have

$$\frac{1}{2^{r+1}} > \left(\frac{c_r}{p-1}\right)\frac{1}{m_1\cdots m_r} - \left(\frac{q_{r+1}}{m_{r+1}}\right)\frac{1}{m_1\cdots m_r} > 0. \tag{4.18}$$

By Theorem 3.2,

$$\left(\frac{c'_{r+2}}{p-1}\right)\frac{1}{m'_1\cdots m'_{r+2}} = \left(\frac{c'_{r+1}}{p-1}\right)\frac{1}{m'_1\cdots m'_{r+1}} - \frac{q'_{r+2}}{m'_1\cdots m'_{r+2}} + \frac{1}{m'_1\cdots m'_{r+2}}$$

and by Theorem 3.3,

$$\frac{c'_{r+3}}{p-1} = \left(\frac{c'_{r+2}}{p-1}\right)m'_{r+3} - m'_{r+3}.$$

We thus have that

$$\left(\frac{c'_{r+3}}{p-1}\right)\frac{1}{m'_1\cdots m'_{r+3}} = \left(\frac{c'_{r+1}}{p-1}\right)\frac{1}{m'_1\cdots m'_{r+1}} - \frac{q'_{r+2}}{m'_1\cdots m'_{r+2}}. \tag{4.19}$$

Equation (4.8) implies

$$\frac{1}{2^{r+2}} > \left(\frac{c'_{r+1}}{p-1}\right)\frac{1}{m'_1\cdots m'_{r+1}} - \frac{q'_{r+2}}{m'_1\cdots m'_{r+2}} > 0. \tag{4.20}$$

Now $J(S_i/R_i) = (x_i^{c_i})$ and $x_0 = x_i^{m_1\cdots m_i}$ so $\omega(J(S_i/R_i)) = \frac{c_i}{m_1\cdots m_i}\omega(x_0)$ and thus by Proposition 3.4, (4.17) and (4.18), we have that

$$-\operatorname{dist}(\omega/\nu) = \frac{1}{p-1}\inf_i\{\omega(J(S_i/R_i))\} = 0.$$

Thus L/K has independent defect.

We have that $J(T_i/S_i) = (z_i^{c_i'})$ and $z_0 = z_i^{m_1'\cdots m_i'}$ so $\omega(J(T_i/S_i)) = \frac{c_i'}{m_1'\cdots m_i'}\omega(z_0)$ and thus by Proposition 3.4, (4.19) and (4.20), we have that

$$-\operatorname{dist}(\mu/\omega) = \frac{1}{p-1}\inf_i\{\omega(J(T_i/S_i))\} = 0.$$

Thus M/L has independent defect. □

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