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Extensions and restrictions of holomorphic foliations [(*)]

Mateus Gomes Figueira [(1)]

Abstract. — We prove an extension criterion for codimension one foliations on projective hypersurfaces based on the degree of the foliation and the degree of the hypersurface, and we ensure, in some instances, an isomorphism between the corresponding spaces of foliations. We also present some examples of foliations that do not satisfy the extension criterion and do not extend.

Résumé. — Nous prouvons un critère d'extension pour les feuilletages de codimension un sur les hypersurfaces projectives, basé sur le degré du feuilletage et sur le degré de l'hypersurface, et nous assurons, dans certains cas, un isomorphisme entre les espaces de feuilletages correspondants. Nous présentons également quelques exemples de feuilletages qui ne satisfont pas le critère d'extension et ne s'étendent pas.

1. Introduction

Let $\mathcal{F}$ be a codimension one singular foliation on a smooth hypersurface X of $\mathbb{P}^n$, $n > 2$. We say that a foliation $\mathcal{G}$ on $\mathbb{P}^n$ is an *extension* to $\mathcal{F}$ if its restriction to X is $\mathcal{F}$. D. Cerveau in [5] proposed the investigation of necessary conditions to guarantee the existence of extensions of foliations on projective hypersurfaces. He further asks whether the unconditional existence of extensions for foliations on a hypersurface characterizes hyperplanes. In this work, we will show the following criterion for the existence of extensions of foliations on smooth projective hypersurfaces.

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Theorem A. — *Let X be a smooth hypersurface in $\mathbb{P}^n$, $n > 3$, and let $\mathcal{F}$ be a codimension one holomorphic foliation on X. If $\deg(X) > 2\deg(\mathcal{F}) + 1$ then $\mathcal{F}$ extends.*

We also establish an analogue criterion for the extension of codimension two distributions on smooth hypersurfaces in $\mathbb{P}^n$, $n > 4$, see Theorem 3.9. The proofs of both criteria rely on the study of the restriction morphisms relating twisted differentials on $\mathbb{P}^n$ to twisted differentials on a projective hypersurface. Despite the simplicity of our arguments, they allow us to recover in Proposition 3.8 bounds for the degree of hypersurfaces invariant by Pfaff equations on projective spaces previously obtained in [4] and [17].

In [3], the authors showed an isomorphism between the space of foliations of degree zero on some cominuscule varieties $X \subset \mathbb{P}^n$ and the space of foliations of degree zero on the projective space. Theorem A allows us to establish a similar result when X is a smooth projective hypersurface on $\mathbb{P}^n$, with $n > 3$ and $\deg(X) > 2\deg(\mathcal{F}) + 1$.

We will show examples of non-extension in some cases not covered by Theorem A. For instance, any non-planar smooth surface $\mathcal{S}$ in $\mathbb{P}^3$ admits a foliation that does not extend. Besides, if $\mathcal{S}$ is a plane, each foliation on it has an extension, and the question of characterizing planes in $\mathbb{P}^3$ proposed by Cerveau turns out to be true.

Proposition 1.1. — *A smooth surface in $\mathbb{P}^3$ is a plane if, and only if, each one of its foliations extends.*

We also prove the existence of a degree one foliation on the three dimensional smooth quadric, which does not extend. We will show this result using the fact that, up to a perturbation by an automorphism of $\mathbb{P}^n$, the restriction of any foliation in the projective space to a smooth hypersurface of degree at least two has an isolated singularity.

This paper is organized as follows. In Section 2, we introduce Pfaff equations, holomorphic foliations, distributions, their invariant hypersurfaces, and the degree of foliations. We also define precisely the space of foliations and the restriction and extension of foliations and distributions. In Section 3, we prove Theorem A and an extension criterion for codimension two distributions. As a corollary, we obtain an isomorphism between the space of foliations on the projective space and the space of foliations on hypersurfaces, under suitable assumptions. In the last part of this section, we prove Proposition 1.1. Finally, in Section 4, the property of always getting Morse singularities on restricted foliations, up to perturbation by an automorphism of the ambient projective space, is proved. This allows us to exhibit an example of foliation on a smooth quadric in $\mathbb{P}^4$ that does not extend.

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2. Preliminaries

2.1. Foliations and Pfaff equations

Let X be a complex projective manifold of dimension n and L be a line bundle over X. A *Pfaff equation* of codimension q and coefficients in L is a global section α of $\Omega^q_X \otimes L$. The *singular set* of α is $\mathrm{Sing}(\alpha) := \{p \in X | \alpha(p) = 0\}$.

A *singular codimension q holomorphic foliation* $\mathcal{F}$ is determined by a line bundle L and a Pfaff equation $\omega \in H^0(X, \Omega^q_X \otimes L)$ such that $\mathrm{codim}(\mathrm{Sing}(\omega)) \geqslant 2$ and ω is *decomposable* and *integrable* in the following sense: let $p \in X \setminus \mathrm{Sing}(\omega)$ be a point, then there are 1-forms $\eta_1, \dots \eta_q$ defined over an open set $U \subset X$ containing p and satisfying

(1) $\omega\big|_U = \eta_1 \wedge \dots \wedge \eta_q$ (decomposability condition)
(2) $\mathrm{d}\eta_i \wedge \eta_1 \wedge \dots \wedge \eta_q = 0$, for all $i = 1, \dots, q$ (integrability condition).

The *singular set of* $\mathcal{F}$ is $\mathrm{Sing}(\mathcal{F}) := \mathrm{Sing}(\omega)$.

Remark 2.1. — If a divisor D is contained in the singular locus of a decomposable and integrable $\omega \in H^0(X, \Omega^q_X \otimes L)$, we can replace ω with $\omega' = \frac{\omega}{f} \in H^0(X, \Omega^q_X \otimes L')$ to ensure that $D \not\subset \mathrm{Sing}(\omega')$, where $f \in H^0(X, \mathcal{O}_X(D))$ vanishes along D and $L' = L \otimes \mathcal{O}_X(-D)$. This process is called *saturation* of ω.

For a Pfaff equation of codimension one with local representative η, the decomposability condition is automatic, and the integrability condition is given by

$$\eta \wedge \mathrm{d}\eta = 0.$$

Remark 2.2. — When $\alpha \in H^0(X, \Omega^q_X \otimes L)$ satisfies only the decomposability condition, we say that α defines a *singular codimension q distribution* $\mathcal{D}$ over X. In particular, if $q = 2$, then $\alpha \in H^0(X, \Omega^q_X \otimes L)$ defines distribution if and only if $\alpha \wedge \alpha = 0$ (see [7, Proposition 1]).

The integrability condition ensures that the kernel of ω defines a subsheaf $T\mathcal{F}$ of TX, called *tangent sheaf of* $\mathcal{F}$, such that in an analytic neighborhood of each non-singular point, $T\mathcal{F}$ is the relative tangent sheaf of a holomorphic fibration. The *leaves* of such a foliation are given by analytic continuation.

2.2. Invariant hypersurfaces

Let $Y \subset X$ be a hypersurface. The inclusion map $i : Y \to X$ induces a restriction map of Pfaff equations projecting $\alpha \in H^0(X, \Omega^q_X \otimes L)$ on $i^*\alpha \in H^0(Y, \Omega^q_Y \otimes L\big|_Y)$.

Let $\omega \in H^0(X, \Omega^q_X \otimes L)$ be a Pfaff equation. We say that a hypersurface $Y \subset X$ is *invariant* by ω if $i^*\omega = 0$. If ω determines a foliation $\mathcal{F}$ and Y is invariant by ω, then we say that Y is *invariant* by $\mathcal{F}$.

2.3. Degree of a foliation

If $\mathrm{Pic}(X) \simeq \mathbb{Z}$, taking a positive generator M of $\mathrm{Pic}(X)$, we define the *degree of a line bundle* L as $\deg(L) = l$, when $L \simeq M^{\otimes l}$. If $X = \mathbb{P}^n$, $n \geqslant 2$, then a Pfaff equation of codimension q can be represented as a degree k homogeneous q-form of $\mathbb{C}^{n+1}$ such that its contraction with the radial vector field is zero, namely a q-form ω that satisfies $i_R(\omega) = 0$, where

$$R = \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}.$$

The *degree of a codimension q foliation* $\mathcal{F}$ on $\mathbb{P}^n$ is the degree of the tangency set of the leaves of $\mathcal{F}$ with a generic q-plane in $\mathbb{P}^n$, and is denoted by $\deg(\mathcal{F})$. If $\mathcal{F}$ is determined by $\omega \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(k))$, then, for instance by [4, Lemma 3.2], one has

$$\deg(\mathcal{F}) = k - q - 1.$$

Supposing now that X is a smooth hypersurface of $\mathbb{P}^n$, $n > 3$, we have, by [12, Corollary II.3.2], $\mathrm{Pic}(X) \simeq \mathbb{Z}$, and we define the *degree* of a foliation $\mathcal{G}$ on X generated by $\omega \in H^0(X, \Omega^q_X(k))$ as $\deg(\mathcal{G}) = k - q - 1$.

2.4. Restrictions and Extensions

A codimension q holomorphic singular foliation $\mathcal{F}$ (resp. a distribution $\mathcal{D}$) on the complex projective space $\mathbb{P}^n$ determined by $\omega \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(k))$ is *transverse* to a smooth hypersurface X if the singular set of $i^*\omega$ has codimension at least two, where $i : X \to \mathbb{P}^n$ is the natural inclusion. In this case, $i^*\omega$ determines a foliation $\mathcal{G}$ (resp. a distribution $\mathcal{D}'$) on X, and we say that $\mathcal{G}$ is the *restriction* of $\mathcal{F}$ (resp. $\mathcal{D}'$ is the *restriction*). If a foliation $\mathcal{G}$ (resp. a distribution $\mathcal{D}'$) over a smooth projective hypersurface $X \subset \mathbb{P}^n$ is the restriction of a foliation $\mathcal{F}$ (resp. a distribution $\mathcal{D}$) on $\mathbb{P}^n$ to X, we say $\mathcal{F}$ is an *extension* to $\mathcal{G}$ (resp. an *extension* to $\mathcal{D}'$).

2.5. Space of foliations

Let $X \subset \mathbb{P}^n$, $n > 3$, be a smooth projective hypersurface. If $\omega \in H^0(X, \Omega^1_X(l+2))$ determines a foliation $\mathcal{F}$, then any non-zero constant multiple of ω determines the same foliation. Thus, we define the space of degree l foliations on X as the quasi-projective variety

$$\mathcal{F}\mathrm{ol}(X, l) := \{[\omega] \in \mathbb{P}H^0(X, \Omega^1_X(l+2) \,|\, d\omega \wedge \omega = 0 \text{ and } \operatorname{codim}(\operatorname{Sing}(\omega)) \geqslant 2\}.$$

Determining the irreducible components of such a variety has been studied in some cases, especially when X is a projective space and the degree l is small (see for example [6] and [8]).

3. Extensions of foliations on projective hypersurfaces

According to Subsection 2.4, in order to find an extension of a degree l codimension one foliation on a projective hypersurface X determined by $\omega \in H^0(X, \Omega^1_X(l+2))$, we need to find an integrable 1-form $\alpha \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(l+2))$ such that $i^*\alpha = \omega$, where $i : X \hookrightarrow \mathbb{P}^n$ is the natural inclusion map. Thus, we need to understand the properties of the restriction map of twisted differential forms.

Lemma 3.1. — *Let $X \subset \mathbb{P}^n$, $n > 3$, be a smooth hypersurface such that* $\deg(X) \geqslant 2$ *and* $1 \leqslant q \leqslant n-1$. *Let*

$$\operatorname{rest}_q : H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(k)) \longrightarrow H^0(X, \Omega^q_X(k))$$

be the restriction map of Pfaff equations. Then

(a) rest_q *is injective if* $k - q + 1 \leqslant \deg(X)$*;*

(b) rest_q *is surjective if* $q < n-1$ *and either* $q \neq 2$ *or* $k \neq \deg(X)$.

Proof. — In [2, Proposition 5.22] it was proved that rest_1 is an isomorphism if $k \leqslant \deg(X)$ and rest_q is injective if $k - q + 1 \leqslant \deg(X)$. Therefore, we only need to check the surjectivity of rest_q when $n-1 > q$ and either $q > 2$ or $k > \deg(X)$. For that, consider the exact sequence

$$0 \longrightarrow \Omega^q_{\mathbb{P}^n}(k-\deg(X)) \longrightarrow \Omega^q_{\mathbb{P}^n}(k) \longrightarrow \Omega^q_{\mathbb{P}^n}(k)\big|_X \longrightarrow 0 \tag{3.1}$$

on $\mathbb{P}^n$ and the exact sequence

$$0 \longrightarrow \Omega^{q-1}_X(k-\deg(X)) \longrightarrow \Omega^q_{\mathbb{P}^n}(k)\big|_X \longrightarrow \Omega^q_X(k) \longrightarrow 0. \tag{3.2}$$

on X.

If $n-1 > q > 2$ or $k \neq \deg(X)$ we have $H^1(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(k-\deg(X))) = 0$ by Bott's Theorem (see [10, Theorem 2.3.2] or [16, page 4]). Furthermore, $H^1(X, \Omega^{q-1}_X(k-\deg(X))) = 0$ by [11, Satz 8.11]. Therefore, the maps

$$H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(k)) \longrightarrow H^0(X, \Omega^q_{\mathbb{P}^n}(k)\big|_X)$$

and

$$H^0(X, \Omega^q_{\mathbb{P}^n}(k)\big|_X) \longrightarrow H^0(X, \Omega^q_X(k))$$

obtained by long sequence in cohomology of (3.1) and (3.2) are surjective. Since the restriction map is obtained by composing these two maps, the result follows. $\square$

Remark 3.2. — If $q = 2$, $k = \deg(X)$ and $\dim(X) > 3$ then $\mathrm{rest}_2 : H^0(\mathbb{P}^n, \Omega^2_{\mathbb{P}^n}(k)) \to H^0(X, \Omega^2_X(k))$ is surjective. In fact, by Bott's Theorem ([10, Theorem 2.3.2]) we have the long exact sequence in cohomology of (3.1)

$$0 \longrightarrow H^0(\mathbb{P}^n, \Omega^2_{\mathbb{P}^n}(k)) \xrightarrow{\ \phi\ } H^0(X, \Omega^2_{\mathbb{P}^n}(k)\big|_X) \longrightarrow 0 \longrightarrow H^1(X, \Omega^2_{\mathbb{P}^n}(k)\big|_X) \xrightarrow{\ \sim\ } H^2(\mathbb{P}^n, \Omega^2_{\mathbb{P}^n}) \simeq \mathbb{C} \longrightarrow 0.$$

Thus, ϕ is an isomorphism, and $H^1(X, \Omega^2_{\mathbb{P}^n}(k)\big|_X) \simeq \mathbb{C}$. Now [2, 5.15] and [2, 5.17] applied to the long sequence in cohomology of the short exact sequence (3.2) gives us

$$0 \longrightarrow H^0(X, \Omega^2_{\mathbb{P}^n}(k)\big|_X) \xrightarrow{\ \psi\ } H^0(X, \Omega^2_X(k)) \longrightarrow H^1(X, \Omega^1_X) \simeq \mathbb{C} \xrightarrow{\ \beta\ } H^1(X, \Omega^2_{\mathbb{P}^n}(k)\big|_X) \simeq \mathbb{C} \longrightarrow H^1(X, \Omega^2_X(k)).$$

If $\dim(X) > 3$ by [11, Satz 8.11] we have $H^1(X, \Omega^2_X(k)) = 0$, and β is a surjective linear application between dimension one spaces, i.e., β is isomorphism. So ψ is isomorphism, thus $\mathrm{rest}_2 = \phi \circ \psi$ is isomorphism.

Remark 3.3. — If $n = 3$ and $k \leqslant \deg(X)$ then rest_1 is injective. Actually, if $k < \deg(X)$ we have

$$H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(k - \deg(X))) = 0 = H^0(X, \mathcal{O}_X(k - \deg(X)))$$

by Bott's Theorem [10, Theorem 2.3.2] and [13, Theorem II.5.1 and Exercise II.5.5]. So the maps $H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(k)) \to H^0(X, \Omega^1_{\mathbb{P}^3}(k)\big|_X)$ and $H^0(X, \Omega^1_{\mathbb{P}^n}(k)\big|_X) \to H^0(X, \Omega^1_X(k))$ are injective.

If $\deg(X) = k$, by Bott's Theorem [10, Theorem 2.3.2] and [13, Theorem II.5.1 and Exercise II.5.5], we have $H^1(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(k)) = 0 = H^1(X, \mathcal{O}_X)$ and the exact sequences

$$0 \longrightarrow H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(k)) \xrightarrow{\phi_1} H^0(X, \Omega^1_{\mathbb{P}^3}(k)\big|_X) \xrightarrow{\alpha} H^1(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}) \simeq \mathbb{C} \longrightarrow 0$$

and

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \simeq \mathbb{C} \xrightarrow{\beta} H^0(X, \Omega^1_{\mathbb{P}^3}(k)\big|_X) \xrightarrow{\psi_1} H^0(X, \Omega^1_X(k)) \longrightarrow 0.$$

Thus, $H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(k))$ and $H^0(X, \Omega^1_X(k))$ are vector spaces of the same dimension. Since ϕ_1 is injective and ψ_1 is surjective, it suffices to show that the kernel of ψ_1 does not intersect the image of ϕ_1 to conclude that $\mathrm{rest}_1 := \psi_1 \circ \phi_1$ is an isomorphism. Let f be a degree k irreducible homogeneous polynomial such that $X := \{f = 0\}$. We have $\mathrm{d}f \notin H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(k))$, because $i_R(\mathrm{d}f) \neq 0$. Then, $\mathrm{d}f$ is not in the image of ϕ_1, and $\alpha(\mathrm{d}f)$ generates $H^1(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}) \simeq \mathbb{C}$. Furthermore, $\mathrm{d}f$ generates $\ker(\psi_1) = H^0(X, \mathcal{O}_X) \simeq \mathbb{C}$, so $\ker(\psi_1) \cap \phi_1(H^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(k))) = \emptyset$ and rest_1 is an isomorphism.

Remark 3.4. — If $n = 3$ and $k - 1 \leqslant \deg(X)$ then rest_2 is injective. In fact, by Bott's Theorem [10, Theorem 2.3.2] and [11, Satz 8.11], we have

$$H^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(k - \deg(X))) = 0 = H^0(X, \Omega^1_X(k - \deg(X))).$$

Thus, the maps

$$\phi_2 : H^0(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(k)) \longrightarrow H^0(X, \Omega^2_{\mathbb{P}^3}(k)\big|_X)$$

and

$$\psi_2 : H^0(X, \Omega^2_{\mathbb{P}^n}(k)\big|_X) \longrightarrow H^0(X, \Omega^2_X(k))$$

are injective, so $\mathrm{rest}_2 := \psi_2 \circ \phi_2$ is injective.

PROPOSITION 3.5. — *Let $X \subset \mathbb{P}^n$, $n > 3$, be a smooth hypersurface, and let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n$. If $\deg(\mathcal{F}) + 2 \leqslant \deg(X)$, then $\mathcal{F}$ is transverse to X.*

Proof. — Let $\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(\deg(\mathcal{F}) + 2))$ be a 1-form that determines $\mathcal{F}$. According to the foliation definition, $\operatorname{codim} \operatorname{Sing}(\omega) \geqslant 2$. Suppose for the sake of contradiction that there is a divisor $D \subset \operatorname{Sing}(\mathrm{rest}_1(\omega))$. As already mentioned in the general definition of degree, for $n > 3$, we have $\operatorname{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$. Therefore, $\mathcal{O}_X(D) \simeq \mathcal{O}_X(k)$, for some $k \in \mathbb{Z}_{>0}$, and there are $f \in H^0(X, \mathcal{O}_X(k))$ and $\eta \in H^0(X, \Omega^1_X(\deg(\mathcal{F}) + 2 - k))$ such that $\mathrm{rest}_1(\omega) = f\eta$.

By [13, Exercise II.5.5], there is $g \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ whose restriction to X is f, and by Lemma 3.1, there is $\alpha \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(\deg(\mathcal{F}) + 2 - k))$ such that $\mathrm{rest}_1(\alpha) = \eta$. So,

$$\mathrm{rest}_1(g\alpha) = \mathrm{rest}_1(\omega),$$

and since $\deg(\mathcal{F}) + 2 \leqslant \deg(X)$, Lemma 3.1 guarantees that rest_1 is injective and $g\alpha = \omega$. It follows that $\mathrm{codim}(\mathrm{Sing}(\omega)) = 1$, contradicting $\mathrm{codim}\,\mathrm{Sing}(\omega) \geqslant 2$. □

3.1. Proof of Theorem A

Let $\omega \in H^0(X, \Omega^1_X(\deg(\mathcal{F}) + 2))$ be a Pfaff equation determining $\mathcal{F}$. Let $\alpha \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(\deg(\mathcal{F}) + 2))$ be an extension of ω given by Lemma 3.1. Now notice that the restriction of $\alpha \wedge \mathrm{d}\alpha \in H^0(\mathbb{P}^n, \Omega^3_{\mathbb{P}^n}(2\deg(\mathcal{F}) + 4))$ to X is zero, because $i^*\alpha = \omega$ is integrable. Again, Lemma 3.1 guarantees the injectivity of the restriction map for 3-forms when $\deg(X) > 2\deg(\mathcal{F}) + 1$, so $\alpha \wedge \mathrm{d}\alpha = 0$ and $\mathcal{F}$ extends. □

3.2. Space of foliation on projective hypersurfaces

Theorem A and Proposition 3.5 allow to prove the following result.

Theorem 3.6. — *Let X be a smooth hypersurface in $\mathbb{P}^n$, $n > 3$. If $\deg(X) > 2l + 1$ then the map*

$$\mathcal{F}\mathrm{ol}(\mathbb{P}^n, l) \longrightarrow \mathcal{F}\mathrm{ol}(X, l)$$

is an isomorphism.

Proof. — Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n$ determined by $\alpha \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(l + 2))$. Proposition 3.5 guarantees that $\mathcal{F}$ is transverse to X, so $\mathrm{rest}_1(\alpha)$ defines a codimension one foliation $\mathcal{G}$ on X and $\deg(\mathcal{G}) = l$ (by very definition of the degree, see Subsection 2.3). Therefore, the map between quasi-projective varieties $\pi : \mathcal{F}\mathrm{ol}(\mathbb{P}^n, l) \to \mathcal{F}\mathrm{ol}(X, l)$ induced by rest_1 is well-defined everywhere. It is injective, since $\deg(X) \geqslant 2l + 1 \geqslant l + 2$ (see Lemma 3.1). Furthermore, by Theorem A, it is surjective. □

A degree zero codimension one foliation on $\mathbb{P}^n$, $n > 3$, is a pencil of hyperplanes, and it has a first integral of the form F/G, where F and G are co-prime homogeneous polynomials of degree one (for a proof of this fact, see [9, Proposition 3.1]). Thus, any two foliations of degree zero on $\mathbb{P}^n$ are conjugated, and the set of all of them is an irreducible projective smooth variety isomorphic to the space of projective lines $\mathbb{G}(1,n)$. This fact and Theorem 3.6 imply the following.

Corollary 3.7. — *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface, $n > 3$. If $\deg(X) \geqslant 2$, then*

$$\mathcal{F}\mathrm{ol}(X,0) \simeq \mathbb{G}(1,n).$$

3.3. Degree of invariant smooth hypersurfaces

The results regarding the injectivity of the restriction map allow us to bound the degree of smooth hypersurfaces invariant by a codimension q Pfaff equation on $\mathbb{P}^n$, $n \geqslant 3$ and $1 \leqslant q \leqslant n-1$, as follows.

Proposition 3.8. — *If a smooth hypersurface $X \subset \mathbb{P}^n$, $n \geqslant 3$, is invariant by a non-trivial Pfaff equation $\alpha \in H^0(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(k))$, $1 \leqslant q \leqslant n-1$, then*

$$\deg(X) \leqslant k-q.$$

Proof. — Since X is invariant by α, we have $\mathrm{rest}_q(\alpha) = 0$. Suppose that $\deg(X) \geqslant k-q+1$. Therefore, rest_q is injective by Lemma 3.1, Remark 3.3 and Remark 3.4. This implies α is zero, which is absurd. □

As already mentioned in the Introduction, this gives an alternative proof of previously known bounds for the degree of hypersurfaces invariant by Pfaff equations on projective spaces, see [4] and [17].

3.4. Extensions of codimension two distribution

Concerning codimension two distributions on projective smooth hypersurfaces, let us now see an example of non-extension. Let $\eta \in H^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(3))$ be the codimension two Pfaff equation given by

$$\eta = i_R(\mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4).$$

We have $\eta \wedge \eta \neq 0$, so that η does not satisfy the decomposability condition given in Remark 2.2. Therefore, it does not define a distribution on $\mathbb{P}^4$.

Let $X \subset \mathbb{P}^4$ be a smooth hypersurface such that $\deg(X) > 2$. By Lemma 3.1, the restriction map

$$\mathrm{rest}_2 : H^0(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}(3)) \longrightarrow H^0(X, \Omega^2_X(3))$$

is injective. Therefore $\omega := \mathrm{rest}_2(\eta) \in H^0(X, \Omega^2_X(3))$ is non-zero and satisfies $\omega \wedge \omega = 0$ in $H^0(X, \Omega^4_X(6))$, since $\dim(X) = 3$. Thus ω defines distribution in X. As $\mathrm{rest}_2^{-1}(\omega) = \{\eta\}$, the distribution determined by ω does not extend.

This non-extension example was possible because the restriction of any Pfaff equation of codimension two in $\mathbb{P}^4$ to a smooth hypersurface X automatically satisfies the decomposability condition. On the other hand, if $\dim(X) > 3$, we have the following extension criterion.

Theorem 3.9. — *Let $X \subset \mathbb{P}^n$, $n > 4$, be a smooth hypersurface and let $\mathcal{D}$ be a codimension two distribution determined by $\alpha \in H^0(X, \Omega_X^2(k))$ such that $2k - 3 \leqslant \deg(X)$. Then $\mathcal{D}$ extends.*

Proof. — If $k < 3$, by [2, Lemma 5.17], we have $H^0(X, \Omega_X^2(k)) = 0$. Suppose $k \geqslant 3$. Thus, by Lemma 3.1 and Remark 3.2, there is $\beta \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^2(k))$ whose restriction to X is α. Since $2k - 3 \leqslant \deg(X)$, the restriction map $\mathrm{rest}_4 : H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^4(2k)) \to H^0(X, \Omega_X^4(2k))$ is injective. Thus $\mathrm{rest}_4(\beta \wedge \beta) = \alpha \wedge \alpha = 0$, so $\beta \wedge \beta = 0$ and, by Remark 2.2, β determines a codimension two distribution on $\mathbb{P}^n$ that extends $\mathcal{D}$. ☐

3.5. Proof of Proposition 1.1

Let $\mathcal{H}$ be a hyperplane in $\mathbb{P}^n$, $n \geqslant 3$, and $\mathcal{G}$ be a codimension q foliation on $\mathcal{H}$. Taking a point $p \in \mathbb{P}^n \setminus \mathcal{H}$, we have a rational map $\pi_p : \mathbb{P}^n \dashrightarrow \mathcal{H}$, called *projection from a point* p, whose definition domain is $\mathbb{P}^n \setminus \{p\}$.

Example 3.10. — If $\mathcal{H} = \{x_n = 0\}$ and taking $p := (0 : 0 : \ldots : 0 : 1) \in \mathbb{P}^n$, π_p is determined by

$$\begin{array}{rcl} \pi_p : \mathbb{P}^n & \dashrightarrow & \mathcal{H} \\ (x_0 : \ldots : x_n) & \longmapsto & (x_0 : \ldots : x_{n-1}) \end{array}.$$

The pull-back of $\mathcal{G}$ by π_p is a codimension q foliation $\pi_p^*\mathcal{G}$ on $\mathbb{P}^n$. It is called *trivial extension*. Therefore, every foliation on a hyperplane of $\mathbb{P}^n$ extends.

Let $\eta \in H^0(\mathbb{P}^3, \Omega_{\mathbb{P}^3}^1(2))$ be the contact 1-form given by

$$\eta = i_R(\mathrm{d}x_0 \wedge \mathrm{d}x_1 + \mathrm{d}x_2 \wedge \mathrm{d}x_3) = x_0\mathrm{d}x_1 - x_1\mathrm{d}x_0 + x_2\mathrm{d}x_3 - x_3\mathrm{d}x_2.$$

This 1-form does not define a foliation on $\mathbb{P}^3$ because it is non-integrable. Furthermore, if $\mathcal{S} \subset \mathbb{P}^3$ is a smooth surface, with $\deg(\mathcal{S}) \geqslant 2$, then Proposition 3.8 implies that $\mathcal{S}$ is non-invariant by η. The integrability of the restriction of η to $\mathcal{S}$ is trivially satisfied, so that $\mathrm{rest}_1(\eta)$ defines a foliation on $\mathcal{S}$.

By Remark 3.3, if $\deg(\mathcal{S}) \geqslant 2$, the contact form η is the only 1-form whose restriction to $\mathcal{S}$ is $\mathrm{rest}_1(\eta)$. Therefore, the foliation generated by $\mathrm{rest}_1(\eta)$ on $\mathcal{S}$ does not extend to $\mathbb{P}^3$. Then, every smooth surface $\mathcal{S}$ on $\mathbb{P}^3$ such that $\deg(\mathcal{S}) \geqslant 2$ has a foliation that does not extend. This fact and the trivial extensions prove Proposition 1.1. ☐

4. A special foliation on the three-dimensional quadric

4.1. Restriction of foliations

Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{P}^n$ and $f \in \mathbb{C}[x_0, \dots, x_n]$ be an irreducible homogeneous polynomial that determines a smooth hypersurface X as its zero set. Let us give some properties on the singular set of the restriction $\mathcal{F}\big|_X$.

The *Gauss map* of $\mathcal{F}$ is the rational map $G_{\mathcal{F}} : \mathbb{P}^n \dashrightarrow \check{\mathbb{P}}^n$ given by $p \mapsto T_p\mathcal{F}$, which is defined outside the singular points of $\mathcal{F}$.

Definition 4.1. — *Let $\mathcal{G}$ be a codimension one foliation on a projective hypersurface $X \subset \mathbb{P}^n$. We say that $p \in \operatorname{Sing}(\mathcal{G})$ is of* Morse type *if there are an open subset U containing p, and a first integral $g : U \to \mathbb{C}$ of $\mathcal{G}$ such that, in local coordinates, we can write $g = x_1^2 + \dots + x_{n-1}^2$.*

By Morse Lemma (see [15, Lemma 2.2]), a singular point p of a function f is of Morse type if and only if the *Hessian matrix* of f in p, given by

$$\operatorname{Hess}_p(f) = \left(\frac{\partial^2 f(p)}{\partial x_i \partial x_j}\right)_{i,j=1,\dots,n},$$

has a non-zero determinant. Furthermore, Morse-type singularities are isolated.

Given a generic hyperplane $\mathcal{H} \subset \mathbb{P}^n$, the singular set of $\mathcal{F}\big|_{\mathcal{H}}$ is given by $(\operatorname{Sing}(\mathcal{F}) \cap \mathcal{H}) \cup G_{\mathcal{F}}^{-1}(\mathcal{H})$. Furthermore, if $G_{\mathcal{F}}^{-1}(\mathcal{H})$ is non-empty, then its points are isolated Morse-type singularities (see [1, Proposition 1.10]).

Definition 4.2. — *Let $X \subset \mathbb{P}^n$ be a hypersurface defined by a homogeneous polynomial f. The* Gauss map *of X is the rational map*

$$\begin{array}{rccc} G_X : & X & \dashrightarrow & \check{\mathbb{P}}^n \\ & p & \longmapsto & \left[\frac{\partial f(p)}{\partial x_0} : \dots : \frac{\partial f(p)}{\partial x_n}\right] \end{array}.$$

The indeterminacy points of G_X coincide with the singularities of X.

If $X \subset \mathbb{P}^n$ is a smooth projective hypersurface of degree at least two, then G_X is a morphism and its rank is generically equal to $n-1$ (See [19, Corollary I.2.5]). The next result tells us that, up to conjugation by a generic automorphism, the restriction of a foliation $\mathcal{F}$ on $\mathbb{P}^n$ to a smooth hypersurface of degree at least two has singularities outside of $\operatorname{Sing}(\mathcal{F}) \cap X$ which are of Morse type.

Theorem 4.3. — *Let $\mathcal{F}$ be codimension one holomorphic foliation on $\mathbb{P}^n$, $n > 2$, and $X \subset \mathbb{P}^n$ be a non-planar smooth hypersurface transverse to $\mathcal{F}$. Then there is an automorphism $h \in \mathrm{Aut}(\mathbb{P}^n)$ such that $h^*\mathcal{F}\big|_X$ has an isolated singularity $p \notin \mathrm{Sing}(h^*\mathcal{F})$.*

Proof. — Since X is smooth, the rank of the associated Gauss map is generically $n-1$. Therefore, we can suppose that the point $p = [1 : 0 : \ldots : 0] \in X$ is a regular point of $\mathcal{F}$, and G_X has maximal rank in p. Let us take an affine coordinate system $(x_1, \ldots, x_n)$ centered on p corresponding to an affine chart U of $\mathbb{P}^n$ such that the first $n-1$ coordinates vectors determine the tangent space of X in p.

Thus, in a neighborhood of p, we have that $x_n\big|_X = x_n(x_1, \ldots, x_{n-1})$ vanishes in second order in $(x_1, \ldots, x_{n-1}) = (0, \ldots, 0)$. Therefore,

$$x_n = \sum_{i,j=1}^{n-1} b_{ij} x_i x_j + \sum_{i \geqslant 3} \widetilde{g_i}(x_1, \ldots, x_{n-1}),$$

in X, where each $\widetilde{g_i}$ is a homogeneous polynomial of degree i. The rank of the matrix (b_{ij}) is identical to the rank of the associated Gauss map. This fact results from the description of the second fundamental form in terms of the rank of the Gauss map (see, for example, the end of Section 6.4B of [18]).

Therefore, in a suitable coordinate system centered on p, we can write

$$x_n = x_1^2 + \cdots + x_{n-1}^2 + \sum_{r \geqslant 3} g_i(x_1, \ldots, x_{n-1}),$$

in X, where each g_i is a homogeneous polynomial of degree i.

As p is a regular point of $\mathcal{F}$, restricting U if necessary, the local first integral of $\mathcal{F}$ defined on U is of the form

$$f = a_1 x_1 + \cdots + a_n x_n + \sum_{j \geqslant 2} p_j(x_1, \ldots, x_n),$$

where each p_j is a homogeneous polynomial of degree j and $a_1, \ldots, a_n \in \mathbb{C}$, with $a_k \neq 0$ for some $k \in \{1, \ldots, n\}$. Then for every $\lambda \in \mathbb{C}^*$, we can find an automorphism $h_\lambda \in \mathrm{Aut}(\mathbb{P}^n)$ such that the local first integral of $h_\lambda^*\mathcal{F}$ on an open set $V \subset U$ of p is

$$f_\lambda = \lambda \cdot x_n + \sum_{j \geqslant 2} h_j(x_1, \ldots, x_n),$$

where h_j is a homogeneous polynomial of degree j and

$$h_2 = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j.$$

For instance, take h_λ the automorphism that preserves the hyperplane at infinity and, on affine coordinates, is defined by $x_i \mapsto x_i$, if $i \neq k$, and

$$x_k \longmapsto \frac{1}{a_k}\left(\left(\sum_{\substack{i=1\\ i\neq k}}^{n} -a_i x_i\right) + \lambda x_n\right).$$

The restriction of $h_\lambda^* \mathcal{F}$ to X has a first integral on p of the form

$$f_\lambda\big|_X = \lambda(x_1^2 + \cdots + x_{n-1}^2) + \sum_{j\geqslant 2} \widetilde{h_j}(x_1, \dots, x_{n-1}),$$

such that each $\widetilde{h_j}$ is a homogeneous polynomial of degree j. Additionally,

$$\widetilde{h_2} = h_2(x_1, \dots, x_{n-1}, 0) = \sum_{j=1}^{n-1}\sum_{i=1}^{n-1} a_{ij} x_i x_j,$$

and the $a_{ij} = a_{ji} \in \mathbb{C}$ do not depend on the choice of λ. In fact, if $\lambda_1, \lambda_2 \in \mathbb{C}^*$, to replace f_{λ_1} by f_{λ_2} just make a pull-back of $h_{\lambda_1}^* \mathcal{F}$ via an automorphism of $\mathbb{P}^n$ that perform $x_i \mapsto x_i$, if $i \neq n$, and $x_n \mapsto \frac{\lambda_2 x_n}{\lambda_1}$, in affine coordinates. As $\frac{\partial f_\lambda|_X}{\partial x_i}(p) = 0$ for all $i = 1, \dots, n-1$, then p is a singularity of $h_\lambda^* \mathcal{F}\big|_X$. We will show that for generic $\lambda \in \mathbb{C}^*$, p is a Morse singularity, so it is an isolated singularity. To do this, it suffices to show that, for a generic λ, the determinant of the Hessian matrix of $f_\lambda\big|_X$ is nonzero at the point p. In fact, we have

$$\frac{\partial^2 f_\lambda\big|_X}{\partial x_i^2}(p) = 2\lambda + 2a_{ii}$$

and, if $i \neq j$, then

$$\frac{\partial^2 f_\lambda\big|_X}{\partial x_i \partial x_j}(p) = 2a_{ij}.$$

Therefore, the determinant of the Hessian matrix

$$\mathrm{Hess}_p(f_\lambda\big|_X) = \det\left(\frac{\partial^2 f_\lambda\big|_X}{\partial x_i \partial x_j}(p)\right)$$

is a polynomial of degree $n-1$ in the variable λ, with leading coefficient equal to 2^{n-1}. Then $\mathrm{Hess}_p(f_\lambda\big|_X)$ is not identically zero and has finite number of roots. This tells us that for a generic $\lambda \in \mathbb{C}^*$, p is a Morse-type singularity of $h_\lambda^* \mathcal{F}\big|_X$. □

4.2. Non-extension of a foliation on the three-dimensional quadric

For a codimension one and degree one foliation over a smooth quadric in $\mathbb{P}^4$, Theorem A does not guarantee its extension. In [14, 5.11], the authors presented a degree one foliation over the smooth quadric on $\mathbb{P}^4$ with interesting properties. We will briefly describe such foliation and prove that this foliation does not extend.

We can realize $\mathbb{P}^4$ as the equivalence class of homogeneous polynomials of degree four in two variables, i.e., four unordered points on the projective line. With this identification, there is a natural action of $\mathrm{Aut}(\mathbb{P}^1) \simeq \mathrm{PGL}(2,\mathbb{C})$ on $\mathbb{P}^4$. The closure of the orbit of $\mathrm{PGL}(2,\mathbb{C})$ on the point of $\mathbb{P}^4$ corresponding to $\{1,-1,i,-i\}$ generates a smooth quadric $Q^3 \subset \mathbb{P}^4$.

The action of the affine subgroup $\mathbb{A}\mathrm{ff}(\mathbb{C}) \subset \mathrm{PGL}(2,\mathbb{C})$ on Q^3 induces a codimension one foliation $\mathcal{A}$ of degree one, whose singular set does not contain isolated points: $\mathrm{Sing}(\mathcal{A})$ is composed of a rational normal curve of degree four, a Veronese curve of degree 3 and a line, which correspond, respectively, to points of the form $\{p,p,p,p\}$, $\{\infty,p,p,p\}$ and $\{p,\infty,\infty,\infty\}$. Also, $\mathcal{A}$ belongs to an irreducible component $\mathbb{A}\mathrm{ff} \subset \mathcal{F}\mathrm{ol}(Q^3,1)$ called *affine component* whose general element is given by conjugation of $\mathcal{A}$ with element of $\mathrm{Aut}(Q^3) \simeq \mathrm{PO}(5,\mathbb{C})$ (see [14, Theorem 5.2]).

Theorem 4.4. — *Let $\mathcal{A}$ be the degree one affine foliation described above. Then $\mathcal{A}$ does not extend.*

Proof. — Suppose by contradiction that there is a foliation $\mathcal{F}$ in $\mathbb{P}^4$ which is the extension of $\mathcal{A}$. We have the following rational map

$$\begin{array}{rcl}\Phi : \mathrm{Aut}(\mathbb{P}^4) & \dashrightarrow & \mathcal{F}\mathrm{ol}(Q,1) \\ g & \longmapsto & g^*\mathcal{F}\big|_Q \end{array}.$$

As $\Phi(\mathrm{id}) = \mathcal{A}$ and $\mathrm{Aut}(\mathbb{P}^4)$ is irreducible, then component $\mathbb{A}\mathrm{ff}$ contains the image of Φ. On the other hand, by Theorem 4.3, there exists an automorphism $h \in \mathrm{Aut}(\mathbb{P}^4)$ such that $h^*\mathcal{F}\big|_{Q^3}$ has an isolated singularity $p \notin \mathrm{Sing}(h^*\mathcal{F})$. Therefore, such foliation can not belong to $\mathbb{A}\mathrm{ff}$, as the elements in this component are a pull-back of $\mathcal{A}$ by an automorphism of Q and does not have isolated singularities, which is absurd. □

Bibliography

[1] T. F. Amaral, "Sobre a Aplicação de Gauss de Folheações Holomorfas em Espaços Projetivos", PhD Thesis, Instituto Nacional de Matemática Pura e Aplicada, 2008, https://impa.br/wp-content/uploads/2017/08/tese_dout_thiago_fassarella_amaral.pdf.

[2] C. Araujo, M. Corrêa & A. Massarenti, "Codimension one Fano distributions on Fano manifolds", *Commun. Contemp. Math.* **20** (2018), no. 5, article no. 1750058 (28 pages).

[3] V. Benedetti, D. Faenzi & A. Muniz, "Codimension one foliations on homogeneous varieties", *Adv. Math.* **434** (2023), article no. 109332 (45 pages).

[4] M. Brunella & L. Gustavo Mendes, "Bounding the degree of solutions to Pfaff equations", *Publ. Mat., Barc.* **44** (2000), no. 2, p. 593-604.

[5] D. Cerveau, "Quelques problèmes en géométrie feuilletée pour les 60 années de l'IMPA", *Bull. Braz. Math. Soc. (N.S.)* **44** (2013), no. 4, p. 653-679.

[6] D. Cerveau & A. Lins Neto, "Irreducible components of the space of holomorphic foliations of degree two in $C\mathrm{P}(n)$, $n \geqslant 3$", *Ann. Math.* **143** (1996), no. 3, p. 577-612.

[7] D. Cerveau & A. L. Neto, "Codimension two holomorphic foliation", *J. Differ. Geom.* **113** (2019), no. 3, p. 385-416.

[8] R. C. da Costa, R. Lizarbe & J. Vitório Pereira, "Codimension one foliations of degree three on projective spaces", *Bull. Sci. Math.* **174** (2022), article no. 103092 (39 pages).

[9] J. Déserti & D. Cerveau, "Feuilletages et actions de groupes sur les espaces projectifs", *Mém. Soc. Math. Fr., Nouv. Sér.* **103** (2005), p. vi+124.

[10] I. Dolgachev, "Weighted projective varieties", in *Group actions and vector fields (Vancouver, B.C., 1981)*, Lecture Notes in Mathematics, vol. 956, Springer, 1982, p. 34-71.

[11] H. Flenner, "Divisorenklassengruppen quasihomogener Singularitäten", *J. Reine Angew. Math.* **328** (1981), p. 128-160.

[12] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Lecture Notes in Mathematics, vol. 156, Springer, 1970, xiii+256 pages.

[13] ———, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer, 1977, xvi+496 pages.

[14] F. Loray, J. V. Pereira & F. Touzet, "Foliations with trivial canonical bundle on Fano 3-folds", *Math. Nachr.* **286** (2013), p. 921-940.

[15] J. W. Milnor, M. Spivak & R. Wells, *Morse Theory*, Annals of Mathematics Studies, vol. 51, Princeton University Press, 1969.

[16] C. Okonek, M. Schneider & H. Spindler, *Vector Bundles on Complex Projective Spaces*, Modern Birkhäuser Classics, Birkhäuser, 1988.

[17] M. G. Soares, "The Poincaré problem for hypersurfaces invariant by one-dimensional foliations", *Invent. Math.* **128** (1997), no. 3, p. 495-500.

[18] E. A. Tevelev, "Projectively Dual Varieties", *J. Math. Sci., New York* **117** (2003), no. 6, p. 4585-4732.

[19] F. L. Zak, *Tangents and Secants of Algebraic Varieties*, Translations of Mathematical Monographs, vol. 127, American Mathematical Society, 1993.