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Shifted Contact Structures and Their Local Theory (*)

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Abstract. — In this paper, we formally define *k-shifted contact structures* on derived (Artin) stacks and study their local properties in the context of derived algebraic geometry. In this regard, for k-shifted contact derived $\mathbb{K}$-schemes, we develop a *Darboux-like theorem* and formulate the notion of *symplectification*.

Résumé. — Dans cet article, nous définissons formellement des *structures contacts k-décalées* sur des champs (d'Artin) dérivés et étudions leurs propriétés locales dans le contexte de la géométrie algébrique dérivée. À cet égard, pour les $\mathbb{K}$-schémas dérivés contacts k-décalés, nous développons un *théorème de type Darboux* et formulons la notion de *symplectification*.

1. Introduction and summary

Derived algebraic geometry (DAG) essentially provides a new setup to deal with non-generic situations in geometry (e.g. non-transversal intersections and "bad" quotients). To this end, it combines higher categorical objects and homotopy theory with many tools from homological algebra.

In brief, DAG can be considered as *a higher categorical/homotopy theoretical refinement of classical algebraic geometry*. In that respect, it offers a new way of organizing information for various purposes. Therefore, it has many interactions with other mathematical domains. For a survey of some directions, we refer to [1, 15].

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In the context of DAG, it is also possible to work with familiar geometric structures, but in more general forms. For instance, k-shifted versions of Symplectic and Poisson geometries have already been described and studied in [6, 12]. In this regard, [3, 4, 9] offer some applications and local constructions.

Throughout this paper, we mainly work within the context of Toën & Vezzosi's version of DAG [15, 16]. We also benefit from Lurie's version [10]. In that respect, we always consider objects with higher structures in a functorial perspective, and we focus on nice representatives for those structures. For instance, by a *derived* $\mathbb{K}$*-stack*, we essentially mean a simplicial presheaf on the category of commutative differential graded $\mathbb{K}$-algebras (cdga) having nice local-to-global properties.

DAG provides an appropriate concept of a *spectrum functor* Spec from cdgas to (higher) spaces. Using this functor, we call a derived space of the form $\mathbf{X} \simeq \operatorname{Spec} A$ for some cdga A an *affine derived* $\mathbb{K}$*-scheme.* As in the classical theory, a general *derived* $\mathbb{K}$*-scheme* $\mathbf{Y}$ is defined to be a space which is locally modeled on $\mathbf{X} \simeq \operatorname{Spec} A$. Note that affine derived schemes are in fact the main objects of interest for us because the concepts to be discussed in this paper are all about the *local structure* of derived schemes. Thus, it is enough to consider the affine case. More details will be given in Section 2.1.

Regarding certain geometric structures on higher spaces; such as k-shifted (closed) p-forms in the sense of [12], it is also known that for sufficiently "nice" cdgas (to be clear later), we can use the A-module Ω^1_A of Kähler differentials as a model for the cotangent complex $\mathbb{L}_A$ of A so that we write $\mathbb{L}_A \simeq \Omega^1_A$. Then, by *a k-shifted p-form on* $\operatorname{Spec} A$ for A a sufficiently nice cdga, we actually mean a k-cohomology class of the complex $(\Lambda^p \Omega^1_A, d)$. Likewise, a *$k$-shifted closed p-form on* $\operatorname{Spec} A$ is just a k-cohomology class of the complex $\prod_{i\geqslant 0}\big(\Lambda^{p+i}\Omega^1_A[-i],\ d_{tot} = d + d_{dR}\big)$.

A reasonable notion of *non-degeneracy* is also available in this framework, which leads to the definition of a *shifted symplectic structure.* Loosely speaking, we are then able to define the notion of a *k-shifted contact form* on $\operatorname{Spec} A$ to be a k-shifted 1-form α on $\operatorname{Spec} A$ with the property that the k-shifted 2-form $d_{dR}\alpha$ satisfies a non-degeneracy condition, which will be formulated later. In fact, we will provide the *general definition of contact data for derived (Artin) stacks*; rather than just for affine derived $\mathbb{K}$-schemes with "nice" local models.

For shifted symplectic structures on derived schemes, it has been shown in [4] that every k-shifted symplectic derived $\mathbb{K}$-scheme $(\mathbf{X}, \omega')$ is Zariski locally equivalent to $(\operatorname{Spec} A, \omega)$ for a pair A, ω in certain symplectic Darboux form. More precisely, Bussi, Brav and Joyce [4, Theorem 5.18] proved that given a k-shifted symplectic derived $\mathbb{K}$-scheme $(\mathbf{X}, \omega')$, one can find a "minimal standard form" cdga A, a Zariski open inclusion $\iota : \operatorname{Spec} A \hookrightarrow \mathbf{X}$, and "coordinates" $x_j^{-i}, y_j^{k+i} \in A$ with $\iota^*(\omega') \sim (\omega^0, 0, 0, \dots)$ such that

$$\omega^0 = \sum_{i,j} d_{dR} x_j^{-i} d_{dR} y_j^{k+i}.$$

We should point out that the expression of ω^0 holds true only for the case where $k < 0$ is an *odd* integer. The other possible cases require some modifications depending on whether $k/2$ is even or odd. However, the underlying idea behind the proofs for each case is the same.

Note also that the case $k < 0$ odd is relatively simple and instructive enough to capture the essential techniques for the constructions of local models under consideration. Therefore, in this paper, we will mainly concentrate on the case with $k < 0$ odd and use it as a *prototype construction*. For the other cases, we will not give all the details. Instead, we will only provide a brief outline. For details, we will always refer to [4, Examples 5.8, 5.9 & 5.10].

Results and the outline

In this paper, we introduce the notion of *a k-shifted contact structure* on a derived Artin stack. In brief, the contact data consist of a morphism $f : \mathcal{K} \to \mathbb{T}_{\mathbf{X}}$ of perfect complexes, a line bundle L such that $\operatorname{Cone}(f) \simeq L[k]$, and a locally defined k-shifted 1-form $\alpha : \mathbb{T}_{\mathbf{X}} \to \mathcal{O}_{\mathbf{X}}[k]$ with certain property and a non-degeneracy condition (cf. Definitions 3.5 & 3.6).

Having provided the formal definitions, the goals are then to develop a Darboux-type model for shifted contact structures and investigate further possible outcomes. The next two theorems summarize the main results of this paper.

A Darboux-type theorem. We first discuss the existence of Darboux-type local models for k-shifted contact derived $\mathbb{K}$-schemes when $k < 0$. More precisely, for a locally finitely presented derived $\mathbb{K}$-scheme $\mathbf{X}$ with a k-shifted contact structure for $k < 0$, we prove the following result (cf. Theorem 3.14):

Theorem 1.1. — *Every k-shifted contact derived $\mathbb{K}$-scheme $\mathbf{X}$ is locally equivalent to $(\operatorname{Spec} A, \alpha_0)$ for A a minimal standard form cdga and α_0 in a contact Darboux form.*

The symplectification. Secondly, we establish a shifted version of the classical connection between contact and symplectic geometries. In classical contact geometry, for a contact manifold M, there is a unique *symplectified space* with a symplectic structure canonically determined by the contact data of M. In this paper, we provide a similar result for shifted contact derived stacks. The upshot is that given a (locally finitely presented) derived $\mathbb{K}$-scheme $\mathbf{X}$ with a k-shifted contact structure for $k < 0$, we define the *symplectification* $\mathcal{S}_{\mathbf{X}}$ of $\mathbf{X}$ as the total space of a certain $\mathbb{G}_m$-bundle over $\mathbf{X}$, constructed via the data of k-shifted contact structure, and provided with a canonical k-shifted symplectic structure for which the $\mathbb{G}_m$-action is of weight 1 (cf. Definition 4.3 & Theorem 4.7). In brief, we have:

Theorem 1.2. — *The space $\mathcal{S}_{\mathbf{X}}$ has the structure of a k-shifted symplectic derived stack with a symplectic form ω which is canonically determined by the shifted contact structure of $\mathbf{X}$. We then call the pair $(\mathcal{S}_{\mathbf{X}}, \omega)$ the* symplectification *of $\mathbf{X}$.*

Now, let us describe the content of this paper in more detail and provide an outline. In Section 2, we review derived symplectic geometry and symplectic Darboux forms. We begin by some background material on derived algebraic geometry and present nice local models for derived $\mathbb{K}$-schemes and their cotangent complexes. In Section 2.2, using these nice local models, we study shifted symplectic structures. Section 2.3 outlines symplectic Darboux forms on derived schemes and presents Darboux-type results given by Bussi, Brav and Joyce [4, Theorem 5.18].

Section 3 discusses contact structures. In Section 3.1, classical contact geometry is briefly revisited, and then in Section 3.2, we introduce *shifted contact structures* and discuss their properties. In Section 3.3, we state a Darboux-type theorem for shifted contact structures on derived $\mathbb{K}$-schemes (Theorem 3.14) and provide the proof of Theorem 1.1.

In Section 4, we discuss the concept of *symplectification* for shifted contact derived schemes and give the proof of Theorem 1.2 (cf. Definition 4.3 & Theorem 4.7).

Section 5 provides some concluding remarks on possible "stacky" generalizations of the main results of this paper. It also advertises possible future directions and ongoing projects.

Conventions. Throughout the paper, $\mathbb{K}$ will be an algebraically closed field of characteristic zero. All cdgas will be graded in nonpositive degrees and over $\mathbb{K}$. All classical $\mathbb{K}$-schemes will be *locally of finite type*, and all derived $\mathbb{K}$-schemes/Artin stacks $\mathbf{X}$ are assumed to be *locally finitely presented.*

2. Shifted symplectic structures

2.1. Some derived algebraic geometry

In this section, we outline the basics of DAG, present some material relevant to this paper, and state some useful results from shifted symplectic geometry. As stressed before, we use both Toën & Vezzosi's version of DAG [15, 16] and the Lurie's version [10]. In what follows, we just intend to give a brief sketch for the objects and constructions that we will be mostly interested in.

Definition 2.1. — *Denote by $cdga_{\mathbb{K}}$ the category of commutative differential graded $\mathbb{K}$-algebras in non-positive degrees, where an* object A *in $cdga_{\mathbb{K}}$ consists of*

(1) *a collection of $\mathbb{K}$-vector spaces $\{A^i\}$, where A^i is a $\mathbb{K}$-vector space of degree i elements for $i = 0, -1, \ldots,$*
(2) *a $\mathbb{K}$-bilinear, associative, supercommutative multiplication operation $A^n \otimes A^m \xrightarrow{\cdot} A^{n+m}$, and*
(3) *a unique square-zero derivation of degree 1 (the differential)* d *on A satisfying the graded Leibniz rule*

$$\mathrm{d}(a \cdot b) = (\mathrm{d}a) \cdot b + (-1)^n a \cdot (\mathrm{d}b)$$

for all $a \in A^n, b \in A^m$.

We denote such objects by (A, d) or just A, such that $A = \bigoplus_i A^i$.

A morphism *in $cdga_{\mathbb{K}}$, on the other hand, is a collection of degree-wise $\mathbb{K}$-linear morphisms $f = \{f^i\} : A \to B$ such that each $f^i : A^i \to B^i$ commutes with all the structures of A, B.*

Definition 2.2. — *Denote by $dSt_{\mathbb{K}}$ the ∞-category of derived stacks, where an object $\mathbf{X}$ of $dSt_{\mathbb{K}}$ is given as a certain ∞-functor*

$$\mathbf{X} : \; cdga_{\mathbb{K}} \longrightarrow \; sSets, \tag{2.1}$$

where $sSets$ denote the ∞-category of simplical sets. More precisely, objects in $dSt_{\mathbb{K}}$ are simplicial presheaves preserving weak equivalences and possessing the descent/local-to-global property w.r.t. the site structure on the source*. For a brief review, we refer to* [17].

We write $cdga_{\mathbb{K}}^{\infty}$ for the associated ∞-category of $cdga_{\mathbb{K}}$ such that the homotopy category $Ho(cdga_{\mathbb{K}}^{\infty})$ can be obtained from $cdga_{\mathbb{K}}$ by formally inverting quasi-isomorphsims.

Note that $Ho(cdga_{\mathbb{K}}^{\infty})$ is just an ordinary category. We should also point out that $cdga_{\mathbb{K}}^{\infty}$, $cdga_{\mathbb{K}}$ and $Ho(cdga_{\mathbb{K}}^{\infty})$ have the same objects; however, lifting properties of morphisms are different. That is, a morphism $f : A \to B$ in $cdga_{\mathbb{K}}$ is also a morphism in $cdga_{\mathbb{K}}^{\infty}$ and $Ho(cdga_{\mathbb{K}}^{\infty})$. But in general, the converse is not true unless A is cofibrant. In the rest of this paper, we will be interested in certain types of cdgas, called *standard form cdgas,* which are in fact "sufficiently cofibrant", and hence suitable for our purposes.

In this framework, there also exists an appropriate concept of a *spectrum functor* [10, Section 4.3]

$$\operatorname{Spec} : (cdga_{\mathbb{K}}^{\infty})^{op} \longrightarrow dSt_{\mathbb{K}},$$

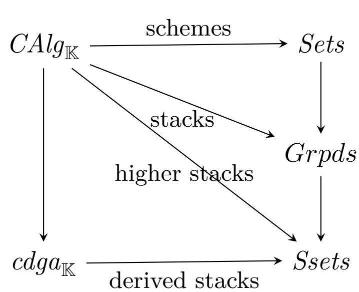
which leads to the following definition.

Definition 2.3. — *An object* $\mathbf{X}$ *in* $dSt_{\mathbb{K}}$ *is called an* affine derived $\mathbb{K}$-scheme *if* $\mathbf{X} \simeq \operatorname{Spec} A$ *for some cdga* $A \in cdga_{\mathbb{K}}$*. An object* $\mathbf{X}$ *in* $dSt_{\mathbb{K}}$ *is then called a* derived $\mathbb{K}$-scheme *if it can be covered by Zariski open affine derived* $\mathbb{K}$*-schemes* $Y \subset X$*.*

Denote by $dSch_{\mathbb{K}} \subset dSt_{\mathbb{K}}$ the full ∞-subcategory of derived $\mathbb{K}$-schemes, and we simply write $dAff_{\mathbb{K}} \subset dSch_{\mathbb{K}}$ for the full ∞-subcategory of affine derived $\mathbb{K}$-schemes. Note that $\operatorname{Spec} : (cdga_{\mathbb{K}}^{\infty})^{op} \to dAff_{\mathbb{K}}$ gives an equivalence of ∞-categories.

We should note that throughout this paper, $\mathbb{K}$ will be an algebraically closed field of characteristic zero. We also assume that all classical $\mathbb{K}$-schemes are *locally of finite type*, and all derived $\mathbb{K}$-schemes $\mathbf{X}$ are *locally finitely presented*, by which we mean that $\mathbf{X}$ can be covered by Zariski open affines $\operatorname{Spec} A$, where A is a finetely presented cdga over $\mathbb{K}$.

Remark 2.4. — Thanks to the Yoneda embedding, one can also realize algebro-geometric objects (like classical $\mathbb{K}$-schemes, stacks, derived spaces, etc.) as *certain functors* in addition to the standard ringed-space formulation. We have the following enlightening diagram from [17] encoding such a functorial interpretation:



Here $CAlg_{\mathbb{K}}$ denotes the category of commutative $\mathbb{K}$-algebras. Denote by $St_{\mathbb{K}}$ the ∞-category of (higher) $\mathbb{K}$-stacks, where objects in $St_{\mathbb{K}}$ are defined via the diagram above. In the underived setup, we have the classical "spectrum functor"

$$\mathrm{spec} : (CAlg_{\mathbb{K}})^{op} \longrightarrow St_{\mathbb{K}}.$$

We then call an object X of $St_{\mathbb{K}}$ an *affine* $\mathbb{K}$*-scheme* if $X \simeq \mathrm{spec}\, A$ for some $A \in CAlg_{\mathbb{K}}$, and a $\mathbb{K}$*-scheme* if it has an open cover by affine $\mathbb{K}$-schemes.

In addition to the spectrum functors Spec, spec above, there is a natural *truncation functor* $\tau : dSt_{\mathbb{K}} \to St_{\mathbb{K}}$, along with a fully faithfull left adjoint *inclusion functor* $\iota : St_{\mathbb{K}} \hookrightarrow dSt_{\mathbb{K}}$, which can be thought of as an embedding of classical algebraic $\mathbb{K}$-spaces into derived spaces.

Note that, for a cdga A there exists an equivalence $\tau \circ \mathrm{Spec}\, A \simeq \mathrm{spec}\, H^0(A)$. This means that if $\mathbf{X}$ is a (affine) derived $\mathbb{K}$-scheme, then its truncation $X = \tau(\mathbf{X})$ is a (affine) $\mathbb{K}$-scheme. Therefore, we can consider a derived $\mathbb{K}$-scheme $\mathbf{X}$ as *an infinitesimal thickening of its truncation* X. It follows that points of a derived $\mathbb{K}$-scheme $\mathbf{X}$ are the same as points of of its truncation X. It means that the main difference between X and $\mathbf{X}$ is in fact encoded by the scheme structure, not by the points!

Nice local models for derived $\mathbb{K}$-schemes. The following result (Theorem 2.5) plays an important role in constructing useful local algebraic models for derived $\mathbb{K}$-schemes and for k-shifted symplectic structures on them.

The upshot is that given a derived $\mathbb{K}$-scheme $\mathbf{X}$ (locally of finite presentation) and a point $x \in \mathbf{X}$, one can always find a "refined" local affine neighborhood $\mathrm{Spec}\, A$ of x that allows us to make more explicit computations over this neighborhood. For example, using such local models, we can identify the cotangent complex $\mathbb{L}_A$ with the module of Kähler differentials Ω^1_A, and then we can provide explicit representatives (rather than just cohomology classes) for (closed) p-forms of degree k. In this regard, Bussi, Brav and Joyce proved the following theorem.

Theorem 2.5 ([4, Theorem 4.1]). — *Every derived $\mathbb{K}$-scheme X is Zariski locally modelled on* $\mathrm{Spec}\, A$ *for some "minimal standard form" cdga A in $cdga_{\mathbb{K}}$.*

More precisely, for each $x \in X$ there is a pair $\big(A, i : \mathrm{Spec}\, A \hookrightarrow X\big)$ and $p \in \mathrm{Spec}\, H^0(A)$ such that i is an open inclusion with $i(p) = x$, where A is a special kind of cdga (cf. Definition 2.6).

Moreover, there is a reasonable way to compare two such local charts $i : \mathrm{Spec}\, A \hookrightarrow X$ and $j : \mathrm{Spec}\, B \hookrightarrow X$ on their overlaps via a third chart. For details, see [4, Theorem 4.1 & 4.2].

In the remainder of this section, we shall elaborate the content of Theorem 2.5, and introduce appropriate notions for the constructions of interest. We will closely follow [4, 9].

Definition 2.6. — *$A \in cdga_{\mathbb{K}}$ is of* standard form *if A^0 is a smooth finitely generated $\mathbb{K}$-algebra, the module $\Omega^1_{A^0}$ of Kähler differentials is free A^0-module of finite rank, and the graded algebra A is freely generated over A^0 by finitely many generators, all in negative degrees.*

In fact, there is a systematic way of constructing such cdgas. [4, Example 2.8] explains how to build these cdgas starting from a smooth $\mathbb{K}$-algebra $A^0 := A(0)$ via applying a sequence of localizations. The upshot is follows: Let $n \in \mathbb{N}$, then a cdga A, as a commutative graded algebra, can be constructed inductively from a smooth $\mathbb{K}$-algebra $A(0)$ by adjoining free finite rank modules M^{-i} of generators in degree $-i$ for $i = 1, 2, \ldots, n$.

More precisely, for any given $n \in \mathbb{N}$, we can inductively construct a sequence of cdgas

$$A(0) \longrightarrow A(1) \longrightarrow \ldots \longrightarrow A(i) \longrightarrow \ldots A(n) =: A, \tag{2.2}$$

where $A^0 := A(0)$, and $A(i)$ is obtained from $A(i-1)$ by adjoining generators in degree $-i$, given by M^{-i}, for all i. Here, each M^{-i} is a free finite rank module (of degree $-i$ generators) over $A(i-1)$. Therefore, the underlying commutative graded algebra of $A = A(n)$ is freely generated over $A(0)$ by finitely many generators, all in negative degrees $-1, -2, \ldots, -n$.

Definition 2.7. — *A standard form cdga A is said to be* minimal *at $p \in \operatorname{Spec} H^0(A)$ if $A = A(n)$ is defined by using the minimal possible numbers of graded generators in each degree $\leqslant 0$ compared to all other cdgas locally equivalent to A near p. (There will be an equivalent definition for minimality later, see Definition 2.16.)*

Definition 2.8. — *Let A be a standard form cdga. $A' \in cdga_{\mathbb{K}}$ is called a* localization *of A if A' is obtained from A by inverting an element $f \in A^0$, by which we mean $A' = A \otimes_{A^0} A^0[f^{-1}]$. A' is then of standard form with $A'^0 \simeq A^0[f]$. If $p \in \operatorname{spec} H^0(A)$ with $f(p) \neq 0$, we say A' is a* localization of A at p.

With these definitions in hand, one has the following observations:

Remark 2.9. — Let A be a standard form cdga. If A' is a *localization* of A, then $\operatorname{Spec} A' \subset \operatorname{Spec} A$ is a Zariski open subset. Likewise, if A' is a *localization of A at $p \in \operatorname{spec} H^0(A) \simeq \tau(\operatorname{Spec} A)$*, then $\operatorname{Spec} A' \subset \operatorname{Spec} A$ is a Zariski open neighborhood of p.

Remark 2.10. — Let $A = A(k)$ be a standard form cdga, then there exist generators $x_1^{-i}, x_2^{-i}, \ldots, x_{m_i}^{-i}$ in A^{-i} (after localization, if necessary) with $i = 1, 2, \ldots, k$ and $m_i \in \mathbb{Z}_{\geqslant 0}$ such that

$$A = A(0)\big[x_j^{-i} : i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, m_i\big], \tag{2.3}$$

where the subscript j in x_j^i labels the generators, and the superscript i indicates the degree of the corresponding element. So, we can consider A as a *graded polynomial algebra over* $A(0)$ *on finitely many generators, all in negative degrees.*

Definition 2.11. — *We then define the* virtual dimension *of A to be the integer* $\operatorname{vdim} A = \sum_i (-1)^i m_i$.

Remark 2.12. — Geometrically, the "smoothness" condition on A^0 implies that the corresponding affine $\mathbb{K}$-scheme $U = \operatorname{spec} A^0$ is smooth together with a local (*étale*) coordinate system

$$(x_1^0, x_2^0, \ldots, x_{m_0}^0) : U \longrightarrow \mathbb{A}_{\mathbb{K}}^{m_0}. \tag{2.4}$$

Nice local models for cotangent complexes of derived schemes. Given $A \in cdga_{\mathbb{K}}$, d on A induces a differential on Ω_A^1, denoted again by d. This makes Ω_A^1 into a dg-module (Ω_A^1, d) with the property that $\delta \circ d = d \circ \delta$, where $\delta : A \to \Omega_A^1$ is the universal derivation of degree 0.

Write the decomposition of Ω_A^1 into graded pieces $\Omega_A^1 = \bigoplus_{k=-\infty}^0 \big(\Omega_A^1\big)^k$ with the differential $d : \big(\Omega_A^1\big)^k \to \big(\Omega_A^1\big)^{k+1}$. Then we define the *de Rham algebra of A* as a double complex

$$DR(A) = \operatorname{Sym}_A(\Omega_A^1[1]) \simeq \bigoplus_{p=0}^{\infty} \bigoplus_{k=-\infty}^{0} \big(\Lambda^p \Omega_A^1\big)^k[p], \tag{2.5}$$

where $DR(A)$ has two gradings: the grading w.r.t. p is called the *weight*, and the grading w.r.t. k is called the *degree*. By construction, there are two differentials, namely the *internal differential* d and the *de Rham differential*

d_{dR}. We diagrammatically have

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \\ \cdots \longrightarrow & & \left(\Lambda^{p+1}\Omega_A^1\right)^k[p+1] & \xrightarrow{d} & \left(\Lambda^{p+1}\Omega_A^1\right)^{k+1}[p+1] & \longrightarrow & \cdots \\ & & d_{dR}\uparrow & & \uparrow d_{dR} & & \\ \cdots \longrightarrow & & \left(\Lambda^{p}\Omega_A^1\right)^k[p] & \xrightarrow[d]{} & \left(\Lambda^{p}\Omega_A^1\right)^{k+1}[p] & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \end{array} \tag{2.6}$$

such that $d_{tot} = d + d_{dR}$, and both differentials satisfy the relations

$$d^2 = d_{dR}^2 = 0, \text{ and } d \circ d_{dR} + d_{dR} \circ d = 0. \tag{2.7}$$

We also have the natural multiplication on $DR(A)$:

$$\left(\Lambda^p\Omega_A^1\right)^k[p] \times \left(\Lambda^q\Omega_A^1\right)^\ell[q] \longrightarrow \left(\Lambda^{p+q}\Omega_A^1\right)^{k+\ell}[p+q]. \tag{2.8}$$

Remark 2.13. — The constructions of Ω_A^1 and $DR(A)$ depend only on the underlying commutative graded algebra of A, *not* on the differential d on A.

Remark 2.14. — When $A = A(k)$ is a minimal standard form cdga, there are two important outcomes:

(1) With such local coordinates $(x_1^0, x_2^0, \ldots, x_{m_0}^0)$, we have

$$\Omega_{A^0}^1 \cong A^0 \otimes_{\mathbb{K}} \langle d_{dR}x_1^0, \ldots, d_{dR}x_{m_0}^0\rangle_{\mathbb{K}}. \tag{2.9}$$

Furthermore, the Kähler differentials is a A-module of the form

$$\Omega_A^1 \cong A \otimes_{\mathbb{K}} \langle d_{dR}x_j^{-i} : i = 0, 1, 2, \ldots, k, \ j = 1, 2, \ldots, m_i\rangle_{\mathbb{K}}. \tag{2.10}$$

(2) Ω_A^1 provides a local model for the cotangent complex $\mathbb{L}_A$. That is, in the case of a minimal standard form cdga, the *cotangent complex* $\mathbb{L}_A$ has the identification

$$\mathbb{L}_A = \Omega_A^1. \tag{2.11}$$

Note that if $D(Mod_A)$ denotes the derived category of Mod_A, then we have $\mathbb{L}_A \in D(Mod_A)$ for standard form cdgas. In general, even if both $\mathbb{L}_A$ and Ω_A^1 are closely related, the identification in (2.11) is not true for an arbitrary $A \in cdga_{\mathbb{K}}$ [4].

When $A = A(n)$ is a standard form cdga as in (2.2), we also have the following description for the restriction of the cotangent complex $\mathbb{L}_A$ to $\operatorname{spec} H^0(A)$.

Proposition 2.15 ([4, Proposition 2.12]). — *If $A = A(n)$, with $n \in \mathbb{N}$, is a standard form cdga constructed inductively as in* (2.2)*, then the restriction of $\mathbb{L}_A$ to* $\operatorname{spec} H^0(A)$ *is represented by a complex of $H^0(A)$-modules*

$$0 \longrightarrow V^{-n} \xrightarrow{d^{-n}} V^{-n+1} \longrightarrow \dots \longrightarrow V^{-1} \xrightarrow{d^{-1}} V^0 \longrightarrow 0, \qquad (2.12)$$

where each V^{-i} can in fact be defined as $V^{-i} = H^{-i}(\mathbb{L}_{A(i)/A(i-1)})$, with $\mathbb{L}_{A(i)/A(i-1)}$ the relative cotangent complex *of the map $A(i-1) \to A(i)$ in* (2.2) *satisfying*

$$\mathbb{L}_{A(i)/A(i-1)} \simeq A(i) \otimes_{A(i-1)} M^{-i}[i].$$

Moreover, the differential $V^{-i} \xrightarrow{d^{-i}} V^{-i+1}$ is identified with the composition

$$H^{-i}(\mathbb{L}_{A(i)/A(i-1)}) \longrightarrow H^{-i+1}(\mathbb{L}_{A(i-1)}) \longrightarrow H^{-i+1}(\mathbb{L}_{A(i-1)/A(i-2)}),$$

which can be obtained from the fiber sequences induced by the morphisms $A(i-1) \to A(i)$ in (2.2)*. Note that for $j > -i$, we have $H^j(\mathbb{L}_{A(i)/A(i-1)}) = 0$. More details and the proof can be found in* [4, Proposition 2.12].

With this result in hand, using local coordinates above, write

$$V^{-i} = \langle d_{dR}x_1^{-i}, d_{dR}x_2^{-i}, \dots, d_{dR}x_{m_i}^{-i} \rangle_{A(0)} \quad \text{for } i = 0, 1, \dots, n.$$

It follows that we have a similar local description for the tangent complex $\mathbb{T}_A = (\mathbb{L}_A)^\vee$ of A when restricted to $\operatorname{spec} H^0(A)$. Also, we have an alternative definition of minimality at a point $p \in \operatorname{spec} H^0(A)$ for a cdga of the form $A = A(n)$.

Definition 2.16. — *Let $A = A(n)$, with $n \in \mathbb{N}$, be a standard form cdga constructed inductively as in* (2.2)*. A is said to be* minimal at $p \in \operatorname{spec} H^0(A)$ *if the internal differential $d^{-i}|_p = 0$ in the complex $\mathbb{L}|_{\operatorname{spec} H^0(A)}$ given in* (2.12)*.*

Note that Definition 2.16 implies $m_i = \dim(H^{-i}(\mathbb{L}_A|_p))$ for each i, and hence A is defined by using the minimum number of graded variables in each degree $\leqslant 0$ compared to all other cdgas locally equivalent to A near p. Therefore, one can recover Definition 2.7.

2.2. PTVV's shifted symplectic geometry on derived schemes

Let $\mathbf{X}$ be a locally finitely presented derived $\mathbb{K}$-scheme with $p \geqslant 0$, $k \in \mathbb{Z}$. Pantev et al. [12] define simplicial sets of p-forms of degree k and closed p-forms of degree k on $\mathbf{X}$. Denote these simplicial sets by $\mathcal{A}^p(X, k)$ and $\mathcal{A}^{(p,cl)}(X, k)$, respectively. These definitions are in fact given first for affine derived $\mathbb{K}$-schemes. Later, both concepts are defined for a general $\mathbf{X}$ in terms of mapping stacks. For a summary of key ideas, see [4, Section 3.4].

In our case, we consider $\mathbf{X} = \operatorname{Spec} A$ with A a standard form cdga, and hence take $\Lambda^p \mathbb{L}_A = \Lambda^p \Omega^1_A$. Therefore, elements of $\mathcal{A}^p(X,k)$ form a simplicial set such that k-cohomology classes of the complex $\big(\Lambda^p \Omega^1_A, d\big)$ correspond to the connected components of this simplicial set. Likewise, the connected components of $\mathcal{A}^{(p,cl)}(X,k)$ are identified with the k-cohomology classes of the complex $\prod_{i\geqslant 0}\big(\Lambda^{p+i}\Omega^1_A[-i], d_{tot}\big)$. We want to work with explicit representatives for these classes.

It should be noted that the results that are cited or to be proven in this paper are all about the *local structure* of derived schemes. Thus, it is enough to consider the affine case. Moreover, we always assume all local models are sufficiently nice by using Theorem 2.5 if necessary.

Definition 2.17. — *Let* $\mathbf{X} = \operatorname{Spec} A$ *be an affine derived* $\mathbb{K}$*-scheme for* A *a minimal standard form cdga. A* p-form of degree k on $\mathbf{X}$ *for* $p \geqslant 0$ *and* $k \leqslant 0$ *is an element*

$$\omega^0 \in \big(\Lambda^p \Omega^1_A\big)^k \text{ with } d\omega^0 = 0. \tag{2.13}$$

Note that an element ω^0 defines a cohomology class as being d-closed. That is,

$$[\omega^0] \in H^k\big(\Lambda^p \Omega^1_A, d\big),$$

where two p-forms ω^0_1, ω^0_2 of degrees k are *equivalent* if there exists $\alpha_{1,2} \in \big(\Lambda^p \Omega^1_A\big)^{k-1}$ so that $\omega^0_1 - \omega^0_2 = d\alpha_{1,2}$.

Remark 2.18. — In the classical "underived" case, for instance when $X = \operatorname{spec} A$ is smooth for a commutative $\mathbb{K}$-algebra A, the cotangent complex $\mathbb{L}_X$ is just a vector bundle over X, and denoted simply by T^*X. Then, a p-form ω on X is defined to be a global section of the bundle $\Lambda^p T^*X$. A careful observation reveals that Definition 2.17 does generalize the definition of a p-form on a smooth space in the following sense: It is clear that any commutative $\mathbb{K}$-algebra A can be realized as an object in $cdga_{\mathbb{K}}$ concentrated in degree 0 with the trivial differential. Thus, in the language of Definition 2.17, a naïve notion of "a p-form ω on a smooth space X" is just a p-form ω of degree 0 on $X \simeq \tau \circ \iota(X)$ in $St_{\mathbb{K}}$ such that $\omega \in \big(\Lambda^p T^*X\big)^0$. Note that the condition $d\omega = 0$ holds trivially, and hence $[\omega] \in H^0\big(\Lambda^p T^*X, d = 0\big)$. Here, $\mathbb{L}_X = T^*X$ is again viewed as graded object concentrated in degree 0, with the zero differential.

In DAG, on the other hand, $\Lambda^p \mathbb{L}_X$ is a (double) complex which possesses a non-trivial internal differential as above, and hence one needs to take into account higher non-trivial cohomology groups as well.

Definition 2.19. — *A 2-form ω^0 of degree k on $\mathbf{X} = \operatorname{Spec} A$ for A a minimal standard form cdga is* non-degenerate *if the induced morphism $\omega^0\cdot : \mathbb{T}_A \to \Omega^1_A[k]$, $Y \mapsto \iota_Y \omega^0$, is a quasi-isomorphism, where $\mathbb{T}_A = (\mathbb{L}_A)^\vee = \operatorname{Hom}_A(\Omega^1_A, A)$ is* the tangent complex of A.

Definition 2.20. — *Let $\mathbf{X} = \operatorname{Spec} A$ be an affine derived $\mathbb{K}$-scheme with A a minimal standard form cdga. A* closed p-form of degree k on $\mathbf{X}$ *for $p \geqslant 0$ and $k \leqslant 0$ is a sequence $\omega = (\omega^0, \omega^1, \dots)$ with $\omega^i \in \left(\Lambda^{p+i}\Omega^1_A\right)^{k-i}$ satisfying the following conditions:*

(1) $d\omega^0 = 0$ *in* $\left(\Lambda^p \Omega^1_A\right)^{k+1}$.
(2) $d_{dR}\omega^i + d\omega^{i+1} = 0$ *in* $\left(\Lambda^{p+i+1}\Omega^1_A\right)^{k-i}$, $i \geqslant 0$.

Remark 2.21. —

(1) From Definition 2.20, there exists a natural projection morphism

$$\pi : \mathcal{A}^{(p,cl)}(X,k) \longrightarrow \mathcal{A}^p(X,k), \quad \omega = (\omega^i)_{i\geqslant 0} \longmapsto \omega^0. \tag{2.14}$$

(2) When we restrict ourselves to the classical case as in Remark 2.18, *the one in which everything is concentrated in degree 0*, we have $d = 0$ and hence $d_{tot} = d_{dR}$. Moreover, the only possible non-trivial component of ω is ω^0. Therefore, using the truncation functor as before, the conditions in Definition 2.20 reduce to

$$\omega^0 \in H^0\left(\Lambda^p T^* X\right) \text{ with } d_{dR}\omega^0 = 0. \tag{2.15}$$

Thus, Definition 2.20 reduces to the usual definition of a (de Rham) closed p-form on smooth spaces.

Definition 2.22. — *A* closed 2-form $\omega = (\omega^i)_{i\geqslant 0}$ *of degree k on an affine derived $\mathbb{K}$-scheme* $\operatorname{Spec} A$ *for a minimal standard form cdga A is called* a k-shifted symplectic structure *if $\pi(\omega) = \omega^0$ is non-degenerate.*

2.3. Shifted symplectic Darboux models

One of the main theorems in [4] provides a k-shifted version of the classical Darboux theorem in symplectic geometry. The statement is as follows.

Theorem 2.23 ([4, Theorem 5.18]). — *Given a derived $\mathbb{K}$-scheme X with a k-shifted symplectic form ω' for $k < 0$ and $x \in X$, there is a local model $\left(A, f : \operatorname{spec} A \hookrightarrow X, \omega\right)$ and $p \in \operatorname{spec} H^0(A)$ such that f is an open inclusion with $f(p) = x$, A is a standard form that is minimal at p, and ω is a k-shifted symplectic form on* $\operatorname{Spec} A$ *such that A, ω are in Darboux form, and $f^*(\omega') \sim \omega$ in the space of k-shifted closed 2-forms.*

To be more precise, it is proven in [4, Theorem 5.18] that such ω can be constructed explicitly depending on the integer $k<0$. Indeed, there are three cases in total:

$$(1)\ k \text{ is odd}, \quad (2)\ k \equiv 0 \bmod 4, \quad (3)\ k \equiv 2 \bmod 4.$$

Equivalently, the cases can be expressed as (1) $k/2 \notin \mathbb{Z}$, (2) $k/2$ is even, and (3) $k/2$ is odd, respectively.

In short, Theorem 2.23 says that every k-shifted symlectic derived $\mathbb{K}$-scheme $(\mathbf{X}, \omega')$ is Zariski locally equivalent to $(\operatorname{Spec} A, \omega)$ for some A, ω, where A is a minimal standard form cdga and ω is a k-shifted symplectic form on $\operatorname{Spec} A$ such that ω is given in a standard way depending on the cases above.

In this paper, for simplicity, we will examine a family of explicit Darboux forms for $k<0$ an *odd* integer [4, Example 5.8]. The other cases can be studied in a similar way, but with some modifications. We will outline the steps. More details can be found in [4, Section 5.3].

We first begin with a useful result that plays a significant role in constructing Darboux-type local models below. The upshot is that one can always simplify the form of a given closed 2-form $\omega = (\omega^0, \omega^1, \omega^2, \dots)$ of degree $k<0$ on $\operatorname{Spec} A$ so that ω^0 can be taken to be exact and $\omega^i = 0$ for all $i>0$. More precisely, we have the following result.

Proposition 2.24 ([4, Proposition 5.7]). — *Let $\omega = (\omega^0, \omega^1, \omega^2, \dots)$ be a closed 2-form of degree $k<0$ on $\operatorname{Spec} A$ for A a standard form cdga over $\mathbb{K}$. Then there exist $H \in A^{k+1}$ and $\phi \in (\Omega^1_A)^k$ such that $dH = 0$ in A^{k+2}, $d_{dR}H + d\phi = 0$ in $(\Omega^1_A)^{k+1}$, and $\omega \sim (d_{dR}\phi, 0, 0, \dots)$.*

Moreover, if (H', ϕ') is another such pair for fixed ω, k, A, then there exist $h \in A^k$ and $\sigma \in (\Omega^1_A)^{k-1}$ such that $H - H' = dh$ and $\phi - \phi' = d_{dR}h + d\sigma$.

The proof of Proposition 2.24 is based on the fact that any such forms can be interpreted in the context of *cyclic homology theory of mixed complexes.* Indeed, any such forms can be viewed as cocycles in the so-called *negative cyclic complex of weight p* on $\operatorname{Spec} A$, which is constructed from the de Rham algebra $DR(A)$ in certain way. When $p=2$, there are some useful short exact sequences and vanishing results, by which one can eventually obtain the desired simplification above. For more details on this cyclic homology perspective, we refer to [4, Section 5.2].

Remark 2.25. — Assume (H, ϕ) is a such pair for fixed ω, k, A, with $d_{dR}\phi = k\omega^0$. Let $f \in \mathbb{K}$ be a non-zero element. Define $H' = fH$ and $\phi' = f\phi$. Then both H', ϕ' satisfy the relations $dH' = 0$ and $d_{dR}H' + d\phi' = 0$. From the choices, we also have $d_{dR}\phi' = kf\omega^0$, and hence $\omega \sim (d_{dR}\phi', 0, 0, \dots)$. By

Proposition 2.24, there exist $h \in A^k$ and $\sigma \in (\Omega^1_A)^{k-1}$ such that $H - H' = dh$ and $\phi - \phi' = d_{dR}h + d\sigma$. It follows that $(1-f)H = dh$. Localizing A by the element $(1-f)$ if necessary, we can write $H = d\big[(1-f)^{-1}h\big]$. It means that we can "locally" take H to be d-exact.

Prototype Darboux model for $k < 0$ odd. Let $k = -2\ell - 1$ for $\ell \in \mathbb{N}$. Then the *local model* consists of the following data:

(1) Let $A^0 = A(0)$ be a smooth $\mathbb{K}$-algebra of dim m_0, choose $x^0_1, \dots, x^0_{m_0}$ such that $d_{dR}x^0_1, \dots, d_{dR}x^0_{m_0}$ form a basis for $\Omega^1_{A^0}$. Then A is defined to be the free graded algebra over A^0 generated by variables

$$\begin{aligned} x_1^{-i}, x_2^{-i}, \dots, x_{m_i}^{-i} &\text{ in degree } (-i) \text{ for } i = 1, 2, \dots, \ell, \\ y_1^{k+i}, y_2^{k+i}, \dots, y_{m_i}^{k+i} &\text{ in degree } (k+i) \text{ for } i = 0, 1, \dots, \ell. \end{aligned} \tag{2.16}$$

(2) Ω^1_A is the free A-module of finite rank given by

$$\Omega^1_A \simeq A \otimes_{\mathbb{K}} \langle d_{dR}x_j^{-i}, d_{dR}y_j^{k+i} : i = 0, 1, \dots, \ell,\ j = 1, 2, \dots, m_i \rangle_{\mathbb{K}}. \tag{2.17}$$

(3) Define an element $\omega^0 \in (\Lambda^2\Omega^1_A)^k$ of degree k and weight 2 in $DR(A)$ to be

$$\omega^0 = \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} d_{dR}x_j^{-i} d_{dR}y_j^{k+i}. \tag{2.18}$$

(4) From Proposition 2.24, there exists a pair $(\phi, H) \in (\Omega^1_A)^k \times A^{k+1}$ satisfying the following properties:

(a) $dH = 0$ in A^{k+2}, $d_{dR}H + d\phi = 0$ in $(\Omega^1_A)^{k+1}$, and $d_{dR}\phi = k\omega^0$.

(b) H satisfies the condition (a.k.a. the *classical master equation*)

$$\sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \frac{\partial H}{\partial x_j^{-i}} \frac{\partial H}{\partial y_j^{k+i}} = 0 \text{ in } A^{k+2}, \tag{2.19}$$

which in fact corresponds to the condition "$dH = 0$". We call H the *Hamiltonian.*

(c) Explicitly, we have

$$\phi := \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} \big[-ix_j^{-i} d_{dR}y_j^{k+i} + (k+i)y_j^{k+i} d_{dR}x_j^{i} \big]. \tag{2.20}$$

Note that we can choose another representatives by replacing H, ϕ by $\phi' = \phi + d_{dR}\theta$ and $H' = H + d\theta$ for any $\theta \in A^k$. This modification will leave ω^0 unchanged, and both H', ϕ' satisfy $dH' = 0$ and $d_{dR}H' + d\phi' = 0$. Letting $\theta = \sum_{i,j} \big[(-1)^i x_j^{-i} y_j^{k+i}\big]$, for instance, we may take $\phi := k \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} y_j^{k+i} d_{dR}x_j^{-i}$.

(d) The internal differential d on A can be defined as

$$d|_{A^0} = 0, \quad dx_j^{-i} = \frac{\partial H}{\partial y_j^{k+i}} \ \text{ and } \ dy_j^{k+i} = \frac{\partial H}{\partial x_j^{-i}}. \tag{2.21}$$

(5) Clearly $d_{dR}\omega^0 = 0$, but it is a little bit cumbersome to check that $d\omega^0 = 0$, and ω^0 defines a non-degenerate pairing. For details, we refer to [4, Example 5.8]. As a result, the sequence $\omega := (\omega^0, 0, 0, \dots)$ defines a k-shifted symplectic structure on $\operatorname{Spec} A$.

Definition 2.26. — *If A, ω are as above, then we say that the pair (A, ω) is in (symplectic) Darboux form.*

In brief, Theorem 2.23 implies that every k-shifted symlectic derived $\mathbb{K}$-scheme $(\mathbf{X}, \omega')$, with $k < 0$ odd, is Zariski locally equivalent to $(\operatorname{Spec} A, \omega)$ for a pair A, ω in Darboux form as above. More precisely, Bussi, Brav and Joyce [4, Theorem 5.18] proved that given a k-shifted symlectic derived $\mathbb{K}$-scheme $(\mathbf{X}, \omega')$, one can find a minimal standard form cdga A with “coordinates” x_j^{-i}, y_j^{k+i}, and a Zariski open inclusion $\iota : \operatorname{Spec} A \hookrightarrow \mathbf{X}$ such that $\iota^*(\omega') \simeq (\omega^0, 0, 0, \dots)$ and $\omega^0 = \sum_{i,j} d_{dR}x_j^{-i} d_{dR}y_j^{k+i}$.

Note that the expression of ω^0 above is valid only for the case $k < 0$ odd, and the other cases require some modifications and extra variables depending on whether $k/2$ is even or odd. But, as mentioned before, the proofs follow the same logic. We now give an outline for the cases.

Darboux forms for the other cases of k. For the sake of completeness, we briefly summarize the cases when $k/2$ is even or odd. Here, the main difference from the case k being odd is about the existence of *middle degree variables*. In fact, when k is odd, there is no such degree. But if $k/2$ is even, there are such variables and 2-forms are *anti-symmetric* in these variables. On the other hand, when $k/2$ is odd, such forms are *symmetric* in the middle degree variables. Let us briefly examine each case:

(a) [4, Example 5.9] When $k/2$ is *even*, say $k = -4\ell$ for $\ell \in \mathbb{N}$, the cdga A is now free over $A(0)$ generated by the new set of variables

$$\begin{aligned} &x_1^{-i}, x_2^{-i}, \dots, x_{m_i}^{-i} \text{ in degree } -i \text{ for } i = 1, 2, \dots, 2\ell - 1, \\ &x_1^{-2\ell}, x_2^{-2\ell}, \dots, x_{m_{2\ell}}^{-2\ell}, y_1^{-2\ell}, y_2^{-2\ell}, \dots, y_{m_{2\ell}}^{-2\ell} \text{ in degree } -2\ell, \\ &y_1^{k+i}, y_2^{k+i}, \dots, y_{m_i}^{k+i} \text{ in degree } k + i \text{ for } i = 0, 1, \dots, 2\ell - 1. \end{aligned} \tag{2.22}$$

Then we define an element $\omega^0 = \sum_{i=0}^{2\ell} \sum_{j=1}^{m_i} d_{dR}x_j^{-i} d_{dR}y_j^{k+i}$ in $(\Lambda^2\Omega_A^1)^k$, and set ω to be $(\omega^0, 0, 0, \dots)$ as before. Choose an element $H \in A^{k+1}$ satisfying the analogue of *classical master equation*, and define d on A as in Equation (2.21) using H. We also define the element $\phi \in (\Omega^1_A)^k$ by the analogue of Equation (2.20).

(b) [4, Example 5.10] When $k/2$ is *odd*, say $k = -4\ell - 2$ for $\ell \in \mathbb{N}$, A is freely generated over $A(0)$ by the variables

$$\begin{aligned} &x_1^{-i}, x_2^{-i}, \dots, x_{m_i}^{-i} \text{ in degree } -i \text{ for } i = 1, 2, \dots, 2\ell,\\ &z_1^{-2\ell-1}, z_2^{-2\ell-1}, \dots, z_{m_{2\ell+1}}^{-2\ell-1} \text{ in degree } -2\ell-1,\\ &y_1^{k+i}, y_2^{k+i}, \dots, y_{m_i}^{k+i} \text{ in degree } k+i \text{ for } i = 0, 1, \dots, 2\ell. \end{aligned} \tag{2.23}$$

We then define an element

$$\omega^0 = \sum_{i=0}^{2\ell}\sum_{j=1}^{m_i} d_{dR}x_j^{-i} d_{dR}y_j^{k+i} + \sum_{j=1}^{m_{2\ell+1}} d_{dR}z_j^{-2\ell-1} d_{dR}z_j^{-2\ell-1}$$

in $(\Lambda^2\Omega^1_A)^k$, and set $\omega := (\omega^0, 0, 0, \dots)$ as before. Choose an element $H \in A^{k+1}$ satisfying the analogue of *classical master equation*

$$\sum_{i=1}^{2\ell}\sum_{j=1}^{m_i} \frac{\partial H}{\partial x_j^{-i}} \frac{\partial H}{\partial y_j^{k+i}} + \frac{1}{4}\sum_{j=1}^{m_{2\ell+1}} \Big(\frac{\partial H}{\partial z_j^{-2\ell-1}}\Big)^2 = 0 \text{ in } A^{k+2}. \tag{2.24}$$

Set d on A as in Equation(2.21) with extra data $dz_j^{-2\ell-1} := \frac{1}{2}\frac{\partial H}{\partial z_j^{-2\ell-1}}$.

Finally, we define the element $\phi \in \Omega^1_A)^k$ by

$$\phi = \sum_{i=0}^{2\ell}\sum_{j=1}^{m_i} \big[-ix_j^{-i} d_{dR}y_j^{k+i} + (-1)^{i+1}(k+i)y_j^{k+i} d_{dR}x_j^{i}\big] + k\sum_{j=1}^{m_{2\ell+1}} z_j^{-2\ell-1} d_{dR}z_j^{-2\ell-1}. \tag{2.25}$$

Remark 2.27. — In either case, the virtual dimension of A is *even*. In fact, for any $k < 0$ we have

$$\operatorname{vdim} A = \begin{cases} 2\sum_i (-1)^i m_i, & k \text{ even},\\ 0, & k \text{ odd}. \end{cases}$$

Remark 2.28. — The classical Darboux theorem states that for a symplectic manifold (X, ω), one can find a chart $(U; x_1, \dots, x_n, y_1, \dots, y_n)$ such that $\omega|_U = \sum_j d_{dR}x_j d_{dR}y_j$. Moreover, we can write $\lambda = \sum_j x_j d_{dR}y_j$, called the *Liouville form*, such that $\omega|_U = d_{dR}\lambda$.

In this derived framework, the element ϕ above may seem to play the role of λ. However, it is important to notice that ϕ is not a 1-form (of degree k) in the sense of Definition 2.17, because $d\phi \neq 0$. Therefore, one needs to modify ϕ to obtain a genuine 1-form of degree k.

3. Shifted contact structures and a Darboux-type theorem

3.1. Basics of classical contact geometry

It is very well-known that contact manifolds are viewed as the odd-dimensional analogues of symplectic manifolds. In that respect, they have a number of common features: there is a Darboux theorem providing a local model for such structures; there is no local invariants; and it is more interesting to study their global properties. For details, we refer to [7].

In this section, we shall revisit the basic aspects of contact geometry. There are in fact equivalent ways of describing the notion of a *contact structure*. We prefer to use the one below.

Definition 3.1. — *Let X be a manifold of dimension* $2n+1$. A contact structure *is a smooth field of tangent hyperplanes $\xi \subset TM$ (of rank $2n$) with the property that for any smooth locally defining 1-form α, i.e. $\xi = \ker(\alpha)$, the 2-form $d_{dR}\alpha|_\xi$ is non-degenerate.*

Remark 3.2. — It is also possible to write $\xi = \ker(\alpha)$ with α a *globally* defined contact form on M if and only if ξ is *coorientable*, by which we mean that the quotient line bundle TM/ξ is trivial. Except some pathological cases, it suffices to work with coorientable contact structures. More details can be found in [7, Sections 1 & 2].

Note that if $d_{dR}\alpha|_\xi$ is non-degenerate, then for each $p \in X$, $\xi_p = \ker(\alpha_p)$ is a symplectic vector space with a symplectic form $\omega_p := d_{dR}\alpha_p|_{\xi_p}$. Therefore, we also call such 2-form $d_{dR}\alpha|_\xi$ *symplectic.*

It follows from the theory of symplectic vector spaces that $\dim \xi_p$ is even and the symplectic form on ξ_p has a canonical form. It means that there exists a symplectic basis $\{e_1, f_1, \dots, e_n, f_n\}$ for ξ_p (and the corresponding dual basis $\{e_1^*, f_1^*, \dots, e_n^*, f_n^*\}$ for ξ_p^*) satisfying

$$\omega_p(e_i, e_j) = 0 = \omega_p(f_i, f_j) \text{ and } \omega_p(e_i, f_j) = -\omega_p(f_j, e_i) = \delta_{ij} \quad \forall\, i, j, \tag{3.1}$$

so that ω_p has the form $\omega_p = \sum_i e_i^* \wedge f_i^*$.

Let $(X, \xi = \ker(\alpha))$ be a contact manifold of dim $2n+1$, and $p \in X$. Then we have a splitting

$$T_pX = \xi_p \oplus \ker d_{dR}\alpha_p|_{\xi_p}, \tag{3.2}$$

where $\dim \xi_p = 2n$ and $\dim \ker d_{dR}\alpha_p|_{\xi_p} = 1$. In fact, as $d_{dR}\alpha|_\xi$ is non-degenerate, one can find a local trivialization $\{e_1, f_1, \dots, e_n, f_n, r\}$ of $TM = \ker \alpha \oplus \text{rest}$ such that

$$\ker \alpha = \text{Span}\{e_1, f_1, \dots, e_n, f_n\} \;\text{ and }\; \text{rest} = \text{Span}\{r\}. \tag{3.3}$$

Moreover, using this splitting, one can find a unique vector field $\mathbf{R}$, called the *Reeb vector field of* α, satisfying $\iota_{\mathbf{R}} d_{dR}\alpha = 0$ and $\iota_{\mathbf{R}}\alpha = 1$.

Example 3.3. — On $\mathbb{R}^{2n+1}$ with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$, the so-called *standard contact form* is given by

$$\alpha_{std} = -d_{dR}z + \sum_{i=1}^{n} y_i d_{dR}x_i. \tag{3.4}$$

Let $\xi \subset T\mathbb{R}^{2n+1}$ be the hyperplane field of rank $2n$ defined by α_{std}, i.e. $\xi = \ker \alpha_{std}$. Then we observe that

$$\ker(\alpha_{std}) = \operatorname{Span}\left\{\frac{\partial}{\partial y_j}, y_j\frac{\partial}{\partial z} + \frac{\partial}{\partial x_j} : j = 1, \dots, n\right\}$$

such that $d_{dR}\alpha_{std} = \sum_i d_{dR}x_i \wedge d_{dR}y_i$.

Write $A_j = \partial/\partial y_j$ and $B_j = y_j\partial/\partial z + \partial/\partial x_j$, then it is enough to observe that

$$d_{dR}\alpha_{std}(A_i, B_j) = \delta_{ij} \text{ and } d_{dR}\alpha_{std}(A_i, A_j) = 0 = d_{dR}\alpha_{std}(B_i, B_j).$$

It follows that $d_{dR}\alpha_{std}|_{\xi}$ is non-degenerate, and hence α_{std} is a contact form. Moreover, the Reeb vector field of α_{std} is $\mathbf{R} = \partial/\partial z$.

As in the symplectic case, there is a Darboux-type theorem for contact structures. It basically says that all contact structures can be locally given as in Equation (3.4). More formally, we have:

Theorem 3.4 (Darboux Theorem for contact structures). — *Let (X, α) be a contact manifold of dimension $2n+1$, and $p \in X$. Then there exists a local coordinate system $(U; x_1, \dots, x_n, y_1, \dots, y_n, z)$ around p such that $p = (0, 0, \dots, 0)$ and*

$$\alpha|_U = -d_{dR}z + \sum_{i=1}^{n} y_i d_{dR}x_i. \tag{3.5}$$

3.2. Shifted contact structures and Darboux forms

In this section, we provide an appropriate analogue of Definition 3.1 for derived spaces and study a local framework for shifted contact structures. Let $\mathbf{X}$ be a locally finitely presented derived (Artin) stack. Then we have:

Definition 3.5. — *A* pre-k-shifted contact structure *on $\mathbf{X}$ consists of a perfect complex $\mathcal{K}$ on $\mathbf{X}$ with a monomorphism $\kappa : \mathcal{K} \to \mathbb{T}_{\mathbf{X}}$ of perfect complexes whose cone* Cone(κ) *is of the form $L[k]$, up to quasi-isomorphism, where L is a line bundle*[1]*. Denote such a structure on $\mathbf{X}$ by $(\mathcal{K}, \kappa, L)$.*

(1) In the spirit of Remark 3.2, we say that a pre-k-shifted contact structure is *coorientable* if L in the data is trivial.

Definition 3.6. — *We say that a pre-k-shifted contact structure* $(\mathcal{K},\kappa,L)$ *on* $\mathbf{X}$ *is a* k-shifted contact structure *if locally on* $\mathbf{X}$*, where* L *is trivial, the induced* k*-shifted 1-form* $\alpha:\mathbb{T}_X\to\mathcal{O}_X[k]$ *is such that* $\mathcal{K}$ *is equivalent to* $\mathrm{Cocone}(\underline{\alpha}:\mathbb{T}_X\to\operatorname{Im}\alpha)$(2) *and the 2-form* $d_{dR}\alpha$ *is non-degenerate on* $\mathcal{K}$*. In that case, the triangle* $\mathcal{K}\to\mathbb{T}_{\mathbf{X}}\to L[k]$ *splits locally. Call such local form a* k-contact form.

Let $\mathbf{X}$ be as above with a k-shifted contact structure $(\mathcal{K},\kappa,L)$. Recall from Yoneda's lemma, $\mathbf{X}(A)\simeq Map_{dPstk}(\operatorname{Spec}A,\mathbf{X})$, and hence any A-point $p\in\mathbf{X}(A)$ can be seen as a morphism $p:\operatorname{Spec}A\to\mathbf{X}$ of derived pre-stacks. Then, let us consider the pair (p,α_p), with $p\in\mathbf{X}(A),\ \alpha_p\in p^*(\mathbb{L}_{\mathbf{X}}[k])$ a k-contact form. For $A\in cdga_{\mathbb{K}}$, there is a $\mathbb{G}_m(A)$-action on the pair (p,α_p) by

$$f\triangleleft(p,\alpha_p):=(p,f\cdot\alpha_p).$$

Denote by H^0 the functor sending $A\mapsto H^0(A)$. Denote the image under H^0 of an element f simply by f^0. Note that localizing A if necessary, w.l.o.g. we may assume that the image f^0 is always invertible. It follows that f^0 lies in $(A^0)^\times$, which is by definition $\mathbb{G}_m(A^0)=(A^0)^\times$.

Remark 3.7. — If $\mathbf{X},(p,\alpha_p)$, and the $\mathbb{G}_m(A)$-action are as above, then for an element $f\in\mathbb{G}_m(A)$, we can obtain $\mathrm{Cocone}(f\cdot\alpha_p)\simeq\mathrm{Cocone}(\alpha_p)$ by using the invertibility of f.

Because our results in this paper are all about the *local structure theory* of contact derived stacks, we will focus on the *refined affine case* in the sense of Theorem 2.5. In this regard, we will always assume that our *refined local models* are given in terms of minimal standard form cdgas.

From Proposition 2.15, on a refined affine neighborhood, say on $\operatorname{Spec}A$ with A a minimal standard form cdga, the perfect complexes $\mathbb{T}_A,\mathbb{L}_A$, when restricted to spec $H^0(A)$, are both free finite complexes of $H^0(A)$-modules. In that case, Definitions 3.5 and 3.6, and Remark 3.7 will reduce to the following local descriptions, where $\mathcal{K},\mathrm{Cone}(\kappa)$ will be equivalent to the ordinary $\ker(\alpha),\operatorname{coker}(\kappa)$, respectively in $D(Mod_A)$; and L in the splitting corresponds to the line bundle generated by the Reeb vector field of the classical case.

3.2.1. Shifted contact structures with (nice) local models.

We first start with some relevant notions. Recall that the *(mapping) cone* of a morphism $f:A\to B$ in some homotopical category is a realization of

(2) In brief, the *image* of a morphism $f:X\to Y$ in a ∞-category is a universal subobject $\operatorname{Im}f$ with the epi-mono factorization $X\xrightarrow[epi]{}\operatorname{Im}f\xhookrightarrow[mono]{}Y$. Here we denote the factor $X\to\operatorname{Im}f$ by $\underline{f}$.

the homotopy cofiber of f. That is, it is the homotopy pushout satisfying universally the homotopy diagram

$$\begin{CD} A @>>> \star \\ @VfVV @VVV \\ B @>>> \mathrm{Cone}(f). \end{CD} \tag{3.6}$$

It is also called the *homotopy cokernel* of f or the *weak quotient* of B by the image of A under f.

As a dual notion, the *(mapping) cocone* of a morphism $f : A \to B$, with B a pointed object, in a homotopical category is a particular realization of the homotopy fiber of f (i.e. of the homotopy pullback of the point along f). In that case, the following diagram homotopy commutes:

$$\begin{CD} \mathrm{Cocone}(f) @>>> \star \\ @VVV @VVV \\ A @>>f> B. \end{CD} \tag{3.7}$$

It is also called the *homotopy kernel* of f.

Remark 3.8. — In the case of a morphism $f : A \to B$ (perfect) of complexes, we have $\star = 0$ (as the initial and terminal objects) and that the complex $\ker(f)$, which is the subcomplex of A formed by the (strict) kernels $\{\ker(f_n)\}$, commutes the diagram

$$\begin{CD} \ker(f) @>>> \star \\ @ViVV @VV0V \\ A @>>f> B, \end{CD} \tag{3.8}$$

which essentially means $f \circ i = 0$. This is in fact a strict pullback diagram. From the universality of the homotopy kernel of f, there is a natural inclusion $\ker(f) \hookrightarrow \mathrm{Cocone}(f)$ from the strict fiber to the homotopy fiber (i.e. from the strict kernel to the homotopy kernel). This map is in fact an equivalence if f is a fibration (i.e. a surjective morphism of complexes).

coker(f), on the other hand, is the quotient complex of Bformed by the cokernels $\{\text{coker}(f_n)\}$. By definition, coker(f) commutes the diagram

$$\begin{array}{ccc} A & \longrightarrow & \star \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 0} \\ B & \longrightarrow & \text{coker}(f). \end{array} \tag{3.9}$$

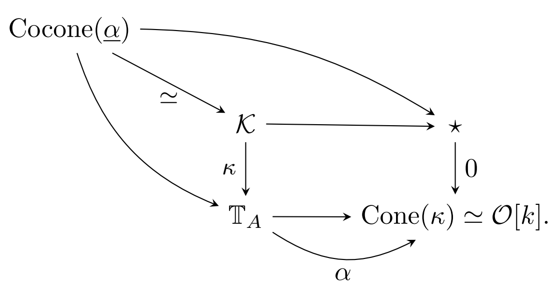
From the universality of the homotopy cokernel of f, there is a natural map $\text{Cone}(f) \to \text{coker}(f)$ from the homotopy cofiber to the strict cofiber (i.e. from the homotopy cokernel to the strict cokernel). This map is in fact an equivalence if f is a cofibration (i.e. an injective morphism of complexes).

Now, in the presence of Remark 3.8, we revisit Definitions 3.5 and 3.6 with (nice) local models.Let $\mathbf{X} = \text{Spec}\, A$ be an affine derived $\mathbb{K}$-scheme for A a minimal standard form cdga, endowed with a k-shifted contact structure $(\mathcal{K}, \kappa, L)$ for $k < 0$. Assume also that L is trivial on Spec A. From definitions, the triangle $\mathcal{K} \to \mathbb{T}_{\mathbf{X}} \to L[k]$ splits and Cone(κ) is of the form a line bundle $L[k]$. In that case, we have the homotopy commuting square

$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & \star \\ {\scriptstyle \kappa}\downarrow & & \downarrow{\scriptstyle 0} \\ \mathbb{T}_{\mathbf{X}} & \longrightarrow & \text{Cone}(\kappa). \end{array} \tag{3.10}$$

Since $\kappa : \mathcal{K} \to \mathbb{T}_{\mathbf{X}}$ is a monomorphism, the homotopy cofibers are equivalent to strict cofibers by Remark 3.8. Thus, Cone(κ) is equivalent to the ordinary coker(κ).

On $\mathbf{X} = \text{Spec}\, A$, where L is trivial, $L[k]$ is of the form $\mathcal{O}[k]$. Recall that since A is a minimal standard form cdga, we have $\mathbb{L}_{\mathbf{X}} \simeq \mathbb{L}_A$ in $D(Mod_A)$ and the perfect complexes $\mathbb{T}_A, \mathbb{L}_A$, when restricted to spec $H^0(A)$, are both free finite complexes of $H^0(A)$-modules. Let α be a k-contact form, with the underlying k-shifted 1-form $\alpha : \mathbb{T}_A \to A[k]$, then we have



(3.11)

Here both outer and inner squares (resp. the homotopy fiber and the homotopy cofiber) commute, and $\mathcal{K} \simeq \mathrm{Cocone}(\underline{\alpha})$, with $\alpha \in \mathcal{A}^1(\mathrm{Spec}\, A, k)$ a map of perfect complexes. From Remark 3.8, there is a natural map $\ker(\underline{\alpha}) \hookrightarrow \mathcal{K}$ from the strict kernel to the homotopy kernel of $\underline{\alpha}$. Since $\underline{\alpha}$ is an epimorphism, the natural map $\ker(\underline{\alpha}) \hookrightarrow \mathcal{K}$ is then an equivalence. Thus, on suitable local models, we can use the strict kernel to represent the homotopy kernel.

Now, we consider an explicit general form of the underlying k-shifted 1-form $\alpha : \mathbb{T}_A \to A[k]$ on $\mathrm{Spec}\, A$ with A as above. Note that it is the minimal (at $p \in \mathrm{Spec}\, H^0(A)$) compared to all other cdgas quasi-isomorphic to A (at $p \in \mathrm{Spec}\, H^0(A)$). Explicitly, Letting $A = A(-k)$, A is the free graded algebra over $A(0)$ generated by the variables $x_1^{-i}, x_2^{-i}, \ldots, x_{m_i}^{-i}$, with $m_i \in \mathbb{Z}$, for $i = 1, \ldots, -k$ such that $d_{dR}x_1^{-i}, d_{dR}x_2^{-i}, \ldots, d_{dR}x_{m_i}^{-i},\ i = 1, \ldots, -k$, is a A-basis for Ω_A^1. Write $\underline{\alpha} = \alpha = \sum_{i,j} \alpha_j^{k+i} d_{dR} x_j^{-i}$ with $\alpha_j^n \in A^n$. Notice that by definition, we have $\ker(\alpha) = \ker(\underline{\alpha})$. Thus, when we consider the strict kernel on such local models, we simply write $\ker(\alpha)$.

Definition 3.9. — *With the discussion above, when restricted to the (nice) local models, we can say w.l.o.g. that a* k-shifted contact structure on $\mathbf{X} = \mathrm{Spec}\, A$ *is a submodule $\mathcal{K}$ of $\mathbb{T}_A$ such that $\mathcal{K} \simeq \ker(\alpha)$ for a k-shifted 1-form α with the property that the k-shifted 2-form $d_{dR}\alpha$ is non-degenerate on* $\ker(\alpha)$ *and the complex* $\mathrm{Cone}(i : \mathcal{K} \hookrightarrow \mathbb{T}_A) \simeq \mathrm{coker}(i)$ *is the quotient complex and of the form $L[k]$, with L a line bundle.*

Remark 3.10. — Let $\mathbf{X} = \mathrm{Spec}\, A$ be an affine derived $\mathbb{K}$-scheme for A a minimal standard form cdga. For any $f \neq 0$ in A and any k-shifted contact form α, one has $\ker(\alpha) \simeq \ker(f\alpha)$. Hence, both define equivalent contact structures on $\mathbf{X}$. In fact, this follows from the fact that the contraction operation ι_Y on the de Rham algebra $DR(A)$ with a homogeneous vector field Y is the unique derivation of degree $|Y| + 1$ such that $\iota_Y g = 0$ and $\iota_Y d_{dR} g = Y(g)$ for all $g \in A$. Therefore,

$$\iota_Y(f\alpha) = (\iota_Y f) \cdot \alpha + (-1)^{|Y|+1} f \cdot \iota_Y \alpha = (-1)^{|Y|+1} f \cdot \iota_Y \alpha.$$

Adopting the classical terminology (in terms of non-integrable distributions), on refined local models, we sometimes call the subcomplex $\ker(\alpha)$ of $\mathbb{T}_{\mathbf{X}}$ a *contact structure* and the corresponding k-shifted 1-form α a *(locally) defining k-contact form.*

3.2.2. A prototype shifted contact model

In what follows, we give a prototype construction for k-shifted contact forms, which is similar to the previous case of shifted symplectic Darboux models.

Example 3.11. — In this example, fixing $\ell \in \mathbb{N}$, we will present how to construct an explicit standard form cdga $A = A(n)$ for $n = 2\ell + 1$ and a k-shifted contact structure α_0 with $k = -2\ell - 1$.

First, we consider a smooth $\mathbb{K}$-algebra $A(0)$ of dimension $m_0 + 1$. We assume that there exist degree 0 variables $x_1^0, x_2^0, \dots, x_{m_0}^0, \widetilde{x}_1^0$ in $A(0)$ such that $d_{dR}x_1^0, \dots, d_{dR}x_{m_0}^0, d_{dR}\widetilde{x}_1^0$ form a basis for $\Omega^1_{A(0)}$ over $A(0)$. This choice can be made by localizing $A(0)$ if necessary.

Next, choosing non-negative integers $m_1, \dots, m_\ell$, define a commutative graded algebra A to be the free graded algebra over $A(0)$ generated by variables

$$x_1^{-i}, x_2^{-i}, \dots, x_{m_i}^{-i} \qquad \text{in degree } -i \text{ for } i = 1, 2, \dots, \ell, \tag{3.12}$$

$$y_1^{k+i}, y_2^{k+i}, \dots, y_{m_i}^{k+i} \qquad \text{in degree } k+i \text{ for } i = 0, 1, \dots, \ell. \tag{3.13}$$

It follows that Ω^1_A is the free A-module of finite rank with an A-basis

$$\left\{d_{dR}x_j^{-i}, d_{dR}y_j^{k+i}, d_{dR}\widetilde{x}_1^0 : i = 0, 1, \dots, \ell,\ j = 1, 2, \dots, m_i\right\}.$$

Choose an element $z \in A^k$ such that $dz = H$ in A^{k+1} and H is the Hamiltonian satisfying the condition (the *classical master equation*)

$$\sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \frac{\partial H}{\partial x_j^{-i}} \frac{\partial H}{\partial y_j^{k+i}} = 0 \text{ in } A^{k+2}.$$

Then we define the internal differential on A by Equation (2.21). As discussed before, the condition on H above is equivalent to saying "$dH = 0$".

By construction, (A, d) is a standard form cdga with $A = A(n = 2\ell + 1)$ defined inductively by adjoining free modules $M^{-i} = \langle x_1^{-i}, x_2^{-i}, \dots, x_{m_i}^{-i}\rangle_{A(i-1)}$ for $i = 1, 2, \dots, \ell$ and $M^{k+i} = \langle y_1^{k+i}, y_2^{k+i}, \dots, y_{m_i}^{k+i}\rangle_{A(-k-i-1)}$ for $i = 0, \dots, \ell$.

Now, we define an element $\alpha_0 \in (\Omega^1_A)^k$ by

$$\alpha_0 = -d_{dR}z + \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} y_j^{k+i} d_{dR}x_j^{-i}. \tag{3.14}$$

Then $d_{dR}\alpha_0$ defines an element, denoted by $\omega_0 \in (\Lambda^2\Omega^1_A)^k$, such that we get $\omega_0 = \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} d_{dR}x_j^{-i} d_{dR}y_j^{k+i}$. From [4, Example 5.8], ω_0 is closed w.r.t. both d and d_{dR} such that

$$dH = 0 \text{ in } A^{k+2},\ \ d_{dR}H + d\phi = 0 \text{ in } (\Omega^1_A)^{k+1},\ \text{ and } \ d_{dR}\phi = k\omega_0. \tag{3.15}$$

We just set $\phi := k \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} y_j^{k+i} d_{dR}x_j^{-i}$. Now write

$$\alpha_0 = -d_{dR}z + (1/k)\phi. \tag{3.16}$$

Using (3.15) and scaling z by the constant k, it is now straightforward to check that α_0 is d-closed, and hence a 1-form of degree k.

Now, it remains to check that $\omega_0|_{\ker\alpha_0}$ is non-degenerate. Denote the vector fields annihilating α_0 by

$$\zeta_j^i = \partial/\partial y_j^{k+i} \quad \text{and} \quad \eta_j^i = \partial/\partial x_j^{-i} + k y_j^{k+i}\partial/\partial z. \tag{3.17}$$

Then $\ker\alpha_0 = \mathrm{Span}\{\zeta_j^i,\ \eta_j^i : \ i = 0, 1, \ldots \ell,\ j = 1, \ldots, m_i\}_{A(0)}$. So we obtain $\omega_0(\zeta_j^i, \eta_{j'}^{i'}) = \delta_{jj'}^{ii'}$ and $\omega_0(\zeta_j^i, \zeta_{j'}^{i'}) = 0 = \omega_0(\eta_j^i, \eta_{j'}^{i'})$. From linear algebra, this is sufficient to ensure that $d_{dR}\alpha_0|_{\ker\alpha_0}$ is non-degenerate in sense of Definition 2.19, and hence α_0 is a k-shifted contact structure on $\mathrm{Spec}\, A$.

Therefore, one has a natural splitting

$$\mathbb{T}_A|_{\mathrm{spec}\, H^0(A)} = \ker\alpha_0|_{\mathrm{spec}\, H^0(A)} \oplus \mathrm{Rest}\,|_{\mathrm{spec}\, H^0(A)}, \tag{3.18}$$

where for each degree i,

$$(\mathbb{T}_A|_{\mathrm{spec}\, H^0(A)})^i = (\ker\alpha|_{\mathrm{spec}\, H^0(A)})^i \oplus \mathrm{Rest}^i\,|_{\mathrm{spec}\, H^0(A)}$$

so that we have

$$\ker\alpha_0 = \mathrm{Span}\{\zeta_j^i,\ \eta_j^i : \ i = 0, 1, \ldots \ell,\ j = 1, \ldots, m_i\}_{A(0)}.$$

$$\mathrm{Rest}\,|_{\mathrm{spec}\, H^0(A)} = \langle \partial/\partial\widetilde{x}_1^0 \rangle_{A(0)}.$$

Note that we can find a vector field $\mathbf{R}$ such that $\iota_{\mathbf{R}} d_{dR}\alpha_0 = 0$ and $\iota_{\mathbf{R}}\alpha_0 = 1$ (i.e. $\mathbf{R} \in \mathrm{Rest}\,|_{\mathrm{spec}\, H^0(A)}$ with scaling). Since $\mathbf{R} \notin \ker\alpha_0|_{\mathrm{spec}\, H^0(A)}$, we get $\iota_{\mathbf{R}}\phi = 0$. Thus, $\iota_{\mathbf{R}} d_{dR} z = -1$ as $\iota_{\mathbf{R}}\alpha_0 = 1$.

It follows that $d_{dR}z, d_{dR}x_1^{-i}, \ldots, d_{dR}x_{m_i}^{-i}, d_{dR}y_1^{k+i}, \ldots, d_{dR}y_{m_i}^{k+i}$ will span $\mathbb{L}_A|_{\mathrm{spec}\, H^0(A)}$ as well. (Otherwise, if $d_{dR}z$ was in the span of $d_{dR}x_1^{-i}, d_{dR}y_j^{k+i}$, then $\iota_{\mathbf{R}} d_{dR} z$ would vanish as $\mathbf{R} \in \mathrm{Rest}\,|_{\mathrm{spec}\, H^0(A)}$.)

Definition 3.12. — *If A and α_0 with the variables x_j^{-i}, y_j^{k+i}, z are as above, we then say A, α_0 are* in contact Darboux form.

Remark 3.13. — Note that the expression in Equation (3.16) will still be valid for the other cases (a) $k \equiv 0 \bmod 4$, and (b) $k \equiv 2 \bmod 4$. Equations (2.22)–(2.25) show that the other cases in fact involve modified versions of H, d, and ϕ with some possible extra terms. In any case, the modified A, α_0 would also serve as the desired contact model. Following the same terminology as above, we again say A, α_0 *are in (contact) Darboux form.*

3.3. A Darboux-type theorem for shifted contact derived schemes

In what follows, we give the proof of Theorem 1.1, which essentially says that every k-shifted contact derived $\mathbb{K}$-scheme $\mathbf{X}$ is locally equivalent to

$(\operatorname{Spec} A, \alpha_0)$ for A a minimal standard form cdga and α_0 as in Section 3.2.2. More precisely, we have:

THEOREM 3.14. — *Let $\mathbf{X}$ be a (locally finitely presented) derived $\mathbb{K}$-scheme with a k-shifted contact structure $(\mathcal{K}, \kappa, L)$ for $k < 0$, and $x \in \mathbf{X}$. Then there is a local contact model $\big(A, \alpha_0\big)$ and $p \in \operatorname{spec} H^0(A)$ such that $i : \operatorname{Spec} A \hookrightarrow \mathbf{X}$ is an open inclusion with $i(p) = x$, A is a standard form cdga that is minimal at p, and α_0 is a k-shifted contact form on* $\operatorname{Spec} A$ *such that A, α_0 are in standard contact Darboux form.*

Note that for $k < 0$ *odd*, for instance, the pair (A, α_0) can be explicitly described in Example 3.11 by Equations (3.12)–(3.16). For the other cases, one should use another sets of variables as in Equations (2.22) and (2.23), and modify H, ϕ, d accordingly (cf. Remark 3.13).

Before giving the proof, we begin by some remarks and simplifying assumptions.

Remark 3.15. — We first note that as $\mathbf{X}$ is a locally finitely presented derived $\mathbb{K}$-scheme, there exists a cover by affine derived $\mathbb{K}$-subschemes of finite presentation. Thus, for $x \in \mathbf{X}$ we can choose a cdga B of finite presentation with a Zariski open inclusion $\iota : \operatorname{Spec} B \hookrightarrow \mathbf{X}$ and a unique $q \in \operatorname{spec} H^0(B)$ such that $q \mapsto x$. In this case, B being of finite presentation implies that $\mathbb{L}_B$ has finite *Tor-amplitude*[(3)] , say in $[-n, 0]$ for some $n \in \mathbb{Z}_+$. Then it follows from [11, Proposition 7.2.4.23] that for any integer $k < 0$, $\mathbb{L}_B[-k-n]$ has Tor-amplitude in $[k, 0]$. Notice that both complexes are equivalent. Since we will be interested in k-shifted structures, for the proof, we equivalently use this shifted complex of B. Therefore, while getting a refined neighborhood, we assume w.l.o.g. that for $k < 0$, the corresponding cdga B is such that its cotangent complex $\mathbb{L}_B$ has Tor-amplitude in $[k, 0]$.

Remark 3.16. — Let $\mathbf{X}, x, B, q$ be as in Remark 3.15. Then the construction given in [4, Theorem 4.1] (cf. Theorem 2.5) ensures that there exists a suitable localization of B at q, which is equivalent to a minimal standard form cdga $A = A(m)$, for some m, constructed inductively as in (2.2) such that there exists $p \in \operatorname{Spec} A$ with $p \mapsto q$. Here the integer m is determined by the Tor-amplitude of $\mathbb{L}_B$, which is by assumption $\leqslant -k$[(4)] . It should also be noted that during induction, each $\mathbb{L}_{A(\ell)}$ has Tor-amplitude in $[-\ell, 0]$. Moreover, as there is an equivalence $A(-k) \to B$, Proposition 2.15 provides a simple description for $\mathbb{L}_A$, with Tor-amplitude $\leqslant -k$. That is, when restricted to $\operatorname{spec} H^0(A)$, $\mathbb{L}_A$ is equivalent to the complex $\Omega^1_A \otimes_A H^0(A)$ of free

(3) We say that a perfect complex E of R-modules has Tor-amplitude in some interval $[a, b]$ if $H^i(E \otimes^L_R N) = 0$ for all $i \notin [a, b]$ and for all R-modules N.

(4) It means the perfect complex has Tor-amplitude in $[k, 0]$

$H^0(A)$-modules

$$0 \longrightarrow V^k \longrightarrow V^{k+1} \longrightarrow \dots \longrightarrow V^{-1} \longrightarrow V^0 \longrightarrow 0,$$

with $d|_p^{-i} = 0$ for $i = 1, 2, \dots, -k$ (due to the minimality of A).

Proof of Theorem 3.14. — Let $k < 0$ and $x \in \mathbf{X}$, apply Theorem 2.5 to get a refined neighborhood $\mathbf{U} = \operatorname{Spec} A$ of x with $p \in \operatorname{spec} H^0(A)$ such that $i : \operatorname{Spec} A \hookrightarrow \mathbf{X}$ is an open inclusion, $i(p) = x$, and A is a standard form cdga that is minimal at p. From Remarks 3.15 & 3.16, we assume that A is in fact constructed inductively as described in (2.2) with $A = A(-k)$ such that $\mathbb{L}_A$ has Tor-amplitude in $[k, 0]$.

W.l.o.g., we also assume that L is trivial on $\mathbf{U}$, and hence, over $\mathbf{U}$, the induced 1-form $\alpha : \mathbb{T}_X \to \mathcal{O}_X[k]$ is such that $\mathcal{K}$ is the cocone of $\underline{\alpha}$, up to quasi-isomorphism, and the 2-form $d_{dR}\alpha$ is non-degenerate on $\mathcal{K}$. In that case, the triangle $\mathcal{K} \to \mathbb{T}_{\mathbf{X}} \to L[k]$ splits over $\mathbf{U}$.

We fix the locally defining 1-form α for the rest of the proof. We denote the restriction of α simply by α_u. From now on, we will use the properties of shifted contact structures when restricted to nice local models (cf. Remark 3.8 and relevant discussions after that). I.e., we consider a simplified (local) description for which $\mathcal{K}, \operatorname{Cone}(\kappa)$ will be equivalent to the ordinary $\ker \alpha, \operatorname{coker}(\kappa)$, respectively in $D(Mod_A)$.

Consider the sequence $\omega_u := (d_{dR}\alpha_u, 0, 0, \dots)$, which defines a closed k-shifted 2-form on $\mathbf{U}$ in the sense of Definition 2.20. Applying Proposition 2.24 to $k\omega_u$, we obtain elements $H \in A^{k+1}$ and $\phi \in (\Omega^1_A)^k$ such that $dH = 0$, $d_{dR}H + d\phi = 0$, and $k\omega_u \sim (d_{dR}\phi, 0, 0, \dots)$.

Notice that we in fact have $d_{dR}\phi = kd_{dR}\alpha_u$, because there is no non-trivial $\beta \in (\Lambda^2\Omega^1_A)^{k-1}$ satisfying the relation $kd_{dR}\alpha_u - d_{dR}\phi = d\beta$ due to degree reasons.

By Proposition 2.15, the tangent complex $\mathbb{T}_A|_{\operatorname{spec} H^0(A)} = (\mathbb{L}_A|_{\operatorname{spec} H^0(A)})^\vee$ is also represented by a complex of free finite rank $H^0(A)$-modules, with Tor-amplitude in $[0, -k]$. Then for any k-shifted 1-form α, the k-shifted 2-form $d_{dR}\alpha$ defines an induced map of complexes via $v \mapsto \iota_v d_{dR}\alpha$:

$$\begin{array}{rccccccccccc}
\mathbb{T}_A|_{\operatorname{spec} H^0(A)} : & 0 \to & (V^0)^* & \to & (V^{-1})^* & \to \cdots \to & (V^{k+1})^* & \to & (V^k)^* & \to 0 \\
\downarrow d_{dR}\alpha & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\mathbb{L}_A|_{\operatorname{spec} H^0(A)} : & 0 \longrightarrow & V^k & \longrightarrow & V^{k+1} & \longrightarrow \cdots \longrightarrow & V^{-1} & \longrightarrow & V^0 & \longrightarrow 0,
\end{array}$$

where both horizontal differentials $d^i, (d^i)^*$ are zero at $p \in \operatorname{spec} H^0(A)$ due to the minimality of A.

By the contactness condition, $d_{dR}\alpha_u$ is non-degenerate on the subcomplex $\ker\alpha|_{\operatorname{spec} H^0(A)}$ of $\mathbb{T}_A|_{\operatorname{spec} H^0(A)}$. Therefore, one has a natural splitting

$$\mathbb{T}_A|_{\operatorname{spec} H^0(A)} = \ker\alpha|_{\operatorname{spec} H^0(A)} \oplus \operatorname{Rest}|_{\operatorname{spec} H^0(A)}. \tag{3.19}$$

Write W for the *dual subcomplex* of $\ker\alpha|_{\operatorname{spec} H^0(A)}$ in $\mathbb{L}_A|_{\operatorname{spec} H^0(A)}$, i.e. we set $W := (\ker\alpha|_{\operatorname{spec} H^0(A)})^*$, then we have the commutative diagram

$$\begin{array}{cccccccccccc} W^*: & 0 \to & (W^0)^* & \to & (W^{-1})^* & \to \cdots \to & (W^{k+1})^* & \to & (W^k)^* & \to 0 \\ {\scriptstyle d_{dR}\alpha_u}\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ W: & 0 \longrightarrow & W^k & \longrightarrow & W^{k+1} & \longrightarrow \cdots \longrightarrow & W^{-1} & \longrightarrow & W^0 & \longrightarrow 0 \end{array} \tag{3.20}$$

such that the vertical maps $d_{dR}\alpha_u : (W^{k+i})^* \to W^{-i}$, $v \mapsto \iota_v d_{dR}\alpha_u$, are all quasi-isomorphisms.

Remark 3.17. — As both horizontal differentials $d^i, (d^i)^*$ are zero at $p \in \operatorname{spec} H^0(A)$ because of the minimality of A, the vertical maps are isomorphisms at p, and hence isomorphisms in a neighborhood of p. By localizing A at p if needed, we may assume that the vertical maps are all *isomorphisms.*

When k is odd. We now focus on the simplest case. Let $k = -2\ell - 1$ for $\ell \in \mathbb{N}$. Localizing A at p if necessary, first choose degree 0 variables $x_1^0, x_2^0, \ldots, x_{m_0}^0, \widetilde{x}_1^0$ in $A(0)$ such that $\{d_{dR}x_j^0 : j = 1, \ldots, m_0\}$ forms a basis for W^0 over $A(0)$, and $\{d_{dR}\widetilde{x}_1^0\}$ forms a $A(0)$-basis for $(\operatorname{Rest}^*)^0$.

Next, for $i = 1, \ldots, \ell$, pick $x_1^{-i}, \ldots, x_{m_i}^{-i} \in A^{-i}$ so that $d_{dR}x_1^{-i}, \ldots, d_{dR}x_{mi}^{-i}$ form a basis of W^{-i} over $A(0)$, and $(\operatorname{Rest}^*)^{-i}$ is trivial over $A(0)$.

From the isomorphism $d_{dR}\alpha_u : (W^{k+i})^* \to W^{-i}$, we have $H^{-i}(\mathbb{L}_A|_W) \simeq H^{k+i}(\mathbb{L}_A|_W)^*$, and hence $\dim H^{-i}(\mathbb{L}_A|_W) = \dim H^{k+i}(\mathbb{L}_A|_W)$. It follows that A is free over $A(0)$ with m_i generators in degree $-i$ for $i = 1, \ldots, \ell$, and m_i generators in degree $k+i$ for $i = 0, \ldots, \ell$. Then choose $y_1^{k+i}, \ldots, y_{m_i}^{k+i}$ in A^{k+i} such that $\{d_{dR}y_j^{k+i} : j = 1, 2, \ldots, m_i\}$ is a basis for W^{k+i} over $A(0)$ which is dual to the basis $\{d_{dR}x_1^{-i}, \ldots, d_{dR}x_{m_i}^{-i}\}$ over $A(0)$. That is, using local coordinates above, for $i = 0, 1, \ldots, \ell$, we get

$$W^{-i} = \langle d_{dR}x_1^{-i}, d_{dR}x_2^{-i}, \ldots, d_{dR}x_{m_i}^{-i}\rangle_{A(0)},$$
$$W^{k+i} = \langle d_{dR}y_1^{k+i}, d_{dR}y_2^{k+i}, \ldots, d_{dR}y_{m_i}^{k+i}\rangle_{A(0)}.$$

By Remark 3.17, the isomorphisms in Diagram (3.20) imply that for each $i = 0, 1, \ldots, \ell$, we have

$$(W^{-i})^* = \langle \partial/\partial x_1^{-i}, \ldots, \partial/\partial x_{m_i}^{-i}\rangle_{A(0)} \xrightarrow{\sim} \langle d_{dR}y_1^{k+i}, \ldots, d_{dR}y_{m_i}^{k+i}\rangle_{A(0)}, \tag{3.21}$$

$$(W^{k+i})^* = \langle \partial/\partial y_1^{k+i}, \ldots, \partial/\partial y_{m_i}^{k+i}\rangle_{A(0)} \xrightarrow{\sim} \langle d_{dR}x_1^{-i}, \ldots, d_{dR}x_{m_i}^{-i}\rangle_{A(0)}. \tag{3.22}$$

Then, from Equation (3.19), when restricted to $\operatorname{spec} H^0(A)$, we get

$$\ker \alpha_u = \left\langle \partial/\partial x_1^{-i}, \dots, \partial/\partial x_{m_i}^{-i}, \partial/\partial y_1^{k+i}, \dots, \partial/\partial y_{m_i}^{k+i} : i = 0, \dots, \ell \ \right\rangle_{A(0)},$$
$$\text{Rest} = \left\langle \partial/\partial \widetilde{x}_1^0 \right\rangle_{A(0)}.$$

Here $\text{Rest}\,|_{\operatorname{spec} H^0(A)}$ is a subcomplex of $\mathbb{T}_A|_{\operatorname{spec} H^0(A)}$ that is concentrated in degree 0. Moreover, we can choose a vector field $R \in \text{Rest}\,|_{\operatorname{spec} H^0(A)}$ of degree 0, up to scaling, such that $\iota_R \alpha_u = 1$. Note that, in this case, we have $\iota_R d_{dR} \alpha_u = 0$.

A is now identified with the standard form cdga over $A(0)$ freely generated by the variables $\widetilde{x}_1^0, x_j^{-i}, y_j^{k+i}$ as in Example 3.11. We also impose suitable differential d as before: d acts on $\widetilde{x}_1^0, x_j^{-i}, y_j^{k+i}$ as in Equation (2.21). Note in particular that $d\widetilde{x}_1^0 = 0$ as $\widetilde{x}_1^0 \in A^0$.

The non-degeneracy condition on $d_{dR}\alpha_u|_{\ker \alpha}$ sending the dual basis of $d_{dR}x_1^{-i}, \dots, d_{dR}x_{m_i}^{-i}$ to the basis $d_{dR}y_1^{k+i}, \dots, d_{dR}y_{m_i}^{k+i}$ (and vice versa) as in Equations (3.21) and (3.22) implies that

$$d_{dR}\alpha_u|_{\ker \alpha_u} = \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} d_{dR}x_j^{-i} d_{dR}y_j^{k+i}. \tag{3.23}$$

Since $d_{dR}\phi = k d_{dR}\alpha_u$, we may take $k\alpha_u = d_{dR}\theta + \phi$ for $\theta \in A^k$. Modifying Equation (2.20), we may explicitly have $\phi = k \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} y_j^{k+i} d_{dR}x_j^{-i}$ using the coordinates above.

Since $R \notin \ker \alpha_u|_{\operatorname{spec} H^0(A)}$, we get $\iota_R \phi = 0$. Thus, $\iota_R d_{dR}\theta = k < 0$ as $\iota_R \alpha_u = 1$. Then $d_{dR}\theta, d_{dR}x_1^{-i}, \dots, d_{dR}x_{m_i}^{-i}, d_{dR}y_1^{k+i}, \dots, d_{dR}y_{m_i}^{k+i}$ span $\mathbb{L}_A|_{\operatorname{spec} H^0(A)}$. (Otherwise, if $d_{dR}\theta$ was in $A(0)$-span of $d_{dR}x_1^{-i}, d_{dR}y_j^{k+i}$, then $\iota_R d_{dR}\theta$ would vanish as $R \in \langle \partial/\partial \widetilde{x}_1^0 \rangle_{A(0)}$.)

Note that the Hamiltonian H is d-closed (as it satisfies the classical master equation). Now, localizing A at p if necessary, choose an element $z \in A^k$ such that $dz = \frac{1}{k}H$ (cf. Remark 2.25). Then replace θ by $-kz$ and write

$$\alpha_u = -d_{dR}z + \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} y_j^{k+i} d_{dR}x_j^{-i}.$$

It is now straightforward to check that α_u is d-closed, and hence a 1-form of degree k, such that $d_{dR}\alpha_u|_{\ker \alpha_u}$ is non-degenerate. Therefore, the graded variables x_j^{-i}, y_j^{k+i}, z on $\mathbf{U}$ serves as the desired local contact Darboux coordinates.

When k is not odd. For the other cases (a) $k \equiv 0 \mod 4,$ and (b) $k \equiv 2 \mod 4,$ one should use another sets of variables as in Equations (2.22) and (2.23), respectively, and modify H, ϕ, d as in Equations (2.22)–(2.25).

This completes the proof of Thm. 3.14, and hence that of Thm. 1.1. □

Remark 3.18. — Let B^0 be the subalgebra of A^0 with basis $x_1^0, \dots, x_{m_0}^0$. Then we define a sub-cdga B of A to be the free algebra over B^0 on generators x_j^{-i}, y_j^{k+i} only, with inclusion $\iota : B \hookrightarrow A$. Observe that the elements ϕ and $\omega^0 := d_{dR}\alpha_u|_{\ker \alpha_u}$ are all images under ι of the elements ϕ_B and $\omega_B^0 := \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} d_{dR} x_j^{-i} d_{dR} y_j^{k+i}$, respectively. As in Section 2.3, B is a minimal standard form cdga which in fact serves as a local symplectic model (for $k < 0$ odd). As noted before, similar local models can be explicitly obtained for the other cases using Equations (2.22)–(2.25).

In any case, suppose that we construct such (A, B) for $k < 0$, then the virtual dimension $\operatorname{vdim} B$ is always *even*, and hence the virtual dimension $\operatorname{vdim} A = \operatorname{vdim} B + 1$ is *odd*. In fact, if a cdga A is in contact Darboux form, we have

$$\operatorname{vdim} A = \begin{cases} 1 + 2\sum_i (-1)^i m_i, & k \text{ even} \\ 1, & k \text{ odd.} \end{cases}$$

4. Symplectification of a shifted contact derived scheme

In this section, we give a formal description of the *symplectification* of a shifted contact derived $\mathbb{K}$-scheme.

In classical contact geometry, for a contact manifold $(M, \xi = \ker(\alpha))$ with a globally defined contact 1-form α, one can define the *symplectification* $\widetilde{M}$ of M as the total space of the bundle $M \times \mathbb{R}^* \to M$ with a canonical symplectic form $\omega := d_{dR}(e^t \alpha)$, where t is the $\mathbb{R}$-coordinate. Most of the standard references [7, 14] use this approach, where the contact structure ξ is in fact assumed to be coorientable (see Remark 3.2 for the definition).

However, for non-coorientible contact structures, the coordinate-dependent description above can no longer be applicable (as no global α and t-variable available); instead, we may use the following description from [2, Appendix 4.E]: Given a contact manifold (M, ξ), we let

$$\widetilde{M} := \big\{ (p, \alpha_p) : p \in M,\ \alpha_p \in T_p^* M,\ \text{s.t. } \ker \alpha_p = \xi_p \big\}. \tag{4.1}$$

Here, $\widetilde{M}$ is just the set of all contact forms on the contact manifold. It should be noted that for a pair $(p, \alpha_p) \in \widetilde{M}$, α_p is not a differential form but just a linear form on one tangent space $T_p M$ at the point of contact of the manifold such that its zero set is the contact plane. From [2, Appendix 4.E], $\widetilde{M}$ is in fact a smooth manifold of even dimension $\dim M + 1$.

Notice that there is a natural $\mathbb{R}^*$-action on $\widetilde{M}$ via $f \cdot (p, \alpha_p) = (p, f\alpha_p)$ such that $\widetilde{M}/\mathbb{R}^* \simeq M$. Therefore, $\widetilde{M}$ can be identified as the total space of the $\mathbb{R}^*$-bundle over M. From this identification, the canonical *symplectic structure* on $\widetilde{M}$ can be defined as $\omega := d_{dR}\lambda$, where the so-called *canonical 1-form* λ is the differential 1-form on $\widetilde{M}$ whose value on any vector $v \in T_x\widetilde{M}$ at a point $x = (p, \alpha_p) \in \widetilde{M}$ is given by

$$\lambda_x(v) := \alpha_p(\pi_{*,x}(v)). \tag{4.2}$$

The construction of a canonical symplectified space associated to a contact space in this manner can be promoted to derived symplectic geometry. This leads to the desired definition of the *symplectification* in our setup. Before the main construction, let us begin by some relevant notions:

Definition 4.1. — *Let* $\mathbf{X} \in dStk_{\mathbb{K}}$ *and* $E \in \mathrm{QCoh}(\mathbf{X})$*, then the* total space $\widetilde{\mathbf{E}}$ of E *is defined as a derived stack sending*

$$A \longmapsto \widetilde{\mathbf{E}}(A) := \{(p, s) : \ p \in \mathbf{X}(A),\ s \in p^*E\},\ \textit{with } A \in cdga_{\mathbb{K}}. \tag{4.3}$$

By Yoneda's lemma, $\mathbf{X}(A) \simeq Map_{dPstk}(\mathrm{Spec}\, A, \mathbf{X})$*, and hence any* A*-point* $p \in \mathbf{X}(A)$ *can be seen as a morphism* $p : \mathrm{Spec}\, A \to \mathbf{X}$ *of derived pre-stacks, and thus its pullback map* $p^* : \mathrm{QCoh}(\mathbf{X}) \to \mathrm{QCoh}(\mathrm{Spec}\, A) \simeq Mod_A$ *sends* $E \mapsto p^*E$*. Hence,* s *is an element of the* A*-module* p^*E*, a "fiber" over* p*.*

Example 4.2. — if $\mathbf{X} \in dStk_{\mathbb{K}}$ admits a cotangent complex (which is always the case when $\mathbf{X}$ is also Artin), we can define the cotangent stack $\mathbf{T}^*X$ to be the total space $\widetilde{\mathbf{L}}_{\mathbf{X}}$ of $\mathbb{L}_{\mathbf{X}} \in \mathrm{QCoh}(\mathbf{X})$ and the n-shifted cotangent stack $\mathbf{T}^*[k]X$ to be the total space $\widetilde{\mathbf{L}_{\mathbf{X}}[k]}$ of $\mathbb{L}_{\mathbf{X}}[k] \in \mathrm{QCoh}(\mathbf{X})$. For more details see [13, Section 2].

Recall from [5] that for a perfect module E over $\mathbf{X}$, its stack of sections $\widetilde{\mathbf{E}}$, defined by $\widetilde{\mathbf{E}}(-) = \mathbb{R}\,\mathrm{Spec}_{\mathbf{X}}(\mathrm{Sym}(E^\vee))(-)$ is acted on by $\mathbb{G}_m$ because $\mathrm{Sym}(E^\vee)$ is graded $\mathcal{O}_{\mathbf{X}}$-algebra. This new grading is then called the *fibre grading.* Note also that the both zero section $\mathbf{X} \to \widetilde{E}$ and the projection $\widetilde{\mathbf{E}} \to X$ are $\mathbb{G}_m$-equivariant for the trivial $\mathbb{G}_m$-action on $\mathbf{X}$. With this terminology, $\mathbf{T}^*[k]X$ is nothing but the stack of k-shifted 1-forms on $\mathbf{X}$ with a natural $\mathbb{G}_m$-action.

More generally, for $E \in \mathrm{QCoh}(\mathbf{X})$, the $\mathbb{G}_m$-action is given as a morphism of derived $\mathbb{K}$-stacks $\triangleleft : \mathbb{G}_m \times_{\mathrm{spec}\,\mathbb{K}} \widetilde{\mathbf{E}} \to \widetilde{E}$ such that for each $A \in cdga_{\mathbb{K}}$, we have a $\mathbb{G}_m(A)$-action $\triangleleft_A$ on $\widetilde{\mathbf{E}}(A)$, where $\mathbb{G}_m$ is the functor that maps $A \mapsto A^\times$. This definition also holds for any derived S-stack.

Now, we are in place of introducing the definition of the *symplectification* of a k-shifted contact derived stack using the machinery above:

Definition 4.3. — *Let* $\mathbf{X}$ *be a locally finitely presented derived* $\mathbb{K}$*-scheme with a* k*-shifted contact structure* $(\mathcal{K}, \kappa, L)$*. The* symplectification *is the total space* $\widetilde{\mathbf{L}}$ *of the* $\mathbb{G}_m$*-bundle of* L*, provided with a canonical* k*-shifted symplectic structure (for which the* $\mathbb{G}_m$*-action is of weight 1) as defined below.*

Step-1: Derived stack of k-contact forms. Let $(\mathbf{X}; \mathcal{K}, \kappa, L)$ be a k-shifted contact derived $\mathbb{K}$-scheme of locally finite presentation. Given $k < 0$ and $p \in \mathbf{X}$, find a minimal standard form cdga A and an affine derived subscheme $\mathbf{U} := \operatorname{Spec} A$ such that $p : \operatorname{Spec} A \to \mathbf{X}$ is Zariski open inclusion. Here, we assume w.l.o.g. that L is trivial on $\mathbf{U}$.

Define a functor $\mathcal{S}_{\mathbf{X}} : cdga_{\mathbb{K}} \to Spcs$ by $A \mapsto \mathcal{S}_{\mathbf{X}}(A)$, where

$$\mathcal{S}_{\mathbf{X}}(A) = \left\{ (p, \alpha, v) : \begin{array}{r} p \in \mathbf{X}(A),\ \alpha : p^*(\mathbb{T}_{\mathbf{X}}) \longrightarrow \mathcal{O}[k], \\ v : \operatorname{Cocone}(\underline{\alpha}) \xrightarrow{\sim} p^*(\mathcal{K}) \end{array} \right\}.$$

Here each v in (p, α, v) is a quasi-isomorphism respecting the natural morphisms $p^*\kappa : p^*\mathcal{K} \to p^*(\mathbb{T}_{\mathbf{X}})$ and $\operatorname{Cocone}(\underline{\alpha}) \to p^*(\mathbb{T}_{\mathbf{X}})$. As before, the perfect complexes $\mathbb{T}_A, \mathbb{L}_A$, when restricted to $\operatorname{spec} H^0(A)$, are both quasi-isomorphic to free finite complexes of $H^0(A)$-modules. For $A \in cdga_{\mathbb{K}}$, we then define a $\mathbb{G}_m(A)$-action on $\mathcal{S}_{\mathbf{X}}(A)$ by $f \triangleleft (p, \alpha, v) := (p, f \cdot \alpha, v)$.

Now, we can endow $\mathcal{S}_{\mathbf{X}}$ with the structure of a derived stack by using the following result:

Proposition 4.4. — $\mathcal{S}_{\mathbf{X}}$ *is equivalent to the total space* $\widetilde{\mathbf{L}}$ *of the* $\mathbb{G}_m$*-bundle of* L[5]*. Therefore, it has the structure of a derived stack together with the projection map* $\pi : \mathcal{S}_{\mathbf{X}} \to \mathbf{X}$*. We then call* $\mathcal{S}_{\mathbf{X}}$ *the* derived stack of k-contact forms.

Proof. — Let (p, α, v) be a point in $\mathcal{S}_{\mathbf{X}}(A)$. From definitions, we have the following homotopy fiber sequences: (i) $\operatorname{Cocone}(\underline{\alpha}) \to p^*(\mathbb{T}_{\mathbf{X}}) \to \operatorname{Im} \alpha$ and (ii) $\operatorname{Cocone}(\alpha) \to p^*(\mathbb{T}_{\mathbf{X}}) \to \mathcal{O}[k]$. Then we get the following observation:

Lemma 4.5. — *There is a triangle* $\operatorname{Cocone}(\underline{\alpha}) \to p^*(\mathbb{T}_{\mathbf{X}}) \to \mathcal{O}[k]$.

Proof of Lemma 4.5. — From the first sequence (i) above, we have the following homotopy commutative diagram, where the left-hand square is the

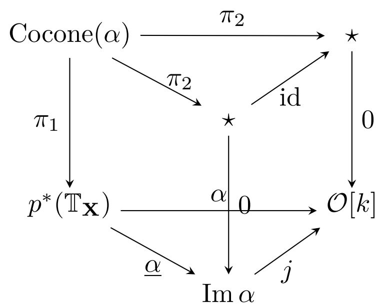
[5] In general, when the group scheme $G = GL_n$, there is an equivalence between locally free sheaves of rank n and GL_n-torsors (hence principal GL_n-bundles). In this regard, a line bundle L is nothing but a $\mathbb{G}_m$-bundle.

homotopy fiber.

$$\begin{array}{ccccc} \mathrm{Cocone}(\underline{\alpha}) & \xrightarrow{\pi_2} & \star & \xrightarrow{\mathrm{id}} & \star \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\ p^*(\mathbb{T}_{\mathbf{X}}) & \xrightarrow{\underline{\alpha}} & \mathrm{Im}\,\alpha & \xrightarrow{j} & \mathcal{O}[k] \end{array} \tag{4.4}$$

There is a homotopy H with $\underline{\alpha} \circ \pi_1 \sim 0$. Using the monomorphism j, we get a homotopy $j \circ H$ such that $j \circ \underline{\alpha} \circ \pi_1 \sim 0$, where $j \circ \underline{\alpha} \sim \alpha$. It follows that $\mathrm{Cocone}(\underline{\alpha})$ homotopy commutes the outer diagram. Then the universality of $\mathrm{Cocone}(\alpha)$ implies that $\exists\, \varphi_1 : \mathrm{Cocone}(\underline{\alpha}) \to \mathrm{Cocone}(\alpha)$.

Likewise, from the second sequence (ii) above, we have the following homotopy commutative diagram.



(4.5)

Here, we have a homotopy such that $\alpha \circ \pi_1 \sim 0$. From the epi-mono factorization of α, we have $j \circ \underline{\alpha} \circ \pi_1 \sim 0$ as well. From $j \circ 0 \sim 0$, we obtain $j \circ \underline{\alpha} \circ \pi_1 \sim j \circ 0$. Since j is a monomorphism, we get the induced homotopy such that $\underline{\alpha} \circ \pi_1 \sim 0$. It follows from the universality of $\mathrm{Cocone}(\underline{\alpha})$ that there exists a map $\varphi_2 : \mathrm{Cocone}(\alpha) \to \mathrm{Cocone}(\underline{\alpha})$.

Using the maps φ_1, φ_2 with the exact triangle (ii), we obtain a triangle $\mathrm{Cocone}(\underline{\alpha}) \to p^*(\mathbb{T}_{\mathbf{X}}) \to \mathcal{O}[k] \to \mathrm{Cocone}(\underline{\alpha})[1]$ as desired. □

Now, using Lemma 4.5, we get an equivalence of triangles of perfect complexes on $\mathrm{Spec}\, A$

$$\begin{array}{ccccc} p^*(\mathcal{K}) & \longrightarrow & p^*(\mathbb{T}_{\mathbf{X}}) & \longrightarrow & p^*(L)[k] \\ \downarrow{\scriptstyle \simeq_v} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \simeq} \\ \mathrm{Cocone}(\underline{\alpha}) & \longrightarrow & p^*(\mathbb{T}_{\mathbf{X}}) & \xrightarrow{\alpha} & \mathcal{O}[k], \end{array} \tag{4.6}$$

and hence an induced isomorphism $p^*(L) \simeq \mathcal{O}$. This map identifies $\mathcal{S}_{\mathbf{X}}$ with the total space of L. We then complete the proof of Proposition 4.4. □

Step-2: The canonical 1-form on $\mathcal{S}_{\mathbf{X}}$. Recall from [5] that a morphism $\mathbf{Y} \to \widetilde{E}$ of derived stacks, with $\widetilde{\mathbf{E}}$ is the total space of $E \in \mathrm{QCoh}(\mathbf{X})$, consists of a morphism $f : \mathbf{Y} \to X$ together with a section s of f^*E. If moreover, $\mathbf{Y}$ is a derived stack equipped with a $\mathbb{G}_m$-action, then a $\mathbb{G}_m$-equivariant morphism $\mathbf{Y} \to \widetilde{E}$ is given by the pair of a $\mathbb{G}_m$-equivariant morphism $f : \mathbf{Y} \to X$ and a $\mathbb{G}_m$-equivariant section s of $f^*E\{1\}$. In particular, the identity map $\widetilde{\mathbf{E}} \to \widetilde{E}$ corresponds to the pair of the projection map $\pi : \widetilde{\mathbf{E}} \to X$ and a ($\mathbb{G}_m$-equivariant) section of $\pi^*E\{1\}$.

Letting $\widetilde{\mathbf{E}} := T^*[k]X$, the identity map $\mathbf{T}^*[k]X \to T^*[k]X$ is then determined by the data of the projection map $\pi_{\mathbf{X}} : \mathbf{T}^*[k]X \to X$ with a section of $\pi_{\mathbf{X}}^*\mathbb{L}_{\mathbf{X}}[k]$ (of weight 1 for the fiber grading). Since we have a natural map $\pi_{\mathbf{X}}^*\mathbb{L}_{\mathbf{X}}[k] \to \mathbb{L}_{\mathbf{T}^*[k]X}[k]$, this section induces a k-shifted 1-from $\lambda_{\mathbf{X}}$ on $\mathbf{T}^*[k]X$ called the *tautological 1-form.* Moreover, [5] shows that the induced closed 2-form $d_{dR}\lambda_{\mathbf{X}}$ of degree k on $\mathbf{T}^*[k]X$ is in fact non-degenerate, and hence gives a k-shifted symplectic structure.

Definition 4.6. — *By construction, we have the natural projection maps $\pi_1 : \mathcal{S}_{\mathbf{X}} \to \mathbf{X}$ and $\pi_2 : \mathcal{S}_{\mathbf{X}} \to \mathbf{T}^*[k]X$. We define the* canonical 1-from λ *on $\mathcal{S}_{\mathbf{X}}$ to be the pullback $\pi_2^*\lambda_{\mathbf{X}}$ of the tautological 1-form on $\mathbf{T}^*[k]X$.*

Step-3: Shifted symplectic structure on $\mathcal{S}_{\mathbf{X}}$. Let $\mathcal{S}_{\mathbf{X}}, \lambda$ be as above. Then $\omega := (d_{dR}\lambda, 0, 0, \dots)$ is a k-shifted closed 2-form on $\mathcal{S}_{\mathbf{X}}$, and hence it defines a pre-k-shifted symplectic structure on $\mathcal{S}_{\mathbf{X}}$. Now, it remains to show that ω is non-degenerate. So, we prove the following result.

Theorem 4.7. — *Let $\mathbf{X}$ be a (locally finitely presented) derived $\mathbb{K}$-scheme with a k-shifted contact structure $(\mathcal{K}, \kappa, L)$. The k-shifted closed 2-form ω described above is non-degenerate, and hence the derived stack $\mathcal{S}_{\mathbf{X}} \to \mathbf{X}$ is k-shifted symplectic.*

We then call the pair $(\mathcal{S}_{\mathbf{X}}, \omega)$ the symplectification of $\mathbf{X}$.

Proof. — The assertion of the theorem is local, so it is enough to prove it in a neighborhood of a point. By definition, locally on $\mathbf{X}$, where L is trivialized, the perfect complex $\mathcal{K}$ in the data of the k-shifted contact structure on $\mathbf{X}$ can be given as a cocone of $\underline{\alpha}$ with α a locally defined k-shifted 1-form $\alpha : \mathbb{T}_{\mathbf{X}} \to \mathcal{O}_{\mathbf{X}}[k]$ with the property that $d_{dR}\alpha|_{\mathcal{K}}$ is non-degenerate; and thus, the triangle $\mathcal{K} \to \mathbb{T}_{\mathbf{X}} \to L[k]$ splits locally. Throughout the proof, we will use "refined" neighborhoods introduced in Section 2; and once the local data is specified, we will fix this k-contact form.

Given $k < 0$ and $p \in \mathbf{X}$, apply Theorem 2.5 to get a refined neighborhood $\mathbf{U} = \operatorname{Spec} A$ of p with $q \in \operatorname{spec} H^0(A)$ such that $p : \operatorname{Spec} A \hookrightarrow \mathbf{X}$ is an open inclusion, with $q \mapsto p$, and A is a minimal standard form cdga. W.l.o.g., we assume that L is trivial on $\mathbf{U}$, and hence, over $\mathbf{U}$, the induced 1-form $\alpha : p^*(\mathbb{T}_{\mathbf{X}}) \to \mathcal{O}[k]$ is such that $\mathrm{Cocone}(\underline{\alpha}) \simeq p^*(\mathcal{K})$ and the 2-form $d_{dR}\alpha$ is non-degenerate on $p^*\mathcal{K}$. In that case, the triangle $p^*\mathcal{K} \to p^*\mathbb{T}_{\mathbf{X}} \to p^*(L[k])$ splits over $\mathbf{U}$. We fix this locally defining 1-form α (and the corresponding isomorphism $u : \mathrm{Cocone}(\underline{\alpha}) \xrightarrow{\sim} p^*\mathcal{K})$ for the rest of the proof.

Denote the pullback of α under the open inclusion p again by $\alpha \in p^*\mathbb{L}_{\mathbf{X}}[k]$. From definitions, the triple (p, α, u) is an element of $\mathcal{S}_{\mathbf{X}}(A)$.

Recall also that we have the distinguished triangle $p^*\mathbb{L}_{\mathbf{X}} \to \mathbb{L}_A \to \mathbb{L}_p$, where $\mathbb{L}_p$ is the relative cotangent complex, such that for refined neighborhoods, the restriction of $\mathbb{L}_A$ to $\mathrm{spec}\, H^0(A)$ is a free finite complex of $H^0(A)$-modules (cf. Proposition 2.15). Also, we have the identification $\mathrm{Cocone}(\underline{\alpha}) \simeq \ker \underline{\alpha}$, and $\ker \underline{\alpha} = \ker \alpha$ in $D(Mod_A)$.

From Remarks 3.7 & 3.10, both α and $f \cdot \alpha$ define equivalent contact structures, up to quasi-isomorphism, for any $f \in H^0(A)$ (after localizing A at q by f, if necessary). It follows that, over $\mathbf{U}$, we can then identify the space $\mathcal{S}_{\mathbf{X}}$ locally as $\mathbf{U} \times^h_{\mathbf{X}} \mathbb{G}_m$,[(6)] with natural projections, where $\mathbb{G}_m = \mathrm{Spec}\, B$ is the affine group scheme, with say $B := \mathrm{Spec}(\mathbb{K}[x, x^{-1}])$. After localizing A at q if necessary, $\varphi \in \mathbb{K}[x, x^{-1}]$ acts on $f \in H^0(A)$ by $f \mapsto \varphi(f, f^{-1})$.

Recall that there is an equivalence $DR(A) \otimes_{\mathbb{K}} DR(B) \simeq DR(A \otimes_{\mathbb{K}} B)$ induced by the identification

$$\mathbb{L}_{A \otimes_{\mathbb{K}} B} \simeq (\mathbb{L}_A \otimes_{\mathbb{K}} B) \oplus (A \otimes_{\mathbb{K}} \mathbb{L}_B). \tag{4.7}$$

Notice that for $q \in \mathbf{U}(\mathbb{K})$, with $f_q \in \mathbb{G}_m(\mathbb{K}) \simeq \mathbb{K}^\times$, $f_q \cdot \alpha$ is also a contact form at q. Thus, on the part of the space $\mathcal{S}_{\mathbf{X}}$ over $\mathbf{U}$, we define a function f with values in $\mathbb{K}^\times$. Then the canonical 1-form in Definition 4.6 can be locally written as[(7)]

$$\lambda = f \cdot \pi^*\alpha. \tag{4.8}$$

Remark 4.8. — This local expression (4.8) follows from [8, Lemma 2.1.4] that the tautological 1-form on the shifted cotangent is universal in the sense that for any k-shifted 1-form β on $\mathbf{Y}$ (viewed as $\beta : \mathbf{Y} \to T^*[k]Y$) we have $\beta^*\lambda_{\mathbf{Y}} = \beta$. Apply this universality for the case above $\mathbf{Y} := \mathbf{U} \times^h_{\mathbf{X}} \mathbb{G}_m$ with $\beta := \pi^*\alpha$ and the $\mathbb{G}_m$-action to get the desired expression. We also have a non-zero factor f as $\mathcal{S}_{\mathbf{X}}$ is identified with the total space of L.

Proposition 4.9. — *Locally on $\mathbf{X}$, for any locally defining 1-form α, the k-shifted 2-form $\omega^0 := d_{dR}\lambda$ is non-degenerate, and hence the sequence $\omega := (\omega^0, 0, 0, \dots)$ defines a k-shifted symplectic structure on $\mathcal{S}_{\mathbf{X}}$.*

(6) This also follows from the fact that $\mathcal{S}_{\mathbf{X}}$ is identified with the total space of L. Therefore, the corresponding homotopy fibers over p are equivalent.

(7) For (non-zero) $\pi^*\alpha, \lambda \in \mathbb{L}_{p^*\mathcal{S}_{\mathbf{X}}}[k]$, the identification of $\mathcal{S}_{\mathbf{X}}$ with the total space of L (i.e. the space of trivializations) implies that there is a non-zero $f \in A$ s.t. $\lambda = f \cdot \pi^*\alpha$.

Proof of Proposition 4.9. — Note first that the non-degeneracy in sense of Definition 2.19 can be equivalently formulated using refined local models as follows:

Lemma 4.10. — *A k-shifted 2-form γ on* $\operatorname{Spec} A$ *for A a minimal standard form cdga is non-degenerate if and only if for any non-zero vector $v \in \mathbb{T}_A|_{\operatorname{spec} H^0(A)}$, there exists a non-zero vector $w \in \mathbb{T}_A|_{\operatorname{spec} H^0(A)}$ such that $\iota_w \iota_v \gamma \neq 0$.*

Let us give a sketch of proof for Lemma 4.10. From Proposition 2.15, when restricted to $\operatorname{spec} H^0(A)$, the induced morphism $\mathbb{T}_A \to \Omega^1_A[k],\ Y \mapsto \iota_Y \gamma$, is just a map of finite complexes of free modules. For non-degeneracy, we require this map to be a (degree-wise) quasi-isomorphism. Recall that localizing A at p if necessary, we may assume that the induced map is indeed an (degree-wise) isomorphism near p. Therefore, Lemma 4.10 follows from an analogous result in linear algebra.

Now, to prove that $\omega^0 := d_{dR}\lambda$ is non-degenerate, we use Lemma 4.10. That is, it suffices to show that, over $\mathbf{U}$, for any non-vanishing (homogeneous) vector field $\sigma \in (\mathbb{L}_{A\otimes_{\mathbb{K}}B})^\vee$, there is a vector field $\eta \in (\mathbb{L}_{A\otimes_{\mathbb{K}}B})^\vee$ such that $\iota_\eta(\iota_\sigma d_{dR}\lambda) \neq 0$.

For the rest of the proof, we will also assume that k is odd, say $k = -2\ell - 1$ for $\ell \in \mathbb{N}$, to provide more explicit representations for vector fields of interest. In fact, our constructions will be independent of k and the corresponding local graded variables.

Localizing A at p if necessary, choose the graded variables $x_j^{-i}, y_j^{k+i}, \widetilde{x}_1^0$ on $\mathbf{U}$ as before so that A is a standard form cdga over $A(0)$ freely generated by these graded variables, such that, when restricted to $\operatorname{spec} H^0(A)$,

$$\begin{aligned}\ker\alpha &= \big\langle \partial/\partial x_1^{-i}, \ldots, \partial/\partial x_{m_i}^{-i}, \partial/\partial y_1^{k+i}, \ldots, \partial/\partial y_{m_i}^{k+i} : i = 0, 1, \ldots, \ell\ \big\rangle_{A(0)},\\ \text{Rest} &= \big\langle \partial/\partial \widetilde{x}_1^0 \big\rangle_{A(0)}.\end{aligned}$$

Remark 4.11. — As $f|_{\operatorname{spec} H^0(A)} \in \mathbb{K}^\times$ and $\mathbb{L}_A = \Omega^1_A$ is a A-module, when restricted to $\operatorname{spec} H^0(A)$, the first summand $\mathbb{L}_A \otimes_{\mathbb{K}} B$ of Equation (4.7) can be equivalently written as

$$\langle d_{dR}x_j^{-i}, d_{dR}y_j^{k+i}, d_{dR}\widetilde{x}_1^0\rangle_A \otimes_{\mathbb{K}} \mathbb{K}[f, f^{-1}] \simeq \langle d_{dR}x_j^{-i}, d_{dR}y_j^{k+i}, d_{dR}\widetilde{x}_1^0\rangle_A.$$

Using a cofibrant replacement of B if necessary, the second summand of Equation (4.7) is just equivalent to $H^0(A) \otimes_{\mathbb{K}} \langle d_{dR} f\rangle_B$.

Now, using the natural splitting 3.19 and Remark 4.11 for the complex in Equation (4.7), we then have, when restricted to $\operatorname{spec} H^0(A)$,

$$(\mathbb{L}_{A\otimes_{\mathbb{K}}B})^\vee \simeq \big(\ker\alpha \oplus \text{Rest}\big) \oplus \big(H^0(A) \otimes_{\mathbb{K}} \langle \partial/\partial f\rangle_B\big). \tag{4.9}$$

Using the splitting in Equation (4.9) with Lemma 4.10, we prove the statement case by case. We first note that for any $\eta, \sigma \in (\mathbb{L}_{A\otimes_{\mathbb{K}}B})^{\vee}$, direct computations give

$$\iota_\eta(\iota_\sigma d_{dR}\lambda) = \mp(d_{dR}f)(\sigma)\alpha(\pi_*\eta) \mp (d_{dR}f)(\eta)\alpha(\pi_*\sigma) \mp d_{dR}\alpha(\pi_*\sigma, \pi_*\eta).$$

From Equation (4.9), it is enough to consider the following cases:

(1) If $\sigma \in \ker\alpha$, then we have $\iota_\eta(\iota_\sigma d_{dR}\lambda) = \mp d_{dR}\alpha(\sigma, \pi_*\eta)$. Since $d_{dR}\alpha|_{\ker\alpha}$ is non-degenerate by the contactness condition on α, it is enough to take η to be any non-zero vector in $\ker\alpha$.

(2) If $\sigma \in \mathrm{Rest}$, then we get $\iota_\eta(\iota_\sigma d_{dR}\lambda) = \mp(d_{dR}f)(\eta)\alpha(\sigma)$. Notice that $\alpha(\sigma) \neq 0$ since $\sigma \in \mathrm{Rest}$. Thus, it is enough to take η to be any non-zero vector in $H^0(A) \otimes_{\mathbb{K}} \langle \partial/\partial f\rangle_B$ so that $(d_{dR}f)(\eta) \neq 0$.

(3) If $\sigma \in H^0(A) \otimes_{\mathbb{K}} \langle \partial/\partial f\rangle_B$, then $\iota_\eta(\iota_\sigma d_{dR}\lambda) = \mp(d_{dR}f)(\sigma)\alpha(\pi_*\eta)$. Observe that $(d_{dR}f)(\sigma) \neq 0$, so it suffices to take η to be any non-zero vector in Rest so that $\alpha(\pi_*\eta) \neq 0$.

This completes the proof of Prop. 4.9, and hence that of Thm. 4.7. □

Remark 4.12. — The proofs of Proposition 4.9 and Theorem 4.7 will still be valid for the other values of k. In fact, it clear to see that coordinates do not play any significant role in the proofs, rather than just providing explicit representations for the splitting.

In short, Proposition 4.9 has indeed a coordinate-free proof, and so does Theorem 4.7. Thus, using the same terminology as before, we say that the pair $(\mathcal{S}_{\mathbf{X}}, \omega)$ above is the *symplectification of the k-shifted contact derived $\mathbb{K}$-scheme* $\mathbf{X}$ for any $k < 0$. Note also that this result is in fact canonical up to quasi-isomorphism by construction.

5. Concluding remarks

We conclude this paper with the following remark on the possible "stacky" generalizations of the main results presented in this paper and more.

Remark 5.1. — It should be noted that "stacky" generalizations of the results in [4] are also available in the literature. Ben-Bassat, Brav, Bussi and Joyce [3] extend the results of [4] from derived schemes to the case of (locally finitely presented) derived Artin $\mathbb{K}$-stacks. In short, Ben-Bassat, Brav, Bussi and Joyce [3] proved that derived Artin $\mathbb{K}$-stacks also have nice local models in some sense. Parts of the results from [3, Theorems 2.8 & 2.9] in fact give the generalization of Theorem 2.5 to the case of derived Artin $\mathbb{K}$-stacks. They also proved that every shifted symplectic derived Artin $\mathbb{K}$-stack admits the so-called "Darboux form atlas" [3, Theorem 2.10]. That is, their result extends Theorem 2.23 from derived $\mathbb{K}$-schemes to derived Artin $\mathbb{K}$-stacks.

In the sequel(s), a work in progress, our goals will be to extend the main results of this paper from derived schemes to the more general case of derived Artin stacks and to discuss more on the theory of shifted contact derived spaces (e.g. introducing *Legendrians* and studying their local behavior). In that respect, we propose:

Conjecture 5.2. — *Theorem 3.14 and Theorem 4.7 still hold for (locally finitely presented) k-shifted contact derived Artin $\mathbb{K}$-stacks with $k < 0$.*

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