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On the decay and Gevrey regularity of the solutions to the Navier–Stokes equations in general two-dimensional domains (*)

Raphaël Danchin (1)

Abstract. — The present paper is devoted to the proof of time decay estimates for derivatives at any order of finite energy global solutions of the Navier–Stokes equations in general two-dimensional domains. These estimates only depend on the order of derivation and on the L^2 norm of the initial data. The same elementary method just based on energy estimates and Ladyzhenskaya inequality also leads to Gevrey regularity results.

Résumé. — On s'intéresse aux propriétés de décroissance temporelle pour les dérivées des solutions globales à énergie finie des équations de Navier–Stokes dans des domaines généraux bidimensionnels. Les estimations obtenues ne dépendent que de l'ordre de dérivation et de la norme L^2 des données initiales. La même méthode élémentaire basée sur les bornes d'énergie et l'inégalité de Ladyzhenskaya conduit également à des résultats de régularité Gevrey.

Introduction

We are concerned with the incompressible Navier–Stokes equations that govern the evolution of the velocity field $u = u(t,x)$ and pressure function $P = P(t,x)$ of homogeneous incompressible viscous flows in a general domain Ω of $\mathbb{R}^2$ or in a two-dimensional periodic box. Adopting standard notation

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Article proposé par Lucia Di Vizio.

these equations read

$$\begin{cases} u_t + \operatorname{div}(u \otimes u) - \Delta u + \nabla P = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u|_{t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{NS}$$

The initial data u_0 is a given divergence free vector-field with normal component vanishing at the boundary $\partial\Omega$ of Ω and we supplement (NS) with homogeneous Dirichlet boundary conditions for u at $\partial\Omega$.

The global existence theory for (NS) originates from the paper [13] by J. Leray in 1934. In the case $\Omega = \mathbb{R}^3$, by combining the energy balance associated to (NS):

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau = \frac{1}{2}\|u_0\|_{L^2}^2, \quad t \in \mathbb{R}_+, \tag{0.1}$$

with compactness arguments, he succeeded in constructing for any divergence free u_0 in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ a global distributional solution of (NS) satisfying (0.1) *with an inequality* (viz. the left-hand side is bounded by the right-hand side).

Leray's result turns out to be very robust and can be adapted to any two or three-dimensional domain: we have the following statement that is proved in e.g. [4]:

Theorem 0.1. — *Let Ω be a domain of $\mathbb{R}^d$ (with $d = 2, 3$) and denote by $L^2_\sigma(\Omega)$ the completion of the set $\mathcal{V}_\sigma$ of smooth divergence free vector-fields compactly supported in Ω for the $L^2(\mathbb{R}^d;\mathbb{R}^d)$ norm. Let $H^1_{0,\sigma}(\Omega)$ be the completion of $\mathcal{V}_\sigma$ for the $H^1(\mathbb{R}^d;\mathbb{R}^d)$ norm.*

Then, for any $u_0 \in L^2_\sigma(\Omega)$ there exists a global distributional solution (u, P) of (NS) *with $u \in L^\infty(\mathbb{R}_+; L^2_\sigma(\Omega)) \cap L^2_{loc}(\mathbb{R}_+; H^1_{0,\sigma}(\Omega))$ satisfying*

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u\|_{L^2}^2 \,\mathrm{d}\tau \leqslant \frac{1}{2}\|u_0\|_{L^2}^2, \qquad t \in \mathbb{R}_+. \tag{0.2}$$

So far, uniqueness of Leray's solutions in dimension three is an open question. In contrast, it holds true in dimension two (see the works by O. A. Ladyzhenskaya in [11], and by J.-L. Lions and G. Prodi in [14]). The key to the proof was the following *Ladyzhenskaya inequality*

$$\|z\|_{L^4}^2 \leqslant C_0 \|z\|_{L^2} \|\nabla z\|_{L^2}, \qquad z \in H^1_0(\Omega) \tag{0.3}$$

that will also play a decisive role in the present paper.

Since the pioneering work by J. Leray, a huge amount of literature has been devoted to the study of (NS) both in two and three dimensional domains. Our goal here is to derive L^2 decay estimates for time derivatives at any order of two-dimensional finite energy global solutions. We shall see that our method actually gives for free Gevrey regularity for short time (or all time if the data are small).

Exhibiting time decay estimates for smooth and small solutions of (NS) goes back to the papers by S. Kawashima, A. Matsumura and T. Nishida [10] and J. G. Heywood [8] devoted to the case $\Omega = \mathbb{R}^3$. In both papers, in addition to be smooth enough, the initial velocity is required to be globally integrable on $\mathbb{R}^3$. An important breakthrough has been made by M. Schonbek [16] in 1985 who observed that *any* weak solution supplemented with an initial velocity u_0 in $L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ satisfies time decay estimates. More accurate decay rates have been obtained shortly after by R. Kajikiya and T. Miyakawa [9] and M. Wiegner [18]. In [17], M. Schonbek pointed out that one cannot expect any generic rate of decay for $\|u(t)\|_{L^2}$ if the initial data is only in L^2.

It is also worth mentioning works pointing out the Gevrey or even analyticity of the solutions to (NS). For exemple, for periodic boundary conditions, C. Foias and R. Temam proved in [7] that H^1 data give rise to solutions with analytic regularity in time, globally in time in dimension two, and locally in time in dimension three. This result has been adapted to the whole space setting and considerably refined by P.-G. Lemarié-Rieusset [12] then by M. Oliver and E. Titi in [15], and translated in the language of critical Besov spaces ($u_0 \in \dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)$ with $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$) by H. Bae, A. Biswas and E. Tadmor in [1]. By a different approach, J.-Y. Chemin in [3] obtained (space) analyticity estimates of L^2 type in the case of small data (see also [5]). The more complicated case of the Navier–Stokes equations with potential forces has been investigated by several authors. The reader may in particular refer to the survey paper by C. Foias, L. Huan and J.-C. Saut [6] where asymptotic expansions for large time are presented.

Most of the aforementioned works dedicated to decay estimates strongly rely on Fourier or spectral analysis. In particular, the *Fourier splitting method* of M. Schonbek [16] can hardly be adapted to general domains (or at the price of complicated arguments that require the domain to be smooth, see [2]). Here we shall see that using only the energy method and Ladyzhenskaya inequality (0.3) leads to optimal time decay estimates.

In order to give an idea of our approach, let us consider the linearized version of (NS) about a null solution, namely the following evolutionary Stokes equations:

$$\begin{cases} u_t - \Delta u + \nabla P = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u|_{t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{0.4}$$

Let us explain how to bound just in terms of $\|u_0\|_{L^2}$ and by elementary arguments (that are valid in any domain) the following quantities for all $k\in\mathbb{N}$:

$$\begin{aligned}\mathcal{L}_{2k}(t){:=}\left\|t^k u_t^{(k)}(t)\right\|_{L^2} \qquad &\text{and } \mathcal{H}_{2k}(t){:=}\left\|t^k\nabla u_t^{(k)}(t)\right\|_{L^2},\\ \mathcal{L}_{2k+1}(t){:=}\left\|t^{k+\frac12}\nabla u_t^{(k)}(t)\right\|_{L^2} \quad &\text{and } \mathcal{H}_{2k+1}(t){:=}\left\|t^{k+\frac12} u_t^{(k+1)}(t)\right\|_{L^2},\end{aligned}\tag{0.5}$$

where $u_t^{(k)}$ stands for the k^{th} time derivative of u.

To handle the case of even exponents, we start from

$$\partial_t\left(t^k u_t^{(k)}\right)-\Delta\left(t^k u_t^{(k)}\right)+\nabla\left(t^k P_t^{(k)}\right)=kt^{k-1}u_t^k$$

then take the L^2 scalar product with $t^k u_t^{(k)}$ and perform an integration by parts to get

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{2k}^2+\mathcal{H}_{2k}^2=k\mathcal{H}_{2k-1}^2.$$

For odd exponents, we rather take the L^2 scalar product with $t^{k+1}u_t^{(k+1)}$ and get

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{2k+1}^2+\mathcal{H}_{2k+1}^2=\left(k+\frac12\right)\mathcal{H}_{2k}^2.$$

In short, we have for all $m\in\mathbb{N}$,

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{m}^2+\mathcal{H}_{m}^2=\frac{m}{2}\mathcal{H}_{m-1}^2$$

which immediately leads after summation on m and time integration to

$$\sum_{m=0}^{\infty}\frac{\mathcal{L}_m^2(t)}{m!}+\int_0^t\sum_{m=0}^{\infty}\frac{\mathcal{H}_m^2(\tau)}{m!}\,\mathrm{d}\tau=\|u_0\|_{L^2}^2,\qquad t\in\mathbb{R}_+.\tag{0.6}$$

We shall proceed in the same way for the Navier–Stokes system, treating the nonlinear term by combination of Hölder and Ladyzhenskaya inequalities (this is the only place where dimension two comes into play). This will lead to the following results:

- Gevrey type regularity (almost as good as (0.6)), that implies time decay estimates for derivatives of arbitrary order in the case of small initial data;
- decay estimates at any order, in terms of $\|u_0\|_{L^2}$ for general finite energy solutions;
- small time Gevrey regularity in the case of large data;
- faster decay for all derivatives in case it is known beforehand that $\|u(t)\|_{L^2}$ has some algebraic decay.

We conclude this introduction pointing out that we here only considered the decay of time derivatives both for simplicity and because proving similar results for the space derivatives requires the fluid domain to have enough smoothness. The reader may refer to Remark 2.2 for a short development on this issue.

1. The case of small data

The main goal of this section is to prove the following theorem.

THEOREM 1.1. — *Let $\alpha > 0$. There exists a constant c_α depending only on α such that for any data u_0 in L^2_σ satisfying $\|u_0\|_{L^2} \leqslant c_\alpha$, the corresponding global finite energy solution u satisfies*

$$\sum_{k=0}^{\infty}\left(\frac{t^{2k}}{2^{2k}(k!)^{2+\alpha}}\left\|u_t^{(k)}(t)\right\|_{L^2}^2+\frac{t^{2k+1}}{2^{2k+1}k!((k+1)!)^{1+\alpha}}\left\|\nabla u_t^{(k)}(t)\right\|_{L^2}^2\right)$$
$$+\frac{1}{2}\sum_{k=0}^{\infty}\int_0^t\left(\frac{\tau^{2k}}{2^{2k}(k!)^{2+\alpha}}\left\|\nabla u_\tau^{(k)}(\tau)\right\|_{L^2}^2\right.$$
$$\left.+\frac{\tau^{2k+1}}{2^{2k+1}k!((k+1)!)^{1+\alpha}}\left\|u_\tau^{(k+1)}(\tau)\right\|_{L^2}^2\right)\mathrm{d}\tau$$
$$\leqslant \|u_0\|_{L^2}^2. \quad (1.1)$$

Proof. — Here and in the following sections, we concentrate on the proof of a priori estimates. The underlying idea is that one can get exactly the same bounds for any approximation that relies on the use of spectral orthogonal projectors (like e.g. the Galerkin method) and that following the compactness procedure that is used for proving Theorem 0.1 ensures that the solution that is constructed in this way satisfies the announced inequalities.

Now, with the notation introduced in (0.5), the energy balance (0.1) translates into

$$\frac{1}{2}\mathcal{L}_0^2(t)+\int_0^t\mathcal{H}_0^2\,\mathrm{d}\tau=\frac{1}{2}\|u_0\|_{L^2}^2. \quad (1.2)$$

To handle $\mathcal{L}_m$ and $\mathcal{H}_m$ in the case of odd index m, we apply ∂_t^{k-1} (for any $k \geqslant 1$) to (NS) and use Leibniz formula, getting:

$$u_t^{(k)}-\Delta u_t^{(k-1)}+\nabla P_t^{(k-1)}=-\sum_{j=0}^{k-1}\binom{k-1}{j}\operatorname{div}\left(u_t^{(j)}\otimes u_t^{(k-1-j)}\right).$$

Taking the scalar product with $t^{2k-1}u_t^{(k)}$ and integrating by parts where needed yields

$$\left\|t^{k-\frac{1}{2}}u_t^{(k)}\right\|_{L^2}^2+\int_\Omega t^{2k-1}\nabla u_t^{(k-1)}\cdot\partial_t\nabla u_t^{(k-1)}\,\mathrm{d}x=-\sum_{j=0}^{k-1}\binom{k-1}{j}\mathcal{R}_{j,2k-1}$$

$$\text{with}\quad \mathcal{R}_{j,2k-1}:=\int_\Omega \mathrm{div}\left(t^ju_t^{(j)}\otimes\left(t^{k-1-j}u_t^{(k-1-j)}\right)\right)\cdot\left(t^ku_t^{(k)}\right)\,\mathrm{d}x,$$

whence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{2k-1}^2+\mathcal{H}_{2k-1}^2=\left(k-\frac{1}{2}\right)\mathcal{H}_{2k-2}^2-\sum_{j=0}^{k-1}\binom{k-1}{j}\mathcal{R}_{j,2k-1}. \tag{1.3}$$

For all $j\in\{0,\ldots,k-1\}$, performing an integration by parts gives

$$\mathcal{R}_{j,2k-1}:=-\int_\Omega\left(t^ju_t^{(j)}\otimes\left(t^{k-1-j}u_t^{(k-1-j)}\right)\right)\cdot\left(t^k\nabla u_t^{(k)}\right)\,\mathrm{d}x,$$

whence using Hölder and Ladyzhenskaya inequality and the definition of $\mathcal{L}_m$ and $\mathcal{H}_m$,

$$\begin{aligned}\mathcal{R}_{j,2k-1}&\leqslant\left\|t^ju_t^{(j)}\right\|_{L^4}\left\|t^{k-1-j}u_t^{(k-1-j)}\right\|_{L^4}\left\|t^k\nabla u_t^{(k)}\right\|_{L^2}\\&\leqslant C_0\mathcal{L}_{2j}^{1/2}\mathcal{H}_{2j}^{1/2}\mathcal{L}_{2k-2-2j}^{1/2}\mathcal{H}_{2k-2-2j}^{1/2}\mathcal{H}_{2k}.\end{aligned}$$

Hence we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{2k-1}^2+\mathcal{H}_{2k-1}^2\leqslant\left(k-\frac{1}{2}\right)\mathcal{H}_{2k-2}^2&\\+C_0\sum_{j=0}^{k-1}\binom{k-1}{j}\mathcal{L}_{2j}^{1/2}\mathcal{H}_{2j}^{1/2}\mathcal{L}_{2k-2-2j}^{1/2}\mathcal{H}_{2k-2-2j}^{1/2}\mathcal{H}_{2k}.&\end{aligned} \tag{1.4}$$

In order to handle even indices, we apply $t^k\partial_t^k$ to (NS). Using Leibniz formula yields:

$$\begin{aligned}\partial_t\left(t^ku_t^{(k)}\right)+\mathrm{div}\left(u\otimes t^ku_t^{(k)}\right)-\Delta\left(t^ku_t^{(k)}\right)+\nabla\left(t^kP_t^{(k)}\right)&\\=kt^{k-1}u_t^{(k)}-\sum_{j=1}^{k}\binom{k}{j}\,\mathrm{div}\left(t^ju_t^{(j)}\otimes\left(t^{k-j}u_t^{(k-j)}\right)\right)\cdot&\end{aligned} \tag{1.5}$$

Hence taking the L^2 scalar product with $t^k u_t^{(k)}$ and performing suitable integration by parts gives:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{2k}^2+\mathcal{H}_{2k}^2=k\mathcal{H}_{2k-1}^2-\sum_{j=1}^{k}\binom{k}{j}\mathcal{R}_{j,2k}$$

$$\text{with}\quad \mathcal{R}_{j,2k}:=\int_\Omega \operatorname{div}\left(t^j u_t^{(j)}\otimes\left(t^{k-j}u_t^{(k-j)}\right)\right)\cdot\left(t^k u_t^{(k)}\right)\,\mathrm{d}x. \quad (1.6)$$

Observe that

$$\mathcal{R}_{j,2k}:=-\int_\Omega\left(t^j u_t^{(j)}\otimes\left(t^{k-j}u_t^{(k-j)}\right)\right)\cdot\left(t^k\nabla u_t^{(k)}\right)\,\mathrm{d}x. \quad (1.7)$$

Therefore, combining Hölder inequality and (0.3) gives

$$\begin{aligned}\mathcal{R}_{j,2k}&\leqslant\left\|t^j u_t^{(j)}\right\|_{L^4}\left\|t^{k-j}\nabla u_t^{(k-j)}\right\|_{L^4}\left\|t^k u_t^{(k)}\right\|_{L^2}\\&\leqslant C_0\mathcal{L}_{2j}^{1/2}\mathcal{H}_{2j}^{1/2}\mathcal{L}_{2k-2j}^{1/2}\mathcal{H}_{2k-2j}^{1/2}\mathcal{H}_{2k}.\end{aligned}$$

Hence we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{2k}^2+\mathcal{H}_{2k}^2\leqslant k\mathcal{H}_{2k-1}^2+C_0\sum_{j=0}^{k-1}\binom{k}{j}\mathcal{L}_{2j}^{1/2}\mathcal{H}_{2j}^{1/2}\mathcal{L}_{2k-2j}^{1/2}\mathcal{H}_{2k-2j}^{1/2}\mathcal{H}_{2k}. \quad (1.8)$$

Let us use renormalize the functionals $\mathcal{L}_m$ and $\mathcal{H}_m$ as follows:

$$\widetilde{L}_{2k}:=\frac{\mathcal{L}_{2k}}{2^k k!},\quad \widetilde{H}_{2k}:=\frac{\mathcal{H}_{2k}}{2^k k!},\quad \widetilde{L}_{2k-1}:=\frac{\sqrt{2}\,\mathcal{L}_{2k-1}}{2^k\sqrt{(k-1)!k!}}$$

$$\text{and}\quad \widetilde{H}_{2k-1}:=\frac{\sqrt{2}\,\mathcal{H}_{2k-1}}{2^k\sqrt{(k-1)!k!}}\cdot \quad (1.9)$$

Then, (1.4) and (1.8) become:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{L}_{2k-1}^2+\widetilde{H}_{2k-1}^2\\&\qquad\leqslant\frac{1}{2}\left(\frac{k-\frac{1}{2}}{k}\right)\widetilde{H}_{2k-2}^2+C_0\left(\sum_{j=0}^{k-1}\widetilde{L}_{2j}^{1/2}\widetilde{H}_{2j}^{1/2}\widetilde{L}_{2k-2-2j}^{1/2}\widetilde{H}_{2k-2-2j}^{1/2}\right)\widetilde{H}_{2k},\\&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{L}_{2k}^2+\widetilde{H}_{2k}^2\\&\qquad\leqslant\frac{1}{2}\widetilde{H}_{2k-1}^2+C_0\left(\sum_{j=1}^{k-1}\widetilde{L}_{2j}^{1/2}\widetilde{H}_{2j}^{1/2}\widetilde{L}_{2k-2j}^{1/2}\widetilde{H}_{2k-2j}^{1/2}\right)\widetilde{H}_{2k}.\end{aligned}$$

In order to get a nice control of the sum, we perform a second renormalization as follows for some suitable nonnegative nondecreasing sequence $(c_m)_{m\in\mathbb{N}}$:

$$\begin{aligned}\widetilde{L}_{2m-1} = c_m L_{2m-1}, \quad \widetilde{H}_{2m-1} = c_m H_{2m-1},\\ \widetilde{L}_{2m} = c_m L_{2m} \quad \text{and} \quad \widetilde{H}_{2m} = c_m H_{2m}.\end{aligned} \tag{1.10}$$

The above inequalities translate into

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}L_{2k-1}^2 + H_{2k-1}^2\\ &\qquad\leqslant \frac{1}{2}\frac{c_{k-1}^2}{c_k^2}H_{2k-2}^2 + C_0\left(\sum_{j=0}^{k-1}\frac{c_j c_{k-1-j}}{c_k}L_{2j}^{1/2}H_{2j}^{1/2}L_{2k-2-2j}^{1/2}H_{2k-2-2j}^{1/2}\right)H_{2k},\\ &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}L_{2k}^2 + H_{2k}^2\\ &\qquad\leqslant \frac{1}{2}H_{2k-1}^2 + C_0\left(\sum_{j=1}^{k}\frac{c_j c_{k-j}}{c_k}L_{2j}^{1/2}H_{2j}^{1/2}L_{2k-2j}^{1/2}H_{2k-2j}^{1/2}\right)H_{2k}.\end{aligned}$$

Let us take $c_j = (j!)^\alpha$ with $\alpha > 0$ so that $\frac{c_j c_{k-j}}{c_k} = \binom{k}{j}^{-\alpha}$. Since $\frac{i}{k-j+i} \leqslant \frac{j}{k}$ for all $i \in \{1,\dots,j\}$, we have $\binom{k}{j} \geqslant (\frac{k}{j})^j$ for all $j \in \{0,\dots,k\}$. Remembering that $\binom{k}{j} = \binom{k}{k-j}$, we get

$$\frac{c_j c_{k-j}}{c_k} \leqslant \min\left(\left(\frac{j}{k}\right)^{\alpha j}, \left(\frac{k-j}{k}\right)^{\alpha(k-j)}\right).$$

Using the obvious bound $j/k \leqslant 1/2$ for $j \leqslant k/2$, we conclude that

$$\frac{c_j c_{k-j}}{c_k} \leqslant \min\left(2^{-j\alpha}, 2^{-(k-j)\alpha}\right) \quad \text{for all} \quad j \in \{0,\dots,k\}. \tag{1.11}$$

Hence we have

$$\begin{aligned}&\sum_{j=0}^{k}\frac{c_j c_{k-j}}{c_k}L_{2j}^{1/2}H_{2j}^{1/2}L_{2k-2j}^{1/2}H_{2k-2j}^{1/2}\\ &\qquad\qquad\leqslant \sum_{j=0}^{k}\sqrt{2^{-j\alpha}L_{2j}H_{2j}}\sqrt{2^{-(k-j)\alpha}L_{2k-2j}H_{2k-2j}}.\end{aligned}$$

Similarly, as $c_{k-1} \leqslant c_k$, we have

$$\sum_{j=0}^{k-1} \frac{c_j c_{k-1-j}}{c_k} L_{2j}^{1/2} H_{2j}^{1/2} L_{2k-2-2j}^{1/2} H_{2k-2-2j}^{1/2} \leqslant \sum_{j=0}^{k-1} \sqrt{2^{-j\alpha} L_{2j} H_{2j}} \sqrt{2^{-(k-1-j)\alpha} L_{2k-2-2j} H_{2k-2-2j}}.$$

Hence, summing up the above two inequalities yields for all $k \geqslant 1$,

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(L_{2k-1}^2 + L_{2k}^2\right) + \frac{1}{2}H_{2k-1}^2 + H_{2k}^2 \\ &\leqslant \frac{1}{2}H_{2k-2}^2 + C_0\left(\sum_{j=0}^{k} \sqrt{2^{-j\alpha} L_{2j} H_{2j}} \sqrt{2^{-(k-j)\alpha} L_{2k-2j} H_{2k-2j}}\right) H_{2k} \\ &\quad + C_0\left(\sum_{j=0}^{k-1} \sqrt{2^{-j\alpha} L_{2j} H_{2j}} \sqrt{2^{-(k-1-j)\alpha} L_{2k-2j-2} H_{2k-2j-2}}\right) H_{2k}. \end{aligned} \tag{1.12}$$

Let us introduce the notation:

$$\mathbb{L}_m^2 := \sum_{k=0}^{m} L_k^2 \quad \text{and} \quad \mathbb{H}_m^2 := \sum_{k=0}^{m} H_k^2.$$

Then summing up (1.2) and (1.12) from $k = 1$ to $k = n$ gives after using the convolution inequality

$$\sum_{k=0}^{n} \sum_{j=0}^{n} a_j b_{k-j} c_k \leqslant \|(a_j)\|_{\ell_n^{4/3}} \|(b_j)\|_{\ell_n^{4/3}} \|(c_j)\|_{\ell_n^2} \quad \text{with} \quad \ell_n^r := \ell^r(\{0, \dots, n\}),$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{L}_{2n}^2 + \frac{1}{2}\mathbb{H}_{2n}^2 + \frac{1}{2}H_{2n}^2 \leqslant 2C_0 \|(2^{-j\alpha} L_{2j} H_{2j})\|_{\ell_n^{2/3}} \mathbb{H}_{2n}. \tag{1.13}$$

Hölder inequality implies that

$$\begin{aligned} \|(2^{-j\alpha} L_{2j} H_{2j})\|_{\ell_n^{2/3}} &\leqslant \|(2^{-j\alpha})\|_{\ell^2} \|(L_{2j})\|_{\ell^2} \|(H_{2j})\|_{\ell^2} \\ &\leqslant C_\alpha \mathbb{L}_{2n} \mathbb{H}_{2n} \quad \text{with} \quad C_\alpha := \sqrt{\frac{1}{1 - 2^{-2\alpha}}}. \end{aligned}$$

Hence, whenever $2C_0 C_\alpha \mathbb{L}_{2n} \leqslant 1/4$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{L}_{2n}^2 + \frac{1}{2}\mathbb{H}_{2n}^2 \leqslant 0.$$

Since $\mathbb{L}_{2n}(0) = \|u_0\|_{L^2}$, a bootstrap argument allows to conclude that if

$$8C_0 C_\alpha \|u_0\|_{L^2} < 1, \tag{1.14}$$

then we have for all time $t \geqslant 0$ and $n \in \mathbb{N}$,

$$\mathbb{L}_{2n}^2(t) + \frac{1}{2}\int_0^t \mathbb{H}_{2n}^2(\tau)\,\mathrm{d}\tau \leqslant \|u_0\|_{L^2}^2.$$

Applying the monotonous convergence theorem then leads to (1.1). □

2. The case of large data

Here we want to establish time decay estimates for derivatives of u at any order, in the case of general, possibly large, finite energy data. The main result is:

Theorem 2.1. — *Let $\alpha > 0$. There exists a constant C_α depending only on α such that for any initial data u_0 in L^2_σ and integer n, we have*

$$\sum_{k=0}^{n}\left(\frac{t^{2k}}{2^{2k}(k!)^{2+\alpha}}\left\|u_t^{(k)}(t)\right\|_{L^2}^2 + \frac{t^{2k+1}}{2^{2k+1}k!((k+1)!)^{1+\alpha}}\left\|\nabla u_t^{(k)}(t)\right\|_{L^2}^2\right)$$
$$+\frac{1}{2}\sum_{k=0}^{n}\int_0^t\left(\frac{\tau^{2k}}{2^{2k+1}(k!)^{2+\alpha}}\left\|\nabla u_\tau^{(k)}(\tau)\right\|_{L^2}^2\right.$$
$$\left.+\frac{\tau^{2k+1}}{2^{2k+1}k!((k+1)!)^{1+\alpha}}\left\|u_\tau^{(k+1)}(t)\right\|_{L^2}^2\right)\mathrm{d}\tau$$
$$\leqslant C_\alpha^{2^n-1}\left(\|u_0\|_{L^2}^2\exp\left(\frac{C_0^2\|u_0\|_{L^2}^2}{2}\right)\right)^{2^n}, \quad (2.1)$$

where C_0 stands for the optimal constant in (0.3).

Proof. — To handle the case of general data, we slightly modify (1.8). In fact, starting from (1.6), we use (1.7) only for $j = 0, \ldots, k-1$ and bound $\mathcal{R}_{k,2k}$ as follows:

$$\mathcal{R}_{k,2k} \leqslant \|\nabla u\|_{L^2}\left\|t^k u_t^{(k)}\right\|_{L^4}^2 \leqslant C_0\mathcal{H}_0\mathcal{L}_{2k}\mathcal{H}_{2k}.$$

This leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{2k}^2 + \mathcal{H}_{2k}^2$$
$$\leqslant k\mathcal{H}_{2k-1}^2 + C_0\sum_{j=0}^{k-1}\binom{k}{j}\mathcal{L}_{2j}^{1/2}\mathcal{H}_{2j}^{1/2}\mathcal{L}_{2k-2j}^{1/2}\mathcal{H}_{2k-2j}^{1/2}\mathcal{H}_{2k} + C_0\mathcal{H}_0\mathcal{L}_{2k}\mathcal{H}_{2k}.$$

Then, adding up (1.4) leads after the same succession of renormalizations as in the previous section to

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(L_{2k}^2+L_{2k+1}^2\right)+H_{2k+1}^2+\frac{1}{2}H_{2k}^2\\ \leqslant \frac{1}{2}H_{2k-1}^2+C_0\mathcal{H}_0L_{2k}H_{2k}\\ +C_0\left(\sum_{j=0}^{k}\sqrt{2^{-j\alpha}L_{2j}H_{2j}}\sqrt{2^{-(k-1-j)\alpha}L_{2k-2-2j}H_{2k-2-2j}}\right)\\ +C_0\left(\sum_{j=1}^{k-1}\sqrt{2^{-j\alpha}L_{2j}H_{2j}}\sqrt{2^{-(k-j)\alpha}L_{2k-2j}H_{2k-2j}}\right).\end{aligned}$$

We may use for each k that $C_0\mathcal{H}_0L_{2k}H_{2k}\leqslant\frac{1}{2}H_{2k}^2+\frac{1}{2}C_0^2\mathcal{H}_0^2L_{2k}^2$ and, after summing for $k=1$ to n, the second and third lines may be bounded by $C_\alpha\mathbb{L}_{2n-2}\mathbb{H}_{2n-2}\mathbb{H}_{2n}$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{L}_{2n}^2+\mathbb{H}_{2n}^2\leqslant C_0^2\mathcal{H}_0^2\mathbb{L}_{2n}^2+4C_0C_\alpha\mathbb{L}_{2n-2}\mathbb{H}_{2n-2}\mathbb{H}_{2n},\quad n\geqslant 1.$$

Performing the change of function:

$$\mathbb{L}_{2n}(t)=e^{\frac{1}{2}\int_0^t C_0^2\mathcal{H}_0^2(\tau)\,\mathrm{d}\tau}\widetilde{\mathbb{L}}_{2n}(t)\quad\text{and}\quad\mathbb{H}_{2n}(t)=e^{\frac{1}{2}\int_0^t C_0^2\mathcal{H}_0^2(\tau)\,\mathrm{d}\tau}\widetilde{\mathbb{H}}_{2n}(t)$$

and using (1.2) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathbb{L}}_{2n}^2+\frac{1}{2}\widetilde{\mathbb{H}}_{2n}^2\leqslant\frac{1}{2}A_{\alpha,u_0}^2\widetilde{\mathbb{L}}_{2n-2}^2\widetilde{\mathbb{H}}_{2n-2}^2$$

and thus, after time integration,

$$\widetilde{\mathbb{L}}_{2n}^2(t)+\frac{1}{2}\int_0^t\widetilde{\mathbb{H}}_{2n}^2\,\mathrm{d}\tau\leqslant\|u_0\|_{L^2}^2+\frac{1}{2}A_{\alpha,u_0}^2\int_0^t\widetilde{\mathbb{L}}_{2n-2}^2\widetilde{\mathbb{H}}_{2n-2}^2\,\mathrm{d}\tau.\qquad(2.2)$$

Let us denote (assuming of course that $u_0\neq 0$)

$$X_n:=\|u_0\|_{L^2}^{-2}\left(\sup_{t\geqslant 0}\widetilde{\mathbb{L}}_{2n}^2(t)+\frac{1}{2}\int_0^\infty\widetilde{\mathbb{H}}_{2n}^2\,\mathrm{d}\tau\right)$$

and use the inequality $2ab\leqslant(a+b)^2$. Then, $X_0=1$ and (2.2) can be rewritten as

$$X_n\leqslant 1+\frac{A_{\alpha,u_0}^2\|u_0\|_{L^2}^2}{2}X_{n-1}^2,\quad n\geqslant 1.$$

Since obviously $X_n\geqslant 1$ for all $n\in\mathbb{N}$, we have

$$X_n\leqslant KX_{n-1}^2\quad\text{with}\quad K:=1+\frac{A_{\alpha,u_0}^2\|u_0\|_{L^2}^2}{2},$$

which implies that

$$\forall\, n\in\mathbb{N},\ X_n\leqslant K^{2^n-1}$$

and thus

$$\widetilde{\mathbb{L}}_{2n}^2(t)+\frac{1}{2}\int_0^t\widetilde{\mathbb{H}}_{2n}^2\,\mathrm{d}\tau\leqslant\|u_0\|_{L^2}^2\left(1+\frac{A_{\alpha,u_0}^2\|u_0\|_{L^2}^2}{2}\right)^{2^n-1}.$$

Clearly, the computations here are relevant only if (1.14) is not satisfied, so that, up to an harmless change of C_α in the definition of A_{α,u_0}, we have

$$1+\frac{A_{\alpha,u_0}^2\|u_0\|_{L^2}^2}{2}\leqslant A_{\alpha,u_0}^2\|u_0\|_{L^2}^2.$$

In the end, we get a constant C_α with behavior $\alpha^{-1/2}$ near 0 such that for all $t\geqslant 0$,

$$\mathbb{L}_{2n}^2(t)+\frac{1}{2}\int_0^t\mathbb{H}_{2n}^2\,\mathrm{d}\tau\leqslant C_\alpha^{2^n-1}\left(\|u_0\|_{L^2}^2\exp\left(\frac{C_0^2\|u_0\|_{L^2}^2}{2}\right)\right)^{2^n}.$$

This gives (2.1). □

Remark 2.2. — As a consequence of the regularity theory for the Stokes system, in the case where the domain Ω is smooth with a "reasonable shape" (like e.g. bounded simply connected or exterior domains), then one can deduce decay estimates at any order for the *space* derivatives of u.

Indeed, we have $u|_{\partial\Omega}=0$,

$$-\Delta u+\nabla P=-u_t-u\cdot\nabla u\quad\text{and}\quad\operatorname{div}u=0\quad\text{in}\quad\Omega.$$

Hence there exists a constant C depending only on Ω (if it is e.g. uniformly C^2 and bounded) such that:

$$\left\|\nabla^2u\right\|_{L^2}+\|\nabla P\|_{L^2}\leqslant C\|u_t+u\cdot\nabla u\|_{L^2}.$$

Multiplying by t and using Hölder and Ladyzhenskaya inequality yields

$$\left\|t\nabla^2u\right\|_{L^2}+\|t\nabla P\|_{L^2}\leqslant C\|tu_t\|_{L^2}+CC_0\|u\|_{L^2}^{1/2}\left\|\sqrt{t}\nabla u\right\|_{L^2}\left\|t\nabla^2u\right\|_{L^2}^{1/2}.$$

Using Young inequality allows to conclude that for some constant still denoted by C,

$$\left\|t\nabla^2u\right\|_{L^2}+\|t\nabla P\|_{L^2}\leqslant C\left(\|tu_t\|_{L^2}+\|u\|_{L^2}\left\|\sqrt{t}\,\nabla u\right\|_{L^2}^2\right).$$

This allows to get a uniform bound of the left-hand side in terms of $\|u_0\|_{L^2}$, due to (2.1) with $n=1$.

By the same token, it is easy to bound $\|t^{k+1}\nabla^2 u_t^{(k)}\|_{L^2}$ for any integer k since $(u_t^{(k)}, P_t^{(k)})$ satisfies $u_t^{(k)}|_{\partial\Omega} = 0$,

$$-\Delta u_t^{(k)} + \nabla P_t^{(k)} = -u_t^{(k+1)} - \sum_{j=0}^{k} \binom{k}{j} u_t^j \cdot \nabla u_t^{(k-j)}$$
$$\text{and} \quad \operatorname{div} u_t^{(k)} = 0 \quad \text{in} \quad \Omega.$$

Hence, requiring only C^2 regularity for Ω gives

$$\begin{aligned}&\left\|t^{k+1}\nabla^2 u_t^{(k)}\right\|_{L^2} + \left\|t^{k+1}\nabla P_t^{(k)}\right\|_{L^2}\\ &\leqslant C\left(\left\|t^{k+1}u_t^{(k+1)}\right\|_{L^2} + \sum_{j=0}^{k}\binom{k}{j}\left\|\left(t^{j+\frac14}u_t^j\right)\cdot\left(t^{k-j+\frac34}\nabla u_t^{(k-j)}\right)\right\|_{L^2}\right)\cdot\end{aligned}$$

The right-hand side may be bounded in terms of $\|u_0\|_{L^2}$ by combining Hölder inequality, (0.3) and (2.1) with $n = k+1$.

In order to bound higher order space derivatives, we use that if Ω is smooth then, for all $j \in \mathbb{N}$, there exists a constant C_j depending only on Ω and j such that

$$\left\|\nabla^{j+2}u\right\|_{L^2} + \left\|\nabla^{j+1}P\right\|_{L^2} \leqslant C_j\left(\left\|\nabla^j u_t\right\|_{L^2} + \left\|\nabla^j(u\cdot\nabla u)\right\|_{L^2}\right)\cdot$$

Similar inequalities at any order may be written for $\nabla^{j+2}u_t^k$. Then using a careful induction argument allows to bound $t^{k+j/2}\|\nabla^j u_t^{(k)}\|_{L^2}$ in terms of u_0 at any order. The (tedious) verifications are left to the reader.

3. Small time Gevrey regularity in the case of large data

In this section we address the question of Gevrey regularity in the case where u_0 is large. Since the solution (ℓ, Q) to the Stokes system (0.4) has analytic regularity (recall (0.6)), it suffices to study the regularity of the fluctuation $f := u - \ell$ that, by definition, satisfies $f|_{\partial\Omega} = 0$, $f|_{t=0} = 0$, and, for some scalar function R,

$$\begin{cases} f_t - \Delta f + \nabla R = -u\cdot\nabla u & \text{in } \mathbb{R}_+\times\Omega,\\ \operatorname{div} f = 0 & \text{in } \mathbb{R}_+\times\Omega.\end{cases} \tag{3.1}$$

The main result of this section reads:

Theorem 3.1. — *Let $\alpha>0$. There exists a positive constant C_α, a positive time T_{α,u_0} and a continuous increasing function $\phi_{\alpha,u_0}:[0,T_{\alpha,u_0}]\to\mathbb{R}_+$ vanishing at 0 such that the fluctuation f satisfies for all $t\in[0,T_{\alpha,u_0}]$:*

$$\sum_{k=0}^{\infty}\left(\frac{t^{2k}}{2^{2k}(k!)^{2+\alpha}}\left\|f_t^{(k)}(t)\right\|_{L^2}^2+\frac{t^{2k+1}}{2^{2k+1}k!((k+1)!)^{1+\alpha}}\left\|\nabla f_t^{(k)}(t)\right\|_{L^2}^2\right)$$
$$+\sum_{k=0}^{\infty}\int_0^t\left(\frac{\tau^{2k}}{2^{2k+1}(k!)^{2+\alpha}}\|\nabla f_\tau^{(k)}(\tau)\|_{L^2}^2\right.$$
$$\left.+\frac{\tau^{2k+1}}{2^{2k+1}k!((k+1)!)^{1+\alpha}}\left\|f_\tau^{(k+1)}(\tau)\right\|_{L^2}^2\right)\mathrm{d}\tau\leqslant\phi_{\alpha,u_0}(t).$$

Proof. — Denote by L_m^ℓ and H_m^ℓ (resp. L_m^f and H_m^f) the quantities L_m and H_m defined in (1.10) pertaining to ℓ (resp. f). According to Leibniz rule, we have for all $k\in\mathbb{N}$,

$$f_t^{(k)}-\Delta f_t^{(k-1)}+\nabla R_t^{k-1}=-\sum_{j=0}^{k-1}\binom{k-1}{j}u_t^{(j)}\cdot\nabla u_t^{(k-1-j)}.$$

Hence taking the scalar product with $t^{2k-1}f_t^{(k)}$ (odd case) or with $t^{2k}f_t^{(k)}$ (even case) and using the following type of inequalities:

$$\left\|t^ju_t^{(j)}\right\|_{L^4}\left\|t^{k-j}u_t^{(k-j)}\right\|_{L^4}$$
$$\leqslant\left\|t^j\ell_t^{(j)}\right\|_{L^4}\left\|t^{k-j}\ell_t^{(k-j)}\right\|_{L^4}+\left\|t^j\ell_t^{(j)}\right\|_{L^4}\left\|t^{k-j}f_t^{(k-j)}\right\|_{L^4}$$
$$+\left\|t^jf_t^{(j)}\right\|_{L^4}\left\|t^{k-j}\ell_t^{(k-j)}\right\|_{L^4}+\left\|t^jf_t^{(j)}\right\|_{L^4}\left\|t^{k-j}f_t^{(k-j)}\right\|_{L^4}$$

which implies, thanks to (0.3) that

$$\left\|t^ju_t^{(j)}\right\|_{L^4}\left\|t^{k-j}u_t^{(k-j)}\right\|_{L^4}$$
$$\leqslant C_0\left(\sqrt{L_{2j}^\ell H_{2j}^\ell L_{2k-2j}^\ell H_{2k-2j}^\ell}+\sqrt{L_{2j}^\ell H_{2j}^\ell L_{2k-2j}^f H_{2k-2j}^f}\right.$$
$$\left.+\sqrt{L_{2j}^f H_{2j}^f L_{2k-2j}^\ell H_{2k-2j}^\ell}+\sqrt{L_{2j}^f H_{2j}^f L_{2k-2j}^f H_{2k-2j}^f}\right),$$

the counterpart of (1.13) now reads

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbb{L}_{2n}^f\right)^2+\frac12\left(\mathbb{H}_{2n}^f\right)^2+\frac12\left(H_{2n}^f\right)^2\leqslant 2C_0\left\|\left(2^{-j\alpha}L_{2j}^fH_{2j}^f\right)\right\|_{\ell_n^{2/3}}\mathbb{H}_{2n}^f$$
$$+4C_0\sqrt{\left\|\left(2^{-j\alpha}L_{2j}^\ell H_{2j}^\ell\right)\right\|_{\ell_n^{2/3}}\left\|\left(2^{-j\alpha}L_{2j}^fH_{2j}^f\right)\right\|_{\ell_n^{2/3}}}\,\mathbb{H}_{2n}^f$$
$$+2C_0\left\|\left(2^{-j\alpha}L_{2j}^\ell H_{2j}^\ell\right)\right\|_{\ell_n^{2/3}}\mathbb{H}_{2n}^f$$

with

$$\mathbb{L}_m^p := \sqrt{\sum_{k=0}^{m} (L_k^p)^2} \quad \text{and} \quad \mathbb{H}_m^p := \sqrt{\sum_{k=0}^{m} (H_k^p)^2} \quad \text{for} \quad p \in \{f, \ell\}.$$

Using the Young inequality to bound the right-hand side, this inequality implies that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} & \left(\mathbb{L}_{2n}^f\right)^2 + \left(\mathbb{H}_{2n}^f\right)^2 + \left(H_{2n}^f\right)^2 \\
& \leqslant 8C_0 \left\| \left(2^{-j\alpha} L_{2j}^f H_{2j}^f\right) \right\|_{\ell_n^{2/3}} \mathbb{H}_{2n}^f + 8C_0 \left\| \left(2^{-j\alpha} L_{2j}^\ell H_{2j}^\ell\right) \right\|_{\ell_n^{2/3}} \mathbb{H}_{2n}^f \\
& \leqslant 8C_0 C_\alpha \mathbb{L}_{2n}^f \left(\mathbb{H}_{2n}^f\right)^2 + 8C_0 C_\alpha \mathbb{L}_{2n}^\ell \mathbb{H}_{2n}^\ell \mathbb{H}_{2n}^f \\
& \leqslant \left(\frac{1}{4} + 8C_0 C_\alpha \mathbb{L}_{2n}^f\right) \left(\mathbb{H}_{2n}^f\right)^2 + 64 C_0^2 C_\alpha^2 \left(\mathbb{L}_{2n}^\ell \mathbb{H}_{2n}^\ell\right)^2.
\end{aligned}$$

Therefore, whenever

$$8C_0 C_\alpha \mathbb{L}_{2n}^f(t) \leqslant 1/4, \tag{3.2}$$

we have

$$\begin{aligned}
\left(\mathbb{L}_{2n}^f(t)\right)^2 + \frac{1}{2} \int_0^t \left(\mathbb{H}_{2n}^f\right)^2 \mathrm{d}\tau & \leqslant 64 C_0^2 C_\alpha^2 \int_0^t \left(\mathbb{L}_{2n}^\ell \mathbb{H}_{2n}^\ell\right)^2 \mathrm{d}\tau \\
& \leqslant 64 C_0^2 C_\alpha^2 \|u_0\|_{L^2}^2 \int_0^t \left(\mathbb{H}_{2n}^\ell\right)^2 \mathrm{d}\tau.
\end{aligned}$$

Since (0.6) guarantees that

$$\int_0^\infty \sum_{k=0}^\infty \left(H_k^\ell\right)^2 \mathrm{d}t < \infty,$$

Lebesgue dominated convergence theorem ensures that there exists $T_0 > 0$ such that

$$8C_0 C_\alpha \|u_0\|_{L^2} \sqrt{\int_0^{T_0} \sum_{k=0}^\infty \left(H_k^\ell\right)^2 \mathrm{d}t} < \frac{1}{32 C_0 C_\alpha} \cdot$$

Reverting to the above inequality and bootstrapping, one can now conclude that (3.2) is satisfied on $[0, T_0]$ for all $n \in \mathbb{N}$, and that we thus have for all $t \in [0, T_0]$,

$$\sum_{k=0}^\infty \left(L_k^f(t)\right)^2 + \frac{1}{2} \int_0^t \sum_{k=0}^\infty \left(H_k^f\right)^2 \mathrm{d}\tau \leqslant 64 C_0^2 C_\alpha^2 \|u_0\|_{L^2}^2 \int_0^t \sum_{k=0}^\infty \left(H_{2k}^\ell\right)^2 \mathrm{d}\tau.$$

As the right-hand side is a continuous nondecreasing function vanishing at zero, this completes the proof of Theorem 3.1. □

4. Faster decay

In this last section, we assume that there exist $K \geqslant 0$ and $\gamma > 0$ such that our reference solution satisfies

$$\|u(t)\|_{L^2} \leqslant Kt^{-\gamma}, \quad t > 0. \tag{4.1}$$

It is known that (4.1) holds true with $\gamma = 1/2$ if u_0 is in L^1 (see [9]). Fix some $\alpha > 0$ and set for all $k \in \mathbb{N}$ and $t \geqslant t_0 \geqslant 0$,

$$L_{2k}(t,t_0) := \frac{\left\|(t-t_0)^k u_t^{(k)}(t)\right\|_{L^2}}{2^k (k!)^{1+\alpha}}$$

and

$$H_{2k}(t,t_0) := \frac{\left\|(t-t_0)^k \nabla u_t^{(k)}(t)\right\|_{L^2}}{2^k (k!)^{1+\alpha}},$$

$$L_{2k+1}(t,t_0) := \frac{\left\|(t-t_0)^{k+\frac{1}{2}} \nabla u_t^{(k)}(t)\right\|_{L^2}}{2^k \sqrt{k!(k+1)}\,((k+1)!)^{\alpha}}$$

and

$$H_{2k+1}(t,t_0) := \frac{\left\|(t-t_0)^{k+\frac{1}{2}} u_t^{(k+1)}(t)\right\|_{L^2}}{2^k \sqrt{k!(k+1)!}\,((k+1)!)^{\alpha}}.$$

Then, repeating the computations leading to (1.1), we arrive at

$$\mathbb{L}_{2n}^2(t,t_0) + \frac{1}{2}\int_0^t \mathbb{H}_{2n}^2(\tau,t_0)\,\mathrm{d}\tau \leqslant \|u(t_0)\|_{L^2}^2$$

with

$$\mathbb{L}_{2n}^2(t,t_0) := \sum_{k=0}^{2n} L_k^2(t,t_0) \quad \text{and} \quad \mathbb{H}_{2n}^2 := \sum_{k=0}^{2n} H_k^2(t,t_0)$$

whenever $8C_0C_\alpha\|u(t_0)\|_{L^2} \leqslant 1$.

Clearly, this latter condition is satisfied for any $t_0 \geqslant 0$ if $8C_0C_\alpha\|u_0\|_{L^2} \leqslant 1$, or, due to (4.1), at $t_0 = t/2$ if $t \geqslant 2(8C_0C_\alpha K)^{1/\gamma}$ in the general case. Consequently, we have proved the following statement:

Theorem 4.1. — *Let $\alpha > 0$. Assume that the considered finite energy global solution u to* (NS) *satisfies* (4.1). *Then there exists $t_0 \geqslant 0$ such that for all $t \geqslant t_0$ we have,*

$$\sum_{k=0}^{\infty}\left(\frac{t^{2k+2\gamma}}{2^{4k}(k!)^{2+\alpha}}\left\|u_t^{(k)}(t)\right\|_{L^2}^2+\frac{t^{2k+1+2\gamma}}{2^{4k+1}k!((k+1)!)^{1+\alpha}}\left\|\nabla u_t^{(k)}(t)\right\|_{L^2}^2\right)$$
$$+\sum_{k=0}^{\infty}\int_0^t\left(\frac{\tau^{2k+2\gamma}}{2^{4k}(k!)^{2+\alpha}}\left\|\nabla u_\tau^{(k)}(\tau)\right\|_{L^2}^2\right.$$
$$\left.+\frac{\tau^{2k+1+2\gamma}}{2^{4k+1}k!((k+1)!)^{1+\alpha}}\left\|u_\tau^{(k+1)}(\tau)\right\|_{L^2}^2\right)\mathrm{d}\tau \leqslant 2^{2\gamma}K^2. \quad (4.2)$$

Bibliography

[1] H. Bae, A. Biswas & E. Tadmor, "Analyticity and decay estimates of the Navier–Stokes equations in critical Besov spaces", *Arch. Ration. Mech. Anal.* **205** (2012), no. 3, p. 963-991.

[2] W. Borchers & T. Miyakawa, "Algebraic L^2 decay for Navier–Stokes flows in exterior domains", *Acta Math.* **165** (1990), no. 3-4, p. 189-227.

[3] J.-Y. Chemin, "Le système de Navier–Stokes incompressible soixante dix ans après Jean Leray", in *Actes des Journées Mathématiques à la Mémoire de Jean Leray*, Séminaires et Congrès, vol. 9, Société Mathématique de France, 2004, p. 99-123.

[4] J.-Y. Chemin, B. Desjardins, I. Gallagher & E. Grenier, *Mathematical geophysics. An introduction to rotating fluids and the Navier–Stokes equations*, Oxford Lecture Series in Mathematics and its Applications, vol. 32, Oxford University Press; Clarendon Press, 2006.

[5] J.-Y. Chemin, I. Gallagher & P. Zhang, "On the radius of analyticity of solutions to semi-linear parabolic systems", *Math. Res. Lett.* **27** (2020), no. 6, p. 1631-1643.

[6] C. Foias, L. Hoang & J.-C. Saut, "Navier and Stokes meet Poincaré and Dulac", *J. Appl. Anal. Comput.* **8** (2018), no. 3, p. 727-763.

[7] C. Foias & R. M. Temam, "Gevrey class regularity for the solutions of the Navier–Stokes equations", *J. Funct. Anal.* **87** (1989), no. 2, p. 359-369.

[8] J. G. Heywood, "The Navier–Stokes equations: on the existence, regularity and decay of solutions", *Indiana Univ. Math. J.* **29** (1980), no. 5, p. 639-681.

[9] R. Kajikiya & T. Miyakawa, "On L^2 decay of weak solutions of the Navier–Stokes equations in $\mathbb{R}^n$", *Math. Z.* **192** (1986), no. 1, p. 135-148.

[10] S. Kawashima, A. Matsumura & T. Nishida, "On the fluid-dynamical approximation to the Boltzmann equation at the level of the Navier–Stokes equation", *Commun. Math. Phys.* **70** (1979), no. 2, p. 97-124.

[11] O. A. Ladyzhenskaya, "Solution "in the large" of the nonstationary boundary value problem for the Navier–Stokes system with two space variables", *Pure Appl. Math.* **12** (1959), p. 427-433.

[12] P. G. Lemarié-Rieusset, "Nouvelles remarques sur l'analyticité des solutions milds des équations de Navier–Stokes dans $\mathbb{R}^3$", *C. R. Math.* **338** (2004), no. 6, p. 443-446.

[13] J. Leray, "Sur le mouvement d'un liquide visqueux emplissant l'espace", *Acta Math.* **63** (1934), no. 1, p. 193-248.

[14] J.-L. LIONS & G. PRODI, "Un théorème d'existence et unicité dans les équations de Navier–Stokes en dimension 2", *C. R. Acad. Sci. Paris* **248** (1959), p. 3519-3521.

[15] M. OLIVER & E. S. TITI, "Remark on the rate of decay of higher order derivatives for solutions to the Navier–Stokes equations in $\mathbb{R}^n$", *J. Funct. Anal.* **172** (2000), no. 1, p. 1-18.

[16] M. E. SCHONBEK, "L^2 decay for weak solutions of the Navier–Stokes equations", *Arch. Ration. Mech. Anal.* **88** (1985), no. 3, p. 209-222.

[17] ———, "Large time behaviour of solutions to the Navier–Stokes equations", *Commun. Partial Differ. Equations* **11** (1986), no. 7, p. 733-763.

[18] M. WIEGNER, "Decay results for weak solutions of the Navier–Stokes equations on $\mathbb{R}^n$,", *J. Lond. Math. Soc. (2)* **35** (1987), no. 2, p. 303-313.