Annales de la Faculté des Sciences de Toulouse

MATHÉMATIQUES

Ryokichi Tanaka

Non-noise sensitivity for word hyperbolic groups

Tome XXXIII, nº 5 (2024), p. 1487–1510.

https://doi.org/10.5802/afst.1803

Publication membre du centre Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2258-7519

Annales de la faculté des sciences de Toulouse Volume XXXIII, nº 5, 2024
pp. 1487-1510

Non-noise sensitivity for word hyperbolic groups (*)

Ryokichi Tanaka (1)

Abstract. — We show that non-elementary random walks on word hyperbolic groups with finite first moment are not noise sensitive in a strong sense for small noise parameters.

Résumé. — Nous montrons que les marches aléatoires non élémentaires sur des groupes hyperboliques au sens de Gromov ayant un premier moment fini ne sont pas sensibles au bruit au sens fort pour de petits paramètres de bruit.

1. Introduction

Let Γ be a countable group, and μ be a probability measure on it. The main interest is in the case when Γ is finitely generated with a finite set of generators S and μ is the uniform distribution on S, but we will also consider the case when μ has an unbounded support. The μ-random walk starting from the identity o is defined by a product of independent sequence of increments with the identical distribution μ. The noise sensitivity question concerning a μ-random walk on Γ asks the following: if we choose some real parameter $\rho \in (0,1)$ and replace each increment by an independent sample with the same law μ with probability ρ or retain it with probability $1-\rho$, independently, then is the resulting random walk asymptotically independent of the original one?

More precisely, the μ-random walk $\{w_n\}_{n=0}^{\infty}$ starting from o is defined by $w_n = x_1 \cdots x_n$ for an independent sequence $\{x_n\}_{n=1}^{\infty}$ with the identical distribution μ and $w_0 := o$. Let μ_n denote the distribution of w_n, which is the

(*) Reçu le 14 novembre 2022, accepté le 26 octobre 2023.
(1) Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan — rtanaka@math.kyoto-u.ac.jp
This work is partially supported by JSPS Grant-in-Aid for Scientific Research JP20K03602.
Article proposé par Vincent Guedj.

n-fold convolutions of μ. For each $\rho \in [0,1]$, let us consider the π^ρ-random walk $\{(w_n^{(1)}, w_n^{(2)})\}_{n=0}^\infty$ on $\Gamma \times \Gamma$ defined by

$$\pi^\rho := \rho\mu \times \mu + (1-\rho)\mu_{\mathrm{diag}} \quad \text{on } \Gamma \times \Gamma,$$

where $\mu \times \mu$ denotes the product measure and $\mu_{\mathrm{diag}}((x,y)) = \mu(x)$ if $x = y$ and 0 otherwise. For any two probability measures ν_1 and ν_2 on a countable set X, the total-variation distance is defined by

$$\|\nu_1 - \nu_2\|_{\mathrm{TV}} := \sup_{A \subset X} |\nu_1(A) - \nu_2(A)|.$$

Definition 1.1. — *The μ-random walk on Γ is called ℓ^1*-noise sensitive *if for all $0 < \rho < 1$,*

$$\|\pi_n^\rho - \mu_n \times \mu_n\|_{\mathrm{TV}} \to 0 \quad \textit{as } n \to \infty.$$

The ℓ^1-noise sensitivity was introduced by Benjamini and Brieussel [1, Definition 2.1]. There it has been shown that this notion in general highly depends not only on Γ but also on μ. Among others, they have proved that if Γ admits a surjective homomorphism onto the infinite cyclic group $\mathbb{Z}$ and the support of μ is a finite set of generators, then a μ-random walk on Γ is *not* ℓ^1-nose sensitive. Moreover, if (Γ, μ) is *non-Liouville*, i.e., there exists a non-constant bounded μ-harmonic function on Γ, then a μ-random walk on Γ is *not* ℓ^1-noise sensitive [1, Theorem 1.1]. It is believed that these two properties are the only possible obstructions for the ℓ^1-noise sensitivity for random walks on groups. We show that non-elementary word hyperbolic groups with large class of μ reveal a strong negation of the ℓ^1-noise sensitivity if ρ is small enough. This also offers in the non-Liouville setting a way to show that random walks are not ℓ^1-noise sensitive in some refined sense for a class of groups possibly without non-trivial homomorphisms onto $\mathbb{Z}$.

Let Γ be a word hyperbolic group. A probability measure μ on Γ is called *non-elementary* if the support generates a non-elementary subgroup $\mathrm{gr}(\mu)$ as a group. In this setting, it is equivalent to say that $\mathrm{gr}(\mu)$ contains a free group of rank greater than one (and Γ is necessarily non-elementary). Furthermore we say that μ has *finite first moment* if

$$\sum_{x \in \Gamma} |x|\mu(x) < \infty,$$

for some (equivalently, every) word norm $|\cdot|$. First we note that all μ-random walk on Γ for a non-elementary μ is not ℓ^1-noise sensitive without any moment condition.

Theorem 1.2. — *Let Γ be a word hyperbolic group and μ be a non-elementary probability measure on Γ. For all $0 \leqslant \rho < 1$, we have*

$$\liminf_{n \to \infty} \|\pi_n^\rho - \mu_n \times \mu_n\|_{\mathrm{TV}} > 0.$$

This in fact follows from the proof of [1, Theorem 4.1] due to the non-Liouville property. We provide a proof in this setting to illustrate our approach. Second we show that under the finite first moment condition if ρ is close enough to 0, then the distribution of a π^ρ-random walk and the joint distribution of independent copies of two μ-random walks are mutually singular at infinity.

Theorem 1.3. — *Let* Γ *be a word hyperbolic group and* μ *be a non-elementary probability measure with finite first moment on* Γ*. There exists some* $0 < \rho_* \leqslant 1$ *such that for all* $0 < \rho < \rho_*$*, we have*

$$\|\pi_n^\rho - \mu_n \times \mu_n\|_{\mathrm{TV}} \to 1 \quad \text{as } n \to \infty.$$

In the case when μ is not a non-elementary probability measure, these statements are no longer true. Indeed, it suffices to consider elementary word hyperbolic groups, which are either finite groups or contain $\mathbb{Z}$ as a finite index subgroup (see e.g., [6, 7] for background). If Γ is a finite group, then for every probability measure μ on it, a μ-random walk on Γ is ℓ^1-noise sensitive. Indeed, π^ρ and $\mu \times \mu$ have the same support on $\Gamma \times \Gamma$ and the distributions of corresponding random walks with the same initial state tend to a common stationary distribution for time with the same parity (cf. [1, Proposition 5.1]). Benjamini and Brieussel have shown that on the infinite dihedral group, a lazy simple random walk on a Cayley graph is ℓ^1-noise sensitive [1, Theorem 1.4].

As it is mentioned above, if Γ has a surjective homomorphism onto $\mathbb{Z}$, then a μ-random walk can not be ℓ^1-noise sensitive. This follows by computing covariance of the random walk on each factor of $\mathbb{Z}^2$ as the image of product of homomorphisms and the classical central limit theorem. The method thus works for μ with finite second moment. We note, however, that so far this and the non Liouville property have been the only known ways to disprove ℓ^1-noise sensitivity. See [1] and [10, Section 3.3.4] for discussions concerning the subject of matters, various interesting notions of noise sensitivity for random walks on groups, questions and conjectures.

The proofs of Theorems 1.2 and 1.3 rely on the boundary of word hyperbolic groups, in particular, the fact that a μ-boundary or the Poisson boundary for (Γ, μ) is realized on a topological boundary of the group. In this setting, we show that if $h(\pi^\rho) \neq h(\pi^{\rho'})$ for $0 \leqslant \rho, \rho' \leqslant 1$, then

$$\left\|\pi_n^\rho - \pi_n^{\rho'}\right\|_{\mathrm{TV}} \to 1 \quad \text{as } n \to \infty, \tag{1.1}$$

where $h(\pi^\rho)$ is the asymptotic entropy for a π^ρ-random walk (see Section 2 for the definition, and Remark 5.1). It is proven by showing that the harmonic measure ν_{π^ρ} on the product of boundaries $(\partial\Gamma)^2$ is *exact dimensional* with a natural (quasi-) metric, i.e.,

$$\frac{\log \nu_{\pi^\rho}(B(\boldsymbol{\xi}, r))}{\log r} \to \frac{h(\pi^\rho)}{l} \quad \text{as } r \to 0 \text{ for } \nu_{\pi^\rho}\text{-almost every } \boldsymbol{\xi} \text{ in } (\partial\Gamma)^2,$$

where l stands for the drift defined by a product metric in $\Gamma \times \Gamma$ (see Theorem 3.1 for the precise statement). The proof follows the methods in [11], adapting to a product of word hyperbolic groups. By (1.1), together with the continuity of $h(\pi^\rho)$ in $\rho \in [0, 1]$ (Corollary 4.2), as it is established by a general result of Erschler and Kaimanovich [5], we show that there exists some $0 < \rho_* \leqslant 1$ with $h(\pi^\rho) < h(\pi^1)$ for all $0 < \rho < \rho_*$, deducing Theorem 1.3.

It would be interesting to determine ρ_* in Theorem 1.3. For example, in the case when μ is a uniform distribution on a finite set of size $m \geqslant 2$, freely generating a free *semi*-group, the asymptotic entropy is explicitly computed as

$$h(\pi^\rho) = \log m - \left(1 - \frac{m-1}{m}\rho\right) \log \left(1 - \frac{m-1}{m}\rho\right) - \frac{m-1}{m}\rho \log \frac{\rho}{m} \qquad \text{for } 0 \leqslant \rho \leqslant 1,$$

and $h(\pi^\rho) < h(\pi^1) = 2 \log m$ if $\rho < 1$. This implies that $\rho_* = 1$ in the special case. It might be expected that $\rho_* = 1$ in many cases, however, we do not know how to show this in general.

Organization of this paper is the following: in Section 2 we review known facts and tools on random walks and word hyperbolic groups, in Section 3 we show that the harmonic measure for π whose marginals are a common non-elementary probability measure μ on Γ with finite first moment is exact dimensional in Theorem 3.1, in Section 4 the continuity of asymptotic entropy in the parameter is established in Corollary 4.2, following [5], in Section 5 we show Theorems 1.2 and 1.3, and in Appendix A, we give a review concerning Poisson boundary for random walks and a proof on continuity of asymptotic entropy for the sake of convenience, mainly for an expository purpose.

Notations

For a real valued function f on the set of non-negative integers, we write $f(n) = o(n)$ if $|f(n)|/n \to 0$ as $n \to \infty$ and $f(n) = O(n)$ if there exists a constant C such that $|f(n)| \leqslant Cn$ for all large enough n. For a set A, we denote by $\#A$ the cardinality, and by A^{c} its complement set. The set of non-negative integers is written as $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. We define $0 \log 0 := 0$.

2. Preliminary

2.1. Random walks on word hyperbolic groups and their products

For a countable group Γ and a probability measure μ on it, the *asymptotic entropy* of a μ-random walk is defined by

$$h(\mu) = \lim_{n\to\infty} \frac{1}{n} H(\mu_n),$$

where $H(\mu) := -\sum_{x\in\Gamma} \mu(x)\log\mu(x)$, the *Shannon entropy* for probability measures μ on Γ, the limit exists by sub-additivity of $n \mapsto H(\mu_n)$ and is finite if $H(\mu) < \infty$.

Let Γ be a word hyperbolic group. For a probability measure μ on Γ, let $\operatorname{supp}\mu$ denote the support and we assume that the group $\operatorname{gr}(\mu)$ generated by $\operatorname{supp}\mu$ as a group is a non-elementary subgroup. We fix a left-invariant word metric d associated with some finite set of generators S closed under inversion $s \mapsto s^{-1}$. The following discussion does not depend on the choice of S. We denote the associated distance function from the identity o by $|x| := d(o, x)$. If μ has finite first moment, then $H(\mu) < \infty$ (cf. [4, Section VII, B]).

Let us consider any probability measure π on $\Gamma \times \Gamma$ such that the push-forward of π on each factor is a fixed μ on Γ. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the probability measure space, where $\Omega = (\Gamma\times\Gamma)^{\mathbb{Z}_+}$, $\mathcal{F}$ is the σ-field generated by the cylinder sets and $\mathbf{P}$ is the distribution of the π-random walk $\{\boldsymbol{w}_n\}_{n=0}^{\infty}$ starting from the identity on $\Gamma \times \Gamma$. The expectation relative to $\mathbf{P}$ is denoted by $\mathbf{E}$. Let π_n be the distribution of $\boldsymbol{w}_n$. Note that for $\boldsymbol{w}_n = (w_n^{(1)}, w_n^{(2)})$, each $\{w_n^{(i)}\}_{n=0}^{\infty}$ gives the μ-random walk on Γ starting from o for $i = 1, 2$. In this setting, we have for all $n \geqslant 0$,

$$H(\mu_n) \leqslant H(\pi_n) \leqslant 2H(\mu_n),$$

and the asymptotic entropy $h(\pi)$ of a π-random walk is finite since $H(\mu) < \infty$, and $h(\pi)$ is positive since $h(\pi) \geqslant h(\mu) > 0$ and $\operatorname{gr}(\mu)$ is a non-elementary subgroup in Γ. We define the metric

$$d_\times((x_1, x_2), (y_1, y_2)) := \max\{d(x_i, y_i), i = 1, 2\} \quad \text{for } (x_1, x_2), (y_1, y_2) \in \Gamma \times \Gamma.$$

Suppose that μ has finite first moment. Then π has finite first moment relative to the distance function $d_\times(o, \cdot\,)$. The *drift* is defined by the limit

$$l := \lim_{n\to\infty} \frac{1}{n} \mathbf{E}\, d_\times(o, \boldsymbol{w}_n),$$

where the limit exists by sub-additivity $n \mapsto \mathbf{E}\, d_\times(o, \boldsymbol{w}_n)$ and is finite. The value l coincides with the drift of a μ-random walk relative to $|\cdot|$, i.e.,

$$l = \lim_{n\to\infty} \frac{1}{n}\left|w_n^{(i)}\right| \quad \text{for } i = 1,2 \text{ and for } \mathbf{P}\text{-almost every } \omega \text{ in } \Omega, \tag{2.1}$$

and also in $L^1(\Omega, \mathcal{F}, \mathbf{P})$ by the Kingman subadditive ergodic theorem, and $l > 0$ since $\mathrm{gr}(\mu)$ is non-elementary [8, Section 7.3].

2.2. Word hyperbolic groups

We refer to [6, 7] for background. Let $\partial\Gamma$ denote the (Gromov) boundary and we endow $\Gamma \cup \partial\Gamma$ with the natural topology which is compact and metrizable. Letting $(x|y)_o$ be the Gromov product for $x, y \in \Gamma \cup \partial\Gamma$ based at o, we define the quasi-metric in $\partial\Gamma$ by

$$q(\xi, \eta) := e^{-(\xi|\eta)_o} \quad \text{for } \xi, \eta \in \partial\Gamma.$$

Note that q satisfies that $q(\xi, \eta) = 0$ if and only if $\xi = \eta$, $q(\xi, \eta) = q(\eta, \xi)$ for $\xi, \eta \in \partial\Gamma$ and the triangle inequality holds up to a multiplicative constant independent of the points. It is known that q is bi-Hölder equivalent to a genuine metric in $\partial\Gamma$ yielding the original topology. We work with the quasi-metric q to define balls. Let

$$B(\xi, r) := \left\{\eta \in \partial\Gamma \ :\ q(\xi, \eta) \leqslant r\right\} \quad \text{for } \xi \in \partial\Gamma \text{ and for real } r \geqslant 0.$$

For any positive real $R > 0$ and $x \in \Gamma$, the *shadow* is defined by

$$\mathcal{O}(x, R) := \left\{\xi \in \partial\Gamma \ :\ (x|\xi)_o \geqslant |x| - R\right\}.$$

By the hyperbolicity of geodesic metric space (Γ, d), for each fixed $T > 0$, there exist constants R_0, $C > 0$ such that for all $R > R_0$, all $\xi \in \partial\Gamma$ and all $x \in \Gamma$ in a T-neighborhood of a geodesic ray from o to ξ, we have

$$B\left(\xi, C^{-1}e^{-|x|+R}\right) \subset \mathcal{O}(x, R) \subset B\left(\xi, Ce^{-|x|+R}\right). \tag{2.2}$$

In the product space $(\partial\Gamma)^2$, we define

$$q_\times((\xi_1, \xi_2), (\eta_1, \eta_2)) := \max\{q(\xi_i, \eta_i), i = 1, 2\} \quad \text{for } (\xi_1, \xi_2), (\eta_1, \eta_2) \in (\partial\Gamma)^2.$$

By the definition, the ball of radius r centered at $\boldsymbol{\xi}$ in $(\partial\Gamma)^2$ relative to $q_\times$ is obtained by

$$B(\boldsymbol{\xi}, r) = B(\xi_1, r) \times B(\xi_2, r) \quad \text{where } \boldsymbol{\xi} = (\xi_1, \xi_2). \tag{2.3}$$

3. The dimension of harmonic measure

The μ-random walk $\{w_n\}_{n=0}^{\infty}$ on a word hyperbolic group Γ converges to a point w_∞ in $\partial\Gamma$ almost surely as $n \to \infty$ in $\Gamma \cup \partial\Gamma$ if $\mathrm{gr}(\mu)$ is non-elementary [8, Theorem 7.6]. This implies that the π-random walk $\{\boldsymbol{w}_n\}_{n=0}^{\infty}$ on $\Gamma \times \Gamma$ converges to a point $\boldsymbol{w}_\infty := (w_\infty^{(1)}, w_\infty^{(2)})$ in $(\partial\Gamma)^2$ almost surely as $n \to \infty$ in the product space $(\Gamma \cup \partial\Gamma)^2$, where $\boldsymbol{w}_n = (w_n^{(1)}, w_n^{(2)})$ and each $w_n^{(i)}$ tends to $w_\infty^{(i)}$ almost surely for $i = 1, 2$. Let ν_μ and ν_π denote the limiting distribution of w_∞ on $\partial\Gamma$ and that of $(w_\infty^{(1)}, w_\infty^{(2)})$ on $(\partial\Gamma)^2$, respectively. We call ν_μ and ν_π *harmonic measures* on $\partial\Gamma$ and on $(\partial\Gamma)^2$, respectively. Note that the push-forward of ν_π on each factor $\partial\Gamma$ coincides with ν_μ. The harmonic measure ν_π (resp. ν_μ) is π-*stationary* (resp. μ-stationary), i.e.,

$$\pi * \nu_\pi = \nu_\pi \quad \text{on } (\partial\Gamma)^2, \tag{3.1}$$

where $\pi * \nu_\pi := \sum_{\boldsymbol{x} \in \Gamma\times\Gamma} \pi(\boldsymbol{x})\boldsymbol{x}\nu_\pi$ and $\boldsymbol{x}\nu_\pi = \nu_\pi \circ \boldsymbol{x}^{-1}$, and similarly,

$$\mu * \nu_\mu = \nu_\mu \quad \text{on } \partial\Gamma. \tag{3.2}$$

The ν_π and ν_μ are unique such measures satisfying (3.1) and (3.2), respectively [8, cf. Theorem 2.4]. Concerning more on background, see [8].

For the harmonic measure ν_π for the π-random walk, we show the following:

Theorem 3.1. — *Let Γ be a word hyperbolic group and μ be a non-elementary probability measure on Γ with finite first moment. If π is a probability measure on $\Gamma\times\Gamma$ such that the push-forward of π on each factor Γ is μ, then the corresponding harmonic measure ν_π on $(\partial\Gamma)^2$ is exact dimensional, i.e.,*

$$\lim_{r\to 0} \frac{\log \nu_\pi(B(\boldsymbol{\xi}, r))}{\log r} = \frac{h(\pi)}{l} \quad \textit{for } \nu_\pi\textit{-almost every } \boldsymbol{\xi} \textit{ in } (\partial\Gamma)^2,$$

where $h(\pi)$ is the asymptotic entropy, l is the drift relative to $d_\times$ for the π-random walk on $\Gamma \times \Gamma$ and the ball $B(\boldsymbol{\xi}, r)$ is defined by $q_\times$ in $(\partial\Gamma)^2$.

Let us define the (upper) Hausdorff dimension of ν_π by

$$\dim_{\mathbb{H}} \nu_\pi = \inf\big\{\dim_{\mathbb{H}} E \ : \ \nu_\pi(E) = 1 \text{ and } E \text{ is Borel}\big\},$$

where $\dim_{\mathbb{H}} E$ stands for the Hausdorff dimension of E relative to $q_\times$ in $(\partial\Gamma)^2$. Theorem 3.1 together with the Frostman-type lemma (cf. [11, Section 2.2]) shows the following:

Corollary 3.2. — *In the setting of Theorem 3.1, we have*

$$\dim_{\mathbb{H}} \nu_\pi = \frac{h(\pi)}{l}.$$

Let us keep the same setting and notations as in Theorem 3.1 throughout this section. We use the following *ray approximation* of a μ-random walk on a word hyperbolic group Γ. Let Π be a Borel measurable map from $\partial\Gamma$ to the space $\mathcal{P}$ of unit speed geodesic rays from o in (Γ, d) endowed with the topology of convergence on compact sets. (Here Π is defined as a Borel measurable map by choosing a total order on a fixed set of generators and the lexicographical minimal geodesic for each point in the boundary.) Letting

$$\gamma_\xi = \Pi(\xi) \quad \text{for } \xi \in \partial\Gamma,$$

we have

$$d\left(w_n(\omega), \gamma_{w_\infty(\omega)}(ln)\right) = o(n) \quad \text{for } \mathbf{P}\text{-almost every } \omega \text{ in } \Omega, \tag{3.3}$$

[8, Section 7.4], where one should write instead $\lfloor ln \rfloor$ (the integer part of ln) here and below, however, we keep "ln" for the sake of simplicity. Let us define the map

$$\Pi_\times : (\partial\Gamma)^2 \to \mathcal{P} \times \mathcal{P}, \quad (\xi, \eta) \mapsto (\gamma_\xi, \gamma_\eta),$$

as a Borel measurable map. For a π-random walk $\{\boldsymbol{w}_n\}_{n=0}^\infty$ on $\Gamma \times \Gamma$, we have by (3.3),

$$d_\times(\boldsymbol{w}_n(\omega), \Pi_\times(\boldsymbol{w}_\infty(\omega))(ln)) = o(n) \quad \text{for } \mathbf{P}\text{-almost every } \omega \text{ in } \Omega. \tag{3.4}$$

Recall that the Shannon theorem for random walks:

$$h(\pi) = \lim_{n\to\infty} -\frac{1}{n} \log \pi_n(\boldsymbol{w}_n(\omega)) \quad \text{for } \mathbf{P}\text{-almost every } \omega \text{ in } \Omega, \tag{3.5}$$

which follows from the Kingman subadditive ergodic theorem ([9, Theorem 2.1] and [3, Section IV] where Y. Derriennic attributes to an observation by J. P. Conze).

First we show the dimension upper bound in the claim.

Lemma 3.3. — *For ν_π-almost all $\boldsymbol{\xi}$ in $(\partial\Gamma)^2$,*

$$\limsup_{r\to 0} \frac{\log \nu_\pi(B(\boldsymbol{\xi}, r))}{\log r} \leqslant \frac{h(\pi)}{l}.$$

Proof. — For all $\varepsilon > 0$ and all interval I in $[0,\infty) \cap \mathbb{Z}$, let

$$A_{\varepsilon,I} := \bigcap_{n\in I} \left\{ \omega \in \Omega \ : \begin{array}{c} \left| d(o, w_n^{(i)}(\omega)) - ln \right| \leqslant \varepsilon n, \\ \left| w_n^{(i)}(\omega)^{-1} w_{n+1}^{(i)}(\omega) \right| \leqslant \varepsilon n \\ \text{for } i = 1,2 \text{ and } \ \pi_n(\boldsymbol{w}_n(\omega)) \geqslant e^{-n(h(\pi)+\varepsilon)} \end{array} \right\}.$$

Since μ has finite first moment, $|w_n^{(i)}(\omega)^{-1} w_{n+1}^{(i)}(\omega)| \leqslant \varepsilon n$ for $i = 1, 2$ for all large n for $\mathbf{P}$-almost every ω in Ω, and by (2.1) and (3.5), there exists an N_ε such that $\mathbf{P}(A_{\varepsilon,[N_\varepsilon,\infty)}) \geqslant 1 - \varepsilon$. Let $A := A_{\varepsilon,[N_\varepsilon,\infty)}$. For each $\omega \in \Omega$, let

$$C_n(\omega) := \left\{ \eta \in \Omega \ : \ \boldsymbol{w}_n(\eta) = \boldsymbol{w}_n(\omega) \right\} \quad \text{for } n \geqslant 0,$$

which defines the event where a π-random walk after time n is $\boldsymbol{w}_n(\omega)$. The conditional probabilities satisfy that

$$\liminf_{n\to\infty} \mathbf{P}(A \mid C_n(\omega)) > 0 \quad \text{for } \mathbf{P}\text{-almost every } \omega \in A. \tag{3.6}$$

Indeed, letting $A_{[N,n)} := A_{\varepsilon,[N_\varepsilon,n)}$ and $A_{[n,\infty)} := A_{\varepsilon,[n,\infty)}$ for simplicity of notation, we have $A = A_{[N,n)} \cap A_{[n,\infty)}$ for $n > N$, and by the Markov property of the π-random walk,

$$\mathbf{P}(A \mid C_n(\omega)) = \mathbf{P}\left(A_{[N,n)} \,\middle|\, C_n(\omega)\right) \mathbf{P}\left(A_{[n,\infty)} \,\middle|\, C_n(\omega)\right).$$

Let $\sigma(w_n, w_{n+1}, \dots)$ denote the σ-algebra generated by $w_n, w_{n+1}, \dots$. Since for $\mathbf{P}$-almost every $\omega \in A = A_{[N,n)} \cap A_{[n,\infty)}$,

$$\begin{aligned}\mathbf{P}\left(A_{[N,n)} \,\middle|\, C_n(\omega)\right) &= \mathbf{P}\left(A_{[N,n)} \,\middle|\, \sigma(w_n, w_{n+1}, \dots)\right)(\omega) \\ &= \mathbf{P}\left(A \,\middle|\, \sigma(w_n, w_{n+1}, \dots)\right)(\omega),\end{aligned}$$

we have by the bounded martingale convergence theorem,

$$\mathbf{P}\left(A_{[N,n)} \,\middle|\, C_n(\omega)\right) \to \mathbf{P}(A \mid \mathcal{T})(\omega) \quad \text{for } \mathbf{P}\text{-almost every } \omega \in A,$$

as $n \to \infty$, where $\mathcal{T} := \bigcap_{n=0}^{\infty} \sigma(w_n, w_{n+1}, \dots)$. Furthermore, since for $\mathbf{P}$-almost every $\omega \in A$,

$$\begin{aligned}\mathbf{P}\left(A_{[n,\infty)} \,\middle|\, C_n(\omega)\right) &= \mathbf{P}\left(A_{[n,\infty)} \,\middle|\, \sigma(w_0, w_1, \dots, w_n)\right)(\omega) \\ &= \mathbf{P}\left(A \,\middle|\, \sigma(w_0, w_1, \dots, w_n)\right)(\omega),\end{aligned}$$

we have

$$\mathbf{P}\left(A_{[n,\infty)} \,\middle|\, C_n(\omega)\right) \to \mathbf{1}_A(\omega) \quad \text{for } \mathbf{P}\text{-almost every } \omega \in A,$$

as $n \to \infty$. Note that $\mathbf{P}(A \mid \mathcal{T})(\omega) > 0$ for $\mathbf{P}$-almost every $\omega \in A$. Indeed, letting $A_{>0} := \{\omega \in \Omega \ : \ \mathbf{P}(A \mid \mathcal{T})(\omega) > 0\}$, we have $\mathbf{P}(A \mid \mathcal{T}) = \mathbf{P}(A \mid \mathcal{T})\mathbf{1}_{A_{>0}}$ almost everywhere in $\mathbf{P}$, whence integrating both sides yields $\mathbf{P}(A) = \mathbf{P}(A \cap A_{>0})$. Thus we obtain (3.6).

For $\mathbf{P}$-almost every $\omega \in A$, we have for each $i = 1, 2$,

$$\left| d\left(o, w_n^{(i)}(\omega)\right) - ln \right| \leqslant \varepsilon n \quad \text{and} \quad d\left(w_n^{(i)}(\omega), w_{n+1}^{(i)}(\omega)\right) \leqslant \varepsilon n \qquad \text{for all } n \geqslant N_\varepsilon,$$

whence $w_\infty^{(i)}$ is defined and

$$\left(w_n^{(i)}(\omega) \,\middle|\, w_\infty^{(i)}(\omega)\right)_o \geqslant (l - 2\varepsilon)n - R \quad \text{for all } n \geqslant N_\varepsilon,$$

for a constant $R \geqslant 0$ independent of ω or n (cf. [8, Section 7.2]). For $\mathbf{P}$-almost every $\eta \in A \cap C_n(\omega)$, since $\boldsymbol{w}_n(\eta) = \boldsymbol{w}_n(\omega)$, by the δ-hyperbolicity we have

$$\left(w_\infty^{(i)}(\eta) \,\middle|\, w_\infty^{(i)}(\omega)\right)_o \geqslant (l - 2\varepsilon)n - R - \delta \quad \text{for each } i = 1, 2,$$

and thus we obtain by (2.3), for $\mathbf{P}$-almost every $\omega \in A$,

$$\boldsymbol{w}_\infty(\eta) \in B\left(\boldsymbol{w}_\infty(\omega), Ce^{-(l-2\varepsilon)n}\right) \quad \text{for } \mathbf{P}\text{-almost every } \eta \in A \cap C_n(\omega),$$

where $C = e^{R+\delta}$ is a positive constant depending only on the metric of the group Γ. Therefore for $\mathbf{P}$-almost every $\omega \in A$,

$$\mathbf{P}(A \cap C_n(\omega)) \leqslant \nu_\pi\left(B\left(\boldsymbol{w}_\infty(\omega), Ce^{-(l-2\varepsilon)n}\right)\right).$$

Moreover, by the definition of A, we have

$$\mathbf{P}(C_n(\omega)) = \pi_n(\boldsymbol{w}_n(\omega)) \geqslant e^{-n(h(\pi)+\varepsilon)}$$

for all $n \geqslant N_\varepsilon$. Invoking (3.6), we obtain

$$\limsup_{n\to\infty} \frac{\log \nu_\pi\left(B\left(\boldsymbol{w}_\infty(\omega), Ce^{-(l-2\varepsilon)n}\right)\right)}{-n} \leqslant h(\pi) + \varepsilon$$
$$\text{for } \mathbf{P}\text{-almost every } \omega \in A.$$

Noting that $r_n := Ce^{-(l-2\varepsilon)n}$ satisfy that $r_n > r_{n+1} = e^{-(l-2\varepsilon)}r_n$ for all $n \geqslant 0$, we have

$$\limsup_{r\to 0} \frac{\log \nu_\pi\left(B\left(\boldsymbol{w}_\infty(\omega), r\right)\right)}{\log r} \leqslant \frac{h(\pi)+\varepsilon}{l-2\varepsilon} \quad \text{for } \mathbf{P}\text{-almost every } \omega \in A.$$

Since $A = A_{\varepsilon,N_\varepsilon}$ and $\mathbf{P}(A_{\varepsilon,N_\varepsilon}) \geqslant 1-\varepsilon$ for all $\varepsilon > 0$, we obtain

$$\limsup_{r\to 0} \frac{\log \nu_\pi\left(B\left(\boldsymbol{\xi}, r\right)\right)}{\log r} \leqslant \frac{h(\pi)}{l} \quad \text{for } \nu_\pi\text{-almost every } \boldsymbol{\xi} \in (\partial\Gamma)^2,$$

as required. ◻

Next we show the dimension lower bound. We use the following lemma.

Lemma 3.4. — *For every $\varepsilon > 0$ there exists a Borel set F_ε in $(\partial\Gamma)^2$ such that $\nu_\pi(F_\varepsilon) \geqslant 1-\varepsilon$ and*

$$\liminf_{r\to 0} \frac{\log \nu_\pi\left(F_\varepsilon \cap B(\boldsymbol{\xi}, r)\right)}{\log r} \geqslant \frac{h(\pi)}{l} - \varepsilon \quad \textit{for } \nu_\pi\textit{-almost every } \boldsymbol{\xi} \textit{ in } (\partial\Gamma)^2.$$

Proof. — Let $\{\mathbf{P}^{\boldsymbol{w}_\infty(\omega)}\}_{\omega\in\Omega}$ be the conditional probability measures associated with $\sigma(\boldsymbol{w}_\infty)$, where we have

$$\mathbf{P} = \int_\Omega \mathbf{P}^{\boldsymbol{w}_\infty(\omega)}\, \mathrm{d}\mathbf{P}(\omega) = \int_{(\partial\Gamma)^2} \mathbf{P}^{\boldsymbol{\xi}}\, \mathrm{d}\nu_\pi(\boldsymbol{\xi}).$$

For all $\varepsilon > 0$ and all positive integer N, if we define

$$A_{\varepsilon,N} := \bigcap_{n\geqslant N} \left\{\omega \in \Omega \ : \ \begin{array}{l} d_\times(\boldsymbol{w}_n(\omega), \Pi_\times(\boldsymbol{w}_\infty(\omega))(ln)) \leqslant \varepsilon n, \\ \pi_n(\boldsymbol{w}_n(\omega)) \leqslant e^{-n(h(\pi)-\varepsilon)} \end{array}\right\},$$

then by (3.4) and (3.5), there exists an N_ε such that $\mathbf{P}(A_{\varepsilon,N_\varepsilon}) \geqslant 1-\varepsilon$. Letting $A := A_{\varepsilon,N_\varepsilon}$, we define

$$F_\varepsilon := \left\{\boldsymbol{\xi} \in (\partial\Gamma)^2 \ : \ \mathbf{P}^{\boldsymbol{\xi}}(A) \geqslant \varepsilon\right\}.$$

Since

$$\begin{aligned}1-\varepsilon \leqslant \mathbf{P}(A) &= \int_{(\partial\Gamma)^2} \mathbf{P}^{\boldsymbol{\xi}}(A)\,\mathrm{d}\nu_\pi(\boldsymbol{\xi}) \\ &= \int_{F_\varepsilon} \mathbf{P}^{\boldsymbol{\xi}}(A)\,\mathrm{d}\nu_\pi(\boldsymbol{\xi}) + \int_{F_\varepsilon^{\mathsf{c}}} \mathbf{P}^{\boldsymbol{\xi}}(A)\,\mathrm{d}\nu_\pi(\boldsymbol{\xi}) \\ &\leqslant \nu_\pi(F_\varepsilon) + \varepsilon\nu_\pi(F_\varepsilon^{\mathsf{c}}),\end{aligned}$$

we have $\nu_\pi(F_\varepsilon) \geqslant 1-2\varepsilon$.

Let $\boldsymbol{z}_n = (z_n^{(1)}, z_n^{(2)})$ be any sequence with $|z_n^{(i)}| = \lfloor ln \rfloor$ for $i=1,2$. Note that for $\mathbf{P}$-almost every $\eta \in A$ and for all $n \geqslant N_\varepsilon$, if

$$\boldsymbol{w}_\infty(\eta) \in \mathcal{O}\left(z_n^{(1)}, R\right) \times \mathcal{O}\left(z_n^{(2)}, R\right),$$

then

$$d_\times(\Pi_\times(\boldsymbol{w}_\infty(\eta))(ln), \boldsymbol{z}_n) \leqslant 2R + C',$$

for a positive constant C' depending only on the hyperbolicity constant of the metric in Γ, and thus

$$\begin{aligned}\eta \in A \quad \text{and} \quad \boldsymbol{w}_\infty(\eta) \in \mathcal{O}\left(z_n^{(1)}, R\right) \times \mathcal{O}\left(z_n^{(2)}, R\right) \\ \Longrightarrow \ \boldsymbol{w}_n(\eta) \in B(\boldsymbol{z}_n, \varepsilon n + C)\end{aligned}$$

where $B(\boldsymbol{z}_n, T) = B(z_n^{(1)}, T) \times B(z_n^{(2)}, T)$ for $T \geqslant 0$ and $C = 2R + C'$ for a fixed $R > 0$. This shows that for all $n \geqslant N_\varepsilon$,

$$\begin{aligned}\mathbf{P}&\left(\boldsymbol{w}_\infty(\eta) \in F_\varepsilon \cap \mathcal{O}\left(z_n^{(1)}, R\right) \times \mathcal{O}\left(z_n^{(2)}, R\right)\right) \\ &\leqslant \mathbf{P}\left(A \cap \{\boldsymbol{w}_n(\eta) \in B(\boldsymbol{z}_n, \varepsilon n + C)\}\right) \\ &\quad + \mathbf{P}\left(A^{\mathsf{c}} \cap \left\{\boldsymbol{w}_\infty(\eta) \in F_\varepsilon \cap \mathcal{O}\left(z_n^{(1)}, R\right) \times \mathcal{O}\left(z_n^{(2)}, R\right)\right\}\right).\end{aligned}$$

The second term is estimated as follows:

$$\begin{aligned}\mathbf{P}&\left(A^{\mathsf{c}} \cap \left\{\boldsymbol{w}_\infty(\eta) \in F_\varepsilon \cap \mathcal{O}\left(z_n^{(1)}, R\right) \times \mathcal{O}\left(z_n^{(2)}, R\right)\right\}\right) \\ &= \int_{F_\varepsilon \cap \mathcal{O}\left(z_n^{(1)}, R\right) \times \mathcal{O}\left(z_n^{(2)}, R\right)} \mathbf{P}^{\boldsymbol{\xi}}\left(A^{\mathsf{c}}\right)\,\mathrm{d}\nu_\pi(\boldsymbol{\xi}) \\ &\leqslant (1-\varepsilon)\nu_\pi\left(F_\varepsilon \cap \mathcal{O}\left(z_n^{(1)}, R\right) \times \mathcal{O}\left(z_n^{(2)}, R\right)\right).\end{aligned}$$

Therefore we obtain

$$\varepsilon\nu_\pi\left(F_\varepsilon\cap\mathcal{O}\left(z_n^{(1)},R\right)\times\mathcal{O}\left(z_n^{(2)},R\right)\right)\\ \leqslant \mathbf{P}\big(A\cap\{\boldsymbol{w}_n(\eta)\in B(\boldsymbol{z}_n,\varepsilon n+C)\}\big), \quad (3.7)$$

for all $n\geqslant N_\varepsilon$. Moreover, since

$$\begin{aligned}\mathbf{P}\big(A\cap\{\boldsymbol{w}_n(\eta)\in B(\boldsymbol{z}_n,\varepsilon n+C)\}\big)&\\ &\leqslant \mathbf{P}\big(\pi_n(\boldsymbol{w}_n(\eta))\leqslant e^{-n(h(\pi)-\varepsilon)},\ \boldsymbol{w}_n(\eta)\in B(\boldsymbol{z}_n,\varepsilon n+C)\big)\\ &\leqslant \#B(\boldsymbol{z}_n,\varepsilon n+C)\cdot e^{-n(h(\pi)-\varepsilon)},\end{aligned}$$

we have for all $n\geqslant N_\varepsilon$,

$$\mathbf{P}\big(A\cap\{\boldsymbol{w}_n(\eta)\in B(\boldsymbol{z}_n,\varepsilon n+C)\}\big)\leqslant e^{2D(\varepsilon n+C)}e^{-n(h(\pi)-\varepsilon)} \quad (3.8)$$

where D is a constant greater than the exponential growth rate of (Γ,d), i.e.,

$$\#B\left(z_n^{(i)},T\right)\leqslant e^{DT}\quad\text{for } i=1,2 \text{ and for all large enough } T.$$

Combining (3.7) and (3.8), we obtain

$$\liminf_{n\to\infty}\frac{\log\nu_\pi\left(F_\varepsilon\cap\mathcal{O}\left(z_n^{(1)},R\right)\times\mathcal{O}\left(z_n^{(2)},R\right)\right)}{-n}\geqslant h(\pi)-\varepsilon-2D\varepsilon.$$

Let us define $\boldsymbol{z}_n:=\Pi_\times(\boldsymbol{w}_\infty(\omega))(ln)$ for $\mathbf{P}$-almost every $\omega\in A_{\delta,N_\delta}$ for all $\delta>0$. By (2.2) and (2.3), as in a similar way in the last part in the proof of Lemma 3.3, for $\mathbf{P}$-almost every $\omega\in A_{\delta,N_\delta}$,

$$\liminf_{r\to 0}\frac{\log\nu_\pi\left(F_\varepsilon\cap B(\boldsymbol{w}_\infty(\omega),r)\right)}{\log r}\geqslant\frac{h(\pi)-\varepsilon-2D\varepsilon}{l}=\frac{h(\pi)}{l}-C'\varepsilon,$$

for a constant $C'>0$. Since $\mathbf{P}(A_{\delta,N_\delta})\geqslant 1-\delta$ for all $\delta>0$, we have

$$\liminf_{r\to 0}\frac{\log\nu_\pi\left(F_\varepsilon\cap B(\boldsymbol{\xi},r)\right)}{\log r}\geqslant\frac{h(\pi)}{l}-C'\varepsilon\quad\text{for } \nu_\pi\text{-almost every } \boldsymbol{\xi} \text{ in } (\partial\Gamma)^2.$$

Replacing ε by a small enough constant yields the claim as stated. ☐

Lemma 3.5. — *For ν_π-almost all $\boldsymbol{\xi}$ in $(\partial\Gamma)^2$,*

$$\liminf_{r\to 0}\frac{\log\nu_\pi\left(B(\boldsymbol{\xi},r)\right)}{\log r}\geqslant\frac{h(\pi)}{l}.$$

Proof. — For all $\varepsilon>0$, let $F:=F_\varepsilon$ be the Borel set in Lemma 3.4. For the hyperbolic metric space (Γ,d) and the boundary $(\partial\Gamma,q)$, we have that for each $0<\alpha<1$, the space $(\partial\Gamma,q^\alpha)$ admits a bi-Lipschitz embedding into the Euclidean space $\mathbb{R}^n$ for some n [2, Theorem 9.2]. Hence there exists a bi-Lipschitz map $\varphi = (\varphi_1, \varphi_2)$,

$$\varphi : \left((\partial\Gamma)^2, q_\times^\alpha\right) \to \mathbb{R}^n \times \mathbb{R}^n, \quad \left(\xi^{(1)}, \xi^{(2)}\right) \mapsto \left(\varphi_1\left(\xi^{(1)}\right), \varphi_2\left(\xi^{(2)}\right)\right),$$

i.e., for a constant $L > 0$,

$$\frac{1}{L} q_\times(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)^\alpha \leqslant \|\varphi(\boldsymbol{\xi}_1) - \varphi(\boldsymbol{\xi}_2)\|_{\mathbb{R}^{2n}} \leqslant L q_\times(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)^\alpha$$

for all $\boldsymbol{\xi}_i \in (\partial\Gamma)^2$, $i = 1, 2$, where $\|\cdot\|_{\mathbb{R}^{2n}}$ denotes the standard Euclidean norm in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. By the Lebesgue density theorem for the Borel measure $\varphi_*\nu_\pi$ in $\mathbb{R}^{2n}$, we have

$$\lim_{r\to 0} \frac{\varphi_*\nu_\pi\left(\varphi(F) \cap B_{\mathbb{R}^{2n}}(\varphi(\boldsymbol{\xi}), r)\right)}{\varphi_*\nu_\pi\left(B_{\mathbb{R}^{2n}}(\varphi(\boldsymbol{\xi}), r)\right)} = 1 \quad \text{for } \nu_\pi\text{-almost every } \boldsymbol{\xi} \in F,$$

where $B_{\mathbb{R}^{2n}}(x, r)$ stands for the standard Euclidean (closed) ball in $\mathbb{R}^{2n}$. This implies that

$$\liminf_{r\to 0} \frac{\nu_\pi\left(F \cap B(\boldsymbol{\xi}, (Lr)^{1/\alpha})\right)}{\nu_\pi\left(B(\boldsymbol{\xi}, (r/L)^{1/\alpha})\right)} \geqslant 1 \quad \text{for } \nu_\pi\text{-almost all } \boldsymbol{\xi} \in F,$$

and for ν_π-almost all $\boldsymbol{\xi} \in (\partial\Gamma)^2$, there exist positive constants $c(\boldsymbol{\xi}) > 0$ and $r(\boldsymbol{\xi}) > 0$ such that

$$\nu_\pi\left(F \cap B(\boldsymbol{\xi}, L^{2/\alpha} r)\right) \geqslant c(\boldsymbol{\xi})\nu_\pi\left(B(\boldsymbol{\xi}, r)\right) \quad \text{for all } 0 < r < r(\boldsymbol{\xi}).$$

Therefore we obtain

$$\liminf_{r\to 0} \frac{\log \nu_\pi\left(B(\boldsymbol{\xi}, r)\right)}{\log r} \geqslant \liminf_{r\to 0} \frac{\log \nu_\pi\left(F \cap B(\boldsymbol{\xi}, r)\right)}{\log r} \qquad \text{for } \nu_\pi\text{-almost all } \boldsymbol{\xi} \in F.$$

Lemma 3.4 implies that

$$\liminf_{r\to 0} \frac{\log \nu_\pi\left(B(\boldsymbol{\xi}, r)\right)}{\log r} \geqslant \frac{h(\pi)}{l} - \varepsilon \quad \text{for } \nu_\pi\text{-almost every } \boldsymbol{\xi} \in F,$$

and since $F = F_\varepsilon$ and $\nu_\pi(F_\varepsilon) \geqslant 1 - \varepsilon$ for all $\varepsilon > 0$,

$$\liminf_{r\to 0} \frac{\log \nu_\pi\left(B(\boldsymbol{\xi}, r)\right)}{\log r} \geqslant \frac{h(\pi)}{l} \quad \text{for } \nu_\pi\text{-almost every } \boldsymbol{\xi} \text{ in } (\partial\Gamma)^2,$$

concluding the claim. □

Proof of Theorem 3.1. — Lemmas 3.3 and 3.5 show the claim. □

4. Continuity of entropy

For a countable group Γ (in particular we discuss a product of word hyperbolic groups), we endow the set of probability measures on Γ with the topology induced by the total variation distance. Note that for all probability measure μ and all sequence of probability measures $\{\mu_{(i)}\}_{i=0}^{\infty}$ we have

$$\|\mu_{(i)} - \mu\|_{\mathrm{TV}} \to 0 \quad \text{as } i \to \infty,$$

if and only if $\mu_{(i)}(x) \to \mu(x)$ as $i \to \infty$ for each $x \in \Gamma$. Fix a left-invariant metric d on Γ with finite exponential growth rate and let $|x| = d(o, x)$ for $x \in \Gamma$. For a finite set K in Γ, let

$$\mathbf{E}_\mu[|x| : \Gamma \setminus K] := \sum_{x \in \Gamma \setminus K} |x|\, \mu(x)$$

Erschler and Kaimanovich have shown that the continuity of $h(\mu)$ in $\mu \in \mathcal{M}$ under some general conditions [5]. We say that a set $\mathcal{M}$ of probability measures on Γ satisfies *uniform first moment condition* if

$$\sup_{\mu \in \mathcal{M}} \mathbf{E}_\mu[|x| : \Gamma \setminus K_n] \to 0 \quad \text{as } n \to \infty, \tag{M}$$

for all sequence of finite sets $\{K_n\}_{n=0}^{\infty}$ with $\bigcup_{n=0}^{\infty} K_n = \Gamma$. We assume that there exists a pair of Borel Γ-spaces B, $\check{B}$ such that the Γ-space $\check{B} \times B$ with the diagonal action admits a Γ-invariant Borel set Λ in $\check{B} \times B$ and a Γ-equivariant map S assigning to $(\check{\xi}, \xi) \in \Lambda$ a proper subset (*strip*) in Γ. Let us say that the strips $S(\check{\xi}, \xi)$ given by the map S satisfy *uniform subexponential growth* if

$$\sup_{(\check{\xi},\xi) \in \Lambda} \frac{1}{n} \log \# \Big(B(o, n) \cap S(\check{\xi}, \xi) \Big) \to 0 \quad \text{as } n \to \infty. \tag{G}$$

For non-negative real R, letting $S_R(\check{\xi}, \xi)$ be the R-neighborhood of $S(\check{\xi}, \xi)$, we define

$$\Lambda_R := \left\{ (\check{\xi}, \xi) \in \Lambda \ : \ o \in S_R(\check{\xi}, \xi) \right\}.$$

Note that the union of Λ_R over R covers Λ. If a pair of Borel Γ-spaces B and $\check{B}$ admits a probability measure ν_μ on B (resp. $\nu_{\check{\mu}}$ on $\check{B}$) for which (B, ν_μ) (resp. $(\check{B}, \nu_{\check{\mu}})$) is a μ- (resp. $\check{\mu}$-) boundary (where $\check{\mu}(x) := \mu(x^{-1})$ for $x \in \Gamma$), and further

$$\inf_{\mu \in \mathcal{M}} \nu_{\check{\mu}} \times \nu_\mu(\Lambda_R) \to 1 \quad \text{as } R \to \infty, \tag{S}$$

then we say that $\mathcal{M}$ satisfies *uniform strip condition.*

Theorem 4.1 ([5, Theorem 1]). — *If a set $\mathcal{M}$ of probability measures on Γ and a map S satisfy the conditions* (M)*,* (G) *and* (S)*, then the function $\mathcal{M} \to \mathbb{R}$, $\mu \mapsto h(\mu)$ is continuous.*

Theorem 4.1 applies to word hyperbolic groups and their products with sequences of probability measures. In the case when Γ is a word hyperbolic group, for a sequence of probability measures $\{\mu_{(i)}\}_{i=0}^{\infty}$ on Γ with uniform first moment converging to a probability measure μ, we have $h(\mu_{(i)}) \to h(\mu)$ as $i \to \infty$ [5, Theorem 2]. Actually, it suffices to consider the case when Γ is a non-elementary word hyperbolic group and the limiting probability measure μ is non-elementary. One may take $B = \check{B} = \partial\Gamma$ the Gromov boundary endowed with the harmonic measures ν_μ and $\nu_{\check{\mu}}$, respectively, and $\Lambda := (\partial\Gamma)^2 \setminus \{\text{diagonal}\}$, which is open in the product $(\partial\Gamma)^2$. Furthermore, for $(\check{\xi}, \xi) \in \Lambda$ the strip $S(\check{\xi}, \xi)$ is defined as the union of bi-infinite geodesics connecting $\check{\xi}$ and ξ in a Cayley graph of Γ. The condition (G) is satisfied since

$$\#\big(B(o,n) \cap S(\check{\xi}, \xi)\big) = O(n),$$

where the implied constant depends only on the Cayley graph. Furthermore the condition (S) is satisfied. Indeed, the harmonic measure ν_μ is the unique μ-stationary measure on $\partial\Gamma$ and the measures $\nu_{\mu_{(i)}}$ weakly converge to ν_μ as $i \to \infty$, and ν_μ is supported on an open set Λ. Letting Λ_R° denote the interior of Λ_R, we have

$$\liminf_{k\to\infty} \check{\nu}_{i_k} \times \nu_{i_k}(\Lambda_R^\circ) \geqslant \nu_{\check{\mu}} \times \nu_\mu(\Lambda_R^\circ),$$

for every subsequence $\check{\nu}_{i_k} \times \nu_{i_k}$ of $\nu_{\check{\mu}_{(i)}} \times \nu_{\mu_{(i)}}$. Noting that Λ_R° increases and exhausts Λ as R grows, we have (S) (cf. [5, Lemma 3]).

In the case when the group is a product $\Gamma \times \Gamma$ of word hyperbolic groups and a probability measure π, one may take $B = \check{B} = (\partial\Gamma)^2$ and

$$\Lambda = \big\{(\check{\boldsymbol{\xi}}, \boldsymbol{\xi}) \in \check{B} \times B \ : \ \check{\xi}^{(i)} \neq \xi^{(i)} \text{ for } i = 1, 2\big\},$$

where $\check{\boldsymbol{\xi}} = (\check{\xi}^{(1)}, \check{\xi}^{(2)})$ and $\boldsymbol{\xi} = (\xi^{(1)}, \xi^{(2)})$, and Λ is open in $\check{B} \times B$. The strip is defined by

$$S\big(\check{\boldsymbol{\xi}}, \boldsymbol{\xi}\big) = S\big(\check{\xi}^{(1)}, \xi^{(1)}\big) \times S\big(\check{\xi}^{(2)}, \xi^{(2)}\big),$$

and we have

$$\#\big(B(o,n) \cap S(\check{\boldsymbol{\xi}}, \boldsymbol{\xi})\big) = O(n^2).$$

This shows that (G) holds. Moreover we have (S) for a sequence of probability measures $\pi_{(i)}$ on $\Gamma \times \Gamma$ since ν_π is the unique π-stationary measure on $(\partial\Gamma)^2$ and supported on an open set Λ as in the case on Γ presented above.

Corollary 4.2. — *For a word hyperbolic group Γ and a non-elementary probability measure μ with finite first moment, the asymptotic entropy $h(\pi^\rho)$ is continuous in $\rho \in [0,1]$.*

Proof. — The set of probability measures $\pi^\rho = \rho\mu \times \mu + (1-\rho)\mu_{\text{diag}}$ on $\Gamma \times \Gamma$ has uniform finite first moment if μ has finite first moment. By the discussion above, the conditions (M), (G) and (S) are satisfied for $\{\pi^{\rho_i}\}_{i=0}^{\infty}$

with every sequence $\{\rho_i\}_{i=0}^{\infty}$ converging to ρ in $[0,1]$ as $i \to \infty$, and thus Theorem 4.1 implies the claim. □

5. Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. — For probability measures ν_1 and ν_2 on $(\Gamma \cup \partial\Gamma)^2$, the total variation distance is defined by

$$\|\nu_1 - \nu_2\|_{\mathrm{TV}} := \sup\left\{|\nu_1(A) - \nu_2(A)| \ : \ A \text{ is Borel in } (\Gamma \cup \partial\Gamma)^2\right\}.$$

For each $0 \leqslant \rho \leqslant 1$, a π^ρ-random walk $\{\boldsymbol{w}_n\}_{n=0}^{\infty}$ converges to $\boldsymbol{w}_\infty$ in $(\partial\Gamma)^2$ as $n \to \infty$ in $(\Gamma \cup \partial\Gamma)^2$, $\mathbf{P}$-almost surely, and the distribution π_n^ρ converges weakly to the harmonic measure ν_{π^ρ} (see the beginning of Section 3). Therefore for $0 \leqslant \rho \leqslant 1$, we have

$$\liminf_{n\to\infty} \|\pi_n^\rho - \mu_n \times \mu_n\|_{\mathrm{TV}} \geqslant \|\nu_{\pi^\rho} - \nu_\mu \times \nu_\mu\|_{\mathrm{TV}}. \tag{5.1}$$

For all $0 \leqslant \rho < 1$, we have $\nu_{\pi^\rho} \neq \nu_\mu \times \nu_\mu$. Indeed, suppose that $\nu_{\pi^\rho} = \nu_\mu \times \nu_\mu$ for some $0 \leqslant \rho < 1$, then we have

$$\pi^\rho * (\nu_\mu \times \nu_\mu) = \nu_\mu \times \nu_\mu$$

since ν_{π^ρ} is the π^ρ-stationary measure on $(\partial\Gamma)^2$ (cf. (3.1) and (3.2) in Section 3). Noting that ν_μ is the μ-stationary measure on $\partial\Gamma$, we have

$$\rho(\nu_\mu \times \nu_\mu) + (1-\rho)\sum_{x \in \Gamma} \mu_{\mathrm{diag}}(x)(x\nu_\mu \times x\nu_\mu) = \nu_\mu \times \nu_\mu,$$

and

$$\mu_{\mathrm{diag}} * (\nu_\mu \times \nu_\mu) = \nu_\mu \times \nu_\mu.$$

This shows that $\nu_\mu \times \nu_\mu$ is the μ_{diag}-stationary (harmonic) measure by the uniqueness. However, the harmonic measure for μ_{diag} is supported on the diagonal in $(\partial\Gamma)^2$ and ν_μ is non-atomic on $\partial\Gamma$ since $\mathrm{gr}(\mu)$ is non-elementary, we have $\nu_{\pi^\rho} \neq \nu_\mu \times \nu_\mu$, yielding a contradiction. Therefore for all $0 \leqslant \rho < 1$, we have $\|\nu_{\pi^\rho} - \nu_\mu \times \nu_\mu\|_{\mathrm{TV}} > 0$, and thus by (5.1),

$$\liminf_{n\to\infty} \|\pi_n^\rho - \mu_n \times \mu_n\|_{\mathrm{TV}} > 0,$$

as claimed. □

Proof of Theorem 1.3. — As in the same way in the beginning of the proof of Theorem 1.2, for all $0 \leqslant \rho, \rho' \leqslant 1$, we have

$$\liminf_{n\to\infty} \left\|\pi_n^\rho - \pi_n^{\rho'}\right\|_{\mathrm{TV}} \geqslant \|\nu_{\pi^\rho} - \nu_{\pi^{\rho'}}\|_{\mathrm{TV}}.$$

Theorem 3.1 shows that for the Borel set

$$E_\rho := \left\{\boldsymbol{\xi} \in (\partial\Gamma)^2 \ : \ \lim_{r\to 0} \frac{\log \nu_{\pi^\rho}(B(\boldsymbol{\xi}, r))}{\log r} = \frac{h(\pi^\rho)}{l}\right\},$$

we have $\nu_{\pi^\rho}(E_\rho) = 1$. By Corollary 4.2, the function $\rho \mapsto h(\pi^\rho)$ for $\rho \in [0,1]$ is continuous. Furthermore,

$$h(\mu) = h(\pi^0) \leqslant h(\pi^\rho) \leqslant h(\pi^1) = 2h(\mu) \quad \text{for all } 0 \leqslant \rho \leqslant 1,$$

and $h(\mu) > 0$ since $\mathrm{gr}(\mu)$ is non-elementary. Hence there exists $0 < \rho_* \leqslant 1$ such that $h(\pi^\rho) < h(\pi^1)$ for all $0 \leqslant \rho < \rho_*$. This shows that for all $0 \leqslant \rho < \rho_*$, we have $\nu_{\pi^\rho}(E_\rho) = 1$ and $(\nu_\mu \times \nu_\mu)(E_\rho) = 0$, implying that ν_{π^ρ} and $\nu_\mu \times \nu_\mu$ are mutually singular and $\|\nu_{\pi^\rho} - \nu_\mu \times \nu_\mu\|_{\mathrm{TV}} = 1$. Therefore we have for all $0 \leqslant \rho < \rho_*$,

$$\liminf_{n\to\infty} \|\pi_n^\rho - \mu_n \times \mu_n\|_{\mathrm{TV}} = 1,$$

and $\lim_{n\to\infty} \|\pi_n^\rho - \mu_n \times \mu_n\|_{\mathrm{TV}} = 1$, as required. □

Remark 5.1. — The proof of Theorem 1.3 shows that if $h(\pi^\rho) \neq h(\pi^{\rho'})$ for $0 \leqslant \rho, \rho' \leqslant 1$, then

$$\left\|\pi_n^\rho - \pi_n^{\rho'}\right\|_{\mathrm{TV}} \to 1 \quad \text{as } n \to \infty,$$

and ν_{π^ρ} and $\nu_{\pi^{\rho'}}$ are mutually singular.

Appendix A. A proof of Theorem 4.1

In this section, $\mathbf{E}_\nu$ denotes the expectation for a probability measure ν. Let Γ be a countable group endowed with a probability measure μ of finite entropy, i.e., $H(\mu) < \infty$. For the μ-random walk $\{w_n\}_{n=0}^\infty$ starting from o on Γ, let us consider the probability measure space $(\Gamma^{\mathbb{Z}_+}, \mathcal{F}, \mathbf{P})$, where $\mathcal{F}$ is the σ-field generated by the cylinder sets and $\mathbf{P}$ is the distribution of $\{w_n\}_{n=0}^\infty$.

For each positive integer n, let α_1^n be the measurable partition on $\Gamma^{\mathbb{Z}_+}$ where $\boldsymbol{\omega} = (\omega_i)_{i=0}^\infty$ and $\boldsymbol{\omega}' = (\omega_i')_{i=0}^\infty$ belong to the same set if and only if $\omega_i = \omega_i'$ for all $0 \leqslant i \leqslant n$. For any sub σ-field $\mathcal{A}$ in $\mathcal{F}$, the conditional entropy is defined by

$$H_{\mathbf{P}}(\alpha_1^n \mid \mathcal{A}) := \mathbf{E}_{\mathbf{P}}\left[-\sum_{B \in \alpha_1^n} \mathbf{P}(B \mid \mathcal{A}) \log \mathbf{P}(B \mid \mathcal{A})\right],$$

where $\mathbf{P}(\,\cdot \mid \mathcal{A})$ stands for the conditional probability measure with respect to $\mathcal{A}$. The *tail* σ-field is defined by $\mathcal{T} := \bigcap_{n=0}^\infty \sigma(w_n, w_{n+1}, \dots)$. In the case when $\mathcal{A} = \mathcal{T}$, letting $\alpha := \alpha_1^1$, we have

$$H_{\mathbf{P}}(\alpha \mid \mathcal{T}) = H(\mu) - h(\mu), \tag{A.1}$$

[9, cf. Proof of Theorem 1.1].

The group Γ acts on $\Gamma^{\mathbb{Z}_+}$ by $x(\omega_n)_{n=0}^\infty = (x\omega_n)_{n=0}^\infty$ for $x \in \Gamma$. The *stationary σ-field* $\mathcal{S}$ is the sub σ-field of $\mathcal{F}$ generated by shift-invariant measurable sets, where the shift is defined by $(\omega_n)_{n=0}^{\infty} \mapsto (\omega_{n+1})_{n=0}^{\infty}$ on $\Gamma^{\mathbb{Z}_+}$. Note that $\mathcal{S}$ is Γ-invariant, i.e., if $A \in \mathcal{S}$, then $xA \in \mathcal{S}$ for all $x \in \Gamma$. By definition, we have $\mathcal{S} \subset \mathcal{T}$, and it is known that their $\mathbf{P}$-completions coincide, i.e., $\mathcal{S} = \mathcal{T}$ mod $\mathbf{P}$ [9, Section 7.0] (where it is crucial that the initial state w_0 is a point). Therefore by (A.1),

$$H_{\mathbf{P}}(\alpha \mid \mathcal{S}) = H(\mu) - h(\mu). \tag{A.2}$$

For each Γ-invariant sub σ-field $\mathcal{A}$ of $\mathcal{S}$, let $\mathbf{P}^{\xi}(\,\cdot\,) = \mathbf{P}(\,\cdot \mid \mathcal{A})(\xi)$ for $\mathbf{P}$-almost every $\xi \in \Gamma^{\mathbb{Z}_+}$, and let

$$\mu_n^{\xi}(x) := \mathbf{P}^{\xi}(w_n = x) \qquad \text{for } x \in \Gamma \text{ and } n \geqslant 0,$$

and we define the entropy of conditional process. For $n \geqslant 0$, let

$$H(\mu_n^{\xi}) := \mathbf{E}_{\mathbf{P}}\left[-\sum_{x \in \Gamma} \mu_n^{\xi}(x) \log \mu_n^{\xi}(x)\right]. \tag{A.3}$$

This yields $H(\mu_n^{\xi}) = H(\mu_n) + n(H(\alpha \mid \mathcal{A}) - H(\mu))$, and in particular, in the case when $\mathcal{A} = \mathcal{S}$, by (A.2), we obtain for all $n \geqslant 0$,

$$H(\mu_n^{\xi}) = H(\mu_n) - nh(\mu), \tag{A.4}$$

[8, Sections 3 and 4].

We write $B_R = B(o, R)$ for simplicity of notations.

Lemma A.1. — *For a set of probability measures $\mathcal{M}$ on Γ with uniform first moment condition, the function $\mathcal{M} \to \mathbb{R}$,*

$$\mu \mapsto H(\mu)$$

is continuous.

Proof. — For the exponential growth rate $v(\Gamma, d)$ for (Γ, d), let us fix $D > v(\Gamma, d)$ and define

$$A := \left\{x \in \Gamma \;:\; \mu(x) \geqslant e^{-D|x|}\right\}.$$

For the ball $K = B_N$ in Γ with N large enough, decomposing the sum

$$H(\mu) = -\sum_{x \in K} \mu(x) \log \mu(x) - \sum_{x \in A \cap K^{\mathrm{c}}} \mu(x) \log \mu(x) - \sum_{x \in A^{\mathrm{c}} \cap K^{\mathrm{c}}} \mu(x) \log \mu(x),$$

we estimate the second and third terms. First, we have

$$-\sum_{x \in A \cap K^{\mathrm{c}}} \mu(x) \log \mu(x) \leqslant -\sum_{x \in K^{\mathrm{c}}} \mu(x) \log e^{-D|x|} \leqslant D \sup_{\mu \in \mathcal{M}} \mathbf{E}_{\mu}[|x| : \Gamma \setminus K].$$

Second, letting $S_k := \{x \in \Gamma \ : \ k < |x| \leqslant k+1\}$ for non-negative integers k,

$$-\sum_{x\,\in\,A^{\mathsf{c}}\cap K^{\mathsf{c}}} \mu(x)\log\mu(x) = -\sum_{k=N}^{\infty}\sum_{x\,\in\,S_k\,\cap\,A^{\mathsf{c}}} \mu(x)\log\mu(x)$$
$$\leqslant \sum_{k=N}^{\infty} \#S_k \cdot D(k+1)e^{-Dk} \leqslant Ce^{-D'N}$$

for constants $0 < D' < D - v(\Gamma, d)$ and $C > 0$, for all large enough N, where in the first inequality we have used $-\mu(x)\log\mu(x) \leqslant -e^{-D|x|}\log e^{-D|x|}$ for $x \in A^{\mathsf{c}}$ and $|x|$ large enough. Finally, we obtain

$$\sup_{\mu\,\in\,\mathcal{M}} \left| H(\mu) + \sum_{x\,\in\,B_N} \mu(x)\log\mu(x)\right| \leqslant D \sup_{\mu\,\in\,\mathcal{M}} \mathbf{E}_\mu\left[|x| : \Gamma\setminus B_N\right] + Ce^{-D'N}.$$

This shows that

$$-\sum_{x\in B_N} \mu(x)\log\mu(x) \to H(\mu) \quad \text{uniformly on } \mu\in\mathcal{M} \text{ as } N\to\infty,$$

implying that $\mu \mapsto H(\mu)$ is continuous on $\mathcal{M}$. □

Lemma A.2. — *In the same setting as in Lemma A.1, for all $L > 4$ and for all positive integer n,*

$$\sup_{\mu\,\in\,\mathcal{M}} \mathbf{E}_{\mu_n}\left[|x| : \Gamma\setminus B_{nL}\right]$$
$$\leqslant n \sup_{\mu\,\in\,\mathcal{M}} \mathbf{E}_\mu\left[|x| : \Gamma\setminus B_{\sqrt{L}}\right] + \frac{2n}{\sqrt{L}} \sup_{\mu\,\in\,\mathcal{M}} \mathbf{E}_\mu |x|, \quad \text{(A.5)}$$

and

$$\sup_{\mu\,\in\,\mathcal{M}} \left(-\sum_{x\,\in\,\Gamma\setminus B_{nL}} \mu_n(x)\log\mu_n(x)\right)$$
$$\leqslant D \sup_{\mu\,\in\,\mathcal{M}} \mathbf{E}_{\mu_n}\left[|x| : \Gamma\setminus B_{nL}\right] + Ce^{-D'nL}, \quad \text{(A.6)}$$

where C, D and D' are positive constants independent of n and L. Moreover, for each $n > 0$, the function $\mathcal{M} \to \mathbb{R}$, $\mu \mapsto H(\mu_n)$ is continuous.

Proof. — We use the same notation as in the proof of Lemma A.1 and obtain (A.6) in the same way for each positive integer n and for all $L > 4$.

Let us show (A.5). Note that

$$\begin{aligned}\mathbf{E}_{\mu_n}\left[|x| : \Gamma\setminus B_{nL}\right] &= \mathbf{E}_{\mathbf{P}}\left[|w_n|\mathbf{1}_{\{|w_n|>nL\}}\right]\\ &\leqslant \mathbf{E}_{\mathbf{P}}\left[\left(\sum_{i=1}^n |x_i|\right)\mathbf{1}_{\left\{\sum_{i=1}^n |x_i|>nL\right\}}\right]\\ &= n\,\mathbf{E}_{\mathbf{P}}\left[|x_1|\mathbf{1}_{\left\{\sum_{i=1}^n |x_i|>nL\right\}}\right],\end{aligned}$$

where the last equality follows since $x_1, \dots, x_n$ are independent and identically distributed. Moreover, we have

$$\begin{aligned}\mathbf{E}_{\mathbf{P}}\left[|x_1|\mathbf{1}_{\left\{\sum_{i=1}^n |x_i|>nL\right\}}\right] &= \mathbf{E}_{\mathbf{P}}\left[|x_1|\mathbf{1}_{\left\{|x_1|>\sqrt{L},\ \sum_{i=1}^n |x_i|>nL\right\}}\right]\\ &\quad+\mathbf{E}_{\mathbf{P}}\left[|x_1|\mathbf{1}_{\left\{|x_1|\leqslant\sqrt{L},\ \sum_{i=1}^n |x_i|>nL\right\}}\right],\end{aligned}$$

where the first term is at most

$$\mathbf{E}_{\mu}\left[|x_1|\mathbf{1}_{\left\{|x_1|>\sqrt{L}\right\}}\right] \leqslant \sup_{\mu\in\mathcal{M}} \mathbf{E}_{\mu}\left[|x| : \Gamma\setminus B_{\sqrt{L}}\right],$$

and the second term is at most

$$\begin{aligned}\sqrt{L}\mathbf{P}\left(\sum_{i=2}^n |x_i| > nL-\sqrt{L}\right) &\leqslant \frac{\sqrt{L}(n-1)}{nL-\sqrt{L}}\,\mathbf{E}_{\mu}|x|\\ &\leqslant \frac{\sqrt{L}}{L-\sqrt{L}/n}\sup_{\mu\in\mathcal{M}}\mathbf{E}_{\mu}|x|,\end{aligned}$$

by the Markov inequality. Therefore for all $L > 4$ and for all $n > 0$, we have $\sqrt{L}/n < L/2$ and

$$\sup_{\mu\in\mathcal{M}}\mathbf{E}_{\mu_n}\left[|x| : \Gamma\setminus B_{nL}\right] \leqslant n\sup_{\mu\in\mathcal{M}}\mathbf{E}_{\mu}\left[|x| : \Gamma\setminus B_{\sqrt{L}}\right] + \frac{2n}{\sqrt{L}}\sup_{\mu\in\mathcal{M}}\mathbf{E}_{\mu}|x|,$$

yielding (A.5). Finally, since $\|\mu_n - \mu'_n\|_{\mathrm{TV}} \leqslant n\|\mu-\mu'\|_{\mathrm{TV}}$ for μ, $\mu' \in \mathcal{M}$, the term

$$-\sum_{x\,\in\,B_{nL}} \mu_n(x)\log\mu_n(x)$$

is continuous in μ for each fixed n, and converges to $H(\mu_n)$ uniformly on μ in $\mathcal{M}$ as $L\to\infty$ by (A.6), the last statement holds. □

Recall the notations from [8, Section 6]. Let $(\Gamma^{\mathbb{Z}}, \overline{\mathbf{P}})$ be the probability measure space of bilateral paths $\overline{\boldsymbol{\omega}} = (\omega_i)_{i\in\mathbb{Z}}$ with $\omega_0 = o$. The space is identified with the product space via the map $\overline{\boldsymbol{\omega}} \mapsto (\check{\boldsymbol{\omega}}, \boldsymbol{\omega})$ from $(\Gamma^{\mathbb{Z}}, \overline{\mathbf{P}})$ to $(\Gamma^{\mathbb{Z}_+}, \check{\mathbf{P}}) \times (\Gamma^{\mathbb{Z}_+}, \mathbf{P})$ where $\check{\boldsymbol{\omega}} = (\omega_{-i})_{i\in\mathbb{Z}_+}$ and $\check{\mathbf{P}}$ is the distribution of $\check{\mu}$-random walk starting from o. We denote by $\overline{U}$ the probability measure preserving transformation on $(\Gamma^{\mathbb{Z}}, \overline{\mathbf{P}})$ induced from the Bernoulli shift in the space of increments, more explicitly,

$$\left(\overline{U}^k \boldsymbol{\omega}\right)_n = \omega_k^{-1}\omega_{n+k} \quad \text{for } \boldsymbol{\omega} = (\omega_i)_{i\in\mathbb{Z}} \in \Gamma^{\mathbb{Z}} \text{ and for } k, n \in \mathbb{Z}.$$

Given Γ-equivariant measurable maps $\mathrm{bnd}_+ : \Gamma^{\mathbb{Z}_+} \to B$ and $\mathrm{bnd}_- : \Gamma^{\mathbb{Z}_+} \to \check{B}$ for the μ-boundary B and for the $\check{\mu}$-boundary $\check{B}$, let us define $\Pi_+ : \Gamma^{\mathbb{Z}} \to B$ by $\overline{\boldsymbol{\omega}} = (\check{\boldsymbol{\omega}}, \boldsymbol{\omega}) \mapsto \mathrm{bnd}_+(\boldsymbol{\omega})$ and $\Pi_- : \Gamma^{\mathbb{Z}} \to \check{B}$ by $\overline{\boldsymbol{\omega}} = (\check{\boldsymbol{\omega}}, \boldsymbol{\omega}) \mapsto \mathrm{bnd}_-(\check{\boldsymbol{\omega}})$. Note that $\nu_\mu = \Pi_{+*}\overline{\mathbf{P}}$ and $\nu_{\check{\mu}} = \Pi_{-*}\overline{\mathbf{P}}$.

Proof of Theorem 4.1. — The condition (S) implies that

$$\varepsilon_R := 1 - \inf_{\mu\in\mathcal{M}} \nu_{\check{\mu}} \times \nu_\mu(\Lambda_R) \to 0 \quad \text{as } R \to \infty,$$

where $\Lambda_R = \{(\check{\xi}, \xi) \in \Lambda \ :\ o \in S_R(\check{\xi}, \xi)\}$ for $R \geqslant 0$, and thus

$$\begin{aligned}\overline{\mathbf{P}}\big(o \in S_R(\Pi_-\overline{\boldsymbol{\omega}}, \Pi_+\overline{\boldsymbol{\omega}})\big) &= \nu_{\check{\mu}} \times \nu_\mu\big((\check{\xi}, \xi) \in \Lambda \ :\ o \in S_R(\check{\xi}, \xi)\big)\\ &= \nu_{\check{\mu}} \times \nu_\mu\left(\Lambda_R\right) \geqslant 1 - \varepsilon_R,\end{aligned}$$

uniformly on $\mu \in \mathcal{M}$. Moreover, since the map S is Γ-equivariant and $\overline{\mathbf{P}}$ is $\overline{U}$-invariant, we have

$$\begin{aligned}\overline{\mathbf{P}}\big(\omega_n \in S_R(\Pi_-\overline{\boldsymbol{\omega}}, \Pi_+\overline{\boldsymbol{\omega}})\big) &= \overline{\mathbf{P}}\left(o \in \omega_n^{-1} S_R(\Pi_-\overline{\boldsymbol{\omega}}, \Pi_+\overline{\boldsymbol{\omega}})\right)\\ &= \overline{\mathbf{P}}\left(o \in S_R\left(\Pi_-\overline{U}^n\overline{\boldsymbol{\omega}}, \Pi_+\overline{U}^n\overline{\boldsymbol{\omega}}\right)\right)\\ &= \overline{\mathbf{P}}\left(o \in S_R(\Pi_-\overline{\boldsymbol{\omega}}, \Pi_+\overline{\boldsymbol{\omega}})\right) \geqslant 1 - \varepsilon_R.\end{aligned}$$

Therefore, disintegrating the measure,

$$\begin{aligned}\overline{\mathbf{P}}\big(\omega_n \in S_R(\Pi_-\overline{\boldsymbol{\omega}}, \Pi_+\overline{\boldsymbol{\omega}})\big) &= \int_\Lambda \mathbf{P}^\xi\big(\omega_n \in S_R(\check{\xi}, \xi)\big)\,\mathrm{d}\nu_{\check{\mu}}\mathrm{d}\nu_\mu\\ &= \int_\Lambda \mu_n^\xi\big(S_R(\check{\xi}, \xi)\big)\mathrm{d}\nu_{\check{\mu}}\mathrm{d}\nu_\mu,\end{aligned}$$

we obtain

$$\int_\Lambda \mu_n^\xi\big(S_R(\check{\xi}, \xi)\big)\mathrm{d}\nu_{\check{\mu}}\mathrm{d}\nu_\mu \geqslant 1 - \varepsilon_R. \tag{A.7}$$

By Lemma A.2 (A.5), for all $L > 4$ and for all $n > 0$,

$$\sup_{\mu\in\mathcal{M}} \mathbf{E}_{\mu_n}\left[|x| : \Gamma \setminus B_{nL}\right] \leqslant n \sup_{\mu\in\mathcal{M}} \mathbf{E}_\mu\left[|x| : \Gamma \setminus B_{\sqrt{L}}\right] + \frac{2n}{\sqrt{L}} \sup_{\mu\in\mathcal{M}} \mathbf{E}_\mu\,|x|,$$

and thus letting

$$\varepsilon_L := \sup_{\mu\in\mathcal{M}} \mathbf{E}_\mu\left[|x| : \Gamma \setminus B_{\sqrt{L}}\right] \quad \text{and} \quad C_\mu := \sup_{\mu\in\mathcal{M}} \mathbf{E}_\mu\,|x|,$$

we have $\varepsilon_L \to 0$ as $L \to \infty$ by the condition (M), and by the Markov inequality,

$$\mathbf{P}(|w_n| > nL) \leqslant \frac{\mathbf{E}_{\mu_n}\left[|x| : \Gamma \setminus B_{nL}\right]}{nL} \leqslant \frac{1}{L}\left(\varepsilon_L + \frac{2C_\mu}{\sqrt{L}}\right) = \frac{\widetilde{\varepsilon}_L}{L},$$
$$\text{where } \widetilde{\varepsilon}_L := \varepsilon_L + \frac{2C_\mu}{\sqrt{L}},$$

and $\widetilde{\varepsilon}_L \to 0$ as $L \to \infty$. Hence for all $n > 0$ and all $\mu \in \mathcal{M}$, we have $\mathbf{P}(|w_n| \leqslant nL) \geqslant 1 - \widetilde{\varepsilon}_L/L$, and this yields by disintegration,

$$\int_B \mu_n^\xi(B_{nL})\,\mathrm{d}\nu_\mu \geqslant 1 - \frac{\widetilde{\varepsilon}_L}{L}. \tag{A.8}$$

For all $\varepsilon > 0$, let us take any L and R satisfying that $\widetilde{\varepsilon}_L < \varepsilon$ and $\varepsilon_R < \varepsilon/L$. Noting that

$$\mu_n^\xi\big(B_{nL} \cap S_R(\check{\xi},\xi)\big) \geqslant \mu_n^\xi(B_{nL}) + \mu_n^\xi\big(S_R(\check{\xi},\xi)\big) - 1,$$

we have by (A.7) and (A.8),

$$\begin{aligned}\int_\Lambda \mu_n^\xi\big(B_{nL} \cap S_R(\check{\xi},\xi)\big)\,\mathrm{d}\nu_{\check{\mu}}\mathrm{d}\nu_\mu \\ &\geqslant \int_B \mu_n^\xi\left(B_{nL}\right)\,\mathrm{d}\nu_\mu + \int_\Lambda \mu_n^\xi\big(S_R(\check{\xi},\xi)\big)\,\mathrm{d}\nu_{\check{\mu}}\mathrm{d}\mu_\mu - 1 \\ &\geqslant 1 - \frac{\widetilde{\varepsilon}_L}{L} + 1 - \varepsilon_R - 1 \geqslant 1 - \frac{2\varepsilon}{L}.\end{aligned} \tag{A.9}$$

Furthermore, the condition (G) implies that for all $R > 0$ and all $L > 0$, there exists a sequence of positive reals $\varphi_{n,R,L}$ such that

$$\sup_{(\check{\xi},\xi)\,\in\,\Lambda} \#\big(S_R(\check{\xi},\xi) \cap B_{nL}\big) \leqslant \varphi_{n,R,L} \quad \text{for all } n > 0, \tag{A.10}$$

and $(1/n)\log\varphi_{n,R,L} \to 0$ as $n \to \infty$.

Let us estimate the conditional entropy $H(\mu_n^\xi)$ in (A.3). For the simplicity of notations, let

$$G_1 := S_R(\check{\xi},\xi) \cap B_{nL}, \quad G_2 := B_{nL} \setminus G_1, \quad \text{and} \quad G_3 := \Gamma \setminus B_{nL}.$$

First we have by the Jensen inequality and by (A.10),

$$\begin{aligned}-\sum_{x\,\in\,G_1} \mu_n^\xi(x)\log\mu_n^\xi(x) &\leqslant \mu_n^\xi(G_1)\log\#G_1 - \mu_n^\xi(G_1)\log\mu_n^\xi(G_1) \\ &\leqslant \log\varphi_{n,R,L} - \mu_n^\xi(G_1)\log\mu_n^\xi(G_1),\end{aligned}$$

and thus,

$$\int_\Lambda -\sum_{x\,\in\, G_1} \mu_n^\xi(x)\log\mu_n^\xi(x)\,\mathrm{d}\nu_{\check\mu}\mathrm{d}\nu_\mu \leqslant \log\varphi_{n,R,L} + e^{-1}, \tag{A.11}$$

where we have used $-x\log x \leqslant e^{-1}$ for $0\leqslant x\leqslant 1$.

Second by (A.9), we have $\mathbf{P}(w_n\in G_2)\leqslant 2\varepsilon/L$, and thus by the Jensen inequality,

$$-\sum_{x\,\in\, G_2}\mu_n^\xi(x)\log\mu_n^\xi(x)\leqslant \mu_n^\xi(G_2)\log\#G_2 - \mu_n^\xi(G_2)\log\mu_n^\xi(G_2),$$

and as in a similar way to (A.11),

$$\begin{aligned}\int_\Lambda -\sum_{x\,\in\, G_2}\mu_n^\xi(x)\log\mu_n^\xi(x)\mathrm{d}\nu_{\check\mu}\mathrm{d}\nu_\mu\qquad&\\ \leqslant \mathbf{P}(w_n\in G_2)\log\#B_{nL}+e^{-1}&\\ \leqslant\left(\frac{2\varepsilon}{L}\right)nLD+e^{-1}=2\varepsilon nD+e^{-1},&\end{aligned}\tag{A.12}$$

for $D>v(\Gamma,d)$ and all large enough n. By (A.11) and (A.12), noting that $G_1\cup G_2=B_{nL}$, we have

$$\int_\Lambda -\sum_{x\,\in\, B_{nL}}\mu_n^\xi(x)\log\mu_n^\xi(x)\,\mathrm{d}\nu_{\check\mu}\mathrm{d}\nu_\mu\leqslant\log\varphi_{n,R,L}+2\varepsilon nD+2e^{-1}, \tag{A.13}$$

for all large enough n.

Finally we obtain on $G_3=B_{nL}^{\mathsf{c}}$,

$$\int_\Lambda -\sum_{x\,\in\, G_3}\mu_n^\xi(x)\log\mu_n^\xi(x)\,\mathrm{d}\nu_{\check\mu}\mathrm{d}\nu_\mu\leqslant -\sum_{x\,\in\, B_{nL}^{\mathsf{c}}}\mu_n(x)\log\mu_n(x),$$

by the Fubini theorem and the Jensen inequality. By Lemma A.2 (A.5) and (A.6), for all large enough $L>4$ and for all positive integer n,

$$-\sum_{x\,\in\, B_{nL}^{\mathsf{c}}}\mu_n(x)\log\mu_n(x)\leqslant\varepsilon nD+Ce^{-D'nL}, \tag{A.14}$$

where C, D and D' are positive constants independent of n and L.

Combining (A.13) and (A.14), we obtain for all $\varepsilon>0$ and for all L,R with $\widetilde\varepsilon_L<\varepsilon$, $\varepsilon_R<\varepsilon/L$ and for all large enough n,

$$\sup_{\mu\,\in\,\mathcal{M}} H(\mu_n^\xi)\leqslant 3\varepsilon nD+\log\varphi_{n,R,L}+O(1),$$

and thus together with (A.10), we have $H(\mu_n^\xi)/n\to 0$ uniformly on $\mu\in\mathcal{M}$ as $n\to\infty$. Since $H(\mu_n^\xi)=H(\mu_n)-nh(\mu)$ by (A.4), and for each $n>0$, the

$H(\mu_n)$ is continuous in $\mu \in \mathcal{M}$ by Lemma A.2, we conclude that $\mu \mapsto h(\mu)$ is continuous on $\mathcal{M}$. $\square$

Acknowledgments

The author would like to thank Jérémie Brieussel for bringing him the problem and discussions, and an anonymous referee for beneficial comments.

Bibliography

[1] I. Benjamini & J. Brieussel, "Noise sensitivity of random walks on groups", *ALEA Lat. Am. J. Probab. Math. Stat.* **20** (2023), no. 2, p. 1139-1164.

[2] M. Bonk & O. Schramm, "Embeddings of Gromov hyperbolic spaces", *Geom. Funct. Anal.* **10** (2000), no. 2, p. 266-306.

[3] Y. Derriennic, "Quelques applications du théorème ergodique sous-additif", in *Conference on Random Walks (Kleebach, 1979)*, Astérisque, vol. 74, Société Mathématique de France, 1980, p. 183-201.

[4] ———, "Entropie, théorèmes limite et marches aléatoires", in *Probability measures on groups, VIII (Oberwolfach, 1985)*, Lecture Notes in Mathematics, vol. 1210, Springer, 1986, p. 241-284.

[5] A. Erschler & V. A. Kaimanovich, "Continuity of asymptotic characteristics for random walks on hyperbolic groups", *Funkts. Anal. Prilozh.* **47** (2013), no. 2, p. 84-89.

[6] É. Ghys & P. de la Harpe (eds.), *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics, vol. 83, Birkhäuser, 1990, papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988.

[7] M. L. Gromov, "Hyperbolic groups", in *Essays in group theory* (S. M. Gersten, ed.), Mathematical Sciences Research Institute Publications, vol. 8, Springer, 1987, p. 75-263.

[8] V. A. Kaimanovich, "The Poisson formula for groups with hyperbolic properties", *Ann. Math. (2)* **152** (2000), no. 3, p. 659-692.

[9] V. A. Kaimanovich & A. M. Vershik, "Random walks on discrete groups: boundary and entropy", *Ann. Probab.* **11** (1983), no. 3, p. 457-490.

[10] G. Kalai, "Three puzzles on mathematics, computation, and games", in *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures*, World Scientific; Sociedade Brasileira de Matemática, 2018, p. 551-606.

[11] R. Tanaka, "Dimension of harmonic measures in hyperbolic spaces", *Ergodic Theory Dyn. Syst.* **39** (2019), no. 2, p. 474-499.