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Minimizing travelling waves for the Gross–Pitaevskii equation on $\mathbb{R} \times \mathbb{T}$ (*)

André de Laire (1), Philippe Gravejat (2) and Didier Smets (3)

Abstract. — We study the Gross–Pitaevskii equation in dimension two with periodic conditions in one direction, or equivalently on the product space $\mathbb{R} \times \mathbb{T}_L$ where $L > 0$ and $\mathbb{T}_L = \mathbb{R}/L\mathbb{Z}$. We focus on the variational problem consisting in minimizing the Ginzburg–Landau energy under a fixed momentum constraint. We prove that there exists a threshold value for L below which minimizers are the one-dimensional dark solitons, and above which no minimizer can be one-dimensional.

Résumé. — Nous considérons l'équation de Gross–Pitaevskii en dimension deux pour des fonctions périodiques dans une direction, soit de façon équivalente dans l'espace produit $\mathbb{R} \times \mathbb{T}_L$, où $L > 0$ et $\mathbb{T}_L = \mathbb{R}/L\mathbb{Z}$. Nous nous intéressons au problème variationnel qui consiste à minimiser l'équation de Ginzburg–Landau à moment fixé. Nous montrons qu'il existe une valeur critique pour la largeur L en dessous de laquelle les minimiseurs sont les solitons sombres à une variable, et au-dessus de laquelle aucun minimiseur ne peut dépendre que d'une seule variable.

1. Introduction

We are interested in the Gross–Pitaevskii equation

$$i\partial_t \Psi = \Delta\Psi + \Psi\big(1 - |\Psi|^2\big). \tag{GP}$$

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In Physics, this equation is a classical model for Bose–Einstein condensates, superfluidity or supraconductivity [11, 15]. It also gives account of the propagation of dark solitons in nonlinear optics [12].

Our attention in this paper is devoted to the case where the spatial domain is the product space $\mathbb{R} \times \mathbb{T}_L$, where $L > 0$ and $\mathbb{T}_L = \mathbb{R}/L\mathbb{Z}$, so that $\Psi \equiv \Psi(x, y, t) : (\mathbb{R} \times \mathbb{T}_L) \times \mathbb{R} \to \mathbb{C}$. Solutions of the 1D equation can be considered as solutions in this 2D setting with a trivial dependence on the variable y.

Dark solitons are special solutions of the 1D Gross–Pitaevskii equation. They are travelling waves of the form

$$\Psi_c(x, t) = \mathfrak{u}_c(x - ct),$$

where c is any subsonic speed, i.e. $|c| < \sqrt{2}$. Their profile $\mathfrak{u}_c$ is solution to the ordinary differential equation

$$i\, c\, \mathfrak{u}_c' + \mathfrak{u}_c'' + \big(1 - |\mathfrak{u}_c|^2\big)\mathfrak{u}_c = 0, \tag{1.1}$$

and is explicitly given by the expression

$$\mathfrak{u}_c(x) = \sqrt{\frac{2 - c^2}{2}} \tanh\Big(\frac{\sqrt{2 - c^2}}{2} x\Big) + i \frac{c}{\sqrt{2}}. \tag{1.2}$$

For $c = 0$, the profile $\mathfrak{u}_0$ vanishes and the corresponding soliton is called the black or kink soliton. The other solitons are called grey solitons.

Variational characterizations of the dark solitons were proved in [1, 3]. These characterizations are based on two conserved quantities. The first one is the 1D Ginzburg–Landau energy

$$E(\psi) := \frac{1}{2} \int_{\mathbb{R}} |\psi'|^2 + \frac{1}{4} \int_{\mathbb{R}} \big(1 - |\psi|^2\big)^2, \tag{1.3}$$

which is the Hamiltonian of the Gross–Pitaevskii equation. Corresponding to this energy is the energy set

$$X(\mathbb{R}) := \big\{\psi \in H^1_{\rm loc}(\mathbb{R}) : \psi' \in L^2(\mathbb{R}) \text{ and } 1 - |\psi|^2 \in L^2(\mathbb{R})\big\}, \tag{1.4}$$

which provides the natural functional framework for analyzing the equation.

The second one is the momentum P, which is formally defined as

$$P(\psi) = \frac{1}{2} \int_{\mathbb{R}} \langle i\psi', \psi\rangle_{\mathbb{C}},$$

where, here as in the sequel, the notation $\langle z_1, z_2\rangle_{\mathbb{C}} := \mathrm{Re}(z_1 \bar{z}_2)$ stands for the canonical scalar product on the 2D real vector space $\mathbb{C}$. The expression of $P(\psi)$ above certainly makes sense if ψ' is compactly supported, but it is generally ill-defined for arbitrary $\psi \in X(\mathbb{R})$ due to the possible lack of integrability of the momentum density $\langle i\psi', \psi\rangle_{\mathbb{C}}$ at infinity. It was shown in [3] (see also Appendix A below) that a notion of momentum can be rigorously defined on the whole energy set $X(\mathbb{R})$ provided its value is understood in the quotient space $\mathbb{R}/\pi\mathbb{Z}$. It was called the untwisted momentum in [3], and denoted by $[P]$. Whenever ψ' has compact support, it holds

$$[P](\psi) = \frac{1}{2} \int_{\mathbb{R}} \langle i\psi', \psi \rangle_{\mathbb{C}} \mod \pi.$$

The characterization of the dark solitons on the line can be phrased as follows.

Proposition 1.1 ([1, 3]). — *Let $p \in \mathbb{R}/\pi\mathbb{Z}$, with $p \neq 0$. The minimizers of the variational problem*

$$\mathfrak{I}(p) := \inf\big\{E(\psi) : \psi \in X(\mathbb{R}) \text{ s.t. } [P](\psi) = p\big\} \tag{1.5}$$

are exactly the dark soliton $\mathfrak{u}_{c_p}$ and the function obtained from $\mathfrak{u}_{c_p}$ by translation and constant phase shift. The value $c_p \in (-\sqrt{2}, \sqrt{2})$ is characterized by the identity $[P](\mathfrak{u}_{c_p}) = p$, and the function $p \mapsto \mathfrak{I}(p)$ has Lipschitz constant equal to $\sqrt{2}$.

In the context of the Gross–Pitaevskii equation on the product space $\mathbb{R} \times \mathbb{T}_L$, we consider the vector space

$$H^1_{\text{loc}}(\mathbb{R} \times \mathbb{T}_L) := \left\{ \psi \in H^1_{\text{loc}}(\mathbb{R}^2) : \begin{array}{r} \psi \text{ is } L\text{-periodic with respect} \\ \text{to its second variable } y \end{array} \right\}. \tag{1.6}$$

For our analysis, it is convenient to work on a fixed domain independently of L. For that purpose, we write $\mathbb{T}$ instead of $\mathbb{T}_L$ when $L = 1$, and given a function $\psi \in H^1_{\text{loc}}(\mathbb{R} \times \mathbb{T})$ and a real parameter $\lambda > 0$, we introduce the rescaled version of the Ginzburg–Landau energy given by

$$E_\lambda(\psi) := \frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}} \big(|\partial_x \psi|^2 + \lambda^2 |\partial_y \psi|^2\big) + \frac{1}{4} \int_{\mathbb{R}\times\mathbb{T}} \big(1 - |\psi|^2\big)^2. \tag{1.7}$$

Up to a multiplicative factor λ, the rescaled energy $E_\lambda(\psi)$ is equal to the Ginzburg–Landau energy of the function $\psi_\lambda(x, y) = \psi(x, \lambda y)$ on the product space $\mathbb{R} \times \mathbb{T}_L$, where $L = 1/\lambda$.

Corresponding to the rescaled Ginzburg–Landau energy E_λ is the energy set

$$X(\mathbb{R} \times \mathbb{T}) := \left\{ \psi \in H^1_{\text{loc}}(\mathbb{R} \times \mathbb{T}) : \begin{array}{r} \nabla \psi \in L^2(\mathbb{R} \times \mathbb{T}) \\ \text{and } 1 - |\psi|^2 \in L^2(\mathbb{R} \times \mathbb{T}) \end{array} \right\}. \tag{1.8}$$

The untwisted momentum $[P]$ along the direction x can be extended to $X(\mathbb{R} \times \mathbb{T})$. We decompose an arbitrary function $\psi \in X(\mathbb{R} \times \mathbb{T})$ as a Fourier series

$$\psi(x,y) = \sum_{k\in\mathbb{Z}} \widehat{\psi}_k(x) e^{2i\pi ky},$$

and check that the Fourier coefficient $\widehat{\psi}_0$ lies in $X(\mathbb{R})$, while the difference $w_0 = \psi - \widehat{\psi}_0$ is in $H^1(\mathbb{R}\times\mathbb{T})$. Due to the orthogonality of the functions $\widehat{\psi}_0$ and w_0, it is natural to define $[P](\psi)$ by the expression

$$[P](\psi) = [P](\widehat{\psi}_0) + \frac{1}{2}\int_{\mathbb{R}} \langle i\partial_x w_0, w_0\rangle_{\mathbb{C}} \mod \pi. \tag{1.9}$$

Note that to any function in $X(\mathbb{R})$, we can associate a function in $X(\mathbb{R}\times\mathbb{T})$, which does not depend on y. By construction, the energy and the momentum of these two functions (either in $X(\mathbb{R})$ or in $X(\mathbb{R}\times\mathbb{T})$) coincide. In the sequel, we shall use the same notation for a function in $X(\mathbb{R})$ and its extension in $X(\mathbb{R}\times\mathbb{T})$, in particular for the dark soliton $\mathfrak{u}_c$.

For $p\in\mathbb{R}/\pi\mathbb{Z}$, we next consider the minimization problem under constraint

$$\mathcal{I}_\lambda(p) := \inf\big\{E_\lambda(\psi) : \psi\in X(\mathbb{R}\times\mathbb{T}) \text{ s.t. } [P](\psi) = p\big\}. \tag{1.10}$$

Our main result is

Theorem 1.2. — *Let $p\in\mathbb{R}/\pi\mathbb{Z}$. There exists $\lambda_p > 0$ such that the following statements hold.*

(i) *For any $\lambda \geqslant \lambda_p$, the minimal value $\mathcal{I}_\lambda(p)$ is equal to*

$$\mathcal{I}_\lambda(p) = \mathfrak{I}(p).$$

The dark soliton $\mathfrak{u}_{c_p}$ is a minimizer of the corresponding minimization problem. When $\lambda > \lambda_p$, it is the unique minimizer up to translation and phase shift.

(ii) *For any $0 < \lambda < \lambda_p$, the minimal value $\mathcal{I}_\lambda(p)$ satisfies*

$$\mathcal{I}_\lambda(p) < \mathfrak{I}(p),$$

and there does not exist any minimizer depending only on the variable x.

Note that, when $0 < \lambda < \lambda_p$, Theorem 1.2 makes no claim about the existence of minimizers for $\mathcal{I}_\lambda(p)$, it only asserts that potential candidates must be truly 2D. The fact that minimizers do exist in such cases will be the object of a future work (see [13]).

Note also that our arguments do not prevent the possible existence of a truly 2D minimizer for $\lambda = \lambda_p$.

We have stated Theorem 1.2 in the case of the spatial domain $\mathbb{R} \times \mathbb{T}$. With minor modifications, the proofs carry over to the case of $\mathbb{R} \times \mathbb{T}^2$, and presumably also to $\mathbb{R} \times M$, where M is any compact Riemannian manifold of dimension $d \leqslant 2$.

Linear transverse instability of solitons for a number of dispersive models, including the Gross–Pitaevskii equation, was proved by F. Rousset and N. Tzvetkov in [17] (see also [16] for the general Hamiltonian framework concerning nonlinear transverse instability). In particular, although they did not consider their variational characterization, it follows from [17, Theorem 3.3] that given a dark soliton $\mathfrak{u}_{c_p}$, there exists $\lambda_p > 0$ such that $\mathfrak{u}_{c_p}$ is not a minimizer for $\mathcal{I}_\lambda$ when $\lambda = \lambda_p/k$ for some $k \geqslant 1$.

In the next section we sketch the main arguments in the proof of Theorem 1.2. We follow a strategy developed by S. Terracini, N. Tzvetkov and N. Visciglia [18] in the different context of the nonlinear Schrödinger equations on product spaces. In Section 3, we provide the full details of our proofs. Note that in some places, it will be convenient to identify $\mathbb{R}/\pi\mathbb{Z}$ with the interval $(-\pi/2, \pi/2]$. A number of properties and ingredients related to the energy spaces $X(\mathbb{R})$ and $X(\mathbb{R} \times \mathbb{T})$ as well as the untwisted momentum $[P]$, which we found of independent interest, are gathered in Appendices A, B and C.

2. Sketch of the proof of Theorem 1.2

The starting point is to check that the minimal energy $\mathcal{I}_\lambda(p)$ tends to the 1D minimal energy $\mathfrak{I}(p)$ as $\lambda \to +\infty$. In this limit, we show that suitable extractions of minimizing sequences tend to the dark soliton $\mathfrak{u}_{c_p}$, up to possible translation and phase shift. The key ingredient of the proof is then to check that these dark solitons are strict local minimizers of the variational problem corresponding to the minimal energy $\mathcal{I}_\lambda(p)$. In this case, the functions in the previous minimizing sequences must be equal to a dark soliton for λ large enough. This property is sufficient to conclude that the minimal energy $\mathcal{I}_\lambda(p)$ is exactly the energy $\mathfrak{I}(p)$ of dark solitons.

We describe now this strategy with additional details. We first observe, by considering test functions depending only on the variable x, that we have

$$\mathcal{I}_\lambda(p) \leqslant \mathfrak{I}(p), \tag{2.1}$$

independently of $\lambda > 0$. We take advantage of this inequality in order to show that

$$\mathcal{I}_\lambda(p) \longrightarrow \mathfrak{I}(p), \text{ as } \lambda \longrightarrow +\infty. \tag{2.2}$$

Indeed, let $\psi \in X(\mathbb{R} \times \mathbb{T})$, and for convenience assume that ψ is smooth and ψ' has compact support. Note first that by definition of the minimal energy $\mathfrak{I}$,

$$E_\lambda(\psi) = \int_{\mathbb{T}} E\big(\psi(\,\cdot\,, y)\big)\,\mathrm{d}y + \frac{\lambda^2}{2}\int_{\mathbb{R}\times\mathbb{T}} |\partial_y \psi|^2$$
$$\geqslant \int_{\mathbb{T}} \mathfrak{I}\big([P](\psi(\,\cdot\,, y)\big)\,\mathrm{d}y + \frac{\lambda^2}{2}\int_{\mathbb{R}\times\mathbb{T}} |\partial_y \psi|^2.$$

Besides, since the function $\mathfrak{I}$ has Lipschitz constant equal to $\sqrt{2}$ (see Proposition 1.1 above), we have, for all $y \in \mathbb{T}$,

$$\mathfrak{I}\big([P](\psi(\,\cdot\,, y))\big) \geqslant \mathfrak{I}\big([P](\psi)\big) - \sqrt{2}\big|[P](\psi(\,\cdot\,, y)) - [P](\psi)\big|.$$

Here for $p \in \mathbb{R}/\pi\mathbb{Z}$, we denote by $|p|$ the distance between p and zero in $\mathbb{R}/\pi\mathbb{Z}$. We shall show that

$$\big|[P](\psi(\,\cdot\,, y)) - [P](\psi)\big| \leqslant \frac{1}{\lambda} E_\lambda(\psi). \tag{2.3}$$

It follows from combining the previous three inequalities that

$$\Big(1 + \frac{\sqrt{2}}{\lambda}\Big) E_\lambda(\psi) \geqslant \mathfrak{I}\big([P](\psi)\big) + \frac{\lambda}{2}\int_{\mathbb{R}\times\mathbb{T}} |\partial_y \psi|^2. \tag{2.4}$$

Considering a minimizing sequence for $\mathcal{I}_\lambda(p)$, this yields in particular

$$\Big(1 + \frac{\sqrt{2}}{\lambda}\Big) \mathcal{I}_\lambda(p) \geqslant \mathfrak{I}(p),$$

and by (2.1), also (2.2).

The term $|\partial_y \psi|^2$ in (2.4), which is weighted by λ, will enforce minimizing functions for $\mathcal{I}_\lambda(p)$ to be essentially 1D. In the next lemma, we formalize the previous claims, and combine them with an additional Pohozaev type property.

Lemma 2.1. — *Let $p \in \mathbb{R}/\pi\mathbb{Z}$ and consider a sequence $(\lambda_n)_{n\geqslant 0}$ such that $\lambda_n \to +\infty$. Then,*

$$\mathcal{I}_{\lambda_n}(p) \longrightarrow \mathfrak{I}(p), \text{ as } n \longrightarrow \infty. \tag{2.5}$$

Moreover, there exists a sequence $(\psi_n)_{n\geqslant 0}$ of smooth functions in $X(\mathbb{R} \times \mathbb{T})$ with compactly supported gradients, which satisfy $[P](\psi_n) = p$,

$$E_{\lambda_n}(\psi_n) - \mathcal{I}_{\lambda_n}(p) \longrightarrow 0, \quad \text{and} \quad \lambda_n^2 \int_{\mathbb{R}\times\mathbb{T}} |\partial_y \psi_n|^2 \longrightarrow 0, \quad \text{as } n \to \infty. \tag{2.6}$$

Besides, we can assume that

$$\frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} |\partial_x \psi_n|^2 = \frac{\lambda_n^2}{2}\int_{\mathbb{R}\times\mathbb{T}} |\partial_y \psi_n|^2 + \frac{1}{4}\int_{\mathbb{R}\times\mathbb{T}} \big(1 - |\psi_n|^2\big)^2. \tag{2.7}$$

We can then establish that a subsequence of the functions ψ_n converges towards a minimizer of the 1D problem $\mathfrak{I}(p)$. More precisely, for $|c| < \sqrt{2}$, we introduce the distance d_c given by

$$d_c(\psi_1, \psi_2)^2 = \big\|\nabla\psi_1 - \nabla\psi_2\big\|_{L^2}^2 + \big\|\eta_c^{\frac{1}{2}}(\psi_1 - \psi_2)\big\|_{L^2}^2 + \big\|(1-|\psi_1|^2) - (1-|\psi_2|^2)\big\|_{L^2}^2,$$

for functions ψ_1 and ψ_2 in $X(\mathbb{R} \times \mathbb{T})$. In the second term, the weight η_c is given by the expression

$$\eta_c(x) = 1 - |\mathfrak{u}_c(x)|^2 = \frac{2 - c^2}{2\cosh\big(\frac{\sqrt{2-c^2}}{2}x\big)^2}. \tag{2.8}$$

The metric d_c is taylored for the study of perturbations of $\mathfrak{u}_c$. Note however that since η_c decays exponentially at infinity all these metrics induce the same topology on $X(\mathbb{R} \times \mathbb{T})$. We refer to Appendix B for more detail about the metric structure corresponding to the distance d_c. Using this distance, we show

Proposition 2.2. — *There exist a sequence of real numbers $(a_n)_{n \geqslant 0}$, a number $\theta \in \mathbb{R}$, and an extraction $\varphi : \mathbb{N} \to \mathbb{N}$ for which*

$$d_{c_p}\big(e^{i\theta}\psi_{\varphi(n)}(\,\cdot - a_{\varphi(n)}, \cdot\,), \mathfrak{u}_{c_p}\big) \longrightarrow 0,$$

as $n \to \infty$.

Given any positive number α, we denote

$$\mathcal{V}_p(\alpha) := \left\{ \psi = \widehat{\psi}_0 + w_0 \in X(\mathbb{R} \times \mathbb{T}) : \begin{array}{r} \inf_{(a,\theta)\in\mathbb{R}^2} d_{c_p}\big(e^{i\theta}\widehat{\psi}_0(\,\cdot - a), \mathfrak{u}_{c_p}\big) < \alpha \\ \text{and } \|w_0\|_{H^1} < \alpha \end{array} \right\}, \tag{2.9}$$

where we have set as before $\widehat{\psi}_0(x) = \int_{\mathbb{T}} \psi(x, y)\, dy$ for any function $\psi \in X(\mathbb{R} \times \mathbb{T})$. We can rephrase Proposition 2.2 (see e.g. statement (i) of Lemma B.4) as the fact that there exists an integer N_α such that

$$\psi_{\varphi(n)} \in \mathcal{V}_p(\alpha), \tag{2.10}$$

for any $n \geqslant N_\alpha$. We next show that the profile $\mathfrak{u}_{c_p}$ minimizes the energy E_λ at fixed momentum p in the open set $\mathcal{V}_p(\alpha)$, provided α is sufficiently small and λ is sufficiently large.

Proposition 2.3. — *Let $p \in \mathbb{R}/\pi\mathbb{Z}$, with $p \neq 0$. There exist two positive numbers α_p and λ_p such that, given any function $\psi \in \mathcal{V}_p(\alpha_p)$, with $[P](\psi) = p$, we have*

$$E_\lambda(\psi) \geqslant E_\lambda(\mathfrak{u}_{c_p}) = \mathfrak{I}(p), \tag{2.11}$$

for any $\lambda \geqslant \lambda_p$. Moreover, equality holds in (2.11) if and only if $\psi(x, y) = e^{-i\theta}\mathfrak{u}_{c_p}(x + a)$ for some $a \in \mathbb{R}$ and $\theta \in \mathbb{R}$.

To derive Proposition 2.3 we use the stability properties of the solitons $\mathfrak{u}_{c_p}$ with respect to the 1D Gross–Pitaevskii flow. In [4, 10], the orbital stability of the dark solitons was derived from the coercivity of the functional $E-c_p[P]$ in the neighbourhood of the profiles $\mathfrak{u}_{c_p}$. Extending this coercivity property to the sets $\mathcal{V}_p(\alpha)$ for α small enough requires to control the dependence on the variable y. For λ large enough, this can be done by using the coercivity provided by the term $\lambda^2 \int_{\mathbb{R}\times\mathbb{T}} |\partial_y \psi|^2$ in the energy $E_\lambda(\psi)$. The functional $E_\lambda - c_p[P]$ is then coercive on the sets $\mathcal{V}_p(\alpha)$ and we obtain (2.11) when the untwisted momentum $[P]$ is moreover fixed.

Combining (2.10) and (2.11), we are finally led to

$$\mathcal{I}_\lambda(p) \geqslant \mathfrak{I}(p),$$

for $\lambda \geqslant \lambda_p$. In view of (2.1), these two quantities are equal as we have claimed in statement (i) of Theorem 1.2.

Before concluding the proof of Theorem 1.2, we need to precise the behaviour of the minimal energy $\mathcal{I}_\lambda(p)$ with respect to the parameter λ when $\lambda \to 0$. This corresponds to the situation where the unscaled initial torus tends to the whole plane $\mathbb{R}^2$, and using some scaling argument from the plane case, we establish

Lemma 2.4. — *Let $p \in \mathbb{R}/\pi\mathbb{Z}$. The function $\lambda \mapsto \mathcal{I}_\lambda(p)$ is non-decreasing and continuous on $\mathbb{R}_+^*$, with*

$$\mathcal{I}_\lambda(p) \longrightarrow 0, \tag{2.12}$$

as $\lambda \to 0$.

With Lemma 2.4 at hand, we are in position to complete the proof of Theorem 1.2.

End of the proof of Theorem 1.2. — Set $\Lambda := \{\lambda \in (0,+\infty) \text{ s.t. } \mathcal{I}_\mu(p) = \mathfrak{I}(p) \text{ for any } \mu \geqslant \lambda\}$ and $\lambda_p = \inf \Lambda$. We have just shown in Proposition 2.3 that Λ is non-empty. Its infimum λ_p cannot be equal to 0 due to (2.12). Hence, λ_p is positive and moreover, a minimum by continuity of the map $\lambda \mapsto \mathcal{I}_\lambda(p)$.

Since this map is also non-decreasing, the minimal value $\mathcal{I}_\lambda(p)$ is strictly less than $\mathfrak{I}(p)$ when $0 < \lambda < \lambda_p$. Moreover, if a function $\psi \in X(\mathbb{R}\times\mathbb{T})$, with $[P](\psi) = p$, only depends on the variable x, then it follows from Proposition 1.1 that

$$E_\lambda(\psi) = E(\psi) \geqslant \mathfrak{I}(p) > \mathcal{I}_\lambda(p).$$

Therefore, a possible minimizer cannot only depend on the variable x.

When $\lambda \geqslant \lambda_p$ instead, we have $\mathcal{I}_\lambda(p) = \mathfrak{I}(p) = E_\lambda(\mathfrak{u}_{c_p})$, so that the profile $\mathfrak{u}_{c_p}$ is a minimizer of the minimization problem (1.10). For $\lambda > \lambda_p$, assume for the sake of a contradiction the existence of a minimizer $\psi \in X(\mathbb{R} \times \mathbb{T})$ such that $E_\lambda(\psi) = \mathcal{I}_\lambda(p) = \mathfrak{I}(p)$, $[P](\psi) = p$, and

$$\int_{\mathbb{R}\times\mathbb{T}} |\partial_y \psi|^2 \neq 0.$$

For $\lambda_p < \mu < \lambda$, we obtain

$$\mathcal{I}_\mu(p) \leqslant E_\mu(\psi) = E_\lambda(\psi) + \frac{\mu^2 - \lambda^2}{2} \int_{\mathbb{R}\times\mathbb{T}} |\partial_y \psi|^2 < E_\lambda(\psi) = \mathfrak{I}(p),$$

which contradicts the definition of the minimum λ_p. Therefore, a possible minimizer cannot depend on the variable y, so that it minimizes the 1D Ginzburg–Landau energy at fixed untwisted momentum. In view of Proposition 1.1, the profile $\mathfrak{u}_{c_p}$ is therefore the unique minimizer of the minimization problem (1.10) up to translation and phase shift. This concludes the proof of Theorem 1.2. □

3. Details in the proof of Theorem 1.2

3.1. Some useful approximation results

In Appendix A, we introduce the set of non-vanishing functions

$$NVX(\mathbb{R}) := \Big\{\psi \in X(\mathbb{R}) \text{ s.t. } \inf_{x\in\mathbb{R}} |\psi(x)| > 0\Big\},$$

and show that, for $\psi = \rho e^{i\theta} \in NVX(\mathbb{R})$, the momentum

$$P(\psi) = \frac{1}{2} \int_{\mathbb{R}} (1 - \rho^2)\theta',$$

is well-defined and satisfies $P(\psi) = [P](\psi)$ modulo π. This set is used throughout for the proof of Theorem 1.2, in particular in Lemma 3.1 below.

Adapting the argument in [2, Lemma 3.3] we first show

Lemma 3.1. — *Let $\lambda > 0$, $p \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ be fixed. There exists a sequence $(\psi_n)_{n\in\mathbb{N}}$ of smooth functions in $NVX(\mathbb{R})$ such that the functions $\psi_n - e^{i\alpha}$ are compactly supported, and with*

$$P(\psi_n) = p \quad \text{and} \quad E_\lambda(\psi_n) = E(\psi_n) \longrightarrow \sqrt{2}\,|p|,$$

as $n \to \infty$.

Proof. — We argue as in the proof of [2, Lemma 3.3]. Assume first that p is positive. Consider a function $\xi \in \mathcal{C}_c^\infty(\mathbb{R})$ and two positive numbers μ and ε such that $\mu\varepsilon\|\partial_x\xi\|_{L^\infty(\mathbb{R})} < 1$. Set

$$\rho(x,y) = 1 - \mu\varepsilon\,\partial_x\xi(\varepsilon x), \quad \theta(x,y) = \alpha + \sqrt{2}\mu\,\xi(\varepsilon x) \quad \text{and} \quad \psi(x,y) = \rho(x,y)e^{i\theta(x,y)},$$

for any $(x,y) \in \mathbb{R}\times\mathbb{T}$. We compute

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} |\partial_x\psi|^2 &= \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} \big((\partial_x\rho)^2 + \rho^2(\partial_x\theta)^2\big)\\ &= \frac{\mu^2\varepsilon^3}{2}\int_{\mathbb{R}}(\partial_{xx}\xi)^2 + \mu^2\varepsilon\int_{\mathbb{R}}\big(1-\mu\varepsilon\,\partial_x\xi\big)^2(\partial_x\xi)^2, \quad (3.1)\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{4}\int_{\mathbb{R}\times\mathbb{T}}\big(1-|\psi|^2\big)^2\\ &\quad = \mu^2\varepsilon\int_{\mathbb{R}\times\mathbb{T}}(\partial_x\xi)^2 - \mu^3\varepsilon^2\int_{\mathbb{R}\times\mathbb{T}}(\partial_x\xi)^3 + \frac{\mu^4\varepsilon^3}{4}\int_{\mathbb{R}\times\mathbb{T}}(\partial_x\xi)^4. \quad (3.2)\end{aligned}$$

The function ψ belongs to $NVX(\mathbb{R})$, so that, by definition (A.5) and Lemma C.4, its momentum is given by

$$P(\psi) = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}}\big(1-\rho^2\big)\partial_x\phi = \sqrt{2}\mu^2\varepsilon\int_{\mathbb{R}\times\mathbb{T}}(\partial_x\xi)^2 - \frac{\mu^3\varepsilon^2}{\sqrt{2}}\int_{\mathbb{R}\times\mathbb{T}}(\partial_x\xi)^3.$$

We now assume that the L^2-norm of the derivative $\partial_x\xi$ is equal to 1 and we choose $\mu_n = n$ for a given integer n. At least when n is large enough, we can find a positive number ε_n such that $P(\psi) = p$. Moreover, we have

$$\varepsilon_n \sim \frac{p}{\sqrt{2}n^2},$$

as $n \to \infty$. In particular, we check that

$$\mu_n\varepsilon_n \longrightarrow 0,$$

as $n \to \infty$, so that the condition $\mu_n\varepsilon_n\|\partial_x\xi\|_{L^\infty(\mathbb{R})} < 1$ is indeed satisfied for n large enough. In view of (3.1) and (3.2), we also obtain

$$E_\lambda(\psi) \underset{n\to\infty}{\longrightarrow} \sqrt{2}p.$$

In conclusion, the functions $\psi_n = \psi$ satisfy all the statements in Lemma 3.1 for p positive. When p is negative, the functions $\widetilde{\psi}_n = e^{2i\alpha}\overline{\psi_n}$ also satisfy these conclusions, while for $p = 0$ it suffices to take $\psi_n = 1$. This completes the proof of Lemma 3.1. □

Combining Lemma 3.1 with Corollary B.5, we prove

LEMMA 3.2. — *Let $\lambda > 0$ be fixed. Given a function $\psi \in X(\mathbb{R}\times\mathbb{T})$, there exists a sequence $(\psi_n)_{n\geqslant 0}$ of smooth functions in $X(\mathbb{R} \times \mathbb{T})$, which satisfies the following properties.*

(i) *Given any integer $n \geqslant 0$, there exist two positive numbers $R_n^\pm$ and two numbers $\theta_n^\pm$ for which*

$$\psi_n(x, y) = e^{i\theta_n^\pm},$$

for any $\pm x \geqslant \pm R_n^\pm$ and any $y \in \mathbb{T}$.

(ii) *We have*

$$[P](\psi_n) = [P](\psi),$$

for any $n \geqslant 0$.

(iii) *We also have*

$$E_\lambda(\psi_n) \longrightarrow E_\lambda(\psi),$$

as $n \to \infty$.

Proof. — From Corollary B.5 and Lemma C.3, we can find a sequence of smooth functions $\widetilde{\psi}_n$ in $X(\mathbb{R} \times \mathbb{T})$, which satisfy statements (i) and (iii) of Lemma 3.2 for numbers $\widetilde{R}_n^\pm$ and $\widetilde{\theta}_n^\pm$, as well as

$$[P](\widetilde{\psi}_n) \longrightarrow [P](\psi), \tag{3.3}$$

in the limit $n \to \infty$. Hence we are reduced to check that we can modify the functions $\widetilde{\psi}_n$, so that their momentum is exactly equal to the momentum of ψ. When these two quantities are actually equal, we simply set $\psi_n = \widetilde{\psi}_n$. When they are not, we invoke Lemma 3.1 with $\alpha = \widetilde{\theta}_n^+$ and $p_n \in (-\pi/2, \pi/2]$ such that $p_n = [P](\widetilde{\psi}_n) - [P](\psi)$ modulo π. This provides a smooth function $\breve{\psi}_n$ such that the function $\breve{\psi}_n - e^{i\tilde{\theta}_n^+}$ is compactly supported in an interval of the form $[-\breve{R}_n^-, \breve{R}_n^+]$, with $P(\breve{\psi}_n) = p_n$ and $E_\lambda(\breve{\psi}_n) \leqslant \sqrt{2}|p_n|$. We next set

$$\psi_n(x, y) = \begin{cases} \widetilde{\psi}_n(x, y) & \text{if } x \leqslant \widetilde{R}_n^+ + 1, \\ \breve{\psi}_n(x - \widetilde{R}_n^+ - \breve{R}_n^- - 2, y) & \text{if } x \geqslant \widetilde{R}_n^+ + 1. \end{cases}$$

By construction, the function ψ_n is smooth, belongs to $X(\mathbb{R}\times\mathbb{T})$ and satisfies statement (i) of Lemma 3.2. Moreover, it follows from Lemmas C.1 and C.5 that

$$[P](\psi_n) = [P](\widetilde{\psi}_n) + P(\breve{\psi}_n) = [P](\psi),$$

modulo π. Finally, we also derive from (3.3) that

$$E_\lambda(\psi_n) = E_\lambda(\widetilde{\psi}_n) + E_\lambda(\breve{\psi}_n) \longrightarrow E_\lambda(\psi),$$

as $n \to \infty$. This concludes the proof. □

3.2. Proof of Lemma 2.1

From Lemma 3.2 and a diagonal argument, we can find a sequence of smooth functions $\psi_n \in X(\mathbb{R} \times \mathbb{T})$ such that

$$E_{\lambda_n}(\psi_n) \leqslant \mathcal{I}_{\lambda_n}(p) + \varepsilon_n, \quad [P](\psi_n) = p, \tag{3.4}$$

and moreover there exist positive numbers $R_n^\pm$ and real numbers $\theta_n^\pm$ such that

$$\psi_n(x, y) = e^{i\theta_n^\pm},$$

for any $\pm x \geqslant \pm R_n^\pm$ and any $y \in \mathbb{T}$. According to Lemma C.5, the momentum $[P](\psi_n)$ is then given by

$$[P](\psi_n) = \int_{\mathbb{T}} p_n(y)\,\mathrm{d}y \mod \pi, \tag{3.5}$$

with

$$p_n(y) := \frac{1}{2}\int_{\mathbb{R}} \langle i\partial_x\psi_n(x, y), \psi_n(x, y)\rangle_{\mathbb{C}}\,\mathrm{d}x + \frac{1}{2}\big(\theta_n^+ - \theta_n^-\big) = [P]\big(\psi_n(\,\cdot\,, y)\big) \mod \pi. \tag{3.6}$$

Since the functions ψ_n are smooth and their derivatives are compactly supported, the functions p_n in the previous definition are well-defined and smooth on $\mathbb{T}$, with

$$p_n'(y) = \int_{\mathbb{R}} \langle i\partial_x\psi_n(x, y), \partial_y\psi_n(x, y)\rangle_{\mathbb{C}}\,\mathrm{d}x,$$

by integration by parts. Hence, we infer from the Cauchy–Schwarz inequality, (2.1) and (3.4) that

$$\int_{\mathbb{T}} |p_n'(y)|\,\mathrm{d}y \leqslant \frac{1}{\lambda_n} E_{\lambda_n}(\psi_n) \leqslant \frac{1}{\lambda_n}\big(\mathfrak{I}(p) + \varepsilon_n\big),$$

so that the Poincaré–Wirtinger inequality in [7] provides

$$\big\|p_n - [P](\psi_n)\big\|_{L^\infty(\mathbb{T})} \leqslant \int_{\mathbb{T}} |p_n'(y)|\,\mathrm{d}y \longrightarrow 0, \tag{3.7}$$

as $n \to \infty$. At this stage, we write

$$E_{\lambda_n}(\psi_n) = \int_{\mathbb{T}} E\big(\psi_n(\,\cdot\,, y)\big)\,\mathrm{d}y + \frac{\lambda_n^2}{2}\int_{\mathbb{R}\times\mathbb{T}} |\partial_y\psi_n|^2 \geqslant \int_{\mathbb{T}} \mathfrak{I}\big([P](\psi_n(\,\cdot\,, y))\big)\,\mathrm{d}y + \frac{\lambda_n^2}{2}\int_{\mathbb{R}\times\mathbb{T}} |\partial_y\psi_n^2|. \tag{3.8}$$

Since the function $\mathfrak{I}$ has Lipschitz constant $\sqrt{2}$, we obtain

$$\big|\mathfrak{I}\big([P](\psi_n(\,\cdot\,, y))\big) - \mathfrak{I}(p)\big| \leqslant \sqrt{2}\,\big|p_n(y) - [P](\psi_n)\big|, \tag{3.9}$$

for n large enough.

In view of (3.8), we conclude that

$$E_{\lambda_n}(\psi_n) \geqslant \mathfrak{I}(p) - \sqrt{2}\big\|p_n - [P](\psi_n)\big\|_{L^\infty(\mathbb{T})} + \frac{\lambda_n^2}{2} \int_{\mathbb{R}\times\mathbb{T}} |\partial_y \psi_n^2|.$$

Combining with (3.4) and (3.7), we first deduce that

$$\lambda_n^2 \int_{\mathbb{R}\times\mathbb{T}} |\partial_y \psi_n^2| \longrightarrow 0, \tag{3.10}$$

as $n \to \infty$. Moreover, it also follows from (2.1) and (3.4) that

$$\mathfrak{I}(p) \geqslant \mathcal{I}_{\lambda_n}(p) \geqslant \mathfrak{I}(p) - \varepsilon_n - \sqrt{2}\big\|p_n - [P](\psi_n)\big\|_{L^\infty(\mathbb{T})} + \frac{\lambda_n^2}{2} \int_{\mathbb{R}\times\mathbb{T}} |\partial_y \psi_n^2|.$$

In view of (3.7) and (3.10), this is enough to obtain the convergence in (2.5).

In order to complete the proof of Lemma 2.1, we now explain how we can assume that ψ_n satisfies the Pohozaev identity in (2.7). It is classical that this identity is based on applying the scaling $(x, y) \mapsto (\tau x, y)$ for positive numbers τ. For a fixed integer $n \geqslant 0$, the functions

$$\xi_\tau(x, y) = \psi_n(\tau x, y)$$

are smooth on $\mathbb{R} \times \mathbb{T}$ and satisfy statement (i) in Lemma 3.2 for the same numbers $\theta_n^\pm$ as the function ψ_n. Arguing as for (3.5), their untwisted momentum $[P](\xi_\tau)$ is given by the formula

$$[P](\xi_\tau) = \frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}} \langle i\partial_x \xi_\tau(x, y), \xi_\tau(x, y)\rangle_\mathbb{C} \, \mathrm{d}x \, \mathrm{d}y + \frac{1}{2}\big(\theta_n^+ - \theta_n^-\big) \mod \pi.$$

By definition of the functions ξ_τ and by (3.5), this quantity reduces to

$$\begin{aligned}[P](\xi_\tau) = \frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}} \langle i\partial_x \psi_n(x, y), \psi_n(x, y)\rangle_\mathbb{C} \, \mathrm{d}x \, \mathrm{d}y + \frac{1}{2}\big(\theta_n^+ - \theta_n^-\big)& \\ = [P](\psi_n) \mod \pi,& \end{aligned} \tag{3.11}$$

for any positive number τ. Similarly, we compute

$$\frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}} |\partial_x \xi_\tau|^2 = \frac{\tau}{2} \int_{\mathbb{R}\times\mathbb{T}} |\partial_x \psi_n|^2 := A_n \tau, \tag{3.12}$$

and

$$\begin{aligned}&\frac{\lambda_n^2}{2} \int_{\mathbb{R}\times\mathbb{T}} |\partial_y \xi_\tau|^2 + \frac{1}{4} \int_{\mathbb{R}\times\mathbb{T}} \big(1 - |\xi_\tau|^2\big)^2 \\ &\qquad = \frac{1}{\tau}\bigg(\frac{\lambda_n^2}{2} \int_{\mathbb{R}\times\mathbb{T}} |\partial_y \psi_n|^2 + \frac{1}{4} \int_{\mathbb{R}\times\mathbb{T}} \big(1 - |\psi_n|^2\big)^2\bigg) := \frac{B_n}{\tau}.\end{aligned} \tag{3.13}$$

Observe here that $A_n \neq 0$. Otherwise, the function ψ_n would not depend on the variable x, so that the numbers $\theta_n^\pm$ would also be equal. As a consequence, the quantity $[P](\psi_n)$ in (3.5) would be equal to 0, and not to p modulo π.

Since $A_n \neq 0$, we can combine (3.12) and (3.13) to derive that the energies $E_{\lambda_n}(\xi_\tau)$ are minimal for τ being chosen as

$$\tau_n = \sqrt{\frac{B_n}{A_n}}.$$

It suffices then to set $\xi_n = \xi_{\tau_n}$ in order to obtain

$$E_{\lambda_n}(\xi_n) \leqslant E_{\lambda_n}(\psi_n), \tag{3.14}$$

by minimality, as well as the Pohozaev identity

$$\frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} |\partial_x \xi_n|^2 = \sqrt{A_n B_n} = \frac{\lambda_n^2}{2}\int_{\mathbb{R}\times\mathbb{T}} |\partial_y \xi_n|^2 + \frac{1}{4}\int_{\mathbb{R}\times\mathbb{T}} \big(1 - |\xi_n|^2\big)^2,$$

by (3.12) and (3.13). In view of (3.11) and (3.14), this completes the proof of Lemma 2.1, replacing ψ_n by ξ_n. □

3.3. Proof of Proposition 2.2

We go on with the notation of the proof of Lemma 2.1. Our first goal is to exhibit a number $y_* \in \mathbb{T}$ such that, up to a possible subsequence, the functions $\psi_n(\,\cdot\,, y_*)$ form an almost minimizing sequence for the 1D minimization problem $\mathfrak{I}(p)$. In view of (3.6), the untwisted momentum $[P](\psi_n(\,\cdot\,, y))$ is equal to $p_n(y)$ modulo π for almost any $y \in \mathbb{T}$, so that our aim is to find a number $y_* \in \mathbb{T}$ such that

$$p_n(y_*) \longrightarrow p \mod \pi, \quad \text{and} \quad e_n(y_*) := E(\psi_n(\,\cdot\,, y_*)) \longrightarrow \mathfrak{I}(p),$$

as $n \to \infty$. In this direction, we first recall that $[P](\psi_n) = p$. Going back to the proof of Lemma 2.1, and more precisely to (3.7), it follows that

$$p_n(y) \longrightarrow p \mod \pi, \tag{3.15}$$

as $n \to \infty$, uniformly with respect to $y \in \mathbb{T}$. We similarly deduce from (3.9) that

$$\int_{\mathbb{T}} |e_n(y) - \mathfrak{I}(p)| \, \mathrm{d}y \leqslant \int_{\mathbb{T}} |e_n(y) - \mathfrak{I}(p_n(y))| \, \mathrm{d}y + \sqrt{2}\int_{\mathbb{T}} |p_n(y) - [P](\psi_n)| \, \mathrm{d}y.$$

Since $e_n(y) \geqslant \mathfrak{I}(p_n(y))$ by definition of the 1D minimal energy $\mathfrak{I}$, we infer again from (3.9) that

$$\int_{\mathbb{T}} |e_n(y) - \mathfrak{I}(p)| \, \mathrm{d}y \leqslant E_{\lambda_n}(\psi_n) - \mathfrak{I}(p) + 2\sqrt{2}\int_{\mathbb{T}} |p_n(y) - [P](\psi_n)| \, \mathrm{d}y.$$

Invoking (2.5) and (3.7), we are led to

$$\int_{\mathbb{T}} |e_n(y) - \mathfrak{I}(p)| \, \mathrm{d}y \longrightarrow 0,$$

as $n \to \infty$. As a consequence, we can find a number $y_* \in \mathbb{T}$ such that, up to a possible subsequence, we have

$$e_n(y_*) \longrightarrow \mathfrak{I}(p),$$

as $n \to \infty$. In view of (3.15), we conclude that the functions $\psi_n(\,\cdot\,, y_*)$ form a minimizing sequence for $\mathfrak{I}(p)$. In particular, we can apply the compactness results in [1, Theorem 3] and [3, Theorem 4] to this sequence. This provides a sequence of real numbers $(a_n)_{n\geqslant 0}$, as well as a number $\theta \in \mathbb{R}$, such that, up to a further subsequence,

$$\begin{aligned} e^{i\theta}\,\psi_n(\,\cdot - a_n, y_*) &\longrightarrow \mathfrak{u}_{c_p} && \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}),\\ 1 - |e^{i\theta}\,\psi_n(\,\cdot - a_n, y_*)|^2 &\longrightarrow 1 - |\mathfrak{u}_{c_p}|^2 && \text{in } L^2(\mathbb{R}),\\ e^{i\theta}\,\partial_x\psi_n(\,\cdot - a_n, y_*) &\longrightarrow \mathfrak{u}_{c_p}' && \text{in } L^2(\mathbb{R}), \end{aligned} \tag{3.16}$$

as $n \to \infty$.

We now extend the convergence to any number $y \in \mathbb{T}$. This follows from the smoothness of the functions ψ_n, which guarantees that

$$e^{i\theta}\,\psi_n(x - a_n, y) - e^{i\theta}\,\psi_n(x - a_n, y_*) = e^{i\theta} \int_{y_*}^{y} \partial_y\psi_n(x - a_n, y')\,\mathrm{d}y'.$$

Invoking the Cauchy–Schwarz inequality, we are led to

$$\int_{-R}^{R} \big|e^{i\theta}\,\psi_n(x-a_n, y) - e^{i\theta}\,\psi_n(x-a_n, y_*)\big|^2\,\mathrm{d}x \leqslant \int_{\mathbb{R}\times\mathbb{T}} |\partial_y\psi_n(x', y')|^2\,\mathrm{d}x'\,\mathrm{d}y',$$

for any positive number R. Combining the convergences in Lemma 2.1 with the first one in (3.16), we deduce that

$$\int_{-R}^{R} \big|e^{i\theta}\,\psi_n(x - a_n, y) - \mathfrak{u}_{c_p}(x)\big|^2\,\mathrm{d}x \longrightarrow 0,$$

as $n \to \infty$, uniformly with respect to $y \in \mathbb{T}$. This is enough to guarantee that the functions $e^{i\theta}\,\psi_n(\,\cdot - a_n, \cdot\,)$ converge to the function $\mathfrak{u}_{c_p}$ in $L^2_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T})$.

At this stage, we again rely on the convergences in Lemma 2.1 in order to claim that both the sequences $(e^{i\theta}\,\nabla\psi_n(\,\cdot - a_n, \cdot\,))_{n\geqslant 0}$ and $(1 - |\psi_n(\,\cdot - a_n, \cdot\,)|^2)_{n\geqslant 0}$ are bounded in $L^2(\mathbb{R} \times \mathbb{T})$. Up to a further subsequence, we can find two functions $\Xi \in L^2(\mathbb{R} \times \mathbb{T})$ and $\eta \in L^2(\mathbb{R} \times \mathbb{T})$ such that

$$\begin{aligned} &e^{i\theta}\,\nabla\psi_n(\,\cdot - a_n, \cdot\,) \rightharpoonup \Xi \text{ in } L^2(\mathbb{R} \times \mathbb{T}),\\ &\qquad\qquad \text{and} \quad 1 - |\psi_n(\,\cdot - a_n, \cdot\,)|^2 \rightharpoonup \eta \text{ in } L^2(\mathbb{R} \times \mathbb{T}), \end{aligned} \tag{3.17}$$

as $n \to \infty$. Since $|z| \leqslant 1 + \big|1 - |z|^2\big|$ for any complex number z, the sequence $(e^{i\theta}\,\psi_n(\,\cdot - a_n, \cdot\,))_{n\geqslant 0}$ is also bounded in $H^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T})$. Applying the Rellich theorem, we can find another function $\psi_\infty \in H^1_{\rm loc}(\mathbb{R} \times \mathbb{T})$ such that, up to a further subsequence,

$$e^{i\theta}\, \psi_n(\,\cdot - a_n, \cdot\,) \longrightarrow \psi_\infty \text{ in } L^q_{\rm loc}(\mathbb{R} \times \mathbb{T}), \tag{3.18}$$

as $n \to \infty$, for any number $1 \leqslant q < +\infty$. Since this convergence holds for $q = 2$, the function ψ_∞ is equal to $\mathfrak{u}_{c_p}$, and we deduce from (3.17) and (3.18) that $\Xi = \nabla \mathfrak{u}_{c_p}$ and $\eta = 1 - |\mathfrak{u}_{c_p}|^2$.

We now transform the weak convergences in (3.17) into strong convergences. We first observe that

$$e^{i\theta}\, \partial_y \psi_n(\,\cdot - a_n, \cdot\,) \longrightarrow \partial_y \mathfrak{u}_{c_p} = 0 \text{ in } L^2(\mathbb{R} \times \mathbb{T}), \tag{3.19}$$

by (2.6). We next rely on the Pohozaev identity (2.7) in order to obtain

$$\begin{aligned} E_{\lambda_n}(\psi_n) &= \int_{\mathbb{R}\times\mathbb{T}} |e^{i\theta}\, \partial_x \psi_n(x - a_n, y)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &= \lambda_n^2 \int_{\mathbb{R}\times\mathbb{T}} |\partial_y \psi_n|^2 + \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} \big(1 - |\psi_n(x - a_n, y)|^2\big)^2 \,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

In view of Lemma 2.1, we note that $E_{\lambda_n}(\psi_n) \to \mathfrak{I}(p)$ as $n \to \infty$. Combining with (2.6), we are led to

$$\begin{aligned} &\int_{\mathbb{R}\times\mathbb{T}} |e^{i\theta}\, \partial_x \psi_n(x - a_n, y)|^2 \,\mathrm{d}x\,\mathrm{d}y \longrightarrow \mathfrak{I}(p), \\ \text{and} \quad &\frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} \big(1 - |\psi_n(x - a_n, y)|^2\big)^2 \,\mathrm{d}x\,\mathrm{d}y \longrightarrow \mathfrak{I}(p), \end{aligned} \tag{3.20}$$

as $n \to \infty$. We finally express the quantity $\mathfrak{I}(p)$ in terms of the travelling-wave profile $\mathfrak{u}_{c_p}$. Recall that this profile solves (1.1) with $c = c_p$. We multiply this equation by the derivative $\mathfrak{u}_{c_p}'$ and integrate it taking into account the exponential decay of the functions $\mathfrak{u}_{c_p}'$ and $1 - |\mathfrak{u}_{c_p}|^2$. This gives

$$\frac{1}{2}|\mathfrak{u}_{c_p}'|^2 = \frac{1}{4}\big(1 - |\mathfrak{u}_{c_p}|^2\big)^2.$$

It is then enough to invoke Proposition 1.1 in order to obtain

$$\mathfrak{I}(p) = E(\mathfrak{u}_{c_p}) = \int_{\mathbb{R}} |\mathfrak{u}_{c_p}'|^2 = \frac{1}{2}\int_{\mathbb{R}} \big(1 - |\mathfrak{u}_{c_p}|^2\big)^2.$$

In view of (3.20), we deduce that

$$\big\| e^{i\theta}\, \partial_x \psi_n(\,\cdot - a_n, \cdot\,) \big\|_{L^2(\mathbb{R}\times\mathbb{T})} \longrightarrow \big\| \mathfrak{u}_{c_p}' \big\|_{L^2(\mathbb{R}\times\mathbb{T})},$$

and

$$\big\| 1 - |\psi_n(\,\cdot - a_n, \cdot\,)|^2 \big\|_{L^2(\mathbb{R}\times\mathbb{T})} \longrightarrow \big\| 1 - |\mathfrak{u}_{c_p}|^2 \big\|_{L^2(\mathbb{R}\times\mathbb{T})},$$

as $n \to \infty$. Combining with (3.19), we conclude that the convergences in (3.17) are actually strong.

In order to complete the proof of Proposition 2.2, it only remains to establish that

$$\int_{\mathbb{R}\times\mathbb{T}} \eta_{c_p} \big| e^{i\theta}\, \psi_n(\,\cdot - a_n, \cdot\,) - \mathfrak{u}_{c_p} \big|^2 \longrightarrow 0, \tag{3.21}$$

as $n \to \infty$. Consider a positive number R and write the decomposition

$$\int_{\mathbb{R}\times\mathbb{T}} \eta_{c_p} \big| e^{i\theta}\, \psi_n(\,\cdot - a_n, \cdot\,) - \mathfrak{u}_{c_p} \big|^2 = I_R + J_R, \tag{3.22}$$

with

$$I_R := \int_{(-R,R)\times\mathbb{T}} \eta_{c_p} \big| e^{i\theta}\, \psi_n(\,\cdot - a_n, \cdot\,) - \mathfrak{u}_{c_p} \big|^2 \longrightarrow 0, \tag{3.23}$$

as $n \to \infty$ by (3.18), and

$$J_R := \int_{(-R,R)^c\times\mathbb{T}} \eta_{c_p} \big| e^{i\theta}\, \psi_n(\,\cdot - a_n, \cdot\,) - \mathfrak{u}_{c_p} \big|^2.$$

Concerning this integral, we have

$$J_R \leqslant 2 \int_{(-R,R)^c\times\mathbb{T}} \eta_{c_p} \Big(2 + \big| e^{i\theta}\, \psi_n(\,\cdot - a_n, \cdot\,) \big|^2 - 1 + \big| \mathfrak{u}_{c_p} \big|^2 - 1 \Big). \tag{3.24}$$

Since $\eta_{c_p} \in L^2(\mathbb{R} \times \mathbb{T})$, we infer from (3.17) that

$$\int_{(-R,R)^c\times\mathbb{T}} \eta_{c_p} \Big(2 + \big| e^{i\theta}\, \psi_n(\,\cdot - a_n, \cdot\,) \big|^2 - 1 + \big| \mathfrak{u}_{c_p} \big|^2 - 1 \Big) \longrightarrow 2 \int_{(-R,R)^c\times\mathbb{T}} \eta_{c_p} \big| \mathfrak{u}_{c_p} \big|^2.$$

in the limit $n \to \infty$. The right-hand side of this limit can be made as small as necessary for R large enough. Combining with (3.22), (3.23) and (3.24) is enough to complete the proof of (3.21). This concludes the proof of Proposition 2.2. □

3.4. Proof of Proposition 2.3 for $p \neq \frac{\pi}{2}$

The proof of Proposition 2.3 is based on a coercivity estimate related to the orbital stability of the dark solitons in dimension one. The technical derivation of this estimate turns out to be different for the grey solitons on the one hand, and the black soliton on the other hand. This claim originates in the fact that we can use the hydrodynamical framework for handling the grey solitons, which is no more possible for the black soliton. This is the reason why we split the proof of Proposition 2.3 into two parts dealing first with the case of the grey solitons for $p \neq \pi/2$.

Given a positive number α, consider a function ψ in $\mathcal{V}_p(\alpha)$. In view of Proposition B.1, we can decompose this function as $\psi = \widehat{\psi}_0 + w_0$, with $\widehat{\psi}_0 \in X(\mathbb{R})$ and $w_0 \in H^1(\mathbb{R} \times \mathbb{T})$. Moreover, it follows from (2.9) that $\|w_0\|_{H^1} < \alpha$ and

$$\inf_{(a,\theta)\in\mathbb{R}^2} d_{c_p}\big(e^{i\theta}\widehat{\psi}_0(\,\cdot - a), \mathfrak{u}_{c_p}\big) < \alpha.$$

We first use this control on the function $\widehat{\psi}_0$ in order to estimate the difference between the energies $E_\lambda(\psi)$ and $E(\widehat{\psi}_0)$. More precisely, we show the following inequality, which is still available for $p = \pi/2$.

Lemma 3.3. — *Let $p \in (-\pi/2, \pi/2]$, with $p \neq 0$. There exists a positive number α_p for which we can find a positive number C_p such that we have*

$$E_\lambda(\psi) \geqslant E(\widehat{\psi}_0) + \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} \Big(|\partial_x w_0|^2 + (\lambda^2 - C_p)|\partial_y w_0|^2 + |w_0|^2\Big), \qquad (3.25)$$

for any function $\psi \in \mathcal{V}_p(\alpha_p)$.

Proof. — The proof relies on the expansion of the energy $E_\lambda(\psi)$ in (B.1). Due to the identity

$$\langle\widehat{\psi}_0, w_0\rangle_{\mathbb{C}}^2 + |w_0|^2\langle\widehat{\psi}_0, w_0\rangle_{\mathbb{C}} + \frac{1}{4}|w_0|^4 = \Big(\langle\widehat{\psi}_0, w_0\rangle_{\mathbb{C}} + \frac{1}{2}|w_0|^2\Big)^2,$$

we indeed deduce from (B.1) that

$$E_\lambda(\psi) - E(\widehat{\psi}_0) \geqslant \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} \big(|\partial_x w_0|^2 + \lambda^2|\partial_y w_0|^2\big) - \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} |w_0|^2(1 - |\widehat{\psi}_0|^2). \qquad (3.26)$$

Invoking Lemma A.2, we can find a positive number α_p such that, when ψ is in $\mathcal{V}_p(\alpha_p)$, we get

$$\big\|(1 - |e^{i\theta}\widehat{\psi}_0(\,\cdot - a)|^2) - (1 - |\mathfrak{u}_{c_p}|^2)\big\|_{L^\infty} < 1,$$

for given numbers $(a, \theta) \in \mathbb{R}^2$. As a consequence, we obtain

$$\frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} |w_0|^2(1 - |\widehat{\psi}_0|^2) \leqslant \frac{1}{2}\Big(1 + \big\|1 - |\mathfrak{u}_{c_p}|^2\big\|_{L^\infty}\Big)\int_{\mathbb{R}\times\mathbb{T}} |w_0|^2,$$

and we can invoke the Poincaré–Wirtinger inequality in order to find a positive number C_p such that

$$\frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} |w_0|^2 + \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} |w_0|^2(1 - |\widehat{\psi}_0|^2) \leqslant \frac{C_p}{2}\int_{\mathbb{R}\times\mathbb{T}} |\partial_y w_0|^2.$$

Combining with (3.26), we obtain (3.25). This completes the proof of Lemma 3.3. □

Our next goal is to provide a similar control for the momentum. When $p \neq \pi/2$, the speed c_p is different from 0, so that it follows from [1, Proposition 1] that the energy $E(\mathfrak{u}_{c_p})$ is strictly less than $2\sqrt{2}/3$. Combining (2.9) and the continuity of the Ginzburg–Landau energy E on $X(\mathbb{R})$ (see Appendix A), we can decrease, if necessary, the value of the number α_p so that the energy $E(\widehat{\psi}_0)$ is strictly less than $2\sqrt{2}/3$ when $\psi \in \mathcal{V}_p(\alpha_p)$. In view of Lemma A.5, this guarantees that the function $\widehat{\psi}_0$ lies in the non-vanishing set $NVX(\mathbb{R})$ defined in (A.1) below. As a consequence, the set $\mathcal{V}_p(\alpha_p)$ is a subset of $Y(\mathbb{R} \times \mathbb{T})$ and the momentum P in statement (ii) of Lemma C.1 is well-defined on this set. Moreover, we can show

Lemma 3.4. — *Let $p \in (-\pi/2, \pi/2)$, with $p \neq 0$. There exist a positive number α_p such that*

$$\big|P(\psi) - P(\widehat{\psi}_0)\big| \leqslant \frac{1}{4} \int_{\mathbb{R}\times\mathbb{T}} |\partial_x w_0|^2 + \frac{1}{2\pi} \int_{\mathbb{R}\times\mathbb{T}} |\partial_y w_0|^2, \tag{3.27}$$

for any function $\psi \in \mathcal{V}_p(\alpha_p)$. Moreover, when $[P](\psi) = p$ modulo π, the momentum $P(\psi)$ in this inequality is equal to

$$P(\psi) = p. \tag{3.28}$$

Proof. — The proof is based on the definition of the momentum $P(\psi)$ in (C.1), which gives

$$\big|P(\psi) - P(\widehat{\psi}_0)\big| \leqslant \frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}} |\partial_x w_0||w_0| \leqslant \frac{1}{4} \int_{\mathbb{R}\times\mathbb{T}} |\partial_x w_0|^2 + \int_{\mathbb{R}\times\mathbb{T}} |w_0|^2.$$

Inequality (3.27) then follows from the Poincaré inequality. Observe that the smallness of the number α_p is only used here in order that the quantity $P(\psi)$ and $P(\widehat{\psi}_0)$ make sense.

Concerning (3.28), we recall that the energy $E(\widehat{\psi}_0)$ is strictly less than $2\sqrt{2}/3$ when α_p is small enough. Hence, it follows from Proposition A.6 that

$$\big|P(\widehat{\psi}_0)\big| < \frac{\pi}{2}. \tag{3.29}$$

Moreover, we know that $[P](\psi) = P(\psi)$ modulo π on the one hand, and $[P](\psi) = p$ modulo π on the other hand. As a consequence, there exists an integer $k \in \mathbb{Z}$ such that $P(\psi) = p + k\pi$. In view of (3.27), we are led to

$$\big|p + k\pi - P(\widehat{\psi}_0)\big| \leqslant \frac{1}{4} \|w_0\|_{H^1}^2 < \frac{\alpha_p^2}{4}.$$

Combining with (3.29), we can decrease the value of the number α_p if necessary so that $k = 0$ and $P(\psi) = p$. This completes the proof of Lemma 3.4. □

Collecting (3.25) and (3.27), we obtain

$$\begin{aligned} &E_\lambda(\psi) - c_p P(\psi) \\ &\geqslant E(\widehat{\psi}_0) - c_p P(\widehat{\psi}_0) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}} \Big(\Big(1 - \frac{|c_p|}{2}\Big) |\partial_x w_0|^2 + \Big(\lambda^2 - C_p - \frac{|c_p|}{\pi}\Big) |\partial_y w_0|^2 + |w_0|^2 \Big). \end{aligned} \tag{3.30}$$

Since $|c_p| < \sqrt{2}$, the last term in this inequality is non-negative for $\lambda^2 > C_p + \sqrt{2}/\pi$. Under this condition, it vanishes if and only if w_0 is identically equal to 0.

Our goal is now to control from below the term $E(\widehat{\psi}_0) - c_p P(\widehat{\psi}_0)$. Since the function $\widehat{\psi}_0$ is in $NVX(\mathbb{R})$, we can rely on the hydrodynamical formulation $\widehat{\psi}_0 = \rho_0 e^{i\theta_0}$ and analyze the quantities $E(\widehat{\psi}_0)$ and $P(\widehat{\psi}_0)$ in terms of the variables $\eta_0 := 1 - \rho_0^2$ and $v_0 := \theta_0'$. In view of (A.2) and (A.5), the energy $E(\widehat{\psi}_0)$ and the momentum $P(\widehat{\psi}_0)$ are then given by

$$E\big(\widehat{\psi}_0\big) = E(\eta_0, v_0) := \frac{1}{8} \int_{\mathbb{R}} \frac{(\eta_0')^2}{1 - \eta_0} + \frac{1}{2} \int_{\mathbb{R}} (1 - \eta_0) v_0^2 + \frac{1}{4} \int_{\mathbb{R}} \eta_0^2,$$

and

$$P\big(\widehat{\psi}_0\big) = P(\eta_0, v_0) := \frac{1}{2} \int_{\mathbb{R}} \eta_0 v_0.$$

Recall also that the pair (η_0, v_0) belongs to the non-vanishing set $NV(\mathbb{R})$ defined in (A.3).

Similarly, we can lift the profile $\mathfrak{u}_{c_p}$ as $\mathfrak{u}_{c_p} = \rho_{c_p} e^{i\theta_{c_p}}$ and introduce the corresponding variables $\eta_{c_p} := 1 - \rho_{c_p}^2$ and $v_{c_p} := \theta_{c_p}'$, which are also in $NV(\mathbb{R})$. With this notation at hand, we can consider the neighbourhoods of the pair (η_{c_p}, v_{c_p}) given by

$$\mathcal{U}_p(\beta) := \left\{ (\eta, v) \in NV(\mathbb{R}) : \begin{aligned} \inf_{a\in\mathbb{R}} \big(&\|\eta_0(\,\cdot - a) - \eta_{c_p}\|_{H^1}^2 \\ &+ \|v_0(\,\cdot - a) - v_{c_p}\|_{L^2}^2 \big) < \beta^2 \end{aligned} \right\}, \tag{3.31}$$

for any positive number β. We first show that the pair (η_0, v_0) lies in one of these neighbourhoods when $\widehat{\psi}_0$ is in $\mathcal{V}_p(\alpha_p)$. More precisely, we show

Lemma 3.5. — *Given any positive number β, there exists a positive number $\alpha \leqslant \alpha_p$ such that*

$$(\eta_0, v_0) \in \mathcal{U}_p(\beta),$$

for any function $\psi \in \mathcal{V}_p(\alpha)$.

Proof. — Consider a positive number α such that $\alpha \leqslant \alpha_p$. Under this condition, the function $\widehat{\psi}_0$ is in $NVX(\mathbb{R})$ when ψ belongs to $\mathcal{V}_p(\alpha)$. In particular, the functions η_0 and v_0 are well-defined. In view of (2.9), we can also find numbers $(a, \theta) \in \mathbb{R}^2$ such that $d_{c_p}(e^{i\theta}\widehat{\psi}_0(\cdot - a), \mathfrak{u}_{c_p}) < \alpha$, so that by (A.8),

$$\big\|\eta_0(\cdot - a) - \eta_{c_p}\big\|_{L^2} \leqslant d_{c_p}\big(e^{i\theta}\widehat{\psi}_0(\cdot - a), \mathfrak{u}_{c_p}\big) < 2\alpha. \tag{3.32}$$

We next write

$$\eta_0'(\cdot - a) - \eta_{c_p}' = -2\langle \widehat{\psi}_0'(\cdot - a) - \mathfrak{u}_{c_p}', \widehat{\psi}_0(\cdot - a)\rangle_{\mathbb{C}} - 2\langle \mathfrak{u}_{c_p}', \widehat{\psi}_0(\cdot - a) - \mathfrak{u}_{c_p}\rangle_{\mathbb{C}}.$$

Invoking Lemma A.2, we can decrease the value of the number α if necessary, so that

$$\big\|\widehat{\psi}_0\big\|_{L^\infty}^2 \leqslant 1 + \big\|\mathfrak{u}_{c_p}\big\|_{L^\infty}^2. \tag{3.33}$$

Moreover, it follows from (1.1) and (2.8) that

$$\big|\mathfrak{u}_{c_p}'\big|^2 = \frac{\eta_{c_p}^2}{2} \leqslant \frac{2 - c_p^2}{4}\eta_{c_p}. \tag{3.34}$$

Hence, we are led to

$$\begin{aligned}\big\|\eta_0'(\cdot - a) - \eta_{c_p}'\big\|_{L^2} \leqslant 2\big(1 + \big\|\mathfrak{u}_{c_p}\big\|_{L^\infty}^2\big)^{\frac{1}{2}} \big\|\widehat{\psi}_0'(\cdot - a) - \mathfrak{u}_{c_p}'\big\|_{L^2} \\ + \big(2 - c_p^2\big)^{\frac{1}{2}} \Big\|\eta_{c_p}^{\frac{1}{2}}\big(\widehat{\psi}_0(\cdot - a) - \mathfrak{u}_{c_p}\big)\Big\|_{L^2},\end{aligned}$$

and there exists a positive number C_p, depending only on p, such that

$$\big\|\eta_0'(\cdot - a) - \eta_{c_p}'\big\|_{L^2} \leqslant C_p\, d_{c_p}\big(e^{i\theta}\widehat{\psi}_0(\cdot - a), \mathfrak{u}_{c_p}\big) < 2C_p\alpha. \tag{3.35}$$

Similarly, we write

$$\begin{aligned}v_0(\cdot - a) - v_{c_p} = \frac{1}{|\psi_0(\cdot - a)|^2}\Bigg(\big\langle i(\mathfrak{u}_{c_p}' - \widehat{\psi}_0(\cdot - a)'), \widehat{\psi}_0(\cdot - a)\big\rangle_{\mathbb{C}} \\ + \big\langle i\mathfrak{u}_{c_p}', \mathfrak{u}_{c_p} - \widehat{\psi}_0(\cdot - a)\big\rangle_{\mathbb{C}} \\ + \frac{\big\langle i\mathfrak{u}_{c_p}', \mathfrak{u}_{c_p}\big\rangle_{\mathbb{C}}}{|\mathfrak{u}_{c_p}|^2}\big(\eta_{c_p} - \eta_0(\cdot - a)\big)\Bigg).\end{aligned}$$

Invoking again Lemma A.2 and using (1.2), we can decrease the value of the number α if necessary, so that

$$\inf_{x \in \mathbb{R}} |\widehat{\psi}_0(x - a)|^2 \geqslant \inf_{x \in \mathbb{R}} |\mathfrak{u}_{c_p}(x)|^2 - \frac{c^2}{4} = \frac{c^2}{4}.$$

Combining with (3.33) and (3.34), we deduce that

$$\big\|v_0(\cdot - a) - v_{c_p}\big\|_{L^2} \leqslant C_p\, d_{c_p}\big(e^{i\theta}\widehat{\psi}_0(\cdot - a), \mathfrak{u}_{c_p}\big) < 2C_p\alpha,$$

for a further positive number C_p. In view of (3.32) and (3.35), we conclude that

$$\big\|\eta_0(\cdot - a) - \eta_{c_p}\big\|_{H^1}^2 + \big\|v_0(\cdot - a) - v_{c_p}\big\|_{L^2}^2 < (4 + 8C_p^2)\alpha^2.$$

It is then enough to fix the choice of $\alpha \leqslant \beta/(4+8C_p)^{1/2}$ in order to complete the proof of Lemma 3.5. □

The sets $\mathcal{U}_p(\beta)$ were already introduced in [4] in order to prove the orbital stability of chains of N solitons. All the results in [4] are stated for an arbitrary integer $N \geqslant 1$, and in particular, hold for a single soliton. We now explicit the results in [4] on which we rely for completing the proof of Proposition 2.3.

We begin by [4, Proposition 2], which provides a decomposition of each pair (η_0, v_0) in $\mathcal{U}_p(\beta)$ as the sum of a modulated soliton plus a remainder term satisfying suitable orthogonality conditions. More precisely, we can rephrase this proposition as

Lemma 3.6 ([4]). — *There exist two positive numbers β_1 and C_1, depending only on c_p, and two functions $\mathfrak{a} \in \mathcal{C}^1(\mathcal{U}_p(\beta_1), \mathbb{R})$ and $\mathfrak{c} \in \mathcal{C}^1(\mathcal{U}_p(\beta_1), (-\sqrt{2}, 0) \cup (0, \sqrt{2}))$ such that, for any pair $(\eta_0, v_0) \in \mathcal{U}_p(\beta_1)$, the function*

$$\varepsilon := (\varepsilon_\eta, \varepsilon_v) := \big(\eta_0(\,\cdot - a) - \eta_c, v_0(\,\cdot - a) - v_c\big), \tag{3.36}$$

with $a := \mathfrak{a}(\eta_0, v_0)$ and $c := \mathfrak{c}(\eta_0, v_0)$, satisfies the orthogonality conditions

$$\big\langle (\varepsilon_\eta, \varepsilon_v), (\eta_c', v_c') \big\rangle_{L^2 \times L^2} = \mathrm{d}P(\eta_c, v_c)(\varepsilon_\eta, \varepsilon_v) = 0. \tag{3.37}$$

Moreover, if there exist numbers $a_* \in \mathbb{R}$ and $\beta \leqslant \beta_1$ such that

$$\|(\eta_0(\,\cdot - a_*), v_0(\,\cdot - a_*)) - (\eta_{c_p}, v_{c_p})\|_{H^1 \times L^2} < \beta,$$

then

$$\big\|\varepsilon\big\|_{H^1 \times L^2} + \big|c - c_p\big| + \big|a - a_*\big| \leqslant C_1 \beta. \tag{3.38}$$

We use the decomposition in Lemma 3.6 to expand the quantities $E(\eta_0, v_0)$ and $P(\eta_0, v_0)$ at second order. Using the invariance by translation of the energy $E(\eta_0, v_0)$, we first obtain

$$\begin{aligned} E\big(\eta_0, v_0\big) &= E\big((\eta_c, v_c) + \varepsilon\big) \\ &= E\big(\eta_c, v_c\big) + \mathrm{d}E\big(\eta_c, v_c\big)(\varepsilon) + \frac{1}{2}\mathrm{d}^2 E\big(\eta_c, v_c\big)(\varepsilon, \varepsilon) + R_c(\varepsilon), \end{aligned} \tag{3.39}$$

with $c = \mathfrak{c}(\eta_0, v_0)$. In this identity, we have set

$$\mathrm{d}E\big(\eta_c, v_c\big)(\varepsilon) := \frac{1}{2} \int_{\mathbb{R}} \Big(\frac{(\eta_c')^2 \varepsilon_\eta}{4(1 - \eta_c)^2} + \frac{\eta_c' \varepsilon_\eta'}{2(1 - \eta_c)} - v_c^2 \varepsilon_\eta + 2(1 - \eta_c) v_c \varepsilon_v + \eta_c \varepsilon_\eta \Big),$$

$$\begin{aligned} \mathrm{d}^2 E\big(\eta_c, v_c\big)(\varepsilon, \varepsilon) := \int_{\mathbb{R}} \Big(\frac{(\varepsilon_\eta')^2}{4(1 - \eta_c)} + \frac{\eta_c' \varepsilon_\eta \varepsilon_\eta'}{2(1 - \eta_c)^2} + \frac{(\eta_c')^2 \varepsilon_\eta^2}{4(1 - \eta_c)^3} \\ - 2 v_c \varepsilon_\eta \varepsilon_v + (1 - \eta_c) \varepsilon_v^2 + \frac{1}{2} \varepsilon_\eta^2 \Big), \end{aligned}$$

and

$$R_c(\varepsilon) := \frac{1}{2} \int_{\mathbb{R}} \bigg(\frac{(\varepsilon_\eta')^2 \varepsilon_\eta}{4(1-\eta_c)(1-\eta_c-\varepsilon_\eta)} + \frac{\eta_c' \varepsilon_\eta^2 \varepsilon_\eta'}{2(1-\eta_c)^2(1-\eta_c-\varepsilon_\eta)} + \frac{(\eta_c')^2 \varepsilon_\eta^3}{4(1-\eta_c)^3(1-\eta_c-\varepsilon_\eta)} - \varepsilon_\eta \varepsilon_v^2 \bigg).$$

Similarly, the invariance by translation of the momentum $P(\eta_0, v_0)$ provides

$$P\big(\eta_0, v_0\big) = P\big((\eta_c, v_c) + \varepsilon\big) = P\big(\eta_c, v_c\big) + \mathrm{d}P\big(\eta_c, v_c\big)(\varepsilon) + \frac{1}{2}\mathrm{d}^2 P\big(\eta_c, v_c\big)(\varepsilon, \varepsilon), \quad (3.40)$$

with

$$\mathrm{d}P\big(\eta_c, v_c\big)(\varepsilon) := \frac{1}{2} \int_{\mathbb{R}} \big(\eta_c \varepsilon_v + v_c \varepsilon_\eta\big), \quad \text{and} \quad \mathrm{d}^2 P\big(\eta_c, v_c\big)(\varepsilon, \varepsilon) := \int_{\mathbb{R}} \varepsilon_\eta \varepsilon_v. \quad (3.41)$$

The previous identities give an expansion at second order of the quantity $E(\eta_0, v_0) - c_p P(\eta_0, v_0)$. We now estimate each term in this expansion in order to bound from below this quantity.

Lemma 3.7. — *Consider a function $(\eta_0, v_0) \in \mathcal{U}_p(\beta_1)$, where β_1 is the positive number in Lemma 3.6, and set $\varepsilon = \big(\eta_0(\,\cdot - a) - \eta_c, v_0(\,\cdot - a) - v_c\big)$, with $a = \mathfrak{a}(\eta_0, v_0)$ and $c = \mathfrak{c}(\eta_0, v_0)$. There exist two positive numbers $\beta_2 \leqslant \beta_1$ and K_2, depending only on c_p, such that*

$$E\big(\eta_c, v_c\big) - c_p P\big(\eta_c, v_c\big) \geqslant E\big(\mathfrak{u}_{c_p}\big) - c_p P\big(\mathfrak{u}_{c_p}\big) - K_2 \big|c - c_p\big|^2, \quad (3.42)$$

$$\mathrm{d}E\big(\eta_c, v_c\big)(\varepsilon) - c_p\, \mathrm{d}P\big(\eta_c, v_c\big)(\varepsilon) = 0, \quad (3.43)$$

$$\mathrm{d}^2 E\big(\eta_c, v_c\big)(\varepsilon, \varepsilon) - c_p\, \mathrm{d}^2 P\big(\eta_c, v_c\big)(\varepsilon, \varepsilon) \geqslant K_2 \Big(\big\|\varepsilon\big\|_{H^1 \times L^2}^2 - \big|c - c_p\big|^2 \Big), \quad (3.44)$$

and

$$R_c(\varepsilon) \geqslant -K_2 \big\|\varepsilon\big\|_{H^1 \times L^2}^3, \quad (3.45)$$

when $(\eta_0, v_0) \in \mathcal{U}_p(\beta_2)$.

Proof. — Concerning (3.42), recall that the modulated speed c lies in $(-\sqrt{2}, 0) \cup (0, \sqrt{2})$ by Lemma 3.6. Hence, it follows from [1, Proposition 1] that the energy $E(\eta_c, v_c)$ and the momentum $P(\eta_c, v_c)$ are given by

$$E(\eta_c, v_c) = \frac{1}{3}\big(2 - c^2\big)^{\frac{3}{2}}, \quad \text{and} \quad P(\eta_c, v_c) = \operatorname{sign}(c)\, \Xi(|c|), \quad (3.46)$$

with

$$\Xi(c) := \frac{\pi}{2} - \arctan\bigg(\frac{c}{\sqrt{2 - c^2}}\bigg) - \frac{c}{2}\sqrt{2 - c^2}, \quad (3.47)$$

for $0 \leqslant c < \sqrt{2}$. In view of (3.38), we can decrease if necessary the value of the number β_1 such that all the modulated speeds c corresponding to pairs in $\mathcal{U}_p(\beta_1)$ are in a compact subset of the interval, either $(-\pi/2, 0)$, or $(0, \pi/2)$, containing the speed c_p. In this case, we can use the smoothness of the maps $c \mapsto E(\eta_c, v_c)$ and $c \mapsto P(\eta_c, v_c)$ on both these intervals in order to find a positive number K, depending only on c_p, such that

$$\begin{aligned} E\big(\eta_c, v_c\big) - c_p P\big(\eta_c, v_c\big) - \Big(E\big(\eta_{c_p}, v_{c_p}\big) - c_p P\big(\eta_{c_p}, v_{c_p}\big)\Big) \\ \geqslant \frac{\mathrm{d}}{\mathrm{d}c}\Big(E\big(\eta_c, v_c\big)\Big)_{|c=c_p} - c_p \frac{\mathrm{d}}{\mathrm{d}c}\Big(P\big(\eta_c, v_c\big)\Big)_{|c=c_p} - K\big(c - c_p\big)^2. \end{aligned}$$

Since $E(\eta_{c_p}, v_{c_p}) - c_p P(\eta_{c_p}, v_{c_p}) = E(\mathfrak{u}_{c_p}) - c_p P(\mathfrak{u}_{c_p})$ by definition, the estimate in (3.42) follows from the property that

$$\frac{\mathrm{d}}{\mathrm{d}c}\Big(E\big(\eta_c, v_c\big)\Big)_{|c=c_p} = -c_p\big(2 - c_p^2\big)^{\frac{1}{2}} = c_p \frac{\mathrm{d}}{\mathrm{d}c}\Big(P\big(\eta_c, v_c\big)\Big)_{|c=c_p},$$

which results from the fact that

$$\Xi'(c) = -\sqrt{2 - c^2}. \tag{3.48}$$

For the proof of (3.43), we first use the second orthogonality condition in (3.37) in order to write

$$\begin{aligned} \mathrm{d}E\big(\eta_c, v_c\big)(\varepsilon) - c_p\, \mathrm{d}P\big(\eta_c, v_c\big)(\varepsilon) &= \mathrm{d}E\big(\eta_c, v_c\big)(\varepsilon) \\ &= \mathrm{d}E\big(\eta_c, v_c\big)(\varepsilon) - c\, \mathrm{d}P\big(\eta_c, v_c\big)(\varepsilon). \end{aligned}$$

We next rephrase the equation satisfied by the profile $\mathfrak{u}_c$ in terms of the hydrodynamic pair (η_c, v_c). In view of (1.1), we are led to the system

$$\begin{cases} \dfrac{\eta_c''}{2(1-\eta_c)} + \dfrac{(\eta_c')^2}{4(1-\eta_c)^2} + c v_c + v_c^2 - \eta_c = 0, \\ (1-\eta_c) v_c = \dfrac{c}{2}\eta_c. \end{cases}$$

It is then enough to multiply the first equation in this system by ε_η, the second one by ε_v, and to integrate by parts in order to obtain

$$\mathrm{d}E\big(\eta_c, v_c\big)(\varepsilon) - c\, \mathrm{d}P\big(\eta_c, v_c\big)(\varepsilon) = 0,$$

and therefore, (3.43).

We now turn to (3.44). We rewrite the second order term as

$$\begin{aligned} &\mathrm{d}^2 E\big(\eta_c, v_c\big)(\varepsilon, \varepsilon) - c_p\, \mathrm{d}^2 P\big(\eta_c, v_c\big)(\varepsilon, \varepsilon) \\ &= \mathrm{d}^2 E\big(\eta_c, v_c\big)(\varepsilon, \varepsilon) - c\, \mathrm{d}^2 P\big(\eta_c, v_c\big)(\varepsilon, \varepsilon) + \big(c - c_p\big) \mathrm{d}^2 P\big(\eta_c, v_c\big)(\varepsilon, \varepsilon). \end{aligned} \tag{3.49}$$

In view of (3.41), we have

$$\begin{aligned} \big(c - c_p\big) \mathrm{d}^2 P\big(\eta_c, v_c\big)(\varepsilon, \varepsilon) &\geqslant -\big|c - c_p\big| \big\|\varepsilon\big\|_{H^1 \times L^2}^2 \\ &\geqslant -\frac{1}{2\delta}\big(c - c_p\big)^2 - \frac{\delta}{2}\big\|\varepsilon\big\|_{H^1 \times L^2}^4, \end{aligned} \tag{3.50}$$

for any positive number δ. Recall that the function ε satisfies the two orthogonal conditions in (3.37), whereas by (3.38), the modulated speeds c lie in a compact subset of the interval $(-\pi/2, 0)$ or $(0, \pi/2)$, containing the speed c_p. As a consequence, we can apply [4, Proposition 1] in order to find a positive number K, depending only on c_p, such that

$$\mathrm{d}^2 E\big(\eta_c, v_c\big)(\varepsilon, \varepsilon) - c\, \mathrm{d}^2 P\big(\eta_c, v_c\big)(\varepsilon, \varepsilon) \geqslant K \big\|\varepsilon\big\|_{H^1 \times L^2}^2.$$

Combining with (3.49) and (3.50), we obtain

$$\begin{aligned} &\mathrm{d}^2 E\big(\eta_c, v_c\big)(\varepsilon, \varepsilon) - c_p\, \mathrm{d}^2 P\big(\eta_c, v_c\big)(\varepsilon, \varepsilon) \\ &\qquad\qquad \geqslant K \big\|\varepsilon\big\|_{H^1 \times L^2}^2 - \frac{1}{2\delta}\big(c - c_p\big)^2 - \frac{\delta}{2}\big\|\varepsilon\big\|_{H^1 \times L^2}^4. \end{aligned}$$

At this stage, we can decrease if necessary the value of the number β_2 so that $\|\varepsilon\|_{H^1 \times L^2} \leqslant 1$ by (3.38). It is then enough to choose $\delta = K/2$ in order to obtain (3.44).

Finally, the estimate in (3.45) essentially results from the Sobolev embedding theorem. In view of (2.8), there indeed exists a positive number $\kappa \leqslant 1$, depending only on c_p, such that

$$1 - \eta_c \geqslant \kappa,$$

for any modulated speed c in a compact subset of either $(-\pi/2, 0)$, or $(0, \pi/2)$, containing c_p. Decreasing if necessary the value of the number β_2, we deduce from (3.38) and the Sobolev embedding theorem that

$$1 - \eta_c - \varepsilon_\eta \geqslant \frac{\kappa}{2}.$$

In view of (2.8), the derivative η_c' is also uniformly bounded by a positive number depending only on c_p. Using once again the Sobolev embedding theorem, we are led to

$$R_c(\varepsilon) \geqslant -\frac{K}{\kappa^4}\big\|\varepsilon\big\|_{H^1 \times L^2}^3,$$

where, as before, K only depends on c_p. This completes the proof of (3.45), as well as of Lemma 3.7. □

We are now in position to conclude the proof of Proposition 2.3 when $p \neq \pi/2$.

End of the proof of Proposition 2.3. — Going back to (3.39) and (3.40) and invoking Lemma 3.7, we can write

$$\begin{aligned} &E\big(\widehat{\psi}_0\big) - c_p P\big(\widehat{\psi}_0\big) = E\big(\eta_0, v_0\big) - c_p P\big(\eta_0, v_0\big) \\ &\geqslant E\big(\mathfrak{u}_{c_p}\big) - c_p P\big(\mathfrak{u}_{c_p}\big) + K_2\Big(\big\|\varepsilon\big\|_{H^1 \times L^2}^2 - \big\|\varepsilon\big\|_{H^1 \times L^2}^3 - 2\big|c - c_p\big|^2\Big). \quad (3.51) \end{aligned}$$

In order to estimate the difference $c - c_p$, we rely on the formula in (3.46) for the momentum $P(\eta_c, v_c)$. Since the modulated speed c lives in a compact subset containing c_p by (3.38), we infer from (3.46) and (3.48) the existence of a positive number K, depending only on c_p, such that

$$\big|c - c_p\big| \leqslant K\big|P(\eta_c, v_c) - P(\eta_{c_p}, v_{c_p})\big|. \tag{3.52}$$

Combining (3.40) with (3.37) and (3.41), we check that

$$\big|P(\eta_c, v_c) - P(\eta_0, v_0)\big| \leqslant \frac{1}{2}\big\|\varepsilon\big\|_{H^1\times L^2}^2.$$

On the other hand, it follows from (3.27) and (3.28) that

$$\big|P(\eta_{c_p}, v_{c_p}) - P(\eta_0, v_0)\big| = \big|p - P(\widehat{\psi}_0)\big| \leqslant \frac{1}{4}\big\|\nabla w_0\big\|_{L^2}^2.$$

Hence, we obtain

$$\big|c - c_p\big| \leqslant \frac{K}{2}\Big(\big\|\varepsilon\big\|_{H^1\times L^2}^2 + \big\|\nabla w_0\big\|_{L^2}^2\Big).$$

Introducing this inequality into (3.51), we are led to

$$\begin{aligned} E\big(\widehat{\psi}_0\big) - c_p P\big(\widehat{\psi}_0\big) \geqslant{} & E\big(\mathfrak{u}_{c_p}\big) - c_p P\big(\mathfrak{u}_{c_p}\big) + K_2\big\|\varepsilon\big\|_{H^1\times L^2}^2 \\ & - K_2\big\|\varepsilon\big\|_{H^1\times L^2}^3 - K^2\big\|\varepsilon\big\|_{H^1\times L^2}^4 - K^2\big\|\nabla w_0\big\|_{L^2}^4. \end{aligned}$$

At this stage, we can again decrease the value of the number β_2 so that (3.38) provides the inequality

$$K_2\big\|\varepsilon\big\|_{H^1\times L^2}^2 - K_2\big\|\varepsilon\big\|_{H^1\times L^2}^3 - K^2\big\|\varepsilon\big\|_{H^1\times L^2}^4 \geqslant \frac{K_2}{2}\big\|\varepsilon\big\|_{H^1\times L^2}^2.$$

As a consequence, we obtain

$$E\big(\widehat{\psi}_0\big) - c_p P\big(\widehat{\psi}_0\big) \geqslant E\big(\mathfrak{u}_{c_p}\big) - c_p P\big(\mathfrak{u}_{c_p}\big) + \frac{K_2}{2}\big\|\varepsilon\big\|_{H^1\times L^2}^2 - K^2\big\|\nabla w_0\big\|_{L^2}^4.$$

We next invoke Lemma 3.5 in order to find a number α such that $(\eta_0, v_0) \in \mathcal{U}_p(\beta_2)$ when $\psi \in \mathcal{V}_p(\alpha)$. In this case, we derive from (2.9) and (3.30) that

$$\begin{aligned} & E_\lambda\big(\psi\big) - c_p P\big(\psi\big) \\ & \geqslant E\big(\mathfrak{u}_{c_p}\big) - c_p P\big(\mathfrak{u}_{c_p}\big) + \frac{K_2}{2}\big\|\varepsilon\big\|_{H^1\times L^2}^2 + \frac{1}{2}\Big(1 - \frac{|c_p|}{2} - 2K^2\alpha^2\Big)\big\|\partial_x w_0\big\|_{L^2}^2 \\ & \qquad + \frac{1}{2}\Big(\lambda^2 - C_p - \frac{|c_p|}{\pi} - 2K^2\alpha^2\Big)\big\|\partial_y w_0\big\|_{L^2}^2 + \frac{1}{2}\big\|w_0\big\|_{L^2}^2. \end{aligned}$$

We finally fix the choice of the number α_p so that $1 - \sqrt{2}/2 - 2K^2\alpha_p^2 > 0$, and the choice of the number λ_p so that $\lambda_p^2 - C_p - \sqrt{2}/\pi - 2K^2\alpha_p^2 > 0$. The previous choices guarantee that

$$E_\lambda\big(\psi\big) - c_p P\big(\psi\big) \geqslant E\big(\mathfrak{u}_{c_p}\big) - c_p P\big(\mathfrak{u}_{c_p}\big),$$

when $\psi \in \mathcal{V}_p(\alpha_p)$ and $\lambda \geqslant \lambda_p$. This inequality is exactly (2.11) due to the facts that $P(\psi) = P(\mathfrak{u}_{c_p}) = p$ and $E(\mathfrak{u}_{c_p}) = E_\lambda(\mathfrak{u}_{c_p})$. Moreover, equality holds if and only if

$$\|\varepsilon\|_{H^1\times L^2} = \|w_0\|_{H^1} = 0.$$

In this case, we observe that $(\eta_0, v_0) = (\eta_c(\,\cdot + a), v_c(\,\cdot + a))$, so that there exists a number $\theta \in \mathbb{R}$ for which $\widehat{\psi}_0 = e^{-i\theta}\mathfrak{u}_c(\,\cdot + a)$. As a consequence, we have

$$\psi = \widehat{\psi}_0 + w_0 = e^{-i\theta}\mathfrak{u}_c(\,\cdot + a) + 0 = e^{-i\theta}\mathfrak{u}_c(\,\cdot + a).$$

Since $p = P(\psi)$ by Lemma 3.4, we deduce that $P(\mathfrak{u}_c) = p$, and we conclude that $c = c_p$. This completes the proof of Proposition 2.3 for $p \neq \pi/2$. □

3.5. Proof of Proposition 2.3 for $p = \frac{\pi}{2}$

For $p = \pi/2$, Proposition 2.3 also relies on a coercivity estimate, but for the black soliton $\mathfrak{u}_0$. This estimate was derived in [10, Proposition 1] for revisiting the orbital stability of $\mathfrak{u}_0$. We can rephrase it as

Lemma 3.8 ([10]). — *For $\psi = \mathfrak{u}_0 + \varepsilon \in X(\mathbb{R})$, set $\eta_\varepsilon := -2\langle \mathfrak{u}_0, \varepsilon\rangle_{\mathbb{C}} - |\varepsilon|^2$. There exists a universal positive number Λ_0 such that*

$$E(\psi) - E(\mathfrak{u}_0) \geqslant \Lambda_0\left(\|\varepsilon\|_{H_0}^2 + \|\eta_\varepsilon\|_{L^2}^2\right) - \frac{1}{\Lambda_0}\|\varepsilon\|_{H_0}^3, \tag{3.53}$$

as soon as

$$\int_{\mathbb{R}} \langle \varepsilon, \mathfrak{u}_0'\rangle_{\mathbb{C}} = \int_{\mathbb{R}} \langle \varepsilon, i\,\mathfrak{u}_0'\rangle_{\mathbb{C}} = \int_{\mathbb{R}} \langle \varepsilon, i\,\mathfrak{u}_0\rangle_{\mathbb{C}}\big(1 - |\mathfrak{u}_0|^2\big) = 0. \tag{3.54}$$

The orthogonality conditions in (3.54) are necessary to control one negative and two null directions of the energy E in the neighbourhood of the black soliton $\mathfrak{u}_0$. As in Lemma 3.6, they can be imposed by introducing suitable modulation parameters related to the speed of the solitons and their invariance by translation and phase shift. These properties were already invoked for constructing modulation parameters in [10, Proposition 2]. Setting

$$\mathfrak{U}_0(\beta) := \Big\{\psi \in X(\mathbb{R}) \text{ s.t. } \inf_{(a,\theta)\in\mathbb{R}^2} d_0\big(e^{i\theta}\psi(\,\cdot - a), \mathfrak{u}_0\big) < \beta\Big\},$$

for any positive number β, we can summarize this construction as follows.

Lemma 3.9 ([10]). — *There exist two positive numbers β_0 and A_0, and three continuously differentiable functions $\mathfrak{a} \in \mathcal{C}^1(\mathfrak{U}_0(\beta_0), \mathbb{R})$, $\vartheta \in \mathcal{C}^1(\mathfrak{U}_0(\beta_0), \mathbb{R}/2\pi\mathbb{Z})$ and $\mathfrak{c} \in \mathcal{C}^1(\mathfrak{U}_0(\beta_0), (-\sqrt{2}, \sqrt{2}))$ such that for any $\psi \in \mathfrak{U}_0(\beta_0)$, the function*

$$\varepsilon := e^{i\theta}\psi(\,\cdot - a) - \mathfrak{u}_c,$$

with $a = \mathfrak{a}(\psi)$, $\theta = \vartheta(\psi)$ *and* $c = \mathfrak{c}(\psi)$, *satisfies the orthogonality conditions*

$$\int_{\mathbb{R}} \langle \varepsilon, \mathfrak{u}_c' \rangle_{\mathbb{C}} = \int_{\mathbb{R}} \langle \varepsilon, i\, \mathfrak{u}_c' \rangle_{\mathbb{C}} = \int_{\mathbb{R}} \langle \varepsilon, i \ \mathrm{Re}(\mathfrak{u}_c) \rangle_{\mathbb{C}} \big(1 - |\mathfrak{u}_c|^2\big) = 0. \tag{3.55}$$

Moreover, if

$$\big\| e^{i\theta_*}\, \psi\big(\cdot - a_*\big) - \mathfrak{u}_0 \big\|_{H_0} \leqslant \beta,$$

for numbers $a_* \in \mathbb{R}$, $\theta_* \in \mathbb{R}$ *and* $\beta \leqslant \beta_0$, *then,*

$$\|\varepsilon\|_{H_0} + |c| + |a - a_*| + \big|e^{i\theta} - e^{i\theta_*}\big| \leqslant A_0 \beta. \tag{3.56}$$

Remark 3.10. — Here, the smoothness of the maps $\mathfrak{a}$, ϑ and $\mathfrak{c}$ must be understood with respect to the differential structure provided by the vector space $H(\mathbb{R})$.

The orthogonality conditions in (3.55) differ from the ones in (3.54). However, a coercivity estimate similar to (3.53) remains available under these latest conditions. Corollary 1 in [10] indeed guarantees that

Lemma 3.11 ([10]). — *For* $|c| < \sqrt{2}$ *and* $\psi = \mathfrak{u}_c + \varepsilon \in X(\mathbb{R})$, *set* $\eta_\varepsilon := -2\langle \mathfrak{u}_c, \varepsilon \rangle_{\mathbb{C}} - |\varepsilon|^2$. *Given any number* $0 < \sigma < \sqrt{2}$, *there exists a positive number* Λ_σ, *depending only on* σ, *such that*

$$E(\psi) - E(\mathfrak{u}_0) \geqslant \Lambda_\sigma \left(\|\varepsilon\|_{H_0}^2 + \|\eta_\varepsilon\|_{L^2}^2 \right) - \frac{1}{\Lambda_\sigma} \left(c^2 + \|\varepsilon\|_{H_0}^3 \right), \tag{3.57}$$

as soon as $|c| \leqslant \sigma$, *and* ε *satisfies the orthogonality conditions in* (3.55).

At this stage, consider a function $\psi \in \mathcal{V}_{\pi/2}(\alpha)$ for a number $0 < \alpha < \beta_0$. By definition, the function $\widehat{\psi}_0$ is in the subset $\mathfrak{U}_0(\alpha)$ of $\mathfrak{U}_0(\beta_0)$. Applying Lemma 3.9, we can find numbers $a_0 \in \mathbb{R}$, $\theta_0 \in \mathbb{R}$ and $c_0 \in (-\sqrt{2}, \sqrt{2})$ such that the function $\varepsilon_0 := e^{i\theta_0} \widehat{\psi}_0(\cdot - a_0) - \mathfrak{u}_{c_0}$ satisfies the orthogonality conditions in (3.55). Combining (3.56) and (3.57), and decreasing if necessary the value of the number α, we find a positive number Λ_α, depending only on α, such that

$$E\big(\widehat{\psi}_0\big) - E\big(\mathfrak{u}_0\big) \geqslant \Lambda_\alpha \left(\|\varepsilon_0\|_{H_0}^2 + \|\eta_{\varepsilon_0}\|_{L^2}^2 \right) - \frac{c_0^2}{\Lambda_\alpha},$$

with $\eta_{\varepsilon_0} := -2\langle \mathfrak{u}_{c_0}, \varepsilon_0 \rangle_{\mathbb{C}} - |\varepsilon_0|^2$ as before. Assuming that $\alpha \leqslant \alpha_{\pi/2}$, where the number $\alpha_{\pi/2}$ is given by Lemma 3.3, we infer from this lemma that

$$\begin{aligned} E_\lambda\big(\psi\big) \geqslant E_\lambda\big(\mathfrak{u}_0\big) + \frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}} \Big(|\partial_x w_0|^2 + (\lambda^2 - C_p) |\partial_y w_0|^2 + |w_0|^2 \Big) \\ + \Lambda_\alpha \left(\|\varepsilon_0\|_{H_0}^2 + \|\eta_{\varepsilon_0}\|_{L^2}^2 \right) - \frac{c_0^2}{\Lambda_\alpha}. \end{aligned} \tag{3.58}$$

As a consequence, we are essentially reduced to control the modulated speed c_0 with respect to the various norms of the functions w_0, ε_0 and η_{ε_0}. As in the previous case $p \neq \pi/2$, we derive this control from the property that $[P](\psi) = \pi/2$ modulo π. In this direction, our main tool is the following consequence of Propositions 4 and 5 in [10].

Lemma 3.12 ([10]). — *There exist two positive numbers $\beta_1 < \beta_0$ and A_1 such that any function $\psi \in \mathfrak{U}_0(\beta_1)$ satisfies*

$$[P](\psi) = [P](\mathfrak{u}_c) - \int_{\mathbb{R}} \langle i\mathfrak{u}_c', \varepsilon \rangle_{\mathbb{C}} + R_c(\varepsilon) \mod \pi, \tag{3.59}$$

with

$$\big|R_c(\varepsilon)\big| \leqslant A_1 \Big(\|\varepsilon\|_{H_0}^2 + \|\eta_\varepsilon\|_{L^2}^2 \Big). \tag{3.60}$$

In the previous formulae, we have set, as before, $\varepsilon = e^{i\theta}\psi(\,\cdot - a) - \mathfrak{u}_c$, with $a = \mathfrak{a}(\psi)$, $\theta = \vartheta(\psi)$ and $c = \mathfrak{c}(\psi)$, as well as $\eta_\varepsilon := -2\langle \mathfrak{u}_c, \varepsilon \rangle_{\mathbb{C}} - |\varepsilon|^2$.

With Lemma 3.12 at hand, we are in position to conclude the proof of Proposition 2.3 for $p = \pi/2$.

End of the proof of Proposition 2.3 for $p = \pi/2$. — Decreasing if necessary the value of α, we can apply Lemma 3.12 to the function $\widehat{\psi}_0$. In view of the second orthogonality condition in (3.55), this provides the identity

$$[P]\big(\widehat{\psi}_0\big) = [P]\big(\mathfrak{u}_{c_0}\big) + R_{c_0}\big(\varepsilon_0\big) \mod \pi,$$

with $R_{c_0}(\varepsilon_0)$ satisfying (3.60) for $\varepsilon = \varepsilon_0$ and $\eta_\varepsilon = \eta_{\varepsilon_0}$. Going to (C.1), we deduce that

$$[P]\big(\psi\big) - [P]\big(\mathfrak{u}_{c_0}\big) = \frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}} \langle i\partial_x w_0, w_0 \rangle_{\mathbb{C}} + R_{c_0}\big(\varepsilon_0\big) \mod \pi. \tag{3.61}$$

Recall now that $[P](\psi) = \pi/2 = \Xi(0)$ modulo π, while the value modulo π of $[P](\mathfrak{u}_{c_0})$ is equal to $\operatorname{sign}(c_0)\, \Xi(|c_0|)$ by [1, Proposition 1]. Here, Ξ refers to the function in (3.47). Moreover, for α small enough, the right-hand side of (3.61) is small by (3.60), so as the modulated speed c_0 by (3.56). As a consequence, we derive from the identity modulo π in (3.61) that

$$\big|\Xi(0) - \Xi(|c_0|)\big| = \bigg| \frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}} \langle i\partial_x w_0, w_0 \rangle_{\mathbb{C}} + R_{c_0}\big(\varepsilon_0\big) \bigg|.$$

Using (3.48), we can argue as for (3.52) in order to derive from (3.56) and (3.60) the existence of a positive number A_α, depending only on α, such that

$$|c_0| = \big|0 - |c_0|\big| \leqslant A_\alpha \bigg(\|\varepsilon_0\|_{H_0}^2 + \|\eta_{\varepsilon_0}\|_{L^2}^2 + \delta \|w_0\|_{L^2}^2 + \frac{1}{\delta} \|\partial_x w_0\|_{L^2}^2 \bigg).$$

for any positive number δ. It then remains to introduce this inequality into (3.58) and to choose the number δ large enough in order to deduce from the Poincaré inequality that

$$E_\lambda\big(\psi\big) \geqslant E_\lambda\big(\mathfrak{u}_0\big) + \Lambda\Big(\|\varepsilon_0\|_{H_0}^2 + \|\eta_{\varepsilon_0}\|_{L^2}^2 + \|w_0\|_{H^1}^2\Big) \geqslant E_\lambda\big(\mathfrak{u}_0\big),$$

for α small enough, λ large enough, and a further positive number Λ, depending only on α and λ. This concludes the proof of (2.11).

Moreover, this inequality is an equality if and only if $\varepsilon_0 = 0$ and $w_0 = 0$, that is if and only if $\psi = e^{-i\theta_0}\mathfrak{u}_{c_0}(\,\cdot + a_0)$. In view of (3.46) and (3.48), the only possibility for the untwisted momentum $[P](\psi)$ to be equal to $\pi/2$ modulo π is that $c_0 = 0$. In conclusion, equality can only hold if $\psi = e^{-i\theta_0}\mathfrak{u}_0(\,\cdot + a_0)$. This completes the proof of Proposition 2.3 for $p = \pi/2$. □

3.6. Proof of Lemma 2.4

Consider a function $\psi \in X(\mathbb{R} \times \mathbb{T})$ such that $[P](\psi) = p$ modulo π. Given two positive numbers λ_1 and λ_2, with $\lambda_1 < \lambda_2$, we have

$$E_{\lambda_1}(\psi) \leqslant E_{\lambda_2}(\psi) \leqslant \Big(\frac{\lambda_2}{\lambda_1}\Big)^2 E_{\lambda_1}(\psi).$$

In view of (1.10), we obtain

$$\mathcal{I}_{\lambda_1}(p) \leqslant \mathcal{I}_{\lambda_2}(p) \leqslant \Big(\frac{\lambda_2}{\lambda_1}\Big)^2 \mathcal{I}_{\lambda_1}(p),$$

which is enough to guarantee that the map $\lambda \mapsto \mathcal{I}_\lambda(p)$ is non-decreasing and continuous on $\mathbb{R}_+^*$.

Concerning the proof of (2.12), we rely on the scaling

$$\psi_L(x, y) = \psi(x, \lambda y), \tag{3.62}$$

which transforms a function $\psi \in X(\mathbb{R} \times \mathbb{T})$ in a function $\psi_L \in X(\mathbb{R} \times \mathbb{T}_L)$. Here, we have set $L = 1/\lambda$. The notation $\mathbb{T}_L$ refers to the torus of size L and the energy set $X(\mathbb{R} \times \mathbb{T}_L)$ is defined according to (1.8), with $\mathbb{T}$ replaced by $\mathbb{T}_L$. In the limit $\lambda \to 0$, the length L tends to $+\infty$ and the minimization problem $\mathcal{I}_\lambda(p)$ can be related to the problem of minimizing the Ginzburg–Landau energy in the whole plane $\mathbb{R}^2$ for a fixed large momentum.

Indeed, we can compute

$$E\big(\psi_L\big) := \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}_L} |\nabla\psi_L|^2 + \frac{1}{4}\int_{\mathbb{R}\times\mathbb{T}_L} \big(1 - |\psi_L|^2\big)^2 = L E_\lambda(\psi). \tag{3.63}$$

Going to Lemma C.1, we also check that the definition of the untwisted momentum on the set $X(\mathbb{R} \times \mathbb{T})$ extends literally to the set $X(\mathbb{R} \times \mathbb{T}_L)$, up to the fact that this quantity is now valued into $\mathbb{R}/\pi L\mathbb{Z}$. Moreover, we can derive from Lemma C.1 that the untwisted momentum $[P]_L(\psi_L)$ is equal to

$$[P]_L(\psi_L) = L[P](\psi) \mod \pi L. \tag{3.64}$$

As a consequence, we obtain

$$\mathcal{I}_\lambda(p) = \frac{1}{L} \inf\Big\{ E(\psi_L) : \psi_L \in X(\mathbb{R} \times \mathbb{T}_L) \text{ s.t. } [P]_L(\psi_L) = pL \bmod \pi L \Big\}.$$

At least formally, the previous infimum is related to the limit $q \to +\infty$ of the minimal value of the Ginzburg–Landau energy in $\mathbb{R}^2$ with fixed momentum equal to q. This latter minimization problem was solved in [2]. It follows from [6] that the limit $q \to +\infty$ of this problem is divergent as $2\pi \ln(q)$. This asymptotics is based on the property that the corresponding minimizer is a pair of vortices in uniform translation. We are now going to use this special configuration as a test function in order to show (2.12).

In order to clarify the construction, we now identify the space $\mathbb{R}^2$ to the complex plane $\mathbb{C}$ by setting $z = x + iy$ in the sequel. We introduce the complex-valued function ξ defined on the disc $D(0,2) := \{z \in \mathbb{C} \text{ s.t. } |z| < 2\}$ by

$$\xi(z) = \frac{\overline{z - i}}{|z - i|} \, \frac{z + i}{|z + i|} \, e^{i\varphi(z)}. \tag{3.65}$$

In this expression, φ refers to a real-valued harmonic function on $D(0,2)$ such that $\xi = 1$ on the circle $\partial D(0,2)$. We can check that the value of φ can be fixed so that

$$\varphi(z) = \arctan\left(\frac{2\,\mathrm{Re}(z)}{1 - |z|^2} \right), \tag{3.66}$$

for any $z \in \partial D(0,2)$. Observe that φ is even with respect to the variable $\mathrm{Im}(z)$. Observe also that f has exactly two vortices with opposite degrees at the points $\pm i$. Given a number $R \geqslant 1$, we next introduce the rescaled and regularized version ξ_R of ξ given by

$$\xi_R(z) = \begin{cases} 1 & \text{if } |z| \geqslant 2R, \\ |z \pm iR| \, \xi\big(\frac{z}{R}\big) & \text{if } |z \pm iR| < 1, \\ \xi\big(\frac{z}{R}\big) & \text{otherwise.} \end{cases} \tag{3.67}$$

The function ξ_R is well-defined and continuous on $\mathbb{R}^2$. Given a number $L \geqslant 4R$, we can consider its restriction to the set $\{z \in \mathbb{C} : |\mathrm{Im}(z)| \leqslant L/2\}$ and extend it as a L-periodic function with respect to the variable y. Denote by ψ_L the corresponding extension and define a function $\psi : \mathbb{R} \times \mathbb{T} \to \mathbb{C}$ according to the scaling in (3.62).

The extension ψ_L belongs to $H^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}_L)$, where this set is defined as in (1.6), with 1-periodic functions replaced by L-periodic functions. It is even with respect to the variable y and identically equal to 1 outside the disc $D(0,2R)$. We now estimate the value of its energy $E(\psi_L)$. A direct computation first provides

$$\frac{1}{4}\int_{\mathbb{R}\times\mathbb{T}_L}\big(1-|\psi_L|^2\big)^2=\frac{1}{2}\int_{|z-iR|<1}\big(1-|z-iR|^2\big)^2\,\mathrm{d}x\,\mathrm{d}y=\frac{\pi}{6}. \tag{3.68}$$

Concerning the gradient $\nabla\psi_L$, we next check that

$$\begin{aligned}|\nabla\psi_L(z)|^2=1+\frac{4R^2}{|z+iR|^2}+\frac{1}{R^2}\Big|\nabla\varphi\Big(\frac{z}{R}\Big)\Big|^2|z-iR|^2\\ -\frac{2}{R}\partial_x\varphi\Big(\frac{z}{R}\Big)\Big(R-y+(y+R)\frac{|z-iR|^2}{|z+iR|^2}\Big)\\ -\frac{2}{R}\partial_y\varphi\Big(\frac{z}{R}\Big)\Big(x-x\frac{|z-iR|^2}{|z+iR|^2}\Big),\end{aligned}$$

for any $|z-iR|<1$. Using the inequality $2ab\leqslant a^2+b^2$ and the fact that $|z+iR|\geqslant R\geqslant 1$ for $y\geqslant 0$, we can bound this quantity by

$$|\nabla\psi_L(z)|^2\leqslant 13+\frac{2}{R^2}\Big|\nabla\varphi\Big(\frac{x}{R}\Big)\Big|^2,$$

when $|z-iR|<1$. Hence, we obtain

$$\frac{1}{2}\int_{|z-iR|<1}|\nabla\psi_L|^2\leqslant\frac{13\pi}{2}+\int_{|z-i|<1/R}|\nabla\varphi|^2. \tag{3.69}$$

By symmetry with respect to the axis x, the same inequality is true replacing $|z-iR|<1$ by $|z+iR|<1$ in the left-hand side, and $|z-i|<1/R$ by $|z+i|<1/R$ in the right-hand side. Similarly, we compute

$$\begin{aligned}|\nabla\psi_L(z)|^2=\frac{1}{|z-iR|^2}+\frac{1}{|z+iR|^2}+2\frac{R^2-|z|^2}{|z-iR|^2|z+iR|^2}+\frac{1}{R^2}\Big|\nabla\varphi\Big(\frac{z}{R}\Big)\Big|^2\\ +\frac{2}{R}\partial_x\varphi\Big(\frac{z}{R}\Big)\Big(\frac{y-R}{|z-iR|^2}-\frac{y+R}{|z+iR|^2}\Big)\\ +\frac{2}{R}\partial_y\varphi\Big(\frac{z}{R}\Big)\Big(\frac{x}{|z+iR|^2}-\frac{x}{|z-iR|^2}\Big),\end{aligned}$$

for any $z\in\omega_R:=\big\{z\in D(0,2R)\text{ s.t. }|z-iR|>1\text{ and }|z+iR|>1\big\}$. As a consequence, we can write

$$\frac{1}{2}\int_{\omega_R}|\nabla\psi_L(z)|^2\leqslant I_1+\frac{1}{2}\int_{\omega_1}|\nabla\varphi|^2+I_2, \tag{3.70}$$

with $\omega_1:=\big\{z\in D(0,2)\text{ s.t. }|z-i|>1/R\text{ and }|z+i|>1/R\big\}$. In this inequality, we have set

$$I_1:=\frac{1}{2}\int_{\omega_R}\Big(\frac{1}{|z-iR|^2}+\frac{1}{|z+iR|^2}+2\frac{R^2-|z|^2}{|z-iR|^2|z+iR|^2}\Big)\,\mathrm{d}x\,\mathrm{d}y,$$

and

$$I_2 := \frac{1}{R}\int_{\omega_R}\Big(\partial_x\varphi\Big(\frac{z}{R}\Big)\Big(\frac{y-R}{|z-iR|^2}-\frac{y+R}{|z+iR|^2}\Big) + \partial_y\varphi\Big(\frac{x}{R}\Big)\Big(\frac{x}{|z+iR|^2}-\frac{x}{|z-iR|^2}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}y.$$

We first estimate the integral I_1 using the fact that its integrand is symmetric with respect to the variable y. Setting $\omega_R^+ := \{z\in\omega_R \text{ s.t. } y\geqslant 0\}$, we combine the inequality $|z+iR|\geqslant R\geqslant 1$ for $y\geqslant 0$ and the identity $R^2-|z|^2 = 2R(R-y)-|z-iR|^2$ in order to get

$$I_1 = \int_{\omega_R^+}\Big(\frac{1}{|z-iR|^2}-\frac{1}{|z+iR|^2}+\frac{4R(R-y)}{|z-iR|^2|z+iR|^2}\Big)\,\mathrm{d}x\,\mathrm{d}y \leqslant \int_{\omega_R^+}\Big(\frac{1}{|z-iR|^2}+\frac{4}{R|z-iR|}\Big)\,\mathrm{d}x\,\mathrm{d}y.$$

When $z\in\omega_R^+$ and $|z-iR|\geqslant R$, we have

$$\frac{1}{|z-iR|^2}+\frac{4}{R|z-iR|}\leqslant\frac{5}{R^2},$$

so that

$$I_1\leqslant 5\pi+\int_{D(0,R)\setminus D(0,1)}\Big(\frac{1}{|z|^2}+\frac{4}{R|z|}\Big)\,\mathrm{d}x\,\mathrm{d}y\leqslant 2\pi\ln(R)+13\pi. \tag{3.71}$$

We next integrate by parts the integral I_2 in order to obtain

$$I_2 = \int_{\partial\omega_R}\Big(\nu_x(z)\Big(\frac{y-R}{|z-iR|^2}-\frac{y+R}{|z+iR|^2}\Big) + \nu_y(z)\Big(\frac{x}{|z+iR|^2}-\frac{x}{|z-iR|^2}\Big)\Big)\varphi\Big(\frac{z}{R}\Big)\,\mathrm{d}\gamma(z),$$

where $\nu(z) = (\nu_x(z),\nu_y(z))$ is the outward unit normal vector to $\partial\omega_R$ and $\mathrm{d}\gamma$ is the infinitesimal length element of the curve $\partial\omega_R$. Recall at this stage that the function φ is harmonic on the disc $D(0,2)$. In view of (3.66), it follows from the maximum principle that

$$\|\varphi\|_{L^\infty(D(0,2))}\leqslant\frac{\pi}{2}, \tag{3.72}$$

so that

$$I_2\leqslant\frac{\pi}{2}\Big(\int_{\partial D(0,2R)}\frac{4}{R}\,\mathrm{d}\gamma(z)+2\int_{\partial D(0,1)}4\,\mathrm{d}\gamma(z)\Big)\leqslant 16\pi^2.$$

Combining with (3.69), (3.70) and (3.71), we finally get

$$\frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}_L}|\nabla\psi_L|^2\leqslant 2\pi\ln(R)+16\pi^2+26\pi+\int_{D(0,2)}|\nabla\varphi|^2. \tag{3.73}$$

In view of (3.68), we deduce the existence of a universal positive constant C such that

$$E(\psi_L) \leqslant 2\pi \ln(R) + C. \tag{3.74}$$

Note in particular that the function $\psi_L \in X(\mathbb{R}\times\mathbb{T}_L)$, so that we are allowed to define its untwisted momentum $[P]_L(\psi_L)$ according to Lemma C.1.

In order to compute this quantity, we first rely on (3.67) from which we derive that the function $[\widehat{\psi}_L]_0$ is identically equal to 1 for $|x| \geqslant R$. As a consequence, the function $\theta_0 = 0$ is one of its phase functions on the intervals $I_R^{\pm}$. In view of (A.6) and (C.1), we obtain

$$P_{\theta_0}(\psi_L) = \frac{1}{2}\int_{\mathbb{R}} \langle i[\widehat{\psi}_L]_0', [\widehat{\psi}_L]_0\rangle_{\mathbb{C}} + \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}_L} \langle i\partial_x w_0, w_0\rangle_{\mathbb{C}},$$

with $w_0 = \psi_L - [\widehat{\psi}_L]_0$ as before. Due to the orthogonality of the functions $[\widehat{\psi}_L]_0$ and w_0, and the compactly supported nature of their derivatives, the previous formula can be simplified as

$$P_{\theta_0}(\psi_L) = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}_L} \langle i\partial_x \psi_L, \psi_L\rangle_{\mathbb{C}}.$$

Going back to (3.67), we derive from the local integrability of the map $z \mapsto y/|z|^2$ that

$$P_{\theta_0}(\psi_L) = J_R + J_+ + J_-, \tag{3.75}$$

where we have set

$$J_R := \frac{1}{2}\int_{D(0,2R)} \Big(-\frac{y-R}{|z-iR|^2} + \frac{y+R}{|z+iR|^2} - \frac{1}{R}\partial_x\varphi\Big(\frac{z}{R}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}y,$$

and

$$J_\pm := \frac{1}{2}\int_{D(\pm iR,1)} \big(1-|z\mp iR|^2\big)\Big(\frac{y-R}{|z-iR|^2} - \frac{y+R}{|z+iR|^2} + \frac{1}{R}\partial_x\varphi\Big(\frac{z}{R}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.76}$$

Integrating by parts, we check that

$$\frac{1}{2R}\int_{D(\pm iR,1)} \big(1-|z\mp iR|^2\big)\,\partial_x\varphi\Big(\frac{z}{R}\Big)\,\mathrm{d}x\,\mathrm{d}y = \int_{D(\pm iR,1)} x\,\varphi\Big(\frac{z}{R}\Big)\,\mathrm{d}x\,\mathrm{d}y,$$

so that by (3.72), we obtain

$$\big|J_\pm\big| \leqslant \frac{1}{2}\int_{D(0,1)} \frac{\mathrm{d}x\,\mathrm{d}y}{|z|} + \frac{\pi}{2} + \frac{\pi^2}{2} \leqslant \frac{3\pi}{2} + \frac{\pi^2}{2}. \tag{3.77}$$

On the other hand, a direct scaling provides

$$J_R = RJ_1 := \frac{R}{2}\int_{D(0,2)} \Big(-\frac{y-1}{|z-i|^2} + \frac{y+1}{|z+i|^2} - \partial_x\varphi(z)\Big)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.78}$$

Applying the Fubini theorem, we can write the integral J_1 as

$$J_1 = \frac{1}{2} \int_{-2}^{2} j_1(y) \, \mathrm{d}y,$$

with

$$j_1(y) = \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \Big(- \frac{y-1}{x^2 + (y-1)^2} + \frac{y+1}{x^2 + (y+1)^2} - \partial_x \varphi(x, y) \Big) \, \mathrm{d}x,$$

for $y \neq \pm 1$. In view of (3.66), the integrals $j_1(y)$ are equal to

$$j_1(y) = -2 \arctan\Big(\frac{\sqrt{4-y^2}}{y-1} \Big) + 2 \arctan\Big(\frac{\sqrt{4-y^2}}{y+1} \Big) + 2 \arctan\Big(\frac{2\sqrt{4-y^2}}{3} \Big).$$

At this stage, we can check that

$$j_1'(y) = 0,$$

when $y \neq \pm 1$, so that

$$j_1(y) = \begin{cases} \lim_{y \to 2} j_1(y) = 0 & \text{for } 1 < y \leqslant 2, \\ \lim_{y \to 1^-} j_1(y) = 2\pi & \text{for } -1 < y < 1, \\ \lim_{y \to -2} j_1(y) = 0 & \text{for } -2 \leqslant y < -1. \end{cases}$$

By the Fubini theorem, the integral J_1 is then equal to $J_1 = 2\pi$, so that $J_R = 2\pi R$. In view of (3.75) and (3.77), we obtain

$$\big| P_{\theta_0}(\psi_L) - 2\pi R \big| \leqslant 3\pi + \pi^2. \tag{3.79}$$

On the other hand, we can derive from (3.75), (3.76) and (3.78) that the map $R \mapsto P_{\theta_0}(\psi_L)$ is continuous on $[1, 4L]$. In view of (3.79), the range of this function covers the interval $[5\pi + \pi^2, \pi L/2 - 3\pi - \pi^2]$. In particular, given a fixed number in $(0, \pi/2)$, we can find, for L large enough, a positive number R_L such that $[P]_L(\psi_L) = P_{\theta_0}(\psi_L) = pL$ modulo $L\pi$, and

$$\Big| R_L - \frac{pL}{2\pi} \Big| \leqslant \frac{3+\pi}{2}.$$

In this case, we deduce from (3.73) that

$$E(\psi_L) \leqslant 2\pi \ln(L) + 2\pi \ln(p) + C,$$

where C is a further universal constant. As a consequence, the function ψ corresponding to ψ_L by the scaling in (3.62) lies in $X(\mathbb{R} \times \mathbb{T})$, with $[P](\psi) = p$ modulo π by (3.64). Using (3.63), we are led to

$$\mathcal{I}_\lambda(p) \leqslant E_\lambda(\psi) \leqslant \frac{1}{L} \Big(2\pi \ln(L) + 2\pi \ln(p) + C \Big),$$

so that $\mathcal{I}_\lambda(p)$ tends to 0 when $\lambda = 1/L \to 0$.

Observe next that

$$[P](\overline{\psi}) = -p \mod \pi,$$

so that similarly,

$$\mathcal{I}_\lambda(-p) \leqslant E_\lambda(\overline{\psi}) = E_\lambda(\psi) \leqslant \frac{1}{L}\Big(2\pi \ln(L) + 2\pi \ln(p) + C\Big),$$

and again for $p \in (-\pi/2, 0)$, $\mathcal{I}_\lambda(p)$ tends to 0 when $\lambda \to 0$. This completes the proof of Lemma 2.4 when $p \neq \pi/2$.

For $p = \pi/2$, it follows from the non-negativity and the Lipschitz continuity of the function $\mathcal{I}_\lambda$ that

$$0 \leqslant \mathcal{I}_\lambda\Big(\frac{\pi}{2}\Big) \leqslant \mathcal{I}_\lambda(p) + \sqrt{2}\Big(\frac{\pi}{2} - p\Big),$$

for any $0 < p < \pi/2$. In the limit $\lambda \to 0$, this gives

$$0 \leqslant \liminf_{\lambda \to 0} \mathcal{I}_\lambda\Big(\frac{\pi}{2}\Big) \leqslant \limsup_{\lambda \to 0} \mathcal{I}_\lambda\Big(\frac{\pi}{2}\Big) \leqslant \sqrt{2}\Big(\frac{\pi}{2} - p\Big).$$

Letting $p \to \pi/2$, we conclude that the quantity $\mathcal{I}_\lambda(\pi/2)$ also tends to 0 as $\lambda \to 0$. This completes the proof of Lemma 2.4. □

Appendix A. Energy set and momentum in dimension one

In this section, we collect useful results concerning the energy set $X(\mathbb{R})$ and the momentum P in dimension one. In particular, we recall several statements established in [1, 5, 9, 10].

In dimension one, the energy set is defined as

$$X(\mathbb{R}) = \big\{\psi \in H^1_{\rm loc}(\mathbb{R}) : \psi' \in L^2(\mathbb{R}) \text{ and } 1 - |\psi|^2 \in L^2(\mathbb{R})\big\}.$$

As a consequence of the Sobolev embedding theorem, a function ψ in this set is actually 1/2-Hölder continuous on $\mathbb{R}$. Moreover, this function is bounded (see [9]), so that the energy set is a subset of the Zhidkov space

$$Z^1(\mathbb{R}) := \big\{\psi \in \mathcal{C}^0_b(\mathbb{R}) : \psi' \in L^2(\mathbb{R})\big\}.$$

This property guarantees that the function $\eta := 1 - |\psi|^2$ belongs to the Sobolev space $H^1(\mathbb{R})$, so that it owns a vanishing limit at $\pm\infty$. In particular, we can find a positive number R such that $\rho(x) := |\psi(x)| \geqslant 1/2$ for $|x| \geqslant R$. We can therefore lift the function ψ as $\psi = \rho e^{i\theta}$ on both the intervals $I_R^- = (-\infty, -R]$ and $I_R^+ = [R, +\infty)$. The phase function θ is continuous on these intervals, with a derivative θ' in $L^2(I_R^\pm)$. Note that this phase function is defined up to two factors in $2\pi\mathbb{Z}$, one on each interval $I_R^\pm$.

This double indeterminacy is removed when the function ψ does not vanish on the whole line, that is belongs to the non vanishing energy set

$$NVX(\mathbb{R}) := \Big\{ \psi \in X(\mathbb{R}) \text{ s.t. } \inf_{x \in \mathbb{R}} |\psi(x)| > 0 \Big\}. \tag{A.1}$$

In this case, the phase function θ is defined up to only one phase factor in $2\pi\mathbb{Z}$. Moreover, the energy $E(\psi)$ is given by the hydrodynamical expression

$$E(\psi) = \frac{1}{8} \int_{\mathbb{R}} \frac{(\eta')^2}{1-\eta} + \frac{1}{2} \int_{\mathbb{R}} (1-\eta) v^2 + \frac{1}{4} \int_{\mathbb{R}} \eta^2, \tag{A.2}$$

in which we have set $v := \theta'$. In particular, there is a natural correspondence between the fact that the function ψ is in $NVX(\mathbb{R})$ and the property that the pair (η, v) lies in

$$NV(\mathbb{R}) := \Big\{ (\eta, v) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \text{ s.t. } \inf_{x \in \mathbb{R}} \eta(x) < 1 \Big\}. \tag{A.3}$$

Concerning the definition of the momentum P, it is formally given by the integral

$$\frac{1}{2} \int_{\mathbb{R}} \langle i \partial_x \psi, \psi \rangle_{\mathbb{C}}.$$

Due to a possible lack of integrability at infinity, this quantity is not necessarily well-defined when $\psi \in X(\mathbb{R})$. In order to give it a rigorous meaning, we assume first that the function ψ can be lifted as $\psi = \rho e^{i\theta}$ and write the hydrodynamical expression

$$\frac{1}{2} \langle i \partial_x \psi, \psi \rangle_{\mathbb{C}} = -\frac{1}{2} \rho^2 \theta' = \frac{1}{2} \eta \theta' - \frac{1}{2} \theta'. \tag{A.4}$$

When $(\eta, \theta') \in NV(\mathbb{R})$, the function $\eta\theta'$ is integrable on $\mathbb{R}$, but in general, the derivative θ' is not. We refer to [1, 3] for a discussion about several ways to by-pass this difficulty. A convenient way to define the momentum, in the sense that the quantity defined in this way will satisfy the natural properties of the momentum, is simply to drop the term containing the derivative θ' and to set

$$P(\psi) = \frac{1}{2} \int_{\mathbb{R}} \eta \theta'. \tag{A.5}$$

Once the decision is made to choose this definition, it is necessary to extend it to functions which can vanish. A natural way to perform this extension is to rely on the property that the functions $\psi \in X(\mathbb{R})$ can be lifted at least on intervals of the form $I_R^{\pm}$ for R large enough. Hence, we can expect that the previous formula for the momentum will be available on these intervals.

In order to check this claim, we introduce a smooth cut-off function $\chi : \mathbb{R} \to [0, 1]$ with $\chi(x) = 0$ for $|x| \leqslant 1$ and $\chi(x) = 1$ for $|x| \geqslant 2$, and we set

$\chi_r(x) = \chi(x/r)$ for any positive number r. When the function ψ does not vanish on $\mathbb{R}$, the expression for its momentum in (A.5) can be rephrased as

$$P(\psi) = P_\theta(\psi) := \frac{1}{2}\int_{\mathbb{R}} \big(\langle i\psi', \psi\rangle_{\mathbb{C}} + (\chi_r\,\theta)'\big), \tag{A.6}$$

in view of (A.4). This identity is true for any choice of the positive number r. Given an arbitrary function $\psi \in X(\mathbb{R})$, we can fix this choice so that the right-hand side $P_\theta(\psi)$ of the previous formula makes sense. Note however that this quantity possibly depends on the choice of the phase function θ. This leads to the following definition of the momentum.

Lemma A.1. — *Given a function $\psi \in X(\mathbb{R})$, consider a positive number R such that $|\psi(x)| \geqslant 1/2$ for $|x| \geqslant R$ and a phase function $\theta \in \mathcal{C}^0(I_R^\pm)$ such that $\psi = |\psi|e^{i\theta}$ on $I_R^\pm$. Choose a smooth cut-off function $\chi : \mathbb{R} \to [0,1]$ such that $\chi(x) = 0$ for $|x| \leqslant 1$ and $\chi(x) = 1$ for $|x| \geqslant 2$, and set $\chi_r(x) = \chi(x/r)$ for a number $r > R$.*

(i) *The quantity $P_\theta(\psi)$ given by formula* (A.6) *is well-defined and does not depend on the choice of neither the function χ, nor the number r.*
(ii) *When the function ψ is in $NVX(\mathbb{R})$, the momentum $P_\theta(\psi)$ does not depend on the choice of the phase function θ.*
(iii) *Given an arbitrary function $\psi \in X(\mathbb{R})$, the value modulo π of the quantity $P_\theta(\psi)$ does not depend on the choice of the phase function θ, and it is possible to fix this choice such that $P_\theta(\psi) \in (-\pi/2, \pi/2]$. In particular, the untwisted momentum $[P] : X(\mathbb{R}) \to \mathbb{R}/\pi\mathbb{Z}$ defined by $[P](\psi) = P_\theta(\psi)$ modulo π is well-defined.*

In the sequel, we drop the dependence on the phase function θ of the momentum $P_\theta(\psi)$ when the function ψ is in $NVX(\mathbb{R})$. This quantity is only defined on $NVX(\mathbb{R})$. Since it is the only one to be defined without ambiguity, this is also the only one which we will call momentum.

Proof. — The fact that the quantity $P_\theta(\psi)$ is well-defined follows from the property that ψ belongs to $H^1_{\rm loc}(\mathbb{R})$ and from the identity

$$\langle i\psi', \psi\rangle_{\mathbb{C}} + (\chi_r\,\theta)' = \eta\,\theta', \tag{A.7}$$

which holds on the intervals $I_{2r}^\pm$. In view of (A.2), the derivative θ' indeed lies in $L^2(I_{2r}^\pm)$, while the function η is in $L^2(\mathbb{R})$. This is enough to guarantee that the function in (A.7) is integrable on $I_{2r}^\pm$, so that the quantity $P_\theta(\psi)$ is well-defined. Moreover, its value does not depend on the choice of either the function χ, or the number r, since

$$\frac{1}{2}\int_{\mathbb{R}} \big((\chi_r - \widetilde{\chi}_{\tilde{r}})\,\theta\big)' = 0,$$

when the function $\widetilde{\chi}$ and the number $\widetilde{r}$ satisfy the assumptions of Lemma A.1.

Note finally that

$$\frac{1}{2} \int_{\mathbb{R}_-} \big(2\pi k_- \chi_r\big)' + \frac{1}{2} \int_{\mathbb{R}_+} \big(2\pi k_+ \chi_r\big)' = \pi\big(k_+ - k_-\big),$$

for $(k_-, k_+) \in \mathbb{Z}^2$. Statement (ii) then follows from the fact that the phase function θ is defined up to a single phase factor $2k\pi = 2k_-\pi = 2k_+\pi$, when ψ does not vanish. In the general case, we can add any phase factors $2\pi k_\pm$ to the value of the phase θ on the intervals $I_R^\pm$. The previous computation then guarantees that we can fix this choice such that the quantity $P_\theta(\psi)$ lies in the interval $(-\pi/2, \pi/2]$, but also that this quantity is only known modulo π. This completes the proof of Lemma A.1. $\square$

We now turn to the regularity properties of the momentum P and untwisted momentum $[P]$. In order to establish their continuity, we endow the energy set $X(\mathbb{R})$ with a suitable metric structure. For a fixed number $0 \leqslant c < \sqrt{2}$, we introduce the weighted Sobolev space

$$H_c(\mathbb{R}) := \big\{\psi \in \mathcal{C}^0(\mathbb{R}) \text{ s.t. } \psi' \in L^2(\mathbb{R}) \text{ and } \eta_c^{1/2}\psi \in L^2(\mathbb{R})\big\}.$$

This space is a Hilbert space for the norm given by the formula

$$\|\psi\|_{H_c}^2 := \int_{\mathbb{R}} \big(|\psi'|^2 + \eta_c|\psi|^2\big),$$

where η_c is given, as before, by (2.8). Using the exponential decay of the functions η_c and the 1/2-Hölder continuity of the functions ψ in $H_c(\mathbb{R})$, we can check that all the norms $\|\cdot\|_{H_c}$ are equivalent. As a consequence, the space $H_c(\mathbb{R})$ does not depend on c, and we set $H(\mathbb{R}) := H_c(\mathbb{R})$ for simplicity. The energy set $X(\mathbb{R})$ then appears as the subset of $H(\mathbb{R})$ given by

$$X(\mathbb{R}) = \big\{\psi \in H(\mathbb{R}) \text{ s.t. } \eta = 1 - |\psi|^2 \in L^2(\mathbb{R})\big\},$$

and we can endow it with the metric structure corresponding to the distances

$$d_c(\psi_1, \psi_2) := \Big(\|\psi_1 - \psi_2\|_{H_c}^2 + \|\eta_1 - \eta_2\|_{L^2}^2\Big)^{\frac{1}{2}}. \tag{A.8}$$

This metric structure guarantees the continuity of the Ginzburg–Landau energy E, and it is also very convenient for dealing with the continuity of the momentum and the stability of the dark solitons (see e.g. [1, 3, 9, 10]). On the other hand, it is badly taylored to deal with the differentiability properties of the momentum (see [9]). This is the reason why we use an alternative approach to establish the differentiability of this quantity. This approach is based on the observation that the energy set $X(\mathbb{R})$ is stable by addition of functions in $H^1(\mathbb{R})$ (see [8, Lemma 1]). In particular, given a function $\psi \in X(\mathbb{R})$, the affine space $\psi + H^1(\mathbb{R})$ provides a natural framework for tackling the differentiability of the momentum around the function ψ.

Before going into more details, we observe that the metric structure corresponding to the distances d_c guarantees a uniform control on the modulus of the functions $\psi \in X(\mathbb{R})$.

Lemma A.2. — *Let $0 \leqslant c < \sqrt{2}$ and consider a function $\psi_0 \in X(\mathbb{R})$. Given any positive number ε, there exists a positive number δ such that, if $d_c(\psi, \psi_0) < \delta$, then*

$$\big\| |\psi|^2 - |\psi_0|^2 \big\|_{L^\infty} < \varepsilon. \tag{A.9}$$

Proof. — We aim at establishing an H^1-control on the difference between the functions $\eta = 1 - |\psi|^2$ and $\eta_0 = 1 - |\psi_0|^2$. An L^2-control on this difference is directly provided by (A.8), so that we focus on the differences

$$\eta' - \eta_0' = 2\big(\langle \psi, \psi_0' - \psi' \rangle_{\mathbb{C}} + \langle \psi_0 - \psi, \psi_0' \rangle_{\mathbb{C}}\big). \tag{A.10}$$

Observe first that

$$\big\| 1 - |\psi| \big\|_{L^2} \leqslant \|\eta\|_{L^2} \leqslant \|\eta_0\|_{L^2} + \delta \quad \text{and} \quad \big\| |\psi|' \big\|_{L^2} \leqslant \|\psi'\|_{L^2} \leqslant \|\psi_0'\|_{L^2} + \delta,$$

when $d_c(\psi, \psi_0) < \delta$. Hence, by the Sobolev embedding theorem, there exists a positive number C such that

$$\big\| 1 - |\psi| \big\|_{L^\infty} \leqslant C\big(\|\psi_0'\|_{L^2} + \|\eta_0\|_{L^2} + \delta\big). \tag{A.11}$$

Note in particular that the function $1 - |\psi_0|$ satisfies this inequality. With these bounds at hand, we estimate (A.10) as

$$\begin{aligned} \big\|\eta' - \eta_0'\big\|_{L^2} \leqslant\ & 2\|\psi\|_{L^\infty} \|\psi' - \psi_0'\|_{L^2} + 2\|\psi - \psi_0\|_{L^\infty([-R,R])} \|\psi_0'\|_{L^2} \\ & + 2\big(\|\psi\|_{L^\infty} + \|\psi_0\|_{L^\infty}\big)\big(\|\psi_0'\|_{L^2(I_R^-)} + \|\psi_0'\|_{L^2(I_R^+)}\big). \end{aligned} \tag{A.12}$$

We next fix the choice of the positive number R in this inequality such that

$$\|\psi_0'\|_{L^2(I_R^-)} + \|\psi_0'\|_{L^2(I_R^+)} \leqslant \delta.$$

We then derive from (2.8), (A.8) and the Sobolev embedding theorem the existence of a positive number C, depending only on c and R, such that

$$\|\psi - \psi_0\|_{L^\infty([-R,R])} \leqslant C\|\psi - \psi_0\|_{H_c} \leqslant C d_c(\psi, \psi_0).$$

In view of (A.11) and (A.12), we are led to

$$\big\|\eta' - \eta_0'\big\|_{L^2} \leqslant C\big(1 + \|\psi_0'\|_{L^2} + \|\eta_0\|_{L^2} + \delta\big)\, d_c(\psi, \psi_0).$$

Since $\|\eta - \eta_0\|_{L^2} \leqslant d_c(\psi, \psi_0) < \delta$ by (A.8), we infer from the Sobolev embedding theorem that

$$\big\| |\psi|^2 - |\psi_0|^2 \big\|_{L^\infty} = \big\|\eta - \eta_0\big\|_{L^\infty} \leqslant C\big(1 + \|\psi_0'\|_{L^2} + \|\eta_0\|_{L^2} + \delta\big)\, d_c(\psi, \psi_0).$$

for a further positive number C. In order to obtain (A.9), we finally fix the choice of the positive number δ such that $C(1 + \|\psi_0'\|_{L^2} + \|\eta_0\|_{L^2} + \delta)\delta < \varepsilon$. This completes the proof of Lemma A.2. □

We deduce from Lemma A.2 that $NVX(\mathbb{R})$ is an open subset of $X(\mathbb{R})$. We also infer from this lemma that the momentum P is continuous on this set. We additionally show that its natural differential at a function $\psi \in NVX(\mathbb{R})$ is given by the function $i\psi'$.

Lemma A.3. — *The momentum P is continuous on the non-vanishing energy set $NVX(\mathbb{R})$. Moreover, given a function $\psi \in NVX(\mathbb{R})$, there exists a positive number δ such that the ball $B(\psi, \delta) := \{\psi + h : h \in H^1(\mathbb{R})$ s.t. $\|h\|_{H^1} < \delta\}$ is a subset of $NVX(\mathbb{R})$ on which*

$$P(\psi + h) = P(\psi) + \int_{\mathbb{R}} \langle i\psi', h\rangle_{\mathbb{C}} + \frac{1}{2}\int_{\mathbb{R}} \langle ih', h\rangle_{\mathbb{C}}. \tag{A.13}$$

In particular, the restriction of the momentum P to the ball $B(\psi, \delta)$ is continuously[(1)] *differentiable, with*

$$\mathrm{d}P(\psi)(h) = \int_{\mathbb{R}} \langle i\psi', h\rangle_{\mathbb{C}},$$

for any function $h \in H^1(\mathbb{R})$.

Proof. — Recall that the momentum P is well-defined on $NVX(\mathbb{R})$ by the formula

$$P(\psi) = \frac{1}{2}\int_{\mathbb{R}} \eta\theta',$$

in which we have set, as before, $\psi = \rho e^{i\theta}$ and $\eta = 1 - \rho^2$. In particular, the continuity of this quantity will follow from the continuity from $NVX(\mathbb{R})$ to $L^2(\mathbb{R})$ of the maps $\psi \mapsto \eta$ and $\psi \mapsto \theta'$. Since the continuity of the first one is a direct consequence of (A.8), we focus on the continuity of the latter one.

Given a fixed function $\psi_0 = \rho_0 e^{i\theta_0} \in NVX(\mathbb{R})$, we compute

$$\theta_0' = -\frac{\langle i\psi_0', \psi_0\rangle_{\mathbb{C}}}{\rho_0^2}.$$

Extending this formula to an arbitrary function ψ of $NVX(\mathbb{R})$, we are led to the expression

$$\theta' - \theta_0' = -\frac{\langle i(\psi' - \psi_0'), \psi\rangle_{\mathbb{C}}}{\rho^2} - \langle i\psi_0', \psi\rangle_{\mathbb{C}} \frac{\rho_0^2 - \rho^2}{\rho^2\rho_0^2} - \frac{\langle i\psi_0', \psi - \psi_0\rangle_{\mathbb{C}}}{\rho_0^2}.$$

For a positive number δ small enough, we deduce from Lemma A.2 that

$$\inf_{x\in\mathbb{R}} \rho(x) \geqslant \frac{m_0}{2} := \frac{1}{2}\inf_{x\in\mathbb{R}} \rho_0(x),$$

(1) With respect to the metric structure induced by the H^1-norm.

when $d_c(\psi,\psi_0)<\delta$. Hence, we obtain

$$\big\|\theta'-\theta_0'\big\|_{L^2}\leqslant\frac{1}{m_0^3}\Big(2m_0^2\big\|\psi'-\psi_0'\big\|_{L^2}+2\|\psi_0'\|_{L^2}\big\|\rho^2-\rho_0^2\big\|_{L^\infty}+m_0\big\|\psi_0'(\psi-\psi_0)\big\|_{L^2}\Big).$$

Invoking (A.8) for estimating the first norm in the right-hand side of this inequality, Lemma A.2 for the second one, and arguing as in the proof of Lemma A.2 for the last one, we infer that the map $\psi\mapsto\theta'$ is continuous from $NVX(\mathbb{R})$ to $L^2(\mathbb{R})$.

Concerning differentiability, we deduce from the Sobolev embedding theorem the existence of a positive number C such that

$$\inf_{x\in\mathbb{R}}|\psi(x)+h(x)|\geqslant\inf_{x\in\mathbb{R}}|\psi(x)|-\|h\|_{L^\infty}\geqslant\inf_{x\in\mathbb{R}}|\psi(x)|-C\|h\|_{H^1}>\frac{1}{2}\inf_{x\in\mathbb{R}}|\psi(x)|>0,\quad\text{(A.14)}$$

when $\|h\|_{H^1}<\delta=\inf_{x\in\mathbb{R}}|\psi(x)|/(2C)$. In this case, the function $\psi+h$ belongs to $NVX(\mathbb{R})$, so that the ball $B(\psi,\delta)$ is a subset of $NVX(\mathbb{R})$.

We next consider a function $h\in\mathcal{C}_c^\infty(\mathbb{R})$ such that $\psi+h\in B(\psi,\delta)$. Combining the inequality

$$\left|\frac{\psi+h}{|\psi+h|}-\frac{\psi}{|\psi|}\right|\leqslant\frac{3|h|}{|\psi+h|}+\frac{|h|^2}{|\psi+h|(|\psi|+|\psi+h|)},$$

with (A.14) and the Sobolev embedding theorem, we can find a further positive number C, depending only on ψ, such that

$$\left\|\frac{\psi+h}{|\psi+h|}-\frac{\psi}{|\psi|}\right\|_{L^\infty}\leqslant C\big(1+\|h\|_{H^1}\big)\|h\|_{H^1}.$$

Decreasing the value of δ if necessary, we can assume that

$$\left\|\frac{\psi+h}{|\psi+h|}-\frac{\psi}{|\psi|}\right\|_{L^\infty}<1.\quad\text{(A.15)}$$

In another direction, it follows from the fact that h has compact support that the phase functions θ_h and θ of the functions $\psi+h$, respectively ψ, are equal at $\pm\infty$ up to constants $2k_\pm\pi$, with $k_\pm\in\mathbb{Z}$. We can choose the integer $k_-=0$ and also deduce from (A.15) and a continuation argument that $|\theta_h-\theta|<2\pi$ on $\mathbb{R}$. In this case, we necessarily have $k_+=0$, so that $\theta_h=\theta$ at infinity.

Going back to (A.6), we can choose a cut-off function χ and a number r in this definition such that the support of the functions h and χ_r are disjoint.

Since the values of the phase functions θ_h and θ are equal at $\pm\infty$, we have

$$P(\psi + h) = \frac{1}{2} \int_{\mathbb{R}} \big(\langle i(\psi' + h'), \psi + h \rangle_{\mathbb{C}} + (\chi_r \, \theta)' \big)$$
$$= P(\psi) + \frac{1}{2} \int_{\mathbb{R}} \big(\langle i(\psi' + h'), h \rangle_{\mathbb{C}} + \langle ih', \psi \rangle_{\mathbb{C}} \big),$$

which yields (A.13) by integrating by parts the last term in the right-hand side of the previous formula.

Given an arbitrary function $h \in H^1(\mathbb{R})$, with $\psi + h \in B(\psi, \delta)$, we next introduce a sequence of functions $h_n \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $h_n \to h$ in $H^1(\mathbb{R})$ as $n \to \infty$. At least for n large enough, we have

$$P(\psi + h_n) = P(\psi) + \int_{\mathbb{R}} \langle i\psi', h_n \rangle_{\mathbb{C}} + \frac{1}{2} \int_{\mathbb{R}} \langle ih_n', h_n \rangle_{\mathbb{C}}. \tag{A.16}$$

In the limit $n \to \infty$, the right-hand side of this identity tends to the right-hand side of (A.13). Concerning the left-hand side, we show that $\psi + h_n \to \psi + h$ in $X(\mathbb{R})$ as $n \to \infty$. This convergence holds in $H(\mathbb{R})$ due to the property that $h_n \to h$ in $H^1(\mathbb{R})$ as $n \to \infty$. Moreover, we compute

$$\big(1 - |\psi + h|^2\big) - \big(1 - |\psi + h_n|^2\big) = 2\langle \psi, h_n - h \rangle_{\mathbb{C}} + |h_n|^2 - |h|^2.$$

Since the function ψ is bounded on $\mathbb{R}$, it follows from the Sobolev embedding theorem that

$$\big\| (1 - |\psi + h|^2) - (1 - |\psi + h_n|^2) \big\|_{L^2} \longrightarrow 0,$$

in the limit $n \to \infty$. Now that the convergence in $X(\mathbb{R})$ is proved, we infer from the continuity of the momentum P that the left-hand side of (A.16) tends to $P(\psi + h)$ as $n \to \infty$. This concludes the proof of (A.13). The continuous differentiability of the restriction of P to the ball $B(\psi, \delta)$ is then a direct consequence of the quadratic expansion in (A.13). This completes the proof of Lemma A.3. □

At this stage, it is tempting to extend by continuity the momentum P to the whole set $X(\mathbb{R})$, but this is not possible. Consider indeed two smooth cut-off functions $\chi : \mathbb{R} \to [0, 1]$ and $\theta : \mathbb{R} \to [0, 1]$, with $\chi(x) = 1$ for $|x| \leqslant 1$ and $\chi(x) = 0$ for $|x| \geqslant 2$, respectively $\theta(x) = 0$ for $x \leqslant -2$ and $\theta(x) = 1$ for $x \geqslant 2$. Given a fixed integer $k \in \mathbb{Z}$, set

$$\psi_n^k(x) = \Big(\mathfrak{u}_0(x) + \frac{i}{n} \chi(nx) \Big) e^{2i\pi k \theta(nx)},$$

for any $n \geqslant 1$. The functions ψ_n^k belong to $NVX(\mathbb{R})$ and they satisfy

$$\big(1 - |\psi_n^k(x)|^2\big) - \big(1 - |\mathfrak{u}_0(x)|^2\big) = -\frac{1}{n^2}\chi(nx)^2,$$
$$\big|\psi_n^k(x) - \mathfrak{u}_0(x)\big| \leqslant |\mathfrak{u}_0(x)|\big|e^{2i\pi k\theta(nx)} - 1\big| + \frac{1}{n}\chi(nx),$$

and

$$\big|\big(\psi_n^k\big)'(x) - \mathfrak{u}_0'(x)\big| \leqslant |\mathfrak{u}_0'(x)|\big|e^{2i\pi k\theta(nx)} - 1\big| + |\chi'(nx)| \\ + 2\pi|k|\big(n|\mathfrak{u}_0(x)||\theta'(nx)| + \chi(nx)|\theta'(nx)|\big).$$

Using the inequality $|\mathfrak{u}_0(x)| \leqslant |x|/\sqrt{2}$ and applying the dominated convergence theorem, we deduce from the three previous formulae the convergence in $X(\mathbb{R})$ of the functions ψ_n^k towards the function $\mathfrak{u}_0$ as $n \to \infty$ for any fixed integer $k \in \mathbb{Z}$. On the other hand, we infer from the formula $\psi_n^{k+1}(x) = \psi_n^k(x)\, e^{2i\pi\theta(nx)}$ that

$$P(\psi_n^{k+1}) - P(\psi_n^k) = n\pi \int_{\mathbb{R}} \big(1 - |\psi_n^k(x)|^2\big)\theta'(nx)\,\mathrm{d}x \\ = \pi \int_{\mathbb{R}} \bigg(1 - \Big|\mathfrak{u}_0\Big(\frac{y}{n}\Big)\Big|^2 - \frac{1}{n^2}\chi(y)^2\bigg)\theta'(y)\,\mathrm{d}y \longrightarrow \pi,$$

as $n \to \infty$. As a consequence, the momentum P cannot be extended by continuity for the function $\mathfrak{u}_0$.

However, the previous counter-example fails to contradict the possible continuity of a momentum that would only be defined modulo π, and we can indeed show the continuity of the untwisted momentum $[P]$ on $X(\mathbb{R})$.

Lemma A.4. — *The untwisted momentum $[P]$ is continuous on $X(\mathbb{R})$. Moreover, it satisfies*

$$[P](\psi + h) = [P](\psi) + \int_{\mathbb{R}} \langle i\psi', h\rangle_{\mathbb{C}} + \frac{1}{2}\int_{\mathbb{R}} \langle ih', h\rangle_{\mathbb{C}} \mod \pi, \qquad \text{(A.17)}$$

for any functions $\psi \in X(\mathbb{R})$ and $h \in H^1(\mathbb{R})$.

Proof. — The proof of continuity is based on Lemma A.2. Consider a function $\psi_0 \in X(\mathbb{R})$ and choose a positive number R such that $|\psi_0| \geqslant 1/4$ on $I_R^{\pm}$. Applying Lemma A.2, we can find a positive number δ such that any function ψ satisfies the condition $|\psi| \geqslant 1/2$ on $I_R^{\pm}$, as soon as $d_c(\psi, \psi_0) < \delta$ for a fixed number $0 \leqslant c < \sqrt{2}$. Setting as before $\psi = \rho e^{i\theta}$ and $\psi_0 = \rho_0 e^{i\theta_0}$ on $I_R^{\pm}$, the quantities $P_\theta(\psi)$ and $P_{\theta_0}(\psi_0)$ are then given by formula (A.6) for a suitable cut-off function χ and a number $r > R$, which is independent of the function ψ satisfying the condition $d_c(\psi, \psi_0) < \delta$. In particular, we obtain

$$\begin{aligned} P_\theta(\psi) - P_{\theta_0}(\psi_0) = \frac{1}{2} \int_{|x| \leqslant 2r} \Big(\langle i\psi', \psi \rangle_{\mathbb{C}} - \langle i\psi_0', \psi_0 \rangle_{\mathbb{C}} \Big) \\ + \frac{1}{2} \Big(\theta(2r) - \theta_0(2r) - \theta(-2r) + \theta_0(-2r) \Big) \\ + \frac{1}{2} \int_{|x| \geqslant 2r} \Big(\eta \theta' - \eta_0 \theta_0' \Big), \end{aligned} \tag{A.18}$$

with $\eta = 1 - |\psi|^2$ and $\eta_0 = 1 - |\psi_0|^2$. When $d_c(\psi, \psi_0) \to 0$, the first term in the right-hand side of (A.18) tends to 0 by definition of the $\| \cdot \|_{H_c}$-norm. Arguing as in the proof of Lemma A.3, we check that the third term also tends to 0. Concerning the second one, we derive from the Sobolev embedding theorem that the convergence in $H(\mathbb{R})$ implies the local uniform convergence. In particular, we have $\psi(\pm 2r) \to \psi_0(\pm 2r)$ as $d_c(\psi, \psi_0) \to 0$. Since $|\psi_0(\pm 2r)| \geqslant 1/4$, this in turn implies that $e^{i\theta(\pm 2r)} \to e^{i\theta_0(\pm 2r)}$, so that

$$\theta(\pm 2r) \longrightarrow \theta_0(\pm 2r) \mod 2\pi.$$

In view of (A.18), we conclude that

$$P_\theta(\psi) \longrightarrow P_{\theta_0}(\psi_0) \mod \pi,$$

which is enough to guarantee the continuity of the untwisted momentum $[P]$ on $X(\mathbb{R})$.

Concerning (A.17), we argue as for (A.13). Assume first that h is smooth and compactly supported. With the notation of Lemma A.1, we can choose the number R in the definition of the quantity $P_\theta(\psi)$ such that the support of h is a subset of $[-R, R]$. In this case, the function $\psi + h$ owns the same phase θ as the function ψ on the intervals $I_R^\pm$. Hence the quantity $P_\theta(\psi + h)$ is well-defined by

$$P_\theta(\psi + h) = \frac{1}{2} \int_{\mathbb{R}} \Big(\big\langle i(\psi' + h'), \psi + h \big\rangle_{\mathbb{C}} + \big(\chi_r \theta \big)',$$

which is equal to

$$P_\theta(\psi + h) = P_\theta(\psi) + \int_{\mathbb{R}} \langle i\psi', h \rangle_{\mathbb{C}} + \frac{1}{2} \int_{\mathbb{R}} \langle ih', h \rangle_{\mathbb{C}},$$

by integration by parts. In view of Lemma A.1, this is exactly (A.17). For an arbitrary $h \in H^1(\mathbb{R})$, we argue by density, as in the proof of Lemma A.3, using the continuity of the untwisted momentum and the property that the right-hand side of (A.17) is continuous with respect to the convergence in $H^1(\mathbb{R})$. This completes the proof of Lemma A.4. □

Due to the previous dual definition of the momentum, two strategies are at hand when we aim at minimizing a quantity under a fixed momentum p. The first one is to minimize under a fixed untwisted momentum $[P]$, but in this case, the constraint p must be assumed to be in $\mathbb{R}/\pi\mathbb{Z}$. The second one is to restrict the minimization set to the non-vanishing energy set $NVX(\mathbb{R})$ in case it is possible to define the corresponding minimization problem for any number $p \in \mathbb{R}$. However, this minimization problem does not necessarily own a minimizer due to the fact that a minimizing sequence could converge to a function, which vanishes on $\mathbb{R}$, and so does not remain in $NVX(\mathbb{R})$.

When the goal is to minimize the Ginzburg–Landau energy E, this second strategy leads to the minimization problem

$$\mathcal{I}(p) := \inf\big\{E(\psi) : \psi \in NVX(\mathbb{R}) \text{ s.t. } P(\psi) = p\big\}, \tag{A.19}$$

where the number p can take any arbitrary value in $\mathbb{R}$. Note that this problem is well-defined. Consider indeed a function $\psi = \rho e^{i\theta} \in NVX(\mathbb{R})$, with $P(\psi) \neq 0$ (for instance a dark soliton $\mathfrak{u}_c$ for $c \neq 0$) and set $\psi_\mu = \rho e^{i\mu\theta}$ for any number $\mu \in \mathbb{R}$. The functions ψ_μ remain in $NVX(\mathbb{R})$ and their momentum

$$P(\psi_\mu) = \mu P(\psi),$$

take any arbitrary value in $\mathbb{R}$. Hence, the minimization problems $\mathcal{I}(p)$ do make sense. An important tool in order to solve them is the following lemma.

Lemma A.5 ([1]). — *Let*

$$\mathcal{E}_0 := \inf\Big\{E(\psi) : \psi \in X(\mathbb{R}) \text{ s.t. } \inf_{x\in\mathbb{R}} |\psi(x)| = 0\Big\}.$$

The black soliton $\mathfrak{u}_0$ is the unique minimizer of the minimization problem $\mathcal{E}_0$ up to the invariances by translation and phase shift. In particular, when

$$E(\psi) < \mathcal{E}_0 = E(\mathfrak{u}_0) = \frac{2\sqrt{2}}{3},$$

the function ψ does not vanish on $\mathbb{R}$, so that it belongs to $NVX(\mathbb{R})$.

Given a fixed number $p \in \mathbb{R}$, and provided that there exists a function $\psi \in X(\mathbb{R})$ such that $E(\psi) < 2\sqrt{2}/3$ and $P(\psi) = p$, Lemma A.5 guarantees that the possible limits of a minimizing sequence for the problem $\mathcal{I}(p)$ still belong to $NVX(\mathbb{R})$. This property was invoked in [1] to address the resolution of the minimization problem $\mathcal{I}(p)$ for $|p| < \pi/2$. For an arbitrary choice of p, we have

Proposition A.6. —

(i) *For $|p| < \pi/2$, denote by $\mathfrak{c}_p$ the unique number in $(0, \sqrt{2})$, which solves*

$$\frac{\pi}{2} - \arctan\left(\frac{\mathfrak{c}_p}{\sqrt{2 - \mathfrak{c}_p^2}}\right) - \frac{\mathfrak{c}_p}{2}\sqrt{2 - \mathfrak{c}_p^2} = |p|. \tag{A.20}$$

and set $c_p = \operatorname{sign}(p)\, \mathfrak{c}_p$*. The dark soliton profile* $\mathfrak{u}_{c_p}$ *is the unique minimizer of the variational problem* (A.19) *up to translation and phase shift. Moreover, the corresponding minimal value is given by*

$$\mathfrak{I}(p) = E(\mathfrak{u}_{c_p}) = \frac{1}{3}\big(2 - c_p^2\big)^{\frac{3}{2}}. \tag{A.21}$$

(ii) *For* $|p| \geqslant \pi/2$*, the variational problem* (A.19) *does not own any minimizer, and its minimal value is equal to*

$$\mathcal{I}(p) = \frac{2\sqrt{2}}{3}.$$

Remark A.7. — We can use Proposition A.6 to complement the proof of Proposition 1.1 with respect to [1]. Observe indeed that

$$\mathfrak{I}(p) \leqslant \inf_{k \in \mathbb{Z}} \mathcal{I}(p + k\pi),$$

for any number $p \in (-\pi/2, \pi/2]$. Combining Lemma A.5 and Proposition A.6, we deduce that $\mathfrak{I}(p) = \mathcal{I}(p)$ for $|p| < \pi/2$. In particular, the conclusion in Proposition 1.1 follows from Proposition A.6 for this range of values of p.

Proof of Proposition A.6. — In view of Lemma A.5, statement (i) is exactly [1, Theorem 2]. We turn now to statement (ii). First, it was proved in [14, Theorem 2] that the minimal energy $\mathcal{I}$ is a non-negative, even, continuous function on $\mathbb{R}$, whose restriction to $\mathbb{R}_+$ is concave. Moreover, it was computed in [1, Theorem 2] that

$$\mathcal{I}(p) = \frac{1}{3}\big(2 - c_p^2\big)^{\frac{3}{2}},$$

for $0 \leqslant p \leqslant \pi/2$. Since

$$\frac{\mathrm{d}\mathfrak{c}_p}{\mathrm{d}p} = -\frac{1}{(2 - \mathfrak{c}_p^2)^{\frac{1}{2}}}, \tag{A.22}$$

$\mathcal{I}$ is continuously differentiable on $(0, \pi/2)$ and

$$\mathcal{I}'(p) = \mathfrak{c}_p \longrightarrow 0,$$

as $p \to \pi/2$. Since $\mathcal{I}$ is also concave on $\mathbb{R}$, we deduce that

$$\mathcal{I}(p) \leqslant \mathcal{I}(\pi/2) = \frac{2\sqrt{2}}{3}, \tag{A.23}$$

for any $p \geqslant \pi/2$.

Assume next the existence of a number $p > \pi/2$ such that $\mathcal{I}(p) < 2\sqrt{2}/3$. Since $\mathcal{I}(\pi/2) = 2\sqrt{2}/3$, we again infer from the concavity of the function $\mathcal{I}$ the existence of a number $q > p$ such that $\mathcal{I}(q) < 0$. This inequality contradicts the non-negativity of the function $\mathcal{I}$, so that $\mathcal{I}(p) \geqslant 2\sqrt{2}/3$ for any number $p > \pi/2$. In view of (A.23), this inequality is an equality, and since $\mathcal{I}$ is an even function, it also holds for $p < -\pi/2$.

In order to complete the proof of statement (ii), we next assume the existence of a minimizer ψ_p for the variational problem $\mathcal{I}(p)$ with $p \in \mathbb{R} \setminus (-\pi/2, \pi/2)$ being fixed. In view of Lemma A.3, this minimizer is characterized by the equation $\mathrm{d}E(\psi_p) = \sigma \mathrm{d}P(\psi_p)$ for a suitable Lagrange multiplier $\sigma \in \mathbb{R}$. The differentials $\mathrm{d}E$ and $\mathrm{d}P$ in this identity are chosen acting on the space $H^1(\mathbb{R})$. Again by Lemma A.3, the minimizer ψ_p is then a solution to (1.1) in $X(\mathbb{R})$. Since $P(\psi_p) \neq 0$, this solution is not constant. As a consequence, the minimizer ψ_p is equal to the dark soliton $\mathfrak{u}_\sigma$ up to the invariances by translation and phase shift. In particular, the number σ lies in $(-\sqrt{2}, \sqrt{2})$, with $\sigma \neq 0$ since the black soliton vanishes. However, it follows from [1, Proposition 1] that the momentum $P(\mathfrak{u}_\sigma)$ belongs to the interval $(-\pi/2, \pi/2)$. This contradicts the fact that $|P(\psi_p)| \geqslant \pi/2$, so that there is no minimizer for $|p| \geqslant \pi/2$. ☐

Appendix B. Properties of the energy set $X(\mathbb{R} \times \mathbb{T})$

In this section, we gather some properties of the energy set

$$X(\mathbb{R} \times \mathbb{T}) = \Big\{ \psi \in H^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{T}) : \nabla \psi \in L^2(\mathbb{R} \times \mathbb{T}) \text{ and } 1 - |\psi|^2 \in L^2(\mathbb{R} \times \mathbb{T}) \Big\},$$

which are required for defining properly the momentum and providing a suitable functional framework to solve the minimization problems $\mathcal{I}_\lambda(p)$. The derivation of these properties heavily relies on the following links between the energy sets $X(\mathbb{R})$ and $X(\mathbb{R} \times \mathbb{T})$.

Proposition B.1. — *Let λ be a fixed positive number.*

(i) *Given a function $\psi \in X(\mathbb{R})$, set $\Psi(x, y) = \psi(x)$ for any $(x, y) \in \mathbb{R} \times \mathbb{T}$. The function Ψ is in $X(\mathbb{R} \times \mathbb{T})$, with*

$$E_\lambda(\Psi) = E(\psi).$$

(ii) *Given a function $\psi \in X(\mathbb{R} \times \mathbb{T})$, set $\widehat{\psi}_0(x) = \int_0^1 \psi(x, y) \, \mathrm{d}y$ and $w_0(x, y) = \psi(x, y) - \widehat{\psi}_0(x)$ for almost any $(x, y) \in \mathbb{R} \times \mathbb{T}$. The functions $\widehat{\psi}_0$ and w_0 belong to $X(\mathbb{R})$, respectively $H^1(\mathbb{R} \times \mathbb{T})$, with*

$$\begin{aligned} E_\lambda(\psi) = E(\widehat{\psi}_0) + \frac{1}{2} \int_{\mathbb{R} \times \mathbb{T}} \big(|\partial_x w_0|^2 + \lambda^2 |\partial_y w_0|^2 \big) \\ + \int_{\mathbb{R} \times \mathbb{T}} \bigg(\langle \widehat{\psi}_0, w_0 \rangle_{\mathbb{C}}^2 - \frac{1}{2} |w_0|^2 (1 - |\widehat{\psi}_0|^2) + |w_0|^2 \langle \widehat{\psi}_0, w_0 \rangle_{\mathbb{C}} + \frac{1}{4} |w_0|^4 \bigg). \end{aligned} \tag{B.1}$$

Remark B.2. — In view of statement (i), we have made the choice to use the same notation for all the objects and quantities that are defined identically on $\mathbb{R}$ and $\mathbb{R} \times \mathbb{T}$. With a slight abuse of notation, we have also identified any function in $X(\mathbb{R})$ with the corresponding function in $X(\mathbb{R}\times\mathbb{T})$.

Proof. — Statement (i) is a direct consequence of the property that the torus $\mathbb{T}$ has a finite measure equal to 1 and that the derivative $\partial_y \psi$ of a function $\psi \in X(\mathbb{R} \times \mathbb{T})$ depending only on the variable x is equal to 0.

Concerning statement (ii), we first infer from the Plancherel formula that the gradients $\nabla\widehat{\psi}_0$ and ∇w_0 belong to $L^2(\mathbb{R} \times \mathbb{T})$, with moreover $\partial_y \widehat{\psi}_0 = 0$ and $\partial_y w_0 = \partial_y \psi$. Invoking the Poincaré–Wirtinger inequality, we obtain

$$\big\|w_0\big\|_{L^2(\mathbb{R}\times\mathbb{T})} \leqslant \frac{1}{2\pi} \big\|\partial_y \psi\big\|_{L^2(\mathbb{R}\times\mathbb{T})},$$

so that the function w_0 is indeed in $H^1(\mathbb{R} \times \mathbb{T})$.

By definition, we also compute

$$1 - |\widehat{\psi}_0|^2 = 1 - |\psi|^2 + 2\langle\psi, w_0\rangle_{\mathbb{C}} - |w_0|^2.$$

Using the inequality

$$|\psi| \leqslant \sqrt{2}\,\mathbb{1}_{|\psi|\leqslant\sqrt{2}} + 2\sqrt{|\psi|^2 - 1}\,\mathbb{1}_{|\psi|>\sqrt{2}},$$

we deduce from the Sobolev embedding theorem that the functions $\langle\psi, w_0\rangle_{\mathbb{C}}$, and then $1 - |\widehat{\psi}_0|^2$ are in $L^2(\mathbb{R} \times \mathbb{T})$. Since $\widehat{\psi}_0$ only depends on the variable x, we conclude that this function lies in $X(\mathbb{R})$. Formula (B.1) finally follows from the fact that the functions w_0 and ∇w_0 are orthogonal in $L^2(\mathbb{R} \times \mathbb{T})$ to all the functions depending only on the variable x. □

Remark B.3. — Arguing as for the proof that the function $1 - |\widehat{\psi}_0|^2$ is in $L^2(\mathbb{R} \times \mathbb{T})$, we can show that a function of the form $\psi + w$ belongs to $X(\mathbb{R} \times \mathbb{T})$ when ψ and w are in $X(\mathbb{R} \times \mathbb{T})$, respectively $H^1(\mathbb{R} \times \mathbb{T})$.

Statement (ii) in Proposition B.1 provides a uniquely determined decomposition of an arbitrary function $\psi \in X(\mathbb{R}\times\mathbb{T})$ as a function in $X(\mathbb{R})$ plus a function in $H^1(\mathbb{R}\times\mathbb{T})$. It is natural to take into account this decomposition in order to endow the energy set $X(\mathbb{R} \times \mathbb{T})$ with a metric structure. In this direction, we first set

$$H(\mathbb{R}\times\mathbb{T}) := \Big\{\psi = \widehat{\psi}_0 + w_0 \in H^1_{\rm loc}(\mathbb{R}\times\mathbb{T}) : \widehat{\psi}_0 \in H(\mathbb{R}) \text{ and } w_0 \in H^1(\mathbb{R}\times\mathbb{T})\Big\}.$$

The set $H(\mathbb{R}\times\mathbb{T})$ is then a Hilbert space for the norms given by the formula

$$\|\psi\|_{H_c}^2 = \int_{\mathbb{R}\times\mathbb{T}} \big(|\nabla\psi|^2 + \eta_c|\psi|^2\big), \tag{B.2}$$

for $0 \leqslant c < \sqrt{2}$. This definition is exactly the same as the one of the norm $\|\cdot\|_{H_c}$ in $H(\mathbb{R})$, so that we have kept the same notation. Observe in particular that the norm $\|\psi\|_{H_c}$ in $H(\mathbb{R}\times\mathbb{T})$ of a function $\psi\in H(\mathbb{R})$ is exactly equal to its norm $\|\psi\|_{H_c}$ in $H(\mathbb{R})$.

Note also that the previous norm is equivalent to the norm given by

$$\|\psi\|^2=\|\widehat{\psi}_0\|_{H_c}^2+\|w_0\|_{H^1}^2. \tag{B.3}$$

Due to the orthogonality of the functions $\nabla\widehat{\psi}_0$ and ∇w_0 in $L^2(\mathbb{R}\times\mathbb{T})$, the norm $\|\psi\|_{H_c}$ indeed controls the norms $\|\nabla\widehat{\psi}_0\|_{L^2}$ and $\|\nabla w_0\|_{L^2}$, and then the norm $\|w_0\|_{L^2}$ by the Poincaré inequality. The reverse inequality follows from the property that the norm $\|w_0\|_{H^1}$ controls the norm $\|w_0\|_{H_c}$.

At this stage, it is natural to endow the energy set $X(\mathbb{R}\times\mathbb{T})$ with the metric structure corresponding to the distances

$$d_c(\psi_1,\psi_2):=\Big(\|\psi_1-\psi_2\|_{H_c}^2+\|\eta_1-\eta_2\|_{L^2}^2\Big)^{\frac{1}{2}},$$

with $\eta_1=1-|\psi_1|^2$ and $\eta_2=1-|\psi_2|^2$, as before. This definition is again exactly the same as in $X(\mathbb{R})$, and the distance $d_c(\psi_1,\psi_2)$ in $X(\mathbb{R}\times\mathbb{T})$ between functions ψ_1 and ψ_2 in $X(\mathbb{R})$ remains equal to their distance in $X(\mathbb{R})$. This is the reason why we have again kept the same notation for the two quantities. A useful property of this metric structure is

Lemma B.4. —

(i) *Let $\psi=\widehat{\psi}_0+w_0\in X(\mathbb{R}\times\mathbb{T})$. Consider a sequence of functions $\psi^n\in X(\mathbb{R}\times\mathbb{T})$ such that $\psi^n\to\psi$ in $X(\mathbb{R}\times\mathbb{T})$ as $n\to\infty$ and denote $\psi^n=\widehat{\psi}_0^n+w_0^n$ the decomposition given by Proposition B.1. In the limit $n\to\infty$, we have*

$$\widehat{\psi}_0^n\longrightarrow\widehat{\psi}_0\in X(\mathbb{R})\quad\text{and}\quad w_0^n\longrightarrow w_0\ \text{in}\ H^1(\mathbb{R}\times\mathbb{T}).$$

(ii) *Let $g\in X(\mathbb{R})$, $h\in H^1(\mathbb{R}\times\mathbb{T})$, and set $\psi=g+h$. Consider sequences of functions $g_n\in X(\mathbb{R})$ and $h_n\in H^1(\mathbb{R}\times\mathbb{T})$ such that $g_n\to g$ in $X(\mathbb{R})$, and $h_n\to h$ in $H^1(\mathbb{R}\times\mathbb{T})$, as $n\to\infty$. Then, the functions $\psi_n=g_n+h_n$ satisfy*

$$\psi_n\longrightarrow\psi\ \text{in}\ X(\mathbb{R}\times\mathbb{T}), \tag{B.4}$$

as $n\to\infty$.

Proof. — Concerning statement (i), we deduce from the equivalence between the H_c-norms and the norms in (B.3) that $\widehat{\psi}_0^n\to\widehat{\psi}_0$ in $H(\mathbb{R})$ and $w_0^n\to w_0$ in $H^1(\mathbb{R}\times\mathbb{T})$. The fact that $1-|\widehat{\psi}_0^n|^2\to1-|\widehat{\psi}_0|^2$ in $L^2(\mathbb{R})$ then follows from the identity

$$\begin{aligned}\big(1-|\widehat{\psi}_0^n|^2\big)-\big(1-|\widehat{\psi}_0|^2\big)&=\big(|\psi|^2-|\psi^n|^2\big)+\big(|w_0^n|^2-|w_0|^2\big)\\&\quad+2\langle\widehat{\psi}_0^n-\widehat{\psi}_0,w_0\rangle_{\mathbb{C}}+2\langle\widehat{\psi}_0^n,w_0^n-w_0\rangle_{\mathbb{C}}.\end{aligned} \tag{B.5}$$

The first term in the right-hand side of this expression tends to 0 in $L^2(\mathbb{R}\times\mathbb{T})$ due to the convergence $\psi_n \to \psi$ in $X(\mathbb{R}\times\mathbb{T})$. The second one also tends to 0 in $L^2(\mathbb{R}\times\mathbb{T})$ due to the convergence $w_0^n \to w_0$ in $H^1(\mathbb{R}\times\mathbb{T})$ and the Sobolev embedding theorem. For the third one, we recall that the convergence in $H(\mathbb{R})$ implies the convergence in $L^\infty_{\rm loc}(\mathbb{R})$ by the Sobolev embedding theorem. Since the energy set $X(\mathbb{R})$ is a subset of $Z^1(\mathbb{R})$, the function $\widehat{\psi}_0$ is also bounded on $\mathbb{R}$. In particular, it follows from the dominated convergence theorem that the third term in (B.5) also converges to 0 in $L^2(\mathbb{R} \times \mathbb{T})$. In view of Lemma A.2, the functions $\widehat{\psi}_0^n$ are then uniformly bounded on $\mathbb{R}$. Similarly, the fourth term in (B.5) also tends to 0 in $L^2(\mathbb{R} \times \mathbb{T})$ due to the convergence $w_0^n \to w_0$ in $H^1(\mathbb{R}\times\mathbb{T})$. In conclusion, the left-hand side of (B.5) converges to 0 in $L^2(\mathbb{R} \times \mathbb{T})$, and then in $L^2(\mathbb{R})$ since it only depends on the variable x.

The proof of statement (ii) is very similar. Observe first that the functions ψ and ψ_n are in $X(\mathbb{R} \times \mathbb{T})$ by Proposition B.1 and Remark B.3. The convergence $\psi_n \to \psi$ in $H(\mathbb{R} \times \mathbb{T})$ then follows from the fact that the H^1-norm controls the H_c-norms. Moreover, we compute

$$\begin{aligned}\big(1 - |\psi_n|^2\big) - \big(1 - |\psi|^2\big) \\ = \big(|g|^2 - |g_n|^2\big) + \big(|h|^2 - |h_n|^2\big) + 2\langle g - g_n, h\rangle_{\mathbb{C}} + 2\langle g_n, h - h_n\rangle_{\mathbb{C}}.\end{aligned}$$

The convergence $1 - |\psi_n + w_n|^2 \to 1 - |\psi + w|^2$ in $L^2(\mathbb{R} \times \mathbb{T})$ follows as for (B.5). This completes the proofs of (B.4) and of Lemma B.4. $\square$

Note also that the energies E_λ are continuous with respect to the distances d_c. Moreover, we can show the following density result, which is useful for describing the minimal energy $\mathcal{I}_\lambda$.

Corollary B.5. — *Let λ be a positive number. Consider a function $\psi \in X(\mathbb{R}\times\mathbb{T})$ and decompose it as $\psi = \widehat{\psi}_0 + w_0$ according to Proposition B.1. There exist two sequences of functions $g_n \in X(\mathbb{R})$ and $h_n \in H^1(\mathbb{R}\times\mathbb{T})$, which satisfy the following properties.*

(i) *The functions g_n are smooth on $\mathbb{R}$ and there exist numbers $R_n^\pm > 0$ and $\theta_n^\pm \in \mathbb{R}$ for which $g_n(x) = e^{i\theta_n^\pm}$ for any $\pm x \geqslant \pm R_n^\pm$.*
(ii) *The functions h_n are smooth on $\mathbb{R} \times \mathbb{T}$ and compactly supported in $[-R_n^-, R_n^+] \times \mathbb{T}$.*
(iii) *We have the convergences*

$$g_n \longrightarrow \widehat{\psi}_0 \ \text{in } X(\mathbb{R}) \quad \text{and} \quad h_n \longrightarrow w_0 \ \text{in } H^1(\mathbb{R} \times \mathbb{T}), \tag{B.6}$$

as $n \to \infty$.

(iv) *The functions* $\psi_n = g_n + h_n$ *are in* $X(\mathbb{R} \times \mathbb{T})$*, with*

$$\psi_n \longrightarrow \psi \text{ in } X(\mathbb{R} \times \mathbb{T}) \quad \text{and} \quad E_\lambda(\psi_n) \longrightarrow E_\lambda(\psi),$$

as $n \to \infty$*.*

Proof. — The proof is based on a decomposition of the functions in $X(\mathbb{R})$, which was established by P. Gérard in [9, Theorem 1.8]. Given an arbitrary function $\widetilde{\psi} \in X(\mathbb{R})$, there exist a real-valued function $\widetilde{\phi} \in \mathcal{C}^0(\mathbb{R})$, with $\widetilde{\phi}' \in L^2(\mathbb{R})$, and a complex-valued function $\widetilde{\varpi} \in H^1(\mathbb{R})$ such that

$$\widetilde{\psi} = e^{i\tilde{\phi}} + \widetilde{\varpi}. \tag{B.7}$$

Moreover, the phase function $\widetilde{\phi}$ is determined up to adding a real-valued function $\varphi \in \mathcal{C}^0(\mathbb{R})$, with $\varphi' \in L^2(\mathbb{R})$, and such that there exist $k_\pm \in \mathbb{Z}$ with $\varphi - 2\pi k_\pm \in L^2(\mathbb{R}_\pm)$. Since $\widehat{\psi}_0$ belongs to $X(\mathbb{R})$ by Proposition B.1, we can decompose it as $\widehat{\psi}_0 = e^{i\phi} + \varpi$, with ϕ and ϖ satisfying the previous conditions.

We next invoke the density of smooth, compactly supported functions in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ so as to find two sequences of functions φ_n and ϖ_n in $\mathcal{C}_c^\infty(\mathbb{R})$ such that $\varphi_n \to \phi'$ in $L^2(\mathbb{R})$, and $\varpi_n \to \varpi$ in $H^1(\mathbb{R})$, as $n \to \infty$. Since the function ϕ is continuous, we are then allowed to define functions ϕ_n by the formula

$$\phi_n(x) = \phi(0) + \int_0^x \varphi_n(t)\,\mathrm{d}t,$$

for any $x \in \mathbb{R}$. By the inequality

$$|\phi_n(x) - \phi(x)| \leqslant \left| \int_0^x \big(\varphi_n(t) - \phi'(t)\big)\,\mathrm{d}t \right| \leqslant \sqrt{R}\,\big\|\varphi_n - \phi'\big\|_{L^2([-R,R])},$$

which holds for any positive number R, we obtain that $\phi_n \to \phi$ in $L^\infty_{\mathrm{loc}}(\mathbb{R})$, while in addition $\phi_n' \to \phi'$ in $L^2(\mathbb{R})$, when $n \to \infty$.

At this stage, we set $g_n = e^{i\phi_n} + \varpi_n$. The functions g_n satisfy statement (i) in Corollary B.5. Given a number $0 \leqslant c < \sqrt{2}$, we moreover have

$$\begin{aligned} \eta_c^{\frac{1}{2}}\big(g_n - \widehat{\psi}_0\big) &= \eta_c^{\frac{1}{2}}\big(e^{i\phi_n} - e^{i\phi}\big) + \eta_c^{\frac{1}{2}}\big(\varpi_n - \varpi\big), \\ g_n' - \widehat{\psi}_0' &= i\big(\phi_n' - \phi'\big)e^{i\phi_n} + i\phi'(e^{i\phi_n} - e^{i\phi}) + \varpi_n' - \varpi', \end{aligned}$$

as well as

$$\big(1 - |g_n|^2\big) - \big(1 - |\widehat{\psi}_0|^2\big) = 2\langle e^{i\phi} - e^{i\phi_n}, \varpi\rangle_{\mathbb{C}} + 2\langle e^{i\phi_n}, \varpi - \varpi_n\rangle_{\mathbb{C}} + |w|^2 - |w_n|^2.$$

Invoking the Sobolev embedding theorem, and applying the dominated convergence theorem when necessary, we are led to

$$d_c(g_n, \widehat{\psi}_0) \longrightarrow 0,$$

as $n \to \infty$. Note also that $g_n \to \widehat{\psi}_0$ in $L^\infty_{\rm loc}(\mathbb{R})$ by the Sobolev embedding theorem.

We finally complete the proof of statement (iii), and provide the one of statement (ii), by introducing a further sequence of functions $h_n \in \mathcal{C}^\infty_c(\mathbb{R}\times\mathbb{T})$ such that $h_n \to w_0$ in $H^1(\mathbb{R} \times \mathbb{T})$, as $n \to \infty$.

The convergence of the functions $\psi_n = g_n + h_n$ towards the function ψ in $X(\mathbb{R} \times \mathbb{T})$ is then a direct consequence of statement (iii) and Lemma B.4. The convergence of the energies $E_\lambda(\psi_n)$ towards the energy $E_\lambda(\psi)$ follows by continuity of the energy E_λ on $X(\mathbb{R} \times \mathbb{T})$. This completes the proof of Corollary B.5. □

Appendix C. Definition and properties of the momentum

In this section, we provide the definition of the momentum in the energy set $X(\mathbb{R} \times \mathbb{T})$ and describe its main properties. Our starting point is the decomposition $\psi = \widehat{\psi}_0 + w_0$ of a function $\psi \in X(\mathbb{R} \times \mathbb{T})$, which is given by Proposition B.1. Using this decomposition, the formal density of momentum writes as

$$\langle i\partial_x \psi, \psi\rangle_{\mathbb{C}} = \langle i\partial_x \widehat{\psi}_0, \widehat{\psi}_0\rangle_{\mathbb{C}} + \langle i\partial_x \widehat{\psi}_0, w_0\rangle_{\mathbb{C}} + \langle i\partial_x w_0, \widehat{\psi}_0\rangle_{\mathbb{C}} + \langle i\partial_x w_0, w_0\rangle_{\mathbb{C}}.$$

The first term in the right-hand side of this identity is the formal density of the momentum of a function $\widehat{\psi}_0 \in X(\mathbb{R})$, so that we can define it rigorously by invoking Lemma A.1. The second and third terms are scalar products of functions, which are at least formally orthogonal in $L^2(\mathbb{R} \times \mathbb{T})$. Hence, their integral is at least formally equal to 0. Finally, the last term is integrable on $\mathbb{R} \times \mathbb{T}$ since $w_0 \in H^1(\mathbb{R} \times \mathbb{T})$. As a conclusion, it is natural to define the momentum of the function ψ as

$$P(\psi) = Q(\widehat{\psi}_0) + \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} \langle i\partial_x w_0, w_0\rangle_{\mathbb{C}}.$$

In this expression, the quantity $Q(\widehat{\psi}_0)$ refers to a 1D momentum of the function $\widehat{\psi}_0$, which can be either equal to the quantity $P_{\theta_0}(\widehat{\psi}_0)$ in (A.6), the momentum $P(\widehat{\psi}_0)$ when $\widehat{\psi}_0 \in NVX(\mathbb{R})$, or the untwisted momentum $[P](\widehat{\psi}_0)$. More precisely, we have

Lemma C.1. — *Given a function ψ in $X(\mathbb{R} \times \mathbb{T})$, decompose it as $\psi = \widehat{\psi}_0 + w_0$, with $\widehat{\psi}_0$ and w_0 as in Proposition B.1. Consider a positive number R_0 such that $|\widehat{\psi}_0(x)| \geqslant 1/2$ for $|x| \geqslant R_0$ and a phase function $\theta_0 \in \mathcal{C}^0(I^\pm_{R_0})$ such that $\widehat{\psi}_0 = |\widehat{\psi}_0|e^{i\theta_0}$ on $I^\pm_{R_0}$. Choose a smooth cut-off function $\chi : \mathbb{R} \to [0,1]$ such that $\chi(x) = 0$ for $|x| \leqslant 1$ and $\chi(x) = 1$ for $|x| \geqslant 2$, and set $\chi_r(x) = \chi(x/r)$ for a number $r > R_0$.*

(i) *The quantity*

$$P_{\theta_0}(\psi) = P_{\theta_0}(\widehat{\psi}_0) + \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} \langle i\partial_x w_0, w_0\rangle_{\mathbb{C}}, \tag{C.1}$$

is well-defined and does not depend on the choice of neither the function χ, nor the number r.

(ii) *When the function $\widehat{\psi}_0$ does not vanish on $\mathbb{R}$, the quantity $P_{\theta_0}(\psi)$ does not depend on the choice of the phase function θ_0. In the sequel, this quantity is called momentum and simply denoted by $P(\psi)$.*

(iii) *In the general case, the value modulo π of the quantity $P_{\theta_0}(\psi)$ does not depend on the choice of the phase function θ_0, and it is possible to fix this choice such that $P_{\theta_0}(\psi) \in (-\pi/2, \pi/2]$. In particular, the untwisted momentum $[P] : X(\mathbb{R}\times\mathbb{T}) \to \mathbb{R}/\pi\mathbb{Z}$ defined by $[P](\psi) = P_{\theta_0}(\psi)$ modulo π is well-defined.*

Remark C.2. — In view of Remark B.2, a function $\psi \in X(\mathbb{R})$ is also a function in $X(\mathbb{R}\times\mathbb{T})$, so that we can define its momentum as a function in $X(\mathbb{R})$ or in $X(\mathbb{R}\times\mathbb{T})$. Lemma C.1 guarantees that these definitions are identical whatever is the definition of the momentum ($P_\theta(\psi)$, $P(\psi)$ or $[P](\psi)$) under consideration. In this case, the functions ψ and $\widehat{\psi}_0$ are indeed equal, so that the function w_0 identically vanishes.

Proof. — Lemma C.1 is a direct consequence of Lemma A.1 since the term depending on the function w_0 in (C.1) is well-defined for $w_0 \in H^1(\mathbb{R}\times\mathbb{T})$. □

At this stage, it is natural to introduce the set

$$Y(\mathbb{R}\times\mathbb{T}) := \Big\{\psi = \widehat{\psi}_0 + w_0 \in X(\mathbb{R}\times\mathbb{T}) \text{ s.t. } \widehat{\psi}_0 \in NVX(\mathbb{R})\Big\}.$$

Though this open set plays the role of the set $NVX(\mathbb{R})$ in the context of the product space $\mathbb{R}\times\mathbb{T}$, it is not the subset $NVX(\mathbb{R}\times\mathbb{T})$ of non-vanishing functions in $X(\mathbb{R}\times\mathbb{T})$. With the definition of $Y(\mathbb{R}\times\mathbb{T})$ at hand, we can extend Lemmas A.3 and A.4 as

Lemma C.3. —

(i) *The momentum P is continuous on the subset $Y(\mathbb{R}\times\mathbb{T})$. Moreover, given a function $\psi \in Y(\mathbb{R}\times\mathbb{T})$, there exists a positive number δ such that the ball $B(\psi, \delta) := \{\psi + h : h \in H^1(\mathbb{R}\times\mathbb{T})$ s.t. $\|h\|_{H^1} < \delta\}$ is a subset of $Y(\mathbb{R}\times\mathbb{T})$ on which*

$$P(\psi + h) = P(\psi) + \int_{\mathbb{R}\times\mathbb{T}} \langle i\partial_x\psi, h\rangle_{\mathbb{C}} + \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} \langle i\partial_x h, h\rangle_{\mathbb{C}}. \tag{C.2}$$

In particular, the restriction of the momentum P to the ball $B(\psi, \delta)$ is continuously [2] *differentiable, with*

$$\mathrm{d}P(\psi)(h) = \int_{\mathbb{R}\times\mathbb{T}} \langle i\partial_x \psi, h\rangle_{\mathbb{C}},$$

for any function $h \in H^1(\mathbb{R} \times \mathbb{T})$.

(ii) *The untwisted momentum $[P]$ is continuous on $X(\mathbb{R} \times \mathbb{T})$.*

Proof. — The continuity of the momentum P and untwisted momentum $[P]$ is a direct consequence of Lemmas A.3 and A.4 applying statement (i) of Lemma B.4.

Concerning the proof of (C.2), we consider a function $\psi = \widehat{\psi}_0 + w_0 \in Y(\mathbb{R}\times\mathbb{T})$ and invoke Lemma A.3 in order to exhibit a positive number δ such that the functions $\widehat{\psi}_0 + g$ lie in $NVX(\mathbb{R})$ when $g \in H^1(\mathbb{R})$ with $\|g\|_{H^1} < \delta$. Assume here that $h \in H^1(\mathbb{R} \times \mathbb{T})$ with $\|h\|_{H^1} < \delta$. We can decompose h as $h = \widehat{h}_0 + w$, with $\widehat{h}_0(x) = \int_{\mathbb{T}} h(x, y)\,\mathrm{d}y$ as before, and use the orthogonality of this decomposition in order to check that $\|\widehat{h}_0\|_{H^1} < \delta$. As a consequence, the function $\psi + h = \widehat{\psi}_0 + \widehat{h}_0 + w_0 + w$ lies in $Y(\mathbb{R}\times\mathbb{T})$, which amounts to say that the ball $B(\psi, \delta)$ is a subset of $Y(\mathbb{R} \times \mathbb{T})$. Moreover, we can combine (A.13) and (C.1) in order to develop the quantity $P(\psi + h)$ as

$$\begin{aligned} P(\psi + h) &= P\big(\widehat{\psi}_0 + \widehat{h}_0\big) + \frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}} \langle i\partial_x(w + w_0), w + w_0\rangle_{\mathbb{C}} \\ &= P\big(\widehat{\psi}_0\big) + \frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}} \big\langle i\partial_x w_0, w_0\big\rangle_{\mathbb{C}} + \int_{\mathbb{R}\times\mathbb{T}} \Big(\big\langle i\partial_x \widehat{\psi}_0, \widehat{h}_0\big\rangle_{\mathbb{C}} + \big\langle i\partial_x w_0, w\big\rangle_{\mathbb{C}}\Big) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}\times\mathbb{T}} \Big(\big\langle i\partial_x \widehat{h}_0, \widehat{h}_0\big\rangle_{\mathbb{C}} + \big\langle i\partial_x w, w\big\rangle_{\mathbb{C}}\Big). \end{aligned}$$

Formula (C.2) then follows from the orthogonality conditions between the functions $\widehat{\psi}_0$ and $\widehat{h}_0$ on the one hand, and w and w_0 on the other hand. The value of the differential $\mathrm{d}P(\psi)$ and its continuity are then a direct consequence of this formula. This ends the proof of Lemma C.3. □

We next relate the momentum of a function $\psi \in X(\mathbb{R} \times \mathbb{T})$ with the untwisted momenta of its slices $\psi(\,\cdot\,, y)$ for y ranging in $\mathbb{T}$.

LEMMA C.4. — *Let $\psi = \widehat{\psi}_0 + w_0 \in X(\mathbb{R}\times\mathbb{T})$. Consider a positive number R_0 such that $|\widehat{\psi}_0(x)| \geqslant 1/2$ for $|x| \geqslant R_0$ and a phase function $\theta_0 \in \mathcal{C}^0(I^\pm_{R_0})$ such that $\widehat{\psi}_0 = |\widehat{\psi}_0|\, e^{i\theta_0}$ on $I^\pm_{R_0}$. For almost every $y \in \mathbb{T}$, the functions $w_0(\,\cdot\,, y)$ and $\psi(\,\cdot\,, y)$ are well-defined in $H^1(\mathbb{R})$, respectively in $X(\mathbb{R})$. In*

[2] With respect to the metric structure induced by the H^1-norm.

particular, the quantities

$$p_{\theta_0}\big(\psi(\,\cdot\,,y)\big) := P_{\theta_0}(\widehat{\psi}_0) + \int_{\mathbb{R}} \big\langle i\widehat{\psi}_0', w_0(\,\cdot\,,y)\big\rangle_{\mathbb{C}} + \frac{1}{2}\int_{\mathbb{R}} \big\langle i\partial_x w_0(\,\cdot\,,y), w_0(\,\cdot\,,y)\big\rangle_{\mathbb{C}}, \quad \text{(C.3)}$$

are well-defined for almost any $y \in \mathbb{T}$*, and they satisfy*

$$P_{\theta_0}(\psi) = \int_{\mathbb{T}} p_{\theta_0}\big(\psi(\,\cdot\,,y)\big)\,\mathrm{d}y, \quad \text{(C.4)}$$

as well as

$$p_{\theta_0}\big(\psi(\,\cdot\,,y)\big) = [P]\big(\psi(\,\cdot\,,y)\big) \mod \pi. \quad \text{(C.5)}$$

Proof. — Recall first that w_0 is in $H^1(\mathbb{R}\times\mathbb{T})$, so that the slices $w_0(\,\cdot\,,y)$ belong to $H^1(\mathbb{R})$ for almost any $y \in \mathbb{T}$. Since the energy set $X(\mathbb{R})$ remains stable by addition of functions in $H^1(\mathbb{R})$, the slices $\psi(\,\cdot\,,y)$ are in $X(\mathbb{R})$ for almost any $y \in \mathbb{T}$. The quantity $p_{\theta_0}(\psi(\,\cdot\,,y))$ is also well-defined and depends only on the function $\psi(\,\cdot\,,y)$ due to the uniqueness of the decomposition $\psi(\,\cdot\,,y) = \widehat{\psi}_0 + w_0(\,\cdot\,,y)$.

Going back to the definition of the quantity $P_{\theta_0}(\psi)$ in Lemma C.1 and using the fact that $\widehat{\psi}_0'$ and w_0 are orthogonal in $L^2(\mathbb{R}\times\mathbb{T})$, we next invoke the Fubini theorem in order to write

$$\begin{aligned} P_{\theta_0}(\psi) &= P_{\theta_0}(\widehat{\psi_0}) + \frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}} \langle i\partial_x w_0, w_0\rangle_{\mathbb{C}} \\ &= \int_{\mathbb{T}}\bigg(P_{\theta_0}(\widehat{\psi}_0) + \int_{\mathbb{R}}\big\langle i\widehat{\psi}_0', w_0(\,\cdot\,,y)\big\rangle_{\mathbb{C}} + \frac{1}{2}\int_{\mathbb{R}}\big\langle i\partial_x w_0(\,\cdot\,,y), w_0(\,\cdot\,,y)\big\rangle_{\mathbb{C}}\bigg)\mathrm{d}y. \end{aligned}$$

This is exactly (C.4), so that it only remains to establish (C.5). This latter inequality is a direct consequence of (A.17) since $P_{\theta_0}(\widehat{\psi}_0) = [P](\widehat{\psi}_0)$ modulo π by definition of the untwisted momentum. This completes the proof of Lemma C.4. ☐

Going back to the density result in Corollary B.5, we finally derive the following useful formula for the momentum of smooth functions with compactly supported gradients.

LEMMA C.5. — *Let* g *be a smooth function in* $X(\mathbb{R})$ *such that there exist numbers* $R^\pm > 0$ *and* $\theta^\pm \in \mathbb{R}$ *for which* $g(x) = e^{i\theta^\pm}$ *for any* $\pm x \geqslant R^\pm$*. Consider a function* $h \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{T})$ *with support in* $[-R^-, R^+]\times\mathbb{T}$ *and set* $\psi = g + h$*. Then, the function* $\widehat{\psi}_0$ *writes as* $\widehat{\psi}_0(x) = e^{i\theta_0(x)}$ *for* $\pm x \geqslant R^\pm$*, with* $\theta_0(x) = \theta^+$ *if* $x \geqslant R^+$ *and* $\theta_0(x) = \theta^-$ *for* $x \leqslant -R^-$*. Moreover, the quantities* $p_{\theta_0}(\psi(\,\cdot\,,y))$ *in Lemma C.4 are given by*

$$p_{\theta_0}\big(\psi(\,\cdot\,,y)\big) = \frac{1}{2}\int_{\mathbb{R}} \langle i\partial_x\psi(\,\cdot\,,y), \psi(\,\cdot\,,y)\rangle_{\mathbb{C}} + \frac{1}{2}\big(\theta^+ - \theta^-\big). \quad \text{(C.6)}$$

for almost any $y \in \mathbb{T}$. As a consequence, we have

$$P_{\theta_0}(\psi) = \int_{\mathbb{T}} p_{\theta_0}\big(\psi(\,\cdot\,, y)\big)\,\mathrm{d}y, \tag{C.7}$$

with $P_{\theta_0}(\psi) = [P](\psi)$ modulo π, and $p_{\theta_0}\big(\psi(\,\cdot\,, y)\big) = [P]\big(\psi(\,\cdot\,, y)\big)$ modulo π, for almost any $y \in \mathbb{T}$. When the function $\widehat{\psi}_0$ does not vanish on $\mathbb{R}$, the momentum $P(\psi)$ is also given by (C.7)*.*

Proof. — Observe first that $\psi(x, y) = g(x)$ when $\pm x \geqslant R^{\pm}$, so that

$$\widehat{\psi}_0(x) = \int_{\mathbb{T}} \psi(x, y)\,\mathrm{d}y = g(x) = e^{i\theta^{\pm}} = e^{i\theta_0(x)}.$$

For almost every $y \in \mathbb{T}$, we therefore deduce from (C.3) that

$$p_{\theta_0}\big(\psi(\,\cdot\,, y)\big) = \frac{1}{2}\int_{\mathbb{R}} \Big(\big\langle i\widehat{\psi}_0', \widehat{\psi}_0\big\rangle_{\mathbb{C}} + \big(\chi_r\,\theta_0\big)' + 2\big\langle i\widehat{\psi}_0', w_0(\,\cdot\,, y)\big\rangle_{\mathbb{C}} + \big\langle i\partial_x w_0(\,\cdot\,, y), w_0(\,\cdot\,, y)\big\rangle_{\mathbb{C}}\Big), \tag{C.8}$$

with $w_0 = \psi - \widehat{\psi}_0$ and $r > \max\{R^-, R^+\}$. We next have

$$\int_{\mathbb{R}} (\chi_r\,\theta_0)' = \theta^+ - \theta^-.$$

Since $w_0(x, y) = 0$ for $\pm x \geqslant R^{\pm}$, we also deduce from an integration by parts that

$$\int_{\mathbb{R}} \big\langle i\widehat{\psi}_0', w_0(\,\cdot\,, y)\big\rangle_{\mathbb{C}} = \frac{1}{2}\int_{\mathbb{R}} \Big(\big\langle i\widehat{\psi}_0', w_0(\,\cdot\,, y)\big\rangle_{\mathbb{C}} + \big\langle i\partial_x w_0(\,\cdot\,, y), \widehat{\psi}_0\big\rangle_{\mathbb{C}}\Big).$$

Formula (C.6) then follows from (C.8). Formula (C.7), as well as the other statements in Lemma C.5, then result from the definitions in Lemma C.1 and the properties in Lemma C.4. This concludes the proof of Lemma C.5. ☐

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