AFST

Annales de la Faculté des Sciences de Toulouse

MATHÉMATIQUES

MICHEL BENAÏM, CLAUDE LOBRY, TEWFIK SARI AND ÉDOUARD STRICKLER

A note on the top Lyapunov exponent of linear cooperative systems

Tome XXXIV, nº 1 (2025), p. 225–241.

https://doi.org/10.5802/afst.1811

Publication membre du centre
Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2258-7519

Annales de la faculté des sciences de Toulouse Volume XXXIV, n° 1, 2025
pp. 225-241

A note on the top Lyapunov exponent of linear cooperative systems (*)

Michel Benaïm (1), Claude Lobry (2), Tewfik Sari (3) and Édouard Strickler (4)

Abstract. — In a recent paper [6], P. Carmona gives an asymptotic formula for the top Lyapunov exponent of a linear T-periodic cooperative differential equation, in the limit $T \to \infty$. This short note discusses and extends this result. The assumption that the system is T-periodic is replaced by the more general assumption that it is driven by a continuous time uniquely ergodic Feller Markov process $(\omega_t)_{t>0}$. When ω_t is replaced by $\omega_t^T = \omega_{t/T}$, asymptotic formulas for the top Lyapunov exponent in the fast (i.e. $T \to \infty$) and slow ($T \to 0$) regimes are given.

Résumé. — Dans un article récent [6], P. Carmona donne une formule asymptotique pour l'exposant de Lyapunov maximal d'une équation différentielle coopérative linéaire T-périodique, dans la limite $T \to \infty$. Cette note discute et étend ce résultat. L'hypothèse que le système est T-périodique est remplacée par l'hypothèse plus générale qu'il est piloté par un processus de Markov à temps continu $(\omega_t)_{t>0}$ Feller et uniquement ergodique. Lorsque ω_t est remplacé par $\omega_t^T = \omega_{t/T}$, des formules asymptotiques pour l'exposant de Lyapunov maximal dans les régimes rapide (c.-à-d. $T \to \infty$) et lent ($T \to 0$) sont données.

1. Notation and main results

Let $d \geqslant 1$ be an integer. Let $\mathcal{M}$ denote the closed convex cone consisting of real $d \times d$ matrices having off diagonal nonnegative entries. Elements of $\mathcal{M}$ are usually called *Metzler* matrices. As usual, a matrix $M \in \mathcal{M}$ is called *irreducible* if for all $i, j \in \{1, \ldots, d\}$ there exist $n \in \mathbb{N}$ and a sequence

(*) Reçu le 9 février 2023, accepté le 8 décembre 2023.

(1) Institut de Mathématiques, Université de Neuchâtel, Switzerland
(2) C.R.H.I, Université Nice Sophia Antipolis, France
(3) ITAP, University of Montpellier, INRAE, Institut Agro, Montpellier, France
(4) Université de Lorraine, CNRS, Inria, IECL, Nancy, France

This research is supported by the Swiss National Foundation grants 200020 196999 and 200020 219913.

Article proposé par Pascal Maillard.

$i_1 = i, i_2, \ldots, i_n = j$ such that $M_{i_l, i_{l+1}} > 0$ for $l = 1, \ldots, n-1$. Equivalently e^M has positive entries. Throughout, we let S denote a compact metric space and

$$A : S \to \mathcal{M},$$

a continuous mapping. We consider the linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = A(\omega_t)y \tag{1.1}$$

with initial condition $y(0) = x \in \mathbb{R}^d_+ \setminus \{0\}$, under the following assumptions:

(i) The process $(\omega_t)_{t\geqslant 0}$ is a continuous time Feller Markov process[(1)] on S and is *uniquely ergodic*. By this, we mean that $(\omega_t)_{t\geqslant 0}$ has a unique invariant probability measure denoted μ.
(ii) The average matrix $\bar{A} = \int_S A(s)\mu(\mathrm{d}s)$ is irreducible.

Remark 1.1. — A sufficient (but non necessary) condition ensuring that $\bar{A}$ is irreducible is that $A(s)$ is irreducible for some s in the topological support of μ. The (easy) proof is left to the reader.

The assumption that $A(s)$ is Metzler for all $s \in S$, makes the non-autonomous differential equation (1.1) *cooperative* in the sense that $\frac{\partial \dot{y}_i}{\partial y_j} \geqslant 0$ for all $i \neq j$ (we refer the reader to [9] for a comprehensive introduction to the theory of deterministic cooperative systems). Systems of this form naturally occur in population dynamics where individuals can migrate between different patches (see e.g. [3, 4] and references therein) or different states (see e.g. [6, 12]). They also occur as linearized systems of non-linear cooperative systems (for instance in certain epidemic models [5]). In all these settings, the process $(\omega_t)_{t\geqslant 0}$ represents the time fluctuations of the environment. The top Lyapunov exponent of the system characterizes the population growth rate, and its sign determines whether the population persists or dies out.

The following examples illustrate the fact that the process $(\omega)_{t\geqslant 0}$ can be deterministic (Examples 1.2 and 1.3), or stochastic (Examples 1.4 and 1.5).

Example 1.2 (Periodic case). — Suppose $S = \mathbb{R}/\mathbb{Z}$ identified with the unit circle and

$$\omega_t = s + t \pmod 1$$

for some $s \in S$. This is the case considered in [6]. Observe that here μ is the Lebesgue normalized measure on S.

Example 1.3 (Quasi-periodic case). — A natural generalization of Example 1.2 is as follows. Suppose $S = (\mathbb{R}/\mathbb{Z})^n$ is the n-torus and

$$\omega_t = (s_1 + ta_1, s_2 + ta_2, \ldots, s_n + ta_n) \pmod 1$$

(1) The precise definition will be recalled in the beginning of Section 2

for some $s = (s_1, \ldots, s_n) \in S$ and $(a_1, \ldots, a_n)$ rationally independent numbers. That is $\sum_{i=1}^n k_i a_i \neq 0$ for any integers $k_1, \ldots, k_n$ such that $(k_1, \ldots, k_n) \neq (0, \ldots, 0)$. Again $(\omega_t)_{t\geqslant 0}$ is uniquely ergodic with μ the Lebesgue measure on S.

Example 1.4 (Switching). — Suppose $S = \{1, \ldots, n\}$ for some $n \in \mathbb{N}^*$ and $(\omega_t)_{t\geqslant 0}$ is an irreducible continuous time Markov chain on S. In other words, the infinitesimal generator of $(\omega_t)_{t\geqslant 0}$ writes

$$Lf(i) = \sum_{j=1}^n a_{ij}(f(j) - f(i))$$

for all $f : S \mapsto \mathbb{R}$, where (a_{ij}) is an irreducible rate matrix. Then $(\omega_t)_{t\geqslant 0}$ is uniquely ergodic and μ is the unique probability vector solution to

$$\sum_{j=1}^n (\mu_j a_{ji} - \mu_i a_{ij}) = 0$$

for all $i = 1, \ldots, n$. This situation has been considered in [5].

Example 1.5. — Suppose S is a compact connected Riemannian manifold and $(\omega_t)_{t\geqslant 0}$ a Brownian motion (or an elliptic diffusion or more generally, the solution to a uniquely ergodic stochastic differential equation) on S. Then $(\omega_t)_{t\geqslant 0}$ is uniquely ergodic and μ is the normalized volume on S (or a measure absolutely continuous with respect to the volume, in the diffusion case).

We now pass to the analysis of the long term behavior of (1.1).

Let $\Delta := \Delta^{d-1} = \{x \in \mathbb{R}^d_+ : \sum_{i=1}^d x_i = 1\}$ be the unit $d-1$ simplex. Every $y \in \mathbb{R}^d_+ \setminus \{0\}$ can be written as

$$y = \rho\theta,$$

with $\rho = \langle y, \mathbb{1}\rangle = \sum_{i=1}^d y_i > 0$ and $\theta = \frac{y}{\langle y, \mathbb{1}\rangle} \in \Delta$. Here and throughout, $\mathbb{1}$ stands for the vector $(1, \ldots, 1)^t$, and $\langle \cdot, \cdot \rangle$ is the usual Euclidean scalar product on $\mathbb{R}^d$.

Using this decomposition, the differential equation (1.1) rewrites

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \rho\langle A(\omega_t)\theta, 1\rangle \tag{1.2}$$

and

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = F(\omega_t, \theta), \tag{1.3}$$

where for all $(s, \theta) \in S \times \Delta$

$$F(s, \theta) = A(s)\theta - \langle A(s)\theta, \mathbb{1}\rangle\theta. \tag{1.4}$$

The following proposition is proved in [5, Proposition 2.13], in the case corresponding to Example 1.4. It mainly relies on the Random Perron–Frobenius theorem as proved by Ruelle [17] and later by Arnold, Demetrius and Gundlach [1] (see also [13, 15], and the references therein). The proof given in [5] extends to the general situation considered here. Details are given in the next section.

Proposition 1.6. — *Let* (ρ_t, θ_t) *be solution to* (1.2), (1.3). *The process* $(\omega_t, \theta_t)_{t\geqslant 0}$ *is a Feller Markov process uniquely ergodic on* $S \times \Delta$.

Let π *denote its (unique) invariant probability and let*

$$\Lambda = \int_{S\times\Delta} \langle A(s)\theta, \mathbb{1}\rangle \pi(\mathrm{d}s\mathrm{d}\theta).$$

Then, for every initial conditions $\rho(0) > 0, \theta(0) \in \Delta$ *and* $\omega_0 = s$, *with probability one,*

$$\lim_{t\to\infty} \frac{\log(\rho_t)}{t} = \Lambda.$$

For further notice, we call Λ the *top Lyapunov* exponent[(2)] of the system given by (1.1). For periodic linear differential equations it corresponds to what is sometimes called the principal Lyapunov exponent [13], or the largest Floquet multiplier [6]. That is, the Floquet exponent with the largest real part. For further details we refer the reader to the Section II.2 of the excellent survey [13] by Mierczyński.

The following corollary easily follows from Proposition 1.6. It provides simple estimates of Λ. Other estimates, mainly for periodic systems, can be found in [13] and in [14] for more general systems.

Corollary 1.7. — *The following inequalities hold true:*

(i) $$\int_S \left[\min_{i=1,\dots,d} \sum_{j=1}^{d} A_{ji}(s)\right] \mu(\mathrm{d}s) \leqslant \Lambda \leqslant \int_S \left[\max_{i=1,\dots,d} \sum_{j=1}^{d} A_{ji}(s)\right] \mu(\mathrm{d}s);$$

(ii) $$\int_S \lambda_{\min}\left(\frac{A(s)+A(s)^t}{2}\right)\mu(\mathrm{d}s) \leqslant \Lambda \leqslant \int_S \lambda_{\max}\left(\frac{A(s)+A(s)^t}{2}\right)\mu(\mathrm{d}s),$$

where $\lambda_{\min}$ *(respectively* $\lambda_{\max}$*) stands for the smallest (largest) eigenvalue.*

---

(2) see Remark 2.3 for a justification of this terminology

Proof. —

(i). — For all $\theta \in \Delta$ and $s \in S$

$$\min_{i=1,\dots,d} \sum_{j=1}^{d} A_{ji}(s) \leqslant \langle A(s)\theta, 1\rangle \leqslant \max_{i=1,\dots,d} \sum_{j=1}^{d} A_{ji}(s),$$

and the result follows from the integral representation of Λ.

(ii). — Let $\|y\|_2 = \sqrt{\langle y, y\rangle}$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\log(\|y_t\|_2) = \frac{\langle y, A(\omega_t)y_t\rangle}{\|y_t\|_2^2} = \frac{\langle \theta_t, A(\omega_t)\theta_t\rangle}{\|\theta_t\|_2^2}.$$

Unique ergodicity of $(\omega_t, \theta_t)_{t\geqslant 0}$, then implies (see e.g. [2, Propositions 7.1 and 4.58]) that for every initial condition $y_0 \in \mathbb{R}_+^d \setminus \{0\}$,

$$\Lambda = \lim_{t\to\infty} \frac{\log(\|y_t\|_2)}{t} = \int_{S\times\Delta} \frac{\langle \theta, A(s)\theta\rangle}{\|\theta\|_2^2} \pi(\mathrm{d}s\mathrm{d}\theta)$$

almost surely. Now, for all $u \in \mathbb{R}_+^d$ such that $\|u\|_2 = 1$,

$$\lambda_{\min}\left(\frac{A(s)+A(s)^t}{2}\right) \leqslant \langle u, A(s)u\rangle \leqslant \lambda_{\max}\left(\frac{A(s)+A(s)^t}{2}\right).$$

This proves the result. □

In the particular case of a periodic system (Example 1.2), more can be said.

Proposition 1.8. — *Suppose $S = \mathbb{R}/\mathbb{Z} \backsim [0,1[$ as in Example 1.2. There exists a continuous 1-periodic function $t \in \mathbb{R} \to \theta^*(t) \in \Delta$, such that: For all $s \in S$ and $\omega_t = s + t \pmod 1$, $t \to \theta^*(s+t)$ is the unique 1-periodic solution to (1.3). It is globally asymptotically stable in the sense that*

$$\lim_{t\to\infty} \|\theta(t) - \theta^*(s+t)\| = 0$$

for every solution $(\theta(t))_{t\geqslant 0}$ to (1.3) with $\omega_t = s + t \pmod 1$. In particular,

$$\pi(\mathrm{d}s\mathrm{d}\theta) = \mathrm{d}s\,\delta_{\theta^*(s)}(\mathrm{d}\theta)$$

and

$$\Lambda = \int_0^1 \langle A(s)\theta^*(s), \mathbb{1}\rangle\, \mathrm{d}s.$$

1.1. Slow and fast regimes

For all $T > 0$, let $\omega_t^T = \omega_{t/T}$. Like $(\omega_t)_{t\geqslant 0}$, $(\omega_t^T)_{t\geqslant 0}$ is a Feller Markov process on S, uniquely ergodic with invariant probability μ. The parameter $1/T$ can be understood as a velocity parameter. For instance, in the context of Example 1.2, $(\omega_t^T)_{t\geqslant 0}$ is a T-periodic signal. In the context of Example 1.4, its mean sojourn time in each state $i \in S$ is proportional to T.

Consider the differential equation (1.1) with $(\omega_t)_{t\geqslant 0}$ replaced by $(\omega_t^T)_{t\geqslant 0}$. We let π^T and Λ^T denote the corresponding invariant probabilities on $S \times \Delta$ and top Lyapunov exponent as defined in Proposition 1.6. This section considers the fast and slow regimes obtained as $T \to 0$ and $T \to \infty$.

For a $d \times d$ real matrix M, we let $\lambda_{\max}(M)$ denote the largest real part of its eigenvalues (sometimes called the *spectral abscissa* of M; see e.g. [6]).

For $r > 0$ sufficiently large, $\overline{A} + rI$ has nonnegative entries and is irreducible. Hence, by Perron–Frobenius theorem (applied to $\overline{A}+rI$), $\lambda_{\max}(\overline{A})$ is an eigenvalue and there exists a unique vector, the Perron–Frobenius vector of $\overline{A}$, $\theta^* \in \Delta$, such that

$$\overline{A}\theta^* = \lambda_{\max}(\overline{A})\theta^*.$$

Proposition 1.9 (Fast regime). —

$$\lim_{T\to 0} \pi^T = \mu \otimes \delta_{\theta^*}$$

(for the weak topology) and*

$$\lim_{T\to 0} \Lambda^T = \lambda_{\max}(\overline{A}).$$

Note that Proposition 1.9 has been proven for Example 1.4 in ([5, Corollary 2.15]). The next result generalizes [6] beyond Example 1.2. Let $\operatorname{supp}(\mu)$ be the topological support of μ. Assume that for all $s \in \operatorname{supp}(\mu)$, $A(s)$ is irreducible. Then, under this assumption, there exists for all $s \in \operatorname{supp}(\mu)$ a unique Perron–Frobenius vector for $A(s), \theta^*(s) \in \Delta$ characterized by

$$A(s)\theta^*(s) = \lambda_{\max}(A(s))\theta^*(s).$$

Proposition 1.10 (Slow regime). — *Assume that for all $s \in \operatorname{supp}(\mu)$, $A(s)$ is irreducible. Then*

$$\lim_{T\to\infty} \pi^T = \mu(\mathrm{d}s)\delta_{\theta^*(s)}$$

(for the weak topology) and*

$$\lim_{T\to\infty} \Lambda^T = \int_S \lambda_{\max}(A(s))\mu(\mathrm{d}s).$$

2. Proofs

Notation and Background

If X is a metric space (such as $S, \Delta, S \times \Delta$) we let $B(X)$ denote the space of real valued Borel bounded functions on X and $C(X) \subset B(X)$ the subspace of bounded continuous functions. For all $f \in B(X)$ we let $\|f\|_\infty = \sup_{x \in X} |f(x)|$. If ν is a probability on X and $f \in B(X)$ we write $\nu(f)$ for $\int_X f \mathrm{d}\nu$.

Our main assumption that $(\omega_t)_{t \geqslant 0}$ is a Feller Markov process on S, means, as usual, that $(\omega_t)_{t \geqslant 0}$ is a Markov process whose transition semigroup $(P_t)_{t \geqslant 0}$ is Feller. That is:

(a) $P_t(C(S)) \subset C(S)$;
(b) $\lim_{t \to 0} P_t f(s) = f(s)$ for all $f \in C(S)$ and $s \in S$.

It turns out (see e.g. [10, Theorem 19.6]) that (a) and (b) make $(P_t)_{t \geqslant 0}$ strongly continuous in the sense that $\lim_{t \to 0} \|P_t f - f\|_\infty = 0$ for all $f \in C(S)$.

An *invariant probability* for $(\omega_t)_{t \geqslant 0}$ (or $(P_t)_{t \geqslant 0}$) is a probability μ on S such that for all $t \geqslant 0$, $\mu P_t = \mu$ (i.e. $\mu(P_t f) = \mu(f)$ for all $f \in B(S)$). Feller continuity and compactness of S imply that such a μ always exists (see e.g. [2, Corollary 4.21]). Our assumption that $(\omega_t)_{t \geqslant 0}$ is *uniquely ergodic* means that μ is unique.

A useful consequence of Feller continuity is that we can assume without loss of generality that $(\omega_t)_{t \geqslant 0}$ is defined on the space Ω consisting of *càdlàg* (right-continuous, left limit) paths $\omega : \mathbb{R}_+ \to S$ equipped with the Skorohod topology and associated Borel sigma field (see e.g. [10, Theorem 19.15]). As usual, for all $s \in S$ we let $\mathbb{P}_s$ denote the law of $(\omega_t)_{t \geqslant 0}$ starting from $\omega_0 = s$ and $\mathbb{P}_\mu = \int_S \mathbb{P}_s \mu(\mathrm{d}s)$. The associated expectations are denoted $\mathbb{E}_s$ and $\mathbb{E}_\mu$.

For all $\omega \in \Omega$ and $t \geqslant 0$ we let $\boldsymbol{\Theta}_t(\omega)$ denote the shifted path defined as $\boldsymbol{\Theta}_t(\omega)(s) = \omega(t+s)$. Ergodicity of μ for the Markov process $(\omega_t)_{t \geqslant 0}$ makes $\mathbb{P}_\mu$ ergodic (but not uniquely ergodic) for the dynamical system $(\boldsymbol{\Theta}_t)_{t \geqslant 0}$ on Ω (see e.g. [2, Proposition 4.49]).

2.1. Proof of Propositions 1.6 and 1.8

For $\omega \in \Omega$ the solution to (1.1) writes $y(t) = \Phi(t, \omega)x$ where $(\Phi(t, \omega))_{t \geqslant 0}$ is solution to the matrix valued differential equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} = A(\omega_t)M, M(0) = \mathrm{Id}\,.$$

Let $\mathcal{M}_+ \subset \mathcal{M}$ denote the set of $d \times d$ Metzler matrices having positive diagonal entries and $\mathcal{M}_{++} \subset \mathcal{M}_+$ the set of matrices having positive entries. Observe that

$$\Phi(t,\omega) \in \mathcal{M}_+$$

for all $t \geqslant 0$. Indeed, for r large enough and all $s \in S$, $A(s) + 2r\,\mathrm{Id} \geqslant r\,\mathrm{Id}$ so that $e^{2rt}\Phi(t,\omega) \geqslant e^{Rt}\,\mathrm{Id}$ (componentwise).

For all $\theta \in \Delta$, the solution to (1.3) with initial condition $\theta(0) = \theta$, writes

$$\theta(t) = \Psi(t,\omega)\theta := \frac{\Phi(t,\omega)\theta}{\langle \Phi(t,\omega)\theta, \mathbb{1}\rangle}.$$

Lemma 2.1. — *For $\mathbb{P}_\mu$ almost all $\omega \in \Omega$:*

(i) *There exists $N \in \mathbb{N}$ such that $\Phi(t,\omega) \in \mathcal{M}_{++}$ for all $t \geqslant N$;*
(ii) *For all $\theta, \theta' \in \Delta$*

$$\lim_{t\to\infty} \|\Psi(t,\omega)\theta - \Psi(t,\omega)\theta'\| = 0.$$

Proof. —

(i). — First observe that $\Phi(t,\omega) \in \mathcal{M}_{++} \Leftrightarrow e^{rt}\Phi(t,\omega) \in \mathcal{M}_{++}$ for all $r > 0$. Therefore, replacing $A(s)$ by $A(s) + r\,\mathrm{Id}$ for $r > \|A\|_\infty$, we can assume without loss of generality that $A(s) \in \mathcal{M}_+$ for all $s \in S$.

Let $x(t) = \Phi(t,\omega)x$ with $x \in \mathbb{R}^d_+ \setminus \{0\}$. Suppose $x_i(0) > 0$. Then $x_i(t) > 0$ because $\dot{x}_i(t) \geqslant A_{ii}(\omega_t)x_i(t) \geqslant 0$. By irreducibility of $\overline{A}$, for all $j \neq i$ there exists a sequence $i_0 = i, i_1, \ldots, i_n = j$ such that $\overline{A}_{i_k i_{k-1}} > 0$ for $k = 1, \ldots n$. By ergodicity, there exists a Borel set $\widetilde{\Omega} \subset \Omega$ with $\mathbb{P}_\mu(\widetilde{\Omega}) = 1$ such that for all $\omega \in \widetilde{\Omega}$

$$\frac{1}{t}\int_0^t A(\omega_u)\mathrm{d}u \to \overline{A}.$$

Therefore, for all $\omega \in \widetilde{\Omega}$, there exists a sequence $t_1 > t_2 > \cdots > t_n$ with

$$A_{i_k i_{k-1}}(\omega_{t_k}) > 0.$$

By right continuity of (ω_t) we also have $A_{i_k i_{k-1}}(\omega_t) > 0$ for $t_k \leqslant t \leqslant t_k + \varepsilon$ for some $\varepsilon > 0$. It follows that $\dot{x}_{i_1}(t) \geqslant A_{i_1,i}(\omega_t)x_1(t) > 0$ for all $t_1 \leqslant t \leqslant t_1 + \varepsilon$. Hence $x_{i_1}(t) > 0$ for all $t > t_1$. Similarly $x_{i_2}(t) > 0$ for all $t > t_2$ and, by recursion, $x_j(t) > 0$ for all $t > t_n$. In summary, we have shown that for all $i, j \in \{1, \ldots, d\}$ and $\omega \in \widetilde{\Omega}$, there exists a time t_n depending on i, j, ω such that for all $t \geqslant t_n$ $x_j(t) > 0$ whenever $x_i(0) > 0$. This proves (i).

(ii). — Let $\mathbb{R}^d_{++} = \{x \in \mathbb{R}^d \ : x_i > 0, \text{ for all } i = 1 \ldots d\}$ and $\dot{\Delta} = \Delta \cap \mathbb{R}^d_{++}$ be the relative interior of Δ. The *projective* or *Hilbert metric* d_H

on $\mathbb{R}^d_{++}$ (see Seneta [18]) is defined by

$$d_H(x,y) = \log \frac{\max_{1\leqslant i\leqslant d} x_i/y_i}{\min_{1\leqslant i\leqslant d} x_i/y_i}.$$

Note that for all $\alpha, \beta > 0$, $d_H(\alpha x, \beta y) = d_H(x,y)$ so that d_H is not a distance on $\mathbb{R}^d_{++}$. However its restriction to $\dot{\Delta}$ is. Furthermore, for all θ, $\theta' \in \dot{\Delta}$,

$$\max_{1\leqslant i\leqslant d} |\theta_i - \theta_i'| \leqslant e^{d_H(\theta,\theta')} - 1. \tag{2.1}$$

By a theorem of Birkhoff (see e.g. [18, Section 3.4]), for all $M \in \mathcal{M}_+$,

$$\sup_{\{x,y\in\mathbb{R}^d_{++}\ d_H(x,y)>0\}} \frac{d_H(Mx,My)}{d_H(x,y)} = \tau[M] \tag{2.2}$$

where $0 \leqslant \tau(M) \leqslant 1$ is the number defined as $\tau(M) = \frac{1-\sqrt{r(M)}}{1+\sqrt{r(M)}}$ with $r(M) = \min_{i,j,k,l} \min \frac{M_{ik}M_{jl}}{M_{jk}M_{il}}$ if $M \in \mathcal{M}_{++}$ and $r(M) = 0$ if $M \in \mathcal{M}_+ \backslash \mathcal{M}_{++}$. In particular, for $M \in \mathcal{M}_+$, $\tau(M) < 1$ if and only if $M \in \mathcal{M}_{++}$.

For all $0 \leqslant s \leqslant t$, let

$$F_{s,t}(\omega) = \max\{\log(\tau[\Phi(t-s, \mathbf{\Theta}_s(\omega))]), s-t\} \in [s-t, 0].$$

We claim that $(F_{s,t})_{0\leqslant s\leqslant t}$ is a sub-additive process. That is:

(1) $F_{s,t} \circ \mathbf{\Theta}_v = F_{s+v,t+v}$, and
(2) $F_{s,u} \leqslant F_{s,t} + F_{t,u}$,

for all $s \leqslant t \leqslant u$ and $v \geqslant 0$.

The first assertion is immediate because $\mathbf{\Theta}_s \circ \mathbf{\Theta}_v = \mathbf{\Theta}_{s+v}$. For the second, by the cocycle property

$$\Phi(u-s, \mathbf{\Theta}_s(\omega)) = \Phi(u-t, \mathbf{\Theta}_t(\omega)) \circ \Phi(t-s, \mathbf{\Theta}_s(\omega)).$$

Thus,

$$\log(\tau[\Phi(u-s, \mathbf{\Theta}_s(\omega))] \leqslant \log(\tau[\Phi(u-t, \mathbf{\Theta}_t(\omega))] + \log(\tau[\Phi(t-s, \mathbf{\Theta}_s(\omega))].$$

This proves (ii).

Note also that $t, s \to F_{s,t}(\omega)$ is continuous and that $\sup_{0\leqslant s\leqslant t\leqslant 1} |F_{s,t}| \leqslant 1$, so that the integrability conditions required for the continuous time version of Kingman's subadditive ergodic theorem (as stated in [11, Theorem 5.6]) are satisfied. Therefore, by this theorem,

$$\limsup_{t\to\infty} \frac{\log(\tau[\Phi(t,\omega)])}{t} \leqslant \lim_{t\to\infty} \frac{F_{0,t}(\omega)}{t} = \gamma,$$

$\mathbb{P}_\mu$ almost surely, where

$$\gamma = \inf_{t>0} \mathbb{E}_\mu \frac{F_{0,t}}{t}.$$

Clearly $\gamma < 0$. For otherwise we would have that $\tau[\Phi(n,\omega)] = 1 \Leftrightarrow \Phi(n,\omega) \in \mathcal{M}_+ \setminus \mathcal{M}_{++}$ for all $n \in \mathbb{N}$, $\mathbb{P}_\mu$ almost surely, in contradiction with (1).

Let N be like in assertion (i) of the Lemma. Then, by what precedes, $\mathbb{P}_\mu$ almost surely,

$$\begin{aligned}&\limsup_{t\to\infty} \frac{\log(d_H(\Psi(t+N,\omega)\theta, \Psi(t+N,\omega)\theta'))}{t}\\ &\leqslant \limsup_{t\to\infty} \frac{\log(\tau[\Phi(t,\mathbf{\Theta}_N(\omega))])}{t} + \limsup_{t\to\infty} \frac{\log(d_H(\Psi(N,\omega)\theta, \Psi(N,\omega)\theta'))}{t} = \gamma.\end{aligned}$$

By inequality (2.1), this concludes the proof. □

Let $(Q_t)_{t\geqslant 0}$ denote the semigroup of the process $(\omega_t,\theta_t)_{t\geqslant 0}$. Then, for all $f \in B(S\times\Delta)$ $(s,\theta) \in S\times\Delta$,

$$Q_t f(s,\theta) = \mathbb{E}_s[f(\omega_t, \Psi(t,\omega)(\theta))].$$

Lemma 2.2. — *The semigroup $(Q_t)_{t\geqslant 0}$ is Feller.*

Proof. — We need to show that

(a) $Q_t(C(S\times\Delta) \subset C(S\times\Delta)$ and
(b) $\lim_{t\to 0} Q_t f(s,\theta) = f(s,\theta)$ for all $f \in C(S\times\Delta)$.

(a). — It is easy to verify that there exist constants $c_1, c_2 \geqslant 0$ such that for all $s,s' \in S, \theta,\theta' \in \Delta$

$$\begin{aligned}\|F(s,\theta) - F(s,\theta')\| &\leqslant c_1\|\theta-\theta'\|\\ \|F(s,\theta) - F(s',\theta)\| &\leqslant c_2\|A(s) - A(s')\|\end{aligned} \tag{2.3}$$

where F is defined by (1.4). Fix $\varepsilon > 0$ and let $\widetilde{\omega}$ be the path defined as $\widetilde{\omega}_u = \omega_{k\varepsilon}$ for all $k\varepsilon \leqslant u < (k+1)\varepsilon$. Then, by Gronwall's lemma,

$$\|\Psi(t,\omega)(\theta) - \Psi(t,\widetilde{\omega})(\theta)\| \leqslant c_t \int_0^t \|A(\omega(u)) - A(\widetilde{\omega}(u))\|\mathrm{d}u \tag{2.4}$$

where $c_t = e^{c_1 t}c_2$. Thus, by Jensen inequality,

$$\begin{aligned}\mathbb{E}_s(\|\Psi(t,\omega)\theta - \Psi(t,\widetilde{\omega}))\theta)\|)^2 &\leqslant \mathbb{E}_s(\|\Psi(t,\omega)(\theta) - \Psi(t,\widetilde{\omega})(\theta)\|^2)\\ &\leqslant c_t^2 t \int_0^t \mathbb{E}_s(\|A(\omega_u)) - A(\widetilde{\omega}_u)\|^2)\,\mathrm{d}u\end{aligned}$$

The choice of the norm being arbitrary we can assume that the norm on the right hand side of the preceding inequality is the Euclidean on $\mathbb{R}^{d^2}$. Then, for all $k\varepsilon \leqslant u < (k+1)\varepsilon$,

$$\begin{aligned}
&\mathbb{E}_s(\|A(\omega_u)) - A(\widetilde{\omega}_u)\|^2) \\
&\quad = \mathbb{E}_s\left(\mathbb{E}(\|A(\omega_u)) - A(\widetilde{\omega}_u)\|^2)|\mathcal{F}_{k\varepsilon})\right) \\
&\quad = \mathbb{E}_s(P_{u-k\varepsilon}(\|A\|^2)(\omega_{k\varepsilon}) - 2\langle A(\omega_{k\varepsilon}), P_{u-k\varepsilon}(A)(\omega_{k\varepsilon})\rangle + \|A(\omega_{k\varepsilon})\|^2) \\
&\quad \leqslant \sup_{0\leqslant h\leqslant\varepsilon} \|P_h(\|A\|^2) - \|A\|^2)\|_\infty + 2\|A\|_\infty\|P_hA - A\|_\infty := \delta(\varepsilon).
\end{aligned}$$

Observe that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ by strong continuity of $(P_t)_{t\geqslant 0}$. Combining the two last inequalities, we get

$$\mathbb{E}_s(\|\Psi(t,\omega)(\theta) - \Psi(t,\widetilde{\omega})(\theta)\|^2) \leqslant c_t^2 t^2 \delta(\varepsilon) \tag{2.5}$$

Let now $f \in C(S \times \Delta)$. Then, for every $\delta > 0$ there exists $\alpha > 0$, such that

$$|f(s,\theta) - f(s,\theta')| \leqslant \delta + 2\|f\|\mathbb{1}_{\|\theta-\theta'\|\geqslant\alpha}$$

Thus

$$\begin{aligned}
&|Q_tf(s,\theta) - \mathbb{E}_s(f(\omega_t, \Psi(t,\widetilde{\omega})(\theta))| \\
&\qquad \leqslant \mathbb{E}_s\left(|f(\omega_t, \Psi(t,\omega)(\theta)) - f(\omega_t, \Psi(t,\widetilde{\omega})(\theta))|\right) \leqslant \delta + 2\frac{\|f\|}{\alpha^2} c_t^2 t^2 \delta(\varepsilon).
\end{aligned}$$

This shows that the left hand term goes to 0 uniformly in $(s,\theta) \in S \times \Delta$ as $\varepsilon \to 0$.

In order to conclude it suffices to show that $(s,\theta) \to \mathbb{E}_s\left(f(\omega_t, \Psi(t,\widetilde{\omega})(\theta))\right)$ is continuous. For all $s \in S$, let $(\Psi_t^s)_{t\geqslant 0}$ denote the semi-flow on Δ induced by the autonomous differential equation $\frac{\mathrm{d}\theta}{\mathrm{d}t} = F(s,\theta)$ Then for $k\varepsilon \leqslant t < (k+1)\varepsilon$

$$f(\omega_t, \Psi(t,\widetilde{\omega})(\theta)) = f(\omega_t, \Psi_{t-k\varepsilon}^{\omega_{k\varepsilon}} \circ \dots \Psi_\varepsilon^{\omega_\varepsilon} \circ \Psi_\varepsilon^{\omega_0}(\theta))$$

Now, for every $h \in C(S^{k+2} \times \Delta)$, Feller continuity of $(P_t)_{t\geqslant 0}$, makes the map $(s,\theta) \to \mathbb{E}_s(h(\omega_t, \omega_{k\varepsilon}, \dots, \omega_0, \theta)$ continuous. This is immediate to verify when h is a product function (i.e. $h(s_{k+1}, \dots, s_0, \theta) = h_{k+1}(s_{k+1}) \cdot h_0(s_0) g(\theta)$) and the general case follows by the density in $C(S^{k+2} \times \Delta)$ of the vector space span by product functions. This concludes the proof of (a).

(b). — Let $f \in C(S \times \Delta)$ and $\delta > 0$. Because $\|\Psi(t,\omega)(\theta) - \theta\| \leqslant t\|F\|_\infty$, $\|Q_tf(s,\theta) - \mathbb{E}_s(f(\omega_t,\theta))\| \leqslant \delta$ for all t sufficiently small. By Feller continuity of (P_t) $\lim_{t\to 0}\mathbb{E}_s(f(\omega_t,\theta)) = \lim_{t\to 0} P_t(f(\,\cdot\,,\theta))(s) = f(s,\theta)$. □

We can now conclude the proof of Proposition 1.6. It follows from Lemma 2.1 (ii) that for all $f : S\times\Delta \mapsto \mathbb{R}$ continuous (hence uniformly continuous) and all $\theta, \theta' \in \Delta$,

$$\lim_{t\to\infty} |f(\omega_t, \Psi(t,\omega)\theta) - f(\omega_t, \Psi(t,\omega)\theta')| = 0$$

$\mathbb{P}_s$ almost surely, for μ almost all $s \in S$. Hence, for all $\theta, \theta' \in \Delta$,

$$\lim_{t \to \infty} |Q_t f(s,\theta) - Q_t f(s,\theta')| = 0, \tag{2.6}$$

for μ almost all $s \in S$. Let now π be an invariant probability of $(Q_t)_{t \geqslant 0}$. Such a π always exist because $(Q_t)_{t \geqslant 0}$ if Feller on $S \times \Delta$ compact. To prove that π is unique, assume that π' is another invariant probability. Then, writing $\pi(f)$ for $\int_{S \times \Delta} f(s,\theta)\pi(\mathrm{d}s\mathrm{d}\theta)$,

$$\begin{aligned}\pi(f) - \pi'(f) &= \pi(Q_t f) - \pi'(Q_t f) \\ &= \int_S \left[\int_{\Delta \times \Delta} (Q_t f(s,\theta) - Q_t f(s,\theta'))\pi(\mathrm{d}\theta|s)\pi(\mathrm{d}\theta'|s) \right] \mu(\mathrm{d}s)\end{aligned}$$

where for each $s \in S$, $\pi(\,\cdot\,|s)$ (respectively $\pi'(\,\cdot\,|s)$) is a conditional distribution of π (respectively π') (see [7, Section 10.2]). It then follows from (2.6) and dominated convergence that $\pi(f) = \pi'(f)$. Thus $\pi = \pi'$. This proves unique ergodicity.

Now, unique ergodicity and Feller continuity of $(\omega_s, \theta_s)_{s \geqslant 0}$ imply that for every continuous function $g : S \times \Delta \to \mathbb{R}$

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t g(\omega_s, \theta_s)\,\mathrm{d}s = \int g \mathrm{d}\pi$$

$\mathbb{P}_{s,\theta}$ almost surely for all $s, \theta \in S \times \Delta$. (see e.g. [2, Proposition 7.1] for discrete time chains combined with Proposition 4.58 to handle continuous time). This concludes the proof of Proposition 1.6 with $g(s,\theta) = \langle A(s)\theta, \mathbb{1} \rangle)$.

Remark 2.3. — By the multiplicative ergodic theorem, there exist numbers $\Lambda_1 < \cdots < \Lambda_r, r \leqslant d$, called *Lyapunov exponents*, such that for $\mathbb{P}_\mu$ almost all ω and all $x \in \mathbb{R}^d \setminus \{0\}$,

$$\lim_{t \to \infty} \frac{\log \|\Phi(t,\omega)x\|}{t} := \Lambda(x,\omega) \in \{\Lambda_1, \ldots, \Lambda_r\}.$$

The set of $x \in \mathbb{R}^d$ for which $\Lambda(x,\omega) < \Lambda_r$ is a vector space (depending on ω) having nonzero codimension. On the other hand, by what precedes, $\Lambda(x,\omega) = \Lambda$ for all $x \in \mathbb{R}^d_+ \setminus \{0\}$. It follows that $\Lambda = \Lambda_r$.

Proof of Proposition 1.8

For $s \in S = \mathbb{R}/\mathbb{Z}$, let $\omega[s] \in \Omega$ be the path defined as

$$\omega_t[s] = s + t \pmod 1.$$

By Brouwer fixed point theorem, the map $\Psi(1, \omega[0]) : \Delta \mapsto \Delta$ has a fixed point θ^*. Set $\theta^*(t) = \Psi(t, \omega[0])(\theta^*)$. Then

$$\theta^*(t+1) = \Psi(t, \omega[1]) \circ \Psi(1, \omega[0])(\theta^*) = \theta^*(t)$$

proving that $t \to \theta^*(t)$ is 1-periodic.

For all $s \in S$ and $\theta \in \Delta$

$$\lim_{t\to\infty} \|\Psi(t,\omega[s])(\theta) - \theta^*(t+s)\| = \lim_{t\to\infty} \|\Psi(t,\omega[s])(\theta) - \Psi(t,\omega[s])(\theta^*(s))\| = 0,$$

by Lemma 2.1 applied with $\omega = \omega[s]$ and $\theta' = \theta^*[s]$. Observe here, that the conclusions of Lemma 2.1 hold with $\omega = \omega[s]$ for all $s \in S$ simply because $\omega[s]$ is 1-periodic.

Remark 2.4. — The proof given here can be re-interpreted in the classical framework of Floquet's theory used in [6]. By Floquet's theorem, every solution to $\frac{\mathrm{d}y}{\mathrm{d}t} = A(\omega_t[0])y$ writes $y(t) = P(t)e^{tB}y(0)$, where P is a 1-periodic matrix such that $P(0) = \mathrm{Id}$. The matrix e^B has nonnegative entries, hence, by Perron–Frobenius theorem, a eigenvector $y^* \in \mathbb{R}^d_+ \setminus \{0\}$. The point θ^* in the proof above is the projection of y^* on the simplex, $\theta^* = y^*/|y^*|$.

3. Proof of Propositions 1.9 and 1.10

For all $T > 0$, let $(Q_t^T)_{t\geqslant 0}$ denote the semigroup of $(\omega_t^T, \theta_t)_{t\geqslant 0}$ with $\omega_t^T = \omega_{t/T}$ and $(\theta_t)_{t\geqslant 0}$ is solution to (1.3) when ω_t is replaced by ω_t^T. Using the notation of the preceding section one sees that

$$Q_t^T(f)(s,\theta) = \mathbb{E}_s\left[f(\omega_{t/T}, \Psi(t,\omega^T)(\theta))\right]$$

for all $f \in B(S \times \Delta)$.

Proof of Proposition 1.9

For all $\theta \in \Delta$, let $\overline{F}(\theta) = \int_S F(s,\theta)\mu(\mathrm{d}s)$, where F is defined by (1.4). Let $(\overline{\Psi}_t)_{t\geqslant 0}$ denote the semi-flow on Δ induced by the differential equation $\dot{\theta} = \overline{F}(\theta)$. The following lemma follows from the averaging principle as given in Freidlin and Wentzell [8, Theorem 2.1, Chapter 7].

Lemma 3.1. — *For all $\delta > 0$ and $t \geqslant 0$,*

$$\lim_{T\to 0} \mathbb{P}_\mu\left(\sup_{\theta\in\Delta, 0\leqslant u\leqslant t} \|\Psi(u,\omega^T)(\theta) - \overline{\Psi}_u(\theta)\| \geqslant \delta\right) = 0.$$

In particular, for all $f \in C(\Delta)$ and $t \geqslant 0$,

$$\lim_{T\to 0} \mathbb{E}_\mu\left[\|f \circ \Psi(t,\omega^T) - f \circ \overline{\Psi}_t\|_\infty\right] = 0.$$

Proof. — We claim that

$$\lim_{R\to\infty}\sup_{t\geqslant 0}\mathbb{P}_\mu\left(\left|\frac{1}{R}\int_t^{t+R}F(\omega_s,\theta)\mathrm{d}s-\overline{F}(\theta)\right|\geqslant\delta\right)=0. \tag{3.1}$$

Indeed, by stationarity (invariance of $\mathbb{P}_\mu$ for $(\boldsymbol{\Theta}_t)_{t\geqslant 0}$),

$$\mathbb{P}_\mu\left(\left|\frac{1}{R}\int_t^{t+R}F(\omega_s,\theta)\mathrm{d}s-\overline{F}(\theta)\right|\geqslant\delta\right)$$
$$=\mathbb{P}_\mu\left(\left|\frac{1}{R}\int_0^{R}F(\omega_s,\theta)\mathrm{d}s-\overline{F}(\theta)\right|\geqslant\delta\right)$$

for all $t\geqslant 0$; and the right hand term goes to 0, as $R\to\infty$, by ergodicity of μ.

By the averaging theorem ([8, Theorem 2.1, Chapter 7]), condition (3.1) implies that for all $\delta>0, t\geqslant 0$ and $\theta\in\Delta$,

$$\lim_{T\to 0}\mathbb{P}_\mu\left(\sup_{0\leqslant u\leqslant t}\|\Psi(u,\omega^T)(\theta)-\overline{\Psi}_u(\theta)\|\geqslant\delta\right)=0. \tag{3.2}$$

By Lipschitz continuity (see (2.3)) and Gronwall's lemma,

$$\sup_{0\leqslant u\leqslant t}\|\Psi(u,\omega^T)(\theta)-\Psi(u,\omega^T)(\theta')\|+\|\overline{\Psi}_t(\theta)-\overline{\Psi}_t(\theta')\|\leqslant 2e^{c_1 t}\|\theta-\theta'\|$$

for all $\theta,\theta'\in\Delta$. Fix $\varepsilon<\frac{\delta}{4}e^{-c_1 t}$ and let $\{B(\theta_i,\varepsilon), i=1,\dots,N\}$ be a finite covering of Δ by balls of radius ε. Then

$$\sup_{0\leqslant u\leqslant t,\theta\in\Delta}\|\Psi(u,\omega^T)(\theta)-\overline{\Psi}_u(\theta)\|$$
$$\leqslant\max_{i=1,\dots,N}\sup_{0\leqslant u\leqslant t}\|\Psi(u,\omega^T)(\theta_i)-\overline{\Psi}_u(\theta_i)\|+\delta/2.$$

Hence

$$\mathbb{P}_\mu\left(\sup_{0\leqslant u\leqslant t,\theta\in\Delta}\|\Psi(u,\omega^T)(\theta)-\overline{\Psi}_u(\theta)\|\geqslant\delta\right)$$
$$\leqslant\sum_{i=1}^N\mathbb{P}_\mu\left(\sup_{0\leqslant u\leqslant t}\|\Psi(u,\omega^T)(\theta_i)-\overline{\Psi}_u(\theta_i)\|\geqslant\delta/2\right).$$

The right hand term goes 0 as $T\to 0$ by (3.2). □

We now prove the proposition. Let π^T be the invariant measure of $(Q_t^T)_{t\geqslant 0}$ and let π^0 be a limit point of $(\pi^T)_{T>0}$ for the weak* topology, as $T\to 0$. That is: $\pi^{T_n}f\to\pi^0 f$ for some sequence $T_n\to 0$ and all $f\in C(S\times\Delta)$.

Let $p: S\times\Delta\to\Delta$ be the projection defined as $p(s,\theta)=\theta$ and let $\pi_2^T=\pi^T\circ p^{-1}$ be the second marginal of π^T. Similarly, set $\pi_2^0=\pi^0\circ p^{-1}$.

For all $f \in C(\Delta)$ and $t \geqslant 0$,

$$\pi_2^T f = \pi^T(f \circ p) = \pi^T Q_t^T(f \circ p) = \int_{S \times \Delta} \mathbb{E}_s[f(\Psi(t, \omega^T)(\theta))]\pi^T(\mathrm{d}s\mathrm{d}\theta).$$

Thus,

$$|\pi_2^T f - \pi_2^T(f \circ \overline{\Psi}_t)| = \left| \int_{S \times \Delta} \mathbb{E}_s[f(\Psi(t, \omega^T)(\theta)) - f(\overline{\Psi}_t(\theta))]\pi^T(\mathrm{d}s\mathrm{d}\theta) \right|$$

$$\leqslant \int_S \mathbb{E}_s[\|f \circ \Psi(t, \omega^T) - f \circ \overline{\Psi}_t\|_\infty]\mu(\mathrm{d}s) = \mathbb{E}_\mu[\|f \circ \Psi(t, \omega^T) - f \circ \overline{\Psi}_t\|_\infty].$$

Here we have used the fact that the first marginal of π^T is μ. Using Lemma 3.1, it comes that

$$\pi_2^0 f = \pi_2^0(f \circ \overline{\Psi}_t)$$

for all $t \geqslant 0$. This proves that π_2^0 is invariant for $\{\overline{\Psi}_t\}_{t \geqslant 0}$, but since $\{\overline{\Psi}_t\}_{t \geqslant 0}$ has θ^* as globally asymptotically stable equilibrium, necessarily $\pi_2^0 = \delta_{\theta^*}$. On the other hand, the first marginal of π^0 is μ. Thus $\pi^0 = \mu \otimes \delta_{\theta^*}$. This concludes the proof.

Proof of Proposition 1.10

Recall (see the proof of Lemma 2.2) that for all $s \in S$, we let $(\Psi_t^s)_{t \geqslant 0}$ denote the semi-flow on Δ induced by the differential equation $\dot{\theta} = F(s, \theta)$.

Let $(Q_t^\infty)_{t \geqslant 0})$ denote the Markov semigroup on $S \times \Delta$ defined as

$$Q_t^\infty f(s, \theta) = f(s, \Psi_t^s(\theta))$$

for all $f \in B(S \times \Delta)$.

Lemma 3.2. — *For all $f \in C(S \times \Delta)$ and $t \geqslant 0$*

$$\lim_{T \to \infty} \|Q_t^T f - Q_t^\infty f\|_\infty = 0.$$

Proof. — Let $f \in C(S \times \Delta)$. By uniform continuity of f, for every $t > 0$ and $\delta > 0$ there exists $\alpha > 0$ such that

$$|f(\omega_t^T, \Psi(t, \omega^T)\theta) - f(s, \Psi_t^s(\theta))| \\ \leqslant \delta + 2\|f\|_\infty \mathbb{1}_{\{d(\omega_t^T, s) + \|\Psi(t, \omega^T)(\theta) - \Psi_t^s(\theta)\| \geqslant \alpha\}}.$$

Thus

$$\|Q_t^T f(s, \theta) - Q_t^\infty f(s, \theta)\| \leqslant \mathbb{E}_s\left(|f(\omega_t^T, \Psi(t, \omega^T)(\theta) - f(s, \Psi_t^s(\theta))|\right) \\ \leqslant \delta + 2\|f\|_\infty \frac{\mathbb{E}_s(\|\Psi(t, \omega^T)(\theta) - \Psi_t^s(\theta)\|) + P_{t/T}(d(\,\cdot\,, s))(s)}{\alpha}.$$

By Feller continuity, $P_{t/T}(d(\cdot,s))(s) \to 0$ uniformly in $s \in S$ as $T \to \infty$. This follows for example from Lemma 19.3 (F_3) in [10]. Now the estimate (2.5) applied with ω^T in place of ω, $(P_{t/T})_{t\geqslant 0}$ in place of $(P_t)_{t\geqslant 0}$ and $\varepsilon > t$ gives

$$\sup_{s\in S} \mathbb{E}_s(\|\Psi(t,\omega^T)(\theta) - \Psi_t^s(\theta)\|) \leqslant c^2 t^2 \delta(\varepsilon/T). \tag{3.3}$$

with $\delta(\varepsilon/T) \to 0$ as $T \to \infty$. This concludes the proof. □

We can now prove Proposition 1.10. Let π^T be the invariant measure of (ω_t^T, θ_t) and let π^∞ be a limit point of $(\pi^T)_{T>0}$ for the weak* topology. That is $\pi^{T_n} f \to \pi^\infty f$ for some sequence $T_n \to \infty$ and all $f \in C(S \times \Delta)$. Then,

$$|\pi^T(f) - \pi^T(Q_t^\infty f)| = |\pi^T(Q_t^T(f) - Q_t^\infty(f))| \leqslant \|Q_t^T(f) - Q_t^\infty(f)\|_\infty.$$

Thus, by Lemma 3.2, $\pi^\infty(f) = \pi^\infty(Q_t^\infty(f))$. Now for all $s \in \mathrm{supp}(\mu)$

$$\lim_{t\to\infty} Q_t^\infty(f)(s,\theta) = \lim_{t\to\infty} f(s, \Psi_t^s(\theta)) = f(s, \theta^*(s)).$$

Thus, since $\pi^\infty(\mathrm{supp}(\mu)\times\Delta) = 1$, it comes that

$$\pi^\infty(f) = \int_S \int_\Delta f(s,\theta^*(s))\pi^\infty(\mathrm{d}s\mathrm{d}\theta) = \int_S f(s,\theta^*(s))\mu(\mathrm{d}s).$$

This proves the first part of Proposition 1.10. The second part follows directly from the first one.

4. Concluding remarks

The results and proofs given here all rely on the assumption that $(\omega_t)_{t\geqslant 0}$ is a Markov process. In particular, they do not apply to the case where $t \to \omega_t$ is a deterministic periodic signal with discontinuities. This situation is investigated in the preprint [4]. The recent preprint [16] provides a first order expansion of Λ^T when T goes to 0.

Acknowledgment

We thank Philippe Carmona and an anonymous referee for their careful reading and useful comments.

Bibliography

[1] L. Arnold, V. M. Gundlach & L. Demetrius, "Evolutionary formalism for products of positive random matrices", *Ann. Appl. Probab.* **4** (1994), no. 3, p. 859-901.

[2] M. Benaïm & T. Hurth, *Markov Chains on Metric Spaces, A Short Course*, Universitext, vol. 99, Springer, 2022, xii+197 pages.

[3] M. Benaïm, C. Lobry, T. Sari & E. Strickler, "Untangling the role of temporal and spatial variations in persistence of populations", *Theor. Popul. Biol.* **154** (2023), p. 1-26.

[4] ———, "When can a population spreading across sink habitats persist?", *J. Math. Biol.* **88** (2024), article no. 19 (56 pages).

[5] M. Benaïm & E. Strickler, "Random switching between vector fields having a common zero", *Ann. Appl. Probab.* **29** (2019), no. 1, p. 326-375.

[6] P. Carmona, "Asymptotic of the largest Floquet multiplier for cooperative matrices", *Ann. Fac. Sci. Toulouse, Math. (6)* **31** (2022), no. 4, p. 1213-1221.

[7] R. M. Dudley, *Real analysis and probability*, Cambridge Studies in Advanced Mathematics, vol. 74, Cambridge University Press, 2002, revised reprint of the 1989 original, x+555 pages.

[8] M. Freidlin & A. Wentzell, *Random perturbations of dynamical systems*, third ed., Grundlehren der Mathematischen Wissenschaften, vol. 260, Springer, 2012, translated from the 1979 Russian original by Joseph Szücs, xxviii+458 pages.

[9] M. W. Hirsch & H. Smith, "Chapter 4 Monotone Dynamical Systems", in *Handbook of Differential Equations: Ordinary Differential Equations. Vol. 2*, North-Holland, 2006, p. 239-357.

[10] O. Kallenberg, *Foundations of modern probability*, Probability Theory and Stochastic Modelling, vol. 99, Springer, 2021, third edition of the 1997 original, xii+946 pages.

[11] U. Krengel, *Ergodic Theorems*, De Gruyter Studies in Mathematics, vol. 99, Walter de Gruyter, 1985, xii+349 pages.

[12] T. Malik & H. Smith, "Does Dormancy Increase Fitness of Bacterial Populations in Time-Varying Environments?", *Bull. Math. Biol.* **70** (2008), p. 1140-1162.

[13] J. Mierczyński, "Estimates for Principal Lyapunov Exponents: A Survey", *Nonauton. Dyn. Syst.* **1** (2014), no. 1, p. 137-162.

[14] ———, "Lower estimates of top Lyapunov exponent for cooperative random systems of linear ODEs", *Proc. Am. Math. Soc.* **143** (2015), no. 3, p. 1127-1135.

[15] J. Mierczyński & W. Shen, "Principal Lyapunov exponents and principal Floquet spaces of positive random dynamical systems. II. Finite-dimensional systems", *J. Math. Anal. Appl.* **404** (2013), no. 2, p. 438-458.

[16] P. Monmarché & E. Strickler, "Asymptotic expansion of the invariant measurefor Markov-modulated ODEs at high frequency", *J. Appl. Probab.* (2025), FirstView (27 pages), `doi:10.1017/jpr.2024.107`.

[17] D. Ruelle, "Analyticity properties of the characteristic exponents of random matrix", *Adv. Math.* **32** (1979), no. 3, p. 68-80.

[18] E. Seneta, *Non-negative matrices and Markov chains*, Springer Series in Statistics, Springer, 2006, revised reprint of the second (1981) edition, xvi+287 pages.