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Nakano–Nadel type, Bogomolov–Sommese type vanishing and singular dual Nakano semi-positivity [(*)]

Yuta Watanabe [(1)]

Abstract. — In this article, we get properties for singular (dual) Nakano semi-positivity and obtain vanishing theorems involving L^2-subsheaves on weakly pseudoconvex manifolds by L^2-estimates and L^2-type Dolbeault isomorphisms. As applications, Fujita's conjecture type theorem with singular Hermitian metrics is presented.

Résumé. — Dans cet article, nous obtenons des propriétés de semi-positivité singulière (duale) de Nakano et obtenons des théorèmes de disparition impliquant des sous-faisceaux L^2 sur des variétés faiblement pseudoconvexes par des estimations L^2 et des isomorphismes de Dolbeault de type L^2. En tant qu'applications, un théorème de type conjecture de Fujita avec des métriques hermitiennes singulières est présenté.

1. Introduction

Throughout this paper, we let X be an n-dimensional complex manifold. Let φ be a plurisubharmonic function on X and let $\mathscr{I}(\varphi)$ be the sheaf of germs of holomorphic functions f such that $|f|^2e^{-\varphi}$ is locally integrable which is called the multiplier ideal sheaf. For a singular Hermitian metric h on a holomorphic line bundle, we define the multiplier ideal sheaf by $\mathscr{I}(h) := \mathscr{I}(\varphi)$ where $h = e^{-\varphi}$ locally. As an invariant of the singularities of the plurisubharmonic functions, the multiplier ideal sheaf play important role in the study of the several complex variables and complex algebraic geometry. Here, a function $\psi : X \to [-\infty, +\infty)$ is *exhaustive* if all sublevel sets $X_c := \{x \in X \mid \psi(x) < c\}, \forall\ c < \sup_X \psi$, are relatively compact. A complex

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manifold is said to be *weakly pseudoconvex* if there exists a smooth exhaustive plurisubharmonic function.

Among the vanishing theorems involving multiplier ideal sheaf, the Nadel–Demailly vanishing theorem [6, 28] is well known as an extension of the Kodaira vanishing theorem [23], and the Kawamata–Viehweg vanishing theorem (cf. [20, 37] and [8, Theorem 6.25] is a more detailed judgment of the positivity. While the theorems mentioned above are for (n,q)-forms, a vanishing result for (p,n)-forms was recently presented in [39] and improved (for the line bundle case) in [25].

That is, the following: let X be a projective manifold, ω be a Kähler metric on X and L be a holomorphic line bundle equipped with a singular Hermitian metric h. If (L,h) is big, i.e. $i\Theta_{L,h} \geqslant \varepsilon\omega$ in the sense of currents for some $\varepsilon > 0$. Then we have

$$H^q(X, K_X \otimes L \otimes \mathscr{I}(h)) = 0 \quad \text{for any} \quad q > 0 \quad \text{(see [6, 28])},$$
$$H^n(X, \Omega_X^p \otimes L \otimes \mathscr{I}(h)) = 0 \quad \text{for any} \quad p > 0 \quad \text{(see [39])}.$$

Note that the above vanishing theorem cannot be extended to the same bidegree (p,q) with $p+q > n$ as the Kodaira–Akizuki–Nakano type vanishing theorem by the Ramanujam's counterexample (cf. [32], [39, Remark 2.10]).

Recently, vanishing theorems for (n,q)-forms involving (Demailly) m-positive holomorphic vector bundles and multiplier ideal sheaves were shown in [17] on compact Kähler manifolds by using L^2-Hogde isomorphisms.

In this paper, we introduce the notion of a *dual m-positivity* (see Definition 2.3), which is associated with the positivity corresponding to (p,n)-forms (see Lemma 2.4), and obtain vanishing theorems for (n,q)-cohomology groups (resp. (p,n)-cohomology groups) involving m-positive (resp. dual m-positive) holomorphic vector bundles and multiplier ideal sheaves by using L^2-estimates and L^2-type Dolbeault isomorphisms. The compactness assumption can be relaxed, we show that the vanishing theorem naturally holds for weakly pseudoconvex manifolds.

Theorem 1.1. — *Let X be a weakly pseudoconvex Kähler manifold, F be a holomorphic vector bundle of rank r over X and L be a holomorphic line bundle over X equipped with a singular Hermitian metric h. We assume that F is m-semi-positive and h is singular positive, i.e. the local weights of h coincide with some strictly plurisubharmonic function almost everywhere (see Definition 3.2 below). Then we have the following*

$$H^q(X, K_X \otimes F \otimes L \otimes \mathscr{I}(h)) = 0$$

for $q > 0$ with $m \geqslant \min\{n-q+1, r\}$.

If the singular Hermitian metric on line bundle is semi-positive, then the following holds by assuming m-positivity for the vector bundle.

Theorem 1.2. — *Let X be a weakly pseudoconvex manifold and L be a holomorphic line bundle over X equipped with a singular semi-positive Hermitian metric h, i.e. $i\Theta_{L,h} \geqslant 0$ in the sense of currents. Then we have the following*

(a) *If X has a Kähler metric and A is a k-positive holomorphic line bundle then*

$$H^q(X, K_X \otimes A \otimes L \otimes \mathscr{I}(h)) = 0$$

for any $q \geqslant k$.

(b) *If F is an m-positive holomorphic vector bundle of rank r then*

$$H^q(X, K_X \otimes F \otimes L \otimes \mathscr{I}(h)) = 0$$

for $q > 0$ with $m \geqslant \min\{n - q + 1, r\}$.

Here, Theorem 1.1 and Theorem 1.2 (b) are generalizations to weakly pseudoconvex manifolds of [17, Theorem 1.14 and Theorem 1.9], respectively. Furthermore, we also provide similar vanishing theorems for (p, n)-forms.

Theorem 1.3. — *Let X be a compact Kähler manifold, F be a holomorphic vector bundle of rank r over X and L be a holomorphic line bundle over X equipped with a singular Hermitian metric h. We assume that F is dual m-semi-positive and h is singular positive, i.e. the local weights of h coincide with some strictly plurisubharmonic function almost everywhere (see Definition 3.2 below). Then we have the following*

$$H^n\left(X, \Omega_X^p \otimes F \otimes L \otimes \mathscr{I}(h)\right) = 0$$

for $p > 0$ with $m \geqslant \min\{n - p + 1, r\}$.

Theorem 1.4. — *Let X be a compact manifold and L be a holomorphic line bundle over X equipped with a singular semi-positive Hermitian metric h, i.e. $i\Theta_{L,h} \geqslant 0$ in the sense of currents. Then we have the following*

(a) *If X has a Kähler metric and A is a k-positive holomorphic line bundle then*

$$H^n\left(X, \Omega_X^p \otimes A \otimes L \otimes \mathscr{I}(h)\right) = 0$$

for any $p \geqslant k$.

(b) *If F is a dual m-positive holomorphic vector bundle of rank r then*

$$H^n\left(X, \Omega_X^p \otimes F \otimes L \otimes \mathscr{I}(h)\right) = 0$$

for $p > 0$ with $m \geqslant \min\{n - p + 1, r\}$.

Note, n^{th} cohomology always vanishes on non-compact complex manifolds (cf. [27, 30]). Indeed, Theorem 1.2 and 1.4 hold on weak pseudoconvex manifolds with positive line bundles; however, their novelty is only observed in the compact case.

Notions of singular Hermitian metrics for holomorphic vector bundles were introduced and investigated (cf. [2, 3]) and of positivity for singular Hermitian metrics is very interesting subjects. It is known that we cannot always define the curvature currents with measure coefficients [33]. Therefore, semi-negativity for Griffiths and Nakano, and semi-positivity for Griffiths and dual Nakano (cf. [2, 31, 33, 39]) were defined naturally by using the properties of plurisubharmonicity instead of the curvature currents. Nakano semi-positivity is defined using a characterization based on L^2-estimates for smooth Hermitian metrics (cf. [10, 19]).

We study properties of singular (dual) Nakano semi-positivity and vanishing theorems. Among them we obtain the following dual-type generalization of Demailly and Skoda's theorem (cf. [4, 26]) to singular metrics. Indeed the metric $h \otimes \det h$ is already known to be L^2-type Nakano semi-positive and L^2-type dual Nakano semi-positive by [19, Theorem 1.3] and [39, Theorem 5.3]; here, the notion of dual Nakano semi-positivity is stronger and more natural than that of L^2-type dual Nakano semi-positivity.

Theorem 1.5. — *Let X be a complex manifold and E be a holomorphic vector bundle over X equipped with a singular Hermitian metric h. If h is Griffiths semi-positive then $h \otimes \det h$ is dual Nakano semi-positive.*

We also get the following vanishing theorems which are generalizations of Griffiths type vanishing theorem to singularities, (dual) m-positivity and weakly pseudoconvex Kähler manifolds. Here, $\mathscr{E}(h)$ is the L^2-subsheaf of $\mathcal{O}_X(E)$ with respect to a singular Hermitian metric h on E analogous to multiplier ideal sheaves. In fact, $\mathscr{E}(h) = \mathcal{O}_X(E) \otimes \mathscr{I}(h)$ if E is a holomorphic line bundle. Moreover, if h is Griffiths semi-positive, then it is already known in [16, 19] that $\mathscr{E}(h \otimes \det h)$ is coherent.

Theorem 1.6. — *Let X be a weakly pseudoconvex manifold and E be a holomorphic vector bundle over X equipped with a singular Hermitian metric h. We assume that h is Griffiths semi-positive on X. Then we have the following*

(a) *If X has a positive holomorphic line bundle and A is a k-positive holomorphic line bundle, then we have the vanishing*

$$H^q\left(X, K_X \otimes A \otimes \mathscr{E}(h \otimes \det h)\right) = 0,$$

for any $q \geqslant k$.

(b) *If F is an m-positive holomorphic vector bundle of rank r over X then*

$$H^q\left(X, K_X \otimes F \otimes \mathscr{E}(h \otimes \det\, h)\right) = 0$$

for $q \geqslant 1$ *and* $m \geqslant \min\{n-q+1, r\}$.

Theorem 1.7. — *Let X be a projective manifold and E be a holomorphic vector bundle over X equipped with a singular Hermitian metric h. We assume that h is Griffiths semi-positive on X. Then we have the following*

(a) *If A is a k-positive holomorphic line bundle, then we have*

$$H^n\left(X, \Omega_X^p \otimes A \otimes \mathscr{E}(h \otimes \det\, h)\right) = 0,$$

for any $p \geqslant k$.

(b) *If F is a dual m-positive holomorphic vector bundle of rank r over X then*

$$H^n\left(X, \Omega_X^p \otimes F \otimes \mathscr{E}(h \otimes \det\, h)\right) = 0$$

for $p \geqslant 1$ *and* $m \geqslant \min\{n-p+1, r\}$.

Moreover, in Section 6, we provide vanishing theorems for singular (dual) Nakano semi-positivity twisted by (dual) m-positive vector bundles on weakly pseudoconvex manifolds with a positive holomorphic line bundle (resp. projective manifolds).

Recently, a Fujita Conjecture type theorem involving multiplier ideal sheaves was presented in [35] using vanishing theorems. Finally, we provide a vanishing theorem that is a more detailed judgment of positivity by numerically dimension for nef line bundle on projective manifolds, and is analogous to [8, Theorem 6.25], i.e. the generalized Kawamata–Viehweg vanishing theorem. We obtain Fujita's conjecture type theorem involving the L^2-subsheaf as an application of our vanishing theorems.

Theorem 1.8. — *Let X be a compact Kähler manifold and E be a holomorphic vector bundle equipped with a singular Hermitian metric h. Let N be a nef line bundle which is neither big nor numerically trivial, i.e.* $\mathrm{nd}(N) \notin \{0, n\}$. *If h is Griffiths semi-positive and there exists a smooth ample divisor A such that the Lelong number $\nu(-\log\det\, h|_A, x) < 1$ for all points in A and that* $\mathrm{nd}(N|_A) = \mathrm{nd}(N)$, *then we have*

$$H^q(X, K_X \otimes N \otimes \mathscr{E}(h \otimes \det\, h)) = 0$$

for any $q > n - \mathrm{nd}(N)$.

Theorem 1.9. — *Let X be a compact Kähler manifold and E be a holomorphic vector bundle equipped with a singular Hermitian metric h. Let L be an ample and globally generated line bundle and N be a nef but not numerically trivial line bundle. If h is Griffiths semi-positive and there exists a smooth ample divisor A such that the Lelong number $\nu(-\log\det h|_A, x) < 1$ for all points in A and that $\mathrm{nd}(N|_A) = \mathrm{nd}(N)$, then the adjoint coherent sheaf*

$$K_X \otimes L^{\otimes n} \otimes N \otimes \mathscr{E}(h \otimes \det h)$$

is globally generated.

Here, for any nef line bundle N, if $\mathrm{nd}(N) \neq n$, i.e. N is not big, then we can always select a nonsingular ample divisor A satisfying $\mathrm{nd}(N|_A) = \mathrm{nd}(N)$.

Strategy of the proof

Vanishing theorems involving multiplier ideal sheaves (resp. L^2-subsheaves) reflecting more precisely singularities of singular Hermitian metrics are derived from analytical techniques such as L^2-estimates in Section 4 and L^2-type Dolbeault isomorphisms in Section 5.

Usually, smoothness of Hermitian metrics on vector bundles is required for L^2-estimates. In order to obtain L^2-estimates with singular Hermitian metrics, the existence of approximations in Section 3.2 is crucial. This is because L^2-estimates are preserved by increasing approximations with the same positivity. In the case of singular Hermitian metrics on vector bundles, since the approximation exists only on Stein manifolds, the L^2-estimates were obtained only on projective manifolds. However, by using Takayama's theorem (cf. [36, Theorem 1.2]), this was extended to weakly pseudoconvex manifolds with a positive line bundle.

Furthermore, in the case of singular metrics, curvature currents may not always exist, and the Bochner–Kodaira–Nakano identity may not hold. Therefore, by avoiding the direct use of curvature and employing techniques similar to *optimal* L^2-estimates introduced in [10], we can address this difficulty and ultimately derive vanishing theorems even in this form.

In proving Nakano semi-positivity in Theorem 1.5, deriving it directly from the definition is difficult. Therefore, an effective method for determination is provided instead (see Proposition 3.15). This is based on the observation that convolving with approximations to the identity is well-suited for preserving plurisubharmonicity.

The rough idea behind Theorem 1.8 is to first solve the vanishing theorem in favorable conditions, namely when L is nef and big, and assuming the condition on the Lelong number of h are imposed globally on X. Then, the general case is tackled using induction on dimension and the Ohsawa–Takegoshi L^2-extension theorem. Finally, Theorem 1.9 is proved using Theorems 1.6 and 1.8, along with Castelnuovo–Mumford regularity.

Organization of the paper

In Section 2, we introduce various notions of positivity for smooth Hermitian metrics and provide a generalized characterization of Nakano semi-positivity based on L^2-estimates.

In Section 3, we introduce various notions of positivity for singular Hermitian metrics in Subsection 3.1, and elucidate properties such as the existence of smooth approximations for these positivity notions and the preservation of positivity through tensor products in Subsection 3.1. Finally, we provide the proof of Theorem 1.5.

In Section 4, we establish L^2-estimates for singular Hermitian metrics associated with various notions of positivity on weakly pseudoconvex manifolds.

In Section 5, by effectively handling L^2-estimates we provide L^2 fine resolutions $(\mathscr{L}^{p,\bullet}_{F\otimes E,h_F\otimes h},\bar{\partial})$ for the coherent sheaf $\Omega^p_X\otimes\mathcal{O}_X(F)\otimes\mathscr{E}(h)$ twisted by the L^2-subsheaf $\mathscr{E}(h)$ (resp. $\mathscr{I}(h)$) associated with a semi-positive singular Hermitian metric h on E. Here, (F,h_F) is a holomorphic Hermitian vector bundle. This leads to the establishment of L^2-type Dolbeault isomorphisms.

In Section 6, we provide various vanishing theorems, which are the main results, by using the L^2-estimates from Section 4 and L^2-type Dolbeault isomorphisms from Section 5.

In Section 7, we establish Theorem 1.8 which provides a more detailed judgment of positivity analogous to [8, Theorem 6.25]. As an application, we provide a Fujita's conjecture type global generation theorem (Theorem 1.9) involving L^2-subsheaves.

2. Smooth Hermitian metrics and positivity

In this section, we define various positivity for holomorphic vector bundles and show equivalence relations with Nakano semi-positivity by using L^2-estimates.

Let ω be a Hermitian metric on X and (E,h) be a holomorphic Hermitian vector bundle of rank r over X. Let $D_h = D'_h + \bar\partial$ be the Chern connection of (E,h), and $\Theta_{E,h} = [D'_h, \bar\partial] = D'_h\bar\partial + \bar\partial D'_h$ be the Chern curvature tensor. Let $(U,(z_1,\dots,z_n))$ be local coordinates. Denote an orthonormal frame of E over $U \subset X$ by $(e_1,\dots,e_r)$, then we can write

$$\begin{aligned} i\Theta_{E,h,x_0} &= i\sum_{j,k}\Theta_{jk}\mathrm{d}z_j\wedge\mathrm{d}\bar z_k \\ &= i\sum_{j,k,\lambda,\mu} c_{jk\lambda\mu}\mathrm{d}z_j\wedge\mathrm{d}\bar z_k\otimes e^*_\lambda\otimes e_\mu, \\ \bar c_{jk\lambda\mu} &= c_{kj\mu\lambda}, \end{aligned}$$

at $x_0 \in U$. To $i\Theta_{E,h}$ corresponds a natural Hermitian form $\theta_{E,h}$ on $T_X\otimes E$ defined by

$$\begin{aligned} \theta_{E,h}(u) := \theta_{E,h}(u,u) &= \sum c_{jk\lambda\mu}u_{j\lambda}\bar u_{k\mu} \\ \text{for any } u &= \sum u_{j\lambda}\frac{\partial}{\partial z_j}\otimes e_\lambda \in T_{X,x_0}\otimes E_{x_0}, \\ \text{i.e.}\quad \theta_{E,h} &= \sum c_{jk\lambda\mu}\,(\mathrm{d}z_j\otimes e^*_\lambda)\otimes\overline{(\mathrm{d}z_k\otimes e^*_\mu)}. \end{aligned}$$

Definition 2.1. — *Let L be a holomorphic line bundle over a complex manifold X. We say that L is k-*positive *if there exists a smooth Hermitian metric h such that $i\Theta_{L,h}$ is semi-positive and has at least $n-k+1$ positive eigenvalues at every point of X.*

Definition 2.2 (cf. [9, Chapter VII], [26, Definition 2.1]). — *Let T and E be complex vector spaces of dimensions n, r respectively, and Θ be a hermitian form on $T\otimes E$. Let (E,h) be a holomorphic vector bundle over a complex manifold X.*

- *A tensor $u \in T\otimes E$ is said to be of rank m if m is the smallest $\geqslant 0$ integer such that u can be written $u = \sum_{j=1}^m \xi_j\otimes s_j$, where $\xi_j \in T$, $s_j \in E$.*
- *Θ is m-*positive *(resp. m-*semi-positive*) if $\Theta(u) > 0$ (resp. $\Theta(u) \geqslant 0$) for any tensor $0 \neq u \in T\otimes E$ of rank $\leqslant m$. In this case, we write $\Theta >_m 0$ (resp. $\geqslant_m 0$).*
- *(E,h) is m-*positive *(resp. m-*semi-positive*) if $\theta_{E,h} >_m 0$ (resp. $\theta_{E,h} \geqslant_m 0$).*
- *(E,h) is said to be* Griffiths positive *(resp.* Griffiths semi-positive*) if (E,h) is 1-positive (resp. 1-semi-positive).*
- *(E,h) is said to be* Nakano positive *(resp.* Nakano semi-positive*) if $\theta_{E,h}$ is positive (resp. semi-positive) definite as a Hermitian form on $T_X\otimes E$, i.e. $\theta_{E,h}(u) > 0$ (resp. $\geqslant 0$). Here, Nakano positivity corresponds to $m \geqslant \min\{n,r\}$.*

- (E, h) *is said to be* dual Nakano positive *(resp.* dual Nakano semi-positive*) if* (E^*, h^*) *is Nakano negative (resp. Nakano semi-negative).*

It is clear that the concepts of Griffiths positive, Nakano positive, 1-positive and positive coincide if rank $E = 1$. We introduce another notion about m-positivity that correspond to positivity for (p, n)-forms.

Definition 2.3. — *Let X be a complex manifold of dimension n and (E, h) be a holomorphic Hermitian vector bundle of rank r over X. (E, h) is said to be* dual m-positive *(resp.* dual m-semi-positive*) if (E^*, h^*) is* m-negative *(resp.* m-semi-negative*).*

Here, E is said to be $\bullet$-*positive* (resp. $\bullet$-*semi-positive*) if there exists a smooth Hermitian metric h_E such that (E, h_E) is $\bullet$-*positive* (resp. $\bullet$-*semi-positive*), where $\bullet$ is (dual) m, Griffiths and (dual) Nakano. Notes that 1-positivity and dual 1-positivity are equivalent due to the equivalence between Griffiths-positivity of (E, h) and Griffiths-negativity of (E^*, h^*).

We denote the operator $[i\Theta_{E,h}, \Lambda_\omega]$ on $\bigwedge^{p,q} T^*_X \otimes E$ by $A^{p,q}_{E,h,\omega}$ and we simply write $A^{p,q}_{E,h,\omega} > 0$ (resp. $\geqslant 0$) if the operator $[i\Theta_{E,h}, \Lambda_\omega]$ is positive (resp. semi-positive) definite on $\bigwedge^{p,q} T^*_X \otimes E$. We obtain the following lemma for the relationship between positivity of the operator $A^{p,q}_{E,h,\omega}$ and (dual) m-positivity by using [9, Chapter VII, Lemma 7.2] and [38, Theorems 2.3 and 2.5]. Let $\mathcal{E}^{p,q}(E)$ be the sheaf of germs of $\mathcal{C}^\infty$ sections of $\bigwedge^{p,q} T^*_X \otimes E$.

Lemma 2.4. — *Let (X, ω) be a Hermitian manifold and (E, h) be a holomorphic vector bundle over X. Then we obtain the following*

(a) *If (E, h) is m-positive (resp. m-semi-positive), then we get*

$$A^{n,q}_{E,h,\omega} = [i\Theta_{E,h}, \Lambda_\omega] > 0 \quad (\textit{resp. } \geqslant 0) \qquad \textit{for } q \geqslant 1 \textit{ and } m \geqslant \min\{n - q + 1, r\}.$$

(b) *If (E, h) is dual m-positive (resp. dual m-semi-positive), then we get*

$$A^{p,n}_{E,h,\omega} = [i\Theta_{E,h}, \Lambda_\omega] > 0 \quad (\textit{resp. } \geqslant 0) \qquad \textit{for } p \geqslant 1 \textit{ and } m \geqslant \min\{n - p + 1, r\}.$$

Proposition 2.5. — *Let (X, ω) be a Hermitian manifold of dimension n and p, q be fixed integers. Let (E, h_E) and (F, h_F) be holomorphic vector bundles over X and C_E, C_F be non-negative real numbers.*

If $A^{p,q}_{E,h_E,\omega} \geqslant C_E \cdot \mathrm{id}_E$, then we obtain $B_{h_E,\omega} := [i\Theta_{E,h} \otimes \mathrm{id}_F, \Lambda_\omega] \geqslant C_E \cdot \mathrm{id}_{E\otimes F}$ on $\bigwedge^{p,q} T^*_X \otimes E \otimes F$, and further assuming $A^{p,q}_{F,h_F,\omega} \geqslant C_F \cdot \mathrm{id}_F$ yields $A^{p,q}_{E\otimes F, h_E\otimes h_F,\omega} \geqslant (C_E + C_F) \cdot \mathrm{id}_{E\otimes F}$.

Proof. — It suffices to check it pointwisely. First, we show the case where $A^{p,q}_{F,h_F,\omega} \geqslant C_F \cdot \mathrm{id}_F$ is further assumed. Let $x_0 \in X$ and $(z_1, \dots, z_n)$ be local coordinates such that $(\partial/\partial z_1, \dots, \partial/\partial z_n)$ is an orthonormal basis of (T_X, ω) at x_0. Let $(e_1, \dots, e_r)$ and $(f_1, \dots, f_r)$ be orthonormal bases of E_{x_0} and F_{x_0}, respectively. We can write $\omega_{x_0} = i\sum_{j=1}^n \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$ and

$$\begin{aligned} i\Theta_{E,h_E,x_0} &= i\sum c^E_{jk\lambda\mu}\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k \otimes e^*_\lambda \otimes e_\mu, \\ i\Theta_{F,h_F,x_0} &= i\sum c^F_{jk\lambda\mu}\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k \otimes f^*_\lambda \otimes f_\mu. \end{aligned}$$

Let J, K be ordered multi-indices with $|J| = p$ and $|K| = q$. For any (p,q)-form $u \in \bigwedge^{p,q} T^*_{X,x_0} \otimes E_{x_0} \otimes F_{x_0}$, we can write

$$\begin{aligned} u &= \sum_{|J|=p,|K|=q,\lambda,\tau} u_{JK\lambda\tau}\mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K \otimes e_\lambda \otimes f_\tau \\ &= \sum_\tau u^E_\tau \otimes f_\tau \\ &= \sum_\lambda u^F_\lambda \otimes e_\lambda, \end{aligned}$$

where

$$\begin{aligned} u^E_\tau &= \sum_{|J|=p,|K|=q,\lambda} u_{JK\lambda\tau}\mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K \otimes e_\lambda, \\ u^F_\lambda &= \sum_{|J|=p,|K|=q,\tau} u_{JK\lambda\tau}\mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K \otimes f_\tau. \end{aligned}$$

We have the following calculations (cf. [9, Chapter VII])

$$\begin{aligned} \Lambda_\omega u &= i(-1)^p \sum_{J,K,\lambda,\tau,s} u_{JK\lambda\tau}\left(\frac{\partial}{\partial z_s} \lrcorner\, \mathrm{d}z_J\right) \\ &\qquad \wedge \left(\frac{\partial}{\partial \bar{z}_s} \lrcorner\, \mathrm{d}\bar{z}_K\right) \otimes e_\lambda \otimes f_\tau \\ &= \sum_\tau \left(\Lambda_\omega u^E_\tau\right) \otimes f_\tau = \sum_\lambda \left(\Lambda_\omega u^F_\lambda\right) \otimes e_\lambda, \\ i\Theta_{E,h_E} \otimes \mathrm{id}_F\, u &= i \sum_{j,k,\lambda,\mu,\tau} \left(c^E_{jk\lambda\mu}\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k \otimes e^*_\lambda \otimes e_\mu \otimes f^*_\tau \otimes f_\tau\right) u \\ &= \sum_\tau \left(\left(i\Theta_{E,h_E}\right) \otimes f^*_\tau \otimes f_\tau\right)\left(\sum_\alpha u^E_\alpha \otimes f_\alpha\right) \\ &= \sum_\tau \left(i\Theta_{E,h_E} u^E_\tau\right) \otimes f_\tau. \end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\big[i\Theta_{E,h_E}\otimes \mathrm{id}_F,\Lambda_\omega\big]u\\
&\qquad = \big(i\Theta_{E,h_E}\otimes \mathrm{id}_F\big)\wedge\Lambda_\omega u-\Lambda_\omega\wedge\big(i\Theta_{E,h_E}\otimes \mathrm{id}_F\big)u\\
&\qquad = \sum_\tau \left(\big(i\Theta_{E,h_E}\big)\otimes f_\tau^*\otimes f_\tau\right)\wedge\left(\sum_\alpha\big(\Lambda_\omega u_\alpha^E\big)\otimes f_\alpha\right)\\
&\qquad\qquad -\Lambda_\omega\wedge\left(\sum_\tau\big(i\Theta_{E,h_E}u_\tau^E\big)\otimes f_\tau\right)\\
&\qquad = \sum_\tau\big(i\Theta_{E,h_E}\wedge\Lambda_\omega\big)u_\tau^E\otimes f_\tau-\sum_\tau\big(\Lambda_\omega\wedge i\Theta_{E,h_E}\big)u_\tau^E\otimes f_\tau\\
&\qquad = \sum_\tau\left([i\Theta_{E,h_E},\Lambda_\omega]u_\tau^E\right)\otimes f_\tau.
\end{aligned}$$

Hence, we obtain the following

$$\begin{aligned}
&\left\langle A^{p,q}_{E\otimes F,h_E\otimes h_F,\,\omega}u,u\right\rangle_{h_E\otimes h_F,\,\omega}\\
&= \big\langle[i\Theta_{E,h_E}\otimes\mathrm{id}_F,\Lambda_\omega]u,u\big\rangle_{h_E\otimes h_F,\,\omega}+\big\langle[i\Theta_{F,h_F}\otimes\mathrm{id}_E,\Lambda_\omega]u,u\big\rangle_{h_E\otimes h_F,\,\omega}\\
&= \left\langle\sum_\tau\left([i\Theta_{E,h_E},\Lambda_\omega]u_\tau^E\right)\otimes f_\tau,\sum_\alpha u_\alpha^E\otimes f_\alpha\right\rangle_{h_E\otimes h_F,\,\omega}\\
&\qquad +\left\langle\sum_\lambda\left([i\Theta_{F,h_F},\Lambda_\omega]u_\lambda^F\right)\otimes e_\lambda,\sum_\beta u_\beta^F\otimes e_\beta\right\rangle_{h_E\otimes h_F,\,\omega}\\
&= \sum_\tau\left\langle A^{p,q}_{E,h_E,\omega}u_\tau^E,u_\tau^E\right\rangle_{h_E,\omega}+\sum_\lambda\left\langle A^{p,q}_{F,h_F,\,\omega}u_\lambda^F,u_\lambda^F\right\rangle_{h_F,\,\omega}\\
&\geqslant \sum_\tau C_E\left|u_\tau^E\right|^2_{h_E,\omega}+\sum_\lambda C_F\left|u_\lambda^F\right|^2_{h_F,\,\omega}=(C_E+C_F)|u|^2_{h_E\otimes h_F,\,\omega}.
\end{aligned}$$

This represents $A^{p,q}_{E\otimes F,h_E\otimes h_F,\,\omega}\geqslant(C_E+C_F)\cdot\mathrm{id}_{E\otimes F}$.

Finally, we immediately obtain

$$\big\langle[i\Theta_{E,h_E}\otimes\mathrm{id}_F,\Lambda_\omega]u,u\big\rangle_{h_E\otimes h_F,\,\omega}\geqslant C_E|u|^2_{h_E\otimes h_F,\,\omega},$$

i.e. $B_{h_E,\omega}\geqslant C_E\cdot\mathrm{id}_{E\otimes F}$, from the above calculations. □

Finally we give a characterization of smooth Nakano semi-positive metrics by L^2-estimate. Similar results can be found in the previous works like [10, Theorem 1.1], [19, Proposition 2.8] and [38, Theorem 1.7], and our result (Proposition 2.6) is a generalization of them. Heuristically speaking, the idea is that the tensor product of a Nakano semi-positive vector bundle with an m-semi-positive vector bundle is still m-semi-positive.

Proposition 2.6. — *Let E be a holomorphic vector bundle and h be a smooth Hermitian metric on E. Then the following conditions are equivalent.*

(1) *h is Nakano semi-positive, i.e. $A^{n,q}_{E,h,\omega} \geqslant 0$ for any $q \geqslant 1$ and any Kähler metric ω (see Lemma 2.4,* [10, Lemma 4.7] *and* [4]*).*
(2) *For any positive integer $k \in \{1, \dots, n\}$, any Stein coordinate open subset S, any Kähler metric ω_S on S and any smooth Hermitian metric h_F on any holomorphic vector bundle F such that $A^{n,s}_{F,h_F,\omega_S} > 0$ for $s \geqslant k$, we have that for any $q \geqslant k$ and any $f \in L^2_{n,q}(S, E \otimes F, h \otimes h_F, \omega_S)$ satisfying $\bar\partial f = 0$ and $\int_S \langle B^{-1}_{h_F,\omega_S} f, f\rangle_{h\otimes h_F,\omega_S} dV_{\omega_S} < +\infty$, there exists $u \in L^2_{n,q-1}(S, E\otimes F, h \otimes h_F, \omega_S)$ such that $\bar\partial u = f$ and*
$$\int_S |u|^2_{h\otimes h_F,\omega_S} \mathrm{d}V_{\omega_S} \leqslant \int_S \left\langle B^{-1}_{h_F,\omega_S} f, f\right\rangle_{h\otimes h_F,\omega_S} \mathrm{d}V_{\omega_S},$$
where $B_{h_F,\omega_S} = [i\Theta_{F,h_F} \otimes \mathrm{id}_E, \Lambda_{\omega_S}]$.

Proof. — First, we show (1)⇒(2). Here, h is Nakano semi-positive if and only if $A^{n,q}_{E,h,\omega_S} \geqslant 0$ for $q \geqslant 1$. From the proof of Proposition 2.5, we have
$$\begin{aligned} A^{n,q}_{E\otimes F,h\otimes h_F,\omega_S} &= [i\Theta_{E,h} \otimes \mathrm{id}_F, \Lambda_{\omega_S}] + [i\Theta_{F,h_F} \otimes \mathrm{id}_E, \Lambda_{\omega_S}] \\ &\geqslant [i\Theta_{F,h_F} \otimes \mathrm{id}_E, \Lambda_{\omega_S}] = B_{h_F,\omega_S} > 0 \end{aligned}$$
on S for any $q \geqslant k$. By L^2-estimate for (n,q)-forms and possibly non-complete Kähler metric (see [9, Chapter VIII]), for any $q \geqslant k$ and any $\bar\partial$-closed $f \in L^2_{n,q}(S, E\otimes F, h\otimes h_F, \omega_S)$, there exists $u \in L^2_{n,q-1}(S, E\otimes F, h \otimes h_F, \omega_S)$ such that $\bar\partial u = f$ and that
$$\begin{aligned} \int_S |u|^2_{h\otimes h_F,\omega} \mathrm{d}V_{\omega_S} &\leqslant \int_S \left\langle \left(A^{n,q}_{E\otimes F,h\otimes h_F,\omega_S}\right)^{-1} f, f\right\rangle_{h\otimes h_F,\omega_S} \mathrm{d}V_{\omega_S} \\ &\leqslant \int_S \left\langle B^{-1}_{h_F,\omega_S} f, f\right\rangle_{h\otimes h_F,\omega_S} \mathrm{d}V_{\omega_S}. \end{aligned}$$

Second, we consider (2)⇒(1). We take a Stein coordinate S, a Kähler metric ω on S and a holomorphic vector bundle F. Let h_F be a smooth Hermitian metric on F such that $A^{n,s}_{F,h_F,\omega} > 0$ for $s \geqslant k$. Then for any $q \geqslant k$ and any $\bar\partial$-closed $f \in \mathcal{E}^{n,q}(S, E\otimes F) \subset L^2_{n,q}(S, E\otimes F, h\otimes h_F, \omega)$, there is $u \in L^2_{n,q-1}(S, E\otimes F, h\otimes h_F, \omega)$ such that $\bar\partial u = f$ and
$$\|u\|^2_{h\otimes h_F,\omega} = \int_S |u|^2_{h\otimes h_F,\omega} \mathrm{d}V_\omega \leqslant \int_S \left\langle B^{-1}_{h_F,\omega} f, f\right\rangle_{h\otimes h_F,\omega} \mathrm{d}V_\omega,$$
where we assume that the right-hand side is finite.

From the Bochner–Kodaira–Nakano identity, for any $\alpha \in \mathcal{E}^{n,q}(S, E \otimes F)$ we have that

$$\begin{aligned}
&\left|\langle\!\langle f, \alpha\rangle\!\rangle_{h\otimes h_F,\,\omega}\right|^2 \\
&\qquad = \left|\left\langle\!\!\left\langle \bar\partial u, \alpha\right\rangle\!\!\right\rangle_{h\otimes h_F,\,\omega}\right|^2 = \left|\left\langle\!\!\left\langle u, \bar\partial^*_{h\otimes h_F}\alpha\right\rangle\!\!\right\rangle_{h\otimes h_F,\,\omega}\right|^2 \\
&\qquad \leqslant \|u\|^2_{h\otimes h_F,\,\omega}\left\|\bar\partial^*_{h\otimes h_F}\alpha\right\|^2_{h\otimes h_F,\,\omega} \\
&\qquad \leqslant \int_S \left\langle B^{-1}_{h_F,\,\omega} f, f\right\rangle_{h\otimes h_F,\,\omega} \mathrm{d}V_\omega \cdot \Big(\left\|D'^*_{h\otimes h_F}\alpha\right\|^2_{h\otimes h_F,\,\omega} \\
&\qquad\qquad + \left\langle\!\!\left\langle [i\Theta_{E,h}\otimes \mathrm{id}_F, \Lambda_\omega]\alpha, \alpha\right\rangle\!\!\right\rangle_{h\otimes h_F,\,\omega} + \left\langle\!\!\left\langle B_{h_F,\,\omega}\alpha, \alpha\right\rangle\!\!\right\rangle_{h\otimes h_F,\,\omega}\Big),
\end{aligned}$$

where $D'_{h\otimes h_F}$ is the $(1,0)$ part of the Chern connection on $E\otimes F$ with respect to the metric $h\otimes h_F$. Let $\alpha = B^{-1}_{h_F,\,\omega} f$, i.e. $f = B_{h_F,\,\omega}\alpha$. Then the above inequality becomes

$$\begin{aligned}
&\left|\langle\!\langle B_{h_F,\,\omega}\alpha, \alpha\rangle\!\rangle_{h\otimes h_F,\,\omega}\right|^2 \\
&\leqslant \langle\!\langle \alpha, B_{h_F,\,\omega}\alpha\rangle\!\rangle_{h\otimes h_F,\,\omega}\Big(\left\|D'^*_{h\otimes h_F}\alpha\right\|^2_{h\otimes h_F,\,\omega} \\
&\qquad\qquad + \langle\!\langle [i\Theta_{E,h}\otimes \mathrm{id}_F, \Lambda_\omega]\alpha, \alpha\rangle\!\rangle_{h\otimes h_F,\,\omega} + \langle\!\langle B_{h_F,\,\omega}\alpha, \alpha\rangle\!\rangle_{h\otimes h_F,\,\omega}\Big).
\end{aligned}$$

Therefore we get

$$\left\langle\!\!\left\langle [i\Theta_{E,h}\otimes \mathrm{id}_F, \Lambda_\omega]\alpha, \alpha\right\rangle\!\!\right\rangle_{h\otimes h_F,\,\omega} + \left\|D'^*_{h\otimes h_F}\alpha\right\|^2_{h\otimes h_F,\,\omega} \geqslant 0. \tag{$*$}$$

Using this formula $(*)$, we show the proposition by contradiction.

Suppose that there is $q \in \mathbb{N}$ such that $A^{n,q}_{E,h,\omega}$ is not semi-positive. Then there is $x_0 \in X$ and $\xi_0 \in \bigwedge^{n,q} T^*_{X,x_0} \otimes E_{x_0}$ such that

$$\langle [i\Theta_{E,h}, \Lambda_\omega]\xi_0, \xi_0\rangle_{h,\omega} = -2c$$

for some $c > 0$.

For any $k \in \{1, \dots, n\}$ and any $R > 0$, we define the following Stein subsets of $\mathbb{C}^n$;

$$\Delta^k_R := \left\{ (z_1, \dots, z_{n-k+1}) \in \mathbb{C}^{n-k+1} \;\middle|\; \sum_{j=1}^{n-k+1} |z_j|^2 < R \right\} \subset \mathbb{C}^{n-k+1},$$

$$D^k_R := \left\{ (z_{n-k+2}, \dots, z_n) \in \mathbb{C}^{k-1} \;\middle|\; \sum_{j=n-k+2}^{n} |z_j|^2 < R \right\} \subset \mathbb{C}^{k-1},$$

so that $\Delta_R^1 = \Delta_R$. Let $(\Delta_{2R}, (z_1, \dots, z_n))$ be a holomorphic coordinate in X centered at x_0 and $\omega = i\partial\overline{\partial}|z|^2$ be a Kähler metric on Δ_{2R}. Here, we can choose R such that $E|_{\Delta_{2R}}$ is trivial on Δ_{2R}. For any trivial holomorphic vector bundle $F = \Delta_{2R} \times \mathbb{C}^t$ where $t = \operatorname{rank} F$, let I_F be a trivial Hermitian metric on F. Fixed an integer k with $q \geqslant k$. We define the Stein subset $S_R := \Delta_R^k \times D_R^k \subset \Delta_{2R}$ and define the plurisubharmonic function $\psi_k := \sum_{j=1}^{n-k+1} |z_j|^2 - R^2/4$ which is strongly k-convex and define the smooth Hermitian metric $I_F^{m\psi_k}$ by $I_F e^{-m\psi_k}$. Then for any $s \geqslant k$, we get

$$A^{n,s}_{F, I_F^{m\psi_k}, \omega} = \left[i\Theta_{F, I_F^{m\psi_k}}, \Lambda_\omega \right] = m \left[i\partial\overline{\partial}\psi_k \otimes \mathrm{id}_F, \Lambda_\omega \right] \geqslant m > 0.$$

Let $e = (e_1, \dots, e_r), b = (b_1, \dots, b_t)$ be holomorphic frames of E, F respectively, where b is orthonormal frame with respect to I_F. For any $u = \sum u_{J\lambda} \mathrm{d}z_N \wedge \mathrm{d}\overline{z}_J \otimes e_\lambda \in \mathscr{E}^{n,q}(\Delta_{2R}, E)$ where $dz_N = dz_1 \wedge \cdots \wedge dz_n$, let $u_F = \sum u \otimes b_\tau \in \mathscr{E}^{n,q}(\Delta_{2R}, E \otimes F)$. Then we have the following calculations

$$\begin{aligned} B_{I_F^{m\psi_k}, \omega} u_F &= m \left[i\partial\overline{\partial}\psi_k \otimes \mathrm{id}_{E\otimes F}, \Lambda_\omega \right] u_F \\ &= m \sum_\tau \Big(\left[i\partial\overline{\partial}\psi_k \otimes \mathrm{id}_E, \Lambda_\omega \right] u \Big) \otimes b_\tau \\ &= m \sum_\tau \left(\left(\sum_{j \,\in\, J \cap I_k} 1 \right) u_{J\lambda} \mathrm{d}z_N \wedge \mathrm{d}\overline{z}_J \otimes e_\lambda \right) \otimes b_\tau \\ &= m \sum_{J,\lambda,\tau} |J \cap I_k| u_{J\lambda} \mathrm{d}z_N \wedge \mathrm{d}\overline{z}_J \otimes e_\lambda \otimes b_\tau, \\ \text{and } \ B^{-1}_{I_F^{m\psi_k}, \omega} u_F &= \frac{1}{m} \sum_{J,\lambda,\tau} |J \cap I_k|^{-1} u_{J\lambda} \mathrm{d}z_N \wedge \mathrm{d}\overline{z}_J \otimes e_\lambda \otimes b_\tau, \end{aligned}$$

where $I_k = \{1, \dots, n-k+1\}$ and $J \cap I_k \neq \emptyset$.

Let $\xi = \sum \xi_{J\lambda} \mathrm{d}z_N \wedge \mathrm{d}\overline{z}_J \otimes e_\lambda \in \mathscr{E}^{n,q}(\Delta_{2R}, E)$ with constant coefficients such that $\xi(x_0) = \xi_0$ and let $\xi_F := \sum_\tau \xi \otimes b_\tau$. We may assume

$$\langle [i\Theta_{E,h}, \Lambda_\omega]\xi, \xi \rangle_{h,\omega} < -c$$

on Δ_{2R}, for any small number $R > 0$.

For any ordered multi-index I, we define $\varepsilon(s, I) \in \{-1, 0, 1\}$ (see [38, Definition 2.1]) to be the number that satisfies $\zeta_s \lrcorner \zeta_I^* = \varepsilon(s, I)\zeta_{I\setminus s}^*$, where $(\zeta_1, \dots, \zeta_n)$ is an orthonormal basis of T_X. Here, the symbol $\bullet \lrcorner \bullet$ represents the interior product, i.e. $\zeta_s \lrcorner \zeta_I^* = \iota_{\xi_s}\zeta_I^*$.

Choose a $\mathcal{C}^\infty$ function $\chi_\Delta \geqslant 0$ over Δ_R^k with compact support contained in $\Delta_{3R/4}^k$, i.e. $\chi_\Delta \in \mathscr{D}(\Delta_R^k, \mathbb{R}_{\geqslant 0})$, such that $\chi_\Delta|_{\Delta_{R/2}^k} = 1$. We still denote

$pr_1^*\chi_\Delta$ by χ_Δ, here $pr_1 : S_R \to \Delta_R^k$ is the projection to the first factor. Let

$$v = \sum_{J,\lambda} \sum_{1 \leqslant j \leqslant n-k+1} (-1)^n \varepsilon(j,J) \xi_{J\lambda} \bar{z}_j \chi_\Delta(z) \cdot \mathrm{d}z_N \wedge \mathrm{d}\bar{z}_{J\backslash j} \otimes e_\lambda \in \mathcal{E}^{n,q-1}(S_R, E),$$

then from $(-1)^n\varepsilon(j,J)\mathrm{d}\bar{z}_j \wedge \mathrm{d}z_N \wedge \mathrm{d}\bar{z}_{J\backslash j} = \mathrm{d}z_N \wedge \mathrm{d}\bar{z}_J$, we have

$$\begin{aligned}
\bar{\partial} v_F|_{G_{R/2}} &= \sum_\tau \bar{\partial} v|_{G_{R/2}} \otimes b_\tau \\
&= \sum_{J,\lambda,\tau} \sum_{j=1}^{n-k+1} \sum_{l=1}^{n} (-1)^n \varepsilon(j,J) \xi_{J\lambda} \delta_{jl} \mathrm{d}\bar{z}_l \wedge \mathrm{d}z_N \wedge \mathrm{d}\bar{z}_{J\backslash j} \otimes e_\lambda \otimes b_\tau \\
&= \sum_{J,\lambda,\tau} \sum_{1 \leqslant j \leqslant n-k+1} (-1)^n \varepsilon(j,J) \xi_{J\lambda} \mathrm{d}\bar{z}_j \wedge \mathrm{d}z_N \wedge \mathrm{d}\bar{z}_{J\backslash j} \otimes e_\lambda \otimes b_\tau \\
&= \sum_{J,\lambda,\tau} \sum_{j \in J \cap I_k} \xi_{J\lambda} \mathrm{d}z_N \wedge \mathrm{d}\bar{z}_J \otimes e_\lambda \otimes b_\tau = B_{I_F^{\psi_k},\omega} \xi_F,
\end{aligned}$$

here we define $G_{R/2} := \Delta_{R/2}^k \times D_R^k$ and we use the simple fact that $j \notin J$ then $\varepsilon(j,J) = 0$.

Let $f := \bar{\partial} v \in \mathcal{E}^{n,q}(S_R, E)$ and $f_F := \sum_\tau f \otimes b_\tau = \sum_\tau \bar{\partial} v \otimes b_\tau = \bar{\partial} v_F \in \mathcal{E}^{n,q}(S_R, E \otimes F)$ then we get $\bar{\partial} f_F = 0$ on S_R and $f_F = B_{I_F^{\psi_k},\omega} \xi_F$ with constant coefficients on $G_{R/2}$. We define

$$\alpha_m := B^{-1}_{I_F^{m\psi_k},\omega} f_F = \frac{1}{m} B^{-1}_{I_F^{\psi_k},\omega} f_F \in \mathcal{E}^{n,q}(S_R, E \otimes F)$$

satisfying $\alpha_m|_{G_{R/2}} = \frac{1}{m}\xi_F$. Here, we can write

$$f_F = \chi_\Delta(z) B_{I_F^{\psi_k},\omega} \xi_F + \sum_{J,\lambda,\tau} \sum_{j \in J \cap I_k,\, l \in I_k} \xi_{J\lambda} \bar{z}_j \frac{\partial \chi_\Delta(z)}{\partial \bar{z}_l} (-1)^n \varepsilon(j,J) \, \mathrm{d}\bar{z}_l \wedge \mathrm{d}z_N \wedge \mathrm{d}\bar{z}_{J\backslash j} \otimes e_\lambda \otimes b_\tau.$$

Since v depends only on the variables $z_1, \ldots, z_{n-k+1}$, so is $f_F = \bar{\partial} v_F$ is also depends only on the variables $z_1, \ldots, z_{n-k+1}$. By smoothness of h on X, h is bounded on S_R. Hence, from χ_Δ and ψ_k depend only on the variables $z_1, \ldots, z_{n-k+1}$ and $\operatorname{supp} f_F \subset \operatorname{supp} \chi_\Delta \subset\subset \Delta_{3R/4}^k \times D_R^k$, we obtain

$$\int_{S_R} \left\langle B^{-1}_{I_F^{m\psi_k},\omega} f_F, f_F \right\rangle_{h \otimes I_F,\omega} e^{-m\psi_k} \mathrm{d}V_\omega = \int_{S_R} \frac{1}{m} \left\langle B^{-1}_{I_F^{\psi_k},\omega} f_F, f_F \right\rangle_{h \otimes I_F,\omega} e^{-m\psi_k} \mathrm{d}V_\omega < +\infty,$$

for any $m > 0$. From $i[\Lambda_\omega, \overline{\partial}] = D'^*_{h\otimes I_F^{m\psi_k}}$ (cf. [8, Chapter 4]), we have the following

$$D'^*_{h\otimes I_F^{m\psi_k}}\alpha_m = \frac{1}{m}D'^*_{h\otimes I_F^{m\psi_k}}\xi_F = 0,$$

and

$$\begin{aligned}
&\big\langle [i\Theta_{E,h}\otimes \mathrm{id}_F, \Lambda_\omega]\alpha_m, \alpha_m\big\rangle_{h\otimes I_F,\,\omega} \\
&\qquad = \frac{1}{m^2}\big\langle [i\Theta_{E,h}\otimes \mathrm{id}_F, \Lambda_\omega]\xi_F, \xi_F\big\rangle_{h\otimes I_F,\,\omega} \\
&\qquad = \frac{1}{m^2}\left\langle \sum_\tau \big([i\Theta_{E,h}\otimes \mathrm{id}_F, \Lambda_\omega]\xi\big)\otimes b_\tau, \sum_\sigma \xi\otimes b_\sigma \right\rangle_{h\otimes I_F,\,\omega} \\
&\qquad = \frac{1}{m^2}\langle [i\Theta_{E,h}, \Lambda_\omega]\xi, \xi\rangle_{h,\,\omega}\sum_{\tau,\sigma}\langle b_\tau, b_\sigma\rangle_{I_F} \\
&\qquad < -\frac{c}{m^2}\,\mathrm{rank}\ F
\end{aligned}$$

on $G_{R/2}$, here we use $\langle b_\tau, b_\sigma\rangle_{I_F} = \delta_{\tau,\sigma}$. Since f_F has compact support in S_R, there is a constant C, such that

$$\big|\langle [i\Theta_{E,h}\otimes \mathrm{id}_F, \Lambda_\omega]\alpha_m, \alpha_m\rangle_{h\otimes I_F,\omega}\big| \leqslant \frac{C}{m^2}, \qquad \Big|D'^*_{h\otimes I_F^{m\psi_k}}\alpha_m\Big|^2_{h\otimes I_F,\omega} \leqslant \frac{C}{m^2}$$

on S_R for any $m > 0$.

Then we consider the left-hand side of $(*)$ with respect to $(S, h_F, \alpha) = (S_R, I_F^{m\psi_k}, \alpha_m)$.

$$\begin{aligned}
&m^2\left(\big\langle\!\big\langle [i\Theta_{E,h}\otimes \mathrm{id}_F, \Lambda_\omega]\alpha_m, \alpha_m\big\rangle\!\big\rangle_{h\otimes I_F^{m\psi_k},\omega} + \Big\|D'^*_{h\otimes I_F^{m\psi_k}}\alpha_m\Big\|^2_{h\otimes I_F^{m\psi_k},\omega}\right) \\
&\quad = m^2\left(\int_{G_{R/2}}\big\langle [i\Theta_{E,h}\otimes \mathrm{id}_F, \Lambda_\omega]\alpha_m, \alpha_m\big\rangle_{h\otimes I_F,\omega} e^{-m\psi_k}dV_\omega\right. \\
&\qquad\quad + \int_{S_R\setminus G_{R/2}}\big\langle [i\Theta_{E,h}\otimes \mathrm{id}_F, \Lambda_\omega]\alpha_m, \alpha_m\big\rangle_{h\otimes I_F,\omega} e^{-m\psi_k}dV_\omega \\
&\qquad\quad \left. + \int_{S_R\setminus G_{R/2}}\Big|D'^*_{h\otimes I_F^{m\psi_k}}\alpha_m\Big|^2_{h\otimes I_F,\omega} e^{-m\psi_k}dV_\omega\right) \\
&\quad \leqslant -c\cdot \mathrm{rank}\ F\int_{G_{R/2}} e^{-m\psi_k}\mathrm{d}V_\omega + 2C\int_{S_R\setminus G_{R/2}} e^{-m\psi_k}\mathrm{d}V_\omega \\
&\quad = \mathrm{Vol}(D_R^k)\left(-c\cdot \mathrm{rank}\ F\int_{\Delta^k_{R/2}} e^{-m\psi_k}\mathrm{d}V_{\omega_k} + 2C\int_{\Delta^k_R\setminus\Delta^k_{R/2}} e^{-m\psi_k}\mathrm{d}V_{\omega_k}\right),
\end{aligned}$$

where $\omega_k = i\partial\overline{\partial}\psi_k$. By $\lim_{m\to+\infty} m\psi_k(z) = +\infty$ for $z \in \Delta^k_R \setminus \overline{\Delta}^k_{R/2}$ and $\psi_k(z) \leqslant 0$ for $z \in \Delta^k_{R/2}$, we obtain the inequality

$$\langle\!\langle [i\Theta_{E,h} \otimes \mathrm{id}_F, \Lambda_\omega]\alpha_m, \alpha_m \rangle\!\rangle_{h\otimes I_F^{m\psi_k},\, \omega} + \left\| D'^*_{h\otimes I_F^{m\psi_k}} \alpha_m \right\|^2_{h\otimes I_F^{m\psi_k},\, \omega} < 0$$

for $m \gg 1$, which contradicts to the inequality $(*)$. □

3. Singular Hermitian metrics

In this section, we consider the case where a Hermitian metric of a holomorphic vector bundle has singularities and investigate its approximation and properties.

3.1. Definition of positivity

First, we introduce singular Hermitian metrics on holomorphic line bundles and define its positivity.

Definition 3.1 (cf. [6], [8, Chapter 3]). — *A singular Hermitian metric h on a line bundle L is a metric which is given in any trivialization $\tau : L|_U \xrightarrow{\simeq} U \times \mathbb{C}$ by*

$$\|\xi\|_h = |\tau(\xi)|e^{-\varphi}, \qquad x \in U,\ \xi \in L_x$$

where $\varphi \in \mathcal{L}^1_{loc}(U)$, called the weight of the metric with respect to the trivialization τ.

Definition 3.2. — *Let L be a holomorphic line bundle on a complex manifold X equipped with a singular Hermitian metric h.*

(a) *h is* singular semi-positive *if $i\Theta_{L,h} \geqslant 0$ in the sense of currents, i.e. the weight of h with respect to any trivialization coincides with some plurisubharmonic function almost everywhere.*
(b) *h is* singular positive *if the weight of h with respect to any trivialization coincides with some strictly plurisubharmonic function almost everywhere.*
(c) *Let ω be a Kähler metric on X and $\delta > 0$ be a positive real number. Then h is* strictly δ_ω-positive *if for any open subset U and any Kähler potential φ of ω on U, $he^{\delta\varphi}$ is singular semi-positive.*

Clearly, singular semi-positivity is coincides with pseudo-effective on compact complex manifolds. Furthermore, singular positivity and strictly δ_ω-positivity also coincide with big on compact Kähler manifolds by Demailly's definition and characterization (see [6], [8, Chapter 6]), where ω is a Kähler metric.

The Lelong number of a plurisubharmonic function φ on X is defined by

$$\nu(\varphi,x) := \liminf_{z\to x}\frac{\varphi(z)}{\log|z-x|}$$

for some coordinate $(z_1,\dots,z_n)$ around $x\in X$. For the relationship between the Lelong number of φ and the integrability of $e^{-\varphi}$, the following important result obtained by Skoda in [34] is known. If $\nu(\varphi,x)<1$ then $e^{-2\varphi}$ is integrable around x. From this, particularly if $\nu(-\log h,x)<2$ then $\mathscr{I}(h)=\mathcal{O}_{X,x}$ immediately.

For holomorphic vector bundles, we introduce the definition of singular Hermitian metrics h and the L^2-subsheaf $\mathscr{E}(h)$ of $\mathcal{O}(E)$ analogous to the multiplier ideal sheaf.

Definition 3.3 (cf.[2, Section 3], [31, Definition 2.2.1] and [33, Definition 1.1]). — *We say that h is a* singular Hermitian metric *on E if h is a measurable map from the base manifold X to the space of non-negative Hermitian forms on the fibers satisfying* $0<\det h<+\infty$ *almost everywhere.*

Definition 3.4 (cf. [3, Definition 2.3.1]). — *Let h be a singular Hermitian metric on E. We define the L^2-subsheaf $\mathscr{E}(h)$ of germs of local holomorphic sections of E as follows:*

$$\mathscr{E}(h)_x := \left\{ s_x\in\mathcal{O}(E)_x \;\middle|\; |s_x|_h^2 \textit{ is locally integrable around } x\right\}.$$

Moreover, we introduce the definitions of positivity and negativity, such as Griffiths and Nakano, for singular Hermitian metrics.

Definition 3.5 (cf. [2, Definition 3.1], [31, Definition 2.2.2] and [33, Definition 1.2]). — *We say that a singular Hermitian metric h is*

(1) Griffiths semi-negative *if $\|u\|_h$ is plurisubharmonic for any local holomorphic section $u\in\mathcal{O}(E)$.*
(2) Griffiths semi-positive *if the dual metric h^* on E^* is Griffiths semi-negative.*

Let h be a smooth Hermitian metric on E and $u=(u_1,\dots,u_n)$ be an n-tuple of locally holomorphic sections of E. We define T_u^h, an $(n-1,n-1)$-form through

$$T_u^h = \sum_{1\leqslant j,k\leqslant n}(u_j,u_k)_h \widehat{dz_j\wedge d\bar{z}_k}$$

where $(z_1, \dots, z_n)$ are local coordinates on X and $\widehat{dz_j \wedge d\bar{z}_k}$ satisfying $i dz_j \wedge d\bar{z}_k \wedge \widehat{dz_j \wedge d\bar{z}_k} = dV_{\mathbb{C}^n}$. Then a short computation yields that (E, h) is Nakano semi-negative if and only if T^h_u is plurisubharmonic in the sense that $i\partial\overline{\partial}T^h_u \geqslant 0$ (see [1, 33]). (E, h) is Griffiths semi-negative if and only if $T^h_{\xi u}$ is plurisubharmonic for any local section $u \in \mathcal{O}(E)$ and any $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ satisfying $u_j = \xi_j u$ and written $\xi u = (u_1, \cdots, u_n)$.

Let h be a singular Hermitian metric of E. For any n-tuple of locally holomorphic sections $u = (u_1, \dots, u_n)$, we say that the $(n-1, n-1)$-form T^h_u is plurisubharmonic if $i\partial\overline{\partial}T^h_u \geqslant 0$ in the sense of currents. From the above, we introduce the definitions of Nakano semi-negativity and dual Nakano semi-positivity for singular Hermitian metrics.

Definition 3.6 (cf. [33, Section 1]). — *We say that a singular Hermitian metric h on E is* Nakano semi-negative *if the $(n-1, n-1)$-form T^h_u is plurisubharmonic for any n-tuple of locally holomorphic sections $u = (u_1, \dots, u_n)$ of E.*

Definition 3.7 (cf. [39, Definition 4.5]). — *We say that a singular Hermitian metric h on E is* dual Nakano semi-positive *if the dual metric h^* on E^* is Nakano semi-negative.*

Since the dual of a Nakano negative bundle in general is not Nakano positive, we cannot define Nakano semi-positivity for singular Hermitian metrics as in the case of Griffiths, but this definition of dual Nakano semi-positivity is natural. We already know one definition of Nakano semi-positivity for singular Hermitian metrics in [19] as follows, which is based on the optimal L^2-estimate condition in [11, 16] and is equivalent to the usual definition for the smooth case.

Definition 3.8 (cf. [19, Definition 1.1]). — *Assume that h is a Griffiths semi-positive singular Hermitian metric. We say that h is* Nakano semi-positive *if for any Stein coordinate open subset S such that $E|_S$ is trivial, any Kähler metric ω_S on S, any smooth strictly plurisubharmonic function ψ on S, any positive integer $q \in \{1, \dots, n\}$ and any $f \in L^2_{n,q}(S, E, he^{-\psi}, \omega_S)$ satisfying $\overline{\partial}f = 0$ and $\int_S \langle B^{-1}_{\psi,\omega_S} f, f\rangle_{h,\omega_S} e^{-\psi} dV_{\omega_S} < +\infty$, there exists $u \in L^2_{n,q-1}(S, E, he^{-\psi}, \omega_S)$ satisfying $\overline{\partial}u = f$ and*

$$\int_S |u|^2_{h,\omega_S} e^{-\psi} dV_{\omega_S} \leqslant \int_S \left\langle B^{-1}_{\psi,\omega_S} f, f \right\rangle_{h,\omega_S} e^{-\psi} dV_{\omega_S},$$

where $B_{\psi,\omega_S} = [i\partial\overline{\partial}\psi \otimes \mathrm{id}_E, \Lambda_{\omega_S}]$.

However, this definition has the disadvantage of not being stable under tensor products. Precisely speaking, it is not clear whether, given a smooth

Hermitian metric h_F with m-semi-positivity on a vector bundle F, we can deduce from a singular Hermitian metric h on E with Nakano semi-positivity that the singular Hermitian metric $h \otimes h_F$ satisfies the (n,q)-L^2_ω-estimate condition for any $q > 0$ with $m \geqslant \min\{n-q+1, r\}$ and any Kähler metric ω (see [38, Definition 1.3]). Here, if h is a smooth Nakano semi-positive metric, then $h \otimes h_F$ is also m-semi-positive and we obtain $A^{n,q}_{E\otimes F, h\otimes h_F, \omega} \geqslant 0$, which is equivalent to satisfying the (n,q)-L^2_ω-estimate condition (see Lemma 2.4 and [38, Theorem 1.5]). In order to overcome this drawback, we propose the following definition by using Proposition 2.6.

Definition 3.9. — *Assume that h is a Griffiths semi-positive singular Hermitian metric. We say that h is L^2-type* Nakano semi-positive *if for any positive integer $k \in \{1, \dots, n\}$, any Stein coordinate S, any Kähler metric ω_S on S and any smooth Hermitian metric h_F on any holomorphic vector bundle F such that $A^{n,s}_{F,h_F,\omega_S} > 0$ for $s \geqslant k$, we have that any positive integer $q \geqslant k$ and any $f \in L^2_{n,q}(S, E\otimes F, h\otimes h_F, \omega_S)$ satisfying $\bar\partial f = 0$ and $\int_S \langle B^{-1}_{h_F,\omega_S} f, f\rangle_{h\otimes h_F,\omega_S} dV_{\omega_S} < +\infty$, there exists $u \in L^2_{n,q-1}(S, E\otimes F, h\otimes h_F, \omega_S)$ satisfying $\bar\partial u = f$ and*

$$\int_S |u|^2_{h\otimes h_F,\omega_S} \mathrm{d}V_{\omega_S} \leqslant \int_S \left\langle B^{-1}_{h_F,\omega_S} f, f \right\rangle_{h\otimes h_F,\omega_S} \mathrm{d}V_{\omega_S},$$

where $B_{h_F,\omega_S} = [i\Theta_{F,h_F} \otimes \mathrm{id}_E, \Lambda_{\omega_S}]$.

From Lemmas 3.14 and 4.2, the assumption of triviality for the vector bundle in Definition 3.8 can be excluded. Obviously h is Nakano semi-positive in the sense of Definition 3.8 if it is L^2-type Naknao semi-positive, as follows from taking the metric $e^{-\psi}$ on the trivial line bundle $F = S \times \mathbb{C}$, resulting in $A^{n,q}_{F,e^{-\psi}} > 0$ for any $q \geqslant 1$. However, the converse is not clear. Definitions 3.8 and 3.9 coincide in the case of line bundles. In fact, h becomes singular semi-positive from Griffiths semi-positivity, and L^2-estimates follow from Theorem 4.3 and Corollary 4.4. When defining dual Nakano semi-positivity using L^2-estimates, this positivity is derived from dual Nakano semi-positivity in the sense of Definition 3.7 (see [39, Proposition 4.10]), and the converse is not obvious.

Notes that the above Definitions 3.5–3.9 does not require the use of curvature currents. For singular Hermitian metrics we cannot always define the curvature currents with measure coefficients [33].

In [28], Nadel proved that $\mathscr{I}(h)$ is coherent by using the Hörmander L^2-estimate. After that, as vector bundles case, Hosono and Inayama proved that $\mathscr{E}(h)$ is coherent if h is Nakano semi-positivity in the sense of Definitions 3.8 (or 3.9) in [16, 19].

Finally we introduce the strictly positivity for Griffiths and Nakano is known.

Definition 3.10 (cf. [18, Definition 2.6], [19, Definition 2.16] and [39, Definition 4.11]). — *Let (X,ω) be a Kähler manifold and h be a singular Hermitian metric on E. Let $\delta > 0$ be a positive real number.*

- *We say that h is* strictly Griffiths δ_ω-positive *if for any open subset U and any Kähler potential φ of ω on U, $he^{\delta\varphi}$ is Griffiths semi-positive on U.*
- *We say that h is* L^2-type strictly Nakano δ_ω-positive *if for any open subset U and any Kähler potential φ of ω on U, $he^{\delta\varphi}$ is L^2-type Nakano semi-positive on U in the sense of Definition 3.9.*
- *We say that h is* strictly dual Nakano δ_ω-positive *if for any open subset U and any Kähler potential φ of ω on U, $he^{\delta\varphi}$ is dual Nakano semi-positive on U.*

On projective manifolds, it is known that we can obtain L^2-estimates for strictly Griffiths and (dual) Nakano δ_ω-positive vector bundles (see [19, Theorem 1.4], [39, Theorem 4.12]) and these imply the vanishing theorems involving L^2-subsheaves (see [19, Theorem 1.5], [39, Theorem 1.3]). In this paper, we consider the L^2-estimates and vanishing theorems for singular Hermitian metrics with L^2-type Nakano and dual Nakano semi-positive twisted by smooth (dual) m-positive Hermitian metrics on weakly pseudoconvex Kähler manifolds.

3.2. Approximation and properties of singular Hermite metrics

For singular semi-positivity on line bundles, the following Demailly's approximation is known.

Theorem 3.11 (cf. [7, Theorem 6.1]). — *Let (X,ω) be a complex manifold equipped with a Hermitian metric ω and $\Omega \Subset X$ be an open subset. Assume that $T = \alpha + \frac{i}{\pi}\partial\overline{\partial}\varphi$ is a closed $(1,1)$-current on X, where α is a smooth real $(1,1)$-form in the same $\partial\overline{\partial}$-cohomology class as T and φ is a quasi-plurisubharmonic function. Let γ be a continuous real $(1,1)$-form such that $T \geqslant \gamma$. Suppose that the Chern curvature tensor of T_X satisfies*

$$(i\Theta_{T_X} + \theta \otimes \mathrm{id}_{T_X})(\kappa_1 \otimes \kappa_2, \kappa_1 \otimes \kappa_2) \geqslant 0 \quad \forall\, \kappa_1, \kappa_2 \in T_X \text{ with } \langle \kappa_1, \kappa_2 \rangle = 0$$

on a neighborhood of $\overline{\Omega}$*, for some continuous nonnegative* $(1,1)$*-form* θ *on* X*. Then for every* $c > 0$*, there is a family of closed* $(1,1)$*-currents* $T_{c,\varepsilon} = \alpha + \frac{i}{\pi}\partial\overline{\partial}\varphi_{c,\varepsilon}$ *such that*

(i) $\varphi_{c,\varepsilon}$ *is quasi-plurisubharmonic on a neighborhood of* $\overline{\Omega}$*, smooth on* $X \setminus E_c(T)$*, increasing with respect to* ε *on* Ω*, and converges to* φ *on* Ω *as* $\varepsilon \to 0$*,*
(ii) $T_{c,\varepsilon} \geqslant \gamma - c\theta - \delta_\varepsilon\omega$ *on* Ω*,*

where $\varepsilon \in (0, \varepsilon_0)$*,* $E_c(T) = \{x \in X \mid \nu(T,x) \geqslant c\}$ *is the c-upperlevel set of Lelong numbers and* $(\delta_\varepsilon)_{\varepsilon>0}$ *is an increasing family of positive numbers such that* $\lim_{\varepsilon\to 0}\delta_\varepsilon = 0$*.*

Remark 3.12 ([41, Remark 3.1]*).* — Although Theorem 3.11 is stated in [7] when X is compact, almost the same proof as in [7] shows that Theorem 3.11 holds in the noncompact case while uniform estimates are obtained only on the relatively compact subset.

We consider the approximation of singular Hermitian metrics using convolution by the mollifier. Let S be a Stein manifold. We may assume that S is a closed submanifold of $\mathbb{C}^N$ (cf. [15]). By the theorem of Docquier and Grauert, there exists an open neighborhood $W \subset \mathbb{C}^N$ of S and a holomorphic retraction $\mu : W \to S$ (cf. [15, Chapter V]). Let $\rho : \mathbb{C}^N \to \mathbb{R}_{\geqslant 0}$ be a smooth function depending only on $|z|$ such that supp $\rho \subset \mathbb{B}^N$ and that $\int_{\mathbb{C}^N}\rho(z)\mathrm{d}V = 1$, where $\mathbb{B}^N$ is the unit ball. Define the mollifier $\rho_\nu(z) = \nu^{2N}\rho(\nu z)$ for $\nu > 0$, where $\rho_\nu \to \delta$ delta distribution if $\nu \to +\infty$. For any subset $D \subset \mathbb{C}^N$, let $D^\nu := \{z \in D \mid \mathrm{d}_N(z, \partial D) > 1/\nu\} \Subset D$. Then for any function φ over D, the convolution $\varphi_\nu := \varphi * \rho_\nu$ is a smooth function defined on D^ν.

Here, for any Stein manifold S, we say that the mollifier sequence $(\rho_\nu)_{\nu\in\mathbb{N}}$ is an *approximate identity with respect to* S. The following are known for approximations of singular Hermitian metrics on holomorphic vector bundles.

Proposition 3.13 (cf. [2, Proposition 3.1] and [39, Proposition 4.10]). *Let* S *be a Stein manifold and* E *be a holomorphic vector bundle over* S *equipped with a singular Hermitian metric* h*. We assume that* E *is trivial over* S*. Then we have the following*

(a) h *is Griffiths semi-negative if and only if there exists a sequence of smooth Griffiths semi-negative Hermitian metrics* $(h_\nu)_{\nu\in\mathbb{N}}$ *decreasing to* h *a.e. on any relatively compact Stein subset of* S*.*
(b) *If* h *is Nakano semi-negative then there exists a sequence of smooth Nakano semi-negative Hermitian metrics* $(h_\nu)_{\nu\in\mathbb{N}}$ *decreasing to* h *a.e. on any relatively compact Stein subset of* S*, where* $h_\nu = h * \rho_\nu$*.*

Here, we can always construct smooth Hermite metrics h_ν on E over any relatively compact Stein open subset of S by convolving it with the function ρ_ν, i.e. $h_\nu = h * \rho_\nu$, where $(\rho_\nu)_{\nu\in\mathbb{N}}$ is an approximate identity with respect to S.

The following lemma shows that the difference between the above assumption of triviality for vector bundles and not assuming it is only about a hypersurface.

Lemma 3.14. — *Let S be a Stein manifold and E be a holomorphic vector bundle on S. Then there exists a hypersurface H such that $E|_{S\setminus H}$ is trivial, where $S \setminus H$ is also Stein.*

Proof. — Let r be a rank of E, $f = (\sigma_1, \dots, \sigma_r)$ be a r-tuple of globally holomorphic sections of E, i.e. $\sigma_j \in H^0(S,E)$, for $1 \leqslant j \leqslant r$. We define the hypersurface by $H := \{z \in S \mid \Lambda^r_{j=1}\sigma_j(z) = 0\}$, where $\Lambda^r_{j=1}\sigma_j \in H^0(S, \det E)$. Here, $S\setminus H$ is also Stein (see [11]). We define the holomorphic map $\tau : S\setminus H\times\mathbb{C}^r \to E|_{S\setminus H}$ by $\tau(z,\xi) = f(z)\cdot\xi = \sum_{j=1}^r \xi_j\sigma_j(z)$ where $\xi = {}^t(\xi_1,\dots,\xi_r)$, then it is holomorphic isomorphism by f is globally holomorphic frame on $S \setminus H$. Hence, $E|_{S\setminus H}$ is trivial. □

We propose one effective method of determining singular Nakano seminegativity. This proposition is in some sense the converse of Proposition 3.13 (b).

Proposition 3.15. — *Let S be a Stein manifold and E be a holomorphic vector bundle which is trivial over S equipped with a singular Hermitian metric h. If $(h_\nu)_{\nu\in\mathbb{N}}$ is a sequence of smooth Nakano semi-negative Hermitian metrics then h is Nakano semi-negative, where $h_\nu = h * \rho_\nu$ and $(\rho_\nu)_{\nu\in\mathbb{N}}$ is an approximate identity with respect to S.*

Proof. — First, we show Griffiths semi-negativity of h. By Proposition 3.13 (a), it is sufficient to show that $(h_\nu)_{\nu\in\mathbb{N}}$ decreases to h a.e. By smooth Griffiths semi-negativity of h_δ, for any locally constant section $s \in \mathcal{O}_S(E)$, the smooth function $\|s\|^2_{h_\delta} = \|s\|^2_h * \rho_\delta$ is plurisubharmonic.

For any positive integers $\nu > \mu$, we have

$$\|s\|^2_{h_\delta} * \rho_\nu \geqslant \|s\|^2_{h_\delta} * \rho_\mu \quad \text{and} \quad \|s\|^2_{h_\delta} * \rho_\nu = \|s\|^2_h * \rho_\delta * \rho_\nu = \|s\|^2_{h_\nu} * \rho_\delta.$$

Therefore, we obtain $\|s\|^2_{h_\nu} \geqslant \|s\|^2_{h_\mu}$ by taking the limit of $\|s\|^2_{h_\nu} * \rho_\delta \geqslant \|s\|^2_{h_\mu} * \rho_\delta$ as $\delta \to +\infty$. Hence, $(h_\nu)_{\nu\in\mathbb{N}}$ is decreasing and converges to h a.e.

For any fixed point $x_0 \in S$, there exist an open neighborhood U of x_0 and $\nu_0 \in \mathbb{N}$ such that $U \subset S^{\nu_0} \subset S^\nu$ for any $\nu \geqslant \nu_0$. For any n-tuple locally holomorphic sections $u = (u_1, \dots, u_n)$ of E, i.e. $u_j \in H^0(U,E)$, we have the following

$$(u_j, u_k)_{h_\nu}(z) = \int (u_j, u_k)_{h^{(w)}}(z)\rho_\nu(w)\mathrm{d}V_w,$$
$$T_u^{h_\nu}(z) = \int T_u^{h^{(w)}}(z)\rho_\nu(w)\mathrm{d}V_w,$$

where $h^{(w)}(z) = h(z-w)$. For any nonnegative test function $\phi \in \mathscr{D}(U)_{\geqslant 0}$, we obtain

$$\begin{aligned}
0 \leqslant i\partial\overline{\partial}T_u^{h_\nu}(\phi) &= \int \phi i\partial\overline{\partial}T_u^{h_\nu} = \int T_u^{h_\nu} \wedge i\partial\overline{\partial}\phi \\
&= \int_z \left\{ \int_w T_u^{h^{(w)}}(z)\rho_\nu(w)\mathrm{d}V_w \right\} \wedge i\partial\overline{\partial}\phi \\
&= \int_w \left\{ \int_z T_u^{h^{(w)}}(z) \wedge i\partial\overline{\partial}\phi \right\} \rho_\nu(w)\mathrm{d}V_w \\
&= \int_{w\,\in\,\mathrm{supp}\ \rho_\nu} i\partial\overline{\partial}T_u^{h^{(w)}}(\phi)\rho_\nu(w)\mathrm{d}V_w.
\end{aligned}$$

Define the function $F = i\partial\overline{\partial}T_u^{h^{(\bullet)}}(\phi) : \mathrm{int}(\mathrm{supp}\ \rho_{\nu_0}) \to \mathbb{R}$. In the sequel we will show $F(0) = i\partial\overline{\partial}T_u^{h}(\phi) \geqslant 0$ which implies that h is Nakano semi-negative. For any $\zeta \in \mathbb{C}^n$ enough close to 0, we have the equation

$$\begin{aligned}
F(\zeta) = i\partial\overline{\partial}T_u^{h^{(\zeta)}}(\phi) &= \int_{w\,\in\,U} T_u^{h^{(\zeta)}}(w) \wedge i\partial\overline{\partial}\phi(w) \\
&= \int_U \sum \big(u_j(w), u_k(w)\big)_{h^{(\zeta)}(w)} \widehat{\mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k} \wedge i\partial\overline{\partial}\phi(w) \\
&= \int_U \sum \big(u_j(w), u_k(w)\big)_{h(w-\zeta)} \phi_{jk}(w)\mathrm{d}V_w \\
&= \int_{U-\zeta} \sum \big(u_j(\xi+\zeta), u_k(\xi+\zeta)\big)_{h(\xi)} \phi_{jk}(\xi+\zeta)\mathrm{d}V_\xi \\
&= \int_U \sum \big(u_j(w+\zeta), u_k(w+\zeta)\big)_{h(w)} \phi_{jk}(w+\zeta)\mathrm{d}V_w
\end{aligned}$$

where $\phi_{jk} = \frac{\partial^2\phi}{\partial z_j \partial\overline{z}_k}$, $\xi = w - \zeta$ and we take a enough small ζ satisfying $\mathrm{supp}\ \phi + \zeta \subset U$.

For any $\zeta \in \mathrm{int}(\mathrm{supp}\ \rho_{\nu_0})$ enough close to 0 and any $w \in U$, we define the function

$$g(\zeta, w) = \sum (u_j(w+\zeta), u_k(w+\zeta))_{h(w)} \phi_{jk}(w+\zeta)$$
$$\text{then} \quad F(\zeta) = \int_U g(\zeta, w)\mathrm{d}V_w.$$

Here, by Griffiths semi-negativity of h, each element h_{jk} is bounded (see [31, Lemma 2.2.4]). Therefore, from this and the fact that ϕ_{jk} has compact support, there exists a integrable function $M : U \to \mathbb{R}_{\geqslant 0}$ such that $|g(\zeta, w)| \leqslant M(w)$ for any $w \in U$ and any ζ enough close to 0. Since Lebesgue's dominated convergence theorem, for any ζ_0 enough close to 0 we have the following

$$\begin{aligned}\lim_{\zeta \to +0} F(\zeta_0 + \zeta) &= \lim_{\zeta \to +0} \int_U g(\zeta_0 + \zeta, w) \mathrm{d}V_w \\ &= \int_U \lim_{\zeta \to +0} g(\zeta_0 + \zeta, w) \mathrm{d}V_w \\ &= \int_U g(\zeta_0, w) \mathrm{d}V_w = F(\zeta_0),\end{aligned}$$

where for any $w \in U$, $g(\zeta, w)$ is smooth as to ζ by smoothness of u_j and ϕ_{jk}. Thus, F is continuous near 0. From smooth Nakano semi-negativity of h_ν, we obtain that

$$\begin{aligned}0 \leqslant \lim_{\nu_0 \leqslant \nu \to \nu + \infty} i\partial\overline{\partial} T_u^{h_\nu}(\phi) &= \lim_{\nu \to +\infty} \int_w F(w)\rho_\nu(w) \mathrm{d}V_w \\ &= \lim_{\nu \to +\infty} \langle \rho_\nu, F \rangle \\ &= \langle \delta_0, F \rangle \\ &= F(0),\end{aligned}$$

here the third equal requires continuity. Hence, h is Nakano semi-negative. ☐

We obtain the following basic properties that the tensor product of a Griffiths (resp. L^2-type Nakano, dual Nakano) semi-positive vector bundle with a singular semi-positive line bundle still holds the same positivity.

Theorem 3.16. — *Let X be a complex manifold, L be a holomorphic line bundle over X equipped with a singular Hermitian metric h_L and E be a holomorphic vector bundle over X equipped with a singular Hermitian metric h_E. We have the following*

(a) *If h_L is singular semi-positive and h_E is Griffiths semi-positive, then there exists a singular Hermitian metric $\widetilde{h}_L$ on L with Griffiths semi-positivity and the singular Hermitian metric $h_E \otimes \widetilde{h}_L$ on $E \otimes L$ is also Griffiths semi-positive.*
(b) *If h_L is singular semi-positive and h_E is L^2-type Nakano semi-positive, then $h_E \otimes h_L$ is also L^2-type Nakano semi-positive.*
(c) *If h_L is singular semi-positive and h_E is dual Nakano semi-positive, then $h_E \otimes h_L$ is also dual Nakano semi-positive.*

Note that, for singular Hermitian metrics on line bundles, singular semi-positivity and a.e. Griffiths semi-positivity (see [31, Definition 2.2.2]) coincide, but these lose the upper semi-continuity of local weights. In general, if the singular Hermitian metric on a vector bundle is Griffiths semi-positive, it is also a.e. Griffiths positive. However, the converse is not necessarily true. In particular, even if the singular Hermitian metric on a line bundle is singular semi-positive, it may not be Griffiths positive.

Proof.

(a). — We take an open subset U such that $L|_U$ and $E|_U$ are trivial. Let $\tau : L|_U \to U \times \mathbb{C}$ be a trivialization and φ be the weight of h_L with respect to τ. By $i\Theta_{L,h_L} = i\partial\overline{\partial}\varphi \geqslant 0$ on U in the sense of currents, there exists a plurisubharmonic function $\widetilde{\varphi}$ on U such that $\widetilde{\varphi} = \varphi$ a.e. This function $\widetilde{\varphi}$ is constructed as $\widetilde{\varphi}(z) := \lim_{\nu\to+\infty} \varphi * \rho_\nu(z)$. We define the singular Hermitian metric $\widetilde{h}_L$ on L by $\widetilde{h}_L = e^{-\widetilde{\varphi}}$ on U, then this metric is Griffiths semi-positive, i.e. $\log|\sigma|^2_{\widetilde{h}^*_L} = \log|\sigma|^2 + \widetilde{\varphi}$ is plurisubharmonic for any $\sigma \in \mathcal{O}(L^*)$, by plurisubharmonicity of $\widetilde{\varphi}$.

For any local holomorphic section $u \in \mathcal{O}(E^* \otimes L^*)(U) = \mathcal{O}(E^*)(U)$, the function $\log|u|^2_{h^*_E}$ is plurisubharmonic by Griffiths semi-positivity of h_E. Hence, the function $\log|u|^2_{h^*_E\otimes h^*_L} = \log|u|^2_{h^*_E} + \widetilde{\varphi}$ is also plurisubharmonic.

(b). — We fix a positive integer $k \in \{1, \dots, n\}$, a Stein coordinate S, a Kähler metric ω_S on S and a smooth Hermitian metric h_F on a holomorphic vector bundle F such that $A^{n,s}_{F,h_F,\omega_S} > 0$ for $s \geqslant k$.

By Lemma 3.14, there is a hypersurface H such that $S_H := S \setminus H$ is also Stein and $L|_{S_H}$ is trivial. There is a strictly plurisubharmonic function ψ on S_H which is smooth exhaustive and $\sup_{S_H} \psi = +\infty$. Let $S_H(j) := \{z \in S_H \mid \psi(z) < j\}$ be Stein sublevel sets. Fixed $j \in \mathbb{N}$. There is $\nu_1 \in \mathbb{N}$ such that $S_H(j) \Subset S_H^{\nu_1} \Subset S_H^{\nu}$ for any integer $\nu \geqslant \nu_1$.

By Proposition 3.13, there is a sequence of smooth semi-positive metrics $(h_\nu)_{\nu\in\mathbb{N}}$ increasing to h_L, where $h_\nu := (h^*_L * \rho_\nu)^*$ defined on S^ν_H. For any $\nu \in \mathbb{N}$, we obtain that $A^{n,t}_{L,h_\nu,\omega_S} \geqslant 0$ for $t \geqslant 1$ and $A^{n,s}_{F\otimes L,h_F\otimes h_\nu,\omega_S} \geqslant 0$ for $s \geqslant k$ by Proposition 2.5 and that

$$\begin{aligned} B_{h_F\otimes h_\nu,\omega_S} &= [i\Theta_{F,h_F} \otimes \mathrm{id}_{L\otimes E}, \Lambda_{\omega_S}] + [i\Theta_{L,h_\nu} \otimes \mathrm{id}_{F\otimes E}, \Lambda_{\omega_S}] \\ &\geqslant [i\Theta_{F,h_F} \otimes \mathrm{id}_{L\otimes E}, \Lambda_{\omega_S}] = B_{h_F,\omega_S}, \end{aligned}$$

where $B_{h_F,\omega_S} > 0$ for any (n,q)-forms with $q \geqslant k$.

Here, for any positive integers $q \geqslant k$ and any $f \in L^2_{n,q}(S, E\otimes L\otimes F, h_E\otimes h_L\otimes h_F, \omega_S)$ satisfying $\bar\partial f = 0$ and $\int_S \langle B^{-1}_{h_F,\,\omega_S} f, f\rangle_{h_E\otimes h_L\otimes h_F,\,\omega_S} dV_{\omega_S} < +\infty$, we have

$$\begin{aligned}\int_S \left\langle B^{-1}_{h_F,\,\omega_S} f, f\right\rangle_{h_E\otimes h_L\otimes h_F,\,\omega_S} \mathrm{d}V_{\omega_S} &\\ &\geqslant \int_{S_H(j)} \left\langle B^{-1}_{h_F,\,\omega_S} f, f\right\rangle_{h_E\otimes h_\nu\otimes h_F,\,\omega_S} \mathrm{d}V_{\omega_S}\\ &\geqslant \int_{S_H(j)} \left\langle B^{-1}_{h_F\otimes h_\nu,\omega_S} f, f\right\rangle_{h_E\otimes h_\nu\otimes h_F,\,\omega_S} \mathrm{d}V_{\omega_S}.\end{aligned}$$

By L^2-type Nakano semi-positivity of h_E, there exists

$$u_{j,\nu} \in L^2_{n,q-1}(S_H(j), E\otimes L\otimes F, h_E\otimes h_\nu\otimes h_F, \omega_S)$$

such that $\bar\partial u_{j,\nu} = f$ on $S_H(j)$ and

$$\begin{aligned}\int_{S_H(j)} |u_{j,\nu}|^2_{h_E\otimes h_\nu\otimes h_F,\,\omega_S} \mathrm{d}V_{\omega_S} &\leqslant \int_{S_H(j)} \left\langle B^{-1}_{h_F\otimes h_\nu,\omega_S} f, f\right\rangle_{h_E\otimes h_\nu\otimes h_F,\,\omega_S} \mathrm{d}V_{\omega_S}\\ &\leqslant \int_S \left\langle B^{-1}_{h_F,\,\omega_S} f, f\right\rangle_{h_E\otimes h_L\otimes h_F,\,\omega_S} \mathrm{d}V_{\omega_S} < +\infty.\end{aligned}$$

From the monotonicity with respect to ν of $|\bullet|^2_{h_E\otimes h_\nu\otimes h_F,\,\omega}$ since $(h_\nu)_{\nu\in\mathbb{N}}$ is increasing in ν, the sequence $(u_{j,\nu})_{\nu_1\leqslant\nu\in\mathbb{N}}$ forms a bounded sequence in $L^2_{n,q-1}(S_H(j), E\otimes L\otimes F, h_E\otimes h_{\nu_1}\otimes h_F, \omega_S)$. Thus, we can obtain a weakly convergent subsequence in $L^2_{n,q-1}(S_H(j), E\otimes L\otimes F, h_E\otimes h_{\nu_1}\otimes h_F, \omega_S)$. By using a diagonal argument, we get a subsequence $(u_{j,\nu_k})_{k\in\mathbb{N}}$ of $(u_{j,\nu})_{\nu\in\mathbb{N}}$ converging weakly in $L^2_{n,q-1}(S_H(j), E\otimes L\otimes F, h_E\otimes h_{\nu_1}\otimes h_F, \omega_S)$, where $u_{j,\nu_k} \in L^2_{n,q-1}(S_H(j), E\otimes L\otimes F, h_E\otimes h_{\nu_k}\otimes h_F, \omega_S) \subset L^2_{n,q-1}(S_H(j), E\otimes L\otimes F, h_E\otimes h_{\nu_1}\otimes h_F, \omega_S)$.

Denote the weakly limit of $(u_{j,\nu_k})_{k\in\mathbb{N}}$ by u_j. Then u_j satisfies $\bar\partial u_j = f$ on $S_H(j)$ and

$$\int_{S_H(j)} |u_j|^2_{h_E\otimes h_{\nu_k}\otimes h_F,\,\omega_S} \leqslant \int_S \left\langle B^{-1}_{h_F,\,\omega_S} f, f\right\rangle_{h_E\otimes h_L\otimes h_F,\,\omega_S} \mathrm{d}V_{\omega_S} < +\infty,$$

for any $k \in \mathbb{N}$. Taking weakly limit $k \to +\infty$ and using the monotone convergence theorem, we have the following estimate

$$\int_{S_H(j)} |u_j|^2_{h_E\otimes h_L\otimes h_F,\,\omega_S} \leqslant \int_S \left\langle B^{-1}_{h_F,\,\omega_S} f, f\right\rangle_{h_E\otimes h_L\otimes h_F,\,\omega_S} \mathrm{d}V_{\omega_S} < +\infty.$$

Here, let $\chi_j \in \mathscr{D}(S_H, \mathbb{R}_{\geqslant 0})$ be a cut-off function satisfying $\chi_j \equiv 1$ on $S_H(j-1)$, supp $\chi_j \subset\subset S_H(j)$ and $0 \leqslant \chi_j \leqslant 1$. We define $v_j := \chi_j u_j \in L^2_{n,q-1}(S_H, E\otimes L\otimes F, h_E\otimes h_L\otimes h_F, \omega_S)$ then v_j satisfies $\bar{\partial} v_j = f$ on $S_H(j-1)$ and

$$\begin{aligned}\int_{S_H(j)} |v_j|^2_{h_E\otimes h_L\otimes h_F,\,\omega_S} &\leqslant \int_{S_H(j)} |u_j|^2_{h_E\otimes h_L\otimes h_F,\,\omega_S}\\ &\leqslant \int_S \left\langle B^{-1}_{h_F,\,\omega_S} f, f\right\rangle_{h_E\otimes h_L\otimes h_F,\,\omega_S} \mathrm{d}V_{\omega_S} < +\infty.\end{aligned}$$

Repeating the above argument and taking weak limit $j \to +\infty$, we get the solution $v \in L^2_{n,q-1}(S_H, E \otimes L \otimes F, h_E \otimes h_L \otimes h_F, \omega_S)$ of $\bar{\partial} v = f$ on S_H such that

$$\begin{aligned}\int_{S_H} |v|^2_{h_E\otimes h_L\otimes h_F,\,\omega_S} \mathrm{d}V_{\omega_S} &= \int_S |v|^2_{h_E\otimes h_L\otimes h_F,\,\omega_S} \mathrm{d}V_{\omega_S}\\ &\leqslant \int_S \left\langle B^{-1}_{h_F,\,\omega_S} f, f\right\rangle_{h_E\otimes h_L\otimes h_F,\,\omega_S} \mathrm{d}V_{\omega_S},\end{aligned}$$

where Lebesgue measure of H is zero. By Lemma 4.2 in Subsection 4.1, letting $v = 0$ on H then we have that $\bar{\partial} v = f$ on S. Hence, $h_E \otimes h_L$ is also L^2-type Nakano semi-positive.

(c). — Let h_L^* be singular semi-positive and h_E be Nakano semi-negative. It is equivalent to prove that $h_E \otimes h_L$ is Nakano semi-negative. Since Nakano semi-negativity is locally property, by Proposition 3.15, it is sufficient to show that $(h_E \otimes h_L) * \rho_\nu$ is Nakano semi-negative on any open subset for each $\nu \in \mathbb{N}$.

First, for a smooth semi-negative Hermitian metric h on L, we show that $h \otimes h_E$ is Nakano semi-negative. For any $x_0 \in X$, there exists an open Stein neighborhood U of x_0 such that $E|_U$ and $L|_U$ are trivial. Let $h_E^\nu := h_E * \rho_\nu$, where $(\rho_\nu)_{\nu\in\mathbb{N}}$ is an approximate identity with respect to U. By Proposition 3.13, h_E^ν is smooth Nakano semi-negative Hermitian metric on E over U^ν. For any n-tuple holomorphic sections $u = (u_1, \ldots, u_n)$ of E, i.e. $u_j \in H^0(U, E)$, we get

$$\begin{aligned}(u_j, u_k)_{h_E^\nu}(z) &= \int (u_j, u_k)_{h_E^{(w)}}(z)\rho_\nu(w)\mathrm{d}V_w,\\ T_u^{h_E^\nu}(z) &= \int T_u^{h_E^{(w)}}(z)\rho_\nu(w)\mathrm{d}V_w,\end{aligned}$$

where $h_E^{(w)}(z) = h_E(z-w)$. Since $L|_U$ is trivial, one can regard u as an n-tuple of holomorphic sections of $E \otimes L$, i.e. regard $u_j \in H^0(U, E \otimes L)$.

By Nakano semi-negativity of $h \otimes h_E^\nu$, for any nonnegative test function $\phi \in \mathscr{D}(U)_{\geqslant 0}$, we obtain that $i\partial\overline{\partial}T_u^{h\otimes h_E^\nu}(\phi) \geqslant 0$ and

$$\begin{aligned}
0 \leqslant \lim_{\nu\to+\infty} i\partial\overline{\partial}T_u^{h\otimes h_E^\nu}(\phi) &= \lim \int \phi i\partial\overline{\partial}T_u^{h\otimes h_E^\nu} \\
&= \lim \int T_u^{h\otimes h_E^\nu} \wedge i\partial\overline{\partial}\phi = \lim \int h \cdot T_u^{h_E^\nu} \wedge i\partial\overline{\partial}\phi \\
&= \lim \int_z \left\{ h \cdot \int_w T_u^{h_E^{(w)}}(z)\rho_\nu(w)\mathrm{d}V_w \right\} \wedge i\partial\overline{\partial}\phi \\
&= \lim \int_w \left\{ \int_z h \cdot T_u^{h_E^{(w)}}(z) \wedge i\partial\overline{\partial}\phi \right\} \rho_\nu(w)\mathrm{d}V_w \\
&= \lim \int_w \left\{ \int_z T_u^{h\otimes h_E^{(w)}}(z) \wedge i\partial\overline{\partial}\phi \right\} \rho_\nu(w)\mathrm{d}V_w \\
&= \lim \int_w i\partial\overline{\partial}T_u^{h\otimes h_E^{(w)}}(\phi)\rho_\nu(w)\mathrm{d}V_w \\
&= \lim \left\langle \rho_\nu, i\partial\overline{\partial}T_u^{h\otimes h_E^{(\bullet)}}(\phi) \right\rangle = \left\langle \delta_0, i\partial\overline{\partial}T_u^{h\otimes h_E^{(\bullet)}}(\phi) \right\rangle \\
&= i\partial\overline{\partial}T_u^{h\otimes h_E^{(0)}}(\phi) = i\partial\overline{\partial}T_u^{h\otimes h_E}(\phi), \quad \text{i.e. } i\partial\overline{\partial}T_u^{h\otimes h_E} \geqslant 0.
\end{aligned}$$

Here, the function $F = i\partial\overline{\partial}T_u^{h\otimes h_E^{(\bullet)}}(\phi) : \mathrm{int}(\mathrm{supp}\ \rho_{\nu_0}) \to \mathbb{R}$ is continuous near 0 by smoothness of h, similar to the proof of Proposition 3.15.

Finally, we show that $(h_E \otimes h_L) * \rho_\nu$ is Nakano semi-negative. Let $h_L^\mu := h_L * \rho_\mu$ then $(h_L^\mu)_{\mu\in\mathbb{N}}$ is a sequence of smooth semi-negative Hermitian metrics decreasing to h_L a.e. by Griffiths semi-negativity of h_L and Proposition 3.13. By the above, the sequence of singular Hermitian metrics $(h_E \otimes h_L^\mu)_{\mu\in\mathbb{N}}$ is a sequence of Nakano semi-negative Hermitian metrics decreasing to $h_E \otimes h_L$ a.e.

Therefore, for any locally constant section $s \in \mathcal{O}_x(E\otimes L)$ and any positive integers $\lambda > \mu$, we get the inequality

$$\|s\|^2_{h_E\otimes h_L^\mu} \geqslant \|s\|^2_{h_E\otimes h_L^\lambda}, \quad \text{i.e.} \quad f_{s,\mu,\lambda} := \|s\|^2_{h_E\otimes h^\mu} - \|s\|^2_{h_E\otimes h^\lambda} \geqslant 0.$$

In particular, $f_{s,\mu,+\infty} = \|s\|^2_{h_E\otimes h_L^\mu} - \|s\|^2_{h_E\otimes h_L} \geqslant 0$ a.e. as we let $\lambda \to +\infty$.

We fixed a positive integer ν. For any positive integers $\lambda > \mu$, we have

$$\begin{aligned}
0 \leqslant f_{s,\mu,\lambda} * \rho_\nu &= \left(\|s\|^2_{h_E\otimes h_L^\mu} - \|s\|^2_{h_E\otimes h_L^\lambda} \right) * \rho_\nu \\
&= \|s\|^2_{(h_E\otimes h_L^\mu)*\rho_\nu} - \|s\|^2_{(h_E\otimes h_L^\lambda)*\rho_\nu}
\end{aligned}$$

and $0 \leqslant f_{s,\mu,+\infty} * \rho_\nu = \|s\|^2_{(h_E \otimes h_L^\mu) * \rho_\nu} - \|s\|^2_{(h_E \otimes h_L) * \rho_\nu}$. From reverse Fatou's lemma, the decreasing sequence of smooth semi-positive functions $(f_{s,\mu,+\infty} * \rho_\nu)_{\mu \in \mathbb{N}}$ converges to 0 pointwise as $\mu \to +\infty$. In fact, $f_{s,\mu,+\infty} * \rho_\nu - f_{s,\mu+1,+\infty} * \rho_\nu = f_{s,\mu,\mu+1} * \rho_\nu \geqslant 0$ and

$$\begin{aligned} 0 &\leqslant \lim_{\mu \to +\infty} f_{s,\mu,+\infty} * \rho_\nu(z) = \lim_{\mu \to +\infty} \int f_{s,\mu,+\infty}(z-w)\rho_\nu(w)\mathrm{d}V_w \\ &\leqslant \limsup_{\mu \to +\infty} \int f_{s,\mu,+\infty}(z-w)\rho_\nu(w)\mathrm{d}V_w \\ &\leqslant \int \limsup_{\mu \to +\infty} f_{s,\mu,+\infty}(z-w)\rho_\nu(w)\mathrm{d}V_w = 0, \end{aligned}$$

where $0 \leqslant f_{s,\mu,+\infty} \leqslant f_{s,1,+\infty} \leqslant \|s\|^2_{h_E \otimes h_L^1} = h_L^1 \|s\|^2_{h_E}$ is locally integrable by smoothness of h_L^1 and plurisubharmonicity of $\|s\|^2_{h_E}$. Here, reverse Fatou's lemma is used when interchanging the integral and limit symbols.

Hence, the sequence of smooth Hermitian metrics $((h_E \otimes h_L^\mu) * \rho_\nu)_{\mu \in \mathbb{N}}$ decreases to $(h_E \otimes h_L) * \rho_\nu$ pointwise and each metric $(h_E \otimes h_L^\mu) * \rho_\nu$ is smooth Nakano semi-negative. Thus $(h_E \otimes h_L) * \rho_\nu$ is smooth Nakano semi-negative for each ν (see [38, Corollary 5.6 and Theorem 1.7]). □

Corollary 3.17. — *Let X be a Kähler manifold and ω be a Kähler metric. Let L and E be a holomorphic line bundle and a holomorphic vector bundle over X equipped with singular Hermitian metrics h_L and h_E, respectively. We have the following*

(a) *If h_L is singular semi-positive and h_E is L^2-type strictly Nakano δ_ω-positive then $h_E \otimes h_L$ is also L^2-type strictly Nakano δ_ω-positive.*
(b) *If h_L is strictly δ_ω-positive and h_E is L^2-type Nakano semi-positive then $h_E \otimes h_L$ is L^2-type strictly Nakano δ_ω-positive.*
(c) *If h_L is singular semi-positive and h_E is strictly dual Nakano δ_ω-positive then $h_E \otimes h_L$ is also strictly dual Nakano δ_ω-positive.*
(d) *If h_L is strictly δ_ω-positive and h_E is dual Nakano semi-positive then $h_E \otimes h_L$ is strictly dual Nakano δ_ω-positive.*

We introduce the following useful lemma using the diagonal argument, as can be understood from the proof of Theorem 3.16 (b).

Lemma 3.18. — *Let (X, ω) be a Kähler manifold and E be a holomorphic vector bundle on X with a singular Hermitian metric h. Let p and q be non-negative integer with $q \geqslant 1$ and f be a fiexd element of $L^2_{p,q}(X, E, h, \omega)$ satisfying $\bar{\partial} f = 0$ and $\int_X \langle A^{-1}_{p,q} f, f\rangle_{h,\omega} \mathrm{d}V_\omega < +\infty$, where $A_{p,q}$ is a semi-positive operator on $\bigwedge^{p,q} T^*_X \otimes E$.*

Let $(h_\nu)_{\nu\in\mathbb{N}}$ be a sequence of smooth Hermitian metrics on E increasing to h a.e. If we can take a solution $u_j \in L^2_{p,q-1}(X,E,h_j,\omega)$ to the $\bar\partial$-problem, i.e. $\bar\partial u_j = f$, satisfying

$$\|u_j\|^2_{X,h_j,\omega} = \int_X |u_j|^2_{h_j,\omega} \mathrm{d}V_\omega \leqslant \int_X \left\langle A^{-1}_{p,q} f, f \right\rangle_{h,\omega} \mathrm{d}V_\omega$$

for any $j\in\mathbb{N}$, then there exists a solution $u \in L^2_{p,q-1}(X,E,h,\omega)$ satisfying $\bar\partial u = f$ and

$$\|u\|^2_{X,h,\omega} = \int_X |u|^2_{h,\omega} \mathrm{d}V_\omega \leqslant \int_X \left\langle A^{-1}_{p,q} f, f \right\rangle_{h,\omega} \mathrm{d}V_\omega$$

as the limit of a convergent subsequence.

Let $(X_j)_{j\in\mathbb{N}}$ be a sequence of subsets increasing to X, i.e. $X_j \subset X_{j+1}$ for any $j\in\mathbb{N}$ and $\bigcup_{j\in\mathbb{N}} X_j = X$. If we can take a solution $u_j \in L^2_{p,q-1}(X_j,E,h,\omega)$ of $\bar\partial u_j = f$ on X_j, satisfying $\|u_j\|^2_{X_j,h,\omega} \leqslant \int_X \langle A^{-1}_{p,q} f, f\rangle_{h,\omega} \mathrm{d}V_\omega$ for any $j\in\mathbb{N}$, then there exists a solution $u\in L^2_{p,q-1}(X,E,h,\omega)$ satisfying the same conditions as above.

Since Griffiths and dual Nakano semi-positivity are local properties, we get following.

Proposition 3.19. — *Let X be a Kähler manifold and ω,γ be Kähler metrics on X. Let E be a holomorphic vector bundle over X equipped with a singular Hermitian metric h. We assume that there exists a positive number $c>0$ such that $\omega \geqslant c\gamma$. Then we get*

(a) *If h is strictly Griffiths δ_ω-positive then h is strictly Griffiths $c\delta_\gamma$-positive.*
(b) *If h is strictly dual Nakano δ_ω-positive then h is strictly dual Nakano $c\delta_\gamma$-positive.*

Proof.

(b). — We show that for any open subset U and any Kähler potential ψ of γ on U, $he^{c\delta\psi}$ is dual Nakano semi-positive on U. Since dual Nakano semi-positivity is a local property, it is sufficient to show that for any $x_0\in U$, there exists a neighborhood B of x_0 such that $B\subset U$ and $he^{c\delta\psi}$ is dual Nakano semi-positive on B. Here, we take B such that the Kähler potential φ of ω on B exists. Then $\varphi - c\psi$ is plurisubharmonic by $\omega\geqslant c\gamma$ and $e^{-\delta(\varphi-c\psi)}$ is semi-positive Hermitian metric on trivial line bundle. From the assumption, the singular metric $he^{\delta\varphi}$ is dual Nakano semi-positive on B. By Theorem 3.16 (c), we obtain that $he^{\delta\varphi}\otimes e^{-\delta(\varphi-c\psi)} = he^{c\delta\psi}$ is dual Nakano semi-positive on B.

(a). — It is shown in the same way as (b). □

Finally, we obtain the following dual-type generalization of Demailly and Skoda's theorem [4] to singularities.

Theorem 3.20 (Theorem 1.5). — *Let X be a complex manifold and E be a holomorphic vector bundle over X equipped with a singular Hermitian metric h. If h is Griffiths semi-positive then $h \otimes \det h$ is dual Nakano semi-positive.*

Proof. — It is equivalent to show that if h is Griffiths semi-negative then $h \otimes \det h$ is Nakano semi-negative. By Proposition 3.15, it is sufficient to show that $(h \otimes \det h) * \rho_\nu$ is smooth Nakano semi-negative for each $\nu \in \mathbb{N}$.

By Griffiths semi-negativity of $h^\mu := h * \rho_\mu$, we get smooth Nakano semi-negativity of $h^\mu \otimes \det h^\mu$. Moreover, the sequence of smooth Nakano semi-negative Hermitian metrics $(h^\mu \otimes \det h^\mu)_{\mu \in \mathbb{N}}$ decreases and converges to $h \otimes \det h$ a.e. since $(h^\mu)_{\mu \in \mathbb{N}}$ decreasing to h. Therefore, for any locally constant section s of $E \otimes \det E$ and any two positive integers $\lambda > \mu$, we get the inequality

$$\|s\|^2_{h^\mu \otimes \det h^\mu} \geqslant \|s\|^2_{h^\lambda \otimes \det h^\lambda} \quad \text{i.e.} \quad f_{s,\mu,\lambda} := \|s\|^2_{h^\mu \otimes \det h^\mu} - \|s\|^2_{h^\lambda \otimes \det h^\lambda} \geqslant 0.$$

In particular, $f_{s,\mu,+\infty} = \|s\|^2_{h^\mu \otimes \det h^\mu} - \|s\|^2_{h \otimes \det h} \geqslant 0$ a.e. as the case where $\lambda = +\infty$.

From reverse Fatou's lemma, the decreasing sequence of smooth semi-positive functions $(f_{s,\mu,+\infty} * \rho_\nu)_{\mu \in \mathbb{N}}$ converges to 0 pointwise, by a similar argument as in the proof of Theorem 3.16 (c). Hence, the sequence of smooth Hermitian metrics $((h^\mu \otimes \det h^\mu) * \rho_\nu)_{\mu \in \mathbb{N}}$ decreases to $(h \otimes \det h) * \rho_\nu$ pointwise and each metric $(h^\mu \otimes \det h^\mu) * \rho_\nu$ is smooth Nakano semi-negative. Thus $(h \otimes \det h) * \rho_\nu$ is smooth Nakano semi-negative for each ν (see [38, Corollary 5.6 and Theorem 1.7]). □

Remark 3.21. — If h is Griffiths semi-positive then $h \otimes \det h$ is L^2-type Nakano semi-positive by the same argument as [19, Theorem 1.3].

4. L^2-estimates with singular Hermitian metrics

4.1. L^2-estimates for line bundles possessing singular Hermitian metrics

In this subsection, we show L^2-estimates on weakly pseudoconvex Kähler manifolds when a holomorphic line bundle has a singular (semi)-positive Hermitian metric. First, we give L^2-estimates with respect to a singular semi-positive line bundle using following lemmas and Demailly's approximation.

Lemma 4.1 (cf. [5, Theorem 1.5]). — *Let X be a Kähler manifold and Z be a closed analytic subset of X that cannot be equal to the whole space. Assume that Ω is a relatively compact open subset of X possessing a complete Kähler metric. Then $\Omega \setminus Z$ carries a complete Kähler metric.*

Lemma 4.2 (cf. [5, Lemma 5.1.3]). — *Let Ω be an open subset of $\mathbb{C}^n$ and Z be a closed analytic subset of Ω that cannot be equal to the whole space. Assume that u is a $(p, q-1)$-form with L^2_{loc} coefficients and g is a (p,q)-form with L^1_{loc} coefficients such that $\bar{\partial} u = g$ on $\Omega \setminus Z$. Then $\bar{\partial} u = g$ on Ω.*

Theorem 4.3. — *Let X be a weakly pseudoconvex Kähler manifold and ω be a Kähler metric on X. Let (F, h_F) be a Hermitian holomorphic vector bundle of rank r and L be a holomorphic line bundle equipped with a singular semi-positive Hermitian metric h, i.e. $i\Theta_{L,h} \geqslant 0$ in the sense of currents. Then we have the following*

(a) *If h_F is m-positive, then for any $q \geqslant 1$ with $m \geqslant \min\{n-q+1, r\}$ and any $f \in L^2_{n,q}(X, F \otimes L, h_F \otimes h, \omega)$ satisfying $\bar{\partial} f = 0$ and $\int_X \langle B^{-1}_{h_F,\omega} f, f\rangle_{h_F \otimes h, \omega} \mathrm{d}V_\omega < +\infty$, there exists $u \in L^2_{n,q-1}(X, F \otimes L, h_F \otimes h, \omega)$ such that $\bar{\partial} u = f$ and*
$$\int_X |u|^2_{h_F \otimes h, \omega} \mathrm{d}V_\omega \leqslant \int_X \left\langle B^{-1}_{h_F,\omega} f, f \right\rangle_{h_F \otimes h, \omega} \mathrm{d}V_\omega,$$

(b) *If h_F is dual m-positive, then for any $p \geqslant 1$ with $m \geqslant \min\{n-p+1, r\}$ and any $f \in L^2_{p,n}(X, F \otimes L, h_F \otimes h, \omega)$ satisfying $\bar{\partial} f = 0$ and $\int_X \langle B^{-1}_{h_F,\omega} f, f\rangle_{h_F \otimes h, \omega} \mathrm{d}V_\omega < +\infty$, there exists $u \in L^2_{p,n-1}(X, F \otimes L, h_F \otimes h, \omega)$ such that $\bar{\partial} u = f$ and*
$$\int_X |u|^2_{h_F \otimes h, \omega} \mathrm{d}V_\omega \leqslant \int_X \left\langle B^{-1}_{h_F,\omega} f, f \right\rangle_{h_F \otimes h, \omega} \mathrm{d}V_\omega,$$

where $B_{h_F,\omega} = [i\Theta_{F,h_F} \otimes \mathrm{id}_L, \Lambda_\omega]$.

Proof. — There exists a smooth exhaustive plurisubharmonic function ψ on X such that $\sup_X \psi = +\infty$. Let $X_j := \{x \in X \mid \psi(x) < j\}$ be a sublevel set which is relatively compact. Let h_0 be a smooth Hermitian metric on L then $h = h_0 e^{-\varphi}$, where φ is quasi-plurisubharmonic function on X and $i\Theta_{L,h} = i\Theta_{L,h_0} + i\partial\bar{\partial}\varphi \geqslant 0$ as currents.

By Theorem 3.11, there is a sequence of quasi-plurisubharmonic functions $(\varphi_\nu)_{\nu \in \mathbb{N}}$ defined on X_j such that

(i) φ_ν is smooth in the complement $X_j \setminus Z_\nu$ of an analytic set $Z_\nu \subset X_j$,
(ii) $(\varphi_\nu)_{\nu \in \mathbb{N}}$ is a decreasing sequence and $\varphi|_{X_j} = \lim_{\nu \to +\infty} \varphi_\nu$,
(iii) $i\Theta_{L,h_0} + i\partial\bar{\partial}\varphi_\nu \geqslant -\beta_\nu \omega$, where $\lim_{\nu \to +\infty} \beta_\nu = 0$.

Here, we can find a sequence of Hermitian metric $h_\nu = h_0 e^{-\varphi_\nu}$ on $L|_{X_j}$. Then h_ν is smooth on $X_j \setminus Z_\nu$, $h_\nu \leqslant h$ and $i\Theta_{L,h_\nu} \geqslant -\beta_\nu \omega$.

(a). — By m-positivity of h_F, Lemma 2.4 and Proposition 2.5, we obtain that $B_{h_F,\omega} > 0$ on $\bigwedge^{n,q} T^*_X \otimes F \otimes L$ for $q \geqslant 1$ with $m \geqslant \min\{n-q+1, r\}$. Fix a positive integer $q \geqslant 1$ with $m \geqslant \min\{n-q+1, r\}$. From $B_{h_F,\omega} > 0$ on X, $\lim_{\nu\to+\infty} \beta_\nu = 0$ and relative compact-ness of X_j, there exist $c > 0$ and $\nu_0 \in \mathbb{N}$ such that

$$0 < q\beta_\nu \cdot \mathrm{id}_F < c \cdot \mathrm{id}_F < B_{h_F,\omega}$$

on X_j for any $\nu \geqslant \nu_0$. Then by smooth-ness of h_ν, we get the inequality

$$\begin{aligned} A^{n,q}_{F\otimes L, h_F\otimes h_\nu, \omega} &= [i\Theta_{F,h_F} \otimes \mathrm{id}_L, \Lambda_\omega] + [i\Theta_{L,h_\nu} \otimes \mathrm{id}_F, \Lambda_\omega] \\ &\geqslant B_{h_F,\omega} - \beta_\nu [\omega \otimes \mathrm{id}_F, \Lambda_\omega] \\ &= B_{h_F,\omega} - q\beta_\nu \cdot \mathrm{id}_F \geqslant \left(1 - \frac{q\beta_\nu}{c}\right) B_{h_F,\omega} > 0 \end{aligned}$$

on $X_j \setminus Z_\nu$. Hence, for any $f \in L^2_{n,q}(X, F\otimes L, h_F \otimes h, \omega)$ satisfying $\bar\partial f = 0$ and $\int_X \langle B^{-1}_{h_F,\omega} f, f\rangle_{h_F\otimes h,\omega} \mathrm{d}V_\omega < +\infty$, we have

$$\begin{aligned} &\int_{X_j\setminus Z_\nu} \big\langle [i\Theta_{F\otimes L, h_F\otimes h_\nu}, \Lambda_\omega]^{-1} f, f\big\rangle_{h_F\otimes h_\nu,\omega} \mathrm{d}V_\omega \\ &\qquad\leqslant \frac{c}{c - q\beta_\nu} \int_{X_j\setminus Z_\nu} \left\langle B^{-1}_{h_F,\omega} f, f\right\rangle_{h_F\otimes h_\nu,\omega} \mathrm{d}V_\omega \\ &\qquad\leqslant \frac{c}{c - q\beta_\nu} \int_{X} \left\langle B^{-1}_{h_F,\omega} f, f\right\rangle_{h_F\otimes h,\omega} \mathrm{d}V_\omega < +\infty. \end{aligned}$$

Since X_j is a weakly pseudoconvex Kähler manifold, X_j carries a complete Kähler metric by [9, Chapter VIII, Theorem 5.2]. From Lemma 4.1, $X_j \setminus Z_\nu$ has a complete Kähler metric. By [9, Chapter VIII, Theorem 6.1], i.e. L^2-estimates for (n,q)-forms with possibly non-complete Kähler metric ω, we obtain a solution $u_{j,\nu} \in L^2_{n,q-1}(X_j \setminus Z_\nu, F\otimes L, h_F \otimes h_\nu, \omega)$ of $\bar\partial u_{j,\nu} = f$ on $X_j \setminus Z_\nu$ satisfying

$$\begin{aligned} \int_{X_j\setminus Z_\nu} |u_{j,\nu}|^2_{h_F\otimes h_\nu} \mathrm{d}V_\omega &\leqslant \int_{X_j\setminus Z_\nu} \big\langle [i\Theta_{F\otimes L, h_F\otimes h_\nu}, \Lambda_\omega]^{-1} f, f\big\rangle_{h_F\otimes h_\nu,\omega} \mathrm{d}V_\omega \\ &\leqslant \frac{c}{c - q\beta_\nu} \int_{X} \left\langle B^{-1}_{h_F,\omega} f, f\right\rangle_{h_F\otimes h,\omega} \mathrm{d}V_\omega < +\infty. \end{aligned}$$

By Lemma 4.2, letting $u_{j,\nu} = 0$ on Z_ν then we have that $u_{j,\nu} \in L^2_{n,q-1}(X_j, F\otimes L, h_F \otimes h_\nu, \omega)$, $\bar\partial u_{j,\nu} = f$ on X_j and that

$$\left(1 - \frac{q\beta_\nu}{c}\right) \int_{X_j} |u_{j,\nu}|^2_{h_F\otimes h_\nu} \mathrm{d}V_\omega \leqslant \int_X \left\langle B^{-1}_{h_F,\omega} f, f\right\rangle_{h_F\otimes h,\omega} \mathrm{d}V_\omega.$$

From the monotonicity with respect to ν of $|\bullet|^2_{h_F\otimes h_\nu,\omega}$ by (ii), and Lemma 3.18, we obtain a solution $u\in L^2_{n,q-1}(X,F\otimes L,h_F\otimes h,\omega)$ of $\bar\partial u=f$ on X such that

$$\int_X |u|^2_{h_F\otimes h}\mathrm{d}V_\omega \leqslant \int_X \left\langle B^{-1}_{h_F,\omega}f,f\right\rangle_{h_F\otimes h,\omega}\mathrm{d}V_\omega<+\infty.$$

(b). — It is shown in the same way as above using [39, Theorem 3.7]. □

Clearly, the following follows by a similar argument as in the proof of Theorem 4.3.

Corollary 4.4. — *Let X be a weakly pseudoconvex Kähler manifold and ω be a Kähler metric on X. Let (A,h_A) be a k-positive holomorphic line bundle and L be a holomorphic line bundle equipped with a singular semi-positive Hermitian metric h, i.e. $i\Theta_{L,h}\geqslant 0$ in the sense of currents. Then we have the following*

(a) *For any $q\geqslant k$ and any $f\in L^2_{n,q}(X,A\otimes L,h_A\otimes h,\omega)$ satisfying $\bar\partial f=0$ and $\int_X\langle B^{-1}_{h_A,\omega}f,f\rangle_{h_A\otimes h,\omega}\mathrm{d}V_\omega<+\infty$, there exists $u\in L^2_{n,q-1}(X,A\otimes L,h_A\otimes h,\omega)$ such that $\bar\partial u=f$ and*
$$\int_X |u|^2_{h_A\otimes h,\omega}\mathrm{d}V_\omega \leqslant \int_X \left\langle B^{-1}_{h_A,\omega}f,f\right\rangle_{h_A\otimes h,\omega}\mathrm{d}V_\omega,$$

(b) *For any $p\geqslant k$ and any $f\in L^2_{p,n}(X,A\otimes L,h_A\otimes h,\omega)$ satisfying $\bar\partial f=0$ and $\int_X\langle B^{-1}_{h_A,\omega}f,f\rangle_{h_A\otimes h,\omega}\mathrm{d}V_\omega<+\infty$, there exists $u\in L^2_{p,n-1}(X,A\otimes L,h_A\otimes h,\omega)$ such that $\bar\partial u=f$ and*
$$\int_X |u|^2_{h_A\otimes h,\omega}\mathrm{d}V_\omega \leqslant \int_X \left\langle B^{-1}_{h_A,\omega}f,f\right\rangle_{h_A\otimes h,\omega}\mathrm{d}V_\omega,$$

where $B_{h_A,\omega}=[i\Theta_{A,h_A}\otimes\mathrm{id}_L,\Lambda_\omega]$.

We will provide a brief explanation for the case of (a). By k-positivity of (A,h_A), we already know that $A^{p,q}_{L,h,\omega}>0$ for $p+q\geqslant n+k$. From Proposition 2.5, we obtain $B_{h_A,\omega}>0$ on $\bigwedge^{n,q}T^*_X\otimes A\otimes L$ over X, and by replacing (F,h_F) with (A,h_A), we can prove (a) in the same way as the proof of Theorem 4.3.

Second, we obtain L^2-estimates when singular Hermitian metrics have positivity by using the following proposition.

Proposition 4.5. — *Let X be a weakly pseudoconvex Kähler manifold and ω be a Kähler metric on X. Let L be a holomorphic line bundle over X equipped with a singular positive Hermitian metric h. Then there exists a positive smooth function $c:X\to\mathbb{R}_{>0}$ such that $i\Theta_{L,h}\geqslant c\omega$ in the sense of currents.*

Proof. — We take relatively compact subsets X_j for any $j \in \mathbb{N}$ as in the proof of Theorem 4.3. By compactness of $\overline{X_j}$, there exists a finite open covering $\{\Omega_k\}_{1\leqslant k\leqslant N}$ such that $X_j \subset \bigcup \Omega_k$ and $L|_{\Omega_k}$ is trivial. Since the weight of h on each Ω_k coincides with a strictly plurisubharmonic function almost everywhere, there exists $c_{\Omega_k} > 0$ such that $i\Theta_{L,h} \geqslant c_{\Omega_k}\omega$ in the sense of currents on Ω_k. Let $c_j := \min_k c_{\Omega_k} > 0$, then we can construct a smooth function $c : X \to \mathbb{R}_{>0}$ satisfying $c_j > c(x) > 0$ for any $x \in X_j \setminus \overline{X_{j-1}}$. □

Similar to the proof of Theorem 4.3, we get the following theorem.

Theorem 4.6. — *Let X be a weakly pseudoconvex Kähler manifold and ω be a Kähler metric on X. Let (F, h_F) be a holomorphic vector bundle of rank r and L be a holomorphic line bundle equipped with a singular positive Hermitian metric h. Here, there exists a positive smooth function $c : X \to \mathbb{R}_{>0}$ such that $i\Theta_{L,h} \geqslant 2c\omega$ in the sense of currents by Proposition 4.5. Then we have the following*

(a) *If h_F is m-semi-positive, then for any $q \geqslant 1$ with $m \geqslant \min\{n-q+1, r\}$ and any $f \in L^2_{n,q}(X, F\otimes L, h_F\otimes h, \omega)$ satisfying $\bar\partial f = 0$ and $\int_X \frac{1}{c}|f|^2_{h_F\otimes h,\omega} dV_\omega < +\infty$, there exists $u \in L^2_{n,q-1}(X, F\otimes L, h_F\otimes h, \omega)$ such that $\bar\partial u = f$ and*
$$\int_X |u|^2_{h_F\otimes h,\omega} dV_\omega \leqslant \frac{1}{q}\int_X \frac{1}{c}|f|^2_{h_F\otimes h,\omega} dV_\omega.$$

(b) *If h_F is dual m-semi-positive, then for any $p \geqslant 1$ with $m \geqslant \min\{n-p+1, r\}$ and any $f \in L^2_{p,n}(X, F\otimes L, h_F\otimes h, \omega)$ satisfying $\bar\partial f = 0$ and $\int_X \frac{1}{c}|f|^2_{h_F\otimes h,\omega} dV_\omega < +\infty$, there exists $u \in L^2_{p,n-1}(X, F\otimes L, h_F\otimes h, \omega)$ such that $\bar\partial u = f$ and*
$$\int_X |u|^2_{h_F\otimes h,\omega} dV_\omega \leqslant \frac{1}{p}\int_X \frac{1}{c}|f|^2_{h_F\otimes h,\omega} dV_\omega.$$

We will provide a brief explanation for the case of (a). Similarly to the proof of Theorem 4.3, there exists a sequence of smooth Hermitian metrics $(h_\nu)_{\nu\in\mathbb{N}}$ increasing to h on X_j satisfying $i\Theta_{L,h_\nu} \geqslant c\omega$. Thus, obtaining the inequality $A^{n,q}_{F\otimes L,h_F\otimes h,\omega} \geqslant qc\cdot \mathrm{id}_{F\otimes L}$, the operator $B_{h_F,\omega}$ within the L^2-estimate is replaced by c.

4.2. L^2-estimates with singular (dual) Nakano semi-positivity

In this subsection, we obtain L^2-estimates on weakly pseudoconvex Kähler manifolds with a positive line bundle for two cases where the singular Hermitian metric of holomorphic vector bundles has L^2-type Nakano semi-positivity and dual Nakano semi-positivity.

Theorem 4.7. — *Let (X,ω) be a weakly pseudoconvex Kähler manifold and E be a holomorphic vector bundle equipped with a singular Hermitian metric h. We assume that h is L^2-type Nakano semi-positive on X. Then we have the following*

(a) *If X has a positive holomorphic line bundle and (A,h_A) is a k-positive line bundle, then for any $q \geqslant k$ and any $f \in L^2_{n,q}(X, A\otimes E, h_A\otimes h,\omega)$ satisfying $\bar\partial f = 0$ and $\int_X \langle B^{-1}_{h_A,\omega} f, f\rangle_{h_A\otimes h,\omega} dV_\omega < +\infty$, there exists $u \in L^2_{n,q-1}(X, A\otimes E, h_A\otimes h,\omega)$ satisfies $\bar\partial u = f$ and*

$$\int_X |u|^2_{h_A\otimes h,\omega} dV_\omega \leqslant \int_X \left\langle B^{-1}_{h_A,\omega} f, f\right\rangle_{h_A\otimes h,\omega} dV_\omega,$$

where $B_{h_A,\omega} = [i\Theta_{A,h_A}\otimes \mathrm{id}_E, \Lambda_\omega]$.

(b) *If (F,h_F) is an m-positive holomorphic vector bundle of rank r, then for any $q \geqslant 1$ with $m \geqslant \min\{n-q+1, r\}$ and any $f \in L^2_{n,q}(X, F\otimes E, h_F\otimes h,\omega)$ satisfying $\bar\partial f = 0$ and $\int_X \langle B^{-1}_{h_F,\omega} f, f\rangle_{h_F\otimes h,\omega} dV_\omega < +\infty$, there exists $u \in L^2_{n,q-1}(X, F\otimes E, h_F\otimes h,\omega)$ satisfies $\bar\partial u = f$ and*

$$\int_X |u|^2_{h_F\otimes h,\omega} dV_\omega \leqslant \int_X \left\langle B^{-1}_{h_F,\omega} f, f\right\rangle_{h_F\otimes h,\omega} dV_\omega.$$

where $B_{h_F,\omega} = [i\Theta_{F,h_F}\otimes \mathrm{id}_E, \Lambda_\omega]$.

Proof.

(b). — There exists a smooth exhaustive plurisubharmonic function Ψ on X such that $\sup_X \Psi = +\infty$. Let $X_j := \{x\in X \mid \Psi(x) < j\}$ be a sublevel set. From m-positivity of h_F, a line bundle $(\det F, \det h_F)$ is positive. By [36, Theorem 1.2], there exists a holomorphically embedding map $\Phi: X \to \mathbb{P}^{2n+1}$. Here, we take a general hyperplane H of $\mathbb{P}^{2n+1}$. Then $\mathbb{P}^{2n+1}\setminus H$ is affine thus Stein, and since H is general, it intersects $\Phi(X)$.

In particular, there is a strictly plurisubharmonic function ψ on $\mathbb{P}^{2n+1}\setminus H$ which is smooth and exhaustive, i.e. $\psi(z) \nearrow +\infty$ as z tends to H. Then, the smooth function $\Phi^*\psi$ on $X\setminus \Phi^{-1}(H)$ is also strictly plurisubharmonic and satisfies $\Phi^*\psi(z) \nearrow +\infty$ as z tends to $\Phi^{-1}(H)$. Hence, since the smooth function $\Phi^*\psi - \log(j-\Psi)$ on $X_j\setminus\Phi^{-1}(H)$ is strictly plurisubharmonic and exhaustive, the subset $X_j\setminus\Phi^{-1}(H)$ is Stein submanifold of $\mathbb{P}^{2n+1}\setminus H$.

From m-positivity of h_F and Lemma 2.4, we get $A^{n,q}_{F,h_F,\omega} > 0$ on X for $q \geqslant 1$ with $m \geqslant \min\{n-q+1, r\}$. Fix a positive integer $q \geqslant 1$ with $m \geqslant \min\{n-q+1, r\}$. By L^2-type Nakano semi-positivity of h, for any $f \in L^2_{n,q}(X, F\otimes E, h_F\otimes h,\omega)$ satisfying $\bar\partial f = 0$ and $\int_X \langle B^{-1}_{h_F,\omega} f, f\rangle_{h_F\otimes h,\omega} dV_\omega < +\infty$, there exists $u_j \in L^2_{n,q-1}(X_j\setminus\Phi^{-1}(H), F\otimes E, h_F\otimes h,\omega)$ satisfies $\bar\partial u_j = f$

on $X_j \setminus \Phi^{-1}(H)$ and

$$\begin{aligned}\int_{X_j\setminus\Phi^{-1}(H)} |u_j|^2_{h_F\otimes h,\omega}\mathrm{d}V_\omega &\leqslant \int_{X_j\setminus\Phi^{-1}(H)} \left\langle B^{-1}_{h_F,\,\omega}f, f\right\rangle_{h_F\otimes h,\omega}\mathrm{d}V_\omega \\ &\leqslant \int_X \left\langle B^{-1}_{h_F,\,\omega}f, f\right\rangle_{h_F\otimes h,\omega}\mathrm{d}V_\omega.\end{aligned}$$

Let $u_j = 0$ on $\Phi^{-1}(H)$ then by Lemma 4.2, we have that $u_j \in L^2_{n,q-1}(X_j, F\otimes E, h_F\otimes h, \omega)$, $\bar\partial u_j = f$ on X_j and

$$\int_{X_j} |u_j|^2_{h_F\otimes h,\omega}\mathrm{d}V_\omega \leqslant \int_X \left\langle B^{-1}_{h_F,\,\omega}f, f\right\rangle_{h_F\otimes h,\omega}\mathrm{d}V_\omega.$$

Hence, by Lemma 3.18, we obtain a solution $u \in L^2_{n,q-1}(X, F\otimes E, h_F\otimes h, \omega)$ of $\bar\partial u = f$ on X satisfying

$$\int_X |u|^2_{h_F\otimes h,\omega}\mathrm{d}V_\omega \leqslant \int_X \left\langle B^{-1}_{h_F,\,\omega}f, f\right\rangle_{h_F\otimes h,\omega}\mathrm{d}V_\omega.$$

(a). — It is shown from the fact $A^{p,q}_{A,h_A,\omega} > 0$ for $p+q > n+k-1$ as above. □

Theorem 4.8. — *Let (X,ω) be a weakly pseudoconvex Kähler manifold and E be a holomorphic vector bundle equipped with a singular Hermitian metric h. We assume that h is dual Nakano semi-positive on X. Then we have the following*

(a) *If X has a positive holomorphic line bundle and (A,h_A) is a k-positive line bundle, then for any $p \geqslant k$ and any $f \in L^2_{p,n}(X, A\otimes E, h_A\otimes h, \omega)$ satisfying $\bar\partial f = 0$ and $\int_X \langle B^{-1}_{h_A,\omega}f, f\rangle_{h_A\otimes h,\omega}\mathrm{d}V_\omega < +\infty$, there exists $u \in L^2_{p,n-1}(X, A\otimes E, h_A\otimes h, \omega)$ satisfies $\bar\partial u = f$ and*

$$\int_X |u|^2_{h_A\otimes h,\omega}\mathrm{d}V_\omega \leqslant \int_X \left\langle B^{-1}_{h_A,\omega}f, f\right\rangle_{h_A\otimes h,\omega}\mathrm{d}V_\omega,$$

where $B_{h_A,\omega} = [i\Theta_{A,h_A}\otimes \mathrm{id}_E, \Lambda_\omega]$.

(b) *If (F,h_F) is a dual m-positive holomorphic vector bundle of rank r, then for any $p \geqslant 1$ with $m \geqslant \min\{n-p+1, r\}$ and any $f \in L^2_{p,n}(X, F\otimes E, h_F\otimes h, \omega)$ satisfying*

$$\bar\partial f = 0 \textit{ and} \int_X \left\langle B^{-1}_{h_F,\,\omega}f, f\right\rangle_{h_F\otimes h,\omega}\mathrm{d}V_\omega < +\infty,$$

there exists $u \in L^2_{p,n-1}(X, F\otimes E, h_F\otimes h, \omega)$ satisfies $\bar\partial u = f$ and

$$\int_X |u|^2_{h_F\otimes h,\omega}\mathrm{d}V_\omega \leqslant \int_X \left\langle B^{-1}_{h_F,\,\omega}f, f\right\rangle_{h_F\otimes h,\omega}\mathrm{d}V_\omega.$$

where $B_{h_F,\,\omega} = [i\Theta_{F,h_F}\otimes \mathrm{id}_E, \Lambda_\omega]$.

Proof.

(b). — There exists a smooth exhaustive plurisubharmonic function Ψ on X such that $\sup_X \Psi = +\infty$. Let $X_j := \{x \in X \mid \Psi(x) < j\}$ be a sublevel set. From dual m-positivity of h_F, a line bundle $(\det F, \det h_F)$ is positive.

Similarly to the proof of Theorem 4.7, there exists an analytic subset Z such that $X_j \setminus Z$ is Stein submanifold for any $j > 0$. By Lemma 3.14, there exists a hypersurface H_j such that $S_j := (X_j \setminus Z) \setminus H_j$ is also Stein and $E|_{S_j}$ is trivial. From Steinness of S_j, there exists a increasing sequence of open Stein subsets $(S_j(k))_{k \in \mathbb{N}}$ such that $S_j(k)$ is relatively compact and that $\bigcup_k S_j(k) = S_j$. Fixed $k \in \mathbb{N}$: there is $\nu_0 \in \mathbb{N}$ such that $S_j(k) \Subset S_j^{\nu_0} \Subset S_j^{\nu}$ for any $\nu \geqslant \nu_0$, where S_j^{ν} is the notation in Subsection 3.2. For an approximate identity $(\rho_\nu)_{\nu \in \mathbb{N}}$ with respect to S_j, we define the smooth Hermitian metric $h_\nu := (h^* * \rho_\nu)^*$ on E over S_j^{ν}. By Proposition 3.13, h_ν is dual Nakano semi-positive.

From dual m-positivity of h_F, Lemma 2.4 and Proposition 2.5, we have $B_{h_F,\omega} > 0$ on $\bigwedge^{p,n} T_X^* \otimes F \otimes E$ for $p \geqslant 1$ with $m \geqslant \min\{n-p+1, r\}$. Fixed a positive integer $p \geqslant 1$ with $m \geqslant \min\{n-p+1, r\}$. By dual Nakano semi-positivity of h_ν and Lemma 2.4, we obtain that $A^{k,n}_{E,h_\nu,\omega} \geqslant 0$ for $k \geqslant 1$ and

$$\begin{aligned} A^{p,n}_{E\otimes F, h_\nu \otimes h_F, \omega} &= [i\Theta_{E,h_\nu} \otimes \mathrm{id}_F, \Lambda_\omega] + [i\Theta_{F,h_F} \otimes \mathrm{id}_E, \Lambda_\omega] \\ &\geqslant [i\Theta_{F,h_F} \otimes \mathrm{id}_E, \Lambda_\omega] = B_{h_F,\omega} > 0, \end{aligned}$$

i.e. $0 < (A^{p,n}_{E\otimes F, h_\nu \otimes h_F, \omega})^{-1} \leqslant B^{-1}_{h_F,\omega}$ on $S_j(k)$ for any $\nu \geqslant \nu_0$. For any $f \in L^2_{p,n}(X, F \otimes E, h_F \otimes h, \omega)$ satisfying $\bar{\partial} f = 0$ and $\int_X \langle B^{-1}_{h_F,\omega} f, f\rangle_{h_F\otimes h,\omega} \mathrm{d}V_\omega < +\infty$, we have

$$\begin{aligned} &\int_{S_j(k)} \left\langle [i\Theta_{F\otimes E, h_F\otimes h_\nu}, \Lambda_\omega]^{-1} f, f \right\rangle_{h_F\otimes h_\nu,\omega} \mathrm{d}V_\omega \\ &\qquad \leqslant \int_{S_j(k)} \left\langle B^{-1}_{h_F,\omega} f, f \right\rangle_{h_F\otimes h_\nu,\omega} \mathrm{d}V_\omega \\ &\qquad \leqslant \int_X \left\langle B^{-1}_{h_F,\omega} f, f \right\rangle_{h_F\otimes h,\omega} \mathrm{d}V_\omega < +\infty. \end{aligned}$$

By the L^2-estimate for (p,n)-forms with possibly non-complete Kähler metric ω (see [39, Theorem 3.7]), we get a solution $u_{j,k,\nu} \in L^2_{p,n-1}(S_j(k), F \otimes$

$E, h_F \otimes h_\nu, \omega)$ of $\bar{\partial} u_{j,k,\nu} = f$ on $S_j(k)$ satisfying

$$\begin{aligned}\int_{S_j(k)} |u_{j,k,\nu}|^2_{h_F \otimes h_\nu, \omega} \mathrm{d}V_\omega &\leqslant \int_{S_j(k)} \langle [i\Theta_{F\otimes E, h_F \otimes h_\nu}, \Lambda_\omega]^{-1} f, f \rangle_{h_F \otimes h_\nu, \omega} \mathrm{d}V_\omega \\ &\leqslant \int_X \left\langle B^{-1}_{h_F, \omega} f, f \right\rangle_{h_F \otimes h, \omega} \mathrm{d}V_\omega < +\infty.\end{aligned}$$

Here, h_ν increasing to h a.e. as ν tends to $+\infty$ and $S_j(k)$ increasing to S_j. By Lemma 3.18, we obtain a solution $u_j \in L^2_{p,n-1}(S_j, F \otimes E, h_F \otimes h, \omega)$ of $\bar{\partial} u_j = f$ on S_j satisfying

$$\begin{aligned}\int_{S_j} |u_j|^2_{h_F \otimes h, \omega} \mathrm{d}V_\omega &\leqslant \int_{S_j} \left\langle B^{-1}_{h_F, \omega} f, f \right\rangle_{h_F \otimes h, \omega} \mathrm{d}V_\omega \\ &\leqslant \int_X \left\langle B^{-1}_{h_F, \omega} f, f \right\rangle_{h_F \otimes h, \omega} \mathrm{d}V_\omega < +\infty.\end{aligned}$$

From Lemma 4.2, letting $u_j = 0$ on Z and H_j then we obtain that $u_j \in L^2_{p,n-1}(X_j, F \otimes E, h_F \otimes h, \omega)$, $\bar{\partial} u_j = f$ on X_j and

$$\int_{X_j} |u_j|^2_{h_F \otimes h, \omega} \mathrm{d}V_\omega \leqslant \int_X \left\langle B^{-1}_{h_F, \omega} f, f \right\rangle_{h_F \otimes h, \omega} \mathrm{d}V_\omega < +\infty.$$

Repeating, by Lemma 3.18 we have a solution $u \in L^2_{p,n-1}(X, F \otimes E, h_F \otimes h, \omega)$ of $\bar{\partial} u = f$ on X satisfying

$$\int_X |u_j|^2_{h_F \otimes h, \omega} \mathrm{d}V_\omega \leqslant \int_X \left\langle B^{-1}_{h_F, \omega} f, f \right\rangle_{h_F \otimes h, \omega} \mathrm{d}V_\omega < +\infty.$$

(a). — It is shown from the fact $A^{p,q}_{A, h_A, \omega} > 0$ for $p + q > n + k - 1$ as above. □

5. An L^2-type Dolbeault isomorphism

In this section, we provide L^2-type Dolbeault isomorphisms including L^2-subsheaves by using the following lemma and theorem.

Lemma 5.1 (Dolbeault–Grothendieck lemma, cf. [9, Chapter I]). — *Let T be a current of type $(p, 0)$ on some open subset $U \subset \mathbb{C}^n$. If T is $\bar{\partial}$-closed then it is a holomorphic differential form, i.e. a smooth differential form with holomorphic coefficients.*

Theorem 5.2 (cf. [39, Theorem 6.1]). — *Let X be a complex manifold and E be a holomorphic vector bundle equipped with a singular Hermitian metric h. We assume that h is Griffiths semi-positive. Then for any $x_0 \in X$, there exist an open neighborhood U of x_0 and a Kähler metric ω on U*

satisfying that for any $\bar\partial$*-closed* $f \in L^2_{p,q}(U, E \otimes \det E, h \otimes \det h, \omega)$*, there exists* $u \in L^2_{p,q-1}(U, E \otimes \det E, h \otimes \det h, \omega)$ *such that* $\bar\partial u = f$*.*

For singular Hermitian metrics h on E, we define the subsheaf $\mathscr{L}^{p,q}_{E,h}$ of germs of (p,q)-forms u with values in E and with measurable coefficients such that both $|u|^2_h$ and $|\bar\partial u|^2_h$ are locally integrable, here we see that $\mathscr{L}^{p,q}_{E,h}$ is a fine sheaf.

Theorem 5.3. — *Let X be a complex manifold of dimension n and (F, h_F) be a Hermitian holomorphic vector bundle over X. Let L be a holomorphic line bundle over X equipped with a singular Hermitian metric h_L and E be a holomorphic vector bundle over X equipped with a singular Hermitian metric h_E. Then we have the following*

(a) *If h_L is singular semi-positive, then we have an exact sequence of sheaves*

$$0 \longrightarrow \Omega^p_X \otimes \mathcal{O}_X(F \otimes L) \otimes \mathscr{I}(h_L) \longrightarrow \mathscr{L}^{p,\bullet}_{F\otimes L, h_F \otimes h_L}.$$

(b) *If h_E is L^2-type Nakano semi-positive, then we get an exact sequence of sheaves*

$$0 \longrightarrow K_X \otimes \mathcal{O}_X(F) \otimes \mathscr{E}(h_E) \longrightarrow \mathscr{L}^{n,\bullet}_{F\otimes E, h_F \otimes h_E}.$$

(c) *If h_E is Griffiths semi-positive, then we have an exact sequence of sheaves*

$$0 \longrightarrow \Omega^p_X \otimes \mathcal{O}_X(F) \otimes \mathscr{E}(h_E \otimes \det h_E) \longrightarrow \mathscr{L}^{p,\bullet}_{F\otimes E \otimes \det E, h_F \otimes h_E \otimes \det h_E}.$$

In particular, L^2-type Dolbeault isomorphisms are obtained from these. For example, $H^q(X, \Omega^p_X \otimes F \otimes L \otimes \mathscr{I}(h_L)) \cong H^q(\Gamma(X, \mathscr{L}^{p,\bullet}_{F\otimes L, h_F\otimes h_L}))$ in the case of (a)*.*

To simplify the proof, we introduce the following definition.

Definition 5.4. — *Let E be a holomorphic vector bundle on a complex manifold X. Consider two singular Hermitian metrics h_1 and h_2 on E. For any open set U of X, we will write $h_1 \sim h_2$ on U, if there is a constant $C > 0$ such that $C^{-1}h_2 \leqslant h_1 \leqslant Ch_2$.*

Proof of Theorem 5.3. — For any fixed point $x_0 \in X$, there exist a Stein open neighborhood U of x_0 such that F is trivial on U, i.e. $F|_U = \mathbb{C}^r \times U := \underline{\mathbb{C}^r}$, where $r = \operatorname{rank} F$. Let $(U; z_1, \ldots, z_n)$ be a local coordinate, I_F be a trivial Hermitian metric on $F|_U$ and $\omega = \sum_{j=1}^n dz_j \wedge d\bar z_j$ be a Kähler metric. By smoothness of h_F, we get $h_F \sim I_F$ on U.

We consider the following sheaves sequence:

$$0 \longrightarrow \ker \bar{\partial}_0 \hookrightarrow \mathscr{L}^{p,0}_{F\otimes E,h_F\otimes h_E} \xrightarrow{\bar{\partial}_0} \mathscr{L}^{p,1}_{F\otimes E,h_F\otimes h_E} \xrightarrow{\bar{\partial}_1} \dots \xrightarrow{\bar{\partial}_{n-1}} \mathscr{L}^{p,n}_{F\otimes E,h_F\otimes h_E} \longrightarrow 0,$$

where $\bar{\partial}_j = \bar{\partial}_{F\otimes E} = \bar{\partial} \otimes \mathrm{id}_{F\otimes E}$. From $h_F \sim I_F$ on U, we get $h_F \otimes h_E \sim I_F \otimes h_E$ on U and $L^2_{p,q}(U, F\otimes E, h_F \otimes h_E, \omega) = L^2_{p,q}(U, \underline{\mathbb{C}^r} \otimes E, I_F \otimes h_E, \omega)$. Therefore, $\mathscr{L}^{p,q}_{F\otimes E,h_F\otimes h_E}(U) = \mathscr{L}^{p,q}_{\underline{\mathbb{C}^r}\otimes E,I_F\otimes h_E}(U)$. By Lemma 5.1, the kernel of $\bar{\partial}_0$ consists of all germs of holomorphic $(p,0)$-forms with values in $F \otimes E$ which satisfy the integrability condition.

We prove that $\bar{\partial}_0 = \Omega^p_X \otimes \mathcal{O}_X(F) \otimes \mathscr{E}(h_E)$. Let $e = (e_1, \dots, e_r)$ and $b = (b_1, \dots, b_s)$ be holomorphic frames of $\underline{\mathbb{C}^r}$ and E on U respectively, where $s = \operatorname{rank} E$, $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ and e is orthonormal with respect to I_F. For any $f \in H^0(U, \Omega^p_X \otimes \underline{\mathbb{C}^r} \otimes E) = H^0(U, \Omega^p_X \otimes F \otimes E)$, we can write

$$f = \sum_{|I|=p,j,\lambda} f_{Ij\lambda} \mathrm{d}z_I \otimes e_j \otimes b_\lambda = \sum_j f_j \otimes e_j = \sum_{|I|=p,j} f_{jI} \mathrm{d}z_I \otimes e_j,$$

where $f_j = \sum_{|I|=p,\lambda} f_{Ij\lambda} \mathrm{d}z_I \otimes b_\lambda = \sum_{|I|=p} f_{jI} \mathrm{d}z_I \in H^0(U, \Omega^p_X \otimes E)$ and $f_{jI} = \sum_\lambda f_{Ij\lambda} \otimes b_\lambda \in H^0(U, E)$. We can calculate the following

$$|f|^2_{I_F\otimes h_E,\omega} = \sum_j |f_j|^2_{h_E,\omega} = \sum_{j,I} |f_{jI}|^2_{h_E},$$
$$\|f\|^2_{I_F\otimes h_E,\omega} = \sum_{j,I} \int_U |f_{jI}|^2_{h_E} \mathrm{d}V_\omega.$$

Therefore, we get $f \in \ker \bar{\partial}_0(U) \iff \|f\|^2_{I_F\otimes h_E,\omega} = \sum_{j,I} \int_U |f_{jI}|^2_{h_E} \mathrm{d}V_\omega < +\infty$, i.e. each $f_{jI} \in H^0(U, E)$ satisfies the condition $f_{jI} \in \mathscr{E}(h_E)(U)$. Hence, we have that $\ker \bar{\partial}_0 = \Omega^p_X \otimes \mathcal{O}_X(F) \otimes \mathscr{E}(h_E)$. In particular, from the fact $\mathscr{E}(h_L) = \mathcal{O}_X(L) \otimes \mathscr{I}(h_L)$ if $(E, h_E) = (L, h_L)$, we obtain $\ker \bar{\partial}_0 = \Omega^p_X \otimes \mathcal{O}_X(F \otimes L) \otimes \mathscr{I}(h_L)$.

From the above, the sheaves sequences of (a)–(c) are exact at $q = 0$. Finally, we prove the exactness of the sheaves sequences of (a)–(c) at $q \geqslant 1$.

(a). — We can retake U such that L is also trivial on U and that U is relatively compact in $\mathbb{C}^n$. By the assumption, there exists a plurisubharmonic function φ on U such that $\varphi = -\log h_L$ a.e. From $\mathscr{L}^{p,q}_{F\otimes L,h_F\otimes h_L}(U) = \mathscr{L}^{p,q}_{\underline{\mathbb{C}^r}\otimes L,I_F\otimes h_L}(U)$, it is sufficient to show that for any $\bar{\partial}$-closed

$$f \in L^2_{p,q}(U, \underline{\mathbb{C}^r} \otimes L, I_F \otimes h_L, \omega),$$

there is $u \in L^2_{p,q-1}(U, \underline{\mathbb{C}^r} \otimes L, I_F \otimes h_L, \omega)$ such that $\bar{\partial} u = f$. Let $\widetilde{e}$ be a holomorphic frame of $L|_U$. We can write

$$f = \sum_{|I|=p,|J|=q,k} f_{IJk} \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J \otimes e_k \otimes \widetilde{e} = \sum_k f_k \otimes e_k,$$
$$f_k = \sum_{|I|=p,|J|=q} f_{IJk} \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J \otimes \widetilde{e}.$$

By $\varphi = -\log h_L$ a.e, we obtain

$$\|f\|^2_{I_F \otimes h_L, \omega} = \sum_k \int_U |f_k|^2 e^{-\varphi} \mathrm{d}V_\omega < +\infty,$$

i.e. $f_k \in L^2_{p,q}(U, L, h_L, \omega) = L^2_{p,q}(U, \varphi, \omega)$ for any k.

From holomorphicity of e, we get $0 = \bar{\partial} f = \bar{\partial} \sum_k f_k \otimes e_k = \sum_k \bar{\partial} f_k \otimes e_k$ and $\bar{\partial} f_k = 0$. By [15, Theorem 4.4.2], there is a solution u_k of $\bar{\partial} u_k = f_k$ satisfying

$$\begin{aligned}\inf_U \left(1+|z|^2\right)^{-2} \int_U |u_k|^2 e^{-\varphi} \mathrm{d}V_\omega &\leqslant \int_U |u_k|^2 e^{-\varphi} \left(1+|z|^2\right)^{-2} \mathrm{d}V_\omega \\ &\leqslant \int_U |f_k|^2 e^{-\varphi} \mathrm{d}V_\omega,\end{aligned}$$

where $\inf_U (1+|z|^2)^{-2} > 0$ and $\overline{U}$ is compact. By defining the $(p, q-1)$-form $u = \sum_k u_k \otimes e_k$, we have the following

$$\bar{\partial} u = \bar{\partial} \sum u_k \otimes e_k = \sum \bar{\partial} u_k \otimes e_k = \sum f_k \otimes e_k = f,$$
$$\begin{aligned}\inf_U \left(1+|z|^2\right)^{-2} \|u\|^2_{I_F \otimes h_L, \omega} &= \inf_U \left(1+|z|^2\right)^{-2} \sum \int_U |u_k|^2 e^{-\varphi} \mathrm{d}V_\omega \\ &\leqslant \sum \int_U |u_k|^2 e^{-\varphi} \left(1+|z|^2\right)^{-2} \mathrm{d}V_\omega \\ &\leqslant \sum \int_U |f_k|^2 e^{-\varphi} \mathrm{d}V_\omega < +\infty,\end{aligned}$$

i.e. $u \in L^2_{p,q-1}(U, \underline{\mathbb{C}^r} \otimes L, I_F \otimes h_L, \omega)$.

(b). — Let $\psi := |z|^2$ be a smooth strictly plurisubharmonic on U then $i\partial\bar{\partial}\psi = \omega$. From $h_F \sim I_F \sim I_F e^{-\psi} := I_F^\psi$ on U, we get

$$\mathscr{L}^{n,q}_{F \otimes E, h_F \otimes h_E}(U) = \mathscr{L}^{n,q}_{\underline{\mathbb{C}^r} \otimes E, I_F^\psi \otimes h_E}(U).$$

Thus, it is sufficient to show that for any $\bar{\partial}$-closed $f \in L^2_{n,q}(U, \underline{\mathbb{C}^r} \otimes E, I^\psi_F \otimes h_E, \omega)$, there exists $u \in L^2_{n,q-1}(U, \underline{\mathbb{C}^r} \otimes E, I^\psi_F \otimes h_E, \omega)$ such that $\bar{\partial} u = f$. We can write

$$f = \sum f_{Jk\lambda} \mathrm{d}z_N \wedge \mathrm{d}\bar{z}_J \otimes e_k \otimes b_\lambda = \sum f_k \otimes e_k,$$
$$f_k = \sum f_{Jk\lambda} \mathrm{d}z_N \wedge \mathrm{d}\bar{z}_J \otimes b_\lambda.$$

Then $f_k \in L^2_{n,q}(U, E, h_E e^{-\psi}, \omega)$ and $\bar{\partial} f_k = 0$ on U for any k. Here, we obtain $B_{\psi,\omega} = q \cdot \mathrm{id}_E$ on $\bigwedge^{n,q} T^*_U \otimes E$. From L^2-type Nakano semi-positivity of h_E, for any $f_k \in L^2_{n,q}(U, E, h_E e^{-\psi}, \omega)$ satisfying $\bar{\partial} f_k = 0$, there exists $u_k \in L^2_{n,q-1}(U, E, h_E e^{-\psi}, \omega)$ satisfying $\bar{\partial} u_k = f_k$ and

$$\begin{aligned}\int_U |u_k|^2_{h_E,\omega} e^{-\psi} \mathrm{d}V_\omega &\leqslant \int_U \left\langle B^{-1}_{\psi,\omega} f_k, f_k \right\rangle_{h_E,\,\omega} e^{-\psi} \mathrm{d}V_\omega \\ &= \frac{1}{q} \int_U |f_k|^2_{h_E,\,\omega} e^{-\psi} \mathrm{d}V_\omega < +\infty.\end{aligned}$$

By defining the $(n, q-1)$-form $u := \sum_k u_k \otimes e_k$, we obtain that $\bar{\partial} u = f$ and

$$\|u\|^2_{I^\psi_F \otimes h_E, \omega} = \int_U |u|^2_{I^\psi_F \otimes h_E,\,\omega} \mathrm{d}V_\omega = \sum_k \int_U |u_k|^2_{h_E,\,\omega} e^{-\psi} \mathrm{d}V_\omega < +\infty,$$

i.e. $u \in L^2_{n,q-1}(U, \underline{\mathbb{C}^r} \otimes E, I^\psi_F \otimes h_E,\, \omega)$.

(c). — It is shown using Theorem 5.2 in the same way as (a). ☐

6. Main results and proofs for vanishing theorems

In this section, we prove main results and additionally give vanishing theorems for the cases where a singular Hermitian metric is L^2-type Nakano semi-positive or dual Nakano semi-positive. Main results can be deduced quite directly from the L^2-estimates and L^2-type Dolbeault isomorphisms established in Section 4 and Section 5 respectively. Since the proofs are similar, we omit them except that of Theorem 1.2 to illustrate the idea. Similarly, Theorem 1.4 can be shown by using Theorem 4.3 and Corollary 4.4, and Theorem 1.1 and 1.3 can be shown by using Theorem 5.3 and 4.6.

Proof of Theorem 1.2. — Let h_F be a smooth Hermitian metric on F. By Theorem 5.3 (a), the complex of sheaves $(\mathscr{L}^{p,\bullet}_{F \otimes L, h_F \otimes h}, \bar{\partial})$ defined by $\bar{\partial}$-operator is a fine resolution of the sheaf $\Omega^p_X \otimes \mathcal{O}_X(F \otimes L) \otimes \mathscr{I}(h)$, thus we have the L^2-type Dolbeault isomorphism

$$H^q\left(X, \Omega^p_X \otimes F \otimes L \otimes \mathscr{I}(h)\right) \cong H^q\left(\Gamma\left(X, \mathscr{L}^{p,\bullet}_{F \otimes L, h_F \otimes h}\right)\right).$$

Let ψ be a smooth exhaustive plurisubharmonic function on X. For any convex increasing function $\chi \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, we define the smooth Hermitian metric $h_F^\chi := h_F e^{-\chi\circ\psi}$.

(b). — By m-positivity of h_F, we have that h_F is Griffiths positive, and $(\det F, \det h_F)$ is positive. Therefore, there exists a Kähler metric ω on X. Since $e^{-\chi\circ\psi}$ is semi-positive Hermitian metric on the trivial line bundle $X\times\mathbb{C} := \underline{\mathbb{C}}$, we obtain $\theta_{\underline{\mathbb{C}},e^{-\chi\circ\psi}} \geqslant 0$ and $\theta_{F,h_F^\chi} = \theta_{F,h_F} + \theta_{\underline{\mathbb{C}},e^{-\chi\circ\psi}} \geqslant \theta_{F,h_F} >_m 0$, thus h_F^χ is also m-positive.

From m-positivity of h_F and h_F^χ, Lemma 2.4 and Proposition 2.5, we have

$$B_{h_F,\,\omega} = [i\Theta_{F,h_F}\otimes \mathrm{id}_L, \Lambda_\omega] > 0 \quad \text{and} \quad B_{h_F^\chi,\omega} = \left[i\Theta_{F,h_F^\chi}\otimes \mathrm{id}_L, \Lambda_\omega\right] > 0$$

on $\bigwedge^{n,q} T_X^*\otimes F\otimes L$ for any positive integer $q \geqslant 1$ with $m \geqslant \min\{n-q+1, r\}$.

Therefore, from the inequality

$$\left\langle B_{h_F^\chi,\omega} v, v\right\rangle_{h_F^\chi\otimes h,\omega} \geqslant \langle B_{h_F,\,\omega} v, v\rangle_{h_F^\chi\otimes h,\omega} = \langle B_{h_F,\,\omega} v, v\rangle_{h_F\otimes h,\omega}\, e^{-\chi\circ\psi} > 0,$$

we get the inequality

$$0 < \left\langle B^{-1}_{h_F^\chi,\omega} v, v\right\rangle_{h_F^\chi\otimes h,\omega} \leqslant \left\langle B^{-1}_{h_F,\,\omega} v, v\right\rangle_{h_F\otimes h,\omega} e^{-\chi\circ\psi},$$

for any (n,q)-forms $u, v \in \bigwedge^{n,q} T_X^*\otimes F\otimes L$. In fact, we obtain

$$\begin{aligned}
\left|\langle v,u\rangle_{h_F^\chi\otimes h,\omega}\right|^2 &= \left|\langle v,u\rangle_{h_F\otimes h,\omega}\right|^2 e^{-2\chi\circ\psi}\\
&\leqslant \left\langle B^{-1}_{h_F,\,\omega} v, v\right\rangle_{h_F\otimes h,\omega} \langle B_{h_F,\,\omega} u, u\rangle_{h_F\otimes h,\omega} e^{-2\chi\circ\psi}\\
&\leqslant \left\langle B^{-1}_{h_F,\,\omega} v, v\right\rangle_{h_F\otimes h,\omega} \left\langle B_{h_F^\chi,\omega} u, u\right\rangle_{h_F^\chi\otimes h,\omega} e^{-\chi\circ\psi},
\end{aligned}$$

and the choose $u = B^{-1}_{h_F^\chi,\omega} v$ implies

$$\left|\left\langle v, B^{-1}_{h_F^\chi,\omega} v\right\rangle_{h_F^\chi\otimes h,\omega}\right|^2 \leqslant \left\langle B^{-1}_{h_F,\,\omega} v, v\right\rangle_{h_F\otimes h,\omega} e^{-\chi\circ\psi}\cdot \left\langle v, B^{-1}_{h_F^\chi,\omega} v\right\rangle_{h_F^\chi\otimes h,\omega}.$$

For any $f \in \Gamma(X, \mathscr{L}^{n,q}_{F\otimes L,h_F\otimes h})$ satisfying $\overline{\partial} f = 0$, the integrals

$$\int_X |f|^2_{h_F^\chi\otimes h,\omega} \mathrm{d}V_\omega = \int_X |f|^2_{h_F\otimes h,\omega} e^{-\chi\circ\psi} \mathrm{d}V_\omega$$

and

$$\int_X \left\langle B^{-1}_{h_F^\chi,\omega} f, f\right\rangle_{h_F^\chi\otimes h,\omega} \mathrm{d}V_\omega \leqslant \int_X \left\langle B^{-1}_{h_F,\,\omega} f, f\right\rangle_{h_F\otimes h,\omega} e^{-\chi\circ\psi} \mathrm{d}V_\omega$$

become convergent if χ grows fast enough. By Theorem 4.3, there exists $u \in L^2_{n,q-1}(X, F \otimes L, h_F^\chi \otimes h, \omega)$ such that $\bar\partial u = f$ and

$$\int_X |u|^2_{h_F \otimes h,\omega} e^{-\chi\circ\psi} \mathrm{d}V_\omega \leqslant \int_X \left\langle B^{-1}_{h_F^\chi,\omega} f, f \right\rangle_{h_F \otimes h,\omega} e^{-\chi\circ\psi} \mathrm{d}V_\omega < +\infty,$$

where $|u|^2_{h_F \otimes h,\omega}$ is locally integrable. Hence, we have that

$$u \in \Gamma\left(X, \mathscr{L}^{n,q-1}_{F\otimes L, h_F\otimes h}\right)$$

and that $H^q(X, K_X \otimes F \otimes L \otimes \mathscr{I}(h)) = 0$.

(a). — This is shown in the same way as above using Corollary 4.4. ☐

Furthermore, by the same argument as above we obtain the following theorem and corollary for L^2-type Nakano semi-positive singular metrics using Theorem 5.3 (b) and Theorem 4.7.

Theorem 6.1. — *Let X be a weakly pseudoconvex manifold and E be a holomorphic vector bundle equipped with a singular Hermitian metric h which is L^2-type Nakano semi-positive on X. Then we have the following*

(a) *If X has a positive holomorphic line bundle and A is a k-positive line bundle, then we have*

$$H^q(X, K_X \otimes A \otimes \mathscr{E}(h)) = 0$$

for any $q \geqslant k$.

(b) *If F is an m-positive holomorphic vector bundle of rank r then*

$$H^q(X, K_X \otimes F \otimes \mathscr{E}(h)) = 0$$

for $q \geqslant 1$ with $m \geqslant \min\{n-q+1, r\}$.

Corollary 6.2. — *Let X be a weakly pseudoconvex manifold and E be a holomorphic vector bundle equipped with a singular Hermitian metric h. We assume that there exists a holomorphic positive line bundle (L, h_L) such that the singular Hermitian metric $h \otimes h_L^*$ on $E \otimes L^*$ is L^2-type Nakano semi-positive on X. Then we have the following vanishing*

$$H^q(X, K_X \otimes \mathscr{E}(h)) = 0$$

for any $q > 0$.

Theorem 1.6 is proved using Theorem 6.1 and Remark 3.21. The following vanishing theorem for strictly dual Nakano positivity on projective manifolds is obtained, which is generalized from Hodge to Kähler, and allows for more singularity then in [39, Theorem 1.2]. In fact, from the definition of strictly dual Nakano δ_{ω_X}-positivity and the proof of L^2-estimates (see [39, Theorem 4.12]), it was necessary to have the existence of a globally defined Kähler potential for ω_X, which is a strictly plurisubharmonic function, on the Stein subset $S := X \setminus D$ obtained by removing the ample divisor D from X. However, this is resolved by Proposition 3.19.

Theorem 6.3. — *Let X be a projective manifold equipped with a Kähler metric ω. Let E be a holomorphic vector bundle over X equipped with a singular Hermitian metric h. We assume that h is strictly dual Nakano δ_ω-positive on X and that $\nu(-\log\det h, x) < 2$ for any point $x \in X$. Then for any $p > 0$, we have the cohomology vanishing*

$$H^n\left(X, \Omega_X^p \otimes E\right) = 0.$$

Proof. — Let $\mathcal{E}^{p,q}(X,E)$ be the space of smooth E-valued (p,q)-forms on X and $\mathcal{U} = \{U_j\}_{j\in I}$ be a locally finite open cover of X such that U_j are biholomorphic to a polydisc. By the assumption, $\det h$ is locally integrable from the results of Skoda (see [34]). Since $h = \det h \cdot \widehat{h^*}$ and each element of $\widehat{h^*}$ is locally bounded (see [31, Lemma 2.2.4]), for any $s \in \mathcal{E}^{p,q}(X,E)$ the function $|s|_h^2$ is also locally integrable. Here, $\widehat{h^*}$ is the adjugate matrix of h^*. Thus, there is an inclusion map $\mathcal{E}^{p,q}(X,E) \hookrightarrow L^2_{\mathrm{loc}(p,q)}(X,E,h,\omega)$.

We know that $U_{j_0}\cap\cdots\cap U_{j_l}$ is a pseudoconvex domain for all $\{j_0,\ldots,j_l\} \subset I$. By [39, Theorem 4.12], we can solve the $\bar\partial$-equation on $U_{j_0}\cap\cdots\cap U_{j_l}$ with respect to h.

Hence, we have the isomorphism

$$H^n\left(X, \Omega_X^p \otimes E\right) \cong \frac{\left\{f \in L^2_{\mathrm{loc}(p,n)}(X,E,h,\omega); \bar\partial f = 0\right\}}{\left\{g \in L^2_{\mathrm{loc}(p,n)}(X,E,h,\omega); \begin{array}{l}\text{there is an } \gamma \in L^2_{\mathrm{loc}(p,n-1)}\\ (X,E,h,\omega) \text{ satisfying } \bar\partial\gamma = g\end{array}\right\}}$$

from the results of sheaf cohomology. This is a singular version of isomorphism theorems (see [29]) and was first mentioned in [18, Corollary 1.2]. By the projectivity of X and Proposition 3.19, h has strictly dual Nakano positivity for a Hodge metric on X. Therefore, we obtain $H^n(X,\Omega_X^p\otimes E) = 0$ from [39, Theorem 4.12]. □

Here, we introduce the following lemma, which is evident from the proofs of this theorem and [18, Corollary 1.2]. This is an extension of Skoda's result to vector bundles.

Lemma 6.4. — *Let X be a complex manifold, E be a holomorphic vector bundle and h be a singular Hermitian metric on E with Griffiths semi-positivity. If $\nu(-\log\det h, x) < 2$ holds at a point $x \in X$, then we have that $\mathscr{E}(h)_x = \mathcal{O}(E)_x$.*

Using Theorem 4.8 and the proof method of Theorem 6.3, we obtain the following theorem for dual Nakano semi-positivity.

Theorem 6.5. — *Let X be a projective manifold, F be a holomorphic vector bundle of rank r and E be a holomorphic vector bundle equipped with a singular Hermitian metric h. We assume that h is dual Nakano semi-positive on X satisfying $\nu(-\log\det h, x) < 2$ for any point $x \in X$. Then we have the following*

(a) *If A is a k-positive line bundle then, for any $p \geqslant k$ we have that*
$$H^n\left(X, \Omega_X^p \otimes A \otimes E\right) = 0.$$

(b) *If F is a dual m-positive holomorphic vector bundle of rank r then*
$$H^n\left(X, \Omega_X^p \otimes F \otimes E\right) = 0$$
for $p \geqslant 1$ with $m \geqslant \min\{n-p+1, r\}$.

Applying Theorems 1.5 and 4.8 and Theorem 5.3 (c), we can prove Theorem 1.7.

Proof of Theorem 1.7. — Let h_F be a smooth Hermitian metric on F. By Theorem 5.3 (c), we obtain
$$H^q\left(X, \Omega_X^p \otimes F \otimes \mathscr{E}(h \otimes \det h)\right) \cong H^q\left(\Gamma\left(X, \mathscr{L}^{p,\bullet}_{F\otimes E\otimes \det E, h_F\otimes h\otimes \det h}\right)\right).$$

(b). — Let ω be a Kähler metric on X. From dual m-positivity of h_F, Lemma 2.4 and Proposition 2.5, we have $A^{p,n}_{F,h_F,\omega} > 0$ and $B_{h_F,\omega} = [i\Theta_{F,h_F} \otimes \mathrm{id}_{E\otimes\det E}, \Lambda_\omega] > 0$ on $\bigwedge^{p,n} T_X^* \otimes F \otimes E \otimes \det E$ for any positive integer $p \geqslant 1$ with $m \geqslant \min\{n-p+1, r\}$.

By compactness of X, for any global section
$$f \in \Gamma\left(X, \mathscr{L}^{p,n}_{F\otimes E\otimes\det E, h_F\otimes h\otimes\det h}\right)$$
we obtain finiteness of the integral $\int_X \langle B^{-1}_{h_F,\omega} f, f\rangle_{h_F\otimes h\otimes\det h,\omega} \mathrm{d}V_\omega < +\infty$. From Theorems 1.5 and 4.8, if f is $\bar\partial$-closed then there exists
$$u \in L^2_{p,n-1}(X, F\otimes E\otimes\det E, h_F\otimes h\otimes\det h, \omega)$$
such that $\bar\partial u = f$ and
$$\int_X |u|^2_{h_F\otimes h\otimes\det h,\omega} dV_\omega \leqslant \int_X \left\langle B^{-1}_{h_F,\omega} f, f\right\rangle_{h_F\otimes h\otimes\det h,\omega} \mathrm{d}V_\omega < +\infty.$$
Since $|u|^2_{h_F\otimes h\otimes\det h,\omega}$ is locally integrable, we obtain
$$u \in \Gamma\left(X, \mathscr{L}^{p,n-1}_{F\otimes E\otimes\det E, h_F\otimes h\otimes\det h}\right).$$

Hence, we have $H^n(X, \Omega_X^p \otimes F \otimes \mathscr{E}(h\otimes\det h)) = 0$.

(a). — This is shown as above using the fact

$$A^{p,q}_{A,h_A,\omega} > 0 \text{ for } p+q > n+k-1. \qquad \square$$

7. Fujita's conjecture type theorem with singular Hermitian metrics

In [13], Fujita proposed the following conjecture which is a open question in classical algebraic geometry. Recall that, X is an n-dimensional complex manifold.

Conjecture 7.1. — *Let X be a smooth projective variety and L be an ample line bundle.*

- $K_X \otimes L^{\otimes(\dim X+1)}$ *is globally generated;*
- $K_X \otimes L^{\otimes(\dim X+2)}$ *is very ample.*

The global generation conjecture has been proved (cf. [12, 21, 40]) up to dimension 5.

Remark 7.2. — Let $\mathbb{K}$ be an algebraically closed field with arbitrary charactristic. Fujita's conjecture is already known for smooth projective varieties over $\mathbb{K}$ under the additional assumption that L is globally generated (see [22, Theorem 1.1]).

Recently, Fujita's conjecture type theorems was obtained in [35] for the case of pseudo-effective involving the multiplier ideal sheaf and for the case of nef involving Nakano semi-positive vector bundles, as follows.

Theorem 7.3 (cf. [35, Theorem 1.3]). — *Let X be a compact Kähler manifold, L be an ample and globally generated line bundle and (B, h) be a pseudo-effective line bundle. If the numerical dimension of (B, h) is not zero, i.e.* $\mathrm{nd}(B, h) \neq 0$*. then*

$$K_X \otimes L^{\otimes n} \otimes B \otimes \mathscr{I}(h)$$

is globally generated.

Theorem 7.4 (cf. [35, Theorem 1.4]). — *Let X be a compact Kähler manifold and L be an ample and globally generated line bundle. Let E be a holomorphic vector bundle which is Nakano semi-positive. If N is a nef but not numerically trivial line bundle, then the adjoint vector bundle $K_X \otimes L^{\otimes n} \otimes N \otimes E$ is globally generated.*

Here, Theorem 7.3 holds with the addition of a Nakano semi-positive vector bundle (see [35, Theorem 4.3]). In this section, as an extension of these theorems to singular Hermitian metric of holomorphic vector bundles, we show a Fujita type global generation theorem for adjoint vector bundles involving the L^2-subsheaf of a L^2-type Nakano semi-positive vector bundle (see Theorem 1.9).

We introduce a concept on numerical dimension for nef line bundles.

Definition 7.5 (cf. [8, Definition 6.20]). — *Let X be a compact Kähler manifold of dimension n and N be a nef line bundle over X. The* numerical dimension $\mathrm{nd}(N)$ *of N is defined as* $\mathrm{nd}(N) = \max\{k = 0, \dots, n \mid c_1^k(N) \neq 0$ *in* $H^{2k}(X, \mathbb{R})\}$.

These global generation conjecture type theorems are shown using the theory of Castelnuovo–Mumford regularity and vanishing theorems.

Definition 7.6 (cf. [24, Definition 1.8.4]). — *Let X be a projective manifold and L be an ample and globally generated line bundle over X. A coherent sheaf $\mathcal{F}$ on X is m*-regular with respect to L *if $H^q(X, \mathcal{F} \otimes L^{\otimes(n-q)}) = 0$ for $q > 0$.*

Lemma 7.7 (Mumford, cf. [24, Theorem 1.8.5]). — *Let $\mathcal{F}$ be a 0-regular coherent sheaf on X with respect to L, then $\mathcal{F}$ is generated by its global sections.*

To prove Theorem 1.9, we show the following vanishing theorem.

Theorem 7.8 (Theorem 1.8). — *Let X be a compact Kähler manifold of dimension n and E be a holomorphic vector bundle equipped with a singular Hermitian metric h. Let N be a nef line bundle which is neither big nor numerically trivial, i.e.* $\mathrm{nd}(N) \notin \{0, n\}$. *If h is Griffiths semi-positive and there exists a smooth ample divisor A such that $\nu(-\log \det h|_A, x) < 1$ for all points in A and that $\mathrm{nd}(N|_A) = \mathrm{nd}(N)$, then we have*

$$H^q(X, K_X \otimes N \otimes \mathscr{E}(h \otimes \det h)) = 0$$

for any $q > n - \mathrm{nd}(N)$.

We first prove the theorem if the condition for the Lelong number holds on whole X. To this end, we need the following proposition. This proposition is an example of when the equality of the subadditivity property (see [8, Theorem 14.2]) to the L^2-subsheaf holds. Actually, it often happens that if the singular metrics h_j on vector bundles E_j, for $j = 1, 2$, satisfy $\mathscr{E}(h_j) = \mathcal{O}(E_j)$, then $\mathscr{E}(h_1 \otimes h_2) \subsetneq \mathcal{O}(E_1 \otimes E_2)$.

Proposition 7.9. — *Let X be a projective manifold and L be a nef and big line bundle. Let E be a holomorphic vector bundle equipped with a singular Hermitian metric h. If h is Griffiths semi-positive and $\nu(-\log \det h, x) < 2$ for all points $x \in X$. Then there exists a singular Hermitian metric h_L on L such that $\mathscr{E}(h \otimes h_L) \cong \mathcal{O}_X(E \otimes L)$.*

Proof. — From Griffiths semi-positivity of h, a line bundle $(\det E, \det h)$ is pseudo-effective. By the assumption and Skoda's result [34], the function $\det h$ is locally integrable, i.e. $1 \in \mathscr{I}(\det h)_x$ for all points $x \in X$. In other words, there exists $R > 0$ such that $\int_{\mathbb{B}^n_R} \det h \, dV_{\mathbb{C}^n} < +\infty$, where $\mathbb{B}^n_R = \{z \in \mathbb{C}^n \mid |z| < R\}$ and $(z_1, \ldots, z_n)$ is a local coordinate around x. By the strongly openness property (see [14]), for some $r \in (0, R)$ there exists $\beta_x > 0$ such that $\int_{\mathbb{B}^n_r} (\det h)^{1+\beta_x} dV_{\mathbb{C}^n} < +\infty$.

By the Hölder inequality, for any singular Hermitian metric h_L on L we get

$$\int_{\mathbb{B}^n_r} \det h \cdot h_L dV_{\mathbb{C}^n} \leqslant \left(\int_{\mathbb{B}^n_r} (\det h)^{1+\beta_x} dV_{\mathbb{C}^n} \right)^{\frac{1}{1+\beta_x}} \left(\int_{\mathbb{B}^n_r} h_L^{1+1/\beta_x} dV_{\mathbb{C}^n} \right)^{\frac{\beta_x}{1+\beta_x}}.$$

Since L is nef and big, for every $\delta > 0$, L has a singular Hermitian metric h_L such that $\max_{x \in X} \nu(-\log h_L, x) < \delta$ and $i\Theta_{L,h_L} \geqslant \varepsilon\omega$ for some $\varepsilon > 0$ (see [8, Corollary 6.19]), where ω is a Kähler metric. Let $\beta = \min_{x \in X} \beta_x > 0$ and $\delta = 2\beta/(1+\beta)$ then for any point $x \in X$, we get $\nu(-\log h_L, x) < \delta \leqslant 2\beta_x/(1+\beta_x)$, i.e. $\nu(-\log h_L^{1+1/\beta_x}, x) < 2$. Therefore, h_L^{1+1/β_x} is locally integrable at x and $\det h \cdot h_L$ is also locally integrable.

From $h = \det h \cdot \widehat{h^*}$ and each element of $\widehat{h^*}$ is locally bounded [31, Lemma 2.2.4], for any local holomorphic section s of $E \otimes L$ the function $|s|^2_{h \otimes h_L}$ is locally integrable. Here, $\widehat{h^*}$ is the adjugate matrix of h^*. Hence, the proof is complete from Definition 3.4. □

Corollary 7.10. — *Let X be a projective manifold and L be a nef and big line bundle. Let E be a holomorphic vector bundle equipped with a singular Hermitian metric h. If h is Griffiths semi-positive and $\nu(-\log \det h, x) < 1$ for all points $x \in X$. Then there exists a singular Hermitian metric h_L on L such that $\mathscr{E}(h \otimes \det h \otimes h_L) \cong \mathcal{O}_X(E \otimes \det E \otimes L)$.*

Proof. — By the assumption, i.e. $\nu(-\log(\det h)^2, x) < 2$, the function $(\det h)^2$ is locally integrable. There exists a singular Hermitian metric h_L such that $(\det h)^2 \cdot h_L$ is locally integrable by Proposition 7.9. From $h \otimes \det h \otimes h_L = (\det h)^2 \cdot h_L \cdot \widehat{h^*}$ and each element of $\widehat{h^*}$ is locally bounded [31, Lemma 2.2.4], for any local holomorphic section s of $E \otimes \det E \otimes L$ the function $|s|^2_{h\otimes \det h \otimes h_L}$ is locally integrable. □

Using Proposition 7.9 and Corollary 7.10, we obtain key lemmas and deduce Theorem 7.8 from them.

Lemma 7.11. — *Let X be a projective manifold, N be a nef line bundle and E be a holomorphic vector bundle equipped with a singular Hermitian metric h. If h is L^2-type Nakano semi-positive and that $\nu(-\log\det h, x) < 2$ for all points $x \in X$, then we have*

$$H^q(X, K_X \otimes E \otimes N) = 0$$

for any $q > n - \mathrm{nd}(N)$.

Proof. — First suppose that $\mathrm{nd}(N) = n$, i.e. N is big. By Proposition 7.9, there exists a singular Hermitian metric h_N such that $i\Theta_{N,h_N} \geqslant \delta\omega$ for some $\delta > 0$ and $\mathscr{E}(h\otimes h_N) \cong \mathcal{O}_X(E\otimes N)$, where ω is a Kähler metric on X. Therefore, $h \otimes h_N$ is L^2-type strictly Nakano δ_ω-positive by Corollary 3.17(b). From [19, Theorem 1.5], for any $q > 0$ we have the following vanishing result

$$0 = H^q(X, K_X \otimes \mathscr{E}(h \otimes h_N)) \cong H^q(X, K_X \otimes E \otimes N).$$

Now, if $\mathrm{nd}(N) < n$, we use hyperplane sections and argue by induction on $n = \dim X$. We can select a nonsingular ample divisor A such that $\mathrm{nd}(N|_A) = \mathrm{nd}(N)$. The line bundle $\mathcal{O}_X(A)\otimes N$ is also ample. Here, we have $\mathscr{E}(h) = \mathcal{O}_X(E)$ by the assumption $\nu(-\log\det h, x) < 2$.

Thus, from Theorem 6.2, we get the following cohomologies vanishing

$$0 = H^q(X, K_X \otimes A \otimes N \otimes \mathscr{E}(h)) \cong H^q(X, K_X \otimes A \otimes N \otimes E)$$

for any $q > 0$. The exact sequence $0 \to K_X \to K_X(\log A) = K_X \otimes \mathcal{O}_X(A) \to K_A \to 0$ twisted by $\mathcal{O}_X(N \otimes E)$ yields an isomorphism

$$H^q(A, K_A \otimes (N \otimes E)|_A) \cong H^{q+1}(X, K_X \otimes N \otimes E)$$

for any $0 < q < n$.

Hence, by the induction hypothesis, i.e. $H^q(A, K_A \otimes (N \otimes E)|_A) = 0$ for $q > n-1-\mathrm{nd}(N|_A)$, we have $H^q(X, K_X \otimes N \otimes E) = 0$ for $q > n - \mathrm{nd}(N|_A) = n - \mathrm{nd}(N)$. □

Lemma 7.12. — *Let X be a projective manifold, N be a nef line bundle and E be a holomorphic vector bundle equipped with a singular Hermitian metric h. If h is Griffiths semi-positive and that $\nu(-\log\det h, x) < 1$ for all points $x \in X$, then we have*

$$H^q(X, K_X \otimes N \otimes E \otimes \det E) = 0$$

for any $q > n - \mathrm{nd}(N)$.

This lemma is shown similarly to the proof of Lemma 7.11, using Corollary 7.10 and Remark 3.21.

Proof of Theorem 7.8. — By ampleness of $\mathcal{O}_X(A)\otimes N$ and Theorem 1.6 (a), for any $q > 0$ we get the cohomology vanishing

$$H^q(X, K_X(\log A) \otimes N \otimes \mathscr{E}(h \otimes \det h)) \\ = H^q(X, K_X \otimes A \otimes N \otimes \mathscr{E}(h \otimes \det h)) = 0.$$

Here, for any point $x \in A$, the function $(\det h)^2|_A$ is locally integrable near x by Skoda's result [34]. From the Ohsawa–Takegoshi L^2-extension theorem, the function $(\det h)^2$ is also locally integrable near x. Then we get $\mathscr{E}(h \otimes \det h)|_A = (E \otimes \det E)|_A$.

Therefore, from this and the short exact sequence

$$0 \longrightarrow K_X \longrightarrow K_X(\log A) = K_X \otimes \mathcal{O}_X(A) \longrightarrow K_A \longrightarrow 0$$

twisted by $\mathcal{O}_X(N \otimes E \otimes \det E)$, the natural map

$$H^q(A, K_A \otimes (N \otimes E \otimes \det E)|_A) \longrightarrow H^{q+1}(X, K_X \otimes N \otimes \mathscr{E}(h \otimes \det h))$$

is an isomorphism for $q \geqslant 1$ and is surjective for $q = 0$.

By properties of plurisubharmonic and Definition 3.5, the singular Hermitian metric $h|_A$ is also Griffiths semi-positive over A. Hence, from Lemma 7.12, we have

$$H^q(A, K_A \otimes (N \otimes E \otimes \det E)|_A) = 0$$

for $q > n-1-\mathrm{nd}(N|_A)$, where $\mathrm{nd}(N|_A) = \mathrm{nd}(N) < n$ since N is not big. □

From the proof of this theorem, we immediately obtain the following.

Corollary 7.13. — *Let X be a compact Kähler manifold of dimension n, N be a nef and big line bundle and E be a holomorphic vector bundle equipped with a singular Hermitian metric h. If h is Griffiths semi-positive and there exists a smooth ample divisor A such that $\nu(-\log\det h|_A, x) < 1$ for all points in A, then for any $q > 1$ we have*

$$H^q(X, K_X \otimes N \otimes \mathscr{E}(h \otimes \det h)) = 0.$$

Finally, the proof of Theorem 1.9 is obtained using the Castelnuovo–Mumford regularity and Theorem 1.6 and 7.8.

Proof of Theorem 1.9. — By Lemma 7.7, we only need to prove $K_X \otimes L^{\otimes n} \otimes N \otimes \mathscr{E}(h \otimes \det h)$ is 0-regular with respect to L. Hence, it suffices to show

$$H^q\left(X, K_X \otimes L^{\otimes(n-q)} \otimes N \otimes \mathscr{E}(h \otimes \det h)\right) = 0 \quad \text{for all} \quad q > 0.$$

For $0<q<n$, by positivity of $L^{\otimes(n-q)}$ and compactness of $X, L^{\otimes(n-q)}\otimes N$ is also positive. Therefore, we have the desired vanishing cohomologies from Theorem 1.6.

When $q=n$, we need to show $H^n(X, K_X\otimes N\otimes \mathscr{E}(h\otimes \det h))=0$. The desired vanishing follows from the assumption that $\mathrm{nd}(N)\neq 0$ and Theorem 7.8 and Corollary 7.13. The case when $q>n$ is obvious and we complete the proof. □

From the above proof and Lemma 7.11, we obtain the following corollary.

Corollary 7.14. — *Let X be a compact Kähler manifold of dimension n and E be a holomorphic vector bundle equipped with a singular Hermitian metric h. Let L be an ample and globally generated line bundle and N be a nef but not numerically trivial line bundle. If h is L^2-type Nakano semi-positive and that $\nu(-\log\det h, x)<2$ for all points $x\in X$, then the adjoint vector bundle $K_X\otimes L^{\otimes n}\otimes N\otimes E$ is globally generated.*

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Bibliography

[1] B. Berndtsson, "Curvature of vector bundles associated to holomorphic fibrations", *Ann. Math. (2)* **169** (2009), no. 2, p. 531-560.

[2] B. Berndtsson & M. Păun, "Bergman kernels and the pseudoeffectivity of relative canonical bundles", *Duke Math. J.* **145** (2008), no. 2, p. 341-378.

[3] M. A. A. de Cataldo, "Singular hermitian metrics on vector bundles", *J. Reine Angew. Math.* **502** (1998), p. 93-122.

[4] J.-P. Demailly, "Relations entre les notions de positivités de P. A. Griffiths et de S. Nakano pour les fibres vectoriels", in *Séminaire P. Lelong – H. Skoda, Analyse, Années 1978/79*, Lecture Notes in Mathematics, vol. 822, Springer, 1980, p. 304-309.

[5] ———, "Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe semi-positif au dessus d'une variété Kählérienne complète", *Ann. Sci. Éc. Norm. Supér. (4)* **15** (1982), p. 457-511.

[6] ———, "A numerical criterion for very ample line bundles", *J. Differ. Geom.* **37** (1993), no. 2, p. 323-374.

[7] ———, "Regularization of closed positive currents of type $(1,1)$ by the flow of a Chern connection", in *Contributions to complex analysis and analytic geometry.*, Aspects of Mathematics, vol. 26, Braunschweig, 1994, p. 105-126.

[8] ———, *Analytic Methods in Algebraic Geometry*, Surveys of Modern Mathematics, vol. 1, International Press; Higher Education Press, 2012.

[9] ———, "Complex analytic and differential geometry", 2012, `https://www-fourier.univ-grenoble-alpes.fr/~demailly/manuscripts/agbook.pdf`.

[10] F. Deng, J. Ning, Z. Wang & X. Zhou, "Positivity of holomorphic vector bundles in terms of L^p-estimates for $\overline{\partial}$", *Math. Ann.* **385** (2023), no. 1-2, p. 575-607.

[11] K. Diederich, "Some aspects of the Levi problem: recent developments", in *Geometric complex analysis. Proceedings of the conference held at the 3rd International Research Institute of the Mathematical Society of Japan*, World Scientific, 1996, p. 163-181.

[12] L. Ein & R. Lazarsfeld, "Global generation of pluricanonical and adjoint linear series on smooth projective threefolds", *J. Am. Math. Soc.* **6** (1993), no. 4, p. 875-903.

[13] T. Fujita, "Contribution to birational geometry of algebraic varieties: open problems", in *23rd International Symposium, August 22-27, 1988, Katata*, Division of Mathematics, the Taniguchi Foundation, 1988.

[14] Q. Guan & X. Zhou, "A proof of Demailly's strong openness conjecture", *Ann. Math. (2)* **182** (2015), no. 2, p. 605-616.

[15] L. Hörmander, *An introduction to complex analysis in several variables*, 3rd revised ed., North-Holland Mathematical Library, vol. 7, North-Holland, 1990.

[16] G. Hosono & T. Inayama, "A converse of Hörmander's L^2-estimate and new positivity notions for vector bundles", *Sci. China, Math.* **64** (2021), no. 8, p. 1745-1756.

[17] C. Huang, "Some Kollár–Enoki type injectivity on compact Kahler manifolds", *J. Geom. Anal.* **34** (2024), no. 1, article no. 4 (17 pages).

[18] T. Inayama, "L^2 estimates and vanishing theorems for holomorphic vector bundles equipped with singular Hermitian metrics", *Mich. Math. J.* **69** (2020), no. 1, p. 79-96.

[19] ———, "Nakano positivity of singular Hermitian metrics and vanishing theorems of Demailly–Nadel–Nakano type", *Algebr. Geom.* **9** (2022), no. 1, p. 69-92.

[20] Y. Kawamata, "A generalization of Kodaira–Ramanujam's vanishing theorem", *Math. Ann.* **261** (1982), p. 43-46.

[21] ———, "On Fujita's freeness conjecture for 3-folds and 4-folds", *Math. Ann.* **308** (1997), no. 3, p. 491-505.

[22] D. S. Keeler, "Fujita's conjecture and Frobenius amplitude", *Am. J. Math.* **130** (2008), no. 5, p. 1327-1336.

[23] K. Kodaira, "On a differential-geometric method in the theory of analytic stacks", *Proc. Natl. Acad. Sci. USA* **39** (1953), p. 1268-1273.

[24] R. Lazarsfeld, *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series. II. Positivity for vector bundles, and multiplier ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 48, 49, Springer, 2004.

[25] Z. Li, X. Meng, J. Ning, Z. Wang & X. Zhou, "On a Bogomolov type vanishing theorem", *Nagoya Math. J.* **257** (2025), p. 170-182.

[26] K. Liu, X. Sun & X. Yang, "Positivity and vanishing theorems for ample vector bundles", *J. Algebr. Geom.* **22** (2013), no. 2, p. 303-331.

[27] B. Malgrange, "Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution", *Ann. Inst. Fourier* **6** (1956), p. 271-355.

[28] A. M. Nadel, "Multiplier ideal sheaves and Kähler–Einstein metrics of positive scalar curvature", *Ann. Math. (2)* **132** (1990), no. 3, p. 549-596.

[29] T. Ohsawa, "Isomorphism theorems for cohomology groups of weakly 1-complete manifolds", *Publ. Res. Inst. Math. Sci.* **18** (1982), p. 191-232.

[30] ———, "Completeness of noncompact analytic spaces", *Publ. Res. Inst. Math. Sci.* **20** (1984), no. 3, p. 683-692.

[31] M. Păun & S. Takayama, "Positivity of twisted relative pluricanonical bundles and their direct images", *J. Algebr. Geom.* **27** (2018), no. 2, p. 211-272.

[32] C. P. Ramanujam, "Remarks on the Kodaira vanishing theorem", *J. Indian Math. Soc., New Ser.* **36** (1972), p. 41-51.

[33] H. Raufi, "Singular Hermitian metrics on holomorphic vector bundles", *Ark. Mat.* **53** (2015), no. 2, p. 359-382.

[34] H. Skoda, "Sous-ensembles analytiques d'ordre fini ou infini dans C^n", *Bull. Soc. Math. Fr.* **100** (1972), p. 353-408.

[35] X. Su & X. Yang, "Global generation and very ampleness for adjoint linear series", *Commun. Anal. Geom.* **27** (2019), no. 7, p. 1639-1663.

[36] S. Takayama, "Adjoint linear series on weakly 1-complete Kähler manifolds. I: Global projective embedding", *Math. Ann.* **311** (1998), no. 3, p. 501-531.

[37] E. Viehweg, "Vanishing theorems", *J. Reine Angew. Math.* **335** (1982), p. 1-8.

[38] Y. Watanabe, "Curvature operator of holomorphic vector bundles and L^2-estimate condition for (n, q) and (p, n)-forms", accepted for publication in *Tôhoku Mathematical Journal*, 2022, `https://arxiv.org/abs/2109.12554v2`.

[39] ———, "Bogomolov–Sommese type vanishing theorem for holomorphic vector bundles equipped with positive singular Hermitian metrics", *Math. Z.* **303** (2023), no. 4, article no. 92 (23 pages).

[40] F. Ye & Z. Zhu, "On Fujita's freeness conjecture in dimension 5", *Adv. Math.* **371** (2020), article no. 107210 (55 pages).

[41] X. Zhou & L. Zhu, "Extension of cohomology classes and holomorphic sections defined on subvarieties", *J. Algebr. Geom.* **31** (2022), no. 1, p. 137-179.