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Geometric optics expansions for quarter-space boundary value problems III: glancing modes and multiple self-interaction (*)

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Abstract. — This article aims to continue the study of geometric optics expansions for hyperbolic boundary value problems in the quarter-space initiated in [2]. The motivations are linked to the range of effective applicability of the theorem establishing the existence of the geometric optics expansions. Compared to [2], we ameliorate the range of applicability by adding two distinct features. The first one is that now we can consider glancing modes in the expansions by using the results of [17]. The second one, which is proper to quarter-space problems, is that we can now consider rather "complicated" self-interaction phenomena. It is a first step in the study of geometric optics expansions in bounded domains. A direct consequence of the first point of amelioration is that no condition on glancing modes is required to intialize the construction of the geometric optics expansion. It seems to indicate that the expected condition characterizing the strong well-posedness of corner problems, established in [14], can be relaxed to the hyperbolic component of the stable subspace only.

Résumé. — Cet article vise à poursuivre l'étude des développements d'optique géométrique pour les problèmes aux limites hyperboliques posés dans un quart d'espace, étude initiée dans [2]. Les motivations sont ici liées au domaine d'applicabilité effective du théorème établissant l'existence de tels développements. Comparé à [2], nous avons amélioré le domaine d'applicabilité de deux façons distinctes. D'abord, nous pouvons maintenant considérer dans les développements des modes rasants en adaptant les résultats de [17]. Ensuite, ceci est propre à la géométrie du quart d'espace, nous pouvons maintenant considérer des phénomènes d'auto-interaction assez « complexes ». Ceci constitue une première étape nécessaire dans la construction de développements d'optique géométrique dans des géométries bornées. Une conséquence directe de notre nouvelle contribution est que, pour son initialisation, la résolution de la cascade d'équations ne nécessite pas de condition sur les modes rasants. Ceci semble indiquer que la condition que l'on croit caractériser les problèmes fortement bien-posés de [14] pourrait être relaxée sur les modes hyperboliques seulement.

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Article proposé par Sylvie Benzoni.

1. Introduction

In this article we revisit the results of [2] about geometric optics expansions for hyperbolic boundary value problems in the quarter-space. The problems considered in this article read under the form

$$\begin{cases} L(\partial)u^{\varepsilon} := \partial_t u^{\varepsilon} + A_1\partial_1 u^{\varepsilon} + A_2\partial_2 u^{\varepsilon} = 0 \\ \qquad\qquad\qquad\qquad\qquad \text{for } (t,x) \in \mathbb{R}\times\mathbb{R}^2_+ := \Omega \\ B_1 u^{\varepsilon}_{|x_1=0} = g^{\varepsilon} \qquad \text{for } (t,x_2) \in \mathbb{R}\times\mathbb{R}_+ := \partial\Omega_1, \\ B_2 u^{\varepsilon}_{|x_2=0} = 0 \qquad \text{for } (t,x_1) \in \mathbb{R}\times\mathbb{R}_+ := \partial\Omega_2, \\ u_{|t\leqslant 0} = 0 \qquad \text{for } x \in \mathbb{R}^2_+ := \Gamma, \end{cases} \tag{1.1}$$

where the interior coefficient matrices A_1, $A_2 \in \mathbf{M}_{N\times N}(\mathbb{R})$, for some fixed $N \geqslant 1$, and where the boundary matrix B_1 (resp. B_2) lies in $\mathbf{M}_{p_1\times N}(\mathbb{R})$ (resp. $\mathbf{M}_{p_2\times N}(\mathbb{R})$). It encodes the good number of boundary conditions. These numbers are made precise in Assumption 2.2.

In the problem (1.1) the real $0 < \varepsilon \ll 1$ stands for a parameter describing the typical wavelength of the boundary term g^{ε}. Constructing a geometric optics expansion aims to give an approximate solution to (1.1) in the high frequencies limit, that is to say $\varepsilon \downarrow 0$. Moreover if the approximate solution is sufficiently good then one can expect to show some qualitative phenomena on it, phenomena that should also be satisfied by the exact solution u^{ε}.

Before to describe precisely the extensions of the present article compared to [2], let us recall briefly the interesting points encountered in the analysis of [2].

However, let us first point that even if the study of hyperbolic boundary value problems is a rather old question starting from [14] for the strong well-posedness question and [15] about formal geometric optics expansions, then such problems remain widely open in spite of the recent works of [1, 6, 8] or [4]. Indeed at present time the full characterization of well-posed boundary value problems in the quarter-space is not achieved yet. Such a well-posedness result is only known to hold for the (particular) class of symmetric problems with (strictly) dissipative boundary conditions (see [4] or [6]).

However having a good idea of the behavior of the (expected to be) approximate solutions given by geometric optics expansions may help in the establishment of such a characterization.

The result of [2] gives geometric optics expansions for (1.1) justifying the seminal work of [15]. Moreover this article highlights some new behaviors of the problem compared to the most classical geometry of the half-space. A new phenomenon of interest is the so-called self-interaction of the phases.

More precisely, for quarter-space problems a phase can be regenerated by iterative reflections against the two sides of the quarter-space. That is to say that we can find at least four phases φ_1, φ_2, φ_3 and φ_4 such that φ_1 generates φ_2, φ_2 generates φ_3, φ_3 generates φ_4 and finally that φ_4 generates the first phase φ_1.

The existence of such phases for quarter-space problems is linked with the geometry of the characteristic variety of the problem (1.1) (we refer to Subsection 3.1 for more details). They may seem to be rather anecdotal in the quarter-space (in the sense that except for constructed toy models, the existence of such phases is not so clear). But as pointed in [3], these phases are in fact generic in the strip geometry where any phase is a self-interacting one. In fact self-interacting phases seem to be generic for problem whose boundary involve several components. In the future we aim to construct geometric optics expansions for problems defined in some (bounded) set whose boundary contains several components. As a consequence, understanding precisely the influence of self-interacting phases for the quarter-space toy model is a good starting point.

Self-interacting phases have a real impact on geometric optics expansions. The existence of such phases complicates a little the construction of the geometric optics expansions. Indeed solving the geometric optics cascade of equations amounts to solve an upper triangular system of equations. In particular, we have to find an equation that can be solved before all the others in order to initialize the whole resolution of the cascade.

Clearly when self-interacting phases come into play, then it is not so clear that we can start by solving the equation for φ_1 (the first generated phase), then solve φ_2 and so on. Indeed, the phase φ_1 depends on itself via its descendants.

However, in [2] it is shown that in order to determine the amplitude u_1, associated to the phase φ_1 it is sufficient to solve some equation reading under the form

$$(I - \mathbf{T})u_1 = \widetilde{g},$$

where $\mathbf{T}$ is some (explicit) linear operator and where $\widetilde{g}$ depends (explicitly) on the boundary source g^ε.

It is quite interesting to point that in his work aiming to characterize strong well-posedness for quarter-space hyperbolic boundary value problems, Osher (see [14]) exhibits a condition reading under the form

$$(I - \widetilde{\mathbf{T}})u_{|x_1=0} = F(g),$$

where $\widetilde{\mathbf{T}}$ and F stand for explicit (but complicated) kind of Fourier integral operators.

Such a phenomenon already occurs in the classical geometry of the half-space where the condition characterizing the strong well-posedness of the problem, namely the uniform Kreiss–Lopatinskii condition of [9], also appears at a microlocalized level when one wants to construct the associated geometric optics expansion. As a consequence, we have good reasons to believe that the condition on the operator $\mathbf{T}$ in [2] is a microlocalized version of the corner condition involving $\widetilde{\mathbf{T}}$ in [14]. So, the better we understand the simplest microlocalized version, the better we will understand the corner condition of [14] from which we can hope to characterize the strong well-posedness of hyperbolic boundary value problems in the quarter-space.

The first extension of the present article compared to [2] is directly linked to this question. Indeed in [2] the geometric optics expansion is constructed under the assumption that in the phase generation process, glancing phases never appear. Without enter into technical details (we refer to [2] for a more precise definition), let us indicate that we have to consider in the expansions three kinds of phases, the elliptic ones associated to some boundary layers, the hyperbolic ones associated to transport phenomena and the glancing ones associated to some tangential (along one of the sides of the boundary) transport phenomena.

The self-interaction operator $\mathbf{T}$ of [2] involves hyperbolic modes and not elliptic modes. Because they are excluded from the assumptions it can, of course, not include glancing mode(s).

In this work we add glancing modes in the geometric optics expansions of (1.1) and we show in particular that the operator $\mathbf{T}$ used to initialize the resolution of the geometric optics expansion cascade do not involve the glancing modes. This phenomenon has already been encounter in the strip geometry [3].

In the author's opinion this fact is a good argument in the direction that the corner condition of [14], if we believe that it is a condition preventing an exponential growth of the solution with respect to time due to iterative reflections against the sides of the domain, may possibly be weakened on some functional space only involving the hyperbolic modes. This is however behind of the scope of the present article and it is left for future studies.

Moreover, it is rather fair to say that in [2] the assumption ensuring that glancing modes never appear is a very restrictive assumption which is really difficult to check effectively. Indeed compared to the half-space geometry, the phase generation process for quarter-space problems is much more elaborate. We refer to Section 3 or to [2] for a precise exposition. But, because all the possible iterative reflections against the two sides of the quarter-space of the phase initially included in g^ε have to be considered, then it is clear that starting from a non glancing phase is not sufficient to prevent the appearance of such glancing modes at some iteration. We refer to Section 8 for a precise example.

So that by including glancing modes, the result of the present article is much more applicable than the one of [2].

The other extensions compared to [2] are linked to the nature of self-interaction that is allowed to appear in the phase generation process. Indeed in [2], we only consider the simplest possible self-interaction phenomenon: a single self-interaction phenomenon which only involves four elements φ_1, φ_2, φ_3 and φ_4.

Here we extend the expansions to problems which can admit several self-interaction phenomena with more than four elements.

As pointed before, because self-interacting phases are rather anecdotal[(1)] for quarter-space problems this extension may sound a little artificial and cosmetic. However we believe that it is not. Indeed, if one wants to construct geometric optics expansions in more complex (bounded) geometries than the quarter-space or the strip, then because the self-interaction phenomenon becomes generic then he/she needs to consider such complicated self-interaction phenomena. A good understanding of the problem in the toy-model of the quarter-space can be seen as a first step to consider such more complicated (and possibly more physically relevant) problems.

The paper is organized as follows. In Section 2 we give some notation, recall some classical definitions for geometric optics expansions and state the main result of the article, namely the construction of geometric optics expansions for quarter-space problems with glancing modes and "elaborate" self-interaction phenomena.

In Section 3, we describe precisely the phase generation process and then collect all the expected phases in the geometric optics expansion. In Section 4 we study the obtained set of phases and we show that we can define on this set some kind of partial order relation even if we have several self-interaction phenomena. This relation is then used in Sections 5 and 6 as a natural order of resolution of the geometric optics cascade of equations. We first apply this (partial) order of resolution in the simplest framework where we have uniqueness of the self-interaction phenomenon in Section 5. Then we reach the whole generality of our main result, that is to say that we allow several self-interaction phenomena in Section 6, by using Section 5.

(1) We here mean that it is always possible to construct problems for which such self-interaction phenomena occur (see Section 8). But that they are toy-models and that we have reasons to believe that for a given hyperbolic operator $L(\partial)$ such phenomenon "generically" do not appear.

Section 7 gives some extra materials linked to the justification of the expansion. The first one deals with finite time problems and the consequences on the number of phases in the expansion. The second one is a justification of the expansion if we have a good enough well-posedness theory for the quarter-space problem (1.1).

At last Section 8 gives some toy-models exhibiting the complicate self-interaction phenomena considered above and insists on the possible appearance of glancing modes at any step of the phase generation process.

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2. Notation, definitions and main result

Let us first introduce some generic notations used throughout the text:

- For $a, b \in \mathbb{Z}$ we define $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$.
- The notation $\delta_{\cdot,\cdot}$ stands for Kronecker symbol.
- For some set A, the notation $\#A$ stands for the cardinal of A.
- For $z \in \mathbb{C}$ we write $z := \Re z + i\Im z$, where $\Re z, \Im z \in \mathbb{R}$ denote respectively the real and the imaginary parts of z.

2.1. Definitions and Assumptions

This paragraph recalls some standard definitions which are commonly used in geometric optics expansions and lists the main assumptions used in this article.

2.1.1. About the hyperbolic operator

As in [2] in the following we will consider strictly hyperbolic operators[(2)] in the following sense:

[(2)] The following construction can also probably operates with not a lot of modifications for constantly hyperbolic operators but we choose the strictly hyperbolic ones for simplicity.

Assumption 2.1 (Strictly hyperbolic operator). — *The operator* $L(\partial)$ *is strictly hyperbolic. That is to say that there exist* N *real-valued functions* $\lambda_1, \dots, \lambda_N$ *analytic on* $\mathbb{R}^2 \setminus \{0\}$ *such that*

$$\forall\ \xi \in \mathbb{S}^1,\ \det \mathscr{L}(\tau, \xi) = \prod_{j=1}^{N} (\tau + \lambda_j(\xi)),$$

where $\mathscr{L}(\tau, \xi) := \tau I + \sum_{j=1}^{2} \xi_j A_j$ *stands for the symbol of* $L(\partial)$ *and where the eigenvalues* λ_j *satisfy* $\lambda_1(\xi) < \lambda_2(\xi) < \cdots < \lambda_N(\xi)$.

We also assume, for simplicity, that the two sides of the boundary, $\partial\Omega_1$ and $\partial\Omega_2$ are non characteristic. That is to say, we assume the following:

Assumption 2.2 (Non characteristics boundary). — *The matrices* A_1 *and* A_2 *are non singular meaning that* $\det A_1, \det A_2 \neq 0$. *We also assume that* p_1 *(resp.* p_2*), the number of lines of* B_1 *(resp.* B_2*), equals the number of positive eigenvalues of* A_1 *(resp.* A_2*).*

With Assumptions 2.1 and 2.2 in hand we can perform some frequency analysis of the hyperbolic boundary value problem (1.1). In order to do so, we first introduce the frequency space

$$\Xi := \{\zeta := (\sigma := \gamma + i\tau, \eta) \in \mathbb{C} \times \mathbb{R}, \gamma \geqslant 0\} \setminus \{(0,0)\},$$

and its boundary $\Xi_0 := \Xi \cap \{\gamma = 0\}$.

We will consider the classical half-space problems associated to (1.1) namely

$$\begin{cases} L(\partial)u = 0 & \text{for } (t, x_1, x_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+, \\ B_2 u_{|x_2=0} = g_2 & \text{for } (t, x_1) \in \mathbb{R}^2, \\ u_{|t \leqslant 0} = 0 & \text{for } (x_1, x_2) \in \mathbb{R} \times \mathbb{R}_+, \end{cases}$$
$$\text{and} \tag{2.1}$$
$$\begin{cases} L(\partial)u = 0 & \text{for } (t, x_1, x_2) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}, \\ B_1 u_{|x_1=0} = g_1 & \text{for } (t, x_2) \in \mathbb{R}^2, \\ u_{|t \leqslant 0} = 0 & \text{for } (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}. \end{cases}$$

We perform a Laplace transform $t \leftrightsquigarrow \sigma$ and a Fourier transform with respect to the tangential space variable in (2.1), $x_1 \leftrightsquigarrow \eta$ or $x_2 \leftrightsquigarrow \eta$. It leads us to consider the ordinary differential equations

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x_2}\widehat{u} = \mathscr{A}_2(\zeta)\widehat{u} & \text{for } x_2 \geqslant 0, \\ B_2\widehat{u}_{|x_2=0} = \widehat{g_2}, \end{cases}$$
and
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x_1}\widehat{u} = \mathscr{A}_1(\zeta)\widehat{u} & \text{for } x_1 \geqslant 0, \\ B_1\widehat{u}_{|x_1=0} = \widehat{g_1}, \end{cases} \tag{2.2}$$

where the so-called resolvent matrices $\mathscr{A}_1$ and $\mathscr{A}_2$ are defined by

$$\mathscr{A}_1(\zeta) := -A_1^{-1}(\sigma I + i\eta A_2) \text{ and } \mathscr{A}_2(\zeta) := -A_2^{-1}(\sigma I + i\eta A_1). \tag{2.3}$$

The behavior of the spectrum of the resolvent matrix has a major impact on the solution to (2.2). As long as the Laplace variable σ admits a non vanishing real part, then the following lemma due to Hersh ensures that the elements in the spectrum of the resolvent matrix are signed. More precisely

Lemma 2.3 (Hersh [7]). — *Under Assumptions 2.1 and 2.2, for $j = 1, 2$ and $\underline{\zeta} \in \Xi\backslash\Xi_0$, the resolvent matrix $\mathscr{A}_j(\underline{\zeta})$ does not have any purely imaginary eigenvalue. We denote by $\mathbf{E}_j^s(\underline{\zeta})$ (resp. $\mathbf{E}_j^u(\underline{\zeta})$) the stable (resp. unstable) subspace that is the eigenspace associated to eigenvalues with strictly negative (resp. positive) real part. Then we have $\dim \mathbf{E}_j^s(\underline{\zeta}) = p_j$ and $\dim \mathbf{E}_j^u(\underline{\zeta}) = N - p_j$, in such a way that*

$$\mathbb{C}^N = \mathbf{E}_j^s(\underline{\zeta}) \oplus \mathbf{E}_j^u(\underline{\zeta}). \tag{2.4}$$

In order to define precisely the different kinds of phases that we will have to consider to construct the geometric optics expansion of (1.1), we recall the following theorem which refines Lemma 2.3. This result known as the block structure lemma has first been demonstrated in the seminal work of Kreiss [9] for strictly hyperbolic systems. It has then been extended by Métivier in [12] to constantly hyperbolic systems.

Theorem 2.4 (Block structure [9, 12]). — *Under Assumptions 2.1 and 2.2, for all $\underline{\zeta} \in \Xi$, there exists a neighbourhood $\mathscr{V}$ of $\underline{\zeta}$ in Ξ, strictly positive integers L_1 and L_2, two partitions $N = \mu_{1,1} + \mu_{1,2} + \cdots + \mu_{1,L_1} = \mu_{2,1} + \mu_{2,2} + \cdots + \mu_{2,L_2}$ and two invertible matrices T_1 and T_2, regular on $\mathscr{V}$, such that for all $\zeta \in \mathscr{V}$, we have:*

$$T_1^{-1}(\zeta)\mathscr{A}_1(\zeta)T_1(\zeta) = \operatorname{diag}\big(\mathscr{A}_{1,1}(\zeta), \mathscr{A}_{1,2}(\zeta), \ldots, \mathscr{A}_{1,L_1}(\zeta)\big),$$
$$T_2^{-1}(\zeta)\mathscr{A}_2(\zeta)T_2(\zeta) = \operatorname{diag}\big(\mathscr{A}_{2,1}(\zeta), \mathscr{A}_{2,2}(\zeta), \ldots, \mathscr{A}_{2,L_2}(\zeta)\big),$$

where for $j = 1, 2$, and $l \in [\![1, L_j]\!]$, the block $\mathscr{A}_{j,l}(\zeta) \in \mathbf{M}_{\mu_{j,l}\times\mu_{j,l}}(\mathbb{C})$ satisfies one of the following alternatives:

(1) *all the elements in the spectrum of $\mathscr{A}_{j,l}(\underline{\zeta})$ have negative real part.*

(2) *All the elements in the spectrum of* $\mathscr{A}_{j,l}(\underline{\zeta})$ *have positive real part.*
(3) *We have* $\mu_{j,l} = 1$, $\mathscr{A}_{j,l}(\underline{\zeta}) \in i\mathbb{R}$, $\partial_\gamma \mathscr{A}_{j,l}(\underline{\zeta}) \in \mathbb{R}\setminus\{0\}$ *and finally* $\mathscr{A}_{j,l}(\zeta) \in i\mathbb{R}$, *for all* $\zeta \in \mathscr{V} \cap \Xi_0$.
(4) *We have* $\mu_{j,l} > 1$ *and there exists some* $k_{j,l} \in i\mathbb{R}$, *such that*

$$\mathscr{A}_{j,l}(\underline{\zeta}) = \begin{bmatrix} k_{j,l} & i & 0 \\ & \ddots & i \\ 0 & & k_{j,l} \end{bmatrix},$$

where the coefficient in the lower left corner of $\partial_\gamma \mathscr{A}_{j,l}(\underline{\zeta}) \in \mathbb{R}\setminus\{0\}$. *Moreover, for all* $\zeta \in \mathscr{V} \cap \Xi_0$, *we have* $\mathscr{A}_{j,l}(\zeta) \in \mathbf{M}_{\mu_{j,l}\times\mu_{j,l}}(i\mathbb{R})$.

Thanks to this theorem we can define precisely the four kinds of frequencies that we will consider in the following.

Definition 2.5. — *For* $j = 1, 2$, *the boundary* Ξ_0 *decomposes into*

$$\Xi_0 := \mathbb{E}_j \cup \mathbb{EH}_j \cup \mathbb{H}_j \cup \mathbb{G}_j,$$

where we introduced

(1) $\mathbb{E}_j$, *the set of elliptic frequencies, that is to say the set of boundary frequencies* $\underline{\zeta} \in \Xi_0$, *such that Theorem 2.4 for the matrix* $\mathscr{A}_j$ *is satisfied with blocks of type* (1) *and* (2) *only.*
(2) $\mathbb{EH}_j$, *the set of mixed frequencies, that is to say the set of boundary frequencies* $\underline{\zeta} \in \Xi_0$, *such that Theorem 2.4 for the matrix* $\mathscr{A}_j$ *is satisfied with blocks of type* (1), (2), *and at least one block of type* (3). *But zero block of type* (4).
(3) $\mathbb{H}_j$, *the set of hyperbolic frequencies, that is to say the set of boundary frequencies* $\underline{\zeta} \in \Xi_0$, *such that Theorem 2.4 for the matrix* $\mathscr{A}_j$ *is satisfied with blocks of type* (3) *only.*
(4) $\mathbb{G}_j$, *the set of glancing frequencies, that is to say the set of boundary frequencies* $\underline{\zeta} \in \Xi_0$, *such that Theorem 2.4 for the matrix* $\mathscr{A}_j$ *is satisfied with at least one block of type* (4).

The study made for instance in [9] shows that the stable subspaces $\mathbf{E}_1^s(\zeta)$ and $\mathbf{E}_2^s(\zeta)$ which are well-defined for $\zeta \in \Xi\setminus\Xi_0$ because of Lemma 2.3, can be extended by continuity (without any change of notation for the extension) up to the boundary Ξ_0. In the following we need to describe with enough precision these extended stable subspaces $\mathbf{E}_1^s(\zeta)$ and $\mathbf{E}_2^s(\zeta)$.

We start by the simplest case where $\underline{\zeta} \in \Xi_0 \setminus (\mathbb{G}_1 \cup \mathbb{G}_2)$. In such a framework, the decomposition (2.4) still holds in the limit $\gamma \downarrow 0$ and we have

$$\mathbb{C}^N = \mathbf{E}_1^s(\underline{\zeta}) \oplus \mathbf{E}_1^u(\underline{\zeta}) = \mathbf{E}_2^s(\underline{\zeta}) \oplus \mathbf{E}_2^u(\underline{\zeta}). \tag{2.5}$$

Moreover, if j=1,2, we can decompose

$$\mathbf{E}_j^s(\underline{\zeta}) := \mathbf{E}_j^{s,e}(\underline{\zeta}) \oplus \mathbf{E}_j^{s,h}(\underline{\zeta}) \text{ and } \mathbf{E}_j^u(\underline{\zeta}) := \mathbf{E}_j^{u,e}(\underline{\zeta}) \oplus \mathbf{E}_j^{u,h}(\underline{\zeta}), \tag{2.6}$$

where $\mathbf{E}_j^{s,e}(\underline{\zeta})$ (resp. $\mathbf{E}_j^{u,e}(\underline{\zeta})$) is the generalized eigenspace associated to generalized eigenvalues of $\mathscr{A}_j(\underline{\zeta})$ with negative (resp. positive) real part; and where $\mathbf{E}_j^{s,h}(\underline{\zeta})$ and $\mathbf{E}_j^{u,h}(\underline{\zeta})$ are sums of eigenspaces associated to purely imaginary eigenvalues of $\mathscr{A}_j(\underline{\zeta})$.

We will give a more precise description of the hyperbolic subspaces namely $\mathbf{E}_j^{s,h}(\underline{\zeta})$ and $\mathbf{E}_j^{u,h}(\underline{\zeta})$. Let $i\omega_{m,j}$ be a purely imaginary eigenvalue of $\mathscr{A}_j(\underline{\zeta})$ so that we have $\det(\underline{\tau} I + \underline{\eta} A_1 + \omega_{m,2} A_2) = \det(\underline{\tau} I + \omega_{m,1} A_1 + \underline{\eta} A_2) = 0$. From the hyperbolicity Assumption 2.1, one can find an index $k_{m,j} \in [\![1, N]\!]$, such that we have

$$\underline{\tau} + \lambda_{k_{m,2}}(\underline{\eta}, \omega_{m,2}) = \underline{\tau} + \lambda_{k_{m,1}}(\omega_{m,1}, \underline{\eta}).$$

Because the eigenvalues $\lambda_{\cdot,\cdot}$ are assumed to be regular we introduce

Definition 2.6 (Group velocities). — *We define:*

- *The set of incoming (resp. outgoing) phases for the side $\partial\Omega_1$, denoted by $\mathfrak{I}_1$ (resp. $\mathfrak{O}_1$), is the set of indices m such that the group velocity $\mathbf{v}_m := \nabla\lambda_{k_{m,1}}(\omega_{m,1}, \underline{\eta})$ satisfies $\mathbf{v}_{m,1} = \partial_1\lambda_{k_{m,1}}(\omega_{m,1}, \underline{\eta}) > 0$ (resp. $\mathbf{v}_{m,1} = \partial_1\lambda_{k_{m,1}}(\omega_{m,1}, \underline{\eta}) < 0$).*
- *The set of incoming (resp. outgoing) phases for the side $\partial\Omega_2$, denoted by $\mathfrak{I}_2$ (resp. $\mathfrak{O}_2$), is the set of indices m such that the group velocity $\mathbf{v}_m := \nabla\lambda_{k_{m,2}}(\underline{\eta}, \omega_{m,2})$ satisfies $\mathbf{v}_{m,2} = \partial_2\lambda_{k_{m,2}}(\underline{\eta}, \omega_{m,2}) > 0$ (resp. $\mathbf{v}_{m,2} = \partial_2\lambda_{k_{m,2}}(\underline{\eta}, \omega_{m,2}) < 0$).*
- *The set of glancing modes for the side $\partial\Omega_1$ (resp. $\partial\Omega_2$), denoted by $\mathfrak{G}_1$ (resp. $\mathfrak{G}_2$), is the set of indices m such that the group velocity $\mathbf{v}_m$ satisfies $\mathbf{v}_{m,1} = 0$ (resp. $\mathbf{v}_{m,2} = 0$).*

We can now describe more precisely the hyperbolic subspaces. The following decompositions hold:

Proposition 2.7. — *Let $j = 1, 2$, then for all $\underline{\zeta} \in \mathbb{H}_j \cup \mathbb{EH}_j$, we have the decompositions*

$$\begin{aligned} \mathbf{E}_1^{s,h}(\underline{\zeta}) &= \bigoplus_{m \in \mathfrak{I}_1} \ker \mathscr{L}(\underline{\tau}, \omega_{m,1}, \underline{\eta}), \\ \mathbf{E}_1^{u,h}(\underline{\zeta}) &= \bigoplus_{m \in \mathfrak{O}_1} \ker \mathscr{L}(\underline{\tau}, \omega_{m,1}, \underline{\eta}), \end{aligned} \tag{2.7}$$

$$\mathbf{E}_2^{s,h}(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{I}_2} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \omega_{m,2}),$$
$$\mathbf{E}_2^{u,h}(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{O}_2} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \omega_{m,2}), \tag{2.8}$$

where we recall that $\mathscr{L}(\cdot)$ stands for the symbol of $L(\partial)$.

Because we are working in a quarter-space we have to refine a little the above definition

Definition 2.8 (Kinds of hyperbolic phases). — *Let $\mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2) \in \mathbb{R}^2$ be a placeholder for the group velocity of some index. We say that the group velocity $\mathbf{v}$ (and by extension that the index) is*

- *outgoing-outgoing if $\mathbf{v}_1, \mathbf{v}_2 < 0$.*
- *outgoing-incoming if $\mathbf{v}_1 < 0$ and $\mathbf{v}_2 > 0$.*
- *incoming-outgoing if $\mathbf{v}_1 > 0$ and $\mathbf{v}_2 < 0$.*
- *incoming-incoming if $\mathbf{v}_1, \mathbf{v}_2 > 0$.*
- *glancing for the side $\partial\Omega_1$ if $\mathbf{v}_1 = 0$, independently of $\mathbf{v}_2$.*
- *glancing for the side $\partial\Omega_2$ if $\mathbf{v}_2 = 0$, independently of $\mathbf{v}_1$.*

We now consider the case where the frequency $\underline{\zeta} \in \mathbb{G}_j$, for $j = 1, 2$. In such a situation because we have $\mathbf{E}_j^s(\underline{\zeta}) \cap \mathbf{E}_j^u(\underline{\zeta}) \neq \{0\}$, then the decomposition (2.5) does not hold any more. We thus give the description

$$\mathbf{E}_j^s(\underline{\zeta}) := \mathbf{E}_j^{s,h}(\underline{\zeta}) \oplus \mathbf{E}_j^{s,e}(\underline{\zeta}) \oplus \mathbf{E}_j^{s,g}(\underline{\zeta})$$
$$\text{and} \quad \mathbf{E}_j^u(\underline{\zeta}) := \mathbf{E}_j^{u,h}(\underline{\zeta}) \oplus \mathbf{E}_j^{u,e}(\underline{\zeta}) \oplus \mathbf{E}_j^{u,g}(\underline{\zeta}), \tag{2.9}$$

where $\mathbf{E}_j^{\cdot,h}(\underline{\zeta})$ and $\mathbf{E}_j^{\cdot,e}(\underline{\zeta})$ are as above and where $\mathbf{E}_j^{s,g}(\underline{\zeta})$ and $\mathbf{E}_j^{u,g}(\underline{\zeta})$ are sums of eigenspaces associated to the Jordan block(s) of $\mathscr{A}_j(\underline{\zeta})$. Thus they satisfy $\mathbf{E}_j^{s,g}(\underline{\zeta}) \cap \mathbf{E}_j^{u,g}(\underline{\zeta}) \neq \{0\}$. As for hyperbolic modes the glancing subspaces $\mathbf{E}_j^{s,g}(\underline{\zeta})$ and $\mathbf{E}_j^{u,g}(\underline{\zeta})$ can be described in terms of the group velocities and of the kernel of the symbol of $L(\partial)$. We have

$$\mathbf{E}_1^{s,g}(\underline{\zeta}) := \bigoplus_{m \in \mathfrak{G}_1} \ker \mathscr{L}(\underline{\tau}, \omega_{m,1}, \underline{\eta})$$
$$\text{and} \quad \mathbf{E}_2^{s,g}(\underline{\zeta}) := \bigoplus_{m \in \mathfrak{G}_2} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \omega_{m,2}). \tag{2.10}$$

In this article we will use the same assumption as in [17] about the size of the glancing modes. Indeed for glancing modes of size more than two, then the construction of the geometric optics expansions for half-space problems is a rather open question. It is possibly a rather complicate question because in [17], Williams gives examples of systems with glancing modes of order more than two which behave badly (at least for the L^∞-norm).

So that we assume in the following that the glancing modes are all of size two. More precisely, we assume

Assumption 2.9 (Size of glancing modes). — *Let* $j = 1, 2$, *and consider* $\underline{\zeta} \in \mathbb{G}_j$, *then any block of type* (4) *in Theorem 2.4 is of size two.*

With this assumption in hand, we have that the subspaces $\mathbf{E}_1^{s,g}(\underline{\zeta}) = \mathbf{E}_1^{u,g}(\underline{\zeta})$ and $\mathbf{E}_2^{s,g}(\underline{\zeta}) = \mathbf{E}_2^{u,g}(\underline{\zeta})$ are one dimensional eigenspaces of $\mathscr{A}_1(\underline{\zeta})$ and $\mathscr{A}_2(\underline{\zeta})$ respectively.

In such a framework, using Proposition 2.7, we can precise the decomposition (2.9) as:

Proposition 2.10. — *Let* $j = 1, 2$ *and* $\underline{\zeta} \in \mathbb{G}_j$ *then we have the decompositions*

$$\mathbf{E}_1^s(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{I}_1} \ker \mathscr{L}(\underline{\tau}, \omega_{m,1}, \underline{\eta}) \bigoplus_{m \in \mathfrak{G}_1} \ker \mathscr{L}(\underline{\tau}, \omega_{m,1}, \underline{\eta}) \oplus \mathbf{E}_1^{s,e}(\underline{\zeta}),$$

$$\mathbf{E}_1^u(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{O}_1} \ker \mathscr{L}(\underline{\tau}, \omega_{m,1}, \underline{\eta}) \bigoplus_{m \in \mathfrak{G}_1} \ker \mathscr{L}(\underline{\tau}, \omega_{m,1}, \underline{\eta}) \oplus \mathbf{E}_1^{u,e}(\underline{\zeta}),$$

$$\mathbf{E}_2^s(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{I}_2} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \omega_{m,2}) \bigoplus_{m \in \mathfrak{G}_2} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \omega_{m,2}) \oplus \mathbf{E}_2^{s,e}(\underline{\zeta}),$$

$$\mathbf{E}_2^u(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{O}_2} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \omega_{m,2}) \bigoplus_{m \in \mathfrak{G}_2} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \omega_{m,2}) \oplus \mathbf{E}_2^{u,e}(\underline{\zeta}).$$

2.1.2. About the boundary conditions

Hereinafter we assume that each side of the boundary $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ satisfies the condition ensuring the strong well-posedness of each half-space problems in (2.1). That is to say that the two boundary conditions B_1 and B_2 satisfy the so-called uniform Kreiss–Lopatinskii condition of [9].

The results of [14] indicate that choosing such boundary conditions is a necessary (but not sufficient) condition for the strong well-posedness of the quarter-space problem (1.1). We refer to Section 7 for more details about strongly well-posed boundary value problems in the quarter-space.

Assumption 2.11 (Uniform Kreiss–Lopatinskii condition). — *For all* $\zeta \in \Xi$, *we assume that*

$$\ker B_1 \cap \mathbf{E}_1^s(\zeta) = \ker B_2 \cap \mathbf{E}_2^s(\zeta) = \{0\}.$$

In particular, the restriction of the boundary matrix B_1 *(resp.* B_2*) to the (extended) stable subspace* $\mathbf{E}_1^s(\zeta)$ *(resp.* $\mathbf{E}_2^s(\zeta)$*) is invertible. Its inverse being denoted by* $\phi_1(\zeta) := B_{1_{|\mathbf{E}_1^s(\zeta)}}^{-1}$ *(resp.* $\phi_2(\zeta) := B_{2_{|\mathbf{E}_2^s(\zeta)}}^{-1}$*).*

2.2. Main result

The main results of the article are stated below, see Theorem 2.12 and Corollary 2.13. They extend the results of [2] to geometric optics expansions with glancing modes and with possibly several self-interaction phenomena. The precise requirements that are made on the geometry of the characteristic variety of $\mathscr{L}$ for these results to hold are precisely described in Subsections 4.2 and 4.3.

Clearly, because the structure described in Subsection 4.1 (uniqueness of the self-interaction loop) is a particular structure of the one of Subsection 4.2 (multiple self-interaction loops), then Corollary 2.13 is a direct consequence of Theorem 2.12.

However, because the proof of Theorem 2.12 establishing the result for multiple self-interaction loops uses in a non trivial way the proof for a unique self-interaction loop that is the proof of Corollary 2.13 (see Section 5), we choose to state both of the results. Corollary 2.13 being demonstrated in Section 5, Theorem 2.12 being demonstrated in Section 6, after the suitable modifications of the proof exposed in Section 5.

To state precisely our main result we first need to introduce the following set of profiles. First we define

$$H^{\infty}_{\natural}(\mathbb{R}\times\mathbb{R}_+) := \left\{u \in H^{\infty}(\mathbb{R}\times\mathbb{R}_+)\setminus \forall\, n\in\mathbb{N},\, (\partial_y^n u)_{|y=0} = 0\right\},$$

the set of flat functions at the corner. Then we introduce the following set for hyperbolic profiles

$$H^{\infty}_{\natural}(\Omega) := \left\{u \in H^{\infty}(\Omega)\setminus u_{|x_1=0},\ u_{|x_2=0} \in H^{\infty}_{\natural}(\mathbb{R}\times\mathbb{R}_+)\right\}.$$

For glancing and elliptic modes, because they are linked to boundary layers we define the set of profiles $\mathbf{P}$ as the set of functions $u_p = u(t, x_{3-p}, Y_p)$ with fast decay with respect to the fast variable Y_p. We refer to Definition 5.1 for a more precise statement.

Theorem 2.12. — *Under Assumptions 2.1, 2.2, 2.9 and 2.11 on the problem* (1.1)*.*

- *If the frequencies set, $\mathscr{F}$, associated to* (1.1) *is complete for the reflections and satisfies the structure Assumption 4.10. Finally, if we have the invertibility Assumption 6.2, then for all $n\in\mathbb{N}$, the geometric optics expansion cascade of equations* (5.2)*,* (5.7) *and* (5.8) *admits solutions in a suitable space of profiles.*
- *If $\#\mathscr{F} < \infty$, the ansatz* (5.1) *makes sense (as a finite sum). In particular, it can be truncated at the order $n = N_0$ to define $u^{\varepsilon}_{\mathrm{app},N_0}$*

(see (7.2)*). If moreover the problem* (1.1) *is strongly well-posed in* L^2 *then* $u^\varepsilon_{\text{app},N_0}$ *is an approximate solution to* (1.1) *in the sense that*

$$\forall\ N_0 \in \mathbb{N},\ \left\| u^\varepsilon - u^\varepsilon_{\text{app},N_0} \right\|_{L^2(\Omega)} \leqslant C\sqrt{\varepsilon}^{N_0+1},$$

where we stress that $C > 0$ *does not depend on* ε*.*

Under the stronger structure Assumption 4.8, we have as a direct consequence of Theorem 2.12

Corollary 2.13. — *Under Assumptions 2.1, 2.2, 2.9 and 2.11 on the problem* (1.1)*. If the frequencies set,* $\mathscr{F}$*, associated to* (1.1) *is complete for the reflections and satisfies the structure Assumption 4.8 and if we have the invertibility Assumption 5.14, then for all* $n \in \mathbb{N}$*, the geometric optics expansion cascade of equations* (5.2)*,* (5.7) *and* (5.8) *admits solutions in a suitable space of profiles.*

3. The phase generation process

In this paragraph we recall for a sake of completeness the main ideas in the phase generation process for geometric optics expansion in a quarter-space. More precisely, we will use the analysis of [2] to describe the generation of phases coming from the reflection of an incoming/outgoing or, an outgoing/incoming phase by repeated reflections against the sides of the boundary. This is the subject of Subsection 3.1.

However, compared to [2], we also have to include the possible reflections of glancing modes. In order to do so, we will use the first order approximation of [16] to justify that glancing modes do not create any new phases in the process. This is not clear at first glance because we know from [16, 17] that glancing modes create boundary layer localized along the side of the boundary for which they are glancing modes.

More precisely, if we have a glancing mode for the side $\partial\Omega_1$, then we have to consider in the expansion a term reading $\chi(x_1/\sqrt{\varepsilon})\widetilde{g}(t, x_2)$ where χ has fast decay and where $\widetilde{g}$ depends explicitly on the source g. As a consequence, if we consider the contribution of this term on the side $\partial\Omega_2$ we have two cases to separate:

- On the one hand, because of the fast decay of χ, the boundary term $\chi\widetilde{g}(t, 0)$ can not contribute on the side $\partial\Omega_2$ when $\{x_1 \geqslant C\sqrt{\varepsilon}\}$. Indeed it is $O(\sqrt{\varepsilon}^\infty)$.

- But on the other hand, near the corner that is to say for $\{x_1 < C\sqrt{\varepsilon}\}$, then the boundary term $\chi\widetilde{g}(t,0)$ is *a priori* $O(1)$. We can not exclude at first glance that it gives a non trivial contribution in the boundary on $\partial\Omega_2$. We will however justify that, using the flatness assumption on the boundary datum g, this contribution is zero.

A precise discussion is made in Subsection 3.2.

3.1. The phase generation process of [2]

3.1.1. The phase generation process for hyperbolic phases

To describe the phase generation process we start from the boundary value problem (1.1) in which we fix for oscillating boundary source a term g^{ε} reading under the form

$$g^{\varepsilon}(t,x_2) := e^{\frac{i}{\varepsilon}\psi(t,x_2)}g(t,x_2), \tag{3.1}$$

where the amplitude g is sufficiently regular, vanishes for negative times and let us say that it has its support away from $\{x_2 = 0\}$. In (3.1) the phase function ψ is linear and is given by

$$\psi(t,x_2) := \underline{\tau}t + \underline{\xi}_2 x_2,$$

for given real frequencies numbers $\underline{\tau}$, $\underline{\xi}_2$.

Because $L(\partial)$ is assumed to be hyperbolic, then it comes with some finite speed of propagation property.[(3)] So, the solution turned on by the supported source g^{ε} can not hit the side $\partial\Omega_2$ immediately. As a consequence, at least during a small time, the problem does not see its boundary condition on $\partial\Omega_2$ and we can thus consider the half-space boundary problem

$$\begin{cases} L(\partial)u^{\varepsilon} = 0 & \text{in } \mathbb{R}\times\mathbb{R}_+\times\mathbb{R}, \\ B_1 u^{\varepsilon}_{|x_1=0} = g^{\varepsilon} & \text{on } \mathbb{R}^2, \\ u^{\varepsilon}_{|t\leqslant 0} = 0 & \text{on } \mathbb{R}_+\times\mathbb{R}. \end{cases} \tag{3.2}$$

It is thus natural to choose for first ansatz the one associated to (3.2). Following [16] and [17] this ansatz can contain three different kinds of terms

(3) Indeed, construct geometric optics expansions for linear operators amounts to solve transport equations.

depending on the nature of the initial boundary frequency $(i\underline{\tau},\underline{\xi}_2)\in\Xi_0$ (we refer to Definition 2.5) and it reads

$$u^\varepsilon \sim \sum_{n\geqslant 0}\sqrt{\varepsilon}^n\left(\sum_{k\in\mathscr{I}\cup\mathscr{O}} e^{\frac{i}{\varepsilon}\varphi_k}u_{n,k}+\sum_{k\in\mathscr{G}_1} e^{\frac{i}{\varepsilon}\varphi_k}u_{n,k}+e^{\frac{i}{\varepsilon}\psi}U_{ev,n}\right) \tag{3.3}$$

where $\mathscr{I}$, $\mathscr{O}$ and $\mathscr{G}_1$ stand respectively for the incoming, outgoing and glancing sets of indices for the boundary value problem (3.2). The phases functions φ_k for such k are then defined by $\varphi_k(t,x):=\psi(t,x_2)+\underline{\xi}_1^k x_1$, where the $\underline{\xi}_1^k$ denote the real roots in the ξ_1 variable of the dispersion relation

$$\det\mathscr{L}(\underline{\tau},\xi_1,\underline{\xi}_2)=0. \tag{3.4}$$

Consequently in (3.3) the so-called evanescent amplitudes $U_{ev,n}$ are linked to the (purely) complex roots of the dispersion relation. It gives rise to a boundary layer at scale ε. Similarly following [17] the glancing amplitudes, namely the $u_{n,k}$ for $k\in\mathscr{G}_1$, give rise to boundary layers at scale $\sqrt{\varepsilon}$. The influence of such layers are investigated in Subsection 3.2.

To end up this paragraph, we recall the main ideas to determine the descendants of the hyperbolic amplitudes $u_{n,k}$ for $k\in\mathscr{I}\cup\mathscr{O}$. We refer to [2] for a complete exposition.

In order to determine the future of the amplitudes $u_{n,k}$ for $k\in\mathscr{I}\cup\mathscr{O}$, we have to consider the distinction introduced in Definition 2.8. We thus have four cases to consider

- If $k\in\mathscr{O}$ and if $\mathbf{v}_k$ is outgoing-outgoing, then the associated amplitude is automatically zero without forcing term in the interior. So that such an index can be excluded from the ansatz (3.3).
- If $k\in\mathscr{O}$ and if $\mathbf{v}_k$ is outgoing-incoming, because there is no non trivial source term in the interior or on the boundary $\partial\Omega_2$ such an index can be *initially* excluded from (3.3). However, because of the self-interaction phenomenon nothing prevents that such a phase comes back in the generated phases by the incoming modes which are described below.
- If $k\in\mathscr{I}$ and if $\mathbf{v}_k$ is incoming-incoming, then the transported information will never hit the boundary $\partial\Omega_2$. It spreads to infinity, it will never be reflected back. As a consequence, incoming-incoming group velocities are ending points in the phase generation process.
- If $k\in\mathscr{I}$ and if $\mathbf{v}_k$ is incoming-outgoing, then by definition the transported information hits, after some (strictly) positive time of travel, the boundary $\partial\Omega_2$. It will create some new phases during the reflection and we have to describe these phases.

In order to do so, we fix one incoming-outgoing phase φ_k. Let us remark that because g has its support away from the corner the same property holds for the impacted term (by resolution of a transport equation). Consequently the finite time of propagation argument applies and it leads us to consider the boundary value problem in the upper half-space:

$$\begin{cases} L(\partial)u^{\varepsilon} = 0 & \text{in } \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+, \\ B_2 u^{\varepsilon}_{|x_2=0} = F(g^{\varepsilon}) & \text{on } \mathbb{R}^2, \\ u^{\varepsilon}_{|t\leqslant 0} = 0 & \text{on } \mathbb{R}\times\mathbb{R}_+, \end{cases} \tag{3.5}$$

where $F(g^{\varepsilon})$ is some (explicit and linear in terms of g^{ε}) source term depending on the trace on $\{x_2 = 0\}$ of the considered incoming-outgoing phase. In terms of phase functions, this source oscillates with respect to the new boundary phase $\widetilde{\psi}(t,x_1) := \underline{\tau}t + \underline{\xi}_1 x_1$. So that, the reflected amplitudes oscillate with respect to the phase functions $\varphi_{k'}(t,x)$ where the real parameters $\underline{\xi}_2^{k'}$ are determined as the (real) roots in the ξ_2 variable of the dispersion relation

$$\det \mathscr{L}\left(\underline{\tau}, \underline{\xi}_1^k, \xi_2\right) = 0. \tag{3.6}$$

If this relation admits (purely) complex roots, we also have to consider an evanescent profile during the reflection. Similarly during this reflection glancing modes for the side $\partial\Omega_2$ may appear.

We thus add to (3.3) the terms obtained so far. The ansatz now reads under the form

$$u^{\varepsilon} \sim \sum_{n\geqslant 0} \sqrt{\varepsilon}^n \left[\sum_{k\in\mathscr{I}_{\text{hyp}}} e^{\frac{i}{\varepsilon}\varphi_k} u_{n,k} + \sum_{k\in\mathscr{G}_1\cup\mathscr{G}_2} e^{\frac{i}{\varepsilon}\varphi_k} u_{n,k} \right. \\ \left. + e^{\frac{i}{\varepsilon}\psi} U_{ev,1,n} + e^{\frac{i}{\varepsilon}\widetilde{\psi}} U_{ev,2,n} \right] \tag{3.7}$$

where $\mathscr{I}_{\text{hyp}}$ stands for a shorthand notation for the collection of the above incoming-incoming, incoming-outgoing and outgoing-incoming hyperbolic modes; where $\mathscr{G}_2$ contains the (possible) new glancing modes, and finally where $U_{ev,2,n}$ stands for the amplitude associated to the (possible) new evanescent mode.

Repeating the same arguments as for the boundary value problem (3.2), we have to determine the reflections against the side $\partial\Omega_1$ of the new outgoing-incoming amplitudes in (3.7). We then repeat the procedure until that to some reflection the obtained hyperbolic phases are all incoming-incoming, evanescent or as it will be justified below glancing. This stops the determination of the descendants of the first considered incoming-outgoing phase. We then repeat the same process for all initial incoming/outgoing phases.

In terms of the section of the characteristic variety $\mathscr{V} := \mathscr{V}_{\underline{\tau}}$ defined by

$$\mathscr{V} := \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \det \mathscr{L}(\underline{\tau}, \xi_1, \xi_2) = 0\right\},$$

the phase generation procedure is easy to represent. Indeed, we first consider the roots in the ξ_1 variable of the dispersion relation (3.4). So that we select the intersections of $\mathscr{V}$ with the horizontal line $\{\xi_2 = \underline{\xi}_2\}$. The possible complex roots can be seen as points at infinity.

Then, for each incoming-outgoing[(4)] intersection, we consider the roots in the ξ_2 variable of (3.6). Consequently, we now select the intersections of $\mathscr{V}$ with the vertical line $\{\xi_1 = \underline{\xi}_1^k\}$. This procedure is repeated as long as we obtain incoming-outgoing or outgoing-incoming modes.

We refer to [1, Chapitre 6] for some examples describing in all details this procedure (see also [2]).

3.1.2. Self-interaction and loops in a nutshell

Self-interaction loops. In the above paragraph self-interaction has been totally ignored. However as shown in [2], see also [3], as soon as the boundary of the domain admits several components the phases can regenerate themselves after a suitable number of the reflections described above.

The simplest self-interaction phenomenon is the one described in [2] and it involves only four phases. Let us denote by φ_1 an incoming-outgoing phase turned on by the boundary source term g^ε. Also assume that this source comes with an outgoing-incoming phase φ_4.

We said in the above discussion that we neglect all the outgoing-incoming phases turned on by the source term. As a consequence, in particular we neglect φ_4. In fact, we should not. Indeed, consider that the phase φ_1 is reflected against $\partial\Omega_2$ according to the above phenomenon into an outgoing-incoming phase φ_2. Assume then that φ_2 is reflected against $\partial\Omega_1$ into an incoming-outgoing phase φ_3. Then the amplitude associated to φ_3 travels. It hits the side $\partial\Omega_2$ and nothing prevents that in the reflected phases one recovers the initially excluded outgoing-incoming phase φ_4. It implies that this phase, as a reflection of φ_3, must now be considered in the ansatz. When one studies the reflection of the phase φ_4 against the side $\partial\Omega_1$, then according to the above discussion about reflections he/she recovers the first considered phase φ_1. We say that the phase φ_1 regenerates itself or is self-interacting because it regenerates itself during the reflections against

(4) This characterization can be easily made graphically by considering the outgoing normal to $\mathscr{V}$ at the intersection point.

the sides of the domain. The same terminology applies to the phases φ_2, φ_3 and φ_4.

In terms of the geometry of the characteristic variety $\mathscr{V}$, we can thus find a rectangle whose vertex namely s_1, s_2, s_3 and s_4 are points of $\mathscr{V}$. We illustrate such a configuration in Figure 3.1 where the red points are associated to incoming-outgoing phases, the blue points being associated to outgoing-incoming phases.

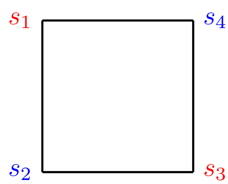

Figure 3.1. The simplest possible loop.

Of course, nothing prevents the initially neglected phase to appear after more than three reflections. Consequently, if we want to consider generic self-interaction phenomena, we should consider in the geometry of $\mathscr{V}$ more generic figures than rectangles that is to say some “stairway” like configurations. Such configurations are called loops and are precisely described in Definition 4.7. We give two illustrations in Figures 3.2 and 3.3.

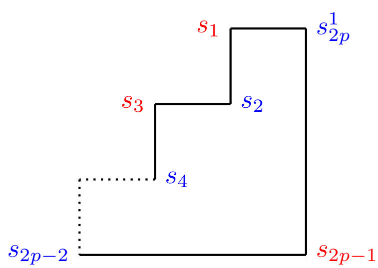

Figure 3.2. A “stairway” like loop

Non self-interacting loops. Self-interaction phases and consequently self-interaction loops are of course the one of main interest, because in particular, they require some new initialization condition to construct the geometric optics expansions. However, let us point that other “stairway like” loops in $\mathscr{V}$ can also appear. These kind of loops was excluded [2] to have the simplest possible proof.

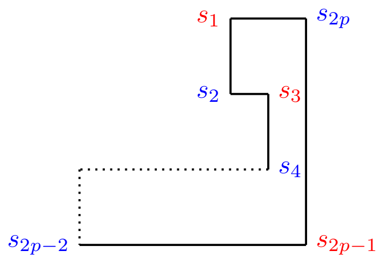

Figure 3.3. An other (not really) “stairway” like loop.

We describe in the following two kinds of such loops. The first one involves an incoming-incoming phase. Let us remark that in the phase generation process such phases can appear during reflections against the side $\partial\Omega_1$ or also $\partial\Omega_2$. As a consequence nothing prevents *a priori* one incoming-incoming amplitude, let us say associated to the phase φ_{ii}, to be generated once against the side $\partial\Omega_1$ and once against the side $\partial\Omega_2$. We illustrate the following situation on Figure 3.4.

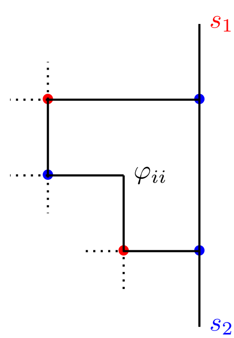

Figure 3.4. Non self-interaction loop with incoming-incoming phase.

On this figure we see that there exist two (distinct) paths of phases leading to the incoming-incoming amplitude φ_{ii}. The first one starts by the first descendant of the self-interacting phase s_1, the second one starts by the second descendant.

We will see that the existence of such multiple paths of phases to generate the considered incoming-incoming phase will be particularly important in the order of determination of the amplitudes. And consequently also in the order of resolution of the whole cascade of equations.

The same kind of behaviour can also arise for non incoming-incoming phases. Indeed, let us consider the two following possible situations.

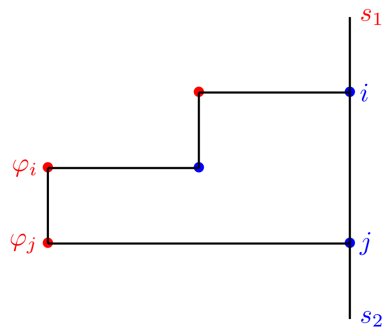

Figure 3.5. Non self-interaction loop with incoming-outgoing phase.

In the first one we consider an incoming-outgoing phase φ_i which has been obtained twice. The first time after three reflections of the first (outgoing-incoming) descendant i of s_1 and the second time as a direct incoming-outgoing descendant, namely φ_j, of the second descendant j of s_1. Then, nothing prevents that the phases φ_i and φ_j verify $\xi_1^i = \xi_1^j$ so that we have again a loop in $\mathscr{V}$. We refer to Figure 3.5 for an illustration.

Let us however point that such a situation is not at all an issue in the resolution of the cascade of equations. Indeed we can easily determine φ_i and φ_j independently the one from the other. It is due to the fact that these phases are both incoming-outgoing, so that for their resolutions, the amplitudes u_i and u_j associated to the above phases only require the trace value on $\{x_1 = 0\}$. On such a trace the coupling condition $\xi_1^i = \xi_1^j$ disappears.

The situation becomes a little more complicated in the following example depicted on Figure 3.6. It is just a slight modification of the previous situation where we only add two points in the (section of) the characteristic variety. It adds an other non self-interacting loop.

In such a configuration, the outgoing-incoming indices j and ℓ are in the situation of Figure 3.5. So the determination of such amplitudes is not really an issue. The point of interest is now the determination of the incoming-outgoing phase φ_j.

Indeed it is now generated by two distinct paths of phases. Firstly, directly as a reflection of j. Secondly we obtain j as a descendant of i via the path (i, k, ℓ). So that, in order to determine φ_j, we will have to determine all

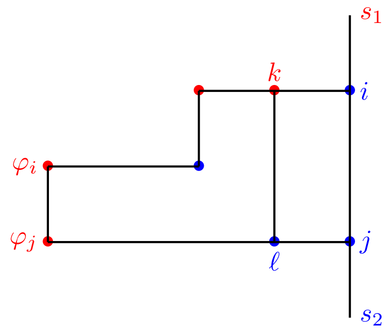

Figure 3.6. Non self-interaction loop with incoming-outgoing phase bis.

the above amplitudes first and thus in the presence of several (non self-interacting) loops the determination of a specific amplitude can necessitate the determination of several (independent) sequences of amplitudes.

The main idea used to overcome this difficulty is that in each of the different paths leading to φ_j the indices are not (yet) concerned with this loop issue. Consequently, they can be determined before to proceed to the resolution of φ_j.

3.2. The work of [16] as a guide for understand the influence of glancing modes

In this paragraph we justify that in the phase generation process described in Subsection 3.1, we can effectively neglect the possible descendants of glancing phases. The main idea for doing this is that these terms are boundary layers. So they can be neglected away from the boundary where they are $O(\varepsilon^\infty)$. Near the boundary they behave as the source g, so that they are zero because of the support assumption on g or by any flatness assumption.

To precise the above remark, we will show the affirmation on the simplified first order approximation proposed by Williams in [16]. We thus consider the crude ansatz

$$u^\varepsilon_{\text{app}} := \left[\sum_{n=0}^{1} \varepsilon^n \sum_{k\,\in\,\mathscr{I}_{\text{hyp}}} e^{\frac{i}{\varepsilon}\varphi_k} u_{n,k} + \sum_{k\,\in\,\mathscr{I}_{g_1}\,\cup\,\mathscr{I}_{g_2}} e^{\frac{i}{\varepsilon}\varphi_k} u_{n,k}\right], \tag{3.8}$$

where $\mathscr{I}_{\text{hyp}}$ stands for the set of all hyperbolic indices and where $\mathscr{I}_{g_1} \cup \mathscr{I}_{g_2}$ collects the glancing indices (we refer to Definition 4.1 for more precise definitions). To simplify the exposition, we also assumed that the ansatz does not contain any evanescent modes. It is not a loss of generality because these modes can be determined independently on the oscillating ones. We refer to Subsection 5.2.2 for more details.

By crude we mean that in order to obtain a higher order approximation the scaling should be refined in $\sqrt{\varepsilon}$ as in [17], and we also need to add some extra correctors.

However, the analysis of [16] shows that such a candidate is a good first order approximation. If we can justify on this first order approximation that glancing boundary layers do not have any descendants, then the same should hold for the higher order approximation ansatz (3.7).

We assume without loss of generality that the initial frequency $\underline{\zeta} := (i\underline{\tau}, \underline{\xi}_2) \in \mathbb{G}_1$, let $\underline{k} \in \mathscr{G}_1$ be a glancing index associated to one of the glancing phases. To save some notations we also assume that it is the only glancing phase appearing in the process. We explain at the end of the paragraph how the discussion can be generalized when several glancing modes appear.

Plugging the ansatz (3.8) in the interior equation of (1.1) leads us to solve (for the considered glancing leading order amplitude) the (usual) equations

$$\begin{cases} \mathscr{L}(d\varphi_{\underline{k}})u_{0,\underline{k}} = 0, \\ i\mathscr{L}(d\varphi_{\underline{k}})u_{1,\underline{k}} + L(\partial)u_{0,\underline{k}} = 0. \end{cases}$$

The first equation is the classical polarization condition while using Lax lemma [10] (see also Subsection 5.2 for more details) the second equation is equivalent to the transport equation

$$(\partial_t + \mathbf{v}_{\underline{k}} \cdot \nabla_x)u_{0,\underline{k}} = 0, \tag{3.9}$$

where $\mathbf{v}_{\underline{k}}$ is the (glancing) group velocity introduced in Definition 2.6. Thus it satisfies $\mathbf{v}_{\underline{k},1} = 0$.

Injecting the ansatz (3.8) in the boundary conditions of (1.1) gives

$$\begin{cases} B_1 \left[u_{0,\underline{k}} + \displaystyle\sum_{k \in \mathscr{I}_{io} \cup \mathscr{I}_{ii}} u_{0,k} \right]_{|x_1=0} = g - B_1 \displaystyle\sum_{k \in \mathscr{I}_{oi}} u_{0,k_{|x_1=0}}, \\ B_2 \left[\displaystyle\sum_{k \in \mathscr{I}_{oi} \cup \mathscr{I}_{ii}} u_{0,k} \right]_{|x_2=0} = -B_2 \displaystyle\sum_{k \in \mathscr{I}_{io}} u_{0,k_{|x_2=0}} \\ \qquad\qquad -B_2 u_{0,\underline{k}_{|x_2=0}}, \end{cases} \tag{3.10}$$

where the sets of indices $\mathscr{I}_\cdot$ are precisely introduced in Definition 4.1. The precise definition is however of little interest for the current discussion. The only thing to keep in mind when we read (3.10) is that the left-hand side of the first (resp. second) of equation (3.10) is sum of elements of $B_1\mathbf{E}_1^s$ (resp. $B_2\mathbf{E}_2^s$).

For glancing modes we are thus face to an extra difficulty compared to hyperbolic modes. Indeed we have some overdetermination issue in the equations. On the one hand, we have from the definition that in (3.9), the normal velocity $\mathbf{v}_{\underline{k},1} = 0$. The transport is tangent to the boundary $\partial\Omega_1$ and consequently no boundary condition on $\partial\Omega_1$ has to be imposed. On the other hand, we have to satisfy the boundary condition on $\partial\Omega_1$ given by (3.10).

In other words, one has to choose between Charybdis and Scylla by solving the interior equation or the boundary condition. The other equation been unsatisfied.

Following [16], it is however not a real choice. Indeed, from the error analysis (see Section 7 for more details) in order that (3.8) approximates the exact solution u^ε, we have to solve the boundary condition exactly. Indeed, if it not solved, then we have an error at scale $O(1)$ in the error estimate and thus the ansatz (3.8) does not give a true approximate solution. While, if we have some error in the interior, then one can construct some corrector that is to say choose $u_{1,\underline{k}}$ in (3.8) in such a way that this $O(1)$ error term in the interior vanishes. We do not give here the precise construction of such a corrector. It can be found in [16].

Using the uniform Kreiss–Lopatinskii condition (see Definition 2.11 and Subsection 5.2.1 for more details) the boundary condition on $\partial\Omega_1$, shows that the trace of the glancing mode is

$$u_{0,\underline{k}_{|x_1=0}} = \Pi^{\underline{k}}\phi_1(\underline{\zeta})\left[g - B_1 \sum_{k\,\in\,\mathscr{I}_{oi}} u_{0,k_{|x_1=0}}\right], \tag{3.11}$$

where $\Pi^{\underline{k}} := \Pi^{\underline{k}}(\underline{\zeta})$ is just a projection selecting the component of the trace associated to the glancing mode. It is clearly introduced in Definition 4.11.

If we assume, in a first time for simplicity, that the frequency $\underline{\zeta}$ is not self-interacting, it implies that there is no outgoing-incoming amplitude in the right-hand side, and (3.11) uniquely determines the trace of the glancing amplitude. Because, we are not interested in solving the equation in the interior we are free to extend this trace in the interior as a boundary layer and we define

$$u_{0,\underline{k}}(t,x) := \chi\left(\frac{x_1}{\sqrt{\varepsilon}}\right)\Pi^{\underline{k}}\phi_1(\underline{\zeta})g(t,x_2), \tag{3.12}$$

where χ has fast decay so that we have $\|u_{0,\underline{k}}\|_{L^2(\Omega)} = O(\varepsilon^{1/4})$, which is sufficient for the error analysis in the interior.

To show that such an amplitude is not reflected, we are now interested in the trace on $\partial\Omega_2$ of such a glancing amplitude. We have, directly from (3.12), using the fact that g vanishes near the corner that $u_{0,\underline{k}}(t, x_1, 0) = 0$, as a consequence this term vanishes in the right-hand side of the second equation of (3.10). It can thus not give rise to some non trivial information on $\partial\Omega_2$.

In other words, the boundary layer turned on by the glancing index $\underline{k}$ can not be reflected against $\partial\Omega_2$ and thus this term has no descendants in the phase generation process by reflections. Exactly as incoming-incoming phases, glancing modes are ending points in the phase generation process.

If now the frequency $\underline{\zeta}$ is self-interacting, then the right-hand side of (3.11) contains a term indexed by $k \in \mathscr{I}_{oi}$. In such a configuration, the same arguments apply because:

- self-interacting amplitudes can be determined before the others amplitudes.
- self-interacting amplitudes inherit the flatness at the origin from the one of the source g.

Once the leading order glancing amplitude $u_{0,\underline{k}}$ has been constructed then following [16] we can define a suitable first order corrector $u_{1,\underline{k}}$ in such a way that $\|u^\varepsilon - u^\varepsilon_{\mathrm{app}}\|_{L^2(\Omega)}$ is $O(\varepsilon^{1/4})$.

The same kind of arguments can then be extended for glancing modes appearing after some reflections against the side $\partial\Omega_1$ or $\partial\Omega_2$. The only required ingredient is that the hyperbolic amplitudes encounter to generate the considered glancing mode have vanishing traces near the corner (or are at least flat at the origin). This is again a consequence of the fact that g vanishes near the corner and that hyperbolic modes are solution to transport equations, so that this property is conserved.

4. Structures of the set of indices

Now that the phase generation process for (1.1) is described and that we have all the expected phases in the ansatz, then we have to find some structure in the set of indices in order to find an ordered way to solve the geometric optics expansion cascade of equations.

In the following because we want to consider several loops, the order of resolution will not be as simple as in [2] for which the uniqueness of the self-interaction loop implies that we can:

(1) Firstly, find some necessary condition to determine the amplitudes of the elements of the loop.

(2) Secondly, determine the amplitudes of the elements in a direct vicinity of the loop's elements.
(3) Finally, define a partition of the remaining indices, partition composed of trees whose roots are in the direct vicinity of the loop and solve inductively in the trees.

This ideal situation is depicted on Figure 4.1. In particular, we see that the situation of Figure 3.4 is a counter-example to the tree structure of Figure 4.1 because in such a setting the trees are intersecting. We do not have a partition any more.

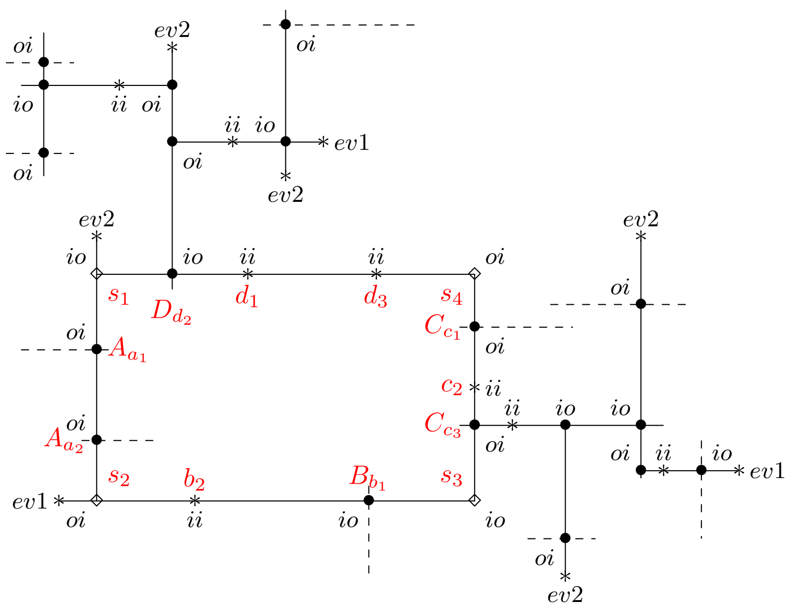

Figure 4.1. Tree structure of the set of indices $\mathscr{I}$ in [2].

The order of determination used in [2] is then not defined any more as soon as the set of indices contains a loop (not necessarily a self-interaction one) in addition of the first considered self-interaction loop (the one turned on by the boundary source term). Indeed, as depicted in Figures 3.4 or 3.6, the determination of an element in the direct vicinity of the loop may require the knowledge of some elements which have been reflected several times.

As a consequence, we will adopt as an order of resolution for the indices an order based on the number of distinct ways to generate the index $\underline{i}$ (the one whose amplitude is considered) from s_1 (namely the self-interaction loop element turned on by the source). From a structure of the set of indices point of view, this leads us to consider all the sequences (we refer to Definition 4.5 for a precise statement) linking s_1^1 to the index $\underline{i}$. We insist on the fact that we do not have as in [2] uniqueness of such sequences.

In order to make this order precise we recall in Subsection 4.1 some elements borrowed from [2]. Then, we adapt these objects to the framework where there only exists a unique self-interaction loop but to multiple non-self-interaction loops in Subsection 4.2. The whole generality of multiple (simple) self-interaction loops (or not) is considered in Subsection 4.3.

4.1. Generic definitions

We recall the following definitions from [2]. The second one is just the generalization of [2, Definition 4.2] to frequencies set containing glancing modes.

Definition 4.1 (Frequencies set).

- *Let* $\mathscr{I} \subset \mathbb{N}$, *and* $\underline{\tau} \in \mathbb{R}$, *we say that a set,* $\mathscr{F}$, *indexed by* $\mathscr{I}$
$$\mathscr{F} := \left\{ f_i := \left(\underline{\tau}, \underline{\xi}_1^i, \underline{\xi}_2^i\right) \backslash\, i \in \mathscr{I} \right\},$$
is a frequencies set for the quarter-space problem (1.1), *if for all* $i \in \mathscr{I}$, *we have first*
$$\det \mathscr{L}(f_i) = 0,\ f_i \neq 0,$$
meaning that $f_i = d\varphi_i$ *solves the eikonale equation of* (1.1). *And if moreover we have one of the following alternatives*
  (1) $\underline{\xi}_1^i, \underline{\xi}_2^i \in \mathbb{R}$.
  (2) $\underline{\xi}_1^i \in \mathbb{C} \setminus \mathbb{R}$, $\underline{\xi}_2^i \in \mathbb{R}$ *and* $\Im \underline{\xi}_1^i > 0$.
  (3) $\underline{\xi}_2^i \in \mathbb{C} \setminus \mathbb{R}$, $\underline{\xi}_1^i \in \mathbb{R}$ *and* $\Im \underline{\xi}_2^i > 0$.
- *For a frequencies set* $\mathscr{F}$, *we define the partition*
$$\mathscr{F} := \mathscr{F}_{os} \cup \mathscr{F}_{ev1} \cup \mathscr{F}_{ev2},$$
where
  - $\mathscr{F}_{os}$ *is the set of* $f_i \in \mathscr{F}$ *satisfying* (1).
  - $\mathscr{F}_{ev1}$ *is the set of* $f_i \in \mathscr{F}$ *satisfying* (2).
  - $\mathscr{F}_{ev2}$ *is the set of* $f_i \in \mathscr{F}$ *satisfying* (3).
- *The set of oscillating frequencies, namely* $\mathscr{F}_{os}$, *is decomposed as follows*
$$\mathscr{F}_{os} := \underbrace{\mathscr{F}_{ii} \cup \mathscr{F}_{io} \cup \mathscr{F}_{oi} \cup \mathscr{F}_{oo}}_{:=\mathscr{F}_{\rm hyp}} \cup \mathscr{F}_{g_1} \cup \mathscr{F}_{g_2}$$
where

– $\mathscr{F}_{ii} := \{f_i \in \mathscr{F}_{os} \setminus \mathbf{v}_{i,1}, \mathbf{v}_{i,2} > 0\}$
– $\mathscr{F}_{io} := \{f_i \in \mathscr{F}_{os} \setminus \mathbf{v}_{i,1} > 0 \quad and \quad \mathbf{v}_{i,2} < 0\}$
– $\mathscr{F}_{oi} := \{f_i \in \mathscr{F}_{os} \setminus \mathbf{v}_{i,1} < 0 \quad and \quad \mathbf{v}_{i,2} > 0\}$
– $\mathscr{F}_{oo} := \{f_i \in \mathscr{F}_{os} \setminus \mathbf{v}_{i,1}, \mathbf{v}_{i,2} < 0\}$
– $\mathscr{F}_{g_1} := \{f_i \in \mathscr{F}_{os} \setminus \mathbf{v}_{i,1} = 0\}$
– $\mathscr{F}_{g_2} := \{f_i \in \mathscr{F}_{os} \setminus \mathbf{v}_{i,2} = 0\}$

where we recall that $\mathbf{v}_{\underline{i}} := (\mathbf{v}_{\underline{i},1}, \mathbf{v}_{\underline{i},2}) \in \mathbb{R}^2$ *stands for the group velocity introduced in Definition 2.6.*

- *Finally, for one of the above subspaces of $\mathscr{F}$, namely $\mathscr{F}_\star$, we define $\mathscr{I}_\star$ the subspace of $\mathscr{I}$ formed by the indices i such that $f_i \in \mathscr{F}_\star$.*

We precise the previous definition by a refinement ensuring that we take into account all the terms in the phases generation process of Section 3.

Definition 4.2 (Complete for reflection frequencies set). — *The corner problem* (1.1) *is said to be complete for the reflections if there exists a set of frequencies $\mathscr{F}$, in the sense of Definition 4.1, satisfying the following properties:*

(1) *The set $\mathscr{F}$ contains all the real roots (in the ξ_1 variable) associated to incoming-outgoing or incoming-incoming group velocities, the real roots associated to glancing modes for the side $\partial\Omega_1$, and finally, the complex roots with positive imaginary part of the (initial) dispersion relation* $\det \mathscr{L}(\underline{\tau}, \xi_1, \underline{\xi}_2) = 0$.
(2) *If $(\underline{\tau}, \xi_1^i, \xi_2^i) \in \mathscr{F}_{io}$, then $\mathscr{F}$ contains all the roots (in the ξ_2 variable), denoted by ξ_2^p, of the dispersion relation* $\det \mathscr{L}(\underline{\tau}, \xi_1^i, \xi_2) = 0$ *satisfying one of the following two alternatives*
   (a) *$\xi_2^p \in \mathbb{R}$ and the frequency $(\underline{\tau}, \xi_1^i, \xi_2^p)$ is associated to an outgoing-incoming group velocity, an incoming-incoming group velocity or is glancing for the side $\partial\Omega_2$.*
   (b) $\Im \xi_2^p > 0$.
(3) *If $(\underline{\tau}, \xi_1^i, \xi_2^i) \in \mathscr{F}_{oi}$, then $\mathscr{F}$ contains all the roots (in the ξ_1 variable), denoted by ξ_1^p, of the dispersion relation* $\det \mathscr{L}(\underline{\tau}, \xi_1, \xi_2^i) = 0$ *satisfying one of the following two alternatives*
   (a) *$\xi_1^p \in \mathbb{R}$ and the frequency $(\underline{\tau}, \xi_1^p, \xi_2^i)$ is associated to an incoming-outgoing group velocity, an incoming-incoming group velocity or is glancing for the side $\partial\Omega_1$.*
   (b) $\Im \xi_1^p > 0$.
(4) *The set $\mathscr{F}$ is minimal (for the inclusion) for the preceding properties.*

As in [2], once that we have a complete for reflections set of frequencies, we define the functions Φ, Ψ which associate to an index $i \in \mathscr{I}$ the indices which are in a "direct vicinity" of i. By direct vicinity of an index $\underline{i}$, we understand both the indices obtained during one single reflection of $\underline{i}$ or the indices containing $\underline{i}$ in their descendants during one single reflection.

More precisely, let $\mathscr{P}_N(\mathscr{I})$ be the power set of $\mathscr{I}$ with at most N elements, then we define Φ, $\Psi : \mathscr{I} \to \mathscr{P}_N(\mathscr{I})$ by the relations: for $i \in \mathscr{I}$ associated to the frequency $f_i = (\underline{\tau}, \xi_1^i, \xi_2^i)$;

$$\Phi(i) := \left\{ j \in \mathscr{I} \setminus \xi_2^j = \xi_2^i \right\} \quad \text{and} \quad \Psi(i) := \left\{ j \in \mathscr{I} \setminus \xi_1^j = \xi_1^i \right\}.$$

The following properties being independent of the existence of loops in the frequencies set, they follow the proofs of [2].

Proposition 4.3. — *If $\mathscr{F}$ is a complete for the reflections frequencies set, in the sense of Definition 4.2, then the applications Φ and Ψ satisfy the properties:*

1. *$\forall\, i \in \mathscr{I}$ we have $i \in \Phi(i)$ and $i \in \Psi(i)$.*
2. *$\forall\, i \in \mathscr{I}$, $\forall\, j \in \Psi(i)$ and $\forall\, k \in \Phi(i)$, we have $\Psi(i) = \Psi(j)$ and $\Phi(k) = \Phi(i)$.*
3. *$\forall\, i \in \mathscr{I}$ we have $\Phi(i) \cap \mathscr{I}_{ev2} = \Phi(i) \cap \mathscr{I}_{g_2} = \emptyset$ and $\Psi(i) \cap \mathscr{I}_{ev1} = \Phi(i) \cap \mathscr{I}_{g_1} = \emptyset$;*
4. *$\forall\, i \in \mathscr{I}_{os}$ we have $\#(\Phi(i) \cap \mathscr{I}_{ev1} \cap \mathscr{I}_{io} \cap \mathscr{I}_{ii} \cap \mathscr{I}_{g_1}) \leqslant p_1$ and $\#(\Psi(i) \cap \mathscr{I}_{ev2} \cap \mathscr{I}_{oi} \cap \mathscr{I}_{ii} \cap \mathscr{I}_{g_2}) \leqslant p_2$.*

Thanks to the applications Φ and Ψ, we can borrow from [2] the notion of linked indices. Let us stress that we add to the definition of [2] the notion of linked indices for glancing modes. Two indices $\underline{i}$ and $\underline{j}$ are linked in $\mathscr{I}$ if the index $\underline{j}$ is obtained from $\underline{i}$ after a suitable number of reflections following the heuristic rules described in Section 3.

Definition 4.4 (Linked indices).

- *Let $\underline{i} \in \mathscr{I}_{io}$ we say that the index*
  - *$j \in \mathscr{I}_{io} \cup \mathscr{I}_{ev1} \cup \mathscr{I}_{g_1}$ (resp. $j \in \mathscr{I}_{oi} \cup \mathscr{I}_{ev2} \cup \mathscr{I}_{g_2}$) is linked to the index $\underline{i}$, if there exists $p \in 2\mathbb{N}+1$ (resp. $p \in 2\mathbb{N}$) and a sequence of indices $\ell := (\ell_1, \ell_2, \dots, \ell_p) \in \mathscr{I}^p$ satisfying the following property:*

    $$\ell_1 \in \Psi(\underline{i}) \cap \mathscr{I}_{oi}, \ell_2 \in \Psi(\ell_1) \cap \mathscr{I}_{io}, \dots, j \in \Phi(\ell_p) \qquad (\alpha)$$

    (resp. $j \in \Psi(\ell_p)$).
  - *$j \in \mathscr{I}_{ii}$ is linked to the index $\underline{i}$, if there exists a sequence $\ell := (\ell_1, \ell_2, \dots, \ell_p) \in \mathscr{I}^p$ such that:*

$$\ell_1 \in \Psi(\underline{i}) \cap \mathscr{I}_{oi}, \ell_2 \in \Phi(\ell_1) \cap \mathscr{I}_{io}, \dots, \begin{cases} j \in \Phi(\ell_p) & p \text{ odd}, \\ j \in \Psi(\ell_p) & p \text{ even}. \end{cases} \qquad (\beta)$$

- *Let* $\underline{i} \in \mathscr{I}_{oi}$ *we say that the index*
  - $j \in \mathscr{I}_{io} \cup \mathscr{I}_{ev1} \cup \mathscr{I}_{g_1}$ *(resp.* $j \in \mathscr{I}_{oi} \cup \mathscr{I}_{ev2} \cup \mathscr{I}_{g_2}$*) is linked to the index* $\underline{i}$*, if there exists* $p \in 2\mathbb{N}$ *(resp.* $p \in 2\mathbb{N}+1$*) and a sequence of indices* $\ell := (\ell_1, \ell_2, \ldots, \ell_p) \in \mathscr{I}^p$ *satisfying the following property:*
  $$\ell_1 \in \Phi(\underline{i}) \cap \mathscr{I}_{io}, \ell_2 \in \Psi(\ell_1) \cap \mathscr{I}_{oi}, \ldots, j \in \Phi(\ell_p) \qquad (\alpha')$$
  (resp. $j \in \Psi(\ell_p)$*).*
  - $j \in \mathscr{I}_{ii}$ *is linked to the index* $\underline{i}$*, if there exists a sequence* $\ell := (\ell_1, \ell_2, \ldots, \ell_p) \in \mathscr{I}^p$ *such that:*
  $$\ell_1 \in \Phi(\underline{i}) \cap \mathscr{I}_{io}, \ell_2 \in \Psi(\ell_2) \cap \mathscr{I}_{oi}, \ldots, \begin{cases} j \in \Psi(\ell_p) & p \text{ odd,} \\ j \in \Phi(\ell_p) & p \text{ even.} \end{cases} \qquad (\beta')$$
- *By convention, we say that every index* $\underline{i} \in \mathscr{I}$ *is linked to itself by the void sequence.*
- *Finally, if* $\underline{i} \in \mathscr{I}_{ev1} \cup \mathscr{I}_{ev2} \cup \mathscr{I}_{g_1} \cup \mathscr{I}_{g_2}$*, then there is no index linked to* $\underline{i}$ *except* $\underline{i}$*.*

With this definition in hand, we can define the notion of type V (for vertical) and type H (for horizontal) sequences. Type V sequences refer to the ones that start by the reflection of an incoming-outgoing phase (namely $\underline{i}$) against the side $\partial\Omega_2$, which is reflected into the outgoing-incoming phase ℓ_1 against $\partial\Omega_2$, and so on until we reach j. Type H sequences that we encountered in [2] will not be used in the following. They refer to sequences that start by the reflection of the outgoing-incoming phase $\underline{i}$ against the side $\partial\Omega_2$, reflected against $\partial\Omega_1$ into the incoming-outgoing phase ℓ_1, and so on until we obtain the phase j.

Definition 4.5 (Type V and type H sequences). — *Let* $i \in \mathscr{I}$ *and* $j \in \mathscr{I}$ *be linked to* i *in the sense of Definition 4.4. We say that the index* $j \in \mathscr{I}$ *is linked to the index* $\underline{i} \in \mathscr{I}$ *by a type* V *(resp.* H*) sequence and we denote* $i \underset{V}{\rightarrowtail} j$ *(resp.* $i \underset{H}{\rightarrowtail} j$*) if the sequence* $(\underline{i}, \ell, j)$ *where* ℓ *is given by Definition 4.4 satisfies* (α) *or* (β) *(resp.* (α') *or* (β')*).*

The following proposition also comes from [2], it is independent of self-interaction so the proof is omitted here. It asserts that the set of indices is the one obtained if one considers all the linked indices to the phases that have been turned on by the source term g^ε. More precisely

Proposition 4.6. — *Let* $\mathscr{F}$ *be a complete for reflections set of frequencies indexed by the set of indices* $\mathscr{I}$*. We introduce* $\mathscr{I}_0$ *the set of indices turned on by the source* g^ε*, that is to say*

$$\mathscr{I}_0 := \Big\{ i \in \mathscr{I}_{io} \cup \mathscr{I}_{ii} \cup \mathscr{I}_{ev1} \cup \mathscr{I}_{g_1} \setminus \det \mathscr{L}\Big(\underline{\tau}, \xi_1^i, \underline{\xi}_2\Big) = 0 \Big\},$$

and $\mathscr{I}_{\mathscr{R}}$ the set of indices linked to the indices in $\mathscr{I}_0$ in the sense of Definition 4.4. Then we have $\mathscr{I} = \mathscr{I}_{\mathscr{R}}$.

At this stage of the description we have justified that any index in $\mathscr{I}$ is in fact linked to one of the phases turned on by the source term. In the following, we will need to be a little more precise about how (or how many times) any index $\underline{i}$ is linked to the "initial" indices. This refinement will be more convenient to state once the notion of self-interacting indices (or loop) has been introduced. This is why in the following paragraphs we state our assumptions governing loops, first by assuming the uniqueness of the self-interaction loop in Subsection 4.2 and then, by considering multiple self-interaction loops in Subsection 4.3.

4.2. Unique self-interaction loop

Let us recall (and modify a little) the notion of loop from [2]. Loops arise when in the section of the characteristic variety $\mathscr{V}$, one can find at least a rectangle and more generically some kind of "stair" whose vertex are in $\mathscr{V}$ and have suitable group velocities (that is to say that the group velocity changes from one vertex to the other from incoming-outgoing to outgoing-incoming) (we refer to Figures 3.1, 3.2 or 3.3).

More precisely we define

Definition 4.7 (Loops). — *Let $i \in \mathscr{I}$, $p \in 2\mathbb{N}+1$ and $\ell := (\ell_1, \ell_2, \dots, \ell_p)$ a sequence of elements of $\mathscr{I}$, we say that*

- *the index $i \in \mathscr{I}$ admits (i, ℓ, i) as a loop if ℓ satisfies*
  $$\ell_1 \in \Phi(i), \ell_2 \in \Psi(\ell_1), \dots, i \in \Psi(\ell_p),$$
  and if moreover the sequence (i, ℓ) does not contain any subsequence periodically repeated.
- *An index $i \in \mathscr{I}_{io}$ (resp. $i \in \mathscr{I}_{oi}$) admits a self-interaction loop if it admits a loop and if moreover the sequence (i, ℓ, i) is of type V (resp. H) in the sense of Definition 4.5.*
- *The self-interaction loop (i, ℓ, i) is said to be simple if the above sequence ℓ is unique. Contrarily, if there exist several sequences such that the above hold, then the loop is said to be complex or composite.*

We note that compared to the definition of loops given in [2], the requirement that the sequence ℓ does not contain periodically repeated subsequences which was referred as simple loop is now stated in the definition of a loop. This requirement is made to avoid to have to consider all the sequences of the form $(i, \ell, i, \ell, i \dots)$ which naturally appear if (i, ℓ, i) is a loop. In the following we will always assume that if $\ell' = (\ell'_1, \dots, \ell'_q)$ is a sequence containing one of the indices of the loop, namely i, (the loop sequence being here denoted by ℓ) at some position, let us say p, then ℓ' has been simplified from $(\ell'_1, \ell'_{p-1}, i, \ell, i, \dots, i, \ell, i, \ell'_{p+1}, \dots, \ell'_q)$ into ℓ'.

Compared to [2] the term "simple" now refers to self-interaction loops for which there exists a unique way to regenerate an index of the loop by repeated reflections against the sides of the quarter-space. We believe that this new use of the word "simple" is more meaningful than in [2]. We have good reasons to believe that composite loops could also be considered with the suitable adaptations of the proofs.

In the remaining of the article we assume that self-interaction loops are always simple in the sense of Definition 4.7. This has the advantage to simplify the analysis. In Section 8 we give an example of system admitting a composite loop. A complete analysis for composite loops is however behind of the scope of this article and it is left for future studies. But let us stress that non self-interaction loops are authorized to be composite.

In this paragraph we assume for simplicity that there exists a unique self-interaction loop. This loop is assumed to be simple, of full size $p \in 2\mathbb{N}$, while it was only of size four in [2]. However, the main difference with [2] is not the size of the loop, it is that Assumption 4.8 authorizes non self-interacting loops that were excluded in [2]. Thus the applicability of the result is wilder. More precisely, we assume the following

Assumption 4.8 (Uniqueness of the self-interaction loop). — *The frequencies set $\mathscr{F}$ indexed by $\mathscr{I}$ admits a unique self-interaction loop of size $p \in 2\mathbb{N}$. This loop is simple in the sense of Definition 4.7. That is to say that the following properties are satisfied:*

(1) *there exists $s_1 \in \mathscr{I}_{io}$ and a unique sequence $s := (s_2, \dots, s_p)$, such that*

$$\forall \; q \in [\![1, p-1]\!] \; s_{2q+1} \in \mathscr{I}_{io} \quad \text{and} \quad \forall \, q \in [\![1, p]\!] \; s_{2q} \in \mathscr{I}_{oi},$$

and

$$s_{2p} \in \Phi(s_1), \; s_{2p-1} \in \Psi(s_{2p}), \; s_{2p-2} \in \Phi(s_{2p-1}), \dots, \; s_1 \in \Psi(s_2),$$

that is to say that $(s_1, s_2, \dots, s_{2p}, s_1)$ is a simple self-interaction loop.

(2) *Let $i \in \mathscr{I}$ be an index admitting a self-interaction loop with sequence $\ell := (\ell_1, \dots \ell_{2q-1})$, then $q = p$ and moreover $\{i, \ell\} = \{s\}$.*

The following proposition is the keystone of the order of determination of the amplitudes in the geometric optics cascade of equations. It claims that every index in $\mathscr{I}$ is linked to the "first" self-interaction index s_1, that is to say the self-interaction index which is turned on by the source term g^ε. In the following, for simplicity, we assume that such self-interaction phenomenon is directly turned on by the source. As a consequence, we choose a source term reading under the form

$$g^\varepsilon(t,x_2) := e^{\frac{i}{\varepsilon}\left(\underline{\tau}t+\underline{\xi}_2^{s_1}x_2\right)} g(t,x_2), \tag{4.1}$$

s_1 being the index of the self-interaction loop given in Assumption 4.8, so that $(i\underline{\tau}, \underline{\xi}_1^{s_1}, \underline{\xi}_2^{s_1}) \in \Xi_0$ is the frequency associated to s_1. This simplifying assumption can however easily be removed (up to the price of the resolution of some extra transport equations).

The proposition of interest is thus the following.

Proposition 4.9. — *For all $i \in \mathscr{I}$, there exists* at least *one type V sequence linking i to s_1. We write $s_1 \underset{V}{\rightarrowtail} i$.*

The proof of the above property is independent of the possible (multiple) loops that we are considering (see [2]) in the section of the characteristic variety. The only difference is that in [2] due to the uniqueness of the loop assumption (that is that we exclude, in particular, non self-interaction loops) by working a little more we can in fact show the uniqueness of the type V sequence. Such a uniqueness then define a natural order to determine the amplitude associated to i.

When non self-interaction loops occur then the above uniqueness of the type V sequence clearly breaks down as we can see on Figure 3.4. Thus we can have uniqueness of the type V sequence or not. This will however still give the order of determination of the amplitude. We first determine the elements for which this uniqueness property holds, then we determine the indices which are linked by two distinct type V sequences and so on.

4.3. Multiple self-interaction loops

In this paragraph we state the assumption dealing with loops in Theorem 2.12. As already mentioned we authorize multiple self-interaction loops but, for simplicity, we require that all these loops are simple in the sense of Definition 4.7. The assumption is then the following:

Assumption 4.10 (Multiple self-interaction loops). — *The frequencies set $\mathscr{F}$ indexed by $\mathscr{I}$ admits $A \in \mathbb{N}$ self-interaction loop each of size $2b_a \in \mathbb{N}$, $b_a \geqslant 2$ and $a \in [\![1, A]\!]$. These loops are simple in the sense of Definition 4.7. That is to say that the following properties are satisfied: we denote by $\mathscr{S}$ the set of self-interacting indices of $\mathscr{I}$. Then*

(1) *for all* $a \in [\![1, A]\!]$, *there exists* $b_a \geqslant 2$, $s_1^a \in \mathscr{I}_{io}$ *and a unique sequence* $s^a := (s_2^a, \dots, s_{2b_a}^a)$ *such that*

$$\forall \ q \in [\![1, 2b_a - 1]\!] \ s_{2q+1}^a \in \mathscr{I}_{io} \quad \text{and} \quad \forall \ q \in [\![1, 2b_a]\!], \quad s_{2q}^a \in \mathscr{I}_{oi},$$

and

$$s_{2b_a}^a \in \Phi(s_1^a), \ s_{2b_a-1}^a \in \Psi(s_{2b_a}^a), \ s_{2b_a-2}^a \in \Phi(s_{2b_a-1}^a), \dots, \ s_1^a \in \Psi(s_2^a),$$

that is to say that $(s_1^a, s_2^a, \dots, s_{2b_a}^a, s_1^a)$ *is a simple self-interaction loop. Let* $\{s^a\} := \{s_1^a, \dots, s_{2b_a}^a\}$, *we then have that the* $\{s^a\}_{a \in [\![1,A]\!]}$ *form a partition of* $\mathscr{S}$.

(2) *Let* $i \in \mathscr{I}$ *be an index admitting a self-interaction loop with sequence* $\ell := (\ell_1, \dots, \ell_{2q-1})$, *then there exists* $\underline{a} \in [\![1, A]\!]$ *such that* $q = 2b_{\underline{a}}$ *and moreover* $\{i, \ell\} = \{s^{\underline{a}}\}$.

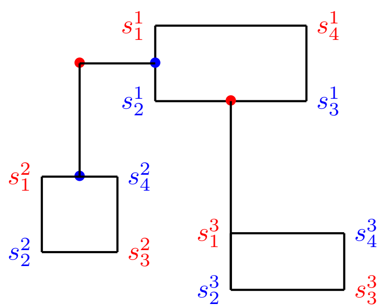

Figure 4.2. An example with several self-interaction loops.

As a consequence, we may now have several self-interaction loops in $\mathscr{V}$, but all these loops need to be separated the ones from the others by some indices which are not self-interacting. We depict this configuration on Figure 4.2. Figure 8.4 gives an example of a characteristic variety $\mathscr{V}$ for which Assumption 4.10 fails.

In such a framework, we assume that the source term g^ε turns on one of the loops elements. We note this index s_1^1 associated to the frequency $(i\underline{\tau}, \underline{\xi}_1^{s_1^1}, \underline{\xi}_2^{s_1^1})$. The source thus reads under the form

$$g^\varepsilon(t, x_2) := e^{\frac{i}{\varepsilon}\left(\underline{\tau}t + \underline{\xi}_2^{s_1^1} x_2\right)} g,$$

where the amplitude $g \in H_\natural^\infty(\mathbb{R} \times \mathbb{R}_+)$ vanishes for negative times.

Under the structure Assumption 4.10 because the proof of Proposition 4.9 is independent of the number of loops (we stress that we do not have the uniqueness of the sequence), this proposition still holds. The number of type V sequences linking an index $\underline{i}$ to s_1^1 will still give a natural order of determination of the amplitudes. However, compared to the uniqueness framework of Assumption 4.8, the order must be refined in the following way:

(1) We first determine the indices linked to s_1^1 by types V sequence(s) which does not contain any self-interaction indices differing from the one of the first loop $\{s^1\}$. We reproduce here the order used under Assumption 4.8. With more details we start by the indices linked by a unique type V sequence, then we proceed to those linked by two sequences and so on.
(2) In a second time, we determine the indices linked to s_1^1 by several type V sequences, but where only one of them contains self-interacting indices differing from the ones of $\{s^1\}$. Let us note here that this sequence is authorized to visit several loops.
(3) We then conclude inductively by determing the indices linked to s_1^1 by several loops, but where two of them contain self-interacting indices differing from the ones of $\{s^1\}$ (not necessarily the same) and so on.

4.4. Some notation to conclude

We conclude the present section with some notation that will be intensively used in the construction of the geometric optics expansions:

- For an index $\underline{i} \in \mathscr{I}$ we write $\Phi^*(\underline{i})$ (resp. $\Psi^*(\underline{i})$) for the sets $\Phi(\underline{i})\setminus\{\underline{i}\}$ (resp. $\Psi(\underline{i}) \setminus \{\underline{i}\}$).
- For an index $\underline{i} \in \mathscr{I}$ associated to some boundary frequency $f^{\underline{i}} := (i\underline{\tau}, \underline{\xi}_1^{\underline{i}}, \underline{\xi}_2^{\underline{i}})$, we will use $\phi_1^{\underline{i}}$ (resp. $\phi_2^{\underline{i}}$) as a shorthand notation for $\phi_1(i\underline{\tau}, \underline{\xi}_2^{\underline{i}})$ (resp. $\phi_2(i\underline{\tau}, \underline{\xi}_2^{\underline{i}})$), where we recall that ϕ_1 and ϕ_2 are the inverses given by the uniform Kreiss–Lopatinskii condition (see Assumption 2.11).
- Then we define the following sets and relations on the set of indices in order to state the ansatz properly. We choose to follow the method of [11] and we aim to treat the evanescent modes in a "monoblock" way. More precisely if $\underline{i} \in \mathscr{I}_{ev1}$ (resp. $\underline{i} \in \mathscr{I}_{ev2}$), then all the indices $j \in \Phi(\underline{i}) \cap \mathscr{I}_{ev1}$ (resp. $j \in \Psi(\underline{i}) \cap \mathscr{I}_{ev2}$) contribute to the same amplitude. This permits to avoid the resolution of transport equation with complex coefficients of [17]. To do so, we first need to define some equivalence relations on the set of indices which regroup the elements in terms of their coordinates in $\mathscr{V}$. From Proposition 4.3,

we can define the equivalence relations on $\mathscr{I}\times\mathscr{I}$

$$i \underset{\Phi}{\sim} j \Leftrightarrow j\in\Phi(i) \quad\text{and}\quad i \underset{\Psi}{\sim} j \Leftrightarrow j\in\Psi(i).$$

We define $\mathfrak{C}_1$ (resp. $\mathfrak{C}_2$) the set of equivalence classes for the relation $\underset{\Phi}{\sim}$ (resp. $\underset{\Psi}{\sim}$) and $\mathscr{R}_1$ (resp. $\mathscr{R}_2$) a set of class representative for $\mathfrak{C}_1$ (resp. $\mathfrak{C}_2$).

As a consequence, $\mathscr{R}_1$ (resp. $\mathscr{R}_2$) is a set of indices describing all the possible values of the ξ_2 (resp. ξ_1) appearing in $\mathscr{F}$. We end up with the definition of the values of the ξ_1 and of the ξ_2 which give rise to some evanescent modes. More precisely we define

$$\mathfrak{R}_1 := \{i\in\mathscr{R}_1 \setminus \Phi(i)\cap\mathscr{I}_{ev1}\neq\emptyset\} \quad\text{and}\quad \mathfrak{R}_2 := \{i\in\mathscr{R}_2 \setminus \Psi(i)\cap\mathscr{I}_{ev2}\neq\emptyset\}.$$

- Finally we define the following projections that will be intensively used in the following:

Definition 4.11. — *Let $\underline{\zeta}$ let a placeholder for a boundary frequency. We define*

- *For $k\in\mathscr{I}_{\text{hyp}}\cup\mathscr{I}_{g_1}\cup\mathscr{I}_{g_2}$ we introduce $\Pi^k=\Pi^k(\underline{\zeta})$ the projection on $\ker\mathscr{L}(d\varphi_k)$ with respect to the decomposition* (2.9)*.*
- *Let $k\in\mathfrak{R}_1$ (resp. $k\in\mathfrak{R}_2$) we introduce $\Pi^k_{s,1}=\Pi^k_{s,1}(\underline{\zeta})$ (resp. $\Pi^k_{s,2}:=\Pi^k_{s,2}(\underline{\zeta})$) the projection on $\mathbf{E}_1^{s,e}(\underline{\zeta})$ (resp. $\mathbf{E}_2^{s,e}(\underline{\zeta})$) with respect to the decomposition* (2.9)*.*
- *For $k\in\mathscr{I}_{\text{hyp}}\cup\mathscr{I}_{g_1}\cup\mathscr{I}_{g_2}$ we define the pseudo-inverse Υ^k of $\mathscr{L}(\underline{\tau},\xi_1^k,\xi_2^k)$ characterized by the relations*

$$\begin{cases}\Upsilon^k\mathscr{L}\big(\underline{\tau},\xi_1^k,\xi_2^k\big)=I-\Pi^k,\\ \Pi^k\Upsilon^k=\Upsilon^k\Pi^k\;=0.\end{cases}$$

5. Construction of the geometric optics expansion under Assumption 4.8

This section is devoted to the construction of geometric optics expansions when we have a unique self-interaction loop in $\mathscr{V}$, that is to say that Assumption 4.8 holds. It thus gives the main part of the proof of Corollary 2.13: the construction part. The justification part is then postponed to Section 7.

5.1. The ansatz and the cascade of equations

With such definitions in hand we choose for an ansatz the following expansion

$$\begin{aligned} u^\varepsilon(t,x) \sim \sum_{n\geqslant 0} \sqrt{\varepsilon}^n \sum_{k\in\mathscr{I}_{\mathrm{hyp}}} & e^{\frac{i}{\varepsilon}\varphi_k(t,x)} u_{n,k}(t,x) \\ & + \sum_{n\geqslant 0} \sqrt{\varepsilon}^n \sum_{k\in\mathscr{I}_{g_1}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} u_{n,k}\left(t,x_2,\frac{x_1}{\sqrt{\varepsilon}}\right) \\ & + \sum_{k\in\mathscr{I}_{g_2}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} u_{n,k}\left(t,x_1,\frac{x_2}{\sqrt{\varepsilon}}\right) \\ & + \sum_{n\geqslant 0} \sqrt{\varepsilon}^n \sum_{k\in\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}(t,x_2)} U_{n,k,1}\left(t,x_2,\frac{x_1}{\varepsilon}\right) \\ & + \sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}(t,x_1)} U_{n,k,2}\left(t,x_1,\frac{x_2}{\varepsilon}\right), \end{aligned} \tag{5.1}$$

where the phases functions are defined by

$$\forall\ k\in\mathscr{I}_{\mathrm{hyp}}\cup\mathscr{I}_{g_1}\cup\mathscr{I}_{g_2},\ \varphi_k(t,x) := \underline{\tau}t+\xi_1^k x_1+\xi_2^k x_2,$$

and

$$\forall\ k\in\mathfrak{R}_1,\ \psi_{k,1}(t,x_2) := \underline{\tau}t+\xi_2^k x_2 \quad \text{and} \quad \forall\ k\in\mathfrak{R}_2,\ \psi_{k,2}(t,x_1) := \underline{\tau}t+\xi_1^k x_1.$$

In the ansatz (5.1) we aim to construct the hyperbolic profiles, namely the $u_{n,k}$ for $k\in\mathscr{I}_{\mathrm{hyp}}$, in the Sobolev space $H^\infty(\Omega)$. We also want them to satisfy some flatness properties for their traces.

The boundary layers linked to the evanescent modes, namely the terms $U_{n,k,1}$ and $U_{n,k,2}$, and glancing modes, namely the $u_{n,k}$ for $k\in\mathscr{I}_{g_1}\cup\mathscr{I}_{g_2}$, will be functions with fast decay with respect to the last variable. More precisely, we introduce the following set of profiles

Definition 5.1 (Boundary layers profiles). — *For $p=1,2$, the set of evanescent and glancing profiles $\mathbf{P}_p$ for the side $\partial\Omega_p$ is defined as the set of functions $f(t,x_{3-p},Y_p)\in H^\infty(\partial\Omega_p\times\mathbb{R}_+)$ for which we can find $\delta_p>0$ such that $e^{\delta_p Y_p} f(t,x_{3-p},Y_p)\in H^\infty(\partial\Omega_p\times\mathbb{R}_+)$.*

As we can see in (5.1), the ansatz includes two different scales for boundary layers. In the following for $p=1,2$, we will denote by $\chi_p := \frac{x_p}{\sqrt{\varepsilon}}$ the "fast" boundary layer scale, associated to glancing modes, and by $X_p := \frac{x_p}{\varepsilon}$ the "slow" one, describing evanescent modes.

We inject the ansatz in the evolution equation of (1.1), we identify in terms of powers of ε and we use the fact that the phases are linearly independent to decouple the equations. It leads us to essentially the same cascade of equations as in [17] (up to the treatment of the evanescent modes). More precisely we obtain

$$\begin{cases} \mathscr{L}(d\varphi_k)u_{0,k} = \mathscr{L}(d\varphi_k)u_{1,k} = 0 & \forall\, k \in \mathscr{I}_{\mathrm{hyp}}, \\ i\mathscr{L}(d\varphi_k)u_{n+2,k} + L(\partial)u_{n,k} = 0 & \forall\, k \in \mathscr{I}_{\mathrm{hyp}}, \\ & \forall\, n \in \mathbb{N}, \\ L_k(\partial_{X_1})U_{0,k,1} = L_k(\partial_{X_1})U_{1,k,1} = 0 & \forall\ k \in \mathfrak{R}_1, \\ L_k(\partial_{X_1})U_{n+2,k,1} + L_1'(\partial)U_{n,k,1} = 0 & \forall\ k \in \mathfrak{R}_1, \\ & \forall\, n \in \mathbb{N}, \\ L_k(\partial_{X_2})U_{0,k,2} = L_k(\partial_{X_2})U_{1,k,2} = 0 & \forall\, k \in \mathfrak{R}_2, \\ L_k(\partial_{X_2})U_{n+2,k,2} + L_2'(\partial)U_{n,k,2} = 0 & \forall\, k \in \mathfrak{R}_2, \\ & \forall\, n \in \mathbb{N}, \\ \mathscr{L}(d\varphi_k)u_{0,k} = i\mathscr{L}(d\varphi_k)u_{1,k} + A_1\partial_{\chi_1}u_{0,k} = 0 & \forall\, k \in \mathscr{I}_{g_1}, \\ i\mathscr{L}(d\varphi_k)u_{n+2,k} + A_1\partial_{\chi_1}u_{n+1,k} + L_1'(\partial)u_{n,k} = 0 & \forall\, k \in \mathscr{I}_{g_1}, \\ & \forall\, n \in \mathbb{N}, \\ \mathscr{L}(d\varphi_k)u_{0,k} = i\mathscr{L}(d\varphi_k)u_{1,k} + A_2\partial_{\chi_2}u_{0,k} = 0 & \forall\, k \in \mathscr{I}_{g_2}, \\ i\mathscr{L}(d\varphi_k)u_{n+2,k} + A_2\partial_{\chi_2}u_{n+1,k} + L_2'(\partial)u_{n,k} = 0 & \forall\, k \in \mathscr{I}_{g_2}, \\ & \forall\, n\, \in \mathbb{N}, \end{cases} \tag{5.2}$$

where the operators of differentiation with respect to the fast variables $L_k(\partial_{X_p})$ are defined by

$$L_k(\partial_{X_1}) := A_1\big(\partial_{X_1} - \mathscr{A}_1(\underline{\tau}, \xi_2^k)\big) \text{ for } k \in \mathfrak{R}_1$$

and

$$L_k(\partial_{X_2}) := A_2\big(\partial_{X_2} - \mathscr{A}_2(\underline{\tau}, \xi^k)\big) \text{ for } k \in \mathfrak{R}_2,$$

where we recall that $\mathscr{A}_p(\cdot)$ stands for the resolvent matrix introduced in (2.3). We also defined the truncated differentiation operators $L_p'(\partial)$ by

$$L_1'(\partial) := \partial_t + A_2\partial_2 \text{ and } L_2'(\partial) := \partial_t + A_1\partial_1.$$

The main difficulty here compared to [17] for the half-space is that the boundary conditions couple the traces of the amplitudes in a rather complicated way. Indeed injecting the ansatz (5.1) in the boundary conditions of the system (1.1) gives the boundary conditions

$$B_1 \left[\begin{array}{l} \displaystyle\sum_{k\,\in\,\mathscr{I}_{\mathrm{hyp}}} e^{\frac{i}{\varepsilon}\left(\underline{\tau}t+\underline{\xi}_2^k x_2\right)} u_{n,k_{|x_1=0}} + \sum_{k\,\in\,\mathscr{I}_{g_1}} e^{\frac{i}{\varepsilon}\left(\underline{\tau}t+\underline{\xi}_2^k x_2\right)} u_{n,k_{|\chi_1=0}} \\ \displaystyle + \sum_{k\,\in\,\mathscr{I}_{g_2}} e^{\frac{i}{\varepsilon}\left(\underline{\tau}t+\underline{\xi}_2^k x_2\right)} u_{n,k_{|x_1=0}} \\ \displaystyle + \sum_{k\,\in\,\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\left(\underline{\tau}t+\underline{\xi}_2^k x_2\right)} U_{n,k,1_{|\chi_1=0}} + \sum_{k\,\in\,\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\underline{\tau}t} U_{n,k,2_{|x_1=0}} \end{array} \right] \\ = \delta_{n,0} e^{\frac{i}{\varepsilon}\left(\underline{\tau}t+\underline{\xi}_2 x_2\right)} g, \qquad (5.3)$$

where $\delta_{\cdot,\cdot}$ stands for Kronecker symbol and

$$B_2 \left[\begin{array}{l} \displaystyle\sum_{k\,\in\,\mathscr{I}_{\mathrm{hyp}}} e^{\frac{i}{\varepsilon}\left(\underline{\tau}t+\underline{\xi}_1^k x_1\right)} u_{n,k_{|x_2=0}} \\ \displaystyle + \sum_{k\,\in\,\mathscr{I}_{g_1}} e^{\frac{i}{\varepsilon}\left(\underline{\tau}t+\underline{\xi}_1^k x_1\right)} u_{n,k_{|x_2=0}} + \sum_{k\,\in\,\mathscr{I}_{g_2}} e^{\frac{i}{\varepsilon}\left(\underline{\tau}t+\underline{\xi}_1^k x_1\right)} u_{n,k_{|\chi_2=0}} \\ \displaystyle + \sum_{k\,\in\,\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\underline{\tau}t} U_{n,k,1_{|x_2=0}} + \sum_{k\,\in\,\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\left(\underline{\tau}t+\underline{\xi}_1^k x_1\right)} U_{n,k,2_{|\chi_2=0}} \end{array} \right] = 0. \qquad (5.4)$$

In particular, the boundary conditions (5.3) and (5.4) are satisfied if we manage to solve the boundary conditions

$$\left\{ \begin{array}{l} B_1 \left[\displaystyle\sum_{k\,\in\,\mathscr{I}_{\mathrm{hyp}}} e^{\frac{i}{\varepsilon}\psi_{1,k}} u_{n,k_{|x_1=0}} + \sum_{k\,\in\,\mathscr{I}_{g_1}} e^{\frac{i}{\varepsilon}\psi_{1,k}} u_{n,k_{|\chi_1=0}} + \sum_{k\,\in\,\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{1,k}} U_{n,k,1_{|\chi_1=0}} \right] \\ \qquad = \delta_{n,0} e^{\frac{i}{\varepsilon}\left(\underline{\tau}t+\underline{\xi}_2 x_2\right)} g, \\ \displaystyle\sum_{k\,\in\,\mathscr{I}_{g_2}} e^{\frac{i}{\varepsilon}\left(\underline{\tau}t+\underline{\xi}_2^k x_2\right)} u_{n,k_{|x_1=0}} = \sum_{k\,\in\,\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\underline{\tau}t} U_{n,k,2_{|x_1=0}} = 0, \end{array} \right. \qquad (5.5)$$

and

$$\left\{ \begin{array}{l} B_2 \left[\displaystyle\sum_{k\,\in\,\mathscr{I}_{\mathrm{hyp}}} e^{\frac{i}{\varepsilon}\psi_{2,k}} u_{n,k_{|x_2=0}} + \sum_{k\,\in\,\mathscr{I}_{g_2}} e^{\frac{i}{\varepsilon}\psi_{2,k}} u_{n,k_{|\chi_2=0}} + \sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{2,k}} U_{n,k,2_{|\chi_1=0}} \right] = 0, \\ \displaystyle\sum_{k\,\in\,\mathscr{I}_{g_1}} e^{\frac{i}{\varepsilon}\left(\underline{\tau}t+\underline{\xi}_1^k x_1\right)} u_{n,k_{|x_2=0}} = \sum_{k\,\in\,\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\underline{\tau}t} U_{n,k,1_{|x_2=0}} = 0. \end{array} \right. \qquad (5.6)$$

Because the phases are linearly independent,[5] the boundary conditions (5.5) and (5.6) amount to solve: $\forall\ n \in \mathbb{N}$,

$$
\begin{cases}
B_1 \left[\sum\limits_{j \in \Phi(s_1) \cap \mathscr{I}_{\mathrm{hyp}}} u_{n,j_{|x_1=0}} + \sum\limits_{j \in \Phi(s_1) \cap \mathscr{I}_{g_1}} u_{n,j_{|X_1=0}} + U_{n,s_1,1_{|X_1=0}} \right] = \delta_{n,0} g \ \text{ if } \ s_1 \in \mathfrak{R}_1, \\
B_1 \left[\sum\limits_{j \in \Phi(s_1) \cap \mathscr{I}_{\mathrm{hyp}}} u_{n,j_{|x_1=0}} + \sum\limits_{j \in \Phi(s_1) \cap \mathscr{I}_{g_1}} u_{n,j_{|X_1=0}} \right] = \delta_{n,0} g \ \text{ if } \ s_1 \notin \mathfrak{R}_1, \\
B_1 \left[\sum\limits_{j \in \Phi(k) \cap \mathscr{I}_{\mathrm{hyp}}} u_{n,j_{|x_1=0}} + \sum\limits_{j \in \Phi(k) \cap \mathscr{I}_{g_1}} u_{n,j_{|X_1=0}} + U_{n,k,1_{|X_1=0}} \right] = 0 \\
\qquad\qquad \forall\ k \in (\mathscr{R}_1 \setminus \{s_1\}) \cap \mathfrak{R}_1, \\
B_1 \left[\sum\limits_{j \in \Phi(k) \cap \mathscr{I}_{\mathrm{hyp}}} u_{n,j_{|x_1=0}} + \sum\limits_{j \in \Phi(k) \cap \mathscr{I}_{g_1}} u_{n,j_{|X_1=0}} \right] = 0 \\
\qquad\qquad \forall\ k \notin (\mathscr{R}_1 \setminus \{s_1\}) \cap \mathfrak{R}_1, \\
B_2 \left[\sum\limits_{j \in \Psi(k) \cap \mathscr{I}_{\mathrm{hyp}}} u_{n,j_{|x_2=0}} + \sum\limits_{j \in \Psi(k) \cap \mathscr{I}_{g_2}} u_{n,j_{|X_2=0}} + U_{n,k,2_{|X_2=0}} \right] = 0\ \forall\ k \in \mathfrak{R}_2, \\
B_2 \left[\sum\limits_{j \in \Psi(k) \cap \mathscr{I}_{\mathrm{hyp}}} u_{n,j_{|x_2=0}} + \sum\limits_{j \in \Psi(k) \cap \mathscr{I}_{g_2}} u_{n,j_{|X_2=0}} \right] = 0\ \forall\ k \notin \mathfrak{R}_2, \\
u_{n,k_{|x_1=0}} = 0\ \forall\ k \in \mathscr{I}_{g_2}, \\
u_{n,k_{|x_2=0}} = 0\ \forall\ k \in \mathscr{I}_{g_1}, \\
U_{n,k,1_{|x_2=0}} = 0\ \forall\ k \in \mathfrak{R}_1, \\
U_{n,k,2_{|x_1=0}} = 0\ \forall\ k \in \mathfrak{R}_2.
\end{cases}
\tag{5.7}
$$

Finally, injecting the ansatz (5.1) in the initial condition leads us to impose the following homogeneous initial conditions

$$
\begin{cases}
u_{n,k_{|t \leqslant 0}} = 0 & \forall\ n \in \mathbb{N},\ \forall\ k \in \mathscr{I}_{\mathrm{hyp}} \cup \mathscr{I}_{g_1} \cup \mathscr{I}_{g_2}, \\
U_{n,k,1_{|t \leqslant 0}} = 0 & \forall\ n \in \mathbb{N},\ \forall\ k \in \mathfrak{R}_1, \\
U_{n,k,2_{|t \leqslant 0}} = 0 & \forall\ n \in \mathbb{N},\ \forall\ k \in \mathfrak{R}_2.
\end{cases}
\tag{5.8}
$$

[5] Let us point that we do not solve here exactly the same boundary conditions as in [2] where in order to use such an independence, we used an extra technical assumption in order to deal with the terms $\sum_{k \in \mathfrak{R}_2} U_{n,k,2_{|x_1=0}}$ and $\sum_{k \in \mathfrak{R}_1} e^{\frac{i}{\varepsilon}\mathcal{T}t} U_{n,k,1_{|x_2=0}}$. Because we impose these terms to vanish in the boundary conditions (5.5) and (5.6), this extra assumption is not required any more. It makes the proof more straightforward.

The aim of the remaining of this section is to show that we can solve the cascades of equations (5.2), (5.7) and (5.8). In order to do so, we first reformulate the interior equations in Subsection 5.2, reformulation in which we pay a special attention to the values of the traces.

The leading order terms are then constructed in Subsection 5.3. Once the leading order terms have been determined, then we can use it in the construction of higher orders terms. This step is really classical, so we feel free to do not give the details.

5.2. Reformulation of the equations

In this paragraph we reformulate the interior cascade of equations (5.2) to determine precisely which trace is required for the determination of each amplitude depending on its kind in the ansatz (5.1).

The reformulation for hyperbolic modes is classical in geometric optics expansions. It relies on Lax lemma [10] and it is made in Subsection 5.2.1. The one for evanescent modes follows the method, based upon Duhamel formula, introduced in [11]. It is described in Subsection 5.2.2. Finally the reformulation of the equations involving glancing modes in (5.2) follows the method of [17]. It is given in Subsection 5.2.3. As a consequence, the reformulations in themselves are rather well-understood and are not new. The main point here is to clearly determine which trace(s) is (are) required for the resolution of the interior equations in the quarter-space geometry.

5.2.1. Reformulation for hyperbolic modes

In this paragraph we consider the equations of (5.2) involving hyperbolic modes. Namely, we consider

$$\begin{cases} \mathscr{L}(d\varphi_k)u_{0,k} = 0 \quad \forall\ k \in \mathscr{I}_{\text{hyp}}, \\ \mathscr{L}(d\varphi_k)u_{1,k} = 0 \quad \forall\ k \in \mathscr{I}_{\text{hyp}}, \\ i\mathscr{L}(d\varphi_k)u_{n+2,k} + L(\partial)u_{n,k} = 0 \quad \forall\ k \in \mathscr{I}_{\text{hyp}}, \forall\ n \in \mathbb{N}. \end{cases} \tag{5.9}$$

The first equations of (5.9) imply that the two first amplitudes for hyperbolic modes are in $\ker \mathscr{L}(d\varphi_k)$. Consequently, we have the well-known polarization conditions:

$$\forall\ k \in \mathscr{I}_{\text{hyp}},\ \Pi^k u_{0,k} = u_{0,k} \quad \text{and} \quad \Pi^k u_{1,k} = u_{1,k}. \tag{5.10}$$

So, a composition of the third equation of (5.9) (written for $n = 0$ and $n = 1$) by Π^k makes the first term vanish and gives

$$\forall\ k \in \mathscr{I}_{\mathrm{hyp}},\ \Pi^k L(\partial)\Pi^k u_{0,k} = \Pi^k L(\partial)\Pi^k u_{1,k} = 0,$$

and we are in position to use Lax lemma which is recalled below for a sake of completeness.

Lemma 5.2 (Lax [10]). — *Under Assumption 2.1, then for all $k \in \mathscr{I}_{\mathrm{hyp}}$, we have the equality*

$$\Pi^k L(\partial)\Pi^k = (\partial_t + \mathbf{v}_k \cdot \nabla_x)\Pi^k,$$

where we recall that $\mathbf{v}_k$ stands for the group velocity introduced in Definition 2.6.

Therefore, as expected, to determine the first orders amplitudes associated to hyperbolic modes we have to solve transport equations. Depending on the kind of the group velocity, these transport equations require boundary condition(s). One on $\partial\Omega_1$ (resp. $\partial\Omega_2$) for incoming-outgoing (resp. outgoing-incoming) modes and one on $\partial\Omega_1$ combined with one on $\partial\Omega_2$ for incoming-incoming modes.

The following proposition then shows that if we know such traces then the transport equations can be explicitly solved by integration along the characteristics in the suitable functional spaces.

Proposition 5.3.

- *Let $k \in \mathscr{I}_{io}$ (resp. $k \in \mathscr{I}_{oi}$) and let $\widetilde{g}$ be a given function in $H^\infty_\natural(\mathbb{R} \times \mathbb{R}_+)$, then the transport equation*

$$\begin{cases} (\partial_t + \mathbf{v}_k \cdot \nabla_x)u = 0 & \text{in } \Omega, \\ u_{|x_1=0} = \widetilde{g} & \text{on } \partial\Omega_1, \\ u_{|t\leqslant 0} = 0 & \text{on } \Gamma, \end{cases}$$
$$\left(\text{resp.} \begin{cases} (\partial_t + \mathbf{v}_k \cdot \nabla_x)u = 0 & \text{in } \Omega, \\ u_{|x_2=0} = \widetilde{g} & \text{on } \partial\Omega_2, \\ u_{|t\leqslant 0} = 0 & \text{on } \Gamma, \end{cases}\right) \tag{5.11}$$

admits a unique solution $u \in H^\infty(\Omega)$ satisfying that $u_{|x_2=0} \in H^\infty_\natural(\mathbb{R} \times \mathbb{R}_+)$ (resp. $u_{|x_1=0} \in H^\infty_\natural(\mathbb{R} \times \mathbb{R}_+)$).

- *Let $k \in \mathscr{I}_{ii}$ and let $(\widetilde{g}_1, \widetilde{g}_2)$ be given in $H^\infty_\natural(\mathbb{R}\times\mathbb{R}_+) \times H^\infty_\natural(\mathbb{R}\times\mathbb{R}_+)$, then the transport equation*

$$\begin{cases} (\partial_t + \mathbf{v}_k \cdot \nabla_x) u = 0 & \textit{in } \Omega, \\ u_{|x_1=0} = \widetilde{g}_1 & \textit{on } \partial\Omega_1, \\ u_{|x_2=0} = \widetilde{g}_2 & \textit{on } \partial\Omega_2, \\ u_{|t\leqslant 0} = 0 & \textit{on } \Gamma, \end{cases}$$

admits a unique solution $u \in H^\infty(\Omega)$. It moreover satisfies $u_{|x_1=0}$, $u_{|x_2=0} \in H^\infty_\natural(\mathbb{R}\times\mathbb{R}_+)$.

Proof. — We consider the case of an incoming-outgoing amplitude. The case of an outgoing-incoming amplitude being similar. We integrate the equation along the characteristics to obtain that

$$u(t,x) = \widetilde{g}\left(t - \frac{1}{\mathbf{v}_{k,1}} x_1, x_2 - \frac{\mathbf{v}_{k,2}}{\mathbf{v}_{k,1}} x_1\right). \tag{5.12}$$

We can read on equation (5.12) that $u \in H^\infty(\Omega)$. Moreover it vanishes for $t \leqslant \frac{x_1}{\mathbf{v}_{k,1}}$. The right-hand side of the above inequality being positive because $k \in \mathscr{I}_{io}$. Then we have

$$u_{|x_2=0}(t,x_1) = \widetilde{g}\left(t - \frac{1}{\mathbf{v}_{k,1}} x_1, -\frac{\mathbf{v}_{k,2}}{\mathbf{v}_{1,k}} x_1\right),$$

so that for $n \geqslant 0$, $\partial_1^n u_{|x_2=0}$ reads

$$\partial_1^n u_{|x_2=0}(t,x_1) = \left[(-1)^n \sum_{p=0}^{n} \binom{n}{p} \left(\frac{1}{\mathbf{v}_{k,1}}\right)^{n-p} \left(\frac{\mathbf{v}_{k,2}}{\mathbf{v}_{k,1}}\right)^p \partial_t^{n-p} \partial_2^p \widetilde{g}\right] \left(t - \frac{1}{\mathbf{v}_{k,1}} x_1, -\frac{\mathbf{v}_{k,2}}{\mathbf{v}_{1,k}} x_1\right).$$

When evaluated at $x_1 = 0$, all the terms in the sum vanish for $p \neq n$, because $\widetilde{g} \in H^\infty_\natural(\mathbb{R}\times\mathbb{R}_+)$. For $n = p$, we thus have $(\partial_t^n \widetilde{g}(t - \frac{1}{\mathbf{v}_{k,1}} x_1, -\frac{\mathbf{v}_{k,2}}{\mathbf{v}_{k,1}} x_1))_{|x_1=0} = \partial_t^n \widetilde{g}(t,0) = 0$, because $\widetilde{g} \in H^\infty_\natural(\mathbb{R}\times\mathbb{R}_+)$. It shows the first claim.

We now consider the incoming-incoming transport equation. By linearity we decompose the solution $u = u_1 + u_2$, where u_1 and u_2 solve respectively the boundary value problems

$$\begin{cases} (\partial_t + \mathbf{v}_k \cdot \nabla_x) u_1 = 0 & \text{in } \Omega, \\ {u_1}_{|x_1=0} = \widetilde{g}_1 & \text{on } \partial\Omega_1, \\ {u_1}_{|x_2=0} = 0 & \text{on } \partial\Omega_2, \\ {u_1}_{|t\leqslant 0} = 0 & \text{on } \Gamma, \end{cases} \quad \text{and} \quad \begin{cases} (\partial_t + \mathbf{v}_k \cdot \nabla_x) u_2 = 0 & \text{in } \Omega, \\ {u_2}_{|x_1=0} = 0 & \text{on } \partial\Omega_1, \\ {u_2}_{|x_2=0} = \widetilde{g}_2 & \text{on } \partial\Omega_2, \\ {u_2}_{|t\leqslant 0} = 0 & \text{on } \Gamma. \end{cases}$$

Following the analysis of the incoming-outgoing case we obtain

$$u(t,x) = \overline{\widetilde{g}_1}\left(t - \frac{1}{\mathbf{v}_{k,1}}x_1, x_2 - \frac{\mathbf{v}_{k,2}}{\mathbf{v}_{k,1}}x_1\right) + \overline{\widetilde{g}_2}\left(t - \frac{1}{\mathbf{v}_{k,2}}x_2, x_1 - \frac{\mathbf{v}_{k,1}}{\mathbf{v}_{k,2}}x_2\right),$$

where $\overline{\widetilde{g}_i}$ stands for the extension of $\widetilde{g}_i$ by zero for $x_{3-i} < 0$. Let us point that because the boundary terms are in $H^\infty_\natural$ such extensions are in H^∞. The traces regularity is then obtained as in the incoming-outgoing case so that we omit the details here. □

For later purposes, we also describe the required modifications to determine the hyperbolic amplitudes of order two and more. The main difference for these terms is that they are not polarized any more. However in a classical setting the unpolarized part depends explicitly on the previous term in the expansion. Indeed, we apply the pseudo-inverse Υ^k (see Definition 4.11) to the third equation of (5.9) (written for $n = 1$). We obtain that

$$(I - \Pi^k)u_{2,k} = i\Upsilon^k L(\partial)u_{1,k}. \tag{5.13}$$

Then, we use this relation by writing $u_{2,k} = \Pi^k u_{2,k} + (I - \Pi^k)u_{2,k}$ in the third equation of (5.9) (written for $n = 2$), we apply Π^k and Lax lemma [10] to obtain that the same, up to a non vanishing source term in the interior, transport equation as before, determines the polarized part $\Pi^k u_{2,k}$:

$$(\partial_t + \mathbf{v}_k \cdot \nabla_x)\Pi^k u_{2,k} = -i\Pi^k L(\partial)\Upsilon^k L(\partial)u_{1,k}. \tag{5.14}$$

The same relations hold at any order $n \geqslant 2$. Depending on the kind of the group velocity $\mathbf{v}_k$, we thus have to consider the transport equation (5.14) with boundary condition(s). This is the subject of the following proposition

Proposition 5.4.

- *Let $k \in \mathscr{I}_{io}$ (resp. $k \in \mathscr{I}_{oi}$) and let $\widetilde{f}, \widetilde{g}$ be given functions in $H^\infty_\natural(\mathbb{R} \times \mathbb{R}_+)$, then the transport equation:*

$$\begin{cases} (\partial_t + \mathbf{v}_k \cdot \nabla_x)u = \widetilde{f}\Big(t - \frac{1}{\mathbf{v}_{k,1}}x_1, x_2 - \frac{\mathbf{v}_{k,2}}{\mathbf{v}_{k,1}}x_1\Big) & \text{in } \Omega, \\ u_{|x_1=0} = \widetilde{g} & \text{on } \partial\Omega_1, \\ u_{|t\leqslant 0} = 0 & \text{on } \Gamma, \end{cases}$$

$$\left(\text{resp.} \begin{cases} (\partial_t + \mathbf{v}_k \cdot \nabla_x)u = \widetilde{f}\Big(t - \frac{1}{\mathbf{v}_{k,2}}x_2, x_1 - \frac{\mathbf{v}_{k,1}}{\mathbf{v}_{k,2}}x_2\Big) & \text{in } \Omega, \\ u_{|x_2=0} = \widetilde{g} & \text{on } \partial\Omega_2, \\ u_{|t\leqslant 0} = 0 & \text{on } \Gamma, \end{cases}\right) \tag{5.15}$$

admits a unique solution $u \in H^\infty(\Omega)$ satisfying that $u_{|x_2=0} \in H^\infty_\natural(\mathbb{R} \times \mathbb{R}_+)$ (resp. $u_{|x_1=0} \in H^\infty_\natural(\mathbb{R} \times \mathbb{R}_+)$).

- *Let $k \in \mathscr{I}_{ii}$ and let $(\widetilde{f}_1, \widetilde{f}_2, \widetilde{g}_1, \widetilde{g}_2)$ be given functions in $H^\infty_\natural(\mathbb{R} \times \mathbb{R}_+)^4$, then the transport equation*

$$\begin{cases} (\partial_t + \mathbf{v}_k \cdot \nabla_x) u = \widetilde{f} & \text{in } \Omega, \\ u_{|x_1=0} = \widetilde{g}_1 & \text{on } \partial\Omega_1, \\ u_{|x_2=0} = \widetilde{g}_2 & \text{on } \partial\Omega_2, \\ u_{|t \leqslant 0} = 0 & \text{on } \Gamma, \end{cases}$$

with

$$\widetilde{f}(t,x) := \widetilde{f}_1\left(t - \frac{1}{\mathbf{v}_{k,1}} x_1, x_2 - \frac{\mathbf{v}_{k,2}}{\mathbf{v}_{k,1}} x_1\right) + \widetilde{f}_2\left(t - \frac{1}{\mathbf{v}_{k,2}} x_2, x_1 - \frac{\mathbf{v}_{k,1}}{\mathbf{v}_{k,2}} x_2\right),$$

admits a unique solution $u \in H^\infty(\Omega)$. Moreover it satisfies $u_{|x_1=0}$, $u_{|x_2=0} \in H^\infty_\natural(\mathbb{R} \times \mathbb{R}_+)$.

Proof. — For the first statement we only consider the incoming-outgoing framework. Once again integrating along the characteristics gives, because of the special form of the interior source, the explicit formula

$$u(t,x) = \widetilde{g}\left(t - \frac{1}{\mathbf{v}_{k,1}} x_1, x_2 - \frac{\mathbf{v}_{k,2}}{\mathbf{v}_{k,1}} x_1\right) + \frac{x_1}{\mathbf{v}_{k,1}} \widetilde{f}\left(t - \frac{1}{\mathbf{v}_{k,1}} x_1, x_2 - \frac{\mathbf{v}_{k,2}}{\mathbf{v}_{k,1}} x_1\right). \tag{5.16}$$

The regularity of u and the one of its trace $u_{|x_2=0}$ are then obtained from (5.16) exactly as in the proof of Proposition 5.3.

To treat the incoming-incoming case, as in the proof of Proposition 5.3, we decompose $u := u_1 + u_2$, where u_i is a solution to the transport equation associated to the sources $(\widetilde{f}_i, \widetilde{g}_i)$ (the boundary condition on $\{x_{3-i} = 0\}$ being homogeneous) and then we use the previous analysis. The details are omitted here. □

As a consequence, the previous discussion states that to determine the values of the hyperbolic amplitudes (at any order), it is in fact sufficient to determine the values of the suitable traces. We thus sum up the previous discussion in the following proposition

Proposition 5.5. — *To solve the cascade of equation* (5.9)*, that is to say to determine the hyperbolic contribution in the cascade* (5.2)*, it is sufficient to determine the following traces: for all $n \geqslant 0$,*

- *The trace $u_{n,k_{|x_1=0}}$ if $k \in \mathscr{I}_{io}$.*
- *The trace $u_{n,k_{|x_2=0}}$ if $k \in \mathscr{I}_{oi}$.*
- *The traces $u_{n,k_{|x_1=0}}$ and $u_{n,k_{|x_2=0}}$ if $k \in \mathscr{I}_{ii}$.*

5.2.2. Reformulation for evanescent modes

In this paragraph we show that to determine the evanescent modes for $\partial\Omega_1$ (resp. $\partial\Omega_2$) it is sufficient to determine the value of their traces on $\partial\Omega_1$ (resp. $\partial\Omega_2$). We recall that for evanescent modes we have to solve the equations:

$$\text{for } p=1,2 \begin{cases} L_k(\partial_{X_p})U_{0,k,p}=0 & \forall\ k\in\mathfrak{R}_p, \\ L_k(\partial_{X_p})U_{1,k,p}=0 & \forall\ k\in\mathfrak{R}_p, \\ L_k(\partial_{X_p})U_{n+2,k,p}+L'_p(\partial)U_{n,k,p}=0 & \forall\ k\in\mathfrak{R}_p, \\ & \forall\ n\in\mathbb{N}, \end{cases} \tag{5.17}$$

The main point is the following lemma due to [11].

Lemma 5.6 (Lescarret [11]). — *Let $p=1,2$ and $\underline{k}\in\mathfrak{R}_p$, we define*

$$\mathbf{P}^{\underline{k}}_{ev,p}U(X_p):=e^{X_p\mathscr{A}_p\left(\underline{\tau},\xi^{\underline{k}}_{3-p}\right)}\Pi^{\underline{k}}_{s,p}U(0), \tag{5.18}$$

$$\begin{aligned}\mathbf{Q}^{\underline{k}}_{ev,p}U(X_p):=&\int_0^{X_p}e^{(X_p-y)\mathscr{A}_p\left(\underline{\tau},\xi^{\underline{k}}_{3-p}\right)}\Pi^{\underline{k}}_{s,p}A_p^{-1}F(y)\,\mathrm{d}y\\ &-\int_{X_p}^{\infty}e^{(X_p-y)\mathscr{A}_p\left(\underline{\tau},\xi^{\underline{k}}_{3-p}\right)}\Pi^{\underline{k}}_{u,p}A_p^{-1}F(y)\,\mathrm{d}y.\end{aligned} \tag{5.19}$$

Then for all $F\in\mathbf{P}_p$ the equation

$$L_{\underline{k}}(\partial_{X_p})U=F,$$

admits a unique solution $U\in\mathbf{P}_p$. It reads under the form

$$U=\mathbf{P}^{\underline{k}}_{ev,p}U+\mathbf{Q}^{\underline{k}}_{ev,p}F.$$

As a consequence, the two first equations of (5.17) imply that we have a kind of polarization condition for the first evanescent modes. They verify $U_{0,k,p}=\mathbf{P}^{k}_{ev,p}U_{0,k,p}$ and $U_{1,k,p}=\mathbf{P}^{k}_{ev,p}U_{1,k,p}$. From the definition of the operator $\mathbf{P}^{k}_{ev,p}$, it is thus sufficient to determine the value of the trace on $\{X_p=0\}$. This trace corresponds to $\{x_p=0\}$.

For higher order evanescent amplitudes, the third equation of (5.17) combined with Lemma 5.6, shows that these amplitudes read under the form

$$\forall\ n\geqslant 2,\ U_{n,k,p}=\mathbf{P}^{\underline{k}}_{ev,p}U_{n,k,p}-\mathbf{Q}^{\underline{k}}_{ev,p}L'_p(\partial)U_{n-2,k,p},$$

where the second term on the right-hand side is a known function. Consequently, to determine the full amplitude $U_{n,k,p}$ it is sufficient to determine $\mathbf{P}^{\underline{k}}_{ev,p}U_{n,k,p}$, that is to say the value of the trace $U_{n,k,p_{|x_p=0}}$.

We sum up the previous discussion in the following proposition

Proposition 5.7. — *To solve the cascade of equations* (5.17) *that is to say to determine the evanescent contributions in the cascade* (5.2)*, it is sufficient to determine the values of the traces* $U_{n,k,p_{|x_p=0}}$ *for* $p = 1, 2$*, for all* $k \in \mathfrak{R}_p$ *and for all* $n \geqslant 0$.

Because we want to solve the extra boundary conditions ensuring that all evanescent modes verify $U_{n,k,1_{|x_2=0}} = U_{n,k,2_{|x_1=0}} = 0$, we also have to justify that the evanescent modes given by Lemma 5.6 satisfy these conditions. It is effectively the case because of the following lemma whose proof is readable from the explicit formulas (5.18) and (5.19).

Lemma 5.8. — *Let* $p = 1, 2$*,* $F \in \mathbf{P}_p$ *and* $U(0)$ *be given functions satisfying* $F_{|x_{3-p}=0} = U(0)_{|x_{3-p}=0} = 0$*. Then the solution* U *to* (5.17) *given by Lemma 5.6 satisfies* $U_{|x_{3-p}=0} = 0$.

5.2.3. Reformulation for glancing modes

Finally, we reformulate the equations for glancing modes in order to show that to determine these amplitudes it is sufficient to know the values of the traces on $\partial\Omega_1$ or $\partial\Omega_2$ depending on the kind of the glancing mode. The analysis exposed below follows closely the one of [17].

We recall that for glancing modes we have the equations: for $p = 1, 2$

$$\begin{cases} \mathscr{L}(d\varphi_k)u_{0,k} = 0 & \forall\ k \in \mathscr{I}_{g_p}, \\ i\mathscr{L}(d\varphi_k)u_{1,k} + A_p\partial_{\chi_p}u_{0,k} = 0 & \forall\ k \in \mathscr{I}_{g_p}, \\ i\mathscr{L}(d\varphi_k)u_{n+2,k} + A_p\partial_{\chi_p}u_{n+1,k} + L'_p(\partial)u_{n,k} = 0 & \forall\ k \in \mathscr{I}_{g_p}, \\ & \forall\ n \in \mathbb{N}. \end{cases} \tag{5.20}$$

The first equation of (5.20) gives the polarization condition $\Pi^k u_{0,k} = u_{0,k}$, for all $k \in \mathscr{I}_{g_p}$. We then consider the second equation of (5.20), we apply Π^k to obtain, thanks to the polarization condition

$$\Pi^k A_p \Pi^k \partial_{\chi_p} u_{0,k} = 0. \tag{5.21}$$

From Lax lemma [10], the matrix $\Pi^k A_p \Pi^k$ simplifies into $\mathbf{v}_{k,p}\Pi^k$. But for glancing modes for the side $\partial\Omega_p$, we have $\mathbf{v}_{k,p} = 0$. As a consequence, (5.21) is trivially satisfied for the polarized part. In order to make it satisfied for the non polarized part, we apply the partial inverse Υ^k to determine the non polarized part of $u_{1,k}$. We have:

$$(I - \Pi^k)u_{1,k} = i\Upsilon^k A_p \Pi^k \partial_{\chi_p} u_{0,k}. \tag{5.22}$$

Finally, we consider the third equation of (5.20) written for $n = 0$. We apply Π^k, we decompose $u_{1,k} = \Pi^k u_{1,k} + (I - \Pi^k)u_{1,k}$, we use $\Pi^k A_p \Pi^k = 0$ and (5.22), in order to obtain the equation governing $u_{0,k}$. We end up with

$$i\Pi^k A_p \Upsilon^k A_p \Pi^k \partial^2_{\chi_p} u_{0,k} + (\partial_t + \mathbf{v}_k \cdot \nabla_x)\Pi^k u_{0,k} = 0,$$

where we used once again Lax lemma.

We have the following result from [17]

Proposition 5.9. — *For $p = 1, 2$, for $k \in \mathscr{I}_{g_p}$, then we have the relation*

$$\Pi^k A_p \Upsilon^k A_p \Pi^k := \frac{1}{c_p}\Pi^k,$$

where $c_p \in \mathbb{R} \setminus \{0\}$.

As a consequence, for glancing modes we are leading to consider the Schrödinger type equation (see [17] for more details about this name)

$$-\partial^2_{\chi_p}\Pi^k u_{0,k} + ic_p(\partial_t + \mathbf{v}_k \cdot \nabla_x)\Pi^k u_{0,k} = 0. \tag{5.23}$$

We can repeat the same procedure for higher order terms, the only difference being that because the amplitudes are not polarized any more some extra source terms involving the non polarized part, which can be expressed using the preceding terms as in (5.22), come into play. We borrow the following proposition to [17]. It gives an explicit solution to the equation (5.23).

Proposition 5.10. — *Let $f \in H^\infty_{\natural,x'}(\mathbb{R} \times \mathbb{R}^2_+)$ be a function with exponential decay with respect to the last variable and $g \in H^\infty_\natural(\mathbb{R} \times \mathbb{R}_+)$ be a boundary term, then for c, $\mathbf{v}' \in \mathbb{R} \setminus \{0\}$, the equation*

$$\begin{cases} -\partial^2_\chi u + ic(\partial_t + \mathbf{v}'\partial_{x'})u = f & \text{for } (t, x', \chi) \in \mathbb{R} \times \mathbb{R}^2_+, \\ u_{|\chi=0} = g & \text{on } \mathbb{R} \times \mathbb{R}_+, \\ u_{|t\leqslant 0} = 0 & \text{on } \mathbb{R}^2_+, \end{cases} \tag{5.24}$$

admits a unique solution $u \in H^\infty_{\natural,x'}(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$. In particular, it satisfies the homogeneous boundary condition $u_{|x'=0} = 0$.

Proof. — The proof relies on the explicit solution given by [17, eq. (8.40)]. We consider $\bar{\cdot}$ the extension of $\cdot$ by zero for $x' < 0$. Because $g \in H^\infty_\natural(\mathbb{R} \times \mathbb{R}_+)$ and $f \in H^\infty_{\natural,x'}(\mathbb{R} \times \mathbb{R}^2_+)$, the extensions are regular i.e. $\bar{g} \in H^\infty(\mathbb{R}^2)$ and $\bar{f} \in H^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$.

We thus consider $\bar{u}$ the solution to

$$\begin{cases} -\partial^2_\chi \bar{u} + ic(\partial_t + \mathbf{v}'\partial_{x'})\bar{u} = \bar{f} & \text{for } (t, x', \chi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+, \\ \bar{u}_{|\chi=0} = \bar{g} & \text{on } \mathbb{R} \times \mathbb{R}_+, \\ \bar{u}_{|t\leqslant 0} = 0 & \text{on } \mathbb{R}^2_+. \end{cases} \tag{5.25}$$

For $\gamma, \mu > 0$, we introduce the new unknown $v := e^{-\gamma t} e^{-\mu x'} \overline{u}$, we perform a Fourier transform with respect to (t, x'). Let $\widehat{\cdot}$ denotes this transform and (τ, η) be the dual variable of (t, x') then (5.25) becomes

$$\begin{cases} -\partial_\chi^2 \widehat{v} + \underbrace{ic(\gamma + \mathbf{v}'\mu + i\tau + i\mathbf{v}'\eta)}_{:=\mathbf{X}(\zeta)} \widehat{v} = \widehat{e^{-\gamma t} e^{-\mu x'} \overline{f}} \\ \qquad\qquad\qquad \text{for } (t, x', \chi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+, \\ \widehat{v}_{|\chi=0} = \widehat{e^{-\gamma t} e^{-\mu x'} \overline{g}} \text{ on } \mathbb{R} \times \mathbb{R}_+, \end{cases} \tag{5.26}$$

where ζ collects the parameters γ, μ and the dual variables. We also define in the following $e_{\gamma,\mu} := e^{-\gamma t} e^{-\mu x'}$.

From [17, eq. (8.40)], , the solution to the interior equation of (5.26) reads

$$\widehat{v}(\zeta, \chi) = e^{\chi\sqrt{\mathbf{X}}} \widehat{v}_{|\chi=0}(\zeta) + \int_0^\chi e^{(\chi-\chi')\sqrt{\mathbf{X}}} \int_{\chi'}^\infty e^{-(\chi'-\chi'')\sqrt{\mathbf{X}}} \widehat{e_{\gamma,\mu} \overline{f}}(\zeta, \chi'') \, \mathrm{d}\chi'' \, \mathrm{d}\chi',$$

where $\sqrt{\mathbf{X}}$ stands for the square root of $\mathbf{X}$ with strictly negative real part, so that $\widehat{v} \in L^2_\chi(\mathbb{R}_+)$. Reversing the Fourier transform then gives the following explicit formula for $\overline{u}$

$$e_{\gamma,\mu} \overline{u}(t, x', \chi) = \mathscr{F}^{-1}_{(\tau,\eta)\to(t,x')} \left(e^{\chi\sqrt{\mathbf{X}}} \widehat{e_{\gamma,\mu} \overline{g}}(\zeta) + \int_0^\chi e^{(\chi-\chi')\sqrt{\mathbf{X}}} \int_{\chi'}^\infty e^{-(\chi'-\chi'')\sqrt{\mathbf{X}}} \widehat{e_{\gamma,\mu} \overline{f}}(\zeta, \chi'') \, \mathrm{d}\chi'' \, \mathrm{d}\chi' \right), \tag{5.27}$$

where $\mathscr{F}^{-1}$ stands for the reverse Fourier transform.

Clearly the restriction of $\overline{u}$ to $x' > 0$ solves the interior equation of (5.24). To conclude, it remains to justify that $\overline{u}$ is regular, that $(\partial_{x'}^n \overline{u})_{|x' \leqslant 0} = 0$, for all $n \in \mathbb{N}$, and finally that the restriction of $\overline{u}$ satisfies the initial condition.

The regularity of $\overline{u}$ can be read directly on the explicit formula (5.27). So that we only consider the traces values. In order to do so we will proceed by causality using the following energy estimate

Lemma 5.11. — *For all γ, μ sufficiently large, chosen in such a way that $\gamma + \mathbf{v}'\mu \neq 0$, and for all $\overline{g} \in H^\infty(\mathbb{R}^2)$ and $\overline{f} \in H^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$ with fast decay with respect to the last variable, we have the energy estimate: there exists*

$C > 0$ such that

$$\begin{aligned}\|e_{\gamma,\mu}\overline{u}\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}_+)} \\ \leqslant \frac{C}{\|\gamma+\mathbf{v}'\mu\|}\left(\frac{1}{\|\gamma+\mathbf{v}'\mu\|}\|e_{\gamma,\mu}\overline{f}\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}_+)}+\|e_{\gamma,\mu}\overline{g}\|^2_{L^2(\mathbb{R}^2)}\right).\end{aligned} \tag{5.28}$$

We also refer to [17, eqs. (8.40) and (8.41)] *for similar estimates.*

With Lemma 5.11 in hand the fact that $\overline{u}_{|x'<0} = 0$, if the sources $\overline{f}$ and $\overline{g}$ vanish on $\{x' < 0\}$ is clear. Indeed, if $\overline{f}_{|x'<0}, \overline{g}_{|x'<0} \equiv 0$ then the right-hand side on (5.28) is $o(e^{\varepsilon\mu})$ for all $\varepsilon > 0$. The same holds for $\|e_{\gamma,\mu}\overline{u}\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}_+)}$ and this implies that $\overline{u}_{|x'<0}$ vanishes. We obtain the desired result by continuity. Moreover, the same arguments apply to the initial condition.

Proof of Lemma 5.11. — Let γ, $\mu > 0$ to be specified below. Using Plancherel identity in (5.27) we have

$$\begin{aligned}\|e_{\gamma,\mu}\overline{u}\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}_+)} \leqslant \left\|e^{\chi\sqrt{\mathbf{X}}}\widehat{e_{\gamma,\mu}\overline{\overline{g}}}(\zeta)\right\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}_+)} \\ + C\left\|\int_0^{\chi} e^{(\chi-\chi')\sqrt{\mathbf{X}}}F(\zeta,\chi')\,\mathrm{d}\chi'\right\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}_+)}\end{aligned} \tag{5.29}$$

where to save some notation we defined

$$F(\zeta,\chi') := \int_{\chi'}^{\infty} e^{-(\chi'-\chi'')\sqrt{\mathbf{X}}}\widehat{e_{\gamma,\mu}\overline{f}}(\zeta,\chi'')\,\mathrm{d}\chi''.$$

The estimate for the first term on the right-hand side of (5.29) is straightforward. Indeed we have

$$\begin{aligned}\left\|e^{\chi\sqrt{\mathbf{X}}}\widehat{e_{\gamma,\mu}\overline{g}}(\zeta)\right\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}_+)} &= \int_{\mathbb{R}^2}\left\|\widehat{e_{\gamma,\mu}\overline{g}}(\zeta)\right\|^2\left(\int_0^{\infty} e^{2\chi\Re\sqrt{\mathbf{X}}}\,\mathrm{d}\chi\right)\mathrm{d}\tau\,\mathrm{d}\eta \\ &= \int_{\mathbb{R}^2} -\frac{1}{2\Re\sqrt{\mathbf{X}}}\left\|\widehat{e_{\gamma,\mu}\overline{g}}(\zeta)\right\|^2\mathrm{d}\tau\,\mathrm{d}\eta \\ &\leqslant \frac{C}{\|\gamma+\mathbf{v}'\mu\|}\|e_{\gamma,\mu}\overline{g}\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

where we used again Plancherel formula to conclude combined with an estimate of the real part of $\sqrt{\mathbf{X}}$ obtained in [5].

We then estimate the second term on the right-hand side of (5.29). We first give an estimate for F. We have

$$\begin{aligned}\|F(\zeta,\chi')\|^2 &= \left|\int_{\chi'}^{\infty} e^{-(\chi'-\chi'')\sqrt{\mathbf{X}}}\widehat{e_{\gamma,\mu}\overline{f}}(\zeta,\chi'')\,\mathrm{d}\chi''\right|^2 \\ &\leqslant \left\|\widehat{e_{\gamma,\mu}\mathbf{f}}(\zeta)\right\|^2 e^{-2\chi'\Re\sqrt{\mathbf{X}}}\left|\int_{\chi'}^{\infty} e^{\chi''(\sqrt{\mathbf{X}}-\delta)}\,\mathrm{d}\chi''\right|^2 \\ &\leqslant \frac{1}{2\left\|\Re\sqrt{\mathbf{X}}-\delta\right\|^2}\left\|\widehat{e_{\gamma,\mu}\mathbf{f}}(\zeta)\right\|^2 e^{-2\chi'\delta}, \\ &\leqslant \frac{C}{\|\gamma+\mathbf{v}'\mu\|^2}\left\|\widehat{e_{\gamma,\mu}\mathbf{f}}(\zeta)\right\|^2 e^{-2\chi'\delta},\end{aligned}$$

where we used the fact that the source $\overline{f}$ has fast decay with respect to χ''. Consequently $\widehat{\overline{f}}$ does the same and we can write $\|\widehat{\overline{f}}(\zeta,\chi'')\| \leqslant \mathbf{f}(\zeta)e^{-\delta\chi''}$, for some square integrable function $\mathbf{f}$. We also use the estimate $\|\Re\sqrt{\mathbf{X}}\| \geqslant \|\gamma+\mathbf{v}'\mu\|$ of [5]. With this estimate in hand from Jensen inequality we thus have

$$\begin{aligned}&\left\|\int_0^{\chi} e^{(\chi-\chi')\sqrt{\mathbf{X}}}F(\zeta,\chi')\,\mathrm{d}\chi'\right\|^2_{L^2(\mathbb{R}^2\times\mathbb{R}_+)} \\ &\qquad = \int_{\mathbb{R}^2}\int_0^{\infty}\left|\int_0^{\chi} e^{(\chi-\chi')\sqrt{\mathbf{X}}}F(\zeta,\chi')\,\mathrm{d}\chi'\right|^2\mathrm{d}\chi\,\mathrm{d}\tau\,\mathrm{d}\eta \\ &\qquad \leqslant \frac{C}{\|\gamma+\mathbf{v}'\mu\|^2}\int_{\mathbb{R}^2}\left\|\widehat{e_{\gamma,\mu}\mathbf{f}}(\zeta)\right\|^2\int_0^{\infty}\chi^2 \\ &\qquad\qquad \int_0^1 e^{2(\chi-\chi\chi')\Re\sqrt{\mathbf{X}}}e^{-2\delta\chi\chi'}\,\mathrm{d}\chi'\,\mathrm{d}\chi\,\mathrm{d}\tau\,\mathrm{d}\eta.\end{aligned}$$

We have

$$\begin{aligned}&\int_0^{\infty}\chi^2\int_0^1 e^{2(\chi-\chi\chi')\Re\sqrt{\mathbf{X}}}e^{-2\delta\chi\chi'}\,\mathrm{d}\chi'\,\mathrm{d}\chi \\ &\qquad = \frac{1}{2(\Re\sqrt{\mathbf{X}}+\delta)}\int_0^{\infty}\chi^2\left(e^{-2\chi\delta}-e^{2\chi\Re\sqrt{\mathbf{X}}}\right)\mathrm{d}\chi. \qquad (5.30)\end{aligned}$$

From the bound $\Re\sqrt{\mathbf{X}} \geqslant -C\|\gamma+\mathbf{v}'\mu\|$, we choose γ,μ large enough such that $\Re\sqrt{\mathbf{X}}+\delta \geqslant -\frac{\delta}{2}$. It implies that the right-hand side of (5.30) is finite and does not depend on the Fourier variables. This ends up the desired estimate for the second term on the right-hand side of (5.29). □

To complete the proof of Proposition 5.10 it only remains to justify that the derivatives $\partial^n_{x'}\overline{u}$ vanish on $\{x' < 0\}$. This is made exactly as for the zero order term $u_{|x'<0}$ by using the explicit formula (5.27) for which we can obtain high order Sobolev estimates in the spirit of Lemma 5.11 (we also refer to [17, eqs. (8.40) and (8.41)]). We omit the details here. □

We thus have justified the following proposition:

Proposition 5.12. — *To solve the cascade of equations* (5.20)*, that is to say to determine the contributions of glancing modes, it is sufficient to determine:*

- *The trace on* $\{x_1 = 0\}$ *if the amplitude is glancing for the side* $\partial\Omega_1$*.*
- *The trace on* $\{x_2 = 0\}$ *if the amplitude is glancing for the side* $\partial\Omega_2$*.*

5.3. Construction of the leading order term

In this paragraph we describe the construction of the leading order term in the geometric optics expansion. From the results of Subsection 5.2, we have justified that the whole determination of the amplitudes of the leading order term amounts to determine the suitable trace values.

Thus, in the following we describe an order of resolution which permits to decouple the boundary conditions cascades (5.5) and (5.6) and thus to perform the construction of the leading order term thanks to Propositions 5.5–5.7 and 5.10.

Because several loops have to be considered the order of resolution will not be as simple as in the unique loop framework of [2]. The main steps of the determination remain however, in some sense, the same. We first find a compatibility condition permitting to determine the elements of the self-interaction loop. We then find some order of resolution to determine the elements coming from reflections of the self-interaction loop's elements.

5.3.1. Determination of the loop elements

In this paragraph we generalize the compatibility condition of [2] to a self-interaction loop with any odd number of elements, possibly greater than four. Consequently, there exists an even number of self-interacting terms. The ideas remain however unchanged.

We assume for a while that the last amplitude of the loop, namely $u_{0,s_{2b}}$, is known. We will recover the values of all the other self-interacting terms in terms of $u_{0,s_{2b}}$ and finally a compatibility condition determining $u_{0,s_{2b}}$.

By definition $s_1 \in \mathscr{I}_{io}$, so from Proposition 5.5, it is sufficient to determine its trace on $\partial\Omega_1$ to determine u_{0,s_1}. From the boundary condition (5.5), this trace satisfies

$$B_1 \left[u_{0,s_1} + \sum_{k \in \Phi^*(s_1) \cap (\mathscr{I}_{ii} \cup \mathscr{I}_{io})} u_{0,k} \right]_{|x_1=0} + B_1 U_{0,s_1,1_{|X_1=0}} + B_1 \sum_{k \in \Phi(s_1) \cap \mathscr{I}_{g_1}} u_{0,k_{|X_1=0}} = g - B_1 \sum_{k \in \Phi(s_1) \cap \mathscr{I}_{oi}} u_{0,k_{|x_1=0}},$$

if $s_1 \in \mathfrak{R}_1$, and

$$B_1 \left[u_{0,s_1} + \sum_{k \in \Phi^*(s_1) \cap (\mathscr{I}_{ii} \cup \mathscr{I}_{io})} u_{0,k} \right]_{|x_1=0} + B_1 \sum_{k \in \Phi(s_1) \cap \mathscr{I}_{g_1}} u_{0,k_{|X_1=0}} = g - B_1 \sum_{k \in \Phi(s_1) \cap \mathscr{I}_{oi}} u_{0,k_{|x_1=0}},$$

if $s_1 \notin \mathfrak{R}_1$. In both cases, we remark that the left-hand side reads under the form $B_1 v$ where $v \in \mathbf{E}_1^s$ so that the uniform Kreiss–Lopatinskii condition (see Assumption 2.11) gives that

$$u_{0,s_1{}_{|x_1=0}} = \Pi^{s_1} \phi_1^{s_1} \left[g - B_1 \sum_{k \in \Phi(s_1) \cap \mathscr{I}_{oi}} u_{0,k_{|x_1=0}} \right], \tag{5.31}$$

where we recall that Π^{s_1} stands for the projection upon $\ker \mathscr{L}(d\varphi_{s_1})$ introduced in Definition 4.11.

We aim to follow the method of resolution of [2]. So we need to justify that $\Phi(s_1) \cap \mathscr{I}_{oi} = \{s_{2b}\}$, to make sure that the right-hand side only depends on the "known" function $u_{0,s_{2b}}$.

Lemma 5.13. — *Consider a complete for reflections frequencies set $\mathscr{F}$ satisfying Assumption 4.8. Then we have $\Phi(s_1) \cap \mathscr{I}_{oi} = \{s_{2b}\}$.*

Proof. — We proceed by contradiction and we assume that there exists $\underline{i} \in \Phi(s_1) \cap \mathscr{I}_{oi}$, $\underline{i} \neq s_{2b}$. Using the fact that the frequencies set is minimal, we obtain that $\underline{i}$ necessarily comes from some reflection. So, there exists some $\underline{j} \in \mathscr{I}_{io}$, such that $\underline{j} \in \Psi(\underline{i})$. From Proposition 4.9, we have that $s_1 \underset{V}{\rightarrowtail} \underline{j}$ so that there exists a type V sequence ℓ with an even number of elements linking s_1 to $\underline{j}$. As a consequence, the sequence $(s_1, \ell, \underline{j}, \underline{i}, s_1)$ is a loop for s_1. It differs from the self-interaction loop $(s_1, s_2, \dots, s_{2b}, s_1)$ because $\underline{i} \neq s_{2b}$ and thus it contradicts Assumption 4.8. ☐

As a consequence, we obtain (see Proposition 5.5) the following value for u_{0,s_1} in terms of the (supposed to be known) term $u_{0,s_{2b}{}_{|x_1=0}}$:

$$u_{0,s_1}(t,x) = \Pi^{s_1}\phi_1^{s_1}\left[g - B_1 u_{0,s_{2b}{}_{|x_1=0}}\right]\left(t - \frac{1}{\mathbf{v}_{s_1,1}}x_1, x_2 - \frac{\mathbf{v}_{s_1,2}}{\mathbf{v}_{s_1,1}}x_1\right). \quad (5.32)$$

In particular

$$u_{0,s_1{}_{|x_2=0}}(t,x_1) = \Pi^{s_1}\phi_1^{s_1}\left[g - B_1 u_{0,s_{2b}{}_{|x_1=0}}\right] \left(t - \frac{1}{\mathbf{v}_{s_1,1}}x_1, -\frac{\mathbf{v}_{s_1,2}}{\mathbf{v}_{s_1,1}}x_1\right). \quad (5.33)$$

We now show that we can determine u_{0,s_2} from (5.33). Because $s_2 \in \mathscr{I}_{oi}$, we only require the trace value on $\partial\Omega_2$. From (5.6) the trace on $\partial\Omega_2$ satisfies the boundary condition

$$B_2\left[u_{0,s_2} + \sum_{k\,\in\,\Psi^*(s_2)\,\cap\,(\mathscr{I}_{ii}\,\cup\,\mathscr{I}_{oi})} u_{0,k}\right]_{|x_2=0} + B_2 U_{0,s_2,2_{|X_2=0}} + B_2 \sum_{k\,\in\,\Psi(s_2)\,\cap\,\mathscr{I}_{g_2}} u_{0,k_{|X_2=0}} = -B_2 \sum_{k\,\in\,\Psi(s_2)\,\cap\,\mathscr{I}_{io}} u_{0,k_{|x_2=0}},$$

if $s_2 \in \mathfrak{R}_2$ and

$$B_2\left[u_{0,s_2} + \sum_{k\,\in\,\Psi^*(s_2)\,\cap\,(\mathscr{I}_{ii}\,\cup\,\mathscr{I}_{oi})} u_{0,k}\right]_{|x_2=0} + B_2 \sum_{k\,\in\,\Psi(s_2)\,\cap\,\mathscr{I}_{g_2}} u_{0,k_{|X_2=0}} = -B_2 \sum_{k\in\,\Phi(s_2)\,\cap\,\mathscr{I}_{io}} u_{0,k},$$

if $s_2 \notin \mathfrak{R}_2$.

Reiterating the same kind of arguments as for u_{0,s_1} (in particular we require some straightforward adaptation of Lemma 5.13), we obtain the trace value

$$u_{0,s_2{}_{|x_2=0}} = -\Pi^{s_2}\phi_2^{s_2} B_2 u_{0,s_1{}_{|x_2=0}},$$

so that

$$u_{0,s_2}(t,x) = \Pi^{s_2}\phi_2^{s_2} B_2 \Pi^{s_1}\phi_1^{s_1}\left[g - B_1 u_{0,s_{2b}{}_{|x_1=0}}\right] \left(t - \frac{1}{\mathbf{v}_{s_1,1}}x_1 - \frac{1}{\mathbf{v}_{s_2,2}}\left(1 - \frac{\mathbf{v}_{s_2,1}}{\mathbf{v}_{s_1,1}}\right)x_2, -\frac{\mathbf{v}_{s_1,2}}{\mathbf{v}_{s_1,1}}\left(x_1 - \frac{\mathbf{v}_{s_2,1}}{\mathbf{v}_{s_2,2}}x_2\right)\right). \quad (5.34)$$

In particular

$$u_{0,s_2|x_1=0}(t,x) = \Pi^{s_2}\phi_2^{s_2}B_2\Pi^{s_1}\phi_1^{s_1}\Big[g - B_1 u_{0,s_{2b}|x_1=0}\Big] \\ \left(t - \frac{1}{\mathbf{v}_{s_2,2}}\left(1-\frac{\mathbf{v}_{s_2,1}}{\mathbf{v}_{s_1,1}}\right)x_2, -\frac{\mathbf{v}_{s_1,2}}{\mathbf{v}_{s_1,1}}\frac{\mathbf{v}_{s_2,1}}{\mathbf{v}_{s_2,2}}x_2\right). \quad (5.35)$$

We can repeat the same computations for all indices of the self-interaction loop. They can all be expressed in terms of $u_{0,s_{2b}|x_1=0}$. At the last step of the process, we determine $u_{0,s_{2b}}$ so that we obtain the value of its trace on $\partial\Omega_1$ in terms of itself. This compatibility condition reads under the form

$$(I-\mathbf{T})u_{0,s_{2b}|x_1=0} = \mathbf{T}g, \quad (5.36)$$

where the operator $\mathbf{T}$ is defined by

$$(\mathbf{T}u)(t,y) := Su(t-\alpha y, \beta y), \quad (5.37)$$

where we defined

$$S := \Pi^{s_{2b}}\phi_2^{s_{2b}}B_2\Pi^{s_{2b-1}}\phi_1^{s_{2b-1}}B_1\cdots\Pi^{s_2}\phi_2^{s_2}B_2\Pi^{s_1}\phi_1^{s_1}B_1,$$
$$\beta := \prod_{k\in\{s\}\cap\mathscr{I}_{io}}\frac{\mathbf{v}_{k,2}}{\mathbf{v}_{k,1}}\prod_{l\in\{s\}\cap\mathscr{I}_{oi}}\frac{\mathbf{v}_{l,1}}{\mathbf{v}_{l,2}},$$

and where $\alpha > 0$ can be made explicit in terms of the group velocities $\mathbf{v}_k$. Its precise value is however of little interest in the following of the discussion. The only point to keep in mind is that it is positive.

In the following in order to determine $u_{0,2b|x_1=0}$ and thus to deduce all the amplitudes for the indices in the self-interaction loop from the above explicit relations (see (5.32) and (5.34)), we will make the following assumption. It is just a generalization of the one of [2] to loops with an arbitrary number of elements.

Assumption 5.14. — *We assume that the operator $I-\mathbf{T}$ where $\mathbf{T}$ is defined in* (5.37) *is invertible from $H_\natural^\infty(\mathbb{R}\times\mathbb{R}_+)$ into $H_\natural^\infty(\mathbb{R}\times\mathbb{R}_+)$.*

Under this assumption, we can use (5.36) in order to obtain $u_{0,s_{2b}|x_1=0}\in H_\natural^\infty(\mathbb{R}\times\mathbb{R}_+)$ in an unique way. We then use the explicit formulas (see (5.32) and (5.34)) to determine the values of the self-interacting elements. In particular, we remark that for $k\in\mathscr{I}_{io}\cap\{s\}$ (resp. $k\in\mathscr{I}_{oi}\cap\{s\}$), $u_{0,k|x_2=0}\in H_\natural^\infty(\mathbb{R}\times\mathbb{R}_+)$ (resp. $u_{0,k|x_1=0}\in H_\natural^\infty(\mathbb{R}\times\mathbb{R}_+)$). As a consequence in the following these terms can effectively be used as boundary terms in Proposition 5.3.

5.3.2. Determination of the other amplitudes

Let $\underline{i} \in \mathscr{I}$ be some index, then we know from the results of Section 4 that $\underline{i}$ is linked to the index s_1 by (possibly many) type V sequences. The order of determination of the amplitudes will be the following:

(1) we start by determining all the indices $\underline{i}$ which are linked to s_1 by exactly one type V sequence.
(2) Then, we proceed to the determination of the ones which are linked by two distinct sequences of type V and so on.

The following proposition, whose proof follows essentially the same lines as the determination of the amplitudes in the so-called trees of [2], states that if $\underline{i} \in \mathscr{I}$ is linked to s_1 by a unique sequence, then it can be determinate from the knowledge of the loop elements.

Proposition 5.15. — *In a complete for reflections frequencies set $\mathscr{F}$ satisfying the uniqueness Assumption 4.8, let $\underline{i} \in \mathscr{I}$ be such that there exists a unique type V sequence ℓ such that $s_1 \underset{V}{\rightarrowtail} \underline{i}$, then the amplitude $u_{0,\underline{i}}$ or $U_{0,\underline{i},p}$, $p = 1, 2$, satisfying the cascade of equations* (5.2), (5.7) *and* (5.8) *can be uniquely determined from the values of the u_{0,s_p} where $s_p \in \{s\}$.*

Proof. — Let $\underline{i} \in \mathscr{I}$ be linked to s_1 by only one sequence of type V. In the following we have to consider several cases depending of the nature of the index $\underline{i}$.

- Firstly, we assume that $\underline{i} \in \mathscr{I}_{io}$. Because of the definition of type V sequence, the index $\underline{i}$ is linked to s_1 by a (unique) sequence ℓ admitting an odd number of elements so that $\ell = (\ell_1, \ell_2, \dots, \ell_{2p+1})$ for some $p \in \mathbb{N}$. By definition of type V sequence, we have $\ell_{2p+1} \in \mathscr{I}_{oi} \cap \Phi(\underline{i})$ and in order to use the uniform Kreiss–Lopatinskii condition in the boundary condition determining $u_{0,\underline{i}_{|x_1=0}}$ (and thus, to determine the whole amplitude $u_{0,\underline{i}}$ from Proposition 5.5), we have to justify that $\Phi(\underline{i}) \cap \mathscr{I}_{oi} = \{\ell_{2p+1}\}$.

  By contradiction, let us assume that there exists some $\underline{j} \in \mathscr{I}_{oi} \cap \Phi(\underline{i})$, $\underline{j} \neq \ell_{2p+1}$. Then because the frequencies set $\mathscr{F}$ is complete for the reflections, it is minimal and thus the index $\underline{j}$ comes from some reflection. As a consequence there exists $\underline{k} \in \mathscr{I}_{io} \cap \Psi(\underline{j})$. From Proposition 4.9, the index $\underline{k}$ is linked to s_1 so that there exists a type V sequence $\ell' = (\ell'_1, \dots, \ell'_{2p'+1})$ such that $s_1 \underset{V}{\rightarrowtail} \underline{k}$. We can not exclude at first glance that we have $\underline{i} \in \ell'$ that is to say that the sequence linking $\underline{k}$ to s_1 passes by $\underline{i}$. In fact, it is not possible. Indeed if we have $\underline{i} = \ell'_{2r+2}$ for some $r \in \mathbb{N}$, then the sequence

$(\ell'_{2r+2}, \ell'_{2r+3}, \ldots, \ell'_{2p'+1}, \underline{k}, \underline{j}, \underline{i})$ forms a self-interaction loop for $\underline{i}$. It contradicts Assumption 4.8. So that we have $\underline{i} \notin \ell'$. The above discussion is summarized in Figure 5.1.

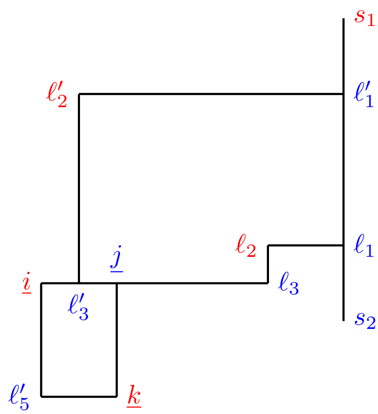

Figure 5.1. Illustration of the proof 1.

Consequently the sequence $(\ell', \underline{k}, \underline{j})$ is a type V sequence which links s_1 to $\underline{i}$. The sequence $(\ell', \underline{k}, \underline{j}, \underline{i})$ can be simplified to $(\ell, \underline{i})$ if and only if $\ell' = (\ell, \underline{i}, q)$ for some sequence q linking $\underline{i}$ to $\underline{k}$. This is however impossible because $\underline{i} \notin \ell'$. We have thus found two distinct sequences of type V linking s_1 to $\underline{i}$ which is a contradiction. We refer to Figure 5.2.

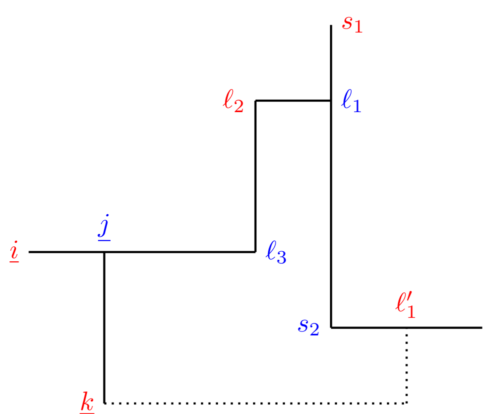

Figure 5.2. Illustration of the proof 2.

At this step of the proof, we have justified that $\Phi(\underline{i}) \cap \mathscr{I}_{oi} = \{\ell_{2p+1}\}$, so that to construct $u_{0,\underline{i}}$, it is sufficient to construct $u_{0,\ell_{2p+1}}$.

To construct $u_{0,\ell_{2p+1}}$ we proceed similarly. We now justify that $\Psi(\ell_{2p+1}) \cap \mathscr{I}_{io} = \{\ell_{2p}\}$. By contradiction assume that there exists $\underline{j} \in \Psi(\ell_{2p+1}) \cap \mathscr{I}_{io}$, $\underline{j} \neq \ell_{2p}$, then the fact that the frequencies set $\mathscr{F}$ is minimal implies that there exists $\underline{k} \in \mathscr{I}_{oi} \cap \Phi(\ell_{2p})$. From Proposition 4.9, this index is linked to s_1 by some type V sequence, namely ℓ'. Reiterating the same arguments as for the first step of the proof, we easily show that $\underline{i} \notin \ell'$, because there exists a unique self-interaction loop in $\mathscr{F}$. As a consequence, there exists two distinct sequences linking s_1 to ℓ_{2p} and thus s_1 to $\underline{i}$. It is excluded from our special choice of $\underline{i}$.

We then proceed inductively for each terms in the sequence ℓ. At the end of the day, we obtain the value of the desired trace $u_{0,\underline{i}}$ in terms of (some) of the u_{0,s_p} for $s_p \in \{s\}$ which have been determined in Subsection 5.3.1. As a consequence, we can construct $u_{0,\underline{i}}$ thanks to Propositions 5.3 and 5.5. The first one applies because we have already justified that the traces of the self-interacting amplitudes admit some flatness at the corner.

- Secondly we consider an index $\underline{i} \in \mathscr{I}_{oi}$. The determination of such $u_{0,\underline{i}}$ is similar to the one where $\underline{i} \in \mathscr{I}_{io}$ except that we require the trace on $\partial\Omega_2$ and that this index is linked to s_1 by a sequence with an even number of terms. We start by showing that $\Psi(\underline{i}) \cap \mathscr{I}_{io} = \{\ell_{2p}\}$ exactly as the property $\Psi(\ell_{2p+1}) \cap \mathscr{I}_{io} = \{\ell_{2p}\}$ has been shown for incoming-outgoing modes. We then proceed inductively for each term of the sequence as for the incoming-outgoing modes. We feel free to skip the details here.
- If $\underline{i} \in \mathscr{I}_{ii}$ then such index can appear after an odd or an even number of reflections. In order to determine the amplitude $u_{0,\underline{i}}$, we have to know the two traces $u_{0,\underline{i}_{|x_1=0}}$ and $u_{0,\underline{i}_{|x_2=0}}$.

  Let $\ell = (\ell_1, \dots, \ell_p)$ be the type V sequence linking $\underline{i}$ to s_1. By definition of such a sequence (see Definition 4.5) we have

$$\begin{cases} \ell_p \in \mathscr{I}_{oi} \quad \text{and} \quad \ell_p \in \Phi(\underline{i}) \quad \text{if } p \text{ is odd,} \\ \ell_p \in \mathscr{I}_{io} \quad \text{and} \quad \ell_p \in \Psi(\underline{i}) \quad \text{if } p \text{ is even.} \end{cases}$$

Consequently if p is odd (resp. even), we can use the boundary condition (5.5) (resp. (5.6)) combined with the uniform Kreiss–Lopatinskii condition to write

$$u_{0,\underline{i}_{|x_1=0}} = -\Pi^{\underline{i}}\phi_1^{\underline{i}} B_1 u_{0,\ell_{p|x_1=0}} - \Pi^{\underline{i}}\phi_1^{\underline{i}} \sum_{k\,\in\,\Phi^*(\ell_p)\,\cap\,\mathscr{I}_{oi}} B_1 u_{0,k_{|x_1=0}},$$
$$\begin{pmatrix}\text{resp. } u_{0,\underline{i}_{|x_2=0}} \\ = -\Pi^{\underline{i}}\phi_2^{\underline{i}} B_2 u_{0,\ell_{p|x_2=0}} - \Pi^{\underline{i}}\phi_2^{\underline{i}} \sum_{k\,\in\,\Psi^*(\ell_p)\,\cap\,\mathscr{I}_{io}} B_1 u_{0,k_{|x_2=0}}\end{pmatrix}. \tag{5.38}$$

We can reiterate the same kind of arguments as for the case where $\underline{i} \in \mathscr{I}_{io}$ in order to show that, because of the uniqueness of the type V sequence linking $\underline{i}$ to s_1, then in both cases $\Psi^*(\ell_p)\cap\mathscr{I}_{io} = \Phi^*(\ell_p)\cap\mathscr{I}_{oi} = \emptyset$. So that, depending on the parity of p one of the traces of $u_{0,\underline{i}}$ is determined in terms of the one of u_{0,ℓ_p}. The amplitude u_{0,ℓ_p} and in particular its required trace can be determined from the case $\underline{i} \in \mathscr{I}_{io}$ or $\underline{i} \in \mathscr{I}_{oi}$. As a consequence, it is sufficient to determine the value of the second trace to construct the whole $u_{0,\underline{i}}$. To fix the ideas, let us assume that p is odd so that $u_{0,\underline{i}_{|x_1=0}}$ is known and $u_{0,\underline{i}_{|x_2=0}}$ has to be determined. We claim that $\Psi(\underline{i}) \cap \mathscr{I}_{io} = \emptyset$. Consequently the boundary condition (5.6) after application of the uniform Kreiss–Lopatinskii condition gives

$$u_{0,\underline{i}_{|x_2=0}} = 0.$$

To show the claim, we proceed by contradiction and we assume that there exists $\underline{j} \in \mathscr{I}_{io} \cap \Psi(\underline{i})$. Then, from Proposition 4.9 there exists a type V sequence ℓ' linking s_1 to $\underline{j}$. Because $\underline{i} \in \mathscr{I}_{ii}$, the sequence $(\ell', \underline{j})$ can not be simplified into ℓ as a consequence $\underline{i}$ is linked to s_1 by two distinct sequences which is impossible because of the choice of $\underline{i}$.

We then have the two values of the traces $u_{0,\underline{i}_{|x_1=0}}$ and $u_{0,\underline{i}_{|x_2=0}}$. We can use Proposition 5.5 to determine the amplitude $u_{0,\underline{i}}$.

- Then we consider evanescent amplitudes $\underline{i} \in \mathfrak{R}_1$ or $\underline{i} \in \mathfrak{R}_2$. We expose here the determination of some $U_{0,ev,\underline{i}}$ with $\underline{i} \in \mathfrak{R}_1$ the determination for $\underline{i} \in \mathfrak{R}_2$ being essentially similar. Let ℓ denotes the type V sequence such that $s_1 \underset{V}{\rightarrowtail} \underline{i}$. Then because of the definition of type V sequence, we have that ℓ contains an odd number of elements. From the boundary condition (5.5) where we applied the uniform Kreiss–Lopatinskii condition, we obtain that the trace of $U_{0,\underline{i},1}$ is given by

$$U_{0,\underline{i},1_{|X_1=0}} = \Pi^{s,e}\phi_1^{\underline{i}} B_1 u_{0,\ell_{2p+1}{}_{|x_1=0}} - \Pi^{s,e}\phi_1^{\underline{i}} B_1 \sum_{k\,\in\,\mathscr{I}_{oi}\,\cap\,\Phi^*(\ell_{2p+1})} u_{0,k_{|x_1=0}},$$

where we recall that $\Pi^{s,e}$ is the projection introduced in Definition 4.11.

We can show, by using the same arguments as in the framework $\underline{i} \in \mathscr{I}_{io}$ that $\mathscr{I}_{oi} \cap \Phi^*(\ell_{2p+1}) = \emptyset$, so that we have determined $U_{0,\underline{i},1}{}_{|X_1=0}$ in terms of $u_{0,\ell_{2p+1}}{}_{|x_1=0}$. Such an amplitude is known from the subcase $\underline{i} \in \mathscr{I}_{oi}$. Proposition 5.7 applies and completes the construction of $U_{0,\underline{i},1}$. Once again from the flatness of the traces of the loop element, we can apply Lemma 5.8 to ensure that the extra boundary conditions in (5.7) are satisfied.

- Finally, we deal with glancing modes $\underline{i} \in \mathscr{I}_{g_1}$ or $\underline{i} \in \mathscr{I}_{g_2}$. The construction is analogous to the one for the elliptic boundary layers. Indeed, from Proposition 5.10, it is sufficient to know $u_{0,\underline{i}}{}_{|x_1=0}$, if $\underline{i} \in \mathscr{I}_{g_1}$, and $u_{0,\underline{i}}{}_{|x_2=0}$, if $\underline{i} \in \mathscr{I}_{g_2}$. From the boundary condition (5.5) (or (5.6)) combined with the uniform Kreiss–Lopatinskii condition we thus have, if $\underline{i} \in \mathscr{I}_{g_1}$

$$u_{0,\underline{i}}{}_{|X_1=0} = \Pi^{\underline{i}} \phi_1^{\underline{i}} B_1 u_{0,\ell_{2p+1}}{}_{|x_1=0},$$

and a similar relation in the case $\underline{i} \in \mathscr{I}_{g_2}$, where we used the fact that $\mathscr{I}_{oi} \cap \Phi^*(\ell_{2p+1}) = \emptyset$ to simplify the right-hand side. Once again we use the case $\underline{i} \in \mathscr{I}_{oi}$ to determine $u_{0,\ell_{2p+1}}{}_{|x_1=0}$. This gives $u_{0,\underline{i}}{}_{|X_1=0}$ and thus the whole amplitude $u_{0,\underline{i}}$ by using Proposition 5.10.

Because we have determined $u_{0,\underline{i}}$ when $s_1 \underset{V}{\rightarrowtail} \underline{i}$ by a unique type V sequence for all possible kinds of the index $\underline{i}$ the proof of Proposition 5.15 is complete. □

From now on, using Proposition 5.15, we can assume that all the indices linked to s_1 by only one type V sequence has been determined. For later purposes, let us remark that in fact the above proof does not really require the uniqueness of the self-interaction loop. Indeed to hold it is sufficient that the sequence ℓ linking $\underline{i}$ to s_1 does not contain any self-interacting indices.

The following proposition states that we can now determine all the indices linked to s_1 by two distinct type V sequences.

Proposition 5.16. — *In a complete for reflections frequencies set satisfying Assumption 4.8, let $\underline{i} \in \mathscr{I}$ be such that there exist two distinct type V sequences ℓ and ℓ' such that $s_1 \underset{V}{\rightarrowtail} \underline{i}$. Then the amplitude $u_{0,\underline{i}}$ or $U_{0,\underline{i},p}$, $p = 1, 2$, solving the cascade of equations* (5.2), (5.7) *and* (5.8) *can be uniquely determined from the values of the u_{0,s_p} where $s_p \in \{s\}$.*

Proof. — Acting as for the proof of Proposition 5.15, we have to separate several cases depending on the nature of the index $\underline{i}$.

- We first consider the case $\underline{i} \in \mathscr{I}_{io}$. Let $\ell := (\ell_1, \dots, \ell_{2p+1})$ and $\ell' := (\ell'_1, \dots, \ell'_{2p'+1})$ with $p, p' \in \mathbb{N}$ be the two sequences such that $s_1 \underset{V}{\rightarrowtail} \underline{i}$. From Proposition 5.5, it is sufficient to determine $u_{0,\underline{i}_{|x_1=0}}$. By definition of type V sequences and the boundary condition (5.5) we have, thanks to the uniform Kreiss–Lopatinskii condition:

$$u_{0,\underline{i}_{|x_1=0}} = -\Pi^{\underline{i}}\phi_1^{\underline{i}} B_1 \Big(u_{0,\ell_{2p+1}{}_{|x_1=0}} + u_{0,\ell'_{2p'+1}{}_{|x_1=0}} \Big) - \Pi^{\underline{i}}\phi_1^{\underline{i}} B_1 \sum_{k \in (\Phi(\underline{i}) \cap \mathscr{I}_{oi}) \setminus \left\{ \ell_{2p+1}, \ell'_{2p'+1} \right\}} u_{0,k_{|x_1=0}}. \quad (5.39)$$

We have several cases to consider to express (5.39) in a suitable way. It depends on the values of the end of the sequences ℓ and ℓ'.

  – First, if $\ell_{2p+1} \neq \ell_{2p'+1}$, then we claim that we have

$$\begin{cases} (\Phi(\underline{i}) \cap \mathscr{I}_{oi}) \setminus \left\{ \ell_{2p+1}, \ell'_{2p'+1} \right\} = \emptyset, \\ s_1 \underset{V}{\rightarrowtail} \ell_{2p+1} \quad \text{and} \quad s_1 \underset{V}{\rightarrowtail} \ell'_{2p'+1} \text{ by exactly one type V sequence,} \end{cases}$$

  so that Proposition 5.15 applies to determine each of the amplitudes $u_{0,\ell_{2p+1}}$ and $u_{0,\ell'_{2p'+1}}$ and thus $u_{0,\underline{i}_{|x_1=0}}$ is known from (5.39). Consequently $u_{0,\underline{i}}$ is constructed if $\ell_{2p+1} \neq \ell'_{2p'+1}$.
  We now prove the claim. We proceed once again by contradiction by assuming that there exists some $\underline{j} \in \Phi(\underline{i}) \cap \mathscr{I}_{oi}$ such that $\underline{j} \neq \ell_{2p+1}, \ell'_{2p'+1}$. By minimality of the frequencies set such a $\underline{j}$ comes from the reflection of some $\underline{k} \in \mathscr{I}_{io} \cap \Psi(\underline{j})$. From Proposition 4.9, such $\underline{k}$ is linked to s_1 by some sequence ℓ''. Using the same arguments as in the proof of Proposition 5.15, we obtain that $\underline{i} \notin \ell''$ (once again we use the uniqueness of the self-interacting loop).
  Consequently the sequence $(\ell'', \underline{k}, \underline{j})$ links s_1 to $\underline{i}$. The fact that $\underline{i} \notin \ell''$ implies that the sequence $(\ell'', \underline{k}, \underline{j}, \underline{i})$ can not be simplified into $(\ell, \underline{i})$ or $(\ell', \underline{i})$. We thus have constructed three type V sequences linking s_1 to $\underline{i}$ which is excluded by definition of $\underline{i}$. It gives the first point of the claim.
  For the second one, we proceed by contradiction and assume that there exists a sequence ℓ'' differing from ℓ and such that $s_1 \underset{V}{\rightarrowtail} \ell_{2p+1}$, then the sequences (ℓ'', ℓ_{2p+1}), ℓ and ℓ' are three distinct type V sequences linking s_1 to $\underline{i}$ which is again a contradiction[6]. The previous proofs are summarized in Figures 5.3 and 5.4.

[6] Indeed the sequence (ℓ'', ℓ_{2p+1}) can be simplified into ℓ if and only if ℓ_{2p+1} admits a self-interaction loop.

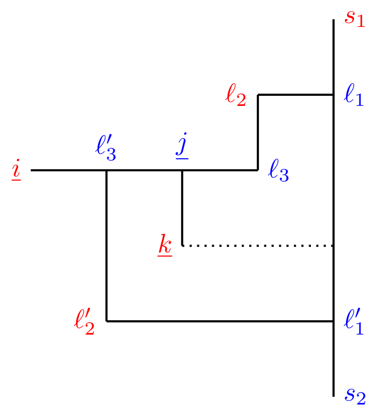

Figure 5.3. Illustration of the first statement of the claim.

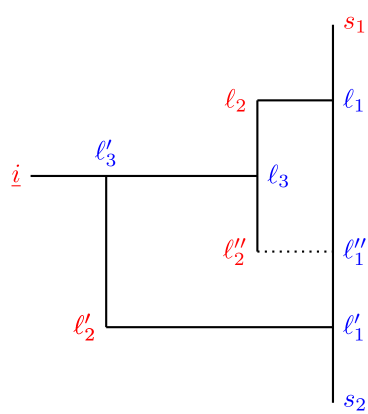

Figure 5.4. Illustration of the second statement of the claim.

– If now we have $\ell_{2p+1} = \ell'_{2p'+1}$, then the above argument fails. Indeed, we can show that we still have $(\Phi(\underline{i}) \cap \mathscr{I}_{oi}) \setminus \{\ell_{2p+1}\} = \emptyset$, but now ℓ_{2p} is linked to s_1 by two type V sequences, namely $(\ell_1, \dots, \ell_{2p-1})$ and $(\ell'_1, \dots, \ell'_{2p'-1})$. Consequently, Proposition 5.15 does not apply directly to determine the right-hand side of (5.39). We refer to Figure 5.5.
We thus need to make the determination of $u_{0,\ell_{2p+1}}$ more precise. Because $\ell_{2p+1} \in \mathscr{I}_{oi}$, using Proposition 5.5 it is sufficient to determine $u_{0,\ell_{2p+1}|_{x_2=0}}$. From the definition of the type V

sequences ℓ and ℓ', the boundary condition (5.6) and the uniform Kreiss–Lopatinskii condition; we have

$$\begin{aligned} u_{0,\ell_{2p+1}|_{x_2=0}} &= -\Pi^{\ell_{2p+1}}\phi_2^{\ell_{2p+1}} B_2\Big(u_{0,\ell_{2p}|_{x_2=0}} + u_{0,\ell'_{2p'}|_{x_2=0}}\Big) \\ &\quad - \Pi^{\ell_{2p+1}}\phi_2^{\ell_{2p+1}} B_2 \sum_{k\,\in\,(\Psi(\underline{i})\,\cap\,\mathscr{I}_{io})\setminus\{\ell_{2p},\ell'_{2p'}\}} u_{0,k|_{x_2=0}}, \end{aligned} \tag{5.40}$$

if $\ell_{2p} \neq \ell'_{2p'}$. We then claim that we have

$$\begin{cases} (\Psi(\underline{i}) \cap \mathscr{I}_{io}) \setminus \{\ell_{2p}, \ell'_{2p'}\} = \emptyset, \\ s_1 \underset{V}{\rightarrowtail} \ell_{2p} \quad \text{and} \quad s_1 \underset{V}{\rightarrowtail} \ell'_{2p'} \text{ by exactly one type V sequence,} \end{cases}$$

so that the right-hand side of (5.40) only depends on $u_{0,\ell_{2p}|_{x_2=0}}+ u_{0,\ell'_{2p'}|_{x_2=0}}$, which is known from Proposition 5.15. The proof of the new claim is totally similar to the previous claim, so that the proof is omitted here. If now $\ell_{2p} = \ell'_{2p}$, we consider the preceding elements ℓ_{2p-1} and $\ell'_{2p'-1}$ and repeat the same arguments until that we find two elements such that $\ell_q \neq \ell'_{q'}$. This necessarily occurs at some step because $\ell \neq \ell'$. This ends up the determination of $u_{0,\underline{i}}$ when $\underline{i} \in \mathscr{I}_{io}$.

- If $\underline{i} \in \mathscr{I}_{oi}$ then the proof is the same *mutatis mutandis* as the one for the case $\underline{i} \in \mathscr{I}_{io}$. We feel free to omit this proof here.
- For all possible layers, that is to say $\underline{i} \in \mathscr{I}_{ev1}$, $\underline{i} \in \mathscr{I}_{ev2}$, $\underline{i} \in \mathscr{I}_{g_1}$ or $\underline{i} \in \mathscr{I}_{g_2}$, then depending on the nature of $\underline{i}$, only one of the traces on $\{Y_1 = 0\}$ or $\{Y_2 = 0\}$ needs to be determined (where Y_p stands

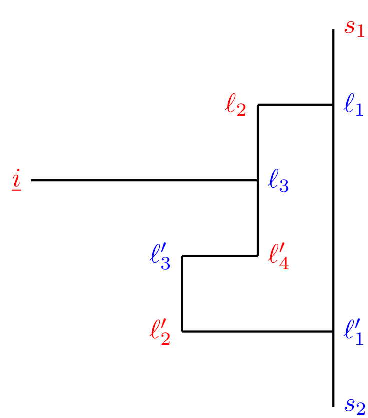

Figure 5.5. Illustration for the case $\ell_{2p+1} = \ell'_{2p'+1}$.

for the associated fast variable of the boundary layer). The boundary conditions (5.5) or (5.6) combined with the uniform Kreiss–Lopatinskii condition permit to express this trace by an equation analogous to (5.39). The end of the determination then follows exactly the one exposed in the framework $\underline{i} \in \mathscr{I}_{io}$.

- Finally, if $\underline{i} \in \mathscr{I}_{ii}$, then the proof differs a little from the one where $\underline{i}$ is linked to s_1 by a unique type V sequence, so we will give more details. Let $\underline{i} \in \mathscr{I}_{ii}$ be linked to s_1 by two distinct sequences $\ell = (\ell_1, \dots, \ell_p)$ and $\ell' = (\ell'_1, \dots, \ell'_{p'})$. From the definition of type V sequences we have $p, p' \in \mathbb{N}$. Let us point that p and p' do not necessarily have the same parity. We thus make the following distinctions depending on the above parities:
  - If p and p' are both odd, then we have $\underline{i} \in \Phi(\ell_p) = \Phi(\ell'_{p'})$ where $\ell_p, \ell_{p'} \in \mathscr{I}_{oi}$. The claim here is that if the end of ℓ differs from the end of ℓ' $(\ell_p \neq \ell'_{p'})$ then

$$\begin{cases} (\Phi(\underline{i}) \cap \mathscr{I}_{oi}) \setminus \{\ell_p, \ell'_{p'}\} = \emptyset, \\ s_1 \underset{V}{\rightarrowtail} \ell_p \quad \text{and} \quad s_1 \underset{V}{\rightarrowtail} \ell'_{p'} \text{ by exactly one sequence of type V.} \end{cases}$$

We are in the same position as in the case $\underline{i} \in \mathscr{I}_{io}$, see Figure 5.6.

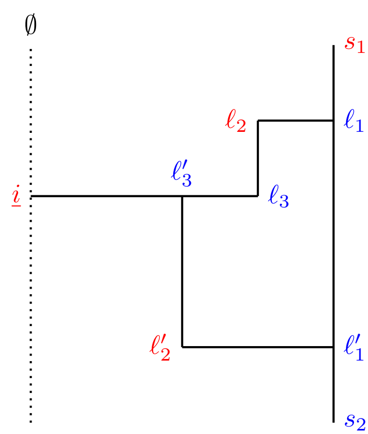

Figure 5.6. Illustration for $\underline{i} \in \mathscr{I}_{ii}$

So, from Proposition 5.15 the amplitudes associated to ℓ_p and $\ell'_{p'}$ are known and then the boundary condition (5.5) combined with the uniform Kreiss–Lopatinskii condition gives the following relation governing $u_{0,\underline{i}|x_1=0}$ in terms of the traces of the above amplitudes:

$$u_{0,\underline{i}_{|x_1=0}} = -\Pi^{\underline{i}}\phi_1^{\underline{i}} B_1 \left(u_{0,\ell_{p\,|x_1=0}} + u_{0,\ell'_{p'\,|x_1=0}} \right). \tag{5.41}$$

To conclude it remains to determine the value of the trace on $\partial\Omega_2$. Reiterating the same arguments as the ones exposed in the proof of Proposition 5.15, we can then show that $\Psi(\underline{i})\cap\mathscr{I}_{io}=\emptyset$. Indeed if such an intersection is not empty, then we can easily construct a third type V sequence linking s_1 to $\underline{i}$. As a consequence, the second trace is given by $u_{0,\underline{i}_{|x_2=0}}=0$. Propositions 5.3 and 5.5 apply and the amplitude $u_{0,\underline{i}}$ is determined.

- If p and p' are both even, then the same arguments apply except that it is now the trace of $u_{0,\underline{i}}$ on $\partial\Omega_2$ which is non trivial and given by the analogous of (5.41), the trace on $\partial\Omega_1$ being trivial.
- If p is odd and p' is even (the other case being similar), then we have on the one hand $\ell_p\in\Phi(\underline{i})$, $\ell_p\in\mathscr{I}_{oi}$ and on the other hand $\ell'_{p'}\in\Psi(\ell'_{p'})$, $\ell'_{p'}\in\mathscr{I}_{io}$. The claim is now that

$$\begin{cases} (\Phi(\underline{i})\cap\mathscr{I}_{oi})\setminus\{\ell_p\}=\emptyset=(\Psi(\underline{i})\cap\mathscr{I}_{io})\setminus\{\ell'_{p'}\}, \\ s_1 \underset{V}{\rightarrowtail} \ell_p \text{ and } s_1 \underset{V}{\rightarrowtail} \ell'_{p'} \text{ by exactly one sequence of type V.} \end{cases}$$

The proof of the claim follows the same lines as the ones of the case $\underline{i}\in\mathscr{I}_{io}$ of Proposition 5.15 and we feel free to omit the details. Consequently, the boundary conditions (5.5) and (5.6) combined with the uniform Kreiss–Lopatinskii condition for each side, give

$$u_{0,\underline{i}_{|x_1=0}} = -\Pi^{\underline{i}}\phi_1^{\underline{i}} B_1 u_{0,\ell_{p\,|x_1=0}} \quad \text{and} \quad u_{0,\underline{i}_{|x_2=0}} = -\Pi^{\underline{i}}\phi_2^{\underline{i}} B_2 u_{0,\ell'_{p'\,|x_2=0}},$$

the right-hand sides being known elements in $H^\infty_\natural(\mathbb{R}\times\mathbb{R}_+)$ from Proposition 5.15. As a consequence, Proposition 5.5 applies and gives $u_{0,\underline{i}}$. The above situation has already been depicted on Figure 3.4.

This ends up the determination of the amplitudes associated to indices $\underline{i}$ linked to s_1 by two distinct type V sequences. □

Using Proposition 5.15, we can assume that all the indices linked to s_1 by at most two sequences of type V have been determined and reiterating the same kind of proof as the one of Proposition 5.16, we can construct inductively all the elements linked to s_1 by at most P sequences of type V for any $P\geqslant 0$. Thus it determines all the leading order amplitudes in the geometric optics expansion.

The proofs are similar to the one where $\underline{i}$ was linked to s_1 by two distinct type V sequences. Indeed, the proofs of the claims only require that by contradiction we obtain $2+1$ type V sequences. We feel free to omit the details for a sake of brevity.

We have thus shown the following concluding proposition:

PROPOSITION 5.17. — *Under Assumptions 2.1, 2.2, 2.9 and 2.11, we consider a complete for reflections frequencies set satisfying the uniqueness Assumption 4.8. Finally, assume that the inversibility Assumption 5.14 holds. Then there exist* $(u_{0,k})_{k\in\mathscr{I}_{os}}$, $(U_{0,1,k})_{k\in\mathfrak{R}_1}$ *and* $(U_{0,k,2})_{k\in\mathfrak{R}_2}$ *satisfying the cascade of equations* (5.2), (5.7) *and* (5.8) *at first order.*

Once that the leading order term of the geometric optics expansion is constructed, then it is a simple and classical exercise to construct the higher order corrector terms. Indeed the only difference being that the hyperbolic and evanescent terms of order more than two are not polarized any more. As a consequence, these terms will be determined by the same equations as in Paragraphs 5.2.1 and 5.2.2 up to some extra interior terms depending on the above non-vanishing unpolarized part. However, the unpolarized part is determined uniquely from the preceding terms so that these source terms are known and Proposition 5.4 and Lemma 5.6 apply. Similarly the construction of high order terms for glancing modes is rather classical. It follows the one given in [17].

Consequently we omit the details here for a sake of brevity. It concludes the construction part of the proof of Corollary 2.13.

6. Construction of the geometric optics expansion under Assumption 4.10

The construction for geometric optics expansion when the set of frequencies admits an arbitrary number of self-interaction loops looks like the one when we have uniqueness of the self-interaction loop. But, we have to be a little more precise about the order of determination of the amplitudes.

Let us first stress that the ansatz when the set of frequencies contains several loops is the same as the one when we have uniqueness of the self-interaction loop that is to say

$$\begin{aligned} u^\varepsilon(t,x) \sim \sum_{n\geqslant 0} \sqrt{\varepsilon}^n \sum_{k\in\mathscr{I}_{\text{hyp}}} & e^{\frac{i}{\varepsilon}\varphi_k(t,x)} u_{n,k}(t,x) \\ & + \sum_{n\geqslant 0} \sqrt{\varepsilon}^n \sum_{k\in\mathscr{I}_{g1}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} u_{n,k}\left(t, x_2, \frac{x_1}{\sqrt{\varepsilon}}\right) \\ & + \sum_{k\in\mathscr{I}_{g2}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} u_{n,k}\left(t, x_1, \frac{x_2}{\sqrt{\varepsilon}}\right) \\ & + \sum_{n\geqslant 0} \sqrt{\varepsilon}^n \sum_{k\in\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}(t,x_2)} U_{n,k,1}\left(t, x_2, \frac{x_1}{\varepsilon}\right) \\ & + \sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}(t,x_1)} U_{n,k,2}\left(t, x_1, \frac{x_2}{\varepsilon}\right). \end{aligned} \tag{6.1}$$

As a consequence, when we inject this ansatz in the interior equation, in the boundary conditions and finally in the initial condition, it still leads us to solve the cascade of equations (5.2)–(5.7) and (5.8). Similarly, the reformulation steps of Subsection 5.2 are unchanged because the cascade of equations is the same. But, let us stress that in this cascade, this will specifically be important for the boundary cascade (5.7), we now have several self-interaction phenomena (all of them being hidden in equation (6.1), in the first sum on $\mathscr{I}_{\text{hyp}}$).

As a consequence, we have to study how the order of determination of the amplitudes is affected by several self-interaction loops. This is described in the following paragraph for the leading order term. We do not describe the determination of higher order terms here. The only point to keep in mind is that because these amplitudes are not polarized any more, we have extra interior source terms compared to the leading order method of resolution. However in Subsection 5.2 we anticipated such a case of study.

In this section we describe how the method described in Section 5 needs to be modified to construct the geometric optics expansions when the frequencies set contains several (simple) self-interaction loops. In order to do so, let us first remark the following important refinements of the construction in the unique self-interaction loop framework of Section 5:

(1) The determination of the elements of the “first” self-interaction loop (namely the one turned on by the source g^ε) only requires that this loop is not a composite loop. So that we can reproduce the

determination of the first loop's amplitudes in order to initialize the resolution.

(2) The determination of the elements away from the loop based on the number of type V sequences only requires that the considered sequence do not contain any self-interaction index.

As a consequence, we can use the same arguments as in Section 5 to determine first the amplitudes associated to the indices in the first loop and then to determine the amplitudes which are linked to s_1^1 by an arbitrary number of type V sequence if all these sequences do not contain any self-interaction indices.

To save some vocabulary we introduce the following definition:

Definition 6.1 (Simply regenerated index). — *Let $\underline{i} \in \mathscr{I}$. We say that $\underline{i}$ is simply regenerated if all the type V sequences linking $\underline{i}$ to s_1^1 do not contain any self-interacting indices except the ones of $\{s^1\}$.*

6.1. Determination of the first self-interaction elements and determination of simply regenerated indices

In this paragraph we first justify that if the first self-interaction loop, namely the one turned on by the source g^ε is a simple loop, then we can reproduce the computations made in Subsection 5.3.1.

The only point to be clarified is Lemma 5.13. However, a careful look at the proof of Lemma 5.13 shows that to conclude we do not really require the uniqueness of the self-interaction loop. We only use that the loop $\{s^1\}$ is simple in the sense of Definition 4.7. It is the case under Assumption 4.10.

As a consequence, we can reproduce the computations made in Subsection 5.3.1. It gives an initialization condition reading under the form (5.36). More precisely we should have

$$(I - \mathbf{T}^1) u_{0,s^1_{2b_1}}{}_{|x_1=0} = \mathbf{T}^1 g,$$

where $\mathbf{T}^1$ is defined by (5.37) (the exponent here only specifies that it is the operator obtained by considering the first interaction loop namely $\{s^1\}$).

We assume that the operator $I - \mathbf{T}^1$ is invertible on the space $H^\infty_\natural(\mathbb{R}\times\mathbb{R}_+)$. It gives $u_{0,s^1_{2b_1}}{}_{|x_1=0}$ and all the amplitudes associated to the loop $\{s^1\}$ are determined.

Because at the end of the day we will have to consider all the self-interaction loops in $\mathscr{I}$ we make the following assumption. This is just a generalization of Assumption 5.14 to the framework of Assumption 4.10.

Assumption 6.2. — *We assume that for all* $a \in [\![1, A]\!]$, *the operator* $I - \mathbf{T}^a$ *defined by* (5.37) *and obtained by repeating the computations of Subsection 5.3.1, to the loop* $\{s^a\}$ *is invertible from* $H^\infty_\natural(\mathbb{R} \times \mathbb{R}_+)$ *into* $H^\infty_\natural(\mathbb{R} \times \mathbb{R}_+)$.

We now describe the determination of simply regenerated indices in $\mathscr{I}$. To do this we use the fact that because the type V sequence(s) linking $\underline{i}$ to s_1^1 do not contain self-interaction indices, then the analysis of Subsection 5.3.2 can be reproduced. Indeed, the only points where we used Assumption 4.8 on self-interaction loop in the proofs of Propositions 5.15 and 5.16 are at the first step of Proposition 5.15 and the first point of Proposition 5.16 where we required that a sequence reading $(\underline{i}, \ell'_{2r+3}, \dots \ell'_{2p'+1}, \underline{k}, \underline{j}, \underline{i})$ can not be a self-interaction loop for $\underline{i}$.

It is the case for simply regenerated indices because in the sequence(s) linking these indices to s_1^1 we do not have any self-interaction indices. As a consequence, Propositions 5.15 and 5.16 apply and it gives the determination of all simply regenerated indices.

At this step of the proof, we thus have determined the amplitudes for the indices in the first loop and the indices which are linked to s_1^1 by type V sequence(s) which do not contain any self-interaction indices except the ones of $\{s^1\}$. In the following, to conclude the whole determination of the amplitudes, we consider the indices linked to s_1^1 by type V sequences containing self-interaction indices (differing from the ones of $\{s^1\}$).

6.2. Determination of the other amplitudes

The determination of the other amplitudes in the geometric optics expansion follows essentially the same sketch of construction as the one performed under Assumption 4.8. We will first determine the amplitudes linked to s_1^1 by a unique type V sequence containing self-interaction indices away from those of $\{s^1\}$ (that is to say that the indices are not simply regenerated any more). It is made in Subsection 6.2.1.

Then, to conclude the whole determination we proceed, inductively by considering indices linked to s_1^1 by two type V sequences containing self-interaction indices (away from the ones of $\{s^1\}$) and so on.

6.2.1. The determination of indices linked by one sequence

In all this paragraph we consider an index $\underline{i} \in \mathscr{I}$. From Proposition 4.9 it is linked to s_1^1 by (possibly many) type V sequences. We assume in the following that $\underline{i}$ is linked to s_1^1 by a unique type V sequence and that, because we have already determined simply regenerated indices, this sequence contains self-interaction indices away from the ones of $\{s^1\}$.

Before to give a precise sketch of construction of the amplitude associated to $\underline{i}$, let us give the following lemma which describes the structure of the considered type V sequence.

LEMMA 6.3. — *Consider a complete for reflections frequencies set satisfying Assumption 4.10. Let* $\underline{i} \in \mathscr{I}$ *and let* ℓ *be a type* V *sequence linking* $\underline{i}$ *to* s_1^1*. Note that we authorize* ℓ *to contain self-interaction indices differing from the ones of* $\{s^1\}$*. Let* $\ell_p := s_{p'}^a$ *and* ℓ_q *be two consecutive self-interacting indices of* ℓ*. Then one of the following alternatives is satisfied:*

(1) *If* $q = p + 1$*, then* $\ell_q = s_{p'+1}^a$ *(with the convention that if* $p' = 2b_a$ *then* $s_{p'+1}^a = s_1^a$*).*
(2) *If* $q > p + 1$*, then* $\ell_q = s_{q'}^{a'}$ *with* $a \neq a'$ *and for some* $q' \in [\![1, 2b_{a'}]\!]$*.*

As a consequence if the sequence ℓ *contains self-interacting indices differing from the ones of* $\{s^1\}$*, then it reads under the form*

$$\ell := \begin{pmatrix} s_2^1, \dots, s_p^1, \ell_{p+1}, \dots, \ell_{q-1}, s_{q'}^a, \dots, s_{q'+r}^a, \\ \ell_{q+r+1}, \dots, \ell_{q+r+l}, s_{q''}^{a'}, \dots, s_{q''+r'}^{a'}, \dots, \ell_f \end{pmatrix},$$

or

$$\ell := \begin{pmatrix} \ell_1, \dots, \ell_{p-1}, s_{p'}^a, \dots, s_{p'+r}^a, \ell_{p+r+1}, \dots, \\ \ell_{q+r+l}, s_{q'}^{a'}, \dots, s_{q'+r'}^{a'}, \dots, \ell_f \end{pmatrix},$$

where the $\ell_\cdot$ *are non self-interacting indices.*

Proof. — To fix the ideas and to simplify the exposition, we assume that $\ell_p := s_2^1 \in \mathscr{I}_{oi}$. This special proof can then be extended to the general framework.

Let us assume that $q = p+1$. We want to show that $\ell_q = s_3^1$. By contradiction let us assume that $\ell_q = s_{q'}^{a'} \in \mathscr{I}_{io} \cap \Phi(s_2^1)$ with $a' \neq a$. Then because $s_{q'}^{a'}$ is self-interacting we can find a sequence ℓ' such that $(s_{q'}^{a'}, \ell', s_{q'}^{a'})$ is a loop for $s_{q'}^{a'}$. In particular the last index of ℓ', namely ℓ'_l, satisfies $\ell'_l \in \Phi(s_{q'}^{a'}) = \Phi(s_2^1)$ and $\ell'_l \in \mathscr{I}_{oi}$. Then by construction the sequence $(s_1^1, s_2^1, s_{q'}^{a'}, \ell', s_3^1, \dots, s_{2b}^1, s_1^1)$ is a loop for the index s_1^1. It differs from the unique self-interaction loop $\{s^1\}$ for s_1^1. Indeed $s_{q'}^{a'}$ is not an element of s^1. As a consequence, $\ell_q \in \{s^1\}$ because $\{s^1\}$ is a simple self-interaction loop and we necessarily have $\ell_q = \ell_{p+1} = s_3^1$.

Let us assume now that $q > p + 1$. By contradiction we thus assume that $\ell_q \in \{s^1\}$. For simplicity, we here justify that we have $\ell_q \notin \{s_3^1, s_4^1, s_5^1, s_1^1\}$, the proof for the other indices follows the same lines.

Assume by contradiction that $\ell_q = s_4^1$. By definition of type V sequences we can find non self-interacting indices $\ell_{p+1}, \ell_{p+2}, \dots, \ell_{p+2r+1}$ such that $(s_1^2, \ell_{p+1}, \ell_{p+2}, \dots, \ell_{p+2r+1})$ forms a type V sequence linking s_4^1 to s_1^1. As a consequence, the sequences $(s_1^1, s_2^1, \dots, s_{2b^1}^1, s_1^1)$ and $(s_1^1, s_2^1, \ell_{p+1}, \dots, \ell_{p+2r+1}, s_4^1, \dots, s_{2b^1}^1, s_1^1)$ are two distinct loops for s_1^1 which is excluded because $\{s^1\}$ is a simple self-interaction loop.

We now justify that we can not have $\ell_q = s_5^1$. Proceeding similarly we can find non self-interacting indices $\ell_{p+1}, \ell_{p+2}, \dots, \ell_{p+2r}$ such that $(s_1^2, \ell_{p+1}, \ell_{p+2}, \dots, \ell_{p+2r})$ forms a type V sequence linking s_5^1 to s_1^1. Once again the existence of such a sequence contradicts the fact that the loop $\{s^1\}$ is simple.

The proof is the same to justify that $\ell_q \neq s_1^1, s_3^1$.

The previous discussion is illustrated on Figures 6.1 and 6.2.

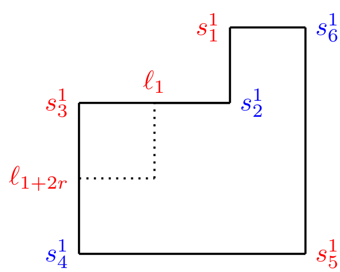

Figure 6.1. Illustration of the structure of type V sequences.

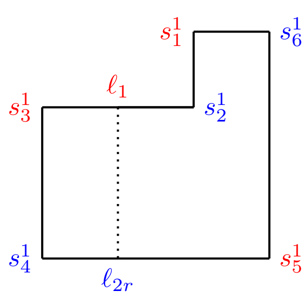

Figure 6.2. Illustration of the structure of type V sequences bis.

□

The simple self-interacting indices being determined from the analysis of Section 5, we now turn to the determination of indices which are not. Such indices are linked to s_1^1 by (at least) one type V sequence(s) containing self-interaction indices differing from the ones of $\{s^1\}$. We start the determination by the indices which are linked by a unique type V sequence containing self-interaction indices which differ from the ones of $\{s^1\}$. We stress that from Lemma 6.3 several loops can be visited in such a sequence.

Proposition 6.4. — *Consider a complete for reflections frequencies set satisfying the loop Assumption 4.10 and the inversibility condition Assumption 6.2. Let $\underline{i} \in \mathscr{I}$ be such that there exists a unique type V sequence linking $\underline{i}$ to s_1^1 and containing self-interaction indices (differing from the ones of $\{s^1\}$). Then $u_{0,\underline{i}}$ or $U_{0,\underline{i},p}$, $p = 1, 2$, solving the cascade of equations* (5.2), (5.7) *and* (5.8) *at first order can be uniquely determined from the source g.*

Proof. — We have to study different cases depending on the nature of the index $\underline{i}$. In all cases, from Lemma 6.3 we know that ℓ, the sequence linking s_1^1 to $\underline{i}$, reads under the form

$$\ell = \begin{pmatrix} \ell_1, \dots, \ell_{p-1}, s^a_{p'}, \dots, s^a_{p'+r}, \ell_{p+r+1}, \\ \dots, \ell_{q+r+l}, s^{a'}_{q'}, \dots, s^{a'}_{q'+r'}, \dots, \ell_f \end{pmatrix}, \tag{6.2}$$

or possibly

$$\ell = \begin{pmatrix} s^1_2, \dots, s^1_p, \ell_{p+1}, \dots, \ell_{q-1}, s^a_{q'}, \dots, s^a_{q'+r}, \\ \ell_{q+r+1}, \dots, \ell_{q+r+l}, s^{a'}_{q''}, \dots, s^{a'}_{q''+r'}, \dots, \ell_f \end{pmatrix}.$$

Because the self-interacting elements in the first loop $\{s^1\}$ are determined from Subsection 6.1, there is no loss of generality by assuming that ℓ reads under the form (6.2). The first terms can possibly depend on the amplitudes indexed by elements of $\{s^1\}$, but such elements are known and depend explicitly on the source g.

Step 1: Entering in the first self-interaction loop after $\{s^1\}$. — We first show that we can reiterate the computations of Subsection 5.3.1, in order to determine the first indices $s^a_{\cdot}$ appearing in ℓ.

We make this justification when $\ell_{p-1} \in \mathscr{I}_{oi}$. The analysis is similar when $\ell_{p-1} \in \mathscr{I}_{io}$. In such a situation we have, because of the definition of type V sequences, $s^a_{p'} \in \mathscr{I}_{io}$ and we want to justify that

$$\begin{cases} \text{any } \Phi(s^a_{p'}) \cap \mathscr{I}_{oi} \setminus \{\ell_{p-1}, s^a_{\underline{p}}\} \text{ is simply regenerated,} \\ \ell_{p-1} \text{ is simply regenerated.} \end{cases} \tag{6.3}$$

where $s^a_{\underline{p}}$ stands for the incoming-outgoing index of s^a such that $s^a_{\underline{p}} \in \Phi(s^a_{p'})$. The second point of the claim (6.3) is clear.

To prove the first part of the claim, we consider $\underline{j} \in \Phi(s^a_{p'}) \cap \mathscr{I}_{oi} \setminus \{\ell_{p-1}, s^a_{\underline{p}}\}$. From Proposition 4.9 this index $\underline{j}$ is linked to s^1_1 by type V sequences and such sequences ℓ' read *a priori* under the form (6.2). However if there exists self-interaction indices in ℓ' then the sequence $(\ell', \underline{j})$ is a sequence (containing self-interaction indices differing from the ones of $\{s^1\}$) linking $\underline{i}$ to s^1_1 and differing from ℓ. This is excluded from the choice of $\underline{i}$ for which we have uniqueness of such a sequence.

As a consequence, any element in $\Phi(s^a_{p'}) \cap \mathscr{I}_{oi} \setminus \{s^a_{\underline{p}}\}$ is simply regenerated and thus it is determinable from the analysis of Section 5. So the boundary condition (5.5) for $s^a_{p'}$ reads after the use of the uniform Kreiss–Lopatinskii condition:

$$u_{0,s^a_{p'}\,|x_1=0} = -\Pi^{s^a_{p'}} \phi_1^{s^a_{p'}} B_1 u_{0,s^a_{\underline{p}}\,|x_1=0} - \Pi^{s^a_{p'}} \phi_1^{s^a_{p'}} B_1 \sum_{j \in \Phi\left(s^a_{p'}\right) \cap \mathscr{I}_{oi} \setminus \left\{s^a_{\underline{p}}\right\}} u_{0,j\,|x_1=0},$$

the second term in the right-hand side being a known function depending explicitly on g. We can not apply Proposition 5.5, because the first term in the right-hand side is a self-interacting index in the loop $\{s^a\}$. We have determined the term which acts as a source term, but we need to reproduce the computations of Subsection 5.3.1 to determine the amplitude $u_{0,s^a_{p'}}$.

Step 2: Solving the first loop. — We can reiterate exactly the same process for all the elements of the considered self-interaction loop $\{s^a\}$ by showing that if $s^a_p \in \mathscr{I}_{io}$ we have

$$u_{0,s^a_p\,|x_1=0} = -\Pi^{s^a_p} \phi_1^{s^a_p} B_1 u_{0,s^a_{p-1}\,|x_1=0} - g_p,$$

and if $s^a_p \in \mathscr{I}_{oi}$ we have

$$u_{0,s^a_p\,|x_2=0} = -\Pi^{s^a_p} \phi_2^{s^a_p} B_2 u_{0,s^a_{p-1}\,|x_2=0} - g_p,$$

where g_p depends on the suitable trace of the known amplitudes $u_{0,\ell_{p-1}}$ (which depend on g from Assumption 4.10). Indeed the proof is exactly the same for all indices s^a_p included in (6.2). For non visited indices we proceed similarly except that we form an other type V sequence by passing through $s^a_{p'}$.

For instance, consider to simplify that $s^a = \{s^a_{p'}, s^a_{p'+1}, s^a_{p'+2}, s^a_{\underline{p}}\}$ and assume that the sequence ℓ only contains $s^a_{p'}$ and $s^a_{p'+1}$. By contradiction, we assume that there exists $\underline{j}$ a non simply regenerated index in $\Psi(s^a_{\underline{p}}) \cap \mathscr{I}_{io} \setminus \{s^a_{p'+2}\}$. Then it is linked to s^1_1 by a sequence ℓ' containing self-interaction indices and thus the sequence $(\ell', \underline{j}, s^a_{p'}, s^a_{p'+1}, \dots)$ is an other type V sequence containing non self-interaction indices sequence linking $\underline{i}$ to s_1^1. It is impossible because such a sequence is assumed to be unique.

As a consequence, we can reproduce the same computations as the ones of Subsection 5.3.1 which give rise to a new compatibility condition reading

$$(I - \mathbf{T}^a)u_{0,s^a_{p'}}{}_{|x_1=0} = \mathbf{T}^a g^a, \tag{6.4}$$

where the operator $\mathbf{T}^a$ is defined by (5.37) applied to the loop $\{s^a\}$ and where g^a depend explicitly on the source g.

From Assumption 6.2, we invert $I - \mathbf{T}^a$ to obtain $u_{0,s^a_{p'}}{}_{|x_1=0}$. We then deduce the values of the visited self-interacting elements of $\{s^a\}$ in ℓ from the explicit formulas of Subsection 5.3.1.

Step 3: Next self-interaction loops. — From the above results the sequence ℓ now reads under the form (6.2) where we now have $a = a'$ and where the $\ell_1, \ldots, \ell_{p-1}$ now depend on the (known) self-interaction indices of $\{s^1\}$ and $\{s^a\}$. We can thus reproduce Steps 1 and 2 where we now used "does not contain self-interaction indices except in $\{s^1\}$ or $\{s^a\}$" for the new concept of simply regenerated indices.

So that pass each loop gives rise to a compatibility condition under the form (6.4) which can be solved uniquely from Assumption 6.2.

Step 4: The end of the sequence. — Using Step 3 as many times as there are distinct self-interacting groups of elements in ℓ we can now assume that ℓ reads under the form

$$\ell = (\ell_1, \ldots, \ell_t, \ell_{t+1}, \ldots, \ell_f), \quad \text{for some } \; t \leqslant f$$

where the first terms $\ell_1, \ldots, \ell_t$ depend on the visited self-interaction indices in ℓ and where $\ell_{t+1}, \ldots, \ell_f$ do not contain any self-interaction indices. Consequently all the amplitudes associated to the indices in the sequence ℓ are known.

Step 5: The determination of $u_{0,\underline{i}}$. — To conclude it remains to determine $u_{0,\underline{i}}$. We distinguish several cases depending on the nature of the index $\underline{i}$.

- If $\underline{i} \in \mathscr{I}_{io}$, then we distinguish two subcases depending on if $\underline{i}$ is self-interacting or not:
  - If $\underline{i} \notin \mathscr{S}$. In such a case the boundary condition (5.5) determining $u_{0,\underline{i}}{}_{|x_1=0}$ reads:

$$u_{0,\underline{i}}{}_{|x_1=0} = -\Pi^{\underline{i}} \phi_1^{\underline{i}} B_1 \sum_{j \in \Phi(\underline{i}) \cap \mathscr{I}_{oi}} u_{0,j}{}_{|x_1=0} \tag{6.5}$$

where the sum in the right-hand side contains the index ℓ_f (whose amplitude is known) and possibly other indices which are simply regenerated (from the uniqueness of the sequence ℓ) so that their amplitudes can be determined. As a consequence, Proposition 5.5 applies and thus $u_{0,\underline{i}}$ is determined from (6.5).

– If $\underline{i} \in \mathscr{S}$, then the boundary condition determining $u_{0,\underline{i}\,|x_1=0}$ reads

$$u_{0,\underline{i}\,|x_1=0} = -\Pi^{\underline{i}}\phi_1^{\underline{i}} B_1 u_{0,\underline{j}\,|x_1=0} - \Pi^{\underline{i}}\phi_1^{\underline{i}} B_1 \sum_{j\,\in\,\Phi(\underline{i})\,\cap\,\mathscr{I}_{oi}\backslash\{\underline{j}\}} u_{0,j\,|x_1=0}, \tag{6.6}$$

where $\underline{j}$ stands for the self-interacting outgoing-incoming index such that $\underline{j} \in \Phi(\underline{i})$. As in the subcase $\underline{i} \notin \mathscr{S}$, the second term in the right-hand side of (6.6) is a known function.

Reiterating the arguments of the above Steps 1 and 2, we can justify that all the boundary conditions involving the elements of the loop containing $\underline{i}$ read under the form (6.6).

This leads us to a compatibility condition $(I - \mathbf{T}^{a_{\underline{i}}})u_{0,\underline{i}\,|x_1=0} = g^{\underline{i}}$, where $a_{\underline{i}}$ is such that $\underline{i} \in \{s^{a_{\underline{i}}}\}$ and where $g^{\underline{i}}$ depends on the right-hand side of (6.6), so on g. Inverting the operator $I - \mathbf{T}^{a_{\underline{i}}}$ thanks to Assumption 6.2 thus gives the desired value of $u_{0,\underline{i}\,|x_1=0}$. As a consequence, Proposition 5.5 applies and thus $u_{0,\underline{i}}$ is determined.

- If $\underline{i} \in \mathscr{I}_{oi}$, then the proof follows the same lines as for the case $\underline{i} \in \mathscr{I}_{io}$. We feel free to omit the details here.
- Similarly when $\underline{i} \in \mathscr{I}_{ii} \cup \mathscr{I}_{ev1} \cup \mathscr{I}_{ev2} \cup \mathscr{I}_{g_1} \cup \mathscr{I}_{g_2}$, then once it is clear that the amplitude associated to the last index of ℓ, namely ℓ_f, has been determined the method of determination of Section 5 applies.

□

As a consequence we have determined all the amplitudes which are linked to s_1^1 by type V sequences where at most one of these sequences contains self-interacting elements away from $\{s^1\}$.

6.2.2. Determination of the other amplitudes

In the spirit of Section 5, we then turn to the indices linked to s_1^1 by two such sequences.

Proposition 6.5. — *Consider a complete for reflections frequencies set satisfying Assumptions 4.10. We also assume that we have Assumption 6.2.*

Let $\underline{i} \in \mathscr{I}$ be an index such that we have two distinct sequences ℓ, ℓ', containing self-interaction indices differing from the ones[7] of $\{s^1\}$ which link s_1^1 to $\underline{i}$. Then $u_{0,\underline{i}}$ or $U_{0,\underline{i},1}$ solving the cascade of equations (5.2)*,* (5.7) *and* (5.8) *can be uniquely determined from the source g.*

Proof. — Using Lemma 6.3, we can assume that the two sequences ℓ, ℓ' read under the form:

$$\ell = \big(\ell_1, \dots, \ell_{p-1}, s_q^a, \dots, s_{q+r}^a, \dots, \ell_f\big),$$

and

$$\ell' = \Big(\ell'_1, \dots, \ell'_{p'-1}, s_{q'}^{a'}, \dots, s_{q'+r'}^{a'}, \dots, \ell'_{f'}\Big),$$

where before to enter into the loop $\{s^a\}$ (resp. $\{s^{a'}\}$) the indices $\ell_1, \dots, \ell_{p-1}$ (resp. $\ell'_1, \dots, \ell'_{p'-1}$) only depend on the self-interacting indices of $\{s^1\}$. So that, we can assume that the associated amplitudes are known from the previous discussion.

The remaining of the proof looks like the one of Proposition 5.16. We have to consider several cases depending on the kind of the index $\underline{i}$. To fix the ideas let us assume that $\underline{i} \in \mathscr{I}_{io}$, the other cases being essentially similar. We have several possibilities:

- Let us assume first that $\ell_f \neq \ell'_{f'}$, then we claim that

$$\begin{cases} \text{every element in } \Phi(\underline{i}) \cap \mathscr{I}_{oi} \setminus \{\ell_f, \ell'_{f'}\} \text{ is simply regenerated,} \\ \ell_f \text{ and } \ell'_{f'} \text{ are linked to } s_1^1 \text{by only one sequence} \\ \text{containing self-interacting indices.} \end{cases} \tag{6.7}$$

  This is a direct consequence of the fact that $\underline{i}$ is linked to s_1^1 by exactly two type V sequences containing self-interacting elements (see the proof of Proposition 5.16). We illustrate the situation on Figure 6.3.

  So that, we can apply Proposition 5.16 to determine the simply regenerated elements in $\Phi(\underline{i}) \cap \mathscr{I}_{oi}$ and Proposition 6.4 to determine the amplitudes u_{0,ℓ_f} and $u_{0,\ell'_{f'}}$. It determines all the required traces to determine $u_{0,\underline{i}}$ by Proposition 5.5.
- We now consider the case where $\ell_f = \ell'_{f'}$. In such a situation the claim is the following:

$$\begin{cases} \text{Any element of } \Phi(\underline{i}) \cap \mathscr{I}_{oi} \setminus \{\ell_f\} \text{ is simply regenerated,} \\ \ell_f \text{ is linked to } s_1^1 \text{ by two type V sequences} \\ \text{containing self-interacting terms.} \end{cases} \tag{6.8}$$

[7] We stress in particular that the visited loops are not necessarily the sames.

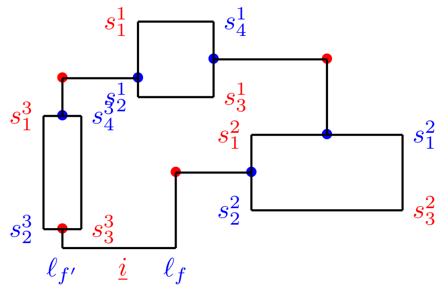

Figure 6.3. First illustration of the situation with several self-interacting loops, $\ell_f \neq \ell'_{f'}$.

The proof of the claim (6.8) is clear. It is a straightforward consequence of the fact that $\underline{i}$ is linked to s_1^1 by exactly two type V sequences containing self-interaction elements. As a consequence, the amplitude associated to any element in $\Phi(\underline{i}) \cap \mathscr{I}_{oi} \setminus \{\ell_f\}$ can be determined from the results of Subsection 6.1.

To construct $u_{0,\underline{i}}$ it is thus sufficient to determine u_{0,ℓ_f}. To do so, we explore the terms composing ℓ and ℓ' until that the two sequences differ. We can then apply the claim (6.3).

The proof operates exactly as the one of Proposition 5.16 (with "type V sequence" replaced by "type V sequence containing self-interaction elements away from the ones of $\{s^1\}$"), so that we feel free to omit the details here. We conclude by Figure 6.4 which illustrates the previously described situation. On this figure, the symbol $\square$ denotes self-interaction loop(s). $\square$

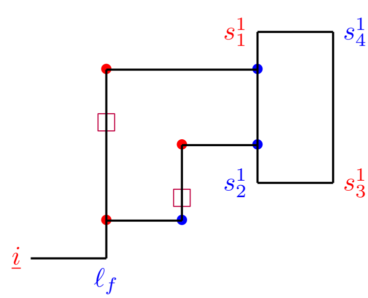

Figure 6.4. First illustration of the situation with several self-interacting loops, $\ell_f = \ell'_{f'}$.

We then generalize inductively Proposition 6.5 to indices linked to s_1^1 by three sequences and so on. This completes the determination of all the amplitudes for the leading order term.

We end up with the following proposition summarizing the above construction.

Proposition 6.6. — *Under Assumptions 2.1, 2.2, 2.9 and 2.11, we consider a complete for reflections frequencies set satisfying the multiple loops Assumption 4.10. Finally, assume that the invertibility Assumption 6.2 holds. Then there exist* $(u_{0,k})_{k\in\mathscr{I}_{os}}$*,* $(U_{0,1,k})_{k\in\mathfrak{R}_1}$ *and* $(U_{0,k,2})_{k\in\mathfrak{R}_2}$ *satisfying the cascade of equations* (5.2)*,* (5.7) *and* (5.8) *at first order.*

Because we do not want to give more details about the construction of higher order terms to complete the proof of Theorem 2.12 we only have to justify that the truncated ansatz $u^{\varepsilon}_{\mathrm{app},N_0}$ makes sense and to justify that it is indeed a good approximation of the solution to (1.1). It is the purpose of the following section.

7. Some extra remarks

To end up the proof of Theorem 2.12, let us justify that the expansion makes sense, that it is effectively an approximate solution and let us also give some more details about Assumption 6.2.

7.1. Some comments about Assumption 6.2

In order to effectively apply Theorem 2.12, we need to verify Assumption 6.2. As a consequence, we have to justify that all the operators of self-interaction are invertible. Because we are considering simple loops, these operators do not interact the ones with the others and it is thus sufficient to consider them separately.

In this paragraph, we will give some more details about the invertibility of some operator reading under the general form:

$$(I-\mathbf{T})u=g, \tag{7.1}$$

where $(\mathbf{T}u)(t,y):=\mathbf{S}u(t-\alpha y,\beta y)$ where α, β are real positive numbers and $\mathbf{S}$ is a given matrix.

Of course one favourable case, where equation (7.1) can be uniquely solved is when the operator $\mathbf{T}$ is a contraction. A simple change of variable shows that we have $\|\mathbf{T}u\|_{L^2} < \|u\|_{L^2}$ when we have the following condition on the parameters $\|\mathbf{S}\| < \sqrt{\beta}$. Then, by direct computations involving Leibniz formula, this condition can also be used as a sufficient condition for invertibility on the subspace H^∞. We here give more precisely the sufficient condition for invertibility obtained in [1, Théorème 6.9.5].

Theorem 7.1 ([1, Théorème 6.9.5]). — *We have the following sufficient condition for solving* (7.1)*:*

- *If $0 < \beta \leqslant 1$ (the ray associated to the loop concentrates at the origin) and if $\|\mathbf{S}\| < \sqrt{\beta}$, then the operator $I - \mathbf{T}$ is invertible on H^∞ to H^∞ with a flat value at the corner.*
- *If $\beta > 1$ (the ray associated to the loop spreads the information to infinity), then let $K \in \mathbb{N}$ and if $\|\mathbf{S}\|\beta^{K-1/2} < 1$, then the operator $I - \mathbf{T}$ is invertible from H^K to H^K with a flat value at the corner.*

We refer the interested reader to [1], Paragraph 6.9.5, where the above condition can be shown to be sufficient on some particular cases (for instance when $\mathbf{S}$ is reduced to a scalar). In the future, we plan to have a more complete study of Assumption 6.2, in particular, we plan to study the influence of compactly supported sources on this assumption.

7.2. Some words about the justification of the expansion

7.2.1. Some comments about the number of phases

The first point in the justification of the expansion is of course to give a precise sense to the formal series defining the ansatz (5.1). In order to do so, as in [2], we can use the assumption that the number of phases obtained during the phase generation process is finite that is that $\#\mathscr{F} < \infty$.

Of course, this assumption is far to be harmless for a given hyperbolic operator. Indeed, when N large, then it is really complicate to effectively apply the phase generation process and to verify effectively this requirement. The reason is simply that the eikonal equations encountered in the phase generation process have "many" roots at each step of the process. Consequently, check effectively that $\#\mathscr{F} < \infty$ is far to be trivial.

Let us, however mention a framework where we can easily show that this assumption is satisfied. Finite time problems can be of some help if we assume moreover that the initial source has its support away from the corner. In such a configuration we can then show that all the required traces in Propositions 5.5, 5.7 and 5.10 have their supports away from the corner (because they are obtained as explicit solution of some transport equation whose source(s) has(ve) this property). Then if we consider a given amplitude $u_{0,\underline{i}} \in \mathscr{I}_{io} \cup \mathscr{I}_{oi}$ its transported trace will have its support away from the origin. Consequently it will require some time to go on the other side. As a consequence, the descendants of $\underline{i}$ appear after the time of appearance of $u_{0,\underline{i}}$ plus some strictly positive time of travel.

The time of resolution being finite we can then not have an infinite number of phases in the process.

Of course, consider finite time problems with some source having its support away from the corner is rather unsatisfactory if we are interested in corner problem. But, it has the advantage to show that the number of phases can effectively be finite. This is however still a rather big remaining obstruction to the applicability of Theorem 2.12.

In a future contribution we aim to study the assumption $\#\mathscr{F} < \infty$ without using any support property. We have two approaches in mind:

- try to give some geometric condition on the characteristic variety $\mathscr{V}$ ensuring that the condition $\#\mathscr{F} < \infty$ holds.
- Try to bypass this assumption by characterizing the boundary conditions which give enough decay and ensure that the series
$$\sum_{k \in \mathscr{I}_{\text{hyp}}} e^{\frac{i}{\varepsilon}\varphi_k} u_{n,k}$$
makes sense even if $\#\mathscr{F} = \infty$. The amplitudes $u_{0,k}$ essentially read under the form
$$u_{0,k}(t,x) = \mathbf{S}^k g_k,$$
where g_k is some explicit evaluation of the source g along the suitable composition of the characteristics and where $\mathbf{S}^k$ describe all the coefficients of reflections $\phi_1^j B_1$ and $\phi_2^j B_2$ encountered to generate the index k. So that, find a way to ensure that $\|\mathbf{S}^k\| \downarrow 0$ sufficiently fast may be a good way to deal with geometric optics expansions without the assumption $\#\mathscr{F} < \infty$.

7.2.2. The justification in itself

We assume that $\#\mathscr{F} < \infty$, so that for $N_0 \in \mathbb{N}$, the following truncated ansatz $u^{\varepsilon}_{\text{app},N_0}$ given by:

$$u^{\varepsilon}_{\text{app},N_0}(t,x) \sim \sum_{n=0}^{N_0} \sqrt{\varepsilon}^n \sum_{k \in \mathscr{I}_{\text{hyp}}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} u_{n,k}(t,x)$$

$$
\begin{aligned}
&+\sum_{n=0}^{N_0}\sqrt{\varepsilon}^n \sum_{k\,\in\,\mathscr{I}_{g_1}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} u_{n,k}\Big(t, x_2, \frac{x_1}{\sqrt{\varepsilon}}\Big)\\
&+\sum_{k\,\in\,\mathscr{I}_{g_2}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} u_{n,k}\Big(t, x_1, \frac{x_2}{\sqrt{\varepsilon}}\Big)\\
&+\sum_{n=0}^{N_0}\sqrt{\varepsilon}^n \sum_{k\,\in\,\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}(t,x_2)} U_{n,k,1}\Big(t, x_2, \frac{x_1}{\varepsilon}\Big)\\
&+\sum_{k\,\in\,\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}(t,x_1)} U_{n,k,2}\Big(t, x_1, \frac{x_2}{\varepsilon}\Big),
\end{aligned}
\tag{7.2}
$$

makes sense as a finite sum. In order to proceed to the error analysis, we require some energy estimates for the solution u to the boundary value problem

$$
\begin{cases}
L(\partial)u = f & \text{in } \Omega,\\
B_1 u_{|x_1=0} = g_1 & \text{on } \partial\Omega_1,\\
B_2 u_{|x_2=0} = g_2 & \text{on } \partial\Omega_2,\\
u_{|t\,\leqslant\,0} = 0 & \text{on } \Gamma.
\end{cases}
\tag{7.3}
$$

We assume that this problem is strongly well-posed in the following (classical) sense. We need to introduce the following weighted L^2-spaces: for $\gamma > 0$ and $X \subset \Omega$ we define

$$
L^2_\gamma(X) := \big\{u \in \mathscr{D}'(X) \setminus e^{-\gamma t}u \in L^2(X)\big\}.
$$

Definition 7.2. — *Let $\gamma > 0$ and let $(f, g_1, g_2) \in L^2_\gamma(\Omega) \times L^2_\gamma(\partial\Omega_1) \times L^2_\gamma(\partial\Omega_2)$, we say that the corner problem* (7.3) *is strongly well-posed, if it admits a unique solution $u \in L^2_\gamma(\Omega)$, with traces in $L^2_\gamma(\partial\Omega_1) \times L^2_\gamma(\partial\Omega_2)$, satisfying the energy estimate: there exists $C > 0$ such that for all $\gamma > 0$,*

$$
\begin{aligned}
\gamma\|u\|^2_{L^2_\gamma(\Omega)} + \big\|u_{|x_1=0}\big\|^2_{L^2_\gamma(\partial\Omega_1)} + \big\|u_{|x_2=0}\big\|^2_{L^2_\gamma(\partial\Omega_2)}&\\
\leqslant C\Big(\frac{1}{\gamma}\|f\|^2_{L^2_\gamma(\Omega)} + \|g_1\|^2_{L^2_\gamma(\partial\Omega_1)} + \|g_2\|^2_{L^2_\gamma(\partial\Omega_2)}\Big)&.
\end{aligned}
\tag{7.4}
$$

The full characterization of the boundary conditions leading to strongly well-posed problems has not been achieved yet in the literature. One of the main advances in such a result is probably the analysis of [14]. This works describes a way to construct a symmetrizer which permits to obtain an *a priori* energy estimate reading under the form (7.4), but with some (non explicit) looses of derivatives.

However, for specific operators and boundary conditions, namely, for symmetric operators with strictly dissipative boundary conditions, then we can show (see for example [4]) that the associated boundary value problem is strongly well-posed in the sense of Definition 7.2.

The aim of the following is to justify that the truncated ansatz (7.2) is a good approximation of the unique solution namely u^ε if the associated boundary value problem is strongly well-posed. More precisely, we have

Proposition 7.3. — *Under the assumptions of Theorem 2.12, assume moreover that $\#\mathscr{F} < \infty$ and that the boundary value problem* (7.3) *is well-posed in the sense of Definition 7.2. Then we have the estimate*

$$\forall\ N_0 \in \mathbb{N},\ \left\| u^\varepsilon - u^\varepsilon_{\mathrm{app},N_0} \right\|_{L^2_\gamma(\Omega)} \leqslant C\sqrt{\varepsilon}^{N_0+1},$$

where u^ε stands for the unique solution to (1.1) *and where $u^\varepsilon_{\mathrm{app},N_0}$ is the truncated ansatz defined in* (7.2).

Proof. — The proof exposed here is rather classical. We first estimate the error $u^\varepsilon - u^\varepsilon_{\mathrm{app},N_0+2}$ and then we conclude by the triangle inequality. The error $u^\varepsilon - u^\varepsilon_{\mathrm{app},N_0+2}$ solves the corner problem

$$\begin{cases} L(\partial)\big(u^\varepsilon - u^\varepsilon_{\mathrm{app},N_0+2}\big) = f^\varepsilon & \text{in } \Omega, \\ B_1\big(u^\varepsilon - u^\varepsilon_{\mathrm{app},N_0+2}\big)_{|x_1=0} = 0 & \text{on } \partial\Omega_1, \\ B_2\big(u^\varepsilon - u^\varepsilon_{\mathrm{app},N_0+2}\big)_{|x_2=0} = 0 & \text{on } \partial\Omega_2, \\ \big(u^\varepsilon - u^\varepsilon_{\mathrm{app},N_0+2}\big)_{|t\leqslant 0} = 0 & \text{on } \Gamma, \end{cases} \tag{7.5}$$

where the source f^ε is defined by

$$\begin{aligned} f^\varepsilon := {} & \sqrt{\varepsilon}^{N_0+1} \sum_{k\in\mathscr{I}_{\mathrm{hyp}}} e^{\frac{i}{\varepsilon}\varphi_k} L(\partial) u_{N_0+1,k} \\ & + \sqrt{\varepsilon}^{N_0+2} \sum_{k\in\mathscr{I}_{\mathrm{hyp}}} e^{\frac{i}{\varepsilon}\varphi_k} L(\partial) u_{N_0+2,k} \\ & + \sqrt{\varepsilon}^{N_0+1} \sum_{k\in\mathscr{G}_1} e^{\frac{i}{\varepsilon}\varphi_k} L_1'(\partial) u_{N_0+1,k}\left(t, x_2, \frac{x_1}{\sqrt{\varepsilon}}\right) \\ & + \sqrt{\varepsilon}^{N_0+1} \sum_{k\in\mathscr{G}_1} e^{\frac{i}{\varepsilon}\varphi_k} A_1 \partial_{\chi_1}(u_{N_0+1,k})\left(t, x_2, \frac{x_1}{\sqrt{\varepsilon}}\right) \\ & + \sqrt{\varepsilon}^{N_0+2} \sum_{k\in\mathscr{G}_1} e^{\frac{i}{\varepsilon}\varphi_k} L_1'(\partial) u_{N_0+2,k}\left(t, x_2, \frac{x_1}{\sqrt{\varepsilon}}\right) \\ & + \sqrt{\varepsilon}^{N_0+1} \sum_{k\in\mathscr{G}_2} e^{\frac{i}{\varepsilon}\varphi_k} L_2'(\partial) u_{N_0+1,k}\left(t, x_1, \frac{x_2}{\sqrt{\varepsilon}}\right) \\ & + \sqrt{\varepsilon}^{N_0+1} \sum_{k\in\mathscr{G}_2} e^{\frac{i}{\varepsilon}\varphi_k} A_2 \partial_{\chi_2}(u_{N_0+1,k})\left(t, x_1, \frac{x_2}{\sqrt{\varepsilon}}\right) \end{aligned}$$

$$+\sqrt{\varepsilon}^{N_0+2}\sum_{k\,\in\,\mathscr{G}_2} e^{\frac{i}{\varepsilon}\varphi_k} L_1'(\partial) u_{N_0+2,k}\left(t,x_1,\frac{x_2}{\sqrt{\varepsilon}}\right)$$
$$+\sqrt{\varepsilon}^{N_0+1}\sum_{k\,\in\,\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}} L_1'(\partial) U_{N_0+1,k,1}\left(t,x_2,\frac{x_1}{\varepsilon}\right)$$
$$+\sqrt{\varepsilon}^{N_0+2}\sum_{k\,\in\,\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}} L_1'(\partial) U_{N_0+2,k,1}\left(t,x_2,\frac{x_1}{\varepsilon}\right)$$
$$+\sqrt{\varepsilon}^{N_0+1}\sum_{k\,\in\,\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}} L_2'(\partial) U_{N_0+1,k,2}\left(t,x_1,\frac{x_2}{\varepsilon}\right)$$
$$+\sqrt{\varepsilon}^{N_0+2}\sum_{k\,\in\,\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}} L_2'(\partial) U_{N_0+2,k,2}\left(t,x_1,\frac{x_2}{\varepsilon}\right).$$

Because of the fast variables, when we take the L^2-norm of the terms depending on the boundary layers on the right-hand side of f^ε, by a simple change of variable, we recover an extra factor ε^α (with $\alpha = 1/2$ for evanescent modes and $\alpha = 1/4$ for glancing modes). Consequently, the limiting term in f^ε is the hyperbolic term $\sqrt{\varepsilon}^{N_0+1}\sum_{k\,\in\,\mathscr{I}_{\text{hyp}}} e^{\frac{i}{\varepsilon}\varphi_k} L(\partial) u_{N_0+1,k}$ which is $O(\sqrt{\varepsilon}^{N_0+1})$. Then, the energy estimate of Definition 7.2 gives that

$$\|u^\varepsilon - u^\varepsilon_{\text{app},N_0+2}\|_{L^2(\Omega)} \leqslant C\sqrt{\varepsilon}^{N_0+1}.$$

The triangle inequality concludes the proof. □

8. Examples

In this last section we give some examples of characteristic varieties for which we have the existence of multiple loops discussed so far. We here only describe the geometry of the characteristic variety and not the operator from which it comes from. As we will see the examples are all based upon an ellipsis and several lines. They can thus be constructed from an operator reading under the form $L(\partial)$ where the coefficients read

$$A_1 := \begin{bmatrix} A_1^\flat & 0 \\ 0 & A_1^\natural \end{bmatrix} \quad \text{and} \quad A_2 := \begin{bmatrix} A_2^\flat & 0 \\ 0 & A_2^\natural \end{bmatrix} \tag{8.1}$$

where $A_1^\flat, A_2^\flat \in \mathbf{M}_{2\times 2}(\mathbb{R})$ are chosen to construct the ellipsis and where $A_1^\natural$, $A_2^\natural$ are diagonal matrices of size M, where M is the number of lines in the characteristic variety. Consequently, the operator $\partial_t + A_1^\flat\partial_1 + A_2^\flat\partial_2$ is a wave type operator which is completed by some (uncoupled) transport phenomena. The coupling may occur at the level of the boundary conditions.

Because the lines associated to the transport phenomena intersect the ellipsis, we are not dealing with a strictly hyperbolic operator. Indeed, the multiplicities of the eigenvalues vary at the intersection points. However, we can show that we have a geometrically regular hyperbolic operator in the sense of [13] and as long as we do not have to consider the phases associated to the intersection points, the previous construction of the geometric optics expansions applies.

The following figures give examples of the possible behaviours encountered so far in the article. More precisely,

- Figure 8.1 gives a simple example of appearance of some glancing mode for a non glancing initial phase.
- Figure 8.2 gives an example with a loop admitting more than four elements.
- Figure 8.3 gives an example with two loops. These loops being simple loops.
- Figure 8.4 gives an example of a composite loop.

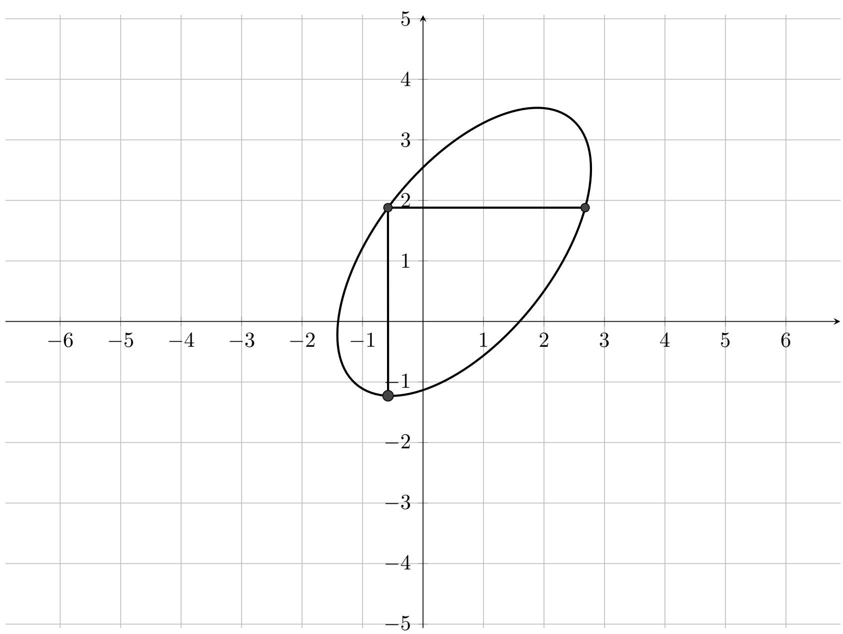

Figure 8.1. A simple example of glancing appearance

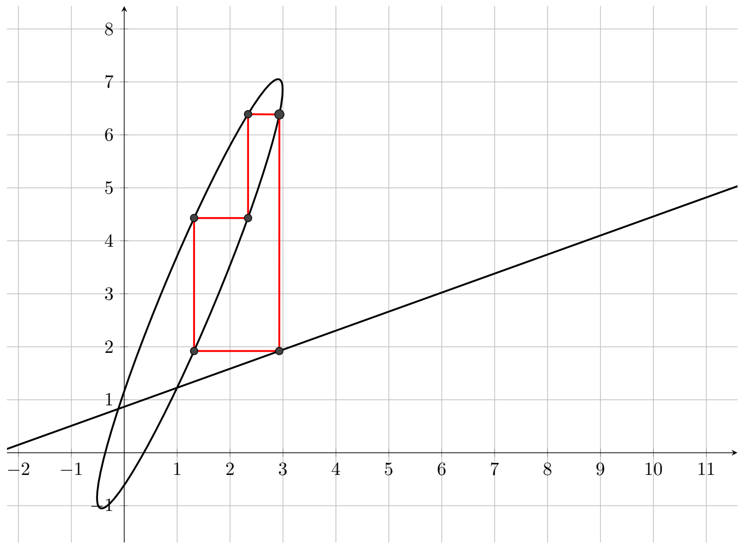

Figure 8.2. A loop with six self-interacting elements.

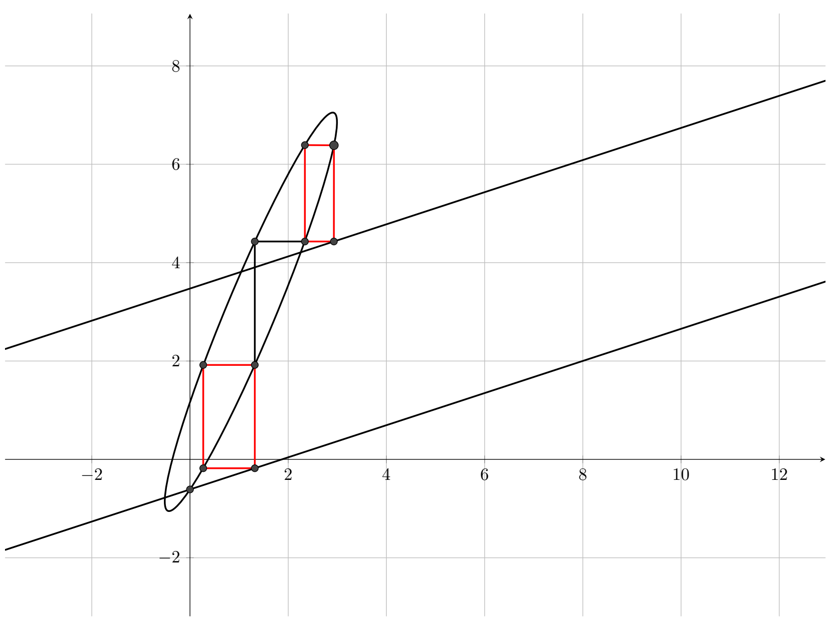

Figure 8.3. A characteristic variety with two simple loops.

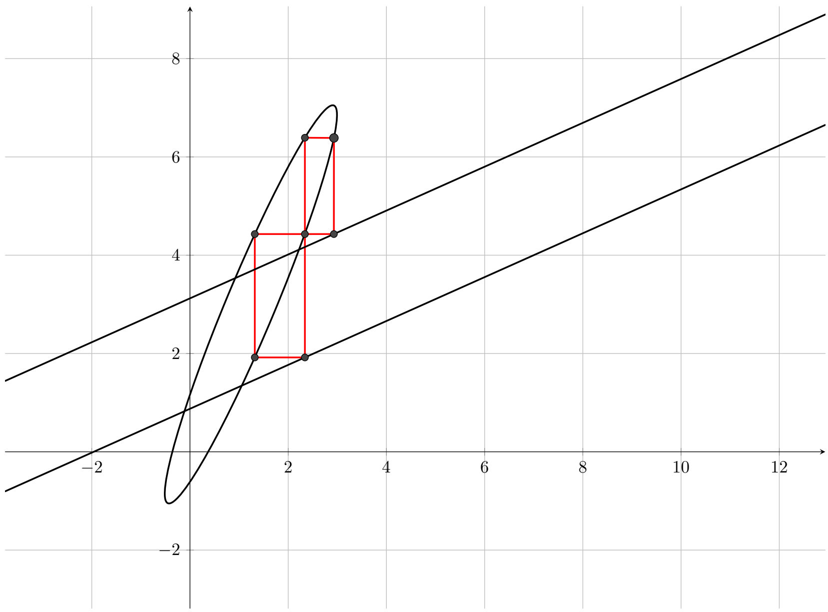

Figure 8.4. A characteristic variety with a composite loop.

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