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Holomorphic bundles framed along a real hypersurface and the Riemann–Hilbert problem ^(*)

ANDREI TELEMAN ⁽¹⁾

ABSTRACT. — Let X be a connected, compact complex manifold, $S \subset X$ a separating real hypersurface, so X decomposes as a union of compact complex manifolds with boundary \bar{X}^\pm with $\bar{X}^+ \cap \bar{X}^- = S$. Let \mathcal{M} be the moduli space of S -framed holomorphic bundles, i.e. of pairs (E, θ) of fixed topological type consisting of a holomorphic bundle E on X and a trivialization θ – belonging to a fixed Hölder regularity class $\mathcal{C}^{\kappa+1}$ – of its restriction to S .

Our problem: compare, via the obvious restriction maps, the moduli space \mathcal{M} to the corresponding Donaldson’s moduli spaces \mathcal{M}^\pm of boundary framed formally holomorphic bundles on \bar{X}^\pm . The restrictions to \bar{X}^\pm of an S -framed holomorphic bundle (E, θ) are boundary framed formally holomorphic bundles (E^\pm, θ^\pm) which induce, via θ^\pm , the same tangential Cauchy–Riemann operator on the trivial bundle on S . Therefore one obtains a natural map from \mathcal{M} into the fiber product $\mathcal{M}^- \times_{\mathcal{C}} \mathcal{M}^+$ over the space \mathcal{C} of Cauchy–Riemann operators on the trivial bundle on S . Our main result states: this map is a homeomorphism for $\kappa \in (0, \infty] \setminus \mathbb{N}$. Note that, by theorems due to S. Donaldson and Z. Xi, the moduli spaces \mathcal{M}^\pm can be further identified with moduli spaces of boundary framed Hermitian Yang–Mills connections.

The proof of our isomorphism theorem is based on a gluing principle for formally holomorphic bundles along a real hypersurface. The same gluing theorem can be used to give a complex geometric interpretation of the space of solutions of a large class of Riemann–Hilbert type problems.

We generalize these results in two directions: first, we will replace the decomposition $X = \bar{X}^- \cup \bar{X}^+$ associated with a separating hypersurface by the manifold with boundary \widehat{X}_S obtained by cutting X along any (not necessarily separating) oriented hypersurface S . Second, instead of vector bundles, we will consider principal G bundles for an arbitrary complex Lie group G .

We give explicit examples of moduli spaces of (boundary) framed holomorphic bundles and explicit formulae for the homeomorphisms provided by the general results.

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RÉSUMÉ. — Soient X une variété complexe compacte connexe et $S \subset X$ une hypersurface réelle séparante; X se décompose donc comme la réunion de deux variétés complexes compactes à bord \bar{X}^\pm telles que $\bar{X}^+ \cap \bar{X}^- = S$. Soit \mathcal{M} l'espace de modules des fibrés holomorphes sur X décorés sur S , c.-à-d. des couples (E, θ) , de type topologique fixé, où E est un fibré *holomorphe* sur X et θ est une trivialisatation appartenant à une classe de régularité fixée $\mathcal{C}^{\kappa+1}$ (au sens de Hölder) de sa restriction sur S .

Notre problème: comparer, via les applications de restriction évidentes, l'espace de modules \mathcal{M} avec les espaces de modules de Donaldson \mathcal{M}^\pm des fibrés formellement holomorphes sur les variétés à bord \bar{X}^\pm , décorés sur le bord. Les restrictions (E^\pm, θ^\pm) sur \bar{X}^\pm d'un fibré holomorphe (E, θ) sur X décoré sur S sont des fibrés formellement holomorphes décorés sur le bord, qui induisent via θ^\pm le même opérateur de Cauchy–Riemann tangentiel sur le fibré trivial sur S . On obtient donc une application naturelle de \mathcal{M} dans le produit fibré $\mathcal{M}^- \times_{\mathcal{C}} \mathcal{M}^+$ des espaces \mathcal{M}^\pm au-dessus de l'espace \mathcal{C} des opérateurs de Cauchy–Riemann sur le fibré trivial sur S .

Notre premier résultat: cette application est un homéomorphisme pour $\kappa \in (0, \infty] \setminus \mathbb{N}$. Notons que, d'après des théorèmes dûs à S. Donaldson et Z. Xi, les espaces \mathcal{M}^\pm s'identifient à des espaces de modules de connexions de Yang–Mills hermitiennes décorées sur le bord.

La démonstration de notre théorème d'isomorphisme s'appuie sur un principe de recollement des fibrés formellement holomorphes le long d'une hypersurface réelle. Le même principe de recollement peut être utilisé pour donner une interprétation géométrique, au sens de la géométrie complexe, de l'espace des solutions d'un problème de type Riemann–Hilbert dans un sens très général.

Nous généralisons ces résultats dans deux directions: premièrement, on va remplacer la décomposition $X = \bar{X}^- \cup \bar{X}^+$ associée à une hypersurface séparante par la variété à bord \widehat{X}_S obtenue en découpant X le long d'une hypersurface orientée arbitraire (pas nécessairement séparante). Deuxièmement, au lieu de se limiter aux fibrés vectoriels, nous allons considérer des fibrés principaux de groupe structural G , un groupe de Lie complexe arbitraire.

Nous donnons des exemples explicites d'espaces de modules de fibrés (formellement) holomorphes décorés sur une hypersurface réelle dans une variété fermée (ou sur le bord d'une variété compacte à bord) ainsi que des formules explicites pour les homéomorphismes fournis par les résultats généraux.

1. Introduction

A fundamental problem in the theory of holomorphic bundles on compact complex manifolds is: understand, in the general (non-necessarily algebraic or Kählerian) framework, the relation between convergence in the space of singular Hermitian–Einstein connections (Donaldson, Tian) and convergence of sheaves in the sense of complex geometric deformation theory. Working on this problem in collaboration with Matei Toma, I noticed that Donaldson's article [6] (which deals with the correspondence between Hermitian–Einstein

connections and holomorphic bundles on compact complex manifolds with boundary) is relevant for our problem. Donaldson's article comes with a fundamental new idea: in the presence of a boundary, it's natural to consider infinite dimensional moduli spaces of *boundary framed* Hermitian Yang-Mills connections, respectively holomorphic bundles.

A boundary framed Hermitian Yang-Mills connection on \bar{X} is a triple (E, A, θ) , where E is a Hermitian vector bundle on \bar{X} , A a Hermitian Yang-Mills connection on E , and θ a *unitary* trivialization of $E_{\partial\bar{X}}$. A boundary framed formally holomorphic vector bundle on \bar{X} is a triple (E, δ, θ) , where E is a differentiable vector bundle on \bar{X} , δ is a Dolbeault operator on E satisfying the *formal integrability* condition $\delta^2 = 0$ (see [32] and Section A.5.1 in this article), and θ is a *differentiable* trivialization of $E_{\partial\bar{X}}$.

Donaldson's theorem [6, Theorem 1'] yields an isomorphism between moduli spaces of gauge theoretical, respectively complex geometric boundary framed objects. An interesting application of this isomorphism theorem: a new proof of a fundamental factorization theorem in loop group theory (see [6, p. 100]).

The manifolds with boundary which appear naturally in our complex geometry project are of the form \bar{X}^\pm where $X^\pm \subset X$ are the open submanifolds obtained by cutting the given closed complex manifold X along a separating real hypersurface $S \subset X$. In our original joint project we focus on the case when S is the boundary of a neighborhood of the bubbling locus of a weakly convergent sequence of Hermitian–Einstein connections. Relevant for the present article: in the presence of a real hypersurface S of a closed complex manifold X it's natural to consider moduli spaces of S -framed holomorphic bundles *on the whole closed manifold* X , i.e. of holomorphic bundles E on X endowed with a differentiable trivialization θ on S . One should of course fix the topological type of the pair (E, θ) .

Although infinite dimensional, such a moduli space can be constructed explicitly and studied using techniques and methods from the classical deformation theory for analytic objects on compact complex spaces. A joint article in preparation [34] is dedicated to these moduli spaces and their role in our initial project.

The starting point of the present article is the natural problem: supposing that S separates X , compare, via the obvious restriction maps, the moduli space \mathcal{M} of S -framed holomorphic bundles (of fixed topological type) on X , with the corresponding Donaldson's moduli spaces \mathcal{M}^\pm of boundary framed holomorphic bundles on \bar{X}^\pm . The restrictions to \bar{X}^\pm of an S -framed holomorphic bundle (E, θ) of rank r are boundary framed formally holomorphic bundles (E^\pm, θ^\pm) which induce, via θ^\pm , the same tangential Cauchy–Riemann

operators on the trivial bundle of rank r on S . Therefore one obtains a natural comparison map from \mathcal{M} into the fiber product $\mathcal{M}^- \times_{\mathcal{C}} \mathcal{M}^+$ over the space \mathcal{C} of Cauchy–Riemann operators on the trivial bundle of rank r on S .

At this point note that we will work in a more general framework: in the definitions of our framed objects we will fix a regularity parameter $\kappa \in [0, \infty]$ and we will require \mathcal{C}^κ -regularity (see Section A.1) for δ in the definition of a boundary framed formally holomorphic vector bundle, and $\mathcal{C}^{\kappa+1}$ -regularity for θ in the definitions of a (formally) holomorphic S -framed (boundary framed) vector bundle. The moduli spaces \mathcal{M} , \mathcal{M}^\pm and the comparison map are defined in this more general framework.

The isomorphism Theorem 2.16 proved in this article states: *the comparison map $\mathcal{M} \rightarrow \mathcal{M}^- \times_{\mathcal{C}} \mathcal{M}^+$ is a homeomorphism for $\kappa \in (0, +\infty) \setminus \mathbb{N}$* . The results also holds for $\kappa = \infty$ if the considered moduli spaces are endowed with suitable topologies, see Remark 2.18. The meaning of this isomorphism theorem can be intuitively expressed as a general principle: in the moduli theory for holomorphic bundles on closed complex manifolds, *framing* on a real hypersurface S is equivalent to *cutting* along S . Note that for $\dim_{\mathbb{C}}(X) = 1$ the compatibility condition on the induced Cauchy–Riemann operators becomes void so, on Riemann surfaces, the principle “framing on S is equivalent to cutting along S ” becomes simply $\mathcal{M} = \mathcal{M}^- \times \mathcal{M}^+$.

The difficult part of the isomorphism theorem is the surjectivity, which follows from the gluing principle given by the crucial Theorem 3.11: let U be a (not necessarily compact) complex manifold, $S \subset U$ a closed, separating, smooth, real hypersurface, \bar{U}^\pm be the corresponding manifolds with boundary, E a \mathcal{C}^∞ -bundle on U and δ^\pm be formally integrable Dolbeault operators on $E_{\bar{U}^\pm}$ with coefficients in \mathcal{C}^κ inducing the same tangential Cauchy–Riemann operators on S . There exists an automorphism f_+ of class $\mathcal{C}^{\kappa+1}$ of $E_{\bar{U}^+}$ which is the identity on S such that δ^- and $f_+(\delta^+)$ glue together and give an integrable Dolbeault operator (so a holomorphic structure) on E . For $\kappa \in (0, +\infty) \setminus \mathbb{N}$ the proof makes use of Whitney’s extension theorem for Lipschitz spaces, which allows us to prove that f_+ can be chosen to depend continuously on (δ^-, δ^+) . For $\kappa = \infty$ we use the \mathcal{C}^∞ version of Whitney’s extension theorem, which does *not* provide a continuous extension operator.

Our gluing principle has other consequences: let E^\pm be \mathcal{C}^∞ complex vector bundles on \bar{U}^\pm and δ^\pm formally integrable Dolbeault operators with coefficients in \mathcal{C}^κ on E^\pm , and let $v : E_S^- \rightarrow E_S^+$ be a bundle isomorphism of class $\mathcal{C}^{\kappa+1}$ (with $\kappa \in (0, +\infty] \setminus \mathbb{N}$) such that the tangential Cauchy–Riemann operators δ_S^\pm induced by δ^\pm on S agree via v . Theorem 2.1 shows, that, under these assumptions, the *topological* bundle $E^v = E^- \amalg_v E^+$ on U comes with a *canonical* holomorphic structure which extends the holomorphic structures

defined by δ_S^\pm on $E_{U^\pm}^\pm$. Therefore, although the gluing isomorphism v is only of class $\mathcal{C}^{\kappa+1}$, if the above compatibility condition is satisfied, one can glue the *formally holomorphic* bundles E^- , E^+ via v , and obtain a canonically defined *holomorphic* bundle on U . In particular, on Riemann surfaces, one can always (no compatibility condition needed) glue formally holomorphic bundles E^\pm on U^\pm via a $\mathcal{C}^{\kappa+1}$ bundle isomorphism $v : E_S^- \rightarrow E_S^+$, and obtain a *holomorphic* vector bundle on U .

Consider the special case where $U = \mathbb{P}_\mathbb{C}^1 = \mathbb{C} \cup \{\infty\}$, $S \subset \mathbb{C}$ is a closed curve, and E^\pm are the trivial bundles on \bar{U}^\pm (endowed with the standard Dolbeault operator $\bar{\partial}$). An isomorphism v as above is precisely the input data of the Riemann–Hilbert problem as stated in [14, Kapitel X]. Using this remark we show that a large class⁽¹⁾ of Riemann–Hilbert type problems, including Hilbert’s original problem and matrix factorization problems (see Problem 4 in Section 2.2.1), can be reduced to a complex geometric problem for holomorphic vector bundles on $\mathbb{P}_\mathbb{C}^1$ (see Corollary 2.10).

Theorem 2.1 can be easily extended to possibly non-separating closed, oriented real hypersurfaces S : one just replaces the disjoint union $\bar{U}^- \amalg \bar{U}^+$ by the manifold with boundary \hat{U}_S obtained by cutting U along S (see Section 2.1.2 and Figure 2.1). This generalization is Theorem 2.4; it applies for instance when S is a non-separating circle on an elliptic curve. This leads us to a general Riemann–Hilbert type problem associated to a closed Riemann surface X and an arbitrary (non-necessarily connected, non-necessarily separating) smooth oriented closed curve $S \subset X$ (see Problem 5 in Section 2.2.2) and to a complex geometric approach to solve it (Corollary 2.12). In Section 2.2.3 we formulate and study a generalization of the Riemann–Hilbert problem for n dimensional complex manifolds noting that, for $n \geq 2$, the above compatibility condition is needed.

Similarly, the moduli space isomorphism $\mathcal{M} \simeq \mathcal{M}^- \times_{\mathcal{C}} \mathcal{M}^+$ can be generalized to the case of an oriented, not necessarily connected, not necessarily separating, real hypersurface $S \subset X$. The boundary \hat{S} of \hat{X}_S decomposes as a disjoint union $S^- \cup S^+$ and comes with a canonical identification map $b : S^- \rightarrow S^+$. Let E be a vector bundle on X and \hat{E} its pull back to \hat{X}_S . A formally integrable Dolbeault operator \mathfrak{d} on \hat{E} will be called *descendable*, if the tangential Cauchy–Riemann operators on $S^\pm \times \mathbb{C}^r$ induced via θ_{S^\pm} by \mathfrak{d} agree via b . The first part of Theorem 2.16 identifies the moduli space of S -framed holomorphic bundles (of a fixed topological type) on X with the moduli space of descendable boundary framed formally holomorphic bundles

⁽¹⁾ Several authors state and study more general Riemann–Hilbert problems on $\mathbb{P}_\mathbb{C}^1$, where S is replaced by a piecewise differentiable, non-necessarily closed, “contour” in \mathbb{C} . These generalizations are related to Hilbert’s 21-st problem [5].

(of the corresponding topological type) on \widehat{X}_S . The intuitive interpretation of this isomorphism is the same as in the separating case: framing on S is equivalent to cutting along S .

If X is a Riemann surface, *any* boundary framed holomorphic bundle on \widehat{X}_S is descendable. Therefore, *if X is a closed Riemann surface, the moduli space of S -framed holomorphic bundles on X (of a fixed topological type) can be identified with the corresponding moduli space of boundary framed formally holomorphic bundles on \widehat{X}_S .*

Suppose now that the closed Riemann surface X has been endowed with a Hermitian metric. By Donaldson's isomorphism theorem [6, Theorem 1'], the latter moduli space, in its turn, can be identified with the corresponding moduli space of boundary framed Hermitian Yang-Mills connections on \widehat{X}_S . Composing the two isomorphisms, one obtains an identification between the considered moduli space of S -framed holomorphic bundles on X and the corresponding moduli space of boundary framed Hermitian Yang-Mills unitary connections on \widehat{X}_S . Theorem 2.6 generalizes Theorems 2.1, 2.4 to principal G -bundles P endowed with (formally) integrable bundle almost complex structures (see Section A.5.2), where G is an arbitrary complex Lie group. In this general framework the role of the tangential Cauchy–Riemann operator δ_S is played by the almost complex structure J_S induced by a bundle almost complex structure J on the pull back $\mathfrak{T}_{P_S} \subset T_{P_S}$ of the canonical distribution $\mathfrak{T}_S := T_S \cap J_U T_S$ of S .

The above results concerning Riemann–Hilbert problems and isomorphisms between moduli spaces of S -framed and boundary framed holomorphic bundles extend to the framework of principal G -bundles. Moreover, in the definition of our moduli spaces, one can use as framings on S (or as boundary framings) differentiable bundle isomorphisms $\theta : \Phi \rightarrow P_S$ ($\theta : \Phi \rightarrow P_{\partial\overline{X}}$), where Φ is a fixed, not necessarily trivial, differentiable G -bundle on S (on $\partial\overline{X}$), see Section 4.2. In particular the isomorphism Theorem 2.17 shows that the principle “framing on S is equivalent to cutting along S ” generalizes to this framework. In Section 5 we give explicit examples of isomorphisms provided by this theorem on Riemann surfaces and, in some cases, using classical theorems in complex analysis, we give explicit formulae for their inverses.

Of special interest is the case when G is a complex reductive group, because, for such groups, we also have an analogue of Donaldson's isomorphism [6, Theorem 1']: one just replaces the moduli space of boundary framed Hermitian Yang-Mills unitary connections by the moduli space of boundary framed Hermitian Yang-Mills K -connections, where K is a fixed maximal compact subgroup of G . Therefore, in this case one can further identify

the two moduli spaces intervening in Theorem 2.17 with a moduli space of boundary framed Hermitian Yang-Mills K -connections. Explicit examples of such identifications are given in Section 5.

Notation. — For a differentiable manifold (possible with boundary) M , a finite dimensional normed space T , a \mathcal{C}^∞ vector bundle E on M and a locally trivial fiber bundle Φ on M we will use the following notations:

- $\mathcal{C}^\kappa(M, T)$: the space of T -valued maps of class \mathcal{C}^κ on M , see Section A.1.
- $\Gamma^\kappa(M, E)$: the space of sections of class \mathcal{C}^κ in E , see Section A.1.
- $\Gamma^\kappa(M, \Phi)$: the space of sections of class \mathcal{C}^κ in Φ in the sense of [26, p. 38].
- \bigwedge_M^d : the bundle of forms of degree d on M .
- $\bigwedge_M^{p,q}$: the bundle of forms of bidegree (p, q) on a complex manifold M .
- $A^d(M, E) := \Gamma^\infty(M, \bigwedge_M^d \otimes E)$, $A^{p,q}(M, E) := \Gamma^\infty(M, \bigwedge_M^{p,q} \otimes E)$.
- $A^d(M, E)_\kappa := \Gamma^\kappa(M, \bigwedge_U^d \otimes E)$, $A^{p,q}(M, E)_\kappa := \Gamma^\kappa(M, \bigwedge_U^{p,q} \otimes E)$.

Acknowledgements

The problems investigated in this article originate from a joint project in collaboration with Matei Toma devoted to the boundedness condition for torsion free coherent sheaves in the complex geometric (not necessarily algebraic) framework. I am indebted to him for the interesting questions he formulated, and also for his useful remarks on preliminary versions of this article.

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2. Statement of results

2.1. Gluing holomorphic bundles along a real hypersurface

Let U be a differentiable manifold, and let $S \subset U$ be a closed real smooth hypersurface.

2.1.1. Gluing holomorphic bundles along a separating real hyper-surface

Let $\kappa \in [0, +\infty]$. We will use the notation \mathcal{C}^κ for the usual k -th differentiability class when $\kappa \in \mathbb{N} \cup \{\infty\}$, and the Hölder class $\mathcal{C}^{[\kappa], \kappa - [\kappa]}$ when $\kappa \notin \mathbb{N} \cup \{\infty\}$ (see Section A.1). Suppose that S separates U , i.e. $U \setminus S$ decomposes as a disjoint union $U \setminus S = U^- \cup U^+$ with $\bar{U}^\pm = U^\pm \cup S$. Therefore \bar{U}^\pm are manifolds with boundary and $\partial \bar{U}^+ = \partial \bar{U}^- = S$. Let E^\pm be a \mathcal{C}^∞ complex vector bundle of rank r on \bar{U}^\pm and let E_S^\pm be its restriction to S . Let $v : E_S^- \rightarrow E_S^+$ be a bundle isomorphism of class $\mathcal{C}^{\kappa+1}$ and $E^v := E^- \amalg_v E^+$ the *topological* bundle obtained by gluing E^\pm along S via v .

Suppose now that U is a complex manifold and E^\pm have been endowed with Dolbeault operators

$$\delta^\pm : \Gamma^{\kappa+1}(\bar{U}^\pm, E^\pm) \longrightarrow \Gamma^\kappa(\bar{U}^\pm, \bigwedge_{\bar{U}^\pm}^{0,1} \otimes E^\pm)$$

with coefficients in \mathcal{C}^κ which satisfy the *formal* integrability condition $F_{\delta^\pm} = 0$, where F_{δ^\pm} is the $\text{End}(E^\pm)$ -valued $(0,2)$ -form on \bar{U}^\pm associated with δ^2 . When $\kappa \in [0, 1)$, F_{δ^\pm} is a distribution supported by \bar{U}^\pm in the sense of [24, Section I.1], see Section A.5.3 in the appendix.

From now on throughout this section we will suppose $\kappa \in (0, +\infty] \setminus \mathbb{N}$. This condition is required in several crucial arguments where we make use of the standard elliptic regularity for Hölder spaces, or of the Hölder version of the Newlander–Nirenberg theorem for principal bundles (see [33] and Section A.5.2 in this article).

THEOREM 2.1. — *Let δ^\pm be a formally integrable Dolbeault operator with coefficients in \mathcal{C}^κ on E^\pm and let \mathfrak{h}_\pm be the corresponding holomorphic structure on the underlying $\mathcal{C}^{\kappa+1}$ bundle of the restrictions $E_{U^\pm}^\pm$ to U^\pm . Suppose that the tangential Cauchy–Riemann operators δ_S^\pm induced by δ^\pm agree via v . Then*

- (1) *The topological bundle E^v on U admits a unique holomorphic reduction \mathfrak{h}^v extending \mathfrak{h}^\pm .*
- (2) *For any local \mathfrak{h}^v -holomorphic section $U \supset V \xrightarrow{\text{open}} E^v$, we have*

$$\sigma|_{V \cap \bar{U}^\pm} \in \Gamma^{\kappa+1}(V \cap \bar{U}^\pm, E^\pm),$$

i.e. the restrictions $\sigma|_{V \cap \bar{U}^\pm}$ of σ are of class $\mathcal{C}^{\kappa+1}$ up to the boundary.

Therefore, although the gluing bundle isomorphism v is supposed to be only of class $\mathcal{C}^{\kappa+1}$ and the required compatibility condition concerns only the

tangential operators δ_S^\pm , we can glue together the two *formally* holomorphic bundles (E^\pm, δ^\pm) along S via v and obtain a holomorphic bundle on U .

Remark 2.2. — For a Dolbeault operator δ on a bundle E^+ on a manifold with boundary \bar{U}^+ , the formal integrability condition $\delta^2 = 0$ does *not* imply integrability (existence of local frames solving the δ -equation) at non pseudo-convex boundary points. In [32] we gave an example of a bundle E^+ on a compact manifold \bar{U}^+ with pseudo-concave boundary with the property that a *generic* formally integrable Dolbeault operator on E^+ is integrable at *no* boundary point. Theorem 2.1 shows that the compatibility condition required in its hypothesis implies local integrability of *both* δ^\pm at *all* points of S , without any pseudo-convexity condition.

Theorem 2.1 gives:

COROLLARY 2.3. — *Under the assumptions of Theorem 2.1, the \mathcal{O}_U -module \mathcal{E} defined by*

$$W \mapsto \left\{ \begin{array}{l} \left(\begin{smallmatrix} f^- \\ f^+ \end{smallmatrix} \right) \in \Gamma^0(W \cap \bar{U}^-, E^-) \\ \times \Gamma^0(W \cap \bar{U}^+, E^+) \end{array} \middle| \begin{array}{l} f^+|_{W \cap S} = v f^-|_{W \cap S}, \\ f^\pm \text{ is } \mathfrak{h}^\pm\text{-holomorphic on } W \cap U^\pm \end{array} \right\} \quad (2.1)$$

is locally free of rank r , and coincides with the apparently smaller sheaf

$$W \mapsto \left\{ \begin{array}{l} \left(\begin{smallmatrix} f^- \\ f^+ \end{smallmatrix} \right) \in \Gamma^{k+1}(W \cap \bar{U}^-, E^-) \\ \times \Gamma^{k+1}(W \cap \bar{U}^+, E^+) \end{array} \middle| \begin{array}{l} f^+|_{W \cap S} = v f^-|_{W \cap S}, \\ f^\pm \text{ is } \mathfrak{h}^\pm\text{-holomorphic on } W \cap U^\pm \end{array} \right\}. \quad (2.2)$$

2.1.2. Gluing holomorphic bundles along an oriented real hypersurface

Theorem 2.1, Corollary 2.3 can be extended to oriented, non-necessarily separating, non-necessarily connected, real hypersurfaces. Let U be a complex manifold and $S \subset U$ be a closed, *oriented* real hypersurface. The normal bundle $n_S := T_{U|S}/T_S$ of S in U comes with a distinguished orientation induced by the complex orientation of U and the fixed orientation of S . Let 0_{n_S} be the zero section of n_S . The quotient $\hat{S} := (n_S \setminus 0_{n_S})/\mathbb{R}_{>0}$ is a trivial double cover of S , so it decomposes as a disjoint union $\hat{S} = S^+ \cup S^-$, where S^\pm are identified with S via the cover map $\hat{S} \rightarrow S$. Therefore we have an obvious identification $b : S^- \xrightarrow{\sim} S^+$. The union

$$\hat{U}_S := (U \setminus S) \cup \hat{S}$$

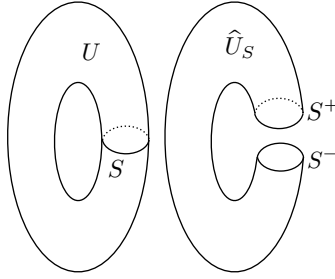


Figure 2.1. U and \widehat{U}_S .

has a canonical structure of a complex manifold with boundary whose boundary is

$$\partial \widehat{U}_S = \widehat{S} = S^- \cup S^+,$$

and comes with an obvious surjective smooth map $p_S^U : \widehat{U}_S \rightarrow U$ extending the biholomorphic identification $\widehat{U}_S \setminus \widehat{S} = U \setminus S$; it will be called *the manifold with boundary obtained by cutting U along S* (see Figure 2.1).

In the special case considered above (when S separates U) we have $\widehat{U}_S = \overline{U}^- \amalg \overline{U}^+$.

Let E be a complex vector bundle of class \mathcal{C}^∞ on \widehat{U}_S . We will denote by $E_{U \setminus S}$ the restriction of E to $\widehat{U}_S \setminus \widehat{S} = U \setminus S$. Let $v : E_{S^-} \rightarrow b^*(E_{S^+})$ be a bundle isomorphism of class $\mathcal{C}^{\kappa+1}$. Identifying E_{S^-} with E_{S^+} via v we obtain a topological bundle E^v on U whose pull back to \widehat{U}_S is tautologically identified with E .

Taking into account that Theorem 2.1 has a local character with respect to S , we obtain:

THEOREM 2.4. — *Let E be a \mathcal{C}^∞ complex vector bundle on \widehat{U}_S , δ a formally integrable Dolbeault operator with coefficients in \mathcal{C}^κ on E , and \mathfrak{h} the corresponding holomorphic structure on the underlying $\mathcal{C}^{\kappa+1}$ bundle of $E_{U \setminus S}$. Let $v : E_{S^-} \rightarrow b^*(E_{S^+})$ be a bundle isomorphism of class $\mathcal{C}^{\kappa+1}$. Suppose that the tangential Cauchy–Riemann operators $\delta_{S^\pm}^\pm$ induced by δ agree via v . Then*

- (1) *The topological bundle E^v on U admits a unique holomorphic reduction \mathfrak{h}^v extending \mathfrak{h} .*
- (2) *For any local \mathfrak{h}^v -holomorphic section $U \xrightarrow{\text{open}} V \xrightarrow{\sigma} E^v$, we have*

$$\widehat{\sigma} := \sigma \circ p_S^V \in \Gamma^{\kappa+1}(\widehat{V}_{V \cap S}, E),$$

i.e. the pull back $\widehat{\sigma}$ of σ via p_S^V is of class $\mathcal{C}^{\kappa+1}$ up to the boundary.

This can also be reformulated in terms of sheaves:

COROLLARY 2.5. — *Under the assumptions of Theorem 2.4, the \mathcal{O}_U -module \mathcal{E} defined by*

$$W \mapsto \left\{ f \in \Gamma^0(\widehat{W}_{W \cap S}, E) \left| \begin{array}{l} (f|_{(S \cap W)^+}) \circ b|_{(W \cap S)^-} = v \circ (f|_{(W \cap S)^-}), \\ f \text{ is } \delta\text{-holomorphic on } \widehat{W}_{W \cap S} \setminus \widehat{S} \end{array} \right. \right\}$$

is locally free of rank r , and coincides with the apparently smaller sheaf

$$W \mapsto \left\{ f \in \Gamma^{\kappa+1}(\widehat{W}_{S \cap W}, E) \left| \begin{array}{l} (f|_{(W \cap S)^+}) \circ b|_{(W \cap S)^-} = v \circ (f|_{(W \cap S)^-}), \\ f \text{ is } \delta\text{-holomorphic on } \widehat{W}_{W \cap S} \setminus \widehat{S} \end{array} \right. \right\}.$$

Let now G be an arbitrary complex Lie group. In the presence of a principal G -bundle P on \widehat{U}_S and a bundle isomorphism $v : P_{S^-} \rightarrow b^*(P_{S^+})$ of class $\mathcal{C}^{\kappa+1}$, one can define the topological bundle P^v as in the vector bundle case. Let $\mathfrak{T}_{P_{S^\pm}} \subset T_{P_{S^\pm}}$ be the pull-back of the canonical almost complex distribution $\mathfrak{T}_S := T_S \cap J_U(T_S)$ of S . Using the definitions and notations explained in Section A.5.2 (see also [33]) we have:

THEOREM 2.6. — *Let $p : P \rightarrow \widehat{U}_S$ be a principal G -bundle on \widehat{U}_S and J a formally integrable bundle almost complex structure (bundle ACS) of class \mathcal{C}^κ on P . Let $v : P_{S^-} \rightarrow b^*(P_{S^+})$ be a bundle isomorphism of class $\mathcal{C}^{\kappa+1}$. Suppose that the tangential almost complex structures J_{S^\pm} induced by J on the distributions $\mathfrak{T}_{P_{S^\pm}}$ agree via v . Then*

- (1) *The topological bundle P^v admits a unique holomorphic reduction \mathfrak{h}^v extending the holomorphic structure \mathfrak{h} induced by J on $P_{X \setminus S}$.*
- (2) *The pull-back $\widehat{\tau}$ of any local \mathfrak{h}^v -holomorphic section $\tau : V \rightarrow P^v$ is of class $\mathcal{C}^{\kappa+1}$ up to the boundary.*

Note that v can be regarded as a section in a locally trivial fiber bundle over S^- . The $\mathcal{C}^{\kappa+1}$ condition on v in Theorem 2.6 is meant in the sense of [26, p. 38].

Remark 2.7. — If G is a closed complex subgroup of $\mathrm{GL}(r, \mathbb{C})$, any bundle ACS on P will induce a Dolbeault operator on the associated rank r vector bundle, and the compatibility condition “the bundle ACS J_{S^\pm} agree via v ” required in Theorem 2.6 can be replaced by a compatibility condition for tangential Cauchy–Riemann operators as in Theorem 2.4. We preferred a formulation which is general and intrinsic in terms of abstract complex Lie groups G and principal G -bundles.

In the special case when S separates U , we obtain as a special case the following generalization of Theorem 2.1:

THEOREM 2.8. — *Let P^\pm be a \mathcal{C}^∞ principal G -bundle on \bar{U}^\pm and let P_S^\pm be its restriction to S . Let $v : P_S^- \rightarrow P_S^+$ be a bundle isomorphism of class $\mathcal{C}^{\kappa+1}$. Let J^\pm be a formally integrable bundle ACS of class \mathcal{C}^κ on P^\pm and let \mathfrak{h}_\pm be the corresponding holomorphic structure on the underlying $\mathcal{C}^{\kappa+1}$ bundle of the restrictions $P_{U^\pm}^\pm$ to U^\pm . Suppose that the tangential almost complex structures J_S^\pm induced by J^\pm on $\mathfrak{I}_{P_S^\pm}$ agree via v . Then*

- (1) *The topological bundle P^v on U admits a unique holomorphic reduction \mathfrak{h}^v extending \mathfrak{h}^\pm .*
- (2) *For any local \mathfrak{h}^v -holomorphic section $U \xrightarrow{\text{open}} V \xrightarrow{\tau} P^v$, we have*

$$\tau|_{V \cap \bar{U}^\pm} \in \Gamma^{\kappa+1}(V \cap \bar{U}^\pm, P^\pm),$$

i.e. the restrictions $\tau|_{V \cap \bar{U}^\pm}$ of τ are of class $\mathcal{C}^{\kappa+1}$ up to the boundary.

Remark 2.9. — The compatibility conditions on the tangential Cauchy–Riemann operators or tangential almost complex structures in Theorems 2.1, 2.4, 2.6, 2.8 are void when $\dim_{\mathbb{C}}(U) = 1$ (i.e. when U is a Riemann surface).

2.2. First applications: The Riemann–Hilbert problem

The first applications of Theorems 2.1, 2.4, 2.6 and their corollaries concern generalizations of the classical Riemann–Hilbert problem.

2.2.1. The Riemann–Hilbert problem on $\mathbb{P}_{\mathbb{C}}^1$

We first illustrate Theorem 2.1 in a simple special case: let $U = \mathbb{P}_{\mathbb{C}}^1 = \mathbb{C} \cup \{\infty\}$ and U^+ (U^-) be the connected component of $\mathbb{P}_{\mathbb{C}}^1 \setminus S$ which contains (does not contain) ∞ , where $S \subset \mathbb{C}$ is a compact, connected smooth curve. Let $v : S \rightarrow \mathrm{GL}(r, \mathbb{C})$ be a map of class $\mathcal{C}^{\kappa+1}$. Such a pair (S, v) is the input data of a Riemann–Hilbert problem.

In the renowned book chapter [14, Kapitel X. Riemanns Probleme in der Theorie der Funktionen einer komplexen Veränderlichen], Hilbert states and studies the following problem⁽²⁾:

⁽²⁾ In Hilbert’s original problem, as stated in loc. cit, v is supposed to be of class \mathcal{C}^2 and S real analytic.

PROBLEM 1 (Riemann–Hilbert). — *Find the space of pairs (f^-, f^+) of continuous maps $f^\pm : \bar{U}^\pm \rightarrow \mathbb{C}^r$ which are holomorphic on U^\pm and whose restrictions to S satisfy the condition $f_S^+ = v f_S^-$.*

Hilbert also states and studies a meromorphic version of the problem: f^+ is still required to be holomorphic, but f^- is allowed to be meromorphic with poles in U^- .

Several authors have stated interesting versions of the Riemann–Hilbert problem; for instance in [5, 28] one can find:

PROBLEM 2. — *Find the space of solutions (f^-, f^+) of the Riemann–Hilbert problem with f^- holomorphic on U^- and f^+ meromorphic on U^+ with a single pole with prescribed singularity type (Laurent coefficients of non-positive index) at ∞ .*

Other authors (see for instance [17]) are interested in matrix factorisation problems of the form:

PROBLEM 3. — *Find the space of pairs (Y^-, Y^+) of continuous maps $Y^\pm : \bar{U}^\pm \rightarrow \mathrm{gl}(r, \mathbb{C})$ which are holomorphic on U^\pm , whose restrictions to S satisfy the condition $Y_S^+ = v Y_S^-$ and such that $Y_+(\infty) = I_r$.*

More generally, let $\rho : G \rightarrow \mathrm{GL}(V)$ be a representation of G on a finitely dimensional complex vector space V , $m \in \mathbb{Z}$, and $\gamma \in V[z] = \sum_{s \geq 0} \gamma_s z^s$ a V -valued polynomial. Put $d := \deg(\gamma) \in \mathbb{Z}_{\geq -1}$ (we use the convention $\deg(\gamma) = -1$ for $\gamma = 0$). Let $\zeta : \mathbb{P}_{\mathbb{C}}^1 \setminus \{0\} \rightarrow \mathbb{C}$ be the standard coordinate of $\mathbb{P}_{\mathbb{C}}^1$ around ∞ ; replacing formally z by ζ^{-1} in the expression of γ , we obtain a Laurent polynomial $\tilde{\gamma} = \sum_{s=-d}^0 \tilde{\gamma}_s \zeta^s \in V[\zeta^{-1}]$ with $\tilde{\gamma}_s = \gamma_{-s}$. Regarding ∞ as an effective divisor on $\mathbb{P}_{\mathbb{C}}^1$, $\tilde{\gamma}$ can be interpreted as an element of $H^0(\mathcal{O}(d\infty)_{(d+1)\infty} \otimes V)$ with empty zero locus on the effective divisor $(d+1)\infty$. Let $v : S \rightarrow G$ be a map of class $\mathcal{C}^{\kappa+1}$. We ask:

PROBLEM 4. — *Find the space of pairs (Y^-, Y^+) of continuous maps*

$$Y^- : \bar{U}^- \longrightarrow V, \quad Y^+ : \bar{U}^+ \setminus \{\infty\} \longrightarrow V$$

with Y^- holomorphic on U^- , Y^+ holomorphic on $U^+ \setminus \{\infty\}$ such that $Y_S^+ = \rho(v)Y_S^-$ and

$$\lim_{z \rightarrow \infty} (z^{d-m} Y^+(z) - \gamma(z)) = 0. \quad (C_\infty)$$

The latter condition implies that ∞ is a non-essential singularity of Y_+ ; it is equivalent to the following condition on the Laurent series $\sum_{k \in \mathbb{Z}} b_k \zeta^k$ of Y_+ at ∞ :

$$b_s = \tilde{\gamma}_{m-d+s} \text{ for } s \leq d - m. \quad (C'_\infty)$$

Therefore the analytic condition (C_∞) has a purely complex geometric interpretation:

- For $d \geq 0$ (i.e. $\gamma \neq 0$) it requires that Y^+ extends as a section of the sheaf $\mathcal{O}(m\infty) \otimes_{\mathbb{C}} V$ on U^+ whose image in $H^0(\mathcal{O}(m\infty)_{(d+1)\infty} \otimes V)$ via the obvious morphism is $z^{m-d} \otimes \tilde{\gamma}$.
- For $d = -1$ (i.e. $\gamma = 0$) it just requires that \tilde{Y}^+ extends as a section of the sheaf $\mathcal{O}(m\infty) \otimes_{\mathbb{C}} V$ on U^+ . This is the “homogenous case”, the case when the solution space is naturally a vector space.

Hilbert’s original problem is obtained taking ρ to be the canonical representation of $\mathrm{GL}(r, \mathbb{C})$ on \mathbb{C}^r , $m = 0$, and $\gamma = 0$. Problem 2 is obtained with the same ρ taking $m = d$, and problem 3 corresponds to the representation of $\mathrm{GL}(r, \mathbb{C})$ on $\mathfrak{gl}(r, \mathbb{C})$ given by left multiplication, taking $m = 0$ and $\gamma =$ the degree 0 polynomial I_r .

Let $P^\pm := \bar{U}^\pm \times G$ be the trivial G -bundle endowed with the standard (trivial) bundle ACS. A map $v : S \rightarrow G$ of class $\mathcal{C}^{\kappa+1}$ can be regarded as a bundle isomorphism $P_S^- \rightarrow P_S^+$ of this class. By Theorem 2.8 and Remark 2.9, for any such v we have a well defined holomorphic structure \mathfrak{h}^v on the bundle P^v over $\mathbb{P}_{\mathbb{C}}^1$. The obtained holomorphic bundle, which will still be denoted by P^v to save on notations, comes with $\mathcal{C}^{\kappa+1}$ -trivializations θ_v^\pm on \bar{U}^\pm which are holomorphic on U^\pm . By Corollary 2.3, the locally free sheaf \mathcal{V}^v associated with the holomorphic vector bundle $P^v \times_\rho V$ is given by the *equivalent* formulae

$$\begin{aligned} W &\mapsto \left\{ \begin{pmatrix} f^- \\ f^+ \end{pmatrix} \in \Gamma^0(W \cap \bar{U}^-, V) \times \Gamma^0(W \cap \bar{U}^+, V) \left| \begin{array}{l} f^+|_S = \rho(v)f^-|_S, \\ f^\pm \text{ is holomorphic on } W \cap U^\pm \end{array} \right. \right\} \\ &= \left\{ \begin{pmatrix} f^- \\ f^+ \end{pmatrix} \in \Gamma^{k+1}(W \cap \bar{U}^-, V) \times \Gamma^{k+1}(W \cap \bar{U}^+, V) \left| \begin{array}{l} f^+|_S = \rho(v)f^-|_S, \\ f^\pm \text{ is holomorphic on } W \cap U^\pm \end{array} \right. \right\}. \end{aligned} \quad (2.3)$$

Note also that the trivialization θ^+ induces isomorphisms

$$H^0(\mathcal{O}(d\infty)_{(d+1)\infty} \otimes V) \xrightarrow{\cong} H^0(\mathcal{V}^v(d\infty)_{(d+1)\infty}),$$

so $\tilde{\gamma}$ gives an element $\nu_\gamma^v \in H^0(\mathcal{V}^v(d\infty)_{(d+1)\infty})$.

With these remarks we obtain:

COROLLARY 2.10. — *Let $S \subset \mathbb{C}$ be a compact, connected smooth curve.*

- (1) *The map $v \mapsto (P^v, \theta_v^-, \theta_v^+)$ gives a bijection between the group $\mathcal{C}^{\kappa+1}(S, G)$ and the set of isomorphism classes of triples (Q, θ^-, θ^+) consisting of a holomorphic principal G -bundle Q on $\mathbb{P}_{\mathbb{C}}^1$ and $\mathcal{C}^{\kappa+1}$ -trivializations θ^\pm of Q on \bar{U}^\pm which are holomorphic on U^\pm .*

- (2) If $\gamma = 0$, the space of solutions of the general Riemann–Hilbert Problem 4 can be naturally identified with $H^0(\mathbb{P}_{\mathbb{C}}^1, \mathcal{V}^v(m\infty))$.
- (3) If $\gamma \neq 0$, the space of solutions of the general Riemann–Hilbert Problem 4 is non-empty if and only if the image of $z^{m-d} \otimes \nu_{\gamma}^v$ via the connecting morphism

$$\begin{aligned} H^0(\mathcal{V}^v(m\infty)_{(d+1)\infty}) &= H^0(\mathbb{P}_{\mathbb{C}}^1, \mathcal{V}^v(m\infty)/\mathcal{V}^v((m-d-1)\infty)) \\ &\longrightarrow H^1(\mathbb{P}^1, \mathcal{V}^v((m-d-1)\infty)) \end{aligned}$$

vanishes. If this is the case, this space has the natural structure of an affine space with model space $H^0(\mathbb{P}_{\mathbb{C}}^1, \mathcal{V}^v((m-d-1)\infty))$, and can be naturally identified with the pre-image of $z^{m-d} \otimes \nu_{\gamma}^v$ via the natural morphism

$$H^0(\mathbb{P}_{\mathbb{C}}^1, \mathcal{V}^v(m\infty)) \longrightarrow H^0(\mathcal{V}^v(m\infty)_{(d+1)\infty}).$$

In particular, the space of solutions of the original Riemann–Hilbert problem (of Problem 1) is naturally isomorphic to the space $H^0(\mathbb{P}_{\mathbb{C}}^1, \mathcal{V}^v)$ associated with the canonical representation of $\mathrm{GL}(r, \mathbb{C})$ on \mathbb{C}^r . Hilbert’s results [14, Sätze 27-30] follow easily from Corollary 2.10. Taking into account formula (2.3) we also obtain the following general regularity result:

Remark 2.11. — Any solution of a Riemann–Hilbert problem with v of class $C^{\kappa+1}$ is also of class $C^{\kappa+1}$ up to the boundary.

By Grothendieck’s classification theorem [12], the sheaf \mathcal{V}^v splits as a direct sum of invertible sheaves, so we have $\mathcal{V}^v \simeq \bigoplus_{j=1}^r \mathcal{O}(n_j)$ with $n_j \in \mathbb{Z}$ and $\sum_{j=1}^r n_j = \deg(\mathcal{V}^v)$. For the canonical representation of $\mathrm{GL}(r, \mathbb{C})$ on \mathbb{C}^r we have $\deg(\mathcal{V}^v) = -\deg(\det(v))$. Therefore, once in possession of the complex geometric objects (Q, ν_{γ}^v) associated with the input data (ρ, v, γ) , the corresponding Riemann–Hilbert problem can be approached using elementary complex geometric methods. For instance, one can easily give examples of such data for which the space of solutions is empty and, at least for small r and standard representations, one can compute all possible dimensions of the space of solutions for a given Grothendieck decomposition of \mathcal{V}^v .

A difficulty remains: make the bijection provided by Corollary 2.10 effective, i.e., for given v , determine *explicitly* a Grothendieck direct sum decomposition of \mathcal{V}^v and the “position” of ν_{γ}^v with respect to the summands.

2.2.2. The Riemann–Hilbert problem on Riemann surfaces

The formalism and the results of Section 2.1.2 allows us to formulate and approach with complex geometric methods a very general Riemann–Hilbert problem: Let X be a closed Riemann surface, $S \subset X$ an *arbitrary*

(non-necessarily connected, non-necessarily separating) closed, oriented real 1-dimensional submanifold, $v : S^- \rightarrow G$ a map of class $\mathcal{C}^{\kappa+1}$, and $\rho : G \rightarrow \mathrm{GL}(V)$ a representation. Let also D, Δ be divisors on $X \setminus S$, with $\Delta \geq 0$, and fix a section $\gamma \in H^0(\mathcal{O}(D)_\Delta \otimes V)$ which is nowhere vanishing⁽³⁾ on Δ .

PROBLEM 5. — *Find the space of meromorphic maps $Y : \widehat{X}_S \setminus \widehat{S} \dashrightarrow V$ which extend continuously around \widehat{S} , such that:*

- (1) $Y|_{S^+} \circ b = \rho(v)Y|_{S^-}$,
- (2) *Via the obvious identification $\widehat{X}_S \setminus \widehat{S} = X \setminus S$, Y extends as a section of $\mathcal{O}(D) \otimes V$, and the image of this extension in $H^0(\mathcal{O}(D)_\Delta \otimes V)$ via the obvious morphism, is γ .*

The “homogenous case” corresponds to the case $\Delta = 0$ (the empty divisor). For $D = \Delta = 0$ one just obtains the space of continuous maps $Y : \widehat{X}_S \rightarrow V$ which satisfy the v -compatibility condition (1) and are holomorphic on $X \setminus S$.

Taking into account Theorem 2.4, Corollary 2.5 and Remark 2.9, we obtain, as in Corollary 2.10, a map

$$v \mapsto (P^v, \theta^v)$$

which gives a bijection between the group $\mathcal{C}^{\kappa+1}(S^-, G)$ and the set of isomorphism classes of pairs (Q, θ) , where Q is a holomorphic principal G -bundle on the Riemann surface X , and θ is a trivialization of class $\mathcal{C}^{\kappa+1}$ of the pull-back $(p_S^X)^*(Q)$ of Q to \widehat{X}_S , which is holomorphic on $\widehat{X}_S \setminus \widehat{S}$. We define the locally free sheaf \mathcal{V}^v as in the previous section, and note that, via the trivialization θ^v , γ gives an element $\nu_\gamma^v \in H^0(\mathcal{V}^v(D)_\Delta)$. Using the explicit formulae for the sheaf \mathcal{V}^v given by Corollary 2.5, we obtain the following complex geometric interpretation of the space of solutions of the general Riemann–Hilbert Problem 5:

COROLLARY 2.12. — *The space of solutions of Problem 5 is non-empty if and only if the image of ν_γ^v via the connecting morphism*

$$H^0(\mathcal{V}^v(D)_\Delta) = H^0(X, \mathcal{V}^v(D)/\mathcal{V}^v(D - \Delta)) \longrightarrow H^1(X, \mathcal{V}^v(D - \Delta))$$

vanishes. If this is the case, this space has the natural structure of an affine space with model space $H^0(X, \mathcal{V}^v(D - \Delta))$, and can be identified with the pre-image of ν_γ^v via the natural morphism $H^0(X, \mathcal{V}^v(D)) \rightarrow H^0(\mathcal{V}^v(D)_\Delta)$.

An interesting special case:

(3) If γ has non-empty zero locus on Δ , one will obtain an equivalent problem associated with a smaller pair (D, Δ) .

Example 2.13. — Let X be a Riemann surface of genus 1, and S a non-separating circle, as in Figure 2.1. We can assume that $X = \mathbb{C}^*/\langle\alpha\rangle$, where $\alpha \in \mathbb{C}^*$ with $|\alpha| < 1$, and S is the image in X of $\Sigma := \partial\bar{D}$, where $\bar{D} \subset \mathbb{C}$ is a smooth compact disk such that $0 \in D$ and $\alpha\bar{D} \subset D$. Therefore, we can identify \widehat{X}_S , the Riemann surface with boundary obtained by cutting X along S , with the annulus $\bar{\Omega} := \bar{D} \setminus \alpha D$ whose boundary is $\alpha\Sigma \cup \Sigma$. In this case the unknown of the Riemann–Hilbert Problem 5 is a meromorphic map $Y : \Omega \dashrightarrow V$ on the open annulus $\Omega := D \setminus \alpha\bar{D}$ extending continuously around $\partial\bar{\Omega}$ and satisfying the compatibility condition:

$$\forall z \in \Sigma, \quad Y(\alpha z) = \rho(v)Y(z).$$

Note that the holomorphic vector bundles on elliptic curves have been classified [1], so Corollary 2.12 allows one (in principle) to solve any Riemann–Hilbert problem of the considered type on an elliptic curve.

2.2.3. The Riemann–Hilbert problem in arbitrary dimension

Theorem 2.6 suggests a natural generalization of the Riemann–Hilbert problem in arbitrary dimension (again for any complex Lie group G), and also a general complex geometric method to approach it.

Let X be a connected, closed complex manifold and $S \subset X$ a general (non-necessarily connected, non necessarily separating) closed, oriented real hypersurface. Let P be a differentiable principal G -bundle on the manifold with boundary \widehat{X}_S obtained by cutting X along S (see Section 2.1.2), and let J be a formally integrable bundle ACS on P (see Sections A.5.2, A.5.3). Therefore, compared with previous generalizations, we start with an arbitrary, not necessarily trivial, formally holomorphic principal G -bundle (P, J) on \widehat{X}_S .

Let $v : P_{S^-} \rightarrow b^*(P_{S^+})$ be a bundle isomorphism of class $\mathcal{C}^{\kappa+1}$. Suppose that the following compatibility condition holds:

COMPATIBILITY CONDITION. — *The tangential almost complex structures J_{S^\pm} induced by J on the distributions $\mathfrak{T}_{P_{S^\pm}} \subset T_{P_{S^\pm}}$ agree via v .*

By Theorem 2.6 we obtain a holomorphic principal G -bundle P^v on X whose pull back $p_S^{X*}(P^v)$ to \widehat{X}_S comes with a tautological bundle isomorphism $P \xrightarrow{\theta^v} p_S^{X*}(P^v)$ of class $\mathcal{C}^{\kappa+1}$ which is holomorphic on $\widehat{X}_S \setminus \widehat{S}$. The map

$$v \longmapsto (P^v, \theta^v)$$

defines a bijection between the set of bundle isomorphisms $v : P_{S-} \rightarrow b^*(P_{S+})$ of class $\mathcal{C}^{\kappa+1}$ satisfying the above compatibility condition on S and the set of isomorphism classes of pairs (Q, θ) consisting of a holomorphic G -bundle Q on the closed complex manifold X , and a bundle isomorphism $\theta : P \rightarrow p_S^{X*}(Q)$ of class $\mathcal{C}^{\kappa+1}$ which is holomorphic on $\widehat{X}_S \setminus \widehat{S}$.

Let $\rho : G \rightarrow \mathrm{GL}(V)$ be a representation of G on V , and \mathcal{V} the locally free sheaf on $X \setminus S$ corresponding to the associated bundle $P_{X \setminus S} \times_{\rho} V$. Let $Z \subset X \setminus S$ a (possibly empty) compact complex subspace and let $\gamma \in H^0(\mathcal{V}_Z)$.

PROBLEM 6. — *Find the space of continuous sections*

$$Y \in \Gamma^0(\widehat{X}_S, P_E \times_{\rho} V)$$

which are holomorphic on $\widehat{X} \setminus \widehat{S}$ such that

- (1) $Y|_{S+} \circ b = \rho(v)Y|_{S-}$,
- (2) *the image of Y in $H^0(\mathcal{V}_Z)$ via the obvious morphism is γ .*

Denoting by \mathcal{V}^v the locally free sheaf on X associated with the holomorphic bundle $P^v \times_{\rho} V$, we obtain an obvious identification $\mathcal{V}^v|_{X \setminus S} = \mathcal{V}$. With these notations and remarks we obtain:

COROLLARY 2.14. — *Suppose the above compatibility condition holds. The space of solutions of Problem 6 is non-empty if and only if the image of γ via the connecting morphism*

$$H^0(\mathcal{V}_Z^v) = H^0(X, \mathcal{V}^v / \mathcal{V}^v \otimes \mathcal{I}_Z) \longrightarrow H^1(X, \mathcal{V}^v \otimes \mathcal{I}_Z)$$

vanishes. If this is the case, this space has the natural structure of an affine space with model space $H^0(X, \mathcal{V}^v \otimes \mathcal{I}_Z)$, and can be identified with the pre-image of γ via the natural morphism $H^0(X, \mathcal{V}^v) \rightarrow H^0(\mathcal{V}_Z^v)$.

2.3. Gauge theoretical applications: Isomorphisms of moduli spaces

In this article by complex manifold with boundary we will always mean a submanifold with boundary \overline{X} of a complex manifold U . In other words, the complex manifolds with boundary we consider have a collar neighborhood in the sense of [15].

2.3.1. Isomorphisms of moduli spaces of framed vector bundles

For a \mathcal{C}^{∞} vector bundle E on a *compact* complex manifold with boundary \overline{X} , we denote by \mathcal{I}_E^{κ} the space of *formally integrable* Dolbeault operators with

coefficients in \mathcal{C}^κ on E and we define the moduli space

$$\mathcal{M}_{\partial\bar{X}}(E) := \mathcal{I}_E^\kappa / \mathcal{G}_{\partial\bar{X}}^E,$$

where $\mathcal{G}_{\partial\bar{X}}^E$ is the gauge group

$$\mathcal{G}_{\partial\bar{X}}^E := \{f \in \Gamma^{\kappa+1}(X, \mathrm{GL}(E)) \mid f_{\partial\bar{X}} = \mathrm{id}_{E_{\partial\bar{X}}}\}.$$

Let now X be a connected, closed complex manifold, $S \subset X$ an oriented closed, smooth real hypersurface, and E a \mathcal{C}^∞ vector bundle on X . In this case \mathcal{I}_E^κ will stand for the space of *integrable* Dolbeault operators with coefficients in \mathcal{C}^κ on E and we define the moduli space

$$\mathcal{M}_S(E) := \mathcal{I}_E^\kappa / \mathcal{G}_S^E,$$

where \mathcal{G}_S^E is the gauge group

$$\mathcal{G}_S^E := \{f \in \Gamma^{\kappa+1}(X, \mathrm{GL}(E)) \mid f_S = \mathrm{id}_{E_S}\}.$$

We denote by \widehat{E} the pull-back of E to the manifold with boundary \widehat{X}_S obtained by cutting X alongs S via the canonical map $p_S^X : \widehat{X}_S \rightarrow X$ (see Section 2.1.2). In the special case when S separates X , \widehat{X}_S reduces to the disjoint union $\bar{X}^- \amalg \bar{X}^+$ of the corresponding pieces, and \widehat{E} reduces to the disjoint union $E^- \amalg E^+$ of the restrictions $E^\pm := E|_{\bar{X}^\pm}$. The bundle \widehat{E} comes with a canonical bundle isomorphism $v : \widehat{E}_{S^-} \rightarrow b^*(\widehat{E}_{S^+})$ of class \mathcal{C}^∞ induced by the obvious isomorphisms $\widehat{E}_{S^\pm} \rightarrow E_S$ which cover the identifications $S^\pm \xrightarrow{\sim} S$.

DEFINITION 2.15. — *A formally integrable Dolbeault operator $\mathfrak{d} \in \mathcal{I}_E^\kappa$ will be called descendable, if the tangential Cauchy–Riemann operators \mathfrak{d}_{S^\pm} on \widehat{E}_{S^\pm} agree via v (are v -compatible).*

The pull-back $\widehat{\delta}$ to \widehat{E} of any integrable Dolbeault operator $\delta \in \mathcal{I}_E^\kappa$ is obviously descendable. Let $\mathcal{I}_{E\downarrow}^\kappa \subset \mathcal{I}_E^\kappa$ be the (obviously $\mathcal{G}_{\partial\widehat{X}_S}^{\widehat{E}}$ -invariant) subspace of descendable formally integrable Dolbeault operators on \widehat{E} , and let $\mathcal{M}_{\partial\widehat{X}_S}^\downarrow(\widehat{E})$ be the corresponding closed subspace of $\mathcal{M}_{\partial\widehat{X}_S}(\widehat{E})$. In the special case when S separates X , we have $\mathcal{I}_E^\kappa = \mathcal{I}_{E^-}^\kappa \times \mathcal{I}_{E^+}^\kappa$, where $E^\pm := E|_{\bar{X}^\pm}$, and a pair (δ^-, δ^+) is descendable if and only if the equality $\delta_S^- = \delta_S^+$ holds in the space, denoted \mathcal{C} , of Cauchy–Riemann operators with coefficients in \mathcal{C}^κ on E_S .

THEOREM 2.16. — *Suppose $\kappa \in (0, +\infty) \setminus \mathbb{N}$. With the above notations and assumptions, the pull-back map $\delta \mapsto \widehat{\delta}$ induces a homeomorphism*

$$\mathcal{M}_S(E) \longrightarrow \mathcal{M}_{\partial\widehat{X}_S}^\downarrow(\widehat{E}).$$

In the special case when S separates X , the restriction map $\delta \mapsto (\delta^-, \delta^+)$ induces a homeomorphism

$$\mathcal{M}_S(E) \longrightarrow \mathcal{M}_{\partial\bar{X}^-}(E^-) \times_{\mathcal{C}} \mathcal{M}_{\partial\bar{X}^+}(E^+).$$

2.3.2. Isomorphisms of moduli spaces of framed principal bundles

Let G be a complex Lie group. With the notations and under the assumptions of Section 2.3.1 we replace:

- E by a \mathcal{C}^∞ principal G bundle P on X (\bar{X}).
- \mathcal{I}_E^κ by the space \mathcal{I}_P^κ of (formally) integrable bundle ACS of class \mathcal{C}^κ on P .
- \mathcal{G}_S^E ($\mathcal{G}_{\partial\bar{X}}^E$) by respectively the gauge groups

$$\mathcal{G}_S^P := \{f \in \Gamma^{\kappa+1}(X, P \times_{\iota} G) \mid f_S = \text{id}\},$$

$$\mathcal{G}_{\partial\bar{X}}^P := \{f \in \Gamma^{\kappa+1}(\bar{X}, P \times_{\iota} G) \mid f_{\partial\bar{X}} = \text{id}\}.$$

- $\mathcal{M}_S(E)$ ($\mathcal{M}_{\partial\bar{X}}(E)$) by respectively the moduli spaces

$$\mathcal{M}_S(P) := \mathcal{I}_P^\kappa / \mathcal{G}_S^P, \text{ respectively } \mathcal{M}_{\partial\bar{X}}(P) := \mathcal{I}_P^\kappa / \mathcal{G}_{\partial\bar{X}}^P.$$

We also replace \widehat{E} by its pull back \widehat{P} to \widehat{X}_S , and, if S separates X , we replace the restrictions E^\pm by $P^\pm := P_{\bar{X}^\pm}$.

In this principal bundle framework we also have a canonical bundle isomorphism $v : \widehat{P}_{S^-} \rightarrow b^*(\widehat{P}_{S^+})$ of class \mathcal{C}^∞ . An element $\mathfrak{J} \in \mathcal{I}_{\widehat{P}}^\kappa$ will be called descendable, if the induced tangential almost complex structures \mathfrak{J}_{S^\pm} on the distributions $\mathfrak{T}_{\widehat{P}_{S^\pm}} \subset T_{\widehat{P}_{S^\pm}}$ (see Remark 3.7) agree via v . We denote by $\mathcal{I}_{\widehat{P}_\downarrow}^\kappa \subset \mathcal{I}_{\widehat{P}}^\kappa$ the subspace of descendable formally integrable bundle ACS on \widehat{P} and by $\mathcal{M}_{\partial\widehat{X}_S}^\downarrow(\widehat{P})$ its quotient by the gauge group $\mathcal{G}_{\partial\widehat{X}_S}^{\widehat{P}}$.

THEOREM 2.17. — $\kappa \in (0, +\infty) \setminus \mathbb{N}$. *With the above notations and assumptions, the pull-back map $J \mapsto \widehat{J}$ induces a homeomorphism*

$$\mathcal{M}_S(P) \longrightarrow \mathcal{M}_{\partial\widehat{X}_S}^\downarrow(\widehat{P}).$$

In the special case when S separates X , the restriction map $J \mapsto (J_{P^-}, J_{P^+})$ induces a homeomorphism

$$\mathcal{M}_S(P) \longrightarrow \mathcal{M}_{\partial\bar{X}^-}(P^-) \times_{\mathcal{I}} \mathcal{M}_{\partial\bar{X}^+}(P^+).$$

onto the fiber product of the moduli spaces $\mathcal{M}_{\partial\bar{X}^\pm}(P^\pm)$ over the space \mathcal{I} of almost complex structures of class \mathcal{C}^κ on \mathfrak{T}_{P_S} which are G -invariant, make the bundle epimorphism $\mathfrak{T}_{P_S} \rightarrow \mathfrak{T}_S$ \mathbb{C} -linear, and the parametrization of the G -orbits pseudo-holomorphic.

Remark 2.18. — Taking $\kappa = +\infty$ in Theorems 2.16, 2.17 we still obtain homeomorphisms provided any moduli space \mathcal{M} intervening in these theorems (but constructed with objects of class \mathcal{C}^∞) is endowed with the initial topology associated with the family of maps $(\mathcal{M} \rightarrow \mathcal{M}^\kappa)_{\kappa \in (0, +\infty) \setminus \mathbb{N}}$; here \mathcal{M}^κ stands for the similar moduli space constructed using objects of class \mathcal{C}^κ .

As pointed out in Remark 2.9, the required compatibility conditions above S become void on Riemann surfaces, so the isomorphisms Theorems 2.16, 2.17 give:

Remark 2.19. — Suppose $\dim(U) = 1$. With the notations and under the assumptions above we have homeomorphisms of moduli spaces:

$$\begin{aligned} \mathcal{M}_S(E) &\longrightarrow \mathcal{M}_{\partial \widehat{X}_S}(\widehat{E}), \quad \mathcal{M}_S(E) \longrightarrow \mathcal{M}_{\partial \overline{X}}(E^-) \times \mathcal{M}_{\partial \overline{X}^+}(E^+), \\ \mathcal{M}_S(P) &\longrightarrow \mathcal{M}_{\partial \widehat{X}_S}(\widehat{P}), \quad \mathcal{M}_S(P) \longrightarrow \mathcal{M}_{\partial \overline{X}}(P^-) \times \mathcal{M}_{\partial \overline{X}^+}(P^+). \end{aligned}$$

Remark 2.20. — In Section 4.2 we will identify the moduli spaces $\mathcal{M}_S(E)$, $\mathcal{M}_S(P)$, $\mathcal{M}_{\partial \overline{X}}(E)$, $\mathcal{M}_{\partial \overline{X}}(P)$ intervening in Theorems 2.16, 2.17 with moduli spaces of *framed bundles* defined (in an abstract way, see Definition 4.3) as pairs consisting of a holomorphic bundle on X (\overline{X}) and a framing of class $\mathcal{C}^{\kappa+1}$ on S (respectively $\partial \overline{X}$).

In Section 5 we will consider explicit examples of (boundary) framed moduli spaces and give explicit formulae for the homeomorphisms given by Theorem 2.17 and their inverses in the special cases :

- $X = \mathbb{P}_{\mathbb{C}}^1$ and $S \subset \mathbb{C}$ is a closed curve.
- X is an elliptic curve and $S \subset X$ is a non-separating closed curve.

3. Gluing holomorphic bundles along a real hypersurface

3.1. The tangential Cauchy–Riemann operator

Let U be a differentiable manifold, $S \subset U$ a closed real hypersurface and $\eta_S \subset T_{U|S}^{\ast \mathbb{C}}$ be the annihilator of T_S (or, equivalently, of $T_S^{\mathbb{C}}$) in the restriction $T_{U|S}^{\ast \mathbb{C}}$ of the complex cotangent bundle $T_U^{\ast \mathbb{C}}$ of U to S ; η_S can be identified with the complexification of the conormal real line bundle n_S^* of S in U .

Suppose now that U is an n -dimensional complex manifold. The image $\eta_S^{0,1}$ of η_S in $\bigwedge_{U|S}^{0,1}$ is a line subbundle of $\bigwedge_{U|S}^{0,1}$, which can be identified with the annihilator of the canonical distribution

$$\mathfrak{T}_S = T_S \cap J_U(T_S) \subset T_S$$

(or, equivalently, of the hyperplane $\mathfrak{T}_S^{0,1} \subset T_{U|S}^{0,1}$) in $\Lambda_{U|S}^{0,1}$. Here $J_U \in \Gamma(U, \text{End}(T_U))$ stands for the integrable almost complex structure on U induced by its complex manifold structure. The projection $T_{U|S}^{*\mathbb{C}} \rightarrow \Lambda_{U|S}^{0,1}$ induces a line bundle isomorphism $\psi_S : \eta_S \rightarrow \eta_S^{0,1}$.

Put $\Lambda_S^{0,q} := \Lambda^{0,q} \mathfrak{T}_S^{*\mathbb{C}}$. The fiber $\Lambda_{S,x}^{0,q}$ of this bundle over $x \in S$ can be identified with the space of alternate \mathbb{R} -multilinear forms $\mathfrak{T}_{S,x}^q \rightarrow \mathbb{C}$ which are anti-linear with respect to each argument. We have an obvious bundle epimorphism

$$r_{\mathfrak{T}_S} : \Lambda_{U|S}^{0,q} \longrightarrow \Lambda_S^{0,q}$$

induced by the inclusion $\mathfrak{T}_S \subset T_{U|S}$. We obtain a commutative diagram of bundle morphisms on S with exact horizontal rows

$$\begin{array}{ccccccc} & \eta_S & & & & & \\ & \psi_S \downarrow & & & & & \\ 0 & \longrightarrow & \eta_S^{0,1} & \hookrightarrow & \Lambda_{U|S}^{0,1} & \xrightarrow{\quad \sqcap_S \quad} & \eta^{0,1*} \otimes \Lambda_{U|S}^{0,2} \\ & & \parallel & & \parallel & & \uparrow [\sqcap_S] \\ 0 & \longrightarrow & \eta_S^{0,1} & \hookrightarrow & \Lambda_{U|S}^{0,1} & \xrightarrow{\quad r_{\mathfrak{T}_S} \quad} & \Lambda_S^{0,1} \longrightarrow 0, \end{array} \quad (3.1)$$

where, via the identification $\Lambda_{U|S}^{0,1} = \eta^{0,1*} \otimes (\eta^{0,1} \otimes \Lambda_{U|S}^{0,1})$, we have put

$$\sqcap_S := \text{id}_{\eta^{0,1*}} \otimes \wedge,$$

and $[\sqcap_S]$ is induced by \sqcap_S .

Remark 3.1. — By the definition of \sqcap_S we have the identity:

$$\forall x \in S, \forall (a^{0,1}, b^{0,1}) \in \eta_x^{0,1} \times \Lambda_{U,x}^{0,1}, \quad a^{0,1} \otimes \sqcap_S(b^{0,1}) = a^{0,1} \wedge b^{0,1}. \quad (3.2)$$

Taking $a^{0,1} = \psi_S(a)$ with $a \in \eta_x$, we obtain

$$\forall x \in S, \forall (a, b^{0,1}) \in \eta_x \times \Lambda_{U,x}^{0,1}, \quad \psi_S(a) \otimes \sqcap_S(b^{0,1}) = \psi_S(a) \wedge b^{0,1}.$$

This formula can be written as

$$\left(\psi_S \otimes \text{id}_{\Lambda_{U|S}^{0,2}} \right) \circ (\text{id}_{\eta_S} \otimes \sqcap_S) = \wedge \left(\psi_S \otimes \text{id}_{\Lambda_{U|S}^{0,1}} \right), \quad (3.3)$$

where, on the right, \wedge stands for the bundle morphism $\eta_S^{0,1} \otimes \Lambda_{U|S}^{0,1} \rightarrow \Lambda_{U|S}^{0,2}$ induced by the wedge product.

Similarly, for any $q \geq 1$ we obtain a commutative diagram of bundles on S

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \eta_S^{0,1} \wedge \Lambda_{U|S}^{0,q-1} & \hookrightarrow & \Lambda_{U|S}^{0,q} & \xrightarrow{\sqcap_S} & \eta^{0,1*} \otimes \Lambda_{U|S}^{0,q+1} \\
 & & \parallel & & \parallel & & \uparrow [\sqcap_S] \\
 0 & \longrightarrow & \eta_S^{0,1} \wedge \Lambda_{U|S}^{0,q-1} & \hookrightarrow & \Lambda_{U|S}^{0,q} & \xrightarrow{r_{\mathfrak{T}_S}} & \Lambda_S^{0,q} \longrightarrow 0
 \end{array} \tag{3.4}$$

with exact rows.

Let E be a complex vector bundle of class \mathcal{C}^∞ on U . For $\gamma \in [0, +\infty]$ put

$$\begin{aligned}
 \mathcal{I}_S^\gamma(U, \Lambda_U^{0,q} \otimes E) &:= \ker [\Gamma^\gamma(U, \Lambda_U^{0,q} \otimes E) \longrightarrow \Gamma^\gamma(S, \Lambda_{U|S}^{0,q} \otimes E) \longrightarrow \Gamma^\gamma(S, \Lambda_S^{0,q} \otimes E)] \\
 &= \begin{cases} \ker(\Gamma^\gamma(U, E) \rightarrow \Gamma^\gamma(S, E)) & \text{if } q = 0, \\ \{\beta \in \Gamma^\gamma(U, \Lambda_U^{0,q} \otimes E) \mid (\sqcap_S \otimes \text{id}_E)\beta_S = 0\} & \text{if } q > 0. \end{cases} \tag{3.5}
 \end{aligned}$$

Remark 3.2. — The two restriction maps in the definition of $\mathcal{I}_S^\gamma(U, \Lambda_U^{0,q} \otimes E)$ are surjective, so their composition $\Gamma^\gamma(U, \Lambda_U^{0,q} \otimes E) \rightarrow \Gamma^\gamma(S, \Lambda_S^{0,q} \otimes E)$ induces an isomorphism

$$\Gamma^\gamma(U, \Lambda_U^{0,q} \otimes E) / \mathcal{I}_S^\gamma(U, \Lambda_U^{0,q} \otimes E) \xrightarrow{\simeq} \Gamma^\gamma(S, \Lambda_S^{0,q} \otimes E).$$

Proof. — The surjectivity of $\Gamma^\gamma(U, \Lambda_U^{0,q} \otimes E) \rightarrow \Gamma^\gamma(S, \Lambda_{U|S}^{0,q} \otimes E)$ follows taking $m = 0$ in Corollary A.13(1)-(2). The map $\Gamma^\gamma(S, \Lambda_{U|S}^{0,q} \otimes E) \rightarrow \Gamma^\gamma(S, \Lambda_S^{0,q} \otimes E)$ is induced by an epimorphism of \mathcal{C}^∞ bundles on S , so is surjective. \square

Let \mathcal{C}_U^γ (respectively $\mathcal{C}^\gamma(\Lambda_U^{0,q} \otimes E)$, $\mathcal{C}^\gamma(\Lambda_{U|S}^{0,q} \otimes E)$) be the sheaves of locally defined \mathbb{C} -valued functions (sections of the bundles $\Lambda_U^{0,q} \otimes E$, $\Lambda_{U|S}^{0,q} \otimes E$) on U , respectively S , of class \mathcal{C}^γ (see Section A.1). The assignment

$$U \supset^{\text{open}} V \longmapsto \mathcal{I}_{S \cap V}^\gamma(V, \Lambda_V^{0,q} \otimes E)$$

defines a sheaf on U which will be denoted $\mathcal{I}_S^\gamma(\Lambda_U^{0,q} \otimes E)$; it is a \mathcal{C}_U^γ -submodule of $\mathcal{C}^\gamma(\Lambda_U^{0,q} \otimes E)$ which coincides with $\mathcal{C}^\gamma(\Lambda_U^{0,q} \otimes E)$ on the complement of S . Let $x \in X$, $\rho \in \mathcal{C}^\infty(V, \mathbb{R})$ be a local defining function for S around x (i.e. we have $x \in S \cap V = \rho^{-1}(0)$ and $d\rho$ is nowhere vanishing on $S \cap V$), and ρ_x its germ at x .

Remark 3.3. — The stalk $\mathcal{I}_S^\gamma(\bigwedge_U^{0,q} \otimes E)_x$ of $\mathcal{I}_S^\gamma(\bigwedge_U^{0,q} \otimes E)$ at $x \in S$ is given by:

$$\begin{aligned} \mathcal{I}_S^\gamma(\bigwedge_U^{0,q} \otimes E)_x = \ker(\mathcal{C}^\gamma(\bigwedge_U^{0,q} \otimes E)_x \longrightarrow \mathcal{C}^\gamma(\bigwedge_{U|S}^{0,q} \otimes E)_x) \\ + \bar{\partial}\rho_x \wedge \mathcal{C}^\gamma(\bigwedge_U^{0,q-1} \otimes E)_x. \end{aligned}$$

Proof. — Diagram (3.4) shows that a form $\beta \in \Gamma^\gamma(V, \bigwedge_V^{0,q} \otimes E)$ belongs to $\mathcal{I}_S^\gamma(V, \bigwedge_V^{0,q} \otimes E)$ if and only if its restriction to $V \cap S$ is a section of the subbundle

$$\eta_{V \cap S}^{0,1} \wedge \bigwedge_{V|S}^{0,q-1} \otimes E_S \subset \bigwedge_{V|S}^{0,q} \otimes E_S.$$

It suffices to note that the restriction of the sub-bundle $\bar{\partial}\rho \wedge \bigwedge_V^{0,q-1} \subset \bigwedge_V^{0,q}$ to $S \cap V$ coincides with $\eta_{V \cap S}^{0,1} \wedge \bigwedge_{V|S}^{0,q-1}$. \square

Note that, for $\gamma = \infty$, Remark 3.3 gives:

$$\mathcal{I}_S^\infty(\bigwedge_U^{0,q} \otimes E)_x = \rho_x \mathcal{C}^\infty(\bigwedge_U^{0,q} \otimes E)_x + \bar{\partial}\rho_x \wedge \mathcal{C}^\infty(\bigwedge_U^{0,q-1} \otimes E)_x.$$

This description does not extend to the case $\gamma < \infty$. For instance, for $q = 0$, an element of the stalk $\mathcal{C}^\gamma(E)_x$ which vanishes on S is not necessarily divisible by ρ_x in this $\mathcal{C}_{U,x}^\gamma$ -module.

Remark 3.4. — Let $\kappa \in [0, \infty]$, δ a Dolbeault operator with coefficients in \mathcal{C}^κ on E and $0 \leq \gamma \leq \kappa$. Then $\delta \mathcal{I}_S^{\gamma+1}(U, \bigwedge_U^{0,q} \otimes E) \subset \mathcal{I}_S^\gamma(U, \bigwedge_U^{0,q+1} \otimes E)$.

Proof. — Let $\beta \in \mathcal{I}_S^{\gamma+1}(U, \bigwedge_U^{0,q} \otimes E)_x$. With respect to a holomorphic chart of U and a local trivialization of E around x , δ is given by

$$\delta\psi = \bar{\partial}\psi + \alpha\psi$$

for a germ $\alpha \in \mathcal{C}^\kappa(\bigwedge_U^{0,1} \otimes \mathfrak{gl}(r, \mathbb{C}))_x$. By Remark 3.3 we have $\beta = \beta^0 + \bar{\partial}\rho_x \wedge \nu$ where

$$\beta^0 \in \ker(\mathcal{C}^{\gamma+1}(\bigwedge_U^{0,q} \otimes E)_x \longrightarrow \mathcal{C}^{\gamma+1}(\bigwedge_{U|S}^{0,q} \otimes E)_x), \quad \nu \in \mathcal{C}^{\gamma+1}(\bigwedge_U^{0,q-1} \otimes E)_x.$$

Writing $\beta^0 = \sum_{|I|=q} d\bar{z}^I \otimes \beta_I$, where all the germs $\beta_I \in \mathcal{C}_{U,x}^{\gamma+1}$ vanish on S , we have

$$\delta\beta^0 = (-1)^q \sum_{|I|=q} d\bar{z}^I \wedge (\bar{\partial}\beta_I + \alpha\beta_I).$$

Since β_I vanishes on S , it follows that $d\beta_I$ vanishes on T_S around x , so $\bar{\partial}\beta_I$ vanishes on \mathfrak{T}_S around x . This proves that the terms $d\bar{z}^I \wedge \bar{\partial}\beta_I$ belong to $\mathcal{I}_S^\gamma(\bigwedge_U^{0,q+1} \otimes E)_x$. The terms $d\bar{z}^I \wedge \alpha\beta_I$, $\delta(\bar{\partial}\rho \wedge \nu) = -\bar{\partial}\rho \wedge d\nu$ obviously belong to $\mathcal{I}_S^\gamma(\bigwedge_U^{0,q+1} \otimes E)_x$, which completes the proof. \square

Using Remarks 3.2, 3.4 it follows.

COROLLARY 3.5. — *Let δ be a Dolbeault operator with coefficients in \mathcal{C}^κ on E . The associated operator $\Gamma^{\kappa+1}(U, \bigwedge_U^{0,q} \otimes E) \rightarrow \Gamma^\kappa(U, \bigwedge_U^{0,q+1} \otimes E)$ induces a first order differential operator $\Gamma^{\kappa+1}(S, \bigwedge_S^{0,q} \otimes E) \rightarrow \Gamma^\kappa(S, \bigwedge_S^{0,q+1} \otimes E)$ with coefficients in \mathcal{C}^κ .*

Taking $q = 0$, we obtain a first order differential operator

$$\delta_S : \Gamma^{\kappa+1}(S, E) \longrightarrow \Gamma^\kappa(S, \bigwedge_S^{0,1} \otimes E),$$

with coefficients in \mathcal{C}^κ , which is called *the tangential Cauchy–Riemann operator associated with δ* .

Remark 3.6. — For a form $\alpha \in \Gamma^\kappa(U, \bigwedge_U^{0,1} \otimes \text{End}(E))$ we have $(\delta + \alpha)_S = \delta_S + \alpha_S$ where α_S is the image of α under

$$\Gamma^\kappa(U, \bigwedge_U^{0,q} \otimes E) \longrightarrow \Gamma^\kappa(S, \bigwedge_{U|S}^{0,q} \otimes E) \longrightarrow \Gamma^\kappa(S, \bigwedge_S^{0,q} \otimes E).$$

In a similar way one obtains a tangential Cauchy–Riemann operator

$$\delta_{\partial\bar{U}^+} : \Gamma^{\kappa+1}(\partial\bar{U}^+, E) \longrightarrow \Gamma^\kappa(\partial\bar{U}^+, \bigwedge_{\partial\bar{U}^+}^{0,1} \otimes E),$$

associated to any Dolbeault operator with coefficients in \mathcal{C}^κ on a \mathcal{C}^∞ vector bundle E on a submanifold with boundary $\bar{U}^+ \subset U$, where $U^+ \subset U$ is open. Note that the correspondence $\delta \mapsto \delta_{\partial\bar{U}^+}$ plays an important role in [6, Section 3.5].

Remark 3.7. — The tangential Cauchy–Riemann operator has an analogue in the framework of principal bundles (see Section A.5.2 in the appendix): Let $p : P \rightarrow U$ ($p^+ : P^+ \rightarrow \bar{U}^+$) be a principal G -bundle on a complex manifold (with boundary) U ($\bar{U}^+ \subset U$), and let $S \subset U$ be a closed, oriented real hypersurface in U (respectively let $S := \partial\bar{U}^+ = \bar{U}^+ \setminus U^+$). A bundle ACS J of class \mathcal{C}^κ on P defines an ACS J_S of the same class on the pull-back distribution $\mathfrak{T}_{P_S} := p_{S*}^{-1}(\mathfrak{T}_S) \subset T_{P_S}$; J_S is G -invariant, makes the vector bundle epimorphism $\mathfrak{T}_{P_S} \rightarrow p_S^*(\mathfrak{T}_S)$ \mathbb{C} -linear, and the parametrization of the G -orbits pseudo-holomorphic.

3.2. Gluing theorems

Let $\iota : G \rightarrow \text{Aut}(G)$ be the group morphism which assigns to $g \in G$ the interior automorphism ι_g , $\mathcal{C}_\iota^{\kappa+1}(P, G)$ the space of ι -equivariant maps $P \rightarrow G$ of class $\mathcal{C}^{\kappa+1}$, and $A_{\text{Ad}}^{p,q}(P, \mathfrak{g}^{1,0})_\kappa$ the space of tensorial $\mathfrak{g}^{1,0}$ -valued forms of type Ad, bidegree (p, q) and class \mathcal{C}^κ on P . In Section A.5.2 we associated with a bundle ACS $J \in \mathcal{J}_P^\kappa$ the maps

$$\bar{\iota}_J : \mathcal{C}_\iota^{\kappa+1}(P, G) \longrightarrow A_{\text{Ad}}^{0,1}(P, \mathfrak{g}^{1,0})_\kappa = A^{0,1}(U, P \times_{\text{Ad}} \mathfrak{g}^{1,0})_\kappa \simeq A^{0,1}(U, \text{Ad}(P))_\kappa,$$

$$\bar{\mathfrak{k}}_J : A_{\text{Ad}}^{0,1}(P, \mathfrak{g}^{1,0})_\kappa \longrightarrow A_{\text{Ad}}^{0,2}(P, \mathfrak{g}^{1,0})_{\kappa-1} \text{ (for } \kappa \geq 1\text{)}.$$

We identify $\mathfrak{g}^{1,0}$ with \mathfrak{g} and $\theta^{1,0}$ with θ in the standard way (see Section A.5.2), so $\bar{\mathfrak{k}}_J$ becomes a map $A^{0,1}(U, \text{Ad}(P))_\kappa \rightarrow A^{0,2}(U, \text{Ad}(P))_{\kappa-1}$. For $s \in \Gamma^{l+1}(U, \text{Ad}(P))$ put $\bar{\mathfrak{d}}_J(s) := \bar{\mathfrak{l}}_J(\exp(s))$.

LEMMA 3.8. — *Let $0 \leq l \leq k$. Let $s \in \Gamma^{l+1}(U, \text{Ad}(P))$ with $j_S^l s = 0$, so that the intrinsic differential $D_S^{l+1} s \in \Gamma^0(S, \eta_S^{\otimes(l+1)} \otimes \text{Ad}(P))$ is well defined (see Section A.6). Let J be a bundle ACS of class \mathcal{C}^k on P . Then*

- (1) $j_S^{l-1}(\bar{\mathfrak{d}}_J(s)) = 0$ (if $l \geq 1$).
- (2) $D_S^l(\bar{\mathfrak{d}}_J(s)) = (\text{id}_{\eta_S}^{\otimes l} \otimes \psi_S \otimes \text{id}_{\text{Ad}(P)})(D_S^{l+1} s)$.

Proof.

(1). — The section s can be identified with an element, denoted by the same symbol, of $\mathcal{C}_{\text{Ad}}^{l+1}(P, \mathfrak{g})$. Using this interpretation of s , we obtain an element $\sigma = \exp(s) \in \mathcal{C}_l^{l+1}(P, G)$. Let $\tau \in \Gamma^\infty(W, P)$ be a local section of P , and put

$$s_\tau := s \circ \tau \in \mathcal{C}^{l+1}(W, \mathfrak{g}), \quad \sigma_\tau := \sigma \circ \tau = \exp(s_\tau) \in \mathcal{C}^{l+1}(W, G).$$

Using formula (A.18) explained in Section A.5.2, we have:

$$\begin{aligned} \tau^*(\bar{\mathfrak{d}}_J(s)) &= \tau^*(\bar{\mathfrak{l}}_J(\sigma)) = \bar{\mathfrak{l}}_J^\tau(\sigma) = \sigma_\tau^*(\theta)^{0,1} + (\text{Ad}_{\sigma_\tau^{-1}} - \text{id})(\alpha_J^\tau) \\ &= s_\tau^*(\exp^*(\theta))^{0,1} + (\text{Ad}_{\exp(-s_\tau)} - \text{id})(\alpha_J^\tau). \end{aligned} \quad (3.6)$$

We may suppose that U is an open subset of \mathbb{C}^n . Since $j_S^l s = 0$, it follows by the composition Lemma A.31 (2) that

$$j_S^l((\text{Ad}_{\exp(-s_\tau)} - \text{id})(\alpha_J^\tau)) = 0, \quad (3.7)$$

and by Lemma A.33 that $j_S^{l-1}(s_\tau^*(\exp^*(\theta))^{0,1}) = 0$.

(2). — It suffices to prove that for any local section $\tau \in \Gamma^\infty(W, P)$ we have

$$D_S^l(\bar{\mathfrak{d}}_J(s_\tau)) = (\text{id}_{\eta_S}^{\otimes l} \otimes \psi_S \otimes \text{id}_{\text{Ad}(P)})(D_S^{l+1} s_\tau).$$

Taking into account (3.6) and (3.7), it suffices to compute $D_S^l(s_\tau^*(\exp^*(\theta))^{0,1})$. We use formula (A.30) of Lemma A.33, taking $V = F = \mathfrak{g}$, $f = s_\tau : U \rightarrow \mathfrak{g}$, and $\omega := \exp^*(\theta)$, which is a holomorphic $(1, 0)$ form on \mathfrak{g} , because \exp is a holomorphic map. We have to specify the map ω_S^f intervening on the right in (A.30). Regarded as a map $\mathcal{C}^\infty(\mathfrak{g}, \text{Hom}(\mathfrak{g}, \mathfrak{g}))$, ω is given by

$$\omega(a)(v) = (l_{\exp(a)^{-1}})_*((\exp_{*,a}(v))),$$

so, for $x \in U$, we have

$$\omega_S^f(x)(v) = (l_{\exp(f(x))^{-1}})_*((\exp_{*,f(x)}(v))).$$

Since we assumed $j_S^l s = 0$, we have $f(x) = s_\tau(x) = 0$ for any $x \in S$, so $\omega_S^f(x) = \text{id}_{\mathfrak{g}}$ for any $x \in S$. \square

Let $\kappa \in (0, \infty] \setminus \mathbb{N}$ and $k := [\kappa]$.

LEMMA 3.9. — *Let J be a bundle ACS of class \mathcal{C}^κ on P . Let $l \leq k$ be a non-negative integer, and let $\beta \in \Gamma^\kappa(U, \bigwedge_U^{0,1} \otimes \text{Ad}(P))$ be such that*

$$j_S^{l-1}\beta = 0 \text{ (required only if } l \geq 1), \quad (\text{id}_{\eta_S}^{\otimes l} \otimes \Pi_S \otimes \text{id}_{\text{Ad}(P)_S})(D_S^l\beta) = 0.$$

(1) *There exists $s \in \Gamma^{\kappa+1}(U, \text{Ad}(P))$ such that, putting*

$$\beta' := \beta - \bar{\mathfrak{d}}_J(s) \in \Gamma^\kappa(U, \bigwedge_U^{0,1} \otimes \text{Ad}(P)),$$

we have

$$j_S^l s = 0, \quad j_S^l \beta' = 0.$$

(2) *Suppose that the considered pair (J, β) also satisfies:*

$$(i) \quad j_S^{k-1}(\mathfrak{k}_J(\beta)) = 0.$$

$$(ii) \quad j_S^{k-2-l}(\mathfrak{f}_J) = 0 \text{ (required only if } l \leq k-2).$$

Then, for any such s , putting $\tilde{\beta} := \text{Ad}_{\exp(s)}(\beta')$, we have:

$$(a) \quad j_S^l \tilde{\beta} = 0.$$

$$(b) \quad j_S^{k-1}(\mathfrak{k}_J(\tilde{\beta})) = 0.$$

$$(c) \quad \text{If } l \leq k-1, \text{ we also have } (\text{id}_{\eta_S}^{\otimes(l+1)} \otimes \Pi_S \otimes \text{id}_{E_S})(D_S^{l+1}\tilde{\beta}) = 0.$$

(3) *If $\kappa \neq \infty$, s can be chosen to depend continuously on β .*

Proof.

(1). — Using the first horizontal exact sequence in (3.1), we see that the hypothesis implies

$$D_S^l \beta \in \Gamma^{\kappa-l}(S, \eta_S^{\otimes l} \otimes \eta_S^{0,1} \otimes \text{Ad}(P)),$$

so, since ψ_S is a line bundle isomorphism, there exists $b \in \Gamma^{\kappa-l}(S, \eta_S^{\otimes(l+1)} \otimes \text{Ad}(P)_S)$ such that

$$D_S^l \beta = (\text{id}_{\eta_S}^{\otimes l} \otimes \psi_S \otimes \text{id}_E)(b). \quad (3.8)$$

The key argument in the proof: by the extension Corollary A.13, there exists $s \in \Gamma^{\kappa+1}(U, \text{Ad}(P))$ such that

$$j_S^l s = 0, \quad D_S^{l+1} s = b. \quad (3.9)$$

It follows that $\beta' := \beta - \bar{\mathfrak{d}}_J(s)$ belongs to $\Gamma^\kappa(U, \bigwedge_U^{0,1} \otimes \text{Ad}(P))$. Moreover, by Lemma 3.8(1), we have $j_S^{l-1}(\bar{\mathfrak{d}}_J(s)) = 0$, so, since $j_S^{l-1}\beta = 0$, we obtain $j_S^{l-1}\beta' = 0$ and

$$D_S^l \beta' = D_S^l \beta - D_S^l(\bar{\mathfrak{d}}_J(s)).$$

Using (3.8), (3.9) and Lemma 3.8(2), we obtain $D_S^l \beta - D_S^l(\bar{\mathfrak{d}}_J(s)) = 0$, so $j_S^l \beta' = 0$.

(2a). — It follows from $j_S^l(\beta') = 0$ using the composition Lemma A.31(2).

(2b). — By Lemma A.24(2) proved in Section A.5.2, we have:

$$\bar{\mathfrak{f}}_J(\tilde{\beta}) = \bar{\mathfrak{f}}_J(\text{Ad}_{\exp(s)}(\beta - \bar{\mathfrak{d}}_J(s)) = \text{Ad}_{\exp(s)}(\bar{\mathfrak{f}}_J(\beta)) + (\text{Ad}_{\exp(s)} - \text{id})(\mathfrak{f}_J).$$

Since we assumed $j_S^{k-1}(\bar{\mathfrak{f}}_J(\beta)) = 0$, it follows again by Lemma A.31(2) that $j_S^{k-1}(\text{Ad}_{\exp(s)}(\bar{\mathfrak{f}}_J(\beta))) = 0$.

On the other hand, since $j_S^{k-2-l}(\mathfrak{f}_J) = 0$ by hypothesis and $j_S^l s = 0$ by (1), it follows by Lemma A.31(1)-(2) that

$$j_S^{k-1}((\text{Ad}_{\exp(s)} - \text{id})(\mathfrak{f}_J)) = 0.$$

Therefore $j_S^{k-1}(\bar{\mathfrak{f}}_J(\tilde{\beta})) = 0$ as claimed.

(2c). — Suppose $l \leq k-1$. Formula (3.3) shows that

$$(\psi_S \otimes \text{id}_{\wedge_{U|S}^{0,2} E_S}) \circ (\text{id}_{\eta_S} \otimes \sqcap_S \otimes \text{id}_{E_S}) = \wedge(\psi_S \otimes \text{id}_{\wedge_{U|S}^{0,1} E_S})$$

on $\eta_S \otimes \wedge_{U|S}^{0,1} E_S$. Therefore

$$\begin{aligned} & (\text{id}_{\eta_S}^{\otimes l} \otimes \psi_S \otimes \text{id}_{\wedge_{U|S}^{0,2} E_S})((\text{id}_{\eta_S}^{\otimes(l+1)} \otimes \sqcap_S \otimes \text{id}_{E_S})(D_S^{l+1} \tilde{\beta})) \\ &= (\text{id}_{\eta_S}^{\otimes l} \otimes \wedge(\psi_S \otimes \text{id}_{\wedge_{U|S}^{0,1} E_S}))(D_S^{l+1} \tilde{\beta}) = D_S^l(\bar{\partial}_J \tilde{\beta}), \end{aligned} \quad (3.10)$$

where, for the last equality we used formula (A.35) of Lemma A.34 for $q = 1$. Since $l \leq k-1$ and we have $j_S^{k-1}(\mathfrak{f}_j(\tilde{\beta})) = 0$ by (2b), it follows that $j_S^l(\mathfrak{f}_j(\tilde{\beta})) = 0$, in particular $D_S^l(\mathfrak{f}_j(\tilde{\beta})) = 0$.

But

$$\mathfrak{f}_j(\tilde{\beta}) = \bar{\partial}_J \tilde{\beta} + \frac{1}{2}[\tilde{\beta} \wedge \tilde{\beta}]$$

with $j_S^l(\tilde{\beta}) = 0$, which implies $j_S^{2l+1}([\tilde{\beta} \wedge \tilde{\beta}]) = 0$ by Lemma A.31(1). It follows $D_S^l(\bar{\partial}_J \tilde{\beta}) = 0$. Since the linear map $\text{id}_{\eta_S}^{\otimes l} \otimes \psi_S \otimes \text{id}_{\wedge_{U|S}^{0,2} E}$ is injective, formula (3.10) shows that

$$(\text{id}_{\eta_S}^{\otimes(l+1)} \otimes \sqcap_S \otimes \text{id}_{E_S})(D_S^{l+1} \tilde{\beta}) = 0,$$

as claimed.

(3). — For $\kappa \neq \infty$, the extension Corollary A.13 provides a solution $s \in \Gamma^{\kappa+1}(U, \text{Ad}(P))$ of the equations (3.9) which depends continuously on b , so on β . \square

PROPOSITION 3.10. — *Let J be a bundle ACS of class C^κ on P such that $j_S^{k-2}(\mathfrak{f}_J) = 0$. Let $\beta \in \mathcal{I}_S^\kappa(U, \wedge_U^{0,1} \otimes \text{Ad}(P))$ be such that $j_S^{k-1}(\bar{\mathfrak{f}}_J(\beta)) = 0$. Then*

- (1) *There exists $s \in \mathcal{I}_S^{\kappa+1}(U, \text{Ad}(P))$ such that $j_S^k(\beta - \bar{\mathfrak{d}}_J(s)) = 0$.*
- (2) *If $\kappa \neq \infty$, s can be chosen to depend continuously on (J, β) .*

Proof.

(1). — Suppose that $\kappa \neq \infty$. Our assumption $\beta \in \mathcal{I}_S^\kappa(U, \bigwedge_U^{0,1} \otimes E)$ means that β satisfies the hypothesis of Lemma 3.9 for $l = 0$. Applying successively this Lemma to

$$\beta_0 := \beta, \quad \beta_1 := \widetilde{\beta}_0, \quad \dots, \quad \beta_k := \widetilde{\beta}_{k-1},$$

we obtain sequences $(\beta'_i)_{0 \leq i \leq k}$ in $\Gamma^\kappa(U, \bigwedge_U^{0,1} \otimes \text{Ad}(P))$, and $(s_i)_{0 \leq i \leq k}$ in $\Gamma^{\kappa+1}(U, \text{Ad}(P))$ such that

$$j_S^l(s_l) = 0, \quad j_S^l(\beta'_l) = 0 \quad \text{for } 0 \leq l \leq k, \quad (3.11)$$

and, putting $\sigma_i := \exp(s_i)$, one has:

$$\beta_l = \text{Ad}_{\sigma_{l-1}}(\beta'_{l-1}) \quad \text{for } 1 \leq l \leq k, \quad (3.12)$$

$$\beta_l = \tilde{\mathbf{l}}_J(\sigma_l) + \beta'_l \quad \text{for } 0 \leq l \leq k. \quad (3.13)$$

Combining (3.12) and (3.13) we obtain:

$$\begin{aligned} \beta &= \tilde{\mathbf{l}}_J(\sigma_0) + \beta'_0 \\ \beta'_0 &= \text{Ad}_{\sigma_0^{-1}}(\tilde{\mathbf{l}}_J(\sigma_1) + \beta'_1) \\ \beta'_1 &= \text{Ad}_{\sigma_1^{-1}}(\tilde{\mathbf{l}}_J(\sigma_2) + \beta'_2) \\ &\vdots \\ \beta'_{k-1} &= \text{Ad}_{\sigma_{k-1}^{-1}}(\tilde{\mathbf{l}}_J(\sigma_k) + \beta'_k). \end{aligned} \quad (3.14)$$

This implies:

$$\begin{aligned} \beta &= \tilde{\mathbf{l}}_J(\sigma_0) + \beta'_0 \\ \beta'_0 &= \text{Ad}_{\sigma_0^{-1}}(\tilde{\mathbf{l}}_J(\sigma_1) + \beta'_1) \\ \text{Ad}_{\sigma_0^{-1}}(\beta'_1) &= \text{Ad}_{\sigma_0^{-1}}\text{Ad}_{\sigma_1^{-1}}(\tilde{\mathbf{l}}_J(\sigma_2) + \beta'_2) \\ &\vdots \\ \text{Ad}_{\sigma_0^{-1}}\text{Ad}_{\sigma_1^{-1}}\dots\text{Ad}_{\sigma_{k-2}^{-1}}(\beta'_{k-1}) &= \text{Ad}_{\sigma_0^{-1}}\text{Ad}_{\sigma_1^{-1}}\dots\text{Ad}_{\sigma_{k-1}^{-1}}(\tilde{\mathbf{l}}_J(\sigma_k) + \beta'_k). \end{aligned} \quad (3.15)$$

For $0 \leq l \leq k$ put $\mathfrak{s}_l := \sigma_l \dots \sigma_0$. We obtain

$$\begin{aligned} \beta &= \tilde{\mathbf{l}}_J(\sigma_0) + \beta'_0 \\ \beta'_0 &= \text{Ad}_{\mathfrak{s}_0^{-1}}(\tilde{\mathbf{l}}_J(\sigma_1) + \beta'_1) \\ \text{Ad}_{\mathfrak{s}_0^{-1}}(\beta'_1) &= \text{Ad}_{\mathfrak{s}_1^{-1}}(\tilde{\mathbf{l}}_J(\sigma_2) + \beta'_2) \\ &\vdots \\ \text{Ad}_{\mathfrak{s}_{k-2}^{-1}}(\beta'_{k-1}) &= \text{Ad}_{\mathfrak{s}_{k-1}^{-1}}(\tilde{\mathbf{l}}_J(\sigma_k) + \beta'_k). \end{aligned} \quad (3.16)$$

Therefore

$$\beta = \bar{\iota}_J(\sigma_0) + \sum_{l=1}^k \text{Ad}_{\mathfrak{s}_{l-1}^{-1}}(\bar{\iota}_J(\sigma_l)) + \text{Ad}_{\mathfrak{s}_{k-1}^{-1}}(\beta'_k). \quad (3.17)$$

But $\mathfrak{s}_l = \sigma_l \mathfrak{s}_{l-1}$ so, by Lemma A.24 (1), we have

$$\bar{\iota}_J(\mathfrak{s}_l) = \bar{\iota}_J(\mathfrak{s}_{l-1}) + \text{Ad}_{\mathfrak{s}_{l-1}^{-1}}(\bar{\iota}_J(\sigma_l)) \quad \text{for } 1 \leq l \leq k,$$

so

$$\bar{\iota}_J(\mathfrak{s}_k) = \bar{\iota}_J(\sigma_0) + \sum_{l=1}^k \text{Ad}_{\mathfrak{s}_{l-1}^{-1}}(\bar{\iota}_J(\sigma_l)).$$

Formula (3.17) becomes

$$\beta = \bar{\iota}_J(\mathfrak{s}_k) + \text{Ad}_{\mathfrak{s}_{k-1}^{-1}}(\beta'_k). \quad (3.18)$$

Making use of Proposition A.15, let $\text{Ad}(P)_0, \iota(P)_0$ be neighborhoods of the zero section (identity section) in the two bundles such that \exp induces a diffeomorphism $\text{Ad}(P)_0 \rightarrow \iota(P)_0$.

We can assume that \mathfrak{s}_l takes values in $\iota(P)_0$ for $0 \leq l \leq k$, so we can write $\mathfrak{s}_l = \exp(s^l)$ for a section $s^l \in \Gamma^{\kappa+1}(U, \text{Ad}(P)_0) \subset \Gamma^{\kappa+1}(U, \text{Ad}(P))$. It suffices to put $s := s^k$ and to take into account that $j_S^k(\text{Ad}_{\mathfrak{s}_{k-1}^{-1}}(\beta'_k)) = 0$ because $j_S^k(\beta'_k) = 0$.

Suppose $\kappa = \infty$. In this case Lemma 3.9 yields infinite sequences $(\beta'_i)_{i \geq 0}$, $(s_i)_{i \geq 0}$ in $\Gamma^\infty(U, \bigwedge_{U}^{0,1} \otimes \text{Ad}(P))$, $\Gamma^\infty(U, \text{Ad}(P))$ satisfying (3.11) and (3.13) for $l \geq 0$ and (3.12) for $l \geq 1$. We define in the same way $\sigma_l, \mathfrak{s}_l \in \Gamma_l^\infty(U, \iota(P)_0)$, $s^l \in \Gamma^\infty(U, \text{Ad}(P)_0)$.

For $l \geq 1$ put $s'_l := s^l - s^{l-1}$. Denoting by $\log : \iota(P)_0 \xrightarrow{\sim} \text{Ad}(P)_0$ the inverse of the fiber bundle isomorphism $\exp|_{\text{Ad}(P)_0} : \iota(P)_0 \xrightarrow{\sim} \text{Ad}(P)_0$, we have

$$s'_l = \log(\mathfrak{s}_l) - s^{l-1} = \log(\sigma_l \mathfrak{s}_{l-1}) - s^{l-1} = \log(\exp(s_l) \exp(s^{l-1})) - s^{l-1}.$$

Since the map $y \mapsto \log(\exp(y) \exp(s^{l-1})) - s^{l-1}$ vanishes at $y = 0$ and $j_S^l(s_l) = 0$, it follows by Lemma A.31 that $j_S^l(s'_l) = 0$. Recalling that $j_S^0(s) = 0$ and using Corollary A.13 (3), it follows that there exists $s \in \Gamma^\infty(U, \text{Ad}(P))$ such that for any $k \in \mathbb{N}$, $j_S^{k+1}((s_0 + \sum_{l=1}^k s'_l) - s) = 0$, i.e.

$$j_S^{k+1}(s^k - s) = 0.$$

Using formula (3.6) for s^k and s together with the second formula in (A.33), we see that this implies

$$j_S^k(\bar{\iota}_J(\exp(s^k)) - \bar{\iota}_J(\exp(s))) = 0 \quad (3.19)$$

i.e. $j_S^k(\bar{\mathbf{I}}_J(\mathfrak{s}_k) - \bar{\mathbf{I}}_J(\exp(s))) = 0$. By (3.18) we infer

$$j_S^k(\beta - \bar{\mathbf{I}}_J(\exp(s)) - \text{Ad}_{\mathfrak{s}_{k-1}^{-1}}(\beta'_k)) = 0,$$

so $j_S^k(\beta - \bar{\mathbf{I}}_J(\exp(s))) = 0$ because $j_S^k(\beta'_k) = 0$.

(2). — For $l \geq 1$ the form

$$\beta_l = \tilde{\beta}_{l-1} = \text{Ad}_{\exp(s_{l-1})}(\beta'_{l-1}) = \text{Ad}_{\exp(s_{l-1})}(\beta_{l-1} - \bar{\mathbf{d}}_J(s_{l-1}))$$

depends continuously on β_{l-1} , s_{l-1} and J . On the other hand, by Lemma 3.9(3), s_{l-1} can be chosen to depend continuously on β_{l-1} . By induction it follows that all s_l (hence also \mathfrak{s}_l , s^l) can be chosen to depend continuously on the initial data (J, β) . \square

Suppose that S separates U , and let $U = \bar{U}^- \cup \bar{U}^+$ be the corresponding decomposition of U as union of manifolds with boundary.

THEOREM 3.11. — *Let G be a complex Lie group and $p : P \rightarrow U$ a principal G -bundle on U . Let $\kappa \in (0, +\infty] \setminus \mathbb{N}$ and J^\pm formally integrable bundle ACS of class \mathcal{C}^κ on $P_{\bar{U}^\pm}$ such that $J_S^+ = J_S^-$.*

(1) *There exists*

(a) $\sigma_+ \in \Gamma^{\kappa+1}(\bar{U}^+, \iota(P))$ with $\sigma_+|_S = e$, the unit element of G .

(b) an integrable bundle ACS J of class \mathcal{C}^κ on P ,

such that $J|_{P^+} = J^+ \cdot \sigma_+$ and $J|_{P^-} = J^-$.

(2) *For any open neighborhood V of S in U , there exists a pair (σ_+, J) as above such that, moreover, σ_+ is constantly e on $U^+ \setminus V$.*

(3) *If $\kappa \neq \infty$ the pair (σ_+, J) can be chosen to depend continuously on (J^-, J^+) .*

Proof.

(1). — Let J_\pm a (not necessarily integrable) extension of J^\pm to a bundle ACS of class \mathcal{C}^κ on P . The existence of such an extension is obtained using the affine structure with model space $\Gamma^\kappa(\bar{U}^\pm, \bigwedge_{\bar{U}^\pm}^{0,1} \otimes \text{Ad}(P))$ of the space $\mathcal{J}_{P_{\bar{U}^\pm}}^\kappa$ and the extension principle given by Corollary A.14(1). Using the affine space structure of the space \mathcal{J}_P^κ put $\beta := J_- - J_+$ and note that the assumption $J_S^+ = J_S^-$ implies $\beta \in \mathcal{I}_S^\kappa(U, \bigwedge_U^{0,1} \otimes \text{Ad}(P))$.

Case (i): $\kappa > 1$. — By formula (A.15) of Section A.5.2 we have

$$\mathfrak{f}_{J_+}(\beta) = \mathfrak{f}_{J_-} - \mathfrak{f}_{J_+}.$$

We have $\mathfrak{f}_{J_\pm}|_{\bar{U}^\pm} = \mathfrak{f}_{J_\pm} = 0$, because J^\pm are assumed formally integrable. Since \mathfrak{f}_{J_\pm} are of class $\mathcal{C}^{\kappa-1}$, this implies that $j_x^{k-1}(\mathfrak{f}_{J_\pm}) = 0$ for any $x \in \bar{U}^\pm$, in particular for any $x \in S$. This proves that $j_S^{k-1}(\bar{\mathfrak{f}}_{J_+}(\beta)) = 0$ and $j_S^{k-1}(\mathfrak{f}_{J_+}) = 0$, in particular $j_S^{k-2}(\mathfrak{f}_{J_+}) = 0$. Therefore Proposition 3.10 applies to the pair

(J_+, β) and gives a section $s \in \Gamma^{\kappa+1}(U, \text{Ad}(P))$ with $s|_S = 0$ such that, putting $\sigma = \exp(s)$, we have $j_S^k(\beta - \mathfrak{l}_{J_+}(\sigma)) = 0$. On the other hand we have:

$$j_S^k(J_- - J_+ \cdot \sigma) = j_S^k(J_- - J_+ + J_+ - J_+ \cdot \sigma) = j_S^k(\beta - \mathfrak{l}_{J_+}(\sigma)),$$

where, for the last equality we have used formula (A.17) of Section A.5.2. Therefore $j_S^k(J_- - J_+ \cdot \sigma) = 0$. By Corollary A.14(2), there exists $J \in \mathcal{J}_P^\kappa$ which coincides with J_- ((hence with J^-) on \bar{U}^- and with $J_+ \cdot \sigma$ (hence with $J^+ \cdot \sigma|_{\bar{U}^+}$) on \bar{U}^+). J is integrable, because \mathfrak{f}_J coincides with \mathfrak{f}_{J^-} on \bar{U}^- and, by formula (A.16), with $\text{Ad}_{\sigma^{-1}}(\mathfrak{f}_{J^+})$ on \bar{U}^+ . It suffices to put $\sigma_+ := \sigma|_{\bar{U}^+}$.

Case (ii): $\kappa \in (0, 1)$. — In this case the assumption “ J^\pm is a formally integrable bundle ACS on $P_{\bar{U}^\pm}$ ” means that \mathfrak{f}_{J^\pm} vanishes as distribution supported by \bar{U}^\pm , see Section A.5.2. We apply Lemma 3.9(1) for $l = 0$ recalling that the conditions imposed on β in the hypothesis of this lemma reduce to $\beta \in \mathcal{I}_S^\kappa(U, \bigwedge_U^{0,1} \otimes \text{Ad}(P))$. We obtain as above $s \in \Gamma^{\kappa+1}(U, \text{Ad}(P))$ with $s|_S = 0$ such that $j_S^0(\beta - \mathfrak{l}_{J_+}(\sigma)) = 0$ with $\sigma = \exp(s)$. We conclude as in the Case (i), but making use of

- Remark A.27 to show that $J^+ \cdot \sigma|_{\bar{U}^+}$ is formally integrable,
- Proposition A.29 to infer that J is integrable.

(2). — Let V' be an open neighborhoods of S in U such that $\bar{V}' \subset V$. By the smooth version of Urysohn’s lemma [27, Lemma 1.3.2], it follows that there exists a \mathcal{C}^∞ function $\lambda : U \rightarrow [0, 1]$ such that $\lambda|_{\bar{V}'} \equiv 1$ and $\lambda|_{U \setminus V} \equiv 0$. It suffices to replace in the proof of (1) s by λs .

(3). — Using Proposition 3.10(2), Lemma 3.9(3) and the continuity properties of the extension operators in Corollary A.14 it follows that the objects J_\pm , s , J introduced in the proof of (1) can be chosen to depend continuously on the input data (J^-, J^+) . \square

We can now prove Theorem 2.8 stated in the introduction:

Proof of Theorem 2.8.

(1). — Making use of Proposition A.18, let $\mathfrak{S} \in \mathcal{S}_a$ be an admissible \mathcal{C}^∞ -structure on P^v and $P_\mathfrak{S}^v$ the corresponding \mathcal{C}^∞ bundle. The obvious isomorphisms $o^\pm : P^\pm \rightarrow P_{\bar{U}^\pm}^v$ become isomorphisms $P^\pm \rightarrow P_{\mathfrak{S}\bar{U}^\pm}^v$ of class $\mathcal{C}^{\kappa+1}$ between \mathcal{C}^∞ bundles on \bar{U}^\pm , so the given formally integrable bundle ACS J^\pm of class \mathcal{C}^κ on P^\pm induce via o^\pm formally integrable bundle ACS J'^\pm of class \mathcal{C}^κ on $P_{\mathfrak{S}\bar{U}^\pm}^v$. The hypothesis “ J_S^\pm agree via v ” in Theorem 2.8 is equivalent to the condition $J_S'^- = J_S'^+$.

Theorem 3.11 applies and gives $\sigma_+ \in \Gamma^{\kappa+1}(\bar{U}^+, P_{\mathfrak{S}\bar{U}^+}^v)$ with $\sigma_+|_S = e$, and an integrable bundle ACS J of class \mathcal{C}^κ on $P_{\mathfrak{S}}^v$ which coincides with J'^- on $P_{\mathfrak{S}\bar{U}^-}^v$ and with $J'^+ \cdot \sigma_+ = (\tilde{\sigma}_+)^{-1}(J'^+)$ (see Section A.5.2) on $P_{\mathfrak{S}\bar{U}^+}^v$.

By Theorem A.22, J defines a holomorphic structure \mathfrak{h}_J on the underlying $\mathcal{C}^{\kappa+1}$ bundle of $P_{\mathfrak{S}}^v$; a local section is holomorphic with respect to \mathfrak{h}_J if and only if it is J -pseudo-holomorphic (see also [33, Corollary 1.4]).

The pair $(\text{id}_{P^-}, \tilde{\sigma}_+)$ defines an element $f \in \text{Aut}^0(P^v)_a$, so $\mathfrak{S}' := f(\mathfrak{S})$ also belongs to \mathcal{S}_a by Lemma A.18. Therefore f becomes a \mathcal{C}^∞ bundle isomorphism $P_{\mathfrak{S}}^v \rightarrow P_{\mathfrak{S}'}^v$. The direct image $\mathfrak{h}^v := f(\mathfrak{h}_J)$ will be a holomorphic structure on the underlying $\mathcal{C}^{\kappa+1}$ bundle of $P_{\mathfrak{S}'}^v$, which coincides with \mathfrak{h}_\pm on U^\pm via o^\pm , because $f(J)$ coincides with J'^\pm on U^\pm .

We now prove the unicity property claimed in (1): Let \mathfrak{h}' , \mathfrak{h}'' be holomorphic structures (see [33, Definition 1.3]) on the topological bundle P^v which extend \mathfrak{h}^\pm . Let $\tau' : V' \rightarrow P_{V'}^v$, $\tau'' : V'' \rightarrow P_{V''}^v$ be local sections which are holomorphic with respect to \mathfrak{h}' , respectively \mathfrak{h}'' . The restrictions

$$\tau' : V' \cap U^\pm \longrightarrow P_{V' \cap U^\pm}^\pm, \quad \tau'' : V'' \cap U^\pm \longrightarrow P_{V'' \cap U^\pm}^\pm$$

are holomorphic sections of the holomorphic bundle $(P_{U^\pm}^\pm, \mathfrak{h}^\pm)$. The corresponding comparison map

$$g_{\tau'\tau''} : V' \cap V'' \longrightarrow G$$

is continuous on the whole $V' \cap V''$ and holomorphic on both $V' \cap V'' \cap U^\pm$, i.e. on $(V' \cap V'') \setminus S$. By the extension Theorem A.19 it follows that $g_{\tau'\tau''}$ is holomorphic on the whole $V' \cap V''$, so τ' , τ'' are holomorphically compatible, so they belong to the same holomorphic structure on P^v .

(2). — Any \mathfrak{h}^v -holomorphic local section $\tau : V \rightarrow P^v$ is a section of class $\mathcal{C}^{\kappa+1}$ of $P_{\mathfrak{S}'}^v$, so, since $\mathfrak{S}' \in \mathcal{S}_a$, its restrictions to $V \cap \bar{U}^\pm$ will be of class $\mathcal{C}^{\kappa+1}$. \square

Theorem 2.6 follows easily from Theorem 2.8 taking into account that:

- (i) Any oriented smooth hypersurface $S \subset U$ separates a sufficiently small open neighborhood $U_0 \supset S$ of S in U .
- (ii) The problem has a local character with respect to S .

Theorems 2.4, 2.1 follow from Theorems 2.6, respectively 2.8 taking $G = \text{GL}(r, \mathbb{C})$. Using Theorem 2.1 we can prove now Corollary 2.3.

Proof of Corollary 2.3. — By the extension Theorem A.19, the locally free sheaf \mathcal{E}^v of locally defined holomorphic sections of (E^v, \mathfrak{h}^v) coincides with the sheaf

$$U \supset^{\text{open}} W \longmapsto \{f \in \Gamma^0(W, E^v) \mid \sigma|_{W \setminus S} \text{ is } \mathfrak{h}^v - \text{holomorphic}\},$$

which (taking into account that \mathfrak{h}^v extends \mathfrak{h}^\pm and the definition of E^v) coincides with the sheaf defined by formula (2.1). By Theorem 2.1(2), the restrictions of any local \mathfrak{h} -holomorphic section $f \in \Gamma^0(W, E^v)$ to $W \cap \overline{U}^\pm$ are of class $\mathcal{C}^{\kappa+1}$ up to the boundary, so formulae (2.1), (2.2) define the same sheaf, as claimed. \square

Corollary 2.5 follows from Corollary 2.3 taking into account again that any oriented smooth hypersurface $S \subset U$ is locally (with respect to S) separating.

4. Isomorphism theorems. Interpretation in terms of framed bundles

In this section we come back to the objects considered in Section 2.3.2: let X be a *closed* complex manifold, $S \subset X$ a closed, smooth, oriented real hypersurface, P a principal G bundle on X and \hat{P} its pull back to \hat{X}_S .

4.1. The proofs of the isomorphism theorems

We begin with the following remark which will be used in the proof of Theorem 2.17:

Remark 4.1. — Any gauge transformation $\mathfrak{f} \in \mathcal{G}_{\partial\hat{X}}(\hat{P})$ descends to a continuous gauge transformation $\check{\mathfrak{f}}$ on P which is of class $\mathcal{C}^{\kappa+1}$ on $X \setminus S$ and is identity on S .

Proof of Theorem 2.17. — The second claim of the theorem is a special case of the first, so we will prove only the first.

Injectivity. — Let $J_1, J_2 \in \mathcal{I}_P^\kappa$ and $\mathfrak{f} \in \mathcal{G}_{\partial\hat{X}}^{\hat{P}} = \Gamma^{\kappa+1}(\hat{X}, \iota(\hat{P}))$ be such that $\hat{J}_2 = \hat{J}_1 \cdot \mathfrak{f}$, where \hat{J}_i is the pull back of J_i to \hat{P} . It follows that $J_2 = J_1 \cdot \check{\mathfrak{f}}$ on $X \setminus S$.

Let $G \times G$ act on G from the left by

$$\mu((a, b), g) = agb^{-1}$$

and note that $\iota(P) := P \times_\iota G$ can be identified with the associated bundle

$$\mu(P \times_X P) := (P \times_X P) \times_\mu G.$$

The pair (J_1, J_2) defines an integrable bundle ACS of class \mathcal{C}^κ on $P \times_X P$, so a holomorphic structure $\mathfrak{h}_{(J_1, J_2)}$ on the principal $G \times G$ -bundle $P \times_X P$. The known

PROPERTY 1. — $J_2 = J_1 \cdot \check{f}$ on $X \setminus S$.

is equivalent to:

PROPERTY 2. — \check{f} , regarded as a section in the bundle $\mu(P \times_X P)$, is holomorphic with respect to the holomorphic structure (induced by) $\mathfrak{h}_{(J_1, J_2)}$, on $X \setminus S$.

By Remark 4.1, \check{f} is continuous on X , and by Corollary A.20 it follows that \check{f} is in fact holomorphic with respect to $\mathfrak{h}_{(J_1, J_2)}$ on whole X . Using Corollary A.23 we infer that \check{f} is of class $\mathcal{C}^{\kappa+1}$ on X , and the relation $J_2 = J_1 \cdot \check{f}$ extends to X .

Surjectivity. — Let $\mathfrak{J} \in \mathcal{I}_{P\downarrow}^\kappa$ be a descendable formally integrable bundle ACS on \widehat{P} . We have to prove the existence of a pair $(J, \mathfrak{s}) \in \mathcal{I}_P^\kappa \times \mathcal{G}_{\partial\widehat{X}}^{\widehat{P}}$ such that $\mathfrak{J} \cdot \mathfrak{s} = \widehat{J}$.

Let

$$S \times \mathbb{R} \xrightarrow{\nu \simeq} U \hookrightarrow X$$

be a tubular neighborhood of S which is compatible with the orientation of its normal bundle associated with the complex orientation of U and the fixed orientation of S . Put

$$U^\pm = \nu(S \times \mathbb{R}_\pm^*), \quad \overline{U}^\pm = \nu(S \times \mathbb{R}_\pm).$$

The disjoint union $\overline{U}^- \amalg \overline{U}^+ = \widehat{U}_S$ is a neighborhood of $\widehat{S} = \partial\widehat{X}_S$ in \widehat{X}_S , so the restriction of \mathfrak{J} to this neighborhood gives formally integrable bundle ACS J^\pm of class \mathcal{C}^κ on $P_{\overline{U}^\pm}$. The assumption “ \mathfrak{J} is descendable” is equivalent to the condition $J_S^- = J_S^+$.

By Theorem 3.11 there exists $\sigma_+ \in \Gamma^{\kappa+1}(\overline{U}^+, \iota(P_{\overline{U}^+}))$ which is constantly e on $S \cup \nu([1, +\infty))$ and an integrable bundle ACS J_0 of class \mathcal{C}^κ on P_U which coincides with J^- on $E_{\overline{U}^-}$ and with $J^+ \cdot \sigma_+$ on $P_{\overline{U}^+}$.

We define $\sigma_0 \in \Gamma^{\kappa+1}(\widehat{U}_S, \iota(\widehat{P}))$ using the constant section e on \overline{U}^- and σ_+ on \overline{U}^+ . Since σ_0 is constantly e above $\widehat{U}_S \setminus \widehat{\nu(-1, 1)}_S$, it extends to \widehat{X} giving a global section $\mathfrak{s} \in \Gamma^{\kappa+1}(\widehat{X}, \iota(\widehat{P}))$ which is constantly e on S and satisfying

$$\mathfrak{J} \cdot \mathfrak{s} = \widehat{J}$$

for an integrable bundle ACS J of class \mathcal{C}^κ on P which coincides with J_0 on P_U .

The pull-back map $J \mapsto \widehat{J}$ is obviously continuous. Using Theorem 3.11 we also infer that, for $\kappa \neq \infty$, σ_+ (so also σ_0 and \mathfrak{s}) can be chosen to depend

continuously on (J^-, J^+) , so, with this choice, J will depend continuously on \mathfrak{J} . This proves the continuity of the inverse of the pull-back map. \square

Remark 2.18 concerning the case $\kappa = \infty$ follows from the following simple

LEMMA 4.2. — *Let X, Y be sets and, for any $i \in I$, let X_i, Y_i be topological spaces, and $f_i : X_i \rightarrow Y_i$, $a_i : X \rightarrow X_i$, $b_i : Y \rightarrow Y_i$ be maps such that the diagrams*

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ a_i \uparrow & & \uparrow b_i \\ X & \xrightarrow{f} & Y \end{array}$$

are commutative. Endow X (resp. Y) with the coarsest topology which makes all maps a_i (respectively b_i) continuous. Then

- (1) *If all f_i are continuous, then f is continuous.*
- (2) *If all f_i are homeomorphisms and f is bijective, then f is a homeomorphism.*

Proof. — (1) follows from the universal property of the initial topology on Y defined by the family of maps $(b_i)_{i \in I}$. For (2) put $g := f^{-1}$, $g_i := f_i^{-1}$ and note that for any $i \in I$,

$$f_i \circ a_i \circ g = b_i \circ f \circ g = b_i = f_i \circ g_i \circ b_i,$$

so, since f_i is injective, we have $a_i \circ g = g_i \circ b_i$. Therefore, since all g_i are continuous, it follows by (1) that g is continuous. \square

Theorem 2.16 is a special case of Theorem 2.17.

4.2. Interpretation in terms of framed bundles

The moduli spaces $\mathcal{M}_S(P)$, $\mathcal{M}_{\partial\bar{X}}(P)$ intervening in Theorem 2.17 have “abstract” interpretations in terms of isomorphism classes of framed (formally) holomorphic bundles:

DEFINITION 4.3. — *Let X be a closed complex manifold, $S \subset X$ a closed real hypersurface and Φ a fixed C^∞ G -bundle on S (a framing bundle).*

An S -framed G -bundle of type $(\Phi, \kappa + 1)$ on X is a pair (\mathcal{P}, θ) , where \mathcal{P} is a holomorphic G -bundle on X and $\theta : \Phi \rightarrow \mathcal{P}_S$ is a bundle isomorphism of class $C^{\kappa+1}$ on S .

An isomorphism $(\mathcal{P}, \theta) \rightarrow (\mathcal{P}', \theta')$ of S -framed holomorphic bundles of type $(\Phi, \kappa + 1)$ is a holomorphic isomorphism $f : \mathcal{P} \rightarrow \mathcal{P}'$ such that $f_S \circ \theta = \theta'$.

Let Φ be \mathcal{C}^∞ G -bundle on the boundary $\partial\bar{X}$ of a compact complex manifold with boundary \bar{X} .

A boundary framed formally holomorphic bundle of type $(\Phi, \kappa+1)$ on \bar{X} is a triple (P, J, θ) , where P is a \mathcal{C}^∞ G -bundle on \bar{X} , J is a formally integrable bundle ACS of class \mathcal{C}^κ on P , and $\theta : \Phi \rightarrow P_{\partial\bar{X}}$ is a bundle isomorphism of class $\mathcal{C}^{\kappa+1}$ on $\partial\bar{X}$.

An isomorphism $(P, J, \theta) \rightarrow (P', J', \theta')$ of boundary framed formally holomorphic bundles of type $(\Phi, \kappa+1)$ is a pseudo-holomorphic isomorphism $f : (P, J) \rightarrow (P', J')$ of class $\mathcal{C}^{\kappa+1}$ on \bar{X} such that $f_{\partial\bar{X}} \circ \theta = \theta'$.

In the special case when $G = \mathrm{GL}(r, \mathbb{C})$ and $\Phi = S \times \mathbb{C}^r$, one recovers the notions of an S -framed, respectively boundary framed bundle as used in [6, Theorem 1'] and explained in the introduction of this article.

Comparing the two definitions note that, whereas a holomorphic G -bundle on a closed complex manifold has a canonical \mathcal{C}^∞ -structure and any holomorphic isomorphism of holomorphic bundles is \mathcal{C}^∞ , this is no longer true for formally holomorphic bundles and formally holomorphic isomorphisms on manifolds with boundary.

Let P be a \mathcal{C}^∞ G -bundle on X , Φ a \mathcal{C}^∞ G -bundle on S which is isomorphic to P_S , and $\theta_0 : \Phi \rightarrow P_S$ a fixed bundle isomorphism of class \mathcal{C}^∞ .

By Theorem A.22 (see also [33]), a bundle ACS J of class \mathcal{C}^κ on P defines a holomorphic reduction \mathfrak{h}_J of the underlying $\mathcal{C}^{\kappa+1}$ -bundle of P . We obtain a holomorphic bundle $\mathcal{P}_J = (P, \mathfrak{h}_J)$ and the identity isomorphism $\mathrm{id}_P : P \rightarrow \mathcal{P}_J$ is an isomorphism of class $\mathcal{C}^{\kappa+1}$ between \mathcal{C}^∞ -bundles, so $\theta_0 : \Phi \rightarrow P_S$ becomes a bundle isomorphism of class $\mathcal{C}^{\kappa+1}$ if P_S is endowed with the \mathcal{C}^∞ structure induced by the holomorphic structure of \mathcal{P}_J . The pair $(\mathcal{P}_J, \theta_0)$ is an S -framed holomorphic bundle of type $(\Phi, \kappa+1)$ on X .

Similarly, let P be a \mathcal{C}^∞ G -bundle on \bar{X} , Φ be a \mathcal{C}^∞ G -bundle on $\partial\bar{X}$ which is isomorphic to $P_{\partial\bar{X}}$ and $\theta_0 : \Phi \rightarrow P_{\partial\bar{X}}$ a fixed bundle isomorphism of class \mathcal{C}^∞ .

PROPOSITION 4.4. — *With the notations and definitions above*

(1) *Let P be a \mathcal{C}^∞ G -bundle on X . The assignment*

$$J \cdot \mathcal{G}_S^P \longmapsto \text{the isomorphism class of } (\mathcal{P}_J, \theta_0)$$

gives a bijection between the moduli space $\mathcal{M}_S(P)$ and the set $\mathcal{M}_S(P, \theta_0)$ of isomorphism classes of S -framed holomorphic bundles of type $(\Phi, \kappa+1)$ on X which are topologically isomorphic to (P, θ_0) .

(2) Let P be a \mathcal{C}^∞ G -bundle on \bar{X} . The assignment

$$J \cdot \mathcal{G}_{\partial\bar{X}}^P \longmapsto \text{the isomorphism class of } (P, J, \theta_0)$$

gives a bijection between the moduli space $\mathcal{M}_{\partial\bar{X}}(P)$ and the set $\mathcal{M}_{\partial\bar{X}}(P, \theta_0)$ of isomorphism classes of boundary framed holomorphic bundles of type $(\Phi, \kappa + 1)$ on \bar{X} which are topologically isomorphic to (P, θ_0) .

Proof.

(1). — For the injectivity, let $J, J' \in \mathcal{J}_P^\kappa$. An isomorphism $f : (\mathcal{P}_J, \theta_0) \rightarrow (\mathcal{P}_{J'}, \theta_0)$ in the sense of Definition 4.3 is an holomorphic isomorphism $f : \mathcal{P}_J \rightarrow \mathcal{P}_{J'}$ such that $f_S \circ \theta_0 = \theta_0$, i.e. such that $f_S = \text{id}_{P_S}$. On the other hand, using Corollary A.23 as in the proof of Theorem 2.17, we see that, since J, J' are of class \mathcal{C}^κ , f is of class $\mathcal{C}^{\kappa+1}$. Therefore $f \in \mathcal{G}_S^P$. On the other hand, the holomorphy of $f : \mathcal{P}_J \rightarrow \mathcal{P}_{J'}$ means $J = J' \cdot f$, so $J \cdot \mathcal{G}_S^P = J' \cdot \mathcal{G}_S^P$.

For the surjectivity, let (\mathcal{P}, θ) be an S -framed holomorphic G -bundle of type $(\Phi, \kappa + 1)$ on X which is topologically isomorphic to (P, θ_0) . Therefore there exists a topological bundle isomorphism $g : P \rightarrow \mathcal{P}$ such that $g_S \circ \theta_0 = \theta$.

Recall that the differentiable and topological classifications of principal bundles on differentiable manifolds coincide, so P, \mathcal{P} are also isomorphic as differentiable bundles. Let $g_0 : P \rightarrow \mathcal{P}$ be a \mathcal{C}^∞ isomorphism which is sufficiently close to g in the \mathcal{C}^0 -topology such that $(g_0^{-1} \circ g)_S$ takes values in the disk bundle $\iota(P_S)_0$ obtained by applying Proposition A.15 to the bundle P_S . Since $(g_0^{-1} \circ g)_S = g_{0S}^{-1} \circ \theta \circ \theta_0^{-1}$ is of class $\mathcal{C}^{\kappa+1}$, it follows by this proposition that $(g_0^{-1} \circ g)$ can be written as $\exp(\lambda)$ for a section $\lambda \in \Gamma^{\kappa+1}(S, \text{Ad}(P))$.

By Corollary A.13 (for $m = 0$) there exists an extension $\mu \in \Gamma^{\kappa+1}(X, \text{Ad}(P))$ of λ . The bundle isomorphism $f = g_0 \exp(\mu) : P \rightarrow \mathcal{P}$ is of class $\mathcal{C}^{\kappa+1}$ and extends $g_S = \theta \circ \theta_0^{-1}$. The pull back $J := f^{-1}(J_{\mathcal{P}})$ of the canonical bundle ACS $J_{\mathcal{P}}$ of \mathcal{P} is an integrable bundle ACS of class \mathcal{C}^κ on P and f gives an isomorphism $(P_J, \theta_0) \rightarrow (\mathcal{P}, \theta)$ of S -framed holomorphic G -bundles of type $(\Phi, \kappa + 1)$ on X .

(2). — The injectivity is clear. For the surjectivity let (Q, I, θ) be a boundary framed formally holomorphic bundle of type $(\Phi, \kappa + 1)$ on \bar{X} which is topologically isomorphic to (P, θ_0) . Therefore there exists a topological bundle isomorphism $g : P \rightarrow Q$ such that $g_{\partial\bar{X}} \circ \theta_0 = \theta$; in other words g is a continuous extension of $\theta \circ \theta_0^{-1}$. We use the same method as above to replace g by an extension $f : P \rightarrow Q$ of $\theta \circ \theta_0^{-1}$ which is of class $\mathcal{C}^{\kappa+1}$. Putting $J := f^{-1}(I)$ we see that f is an isomorphism $(P, J, \theta_0) \rightarrow (Q, I, \theta)$ of boundary framed formally holomorphic bundles of type $(\Phi, \kappa + 1)$. \square

Remark 4.5. — In terms of abstract boundary framed formally holomorphic bundles, the descendibility condition of Definition 2.15 becomes: Let Φ be a \mathcal{C}^∞ bundle on S and $\Phi_{\widehat{S}} = \Phi_{S^-} \cup \Phi_{S^+}$ its pull-back to $\partial\widehat{X}_S = \widehat{S} = S^- \cup S^+$. A boundary framed formally holomorphic bundle $(\Omega, \mathfrak{I}, \theta)$ of type $(\Phi_{\widehat{S}}, \kappa + 1)$ on \widehat{X}_S is descendable if and only if the tangential almost complex structures \mathfrak{I}_{S^\pm} induced by \mathfrak{I} on Φ_{S^\pm} via θ agree via the obvious bundle isomorphism $\Phi_{S^-} \rightarrow b^*(\Phi_{S^+})$.

Similarly, if S separates X , a pair $((Q_-, I_-, \theta_-), (Q_+, I_+, \theta_+))$ of boundary framed formally holomorphic bundles of type $(\Phi, \kappa + 1)$ on \overline{X}^\pm corresponds to a point in the fiber product $\mathcal{M}_{\partial\overline{X}^-}(P^-) \times_{\mathcal{I}} \mathcal{M}_{\partial\overline{X}^+}(P^+)$ intervening in Theorem 2.17, if and only if I_\pm induce the same tangential almost complex structures on Φ via θ_\pm and the identifications $\partial\overline{X}^\pm = S$.

5. Examples

Throughout this section we fix $\kappa \in (0, +\infty] \setminus \mathbb{N}$ and a connected complex Lie group G . Let X a Riemann surface, and $Y \subset X$ a connected open subset whose closure \overline{Y} is a compact surface with smooth, non-empty boundary $\partial\overline{Y} = \overline{Y} \setminus Y$.

PROPOSITION 5.1. — *For any \mathcal{C}^∞ principal G -bundle P on \overline{Y} and bundle ACS J of class \mathcal{C}^κ on P , there exists a J -pseudo-holomorphic section $\tau_0 \in \Gamma(\overline{Y}, P)_{\kappa+1}$. In other words, for any such pair (P, J) there exists a pseudo-holomorphic bundle isomorphism $(\overline{Y} \times G, J_0) \rightarrow (P, J)$ of class $\mathcal{C}^{\kappa+1}$, where J_0 is the standard bundle ACS on the trivial bundle $\overline{Y} \times G$.*

Proof. — Since $\partial\overline{Y} \neq \emptyset$, \overline{Y} has the homotopy type of a bouquet of circles. Taking into account that G is connected, it follows that any topological (differentiable) G bundle on \overline{Y} is trivial, so we may suppose that $P = \overline{Y} \times G$. Let N be a tubular neighborhood of $\partial\overline{Y}$ in X and $\widetilde{Y} := \overline{Y} \cup N$. Therefore \widetilde{Y} is an open neighborhood of \overline{Y} in X which is homotopically equivalent to \overline{Y} .

The bundle ACS J is defined by a form $\alpha_J \in \Gamma^\kappa(\overline{Y}, \wedge_{\overline{Y}}^{0,1} \otimes \mathfrak{g})$ (see Section A.5.2). By the extension Corollary A.14 there exists an extension $\widetilde{\alpha} \in \Gamma^\kappa(\widetilde{Y}, \wedge_{\widetilde{Y}}^{0,1} \otimes \mathfrak{g})$ of α_J . The form $\widetilde{\alpha}$ corresponds to a bundle ACS \widetilde{J} of class \mathcal{C}^κ on $\widetilde{Y} \times G$ which extends J . By the Newlander–Nirenberg Theorem A.22, \widetilde{J} defines a holomorphic structure on the underlying $\mathcal{C}^{\kappa+1}$ bundle of $\widetilde{Y} \times G$. This structure is trivial by Grauert’s classification theorem of holomorphic bundles on Stein manifolds [9], so it admits a global holomorphic section $\widetilde{\tau}_0$. It suffices to put $\tau_0 := \widetilde{\tau}_0|_{\overline{Y}}$. \square

Note that any topological G -bundle on $\partial\bar{Y}$ is also trivial so, with the notations of Section 4.2, it's natural to take $\Phi = \partial\bar{Y} \times G$ as framing bundle on $\partial\bar{Y}$. In other words, in this section, by a boundary framing of a G bundle on \bar{Y} we will always mean a trivialization, or, equivalently, a section of its restriction to $\partial\bar{Y}$.

Consider now the special case when \bar{Y} is a disk $\bar{D} \subset X$. Isomorphism classes of boundary framed topological G -bundles on \bar{D} correspond bijectively to homotopy classes $\chi \in [\partial\bar{D}, G]$ of maps $\theta : \partial\bar{D} \rightarrow G$. Since $\pi_1(G, e)$ is Abelian, the obvious map $\pi_1(G, e) \rightarrow [\partial\bar{D}, G]$ is injective, so $[\partial\bar{D}, G]$ has a natural Abelian group structure. Endowing $\partial\bar{D}$ with its boundary orientation (induced by the complex orientation of \bar{D}), this set can be further identified with $H_1(G, \mathbb{Z})$ via the map

$$[\theta] \longmapsto \deg(\theta) := H_1(\theta)([\partial\bar{D}]).$$

For a class $h \in H_1(G, \mathbb{Z})$ we will denote by $h_{\bar{D}}$ the corresponding isomorphism class of boundary framed topological G -bundles on \bar{D} and by $\mathcal{M}_{\partial\bar{D}}^{\bar{D}}(h)$ the moduli space of boundary framed formally holomorphic G bundles of class \mathcal{C}^κ in this class. By Proposition 5.1 we obtain:

COROLLARY 5.2. — *Let $h \in H_1(G, \mathbb{Z})$. We have a natural identification*

$$\mathcal{M}_{\partial\bar{D}}^{\bar{D}}(h) \simeq \mathcal{C}_h^{\kappa+1}(\partial\bar{D}, G) /_{H^{\kappa+1}(\bar{D}, G)},$$

where $\mathcal{C}_h^{\kappa+1}(\partial\bar{D}, G)$ is the space of $\mathcal{C}^{\kappa+1}$ maps $\partial\bar{D} \rightarrow G$ of degree h , and $H^\kappa(\bar{D}, G)$ is the group of $\mathcal{C}^{\kappa+1}$ maps $\bar{D} \rightarrow G$ which are holomorphic on D .

Remark 5.3. — Suppose that G is reductive, and let $K \subset G$ be a maximal compact subgroup of G . In this case the canonical map

$$\mathcal{C}_h^{\kappa+1}(\partial\bar{D}, K) /_K \longrightarrow \mathcal{C}_h^{\kappa+1}(\partial\bar{D}, G) /_{H^{\kappa+1}(\bar{D}, G)}$$

is an isomorphism. For the standard disk this is a well known factorization theorem in loop group theory [29, Chapter 8], whereas the general case follows using [6, Theorem 1'].

5.1. Holomorphic bundles framed along a circle in $\mathbb{P}_{\mathbb{C}}^1$

Let now $S \subset \mathbb{C}$ be a smooth closed curve and $\mathbb{P}_{\mathbb{C}}^1 = \bar{U}^- \cup \bar{U}^+$ be the corresponding decomposition of $\mathbb{P}_{\mathbb{C}}^1$ as union of closed disks, where $\bar{U}^- \cap \bar{U}^+ = S$, $0 \in U^-$, $\infty \in U^+$. Note that the identifications $S = \partial\bar{U}^\pm$ induce on S opposite orientations.

Let (P, θ) be a topological S -framed principal G -bundle on $\mathbb{P}_{\mathbb{C}}^1$. Choose sections τ^{\pm} of the restrictions $P^{\pm} := P|_{\bar{U}^{\pm}}$ and let $g : \partial\bar{U}^{-} \rightarrow G$ be the comparison map defined by $\tau_S^{+} = \tau_S^{-}g$.

The homotopy degree $e(P) := \deg(g) \in H_1(G, \mathbb{Z})$ is independent of the pair (τ^{-}, τ^{+}) ; it is a topological invariant of P ; isomorphism classes of topological G -bundles over $\mathbb{P}_{\mathbb{C}}^1$ correspond bijectively to elements $e \in H_1(G, \mathbb{Z})$ via this invariant. For a section $\theta \in \Gamma(S, P)$ we define the maps $f_{\theta}^{\pm} : S \rightarrow G$ by $\theta = \tau_S^{\pm}f_{\theta}^{\pm}$; these maps satisfy the identity $f_{\theta}^{+} = g^{-1}f_{\theta}^{-}$ and $h_{\theta}^{\pm} := \deg(f_{\theta}^{\pm}) \in H_1(G, \mathbb{Z})$ are topological invariants of the framed bundle (P, θ) .

Remark 5.4. — Isomorphism classes of S -framed topological G -bundles on $\mathbb{P}_{\mathbb{C}}^1$ correspond bijectively to pairs $(e, h) \in H_1(G, \mathbb{Z}) \times H_1(G, \mathbb{Z})$ via the map

$$(P, \theta) \longmapsto (e(P), h_{\theta}^{-}).$$

For a pair $(e, h) \in H^1(G, \mathbb{Z}) \times H^1(G, \mathbb{Z})$ we denote by $(e, h)_{\mathbb{P}_{\mathbb{C}}^1}$ the corresponding isomorphism class of S -framed topological bundles on $\mathbb{P}_{\mathbb{C}}^1$, and by $\mathcal{M}_S^{\mathbb{P}_{\mathbb{C}}^1}(e, h)$ the moduli space of S -framed holomorphic bundles of class \mathcal{C}^{κ} on $\mathbb{P}_{\mathbb{C}}^1$ belonging to this isomorphism class. By Theorem 2.17 and Proposition 4.4 we obtain the decomposition:

$$\mathcal{M}_S^{\mathbb{P}_{\mathbb{C}}^1}(e, h) = \mathcal{M}_{\partial\bar{U}^{-}}^{\bar{U}^{-}}(h) \times \mathcal{M}_{\partial\bar{U}^{+}}^{\bar{U}^{+}}(e - h). \quad (5.1)$$

Consider now the case $G = \mathbb{C}^*$ and identify $H_1(\mathbb{C}^*, \mathbb{Z})$ with \mathbb{Z} in the standard way. $\mathcal{M}_S^{\mathbb{P}_{\mathbb{C}}^1}(e, h)$ is just the moduli space of pairs (L, θ) consisting of a holomorphic line bundle L of degree e on $\mathbb{P}_{\mathbb{C}}^1$ and a nowhere vanishing section θ of class $\mathcal{C}^{\kappa+1}$ and degree h (with respect to a trivialization on \bar{U}^{-}) of L_S .

Any holomorphic line bundle of degree e on $\mathbb{P}_{\mathbb{C}}^1$ is isomorphic to $|\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(e)|$. We trivialize over $\mathbb{P}_{\mathbb{C}}^1 \setminus \{\infty\}$ (respectively $\mathbb{P}_{\mathbb{C}}^1 \setminus \{0\}$) the line bundle $|\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(e)|$ using $\varphi_0^{\otimes e}$ (respectively $\varphi_1^{\otimes e}$), where $\varphi_i : \mathbb{C}^2 \rightarrow \mathbb{C}$ is the linear form defined by $\varphi_i(Z_0, Z_1) = Z_i$. Since $\text{Aut}(|\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(e)|) = \mathbb{C}^* \text{id}$, we obtain an obvious identification

$$\mathcal{M}_S^{\mathbb{P}_{\mathbb{C}}^1}(e, h) \simeq \mathcal{C}_h^{\kappa+1}(\partial\bar{U}^{-}, \mathbb{C}^*) / \mathbb{C}^*. \quad (5.2)$$

This isomorphism combined with the decomposition (5.1) and Corollary 5.2 gives the homeomorphism

$$\begin{aligned} \mathcal{C}_h^{\kappa+1}(\partial\bar{U}^{-}, \mathbb{C}^*) / \mathbb{C}^* &\xrightarrow[\simeq]{\Psi_{e, h}} \mathcal{C}_h^{\kappa+1}(\partial\bar{U}^{-}, \mathbb{C}^*) / H^{\kappa+1}(\bar{U}^{-}, \mathbb{C}^*) \\ &\quad \times \mathcal{C}_{e-h}^{\kappa+1}(\partial\bar{U}^{+}, \mathbb{C}^*) / H^{\kappa+1}(\bar{U}^{+}, \mathbb{C}^*) \end{aligned} \quad (5.3)$$

given explicitly by

$$[f]_{\mathbb{C}^*} \mapsto \left([f]_{H^{\kappa+1}(\bar{U}^-, \mathbb{C}^*)}, [(\varphi_1 \varphi_0^{-1})^{-e} f]_{H^{\kappa+1}(\bar{U}^+, \mathbb{C}^*)} \right).$$

We are interested in an explicit formula for the inverse of this map. Note that the map $f \mapsto (\varphi_1 \varphi_0^{-1})^{-e} f$ induces an isomorphism

$$\mathcal{C}_{e-h}^{\kappa+1}(\partial \bar{U}^+, \mathbb{C}^*) / H^{\kappa+1}(\bar{U}^+, \mathbb{C}^*) \longrightarrow \mathcal{C}_{-h}^{\kappa+1}(\partial \bar{U}^+, \mathbb{C}^*) / H^{\kappa+1}(\bar{U}^+, \mathbb{C}^*),$$

so it suffices to consider the case $e = 0$. We will see that the inverse of $\Psi_{0,g}$ can be written down explicitly using the Cauchy transform and classical results in holomorphic function theory. Recall first that the Cauchy transform

$$u \mapsto C^S(u), \quad C^S(u)(z) = \frac{1}{2\pi i} \int_S \frac{u(\zeta)}{\zeta - z} d\zeta$$

associated with the oriented closed curve $S = \partial \bar{U}^-$ defines continuous operators

$$C_{\pm}^S : \mathcal{C}^{\kappa+1}(S, \mathbb{C}) \longrightarrow \mathcal{C}^{\kappa+1}(\bar{U}^{\pm}, \mathbb{C})$$

(see [25, Section 2.22] and [35, Theorem 1.10 p. 22, formula (3.3) p. 23]) satisfying the Plemelj–Privalov formula

$$C_-^S(u)|_S - C_+^S(u)|_S = u \quad (5.4)$$

(see [25, formula (17.3) p. 43]).

Let $f^-, f^+ \in \mathcal{C}^{\kappa+1}(S, \mathbb{C}^*)$ be maps of degree h (respectively $-h$) with respect to 0 (respectively ∞). Therefore $\deg(f^{\pm}) = h$ with respect to 0. Let $\varphi \in \mathcal{C}^{\kappa+1}(S, \mathbb{C})$ be such that $\exp(\varphi) = f^+ / f^-$. By (5.4) we obtain $e^{C_-^S(\varphi)}|_S - e^{C_+^S(\varphi)}|_S = f^+ / f^-$, so

$$e^{C_-^S(\varphi)}|_S f^- = e^{C_+^S(\varphi)}|_S f^+.$$

Noting that $e^{C_{\pm}^S(\varphi)} \in H^{\kappa+1}(\bar{U}^{\pm}, \mathbb{C}^*)$ and putting

$$f := e^{C_-^S(\varphi)}|_S f^- = e^{C_+^S(\varphi)}|_S f^+$$

we see that $\mathbb{C}^* f$ is the pre-image of the pair $([f^-]_{H^{\kappa+1}(\bar{U}^-, \mathbb{C}^*)}, [f^+]_{H^{\kappa+1}(\bar{U}^+, \mathbb{C}^*)})$ via $\Psi_{0,h}$. Therefore $\Psi_{0,h}^{-1}$ is given by the explicit formula:

$$\Psi_{0,h}^{-1}([f^-]_{H^{\kappa+1}(\bar{U}^-, \mathbb{C}^*)}, [f^+]_{H^{\kappa+1}(\bar{U}^+, \mathbb{C}^*)}) = [e^{C_{\pm}^S(\log(f^+ / f^-))}|_S f^{\pm}]_{\mathbb{C}^*}. \quad (5.5)$$

Remark 5.5. — Combining the isomorphism (5.3) with Remark 5.3, we obtain an isomorphism

$$\mathcal{C}_h^{\kappa+1}(\partial \bar{U}^-, \mathbb{C}^*) /_{\mathbb{C}^*} \xrightarrow{\simeq} \mathcal{C}_h^{\kappa+1}(\partial \bar{U}^-, S^1) /_{S^1} \times \mathcal{C}_{e-h}^{\kappa+1}(\partial \bar{U}^+, S^1) /_{S^1}.$$

This is a typical example of identification obtained by combining the isomorphism Theorem 2.17 with Donaldson’s Theorem 1’.

5.2. Holomorphic bundles framed along a circle in an elliptic curve

As in Example 2.13 of Section 2.2.2, let $\alpha \in \mathbb{C}^*$ with $|\alpha| < 1$ and $X = \mathbb{C}^*/\langle \alpha \rangle$ be the associated elliptic curve; let $\bar{D} \subset \mathbb{C}$ be a smooth compact disk such that $\alpha\bar{D} \subset D$, $\Omega := D \setminus \alpha\bar{D}$, $\bar{\Omega} := \bar{D} \setminus \alpha D$. Endow the curves $S^+ := \partial\bar{D}$, $S^- := \alpha S^+ = \partial(\alpha\bar{D})$ with their boundary orientations.

As noticed above, since we assumed G connected, any differentiable G bundle on $\bar{\Omega}$ ($\partial\bar{\Omega}$) is trivial. Taking as in the previous section $\Phi = \partial\bar{\Omega} \times G$ as framing bundle, we see that the data of a topological boundary framing of the trivial bundle $\bar{\Omega} \times G$ is equivalent to the data of a pair (τ^+, τ^-) of continuous maps $\tau^\pm : S^\pm \rightarrow G$.

Remark 5.6. — The formula $[(\bar{\Omega} \times G, \tau^+, \tau^-)] \mapsto \deg(\tau_+) - \deg(\tau_-)$ gives a bijection between isomorphism classes of boundary framed topological G -bundles on $\bar{\Omega}$ and $H_1(G, \mathbb{Z})$.

For a class $h \in H_1(G, \mathbb{Z})$ we denote by $h_{\bar{\Omega}}$ the corresponding isomorphism class of boundary framed topological G -bundles on $\bar{\Omega}$ and by $\mathcal{M}_{\partial\bar{\Omega}}^{\bar{\Omega}}(h)$ the moduli space of boundary framed formally holomorphic G bundles of class \mathcal{C}^κ in this class.

Let $H^\kappa(\bar{\Omega}, G)$ be the group of $\mathcal{C}^{\kappa+1}$ maps $\bar{\Omega} \rightarrow G$ which are holomorphic on Ω and $H_m^\kappa(\bar{\Omega}, G) := \{f \in H^\kappa(\bar{\Omega}, G) \mid \deg(f) = m\}$. Using Proposition 5.1 again we obtain:

COROLLARY 5.7. — *Let $h \in H_1(G, \mathbb{Z})$ and $n \in \mathbb{Z}$. We have natural identifications*

$$\begin{aligned} \mathcal{M}_{\partial\bar{\Omega}}^{\bar{\Omega}}(h) &\simeq \left(\prod_{m \in \mathbb{Z}} \mathcal{C}_m^{\kappa+1}(S^+, G) \times \mathcal{C}_{m-h}^{\kappa+1}(S^-, G) \right) / H^{\kappa+1}(\bar{\Omega}, G) \\ &= (\mathcal{C}_n^{\kappa+1}(S^+, G) \times \mathcal{C}_{n-h}^{\kappa+1}(S^-, G)) / H_0^{\kappa+1}(\bar{\Omega}, G). \end{aligned} \quad (5.6)$$

Suppose now that G is reductive, let $K \subset G$ be a maximal compact subgroup of G and let $M_{\partial\bar{\Omega}}^{\bar{\Omega}}(h)$ be the moduli space of boundary framed flat K -connections of topological type $h_{\bar{\Omega}}$ and class \mathcal{C}^κ modulo the gauge group $\mathcal{C}^{\kappa+1}(\bar{\Omega}, K)$. Using [6, Theorem 1'] it follows that the canonical map $M_{\partial\bar{\Omega}}^{\bar{\Omega}}(h) \rightarrow \mathcal{M}_{\partial\bar{\Omega}}^{\bar{\Omega}}(h)$ is a homeomorphism. The moduli space $M_{\partial\bar{\Omega}}^{\bar{\Omega}}(h)$ can be easily described as follows:

Identify Ω with the quotient $\bar{\mathfrak{D}}/H$, where $c : \bar{\mathfrak{D}} \rightarrow \bar{\Omega}$ is a universal cover of $\bar{\Omega}$ and $H := \text{Aut}_{\bar{\Omega}}(\bar{\mathfrak{D}})$. Let $h_0 \in H$ be the generator of H which corresponds to the generator of positive degree of the fundamental group of $\bar{\Omega}$. For any

$a \in K$ let $\tilde{a} : H \rightarrow K$ be the group morphism which maps h_0 to a . Put $\mathfrak{S}^\pm := c^{-1}(S^\pm)$ and note that the space

$$\mathfrak{A} := \{(a, \mathfrak{t}^+, \mathfrak{t}^-) \mid \mathfrak{t}^\pm \in \mathcal{C}_{\tilde{a}}^{\mathfrak{t}^\pm+1}(\mathfrak{S}^\pm, K)\}$$

comes with a natural free K -action given by:

$$b \cdot (a, \mathfrak{t}^+, \mathfrak{t}^-) = (bab^{-1}, b\mathfrak{t}^+, b\mathfrak{t}^-).$$

Let $\mathfrak{f} : \bar{\mathfrak{S}}^+ \rightarrow \bar{\mathfrak{S}}^-$ be a H -equivariant lift of the diffeomorphism $S^+ \ni z \mapsto \alpha z \in S^-$ and note that, for $\mathfrak{t}^\pm \in \mathcal{C}_{\tilde{a}}^{\mathfrak{t}^\pm+1}(\mathfrak{S}^\pm, K)$, the product $(\mathfrak{t}^- \circ \mathfrak{f})^{-1}\mathfrak{t}^+ : \mathfrak{S}^+ \rightarrow K$ is H -invariant, so it descends to a map $[(\mathfrak{t}^- \circ \mathfrak{f})^{-1}\mathfrak{t}^+] : S^+ \rightarrow K$ whose degree $\deg([(\mathfrak{t}^- \circ \mathfrak{f})^{-1}\mathfrak{t}^+]) \in H_1(K, \mathbb{Z}) = H_1(G, \mathbb{Z})$ is independent of the choice of \mathfrak{f} . The subspace

$$\mathfrak{A}(h) := \{(a, \mathfrak{t}^+, \mathfrak{t}^-) \mid \mathfrak{t}^\pm \in \mathcal{C}_{\tilde{a}}^{\mathfrak{t}^\pm+1}(\mathfrak{S}^\pm, K), \deg([(\mathfrak{t}^- \circ \mathfrak{f})^{-1}\mathfrak{t}^+]) = h\} \subset \mathfrak{A}$$

is K -invariant. Let $(a, \mathfrak{t}^+, \mathfrak{t}^-) \in \mathfrak{A}$. The principal K -bundle $P_a := \bar{\mathfrak{D}} \times_{\tilde{a}} K$ comes with a canonical flat connection A_a and the maps \mathfrak{t}^\pm can be interpreted as sections of class $\mathcal{C}^{\kappa+1}$ of $P_a|_{S^\pm}$.

Remark 5.8. — The map

$$\mathfrak{A}(h)/K \longrightarrow M_{\partial\bar{\Omega}}^{\bar{\Omega}}(h), \quad K \cdot (a, \mathfrak{t}^+, \mathfrak{t}^-) \longmapsto [A_a, \mathfrak{t}^+, \mathfrak{t}^-]$$

is a homeomorphism.

This remark gives a simple description of the Donaldson moduli space $M_{\partial\bar{\Omega}}^{\bar{\Omega}}(h)$ of boundary framed flat K -connections on an annulus.

On the other hand, note that the G -bundle $P_a^{\mathbb{C}} := \bar{\mathfrak{D}} \times_{\tilde{a}} G$ comes also with a flat connection, so with a canonical bundle ACS J_a . Making use of Proposition 5.1, let $\mathfrak{t} \in \Gamma(\bar{\Omega}, P_a^{\mathbb{C}})_{\kappa+1}$ be a J_a -pseudo-holomorphic section and let $\tau^\pm : \bar{\Omega} \rightarrow G$ be the maps defined by the formulae $\mathfrak{t}^\pm = \tau\tau^\pm$. The pair (τ^+, τ^-) is independent of the choice of τ up to the $H^{\kappa+1}(\bar{\Omega}, G)$ action. In conclusion, combining Corollary 5.7 with [6, Theorem 1'] we obtain

Remark 5.9. — We have a natural homeomorphism

$$\mathfrak{A}(h)/K \xrightarrow{\sim} \left(\coprod_{m \in \mathbb{Z}} \mathcal{C}_m^{\kappa+1}(S^+, G) \times \mathcal{C}_{m-h}^{\kappa+1}(S^-, G) \right) / H^{\kappa+1}(\bar{\Omega}, G)$$

given explicitly by $K \cdot (A_a, \mathfrak{t}^+, \mathfrak{t}^-) \mapsto H^{\kappa+1}(\bar{\Omega}, G)(\tau^+, \tau^-)$.

Our next goal is to make explicit the isomorphism given by Theorem 2.17 and its inverse in the special case when X is the elliptic curve $\mathbb{C}^*/\langle \alpha \rangle$, S is the image of S^\pm in X and $G = \mathbb{C}^*$. Note that \widehat{X}_S can be identified with $\bar{\Omega}$. Isomorphism classes of S -framed \mathbb{C}^* -bundles over X correspond bijectively to isomorphism classes of \mathbb{C}^* -bundles on X . This follows taking into account

that the restriction map $\mathcal{C}(X, \mathbb{C}^*) \rightarrow \mathcal{C}(S, \mathbb{C}^*)$ is surjective, so the automorphism group of a topological \mathbb{C}^* -bundle P on X acts transitively on the space of continuous sections of P_S . For $e \in \mathbb{Z}$ let $e_{\mathbb{P}_{\mathbb{C}}^1}$ be the isomorphism class of S -framed topological \mathbb{C}^* -bundles (P, s) with $\deg(P) = e$, and let $\mathcal{M}_S^X(e)$ be the corresponding moduli space.

Putting $\text{Pic}^e(X) := \{[L] \in \text{Pic}(X) \mid \deg(L) = e\}$, we have:

Remark 5.10. — The natural map

$$\mathcal{M}_S^X(e) \longrightarrow \text{Pic}^e(X), \quad [(P, s)] \longmapsto [P \times_{\mathbb{C}^*} \mathbb{C}]$$

is a principal bundle with structure group $\mathcal{C}^{\kappa+1}(S, \mathbb{C}^*)/\mathbb{C}^*$.

Taking into account Theorem 2.17, Proposition 4.4 and Corollary 5.7, we obtain:

COROLLARY 5.11. — *We have a natural homeomorphism*

$$\Psi_e : \mathcal{M}_S^X(e) \xrightarrow{\sim} \left(\prod_{m \in \mathbb{Z}} \mathcal{C}_m^{\kappa+1}(S^+, \mathbb{C}^*) \times \mathcal{C}_{m-e}^{\kappa+1}(S^-, \mathbb{C}^*) \right) / H^{\kappa+1}(\bar{\Omega}, \mathbb{C}^*)$$

defined as follows: for an S -framed holomorphic \mathbb{C}^* -bundle (P, s) on X , let (\hat{P}, \hat{s}) be the pull-back boundary framed formally holomorphic bundle on $\bar{\Omega}$, $\hat{s}^{\pm} := \hat{s}|_{S^{\pm}}$ and let τ be a pseudo-holomorphic section of \hat{P} . Then $\Psi_e([P, s]) = (f_+, f_-)$, where $f^{\pm} : S^{\pm} \rightarrow \mathbb{C}^*$ are defined by the formulae $\hat{s}^{\pm} = \tau f^{\pm}$.

Choosing τ such that $\deg(f^+) = n$ in the definition of ρ , we obtain a homeomorphism

$$\Psi_{e,n} : \mathcal{M}_S^X(e) \xrightarrow{\sim} (\mathcal{C}_n^{\kappa+1}(S^+, \mathbb{C}^*) \times \mathcal{C}_{n-e}^{\kappa+1}(S^-, \mathbb{C}^*)) / H_0^{\kappa+1}(\bar{\Omega}, \mathbb{C}^*), \quad (5.7)$$

which is an analogue for elliptic curves of the homeomorphism (5.3) obtained by applying Theorem 2.17 to $\mathbb{P}_{\mathbb{C}}^1$. We are interested in an explicit formula for the inverse

$$\Psi_{0,n}^{-1} : (\mathcal{C}_n^{\kappa+1}(S^+, \mathbb{C}^*) \times \mathcal{C}_n^{\kappa+1}(S^-, \mathbb{C}^*)) / H_0^{\kappa+1}(\bar{\Omega}, \mathbb{C}^*) \longrightarrow \mathcal{M}_S^X(0)$$

corresponding to the special case $e = 0$. Let $f^{\pm} \in \mathcal{C}_n^{\kappa+1}(S^{\pm}, \mathbb{C}^*)$ and let $\varphi \in \mathcal{C}^{\kappa+1}(S^+, \mathbb{C})$ be such that for any $z \in S^+$, we have $e^{\varphi}(z) = f^+(z)f^-(\alpha z)^{-1}$. With the notations introduced in the previous section, put

$$\mathfrak{C}_-^{S^+}(\varphi) := C_-^{S^+}(\varphi) - C_-^{S^+}(\varphi)(0),$$

and define $\psi : \bar{\Omega} \rightarrow \mathbb{C}^*$ by the formula:⁽⁴⁾

$$\psi(z) = \sum_{k=0}^{\infty} \mathfrak{C}_-^{S^+}(\varphi)(\alpha^k z) + \sum_{k=1}^{\infty} C_+^{S^+}(\varphi)(\alpha^{-k} z).$$

⁽⁴⁾ The idea to define ψ in this way and formula (5.8) are due to Alexander Borichev [4].

Noting that $\mathfrak{C}_-^{S^+}(\varphi)(0) = 0$ by the definition of $\mathfrak{C}_-^{S^+}(\varphi)$, and $C_+^{S^+}(\infty) = 0$ by [25, p. 23], it follows using Lemma 5.12 proved below that both series in the definition of ψ are normally convergent on $\bar{\Omega}$. Moreover, writing

$$\psi(z) = \mathfrak{C}_-^{S^+}(\varphi)(z) + C_+^{S^+}(\varphi)(\alpha^{-1}z) + \sum_{k=1}^{\infty} \mathfrak{C}_-^{S^+}(\varphi)(\alpha^k z) + \sum_{k=2}^{\infty} C_+^{S^+}(\varphi)(\alpha^{-k} z),$$

using the properties of the Cauchy transforms $C_{\pm}^{S^+}$ mentioned in the previous section, and noting that the two sums on the right extend holomorphically to a neighborhood of $\bar{\Omega}$, it follows that $\psi \in H^{\kappa+1}(\bar{\Omega}, \mathbb{C})$.

For any $z \in \bar{\Omega}$ we have

$$\psi(z) - \psi(\alpha z) = \mathfrak{C}_-^{S^+}(\varphi)(z) - C_+^{S^+}(\varphi)(z) = \varphi(z) - C_-^{S^+}(\varphi)(0), \quad (5.8)$$

where, for the last equality, we used the Plemelj–Privalov formula (5.4). Putting $\lambda := e^{C_-^{S^+}(\varphi)(0)} \in \mathbb{C}^*$, $g^{\pm} := e^{-\psi}|_{S^{\pm}} f^{\pm}$, this implies

$$\forall z \in S^+, \quad g^-(\alpha z) = \lambda g^+(z). \quad (5.9)$$

Let P_{λ} be the flat holomorphic \mathbb{C}^* -bundle over $X = \mathbb{C}^*/\langle \alpha \rangle$ defined by

$$P_{\lambda} := \mathbb{C}^* \times \mathbb{C}^* / \langle (\alpha, \lambda) \rangle = \bar{\Omega} \times \mathbb{C}^* / \overset{\lambda}{\sim},$$

where $\overset{\lambda}{\sim}$ is the equivalence relation generated by the set of pairs

$$\{((z^+, \zeta), (\alpha z^+, \lambda \zeta)) \mid z^+ \in S^+, \zeta \in \mathbb{C}^*\}.$$

Formula (5.9) shows that (g^+, g^-) defines a section $g \in \Gamma(S, P_{\lambda})^{\kappa+1}$, and that

$$\Psi_{0,n}([P_{\lambda}, g]) = [g^+, g^-]_{H_0^{\kappa+1}(\bar{\Omega}, \mathbb{C}^*)}.$$

On the other hand, the definition of g^{\pm} gives $(g^+, g^-) = e^{-\psi}(f^+, f^-)$, where $e^{-\psi} \in H_0^{\kappa+1}(\bar{\Omega}, \mathbb{C}^*)$, so

$$[f^+, f^-]_{H_0^{\kappa+1}(\bar{\Omega}, \mathbb{C}^*)} = [g^+, g^-]_{H_0^{\kappa+1}(\bar{\Omega}, \mathbb{C}^*)}.$$

Therefore

$$\Psi_{0,n}^{-1}([f^+, f^-]_{H_0^{\kappa+1}(\bar{\Omega}, \mathbb{C}^*)}) = [P_{\lambda}, g]. \quad (5.10)$$

LEMMA 5.12. — *Let $r > 0$ and u be a holomorphic function defined on a neighborhood of the standard compact disk $\bar{D}_r \subset \mathbb{C}$ such that $u(0) = 0$.*

(1) *For any $z \in \bar{D}_r$ we have*

$$|u(z)| \leq r^{-1} \sup_{\zeta \in S_r} |u(\zeta)| |z|.$$

(2) *Let $\alpha \in \mathbb{C}^*$ with $|\alpha| < 1$. For any $(z, k) \in \mathbb{C} \times \mathbb{N}$ such that $\alpha^k z \in \bar{D}_r$ we have*

$$|u(\alpha^k z)| \leq r^{-1} |\alpha|^k \sup_{\zeta \in S_r} |u(\zeta)| |z|.$$

Proof. — For (1) apply the Maximum Principle to the holomorphic extension of the function $z \mapsto z^{-1}u(z)$ on \overline{D}_r . (2) follows directly from (1). \square

5.3. S -framed holomorphic $\mathrm{SL}(2, \mathbb{C})$ -bundles on $\mathbb{P}_{\mathbb{C}}^1$

We come back to the decomposition $\mathbb{P}_{\mathbb{C}}^1 = \overline{U}^- \cup \overline{U}^+$ associated with a closed curve $S \subset \mathbb{C}$ as considered in Section 5.1. We are interested in the moduli space of S -framed $\mathrm{SL}(2, \mathbb{C})$ -bundles on $\mathbb{P}_{\mathbb{C}}^1$. We will use the vector bundle formalism, so in this section by $\mathrm{SL}(2, \mathbb{C})$ -bundle we mean a holomorphic vector bundle of rank 2 endowed with a trivialization of its determinant line bundle. By Grothendieck's classification theorem [12] the map

$$\mathbb{N} \ni n \longmapsto |\mathcal{O}(n)| \oplus |\mathcal{O}(-n)|$$

is a bijection onto the set of isomorphism classes of $\mathrm{SL}(2, \mathbb{C})$ -bundles on $\mathbb{P}_{\mathbb{C}}^1$. In the above formula we used the notation $|\mathcal{L}|$ for the line bundle associated with an invertible sheaf \mathcal{L} . Denoting by $\mathbb{C}[Z_0, Z_1]_d$ the space of homogeneous polynomials of degree d in Z_0, Z_1 , note that

$$\mathrm{Aut}(|\mathcal{O}(n)| \oplus |\mathcal{O}(-n)|) = \begin{cases} \mathrm{SL}(2, \mathbb{C}) & \text{if } n = 0, \\ \left\{ \begin{pmatrix} a & P \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^*, P \in \mathbb{C}[Z_0, Z_1]_{2n} \right\} & \text{if } n > 0. \end{cases}$$

On the affine line $\mathbb{C} \subset \mathbb{P}_{\mathbb{C}}^1$ we trivialize the line bundles $|\mathcal{O}(1)|, |\mathcal{O}(-1)|$ using respectively the linear form φ_0 defined in Section 5.1 and the meromorphic section ξ_0 of the tautological line bundle $|\mathcal{O}(-1)|$ given by

$$\mathbb{P}^1 \ni \xi = [Z_0, Z_1] \longmapsto \left(1, \frac{Z_1}{Z_0}\right) \in |\mathcal{O}(-1)|_{\xi}.$$

The matrix of $A = \begin{pmatrix} a & P \\ 0 & a^{-1} \end{pmatrix}$ with respect to the basis $(\varphi_0^{\otimes n}, \xi_0^{\otimes n})$ is $\mathfrak{A} = \begin{pmatrix} a & p|_S \\ 0 & a^{-1} \end{pmatrix}$ where $p \in \mathbb{C}[z]_{\leq 2n}$, $p(z) = P(1, z)$ is the dehomogenization of P with respect to Z_0 . We obtain:

PROPOSITION 5.13. — *The moduli space \mathcal{M}_S of S -framed $\mathrm{SL}(2, \mathbb{C})$ -bundles on $\mathbb{P}_{\mathbb{C}}^1$ admits a natural stratification $\mathcal{M}_S = \coprod_{n \in \mathbb{N}} \mathcal{M}_S^n$, where*

$$\begin{aligned} \mathcal{M}_S^0 &= \mathcal{C}^{\kappa+1}(S, \mathrm{SL}(2, \mathbb{C}))/\mathrm{SL}(2, \mathbb{C}), \\ \mathcal{M}_S^n &= \mathcal{C}^{\kappa+1}(S, \mathrm{SL}(2, \mathbb{C}))/\left\{ \begin{pmatrix} a & p \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{C}^*, p \in \mathbb{C}[z]_{\leq 2n} \right\} \text{ for } n \geq 1. \end{aligned}$$

For any $n \in \mathbb{N}$, \mathcal{M}_S^n is open in $\overline{\mathcal{M}_S^n} = \bigcup_{m \geq n} \mathcal{M}_S^m$.

Therefore any stratum \mathcal{M}_S^n is an infinite dimensional homogeneous Banach manifold obtained by factorizing a Banach Lie group by a finite dimensional affine algebraic subgroup. Theorem 2.17 gives a homeomorphism

$$\begin{aligned} \mathcal{M}_S \xrightarrow{\cong} \mathcal{C}^{\kappa+1}(\partial\bar{U}^-, \mathrm{SL}(2, \mathbb{C}))/H^{\kappa+1}(\bar{U}^-, \mathrm{SL}(2, \mathbb{C})) \\ \times \mathcal{C}^{\kappa+1}(\partial\bar{U}^+, \mathrm{SL}(2, \mathbb{C}))/H^{\kappa+1}(\bar{U}^+, \mathrm{SL}(2, \mathbb{C})) \end{aligned}$$

induced by the obvious restriction map. Combining this result with [6, Theorem 1'] applied to the two factors on the right, we obtain:

COROLLARY 5.14. — *The product*

$$\mathcal{C}^{\kappa+1}(\partial\bar{U}^-, \mathrm{SU}(2))/\mathrm{SU}(2) \times \mathcal{C}^{\kappa+1}(\partial\bar{U}^+, \mathrm{SU}(2))/\mathrm{SU}(2)$$

can be identified with the moduli space $\mathcal{M}_S = \coprod_{n \in \mathbb{N}} \mathcal{M}_S^n$ of S -framed holomorphic $\mathrm{SL}(2, \mathbb{C})$ -bundles on $\mathbb{P}_{\mathbb{C}}^1$.

Appendix

A.1. Lipschitz spaces, spaces of maps and sections of class \mathcal{C}^κ

In this section we will introduce the spaces: $\mathrm{Lip}^\kappa(\mathbb{R}^n, T)$, $\mathrm{Lip}_{\mathbb{R}^n}^\kappa(F, T)$, $\mathcal{C}^\kappa(U, T)$, $\mathcal{C}^\kappa(\bar{U}, T)$, $\Gamma^\kappa(U, E)$, $\Gamma^\kappa(\bar{U}, E)$.

Let T be a finite dimensional normed space, $k \in \mathbb{N}$ and $f \in \mathcal{C}^k(\mathbb{R}^n, T)$. The order k remainder of f is the map $\mathbb{R}^n \times \mathbb{R}^n \rightarrow T$ defined by

$$R_f^k(x, y) := f(x) - \sum_{0 \leq |l| \leq k} \frac{1}{l!} \partial^l f(y) (x - y)^l.$$

Using the integral formula for the order $k - 1$ Taylor remainder, we obtain

$$\begin{aligned} R_f^k(x, y) &= R_f^{k-1}(x, y) - \sum_{|l|=k} \frac{1}{l!} \partial^l f(y) (x - y)^l \\ &= k \int_0^1 (1-t)^{k-1} \sum_{|l|=k} \frac{1}{l!} [\partial^l f(y + t(x-y)) - \partial^l f(y)] (x-y)^l dt, \quad (\text{A.1}) \end{aligned}$$

which gives the estimate

$$\|R_f^k(x, y)\| \leq c(n, k) \sup_{\substack{t \in [0,1] \\ |l|=k}} \|\partial^l f(y + t(x-y)) - \partial^l f(y)\| \|x - y\|^k. \quad (\text{A.2})$$

Applied to the $\mathcal{C}^{k-|j|}$ map $\partial^j f$ for $|j| \leq k$, formula (A.2) gives

$$\begin{aligned} & \|R_{\partial^j f}^{k-|j|}(x, y)\| \\ & \leq c(n, k - |j|) \sup_{\substack{t \in [0, 1] \\ |l| = k - |j|}} \|\partial^{j+l} f(y + t(x - y)) - \partial^{j+l} f(y)\| \|x - y\|^{k-|j|} \quad (\text{A.3}) \end{aligned}$$

Let now $\kappa \in (0, +\infty) \setminus \mathbb{N}$. We denote by $\text{Lip}^\kappa(\mathbb{R}^n, T)$ the order κ Lipschitz space in supremum norm, as defined in [18, p. 2], [31, p. 176]:

$$\text{Lip}^\kappa(\mathbb{R}^n, T) := \{f \in \mathcal{C}^{[\kappa]}(\mathbb{R}^n, T) \mid \|f\|_{\text{Lip}^\kappa} < \infty\}, \quad (\text{A.4})$$

where

$$\|f\|_{\text{Lip}^\kappa} := \inf \left\{ M \in \mathbb{R}_+ \left| \begin{array}{l} \sup_{\mathbb{R}^n} \|\partial^j f\| \leq M, \text{ for } |j| \leq [\kappa], \text{ and} \\ \|\partial^j f(x) - \partial^j f(y)\| \leq M \|x - y\|^{\kappa-|j|} \\ \text{for } |j| = [\kappa], \ x, y \in \mathbb{R}^n \end{array} \right. \right\}. \quad (\text{A.5})$$

Using formulae (A.2), (A.3) for $k = [\kappa]$, it follows that

Remark A.1. — For any $f \in \text{Lip}^\kappa(\mathbb{R}^n, T)$ and any $j \in \mathbb{N}^n$ with $|j| \leq [\kappa]$ we have an estimate of the form:

$$\|R_{\partial^j f}^{[\kappa]-|j|}(x, y)\| \leq M_j \|x - y\|^{\kappa-|j|}.$$

This justifies the following definition (see [31, p. 176], [18, p. 22] for \mathbb{R} -valued functions, and [3, Definition B1] for maps with values in a Banach space):

DEFINITION A.2. — Let $\kappa \in (0, +\infty) \setminus \mathbb{N}$ and $F \subset \mathbb{R}^n$ be a closed subset. A T -valued Whitney jet of order κ on F is a family $\mathfrak{f} = (f^{(j)})_{0 \leq |j| \leq [\kappa]}$ of bounded continuous maps $f^{(j)} : F \rightarrow T$ such that, putting

$$R_{j, \mathfrak{f}}^{[\kappa]}(x, y) := f^{(j)}(x) - \sum_{0 \leq |l| \leq [\kappa] - |j|} \frac{1}{l!} f^{(j+l)}(y) (x - y)^l,$$

we have estimates of the form $\|R_{j, \mathfrak{f}}^{[\kappa]}(x, y)\| \leq M_j \|x - y\|^{\kappa-|j|}$ on $F \times F$.

Endowed with the norm

$$\|\mathfrak{f}\|_{\text{Lip}^\kappa} := \inf \left\{ M \in \mathbb{R}_+ \left| \begin{array}{l} \|f^{(j)}(x)\| \leq M, \ \|R_{j, \mathfrak{f}}^{[\kappa]}(x, y)\| \leq M \|x - y\|^{\kappa-|j|} \\ \text{for any } (x, y) \in F \times F, \ j \in \mathbb{N}^n \text{ with } |j| \leq [\kappa] \end{array} \right. \right\},$$

the space $\text{Lip}_{\mathbb{R}^n}^\kappa(F, T)$ of T -valued Whitney jets of order κ on F becomes a Banach space. The role of the subscript \mathbb{R}^n in our notation is to avoid confusion with the space $\text{Lip}^\kappa(F, T)$ in the sense of (A.4) in the special case when F is an affine subspace of \mathbb{R}^n (in which case F can be identified with a space \mathbb{R}^m with $m \leq n$).

We refer to [31, Theorem 4, p. 177], [18, Theorem, p. 23] [3, Theorem B.2] for the following fundamental:

THEOREM A.3 (Whitney extension theorem for jets of order κ). — *Let $\kappa \in (0, +\infty) \setminus \mathbb{N}$ and $F \subset \mathbb{R}^n$ a closed set. There exists a continuous extension operator*

$$\mathcal{E}_\kappa : \text{Lip}_{\mathbb{R}^n}^\kappa(F, T) \longrightarrow \text{Lip}^\kappa(\mathbb{R}^n, T)$$

such that, putting $f := \mathcal{E}_\kappa(\mathfrak{f})$, we have $\partial^j f|_F = f^{(j)}$ for any $j \in \mathbb{N}$ with $|j| \leq [\kappa]$ and $f|_{\mathbb{R}^n \setminus F} \in \mathcal{C}^\infty(\mathbb{R}^n \setminus F, T)$.

For the \mathcal{C}^∞ property of f on $\mathbb{R}^n \setminus F$, see the comments of [31, p. 173, 179].

Let $H \subset \mathbb{R}^n$ be an open half-space bounded by an affine hyperplane $S \subset \mathbb{R}^n$. We endow the space

$$\text{Lip}^\kappa(\overline{H}, T) := \{f \in \mathcal{C}^0(\overline{H}, T) \mid \exists \tilde{f} \in \text{Lip}^\kappa(\mathbb{R}^n, T) \text{ such that } \tilde{f}|_{\overline{H}} = f\}$$

with the quotient norm induced by the obvious linear isomorphism

$$\text{Lip}^\kappa(\mathbb{R}^n, T) / \{\varphi \in \text{Lip}^\kappa(\mathbb{R}^n, T) \mid \varphi|_{\overline{H}} = 0\} \xrightarrow{\sim} \text{Lip}^\kappa(\overline{H}, T).$$

For $f \in \text{Lip}^\kappa(\overline{H}, T)$ and $j \in \mathbb{N}$ with $|j| \leq [\kappa]$ we put $\partial^j f := \partial^j \tilde{f}|_{\overline{H}}$, where $\tilde{f} \in \text{Lip}^\kappa(\mathbb{R}^n, T)$ is an extension of f to \mathbb{R}^n . Note that, by Whitney extension Theorem A.3 and [18, Corollary 1 p. 42]), the space $\text{Lip}^\kappa(\overline{H}, T)$ can also be identified with the space $\text{Lip}_{\mathbb{R}^n}^\kappa(\overline{H}, T)$ of Whitney jets of order κ on \overline{H} via the map

$$\text{Lip}_{\mathbb{R}^n}^\kappa(\overline{H}, T) \ni \mathfrak{f} = (f^{(j)})_{|j| \leq [\kappa]} \longmapsto f^{(0)} = f \in \text{Lip}^\kappa(\mathbb{R}^n, T).$$

Via this identification we have $\partial^j f = f^{(j)}$. Similarly, for $m \in \mathbb{N}$, the Fréchet space $\mathcal{C}^m(\overline{H}, T)$ can be identified with the Fréchet space of T -valued Whitney jets of class \mathcal{C}^m on \overline{H} (see [8, Section 1.1] for the Fréchet structure on the space of Whitney jets of class \mathcal{C}^m). By Whitney extension Theorem for Lipschitz spaces, the original Whitney extension for \mathcal{C}^k -spaces ([8, 36]) and Seeley's extension theorem [30] for \mathcal{C}^∞ -spaces, we have

PROPOSITION A.4. — *For $\kappa \in (0, +\infty) \setminus \mathbb{N}$ there exists a continuous extension operator $\text{Lip}^\kappa(\overline{H}, T) \rightarrow \text{Lip}^\kappa(\mathbb{R}^n, T)$. For $m \in \mathbb{N} \cup \{\infty\}$ there exists a continuous extension operator $\mathcal{C}^m(\overline{H}, T) \rightarrow \mathcal{C}^m(\mathbb{R}^n, T)$.*

Put $\mathbb{R}_\pm^n := \mathbb{R}^{n-1} \times \mathbb{R}_\pm$, where $\mathbb{R}_\pm := \pm[0, +\infty)$.

LEMMA A.5. — *Let $\alpha \in (0, 1)$, $M_\pm \in \mathbb{R}_+$ and let $F_\pm : \mathbb{R}_\pm^n \rightarrow T$ be such that $F_+|_{\mathbb{R}^{n-1} \times \{0\}} = F_-|_{\mathbb{R}^{n-1} \times \{0\}}$ and*

$$\|F_\pm(x) - F_\pm(y)\| \leq M_\pm \|x - y\|^\alpha \quad \forall (x, y) \in (\mathbb{R}_\pm^n)^2.$$

Let F be the mutual extension of F_{\pm} to \mathbb{R}^n . Then

$$\|F(x) - F(y)\| \leq 2^{1-\alpha} \max(M_-, M_+) \|x - y\|^\alpha \quad \forall (x, y) \in (\mathbb{R}^n)^2.$$

Proof. — Note first that

$$\sup_{\substack{(x,y) \in (\mathbb{R}_{\pm}^n)^2 \\ x \neq y}} \frac{1}{\|x - y\|^\alpha} \|F(x) - F(y)\| \leq \max(M_-, M_+).$$

It remains to estimate $\|F(x) - F(y)\|$ in terms of $\|x - y\|^\alpha$ when $x = x_+ \in \mathbb{R}_+^n$ and $y = x_- \in \mathbb{R}_-^n$. Let $x_0 \in [x_-, x_+] \cap (\mathbb{R}^{n-1} \times \{0\})$. We have

$$\begin{aligned} \|F(x_+) - F(x_-)\| &\leq \|F(x_+) - F(x_0)\| + \|F(x_0) - F(x_-)\| \\ &\leq M_+ \|x_+ - x_0\|^\alpha + M_- \|x_0 - x_-\|^\alpha \\ &\leq \max(M_-, M_+) (\|x_+ - x_0\|^\alpha + \|x_0 - x_-\|^\alpha). \end{aligned}$$

Using standard estimates between the norms $\|\cdot\|_p$ ($1 \leq p \leq +\infty$) on \mathbb{R}^n we obtain:

$$\begin{aligned} \|x_+ - x_0\|^\alpha + \|x_0 - x_-\|^\alpha &= \|(\|x_+ - x_0\|^\alpha, \|x_0 - x_-\|^\alpha)\|_1 \\ &\leq 2^{1-\alpha} \|(\|x_+ - x_0\|^\alpha, \|x_0 - x_-\|^\alpha)\|_{\frac{1}{\alpha}} \\ &= 2^{1-\alpha} (\|x_+ - x_0\| + \|x_0 - x_-\|)^\alpha = 2^{1-\alpha} \|x_+ - x_-\|^\alpha. \quad \square \end{aligned}$$

PROPOSITION A.6. — Let $\kappa \in (0, \infty) \setminus \mathbb{N}$. Let $F_{\pm} \in \text{Lip}^\kappa(\mathbb{R}_{\pm}^n, T)$ be such that

$$\partial^j F_-|_{\mathbb{R}^{n-1} \times \{0\}} = \partial^j F_+|_{\mathbb{R}^{n-1} \times \{0\}} \text{ for } |j| \leq [\kappa],$$

and let F be the mutual extension of F_{\pm} to \mathbb{R}^n . Then $F \in \text{Lip}^\kappa(\mathbb{R}^n)$, and

$$\|F\|_{\text{Lip}^\kappa} \leq 2^{1-\alpha} \max(\|F_-\|_{\text{Lip}^\kappa}, \|F_+\|_{\text{Lip}^\kappa}).$$

Proof. — For $j \in \mathbb{N}^n$ with $|j| \leq [\kappa]$ let F^j be the mutual extension of $\partial^j F_{\pm}$ to \mathbb{R}^n . We prove that

CLAIM. — For any $j \in \mathbb{N}^n$ with $|j| \leq [\kappa] - 1$, F^j is differentiable and $\partial_i F^j = F^{j+e_i}$ for $1 \leq i \leq n$.

The claim is clear on $\mathbb{R}^n \setminus (\mathbb{R}^{n-1} \times \{0\})$, so let $y \in \mathbb{R}^{n-1} \times \{0\}$. We know that

$$\left\| F_{\pm}^j(x_{\pm}) - \sum_{|j+l| \leq [\kappa]} \frac{1}{l!} F_{\pm}^{j+l}(y)(x_{\pm} - y)^l \right\| \leq M_{\pm} \|x_{\pm} - y\|^{\kappa - |j|}$$

for $x_{\pm} \in \mathbb{R}_{\pm}^n$, where $M_{\pm} := \|F_{\pm}\|_{\text{Lip}^\kappa}$. Since $\kappa - |j| > 1$, this implies

$$\lim_{\substack{x_{\pm} \rightarrow y \\ x_{\pm} \in \mathbb{R}_{\pm}^n}} \frac{1}{\|x_{\pm} - y\|} \left\| F_{\pm}^j(x_{\pm}) - \sum_{i=1}^n F^{j+e_i}(y)(x_{\pm} - y)^i \right\| = 0.$$

Therefore

$$\lim_{x \rightarrow y} \frac{1}{\|x - y\|} \left\| F^j(x) - \sum_{i=1}^n F^{j+e_i}(y)(x - y)^i \right\| = 0,$$

which proves the claim.

Using the claim it follows by induction that F is $[\kappa]$ times differentiable, and $\partial^j F = F^j$ for $0 \leq |j| \leq [\kappa]$, in particular

$$\sup_{\mathbb{R}^n} \|\partial^j F\| \leq \max \left(\sup_{\mathbb{R}_-^n} \|\partial^j F_-\|, \sup_{\mathbb{R}_+^n} \|\partial^j F_+\| \right) \leq \max(M_+, M_-).$$

To complete the proof it suffices to apply Lemma A.5 to the maps $\partial^j F_\pm$ for $|j| = [\kappa]$. \square

PROPOSITION A.7. — *Let $\kappa \in (0, +\infty) \setminus \mathbb{N}$. There exists continuous operators*

$$E_\kappa : \bigoplus_{0 \leq s \leq [\kappa]} \text{Lip}^{\kappa-s}(\mathbb{R}^{n-1}, T) \longrightarrow \text{Lip}^\kappa(\mathbb{R}^n, T)$$

with the following property: putting $A := E_\kappa((A_s)_{0 \leq s \leq [\kappa]})$ we have

$$\forall x' \in \mathbb{R}^{n-1}, \quad \partial_n^s A(x', 0) = A_s(x') \quad (\text{A.6})$$

for $0 \leq s \leq [\kappa]$. Similarly, for any $k \in \mathbb{N}$ there exists a continuous operator

$$F_k : \bigoplus_{0 \leq s \leq k} \mathcal{C}^{k-s}(\mathbb{R}^{n-1}, T) \longrightarrow \mathcal{C}^k(\mathbb{R}^n, T)$$

such that (A.6) holds for $0 \leq s \leq k$. In both cases A is \mathcal{C}^∞ on $\mathbb{R}^n \setminus (\mathbb{R}^{n-1} \times \{0\})$.

Proof. — Let $(A_s)_{0 \leq s \leq [\kappa]} \in \bigoplus_{0 \leq s \leq [\kappa]} \text{Lip}^{\kappa-s}(\mathbb{R}^{n-1}, T)$. For any $j = (j_1, \dots, j_n) = (j', j_n) \in \mathbb{N}^n$ with $|j| = |j'| + j_n \leq [\kappa]$ let $a^{(j)} \in \mathcal{C}^0(\mathbb{R}^{n-1} \times \{0\}, T)$ be given by

$$a^{(j)}(x', 0) := \partial^{j'} A_{j_n}(x'). \quad (\text{A.7})$$

We prove first that

CLAIM. — *The system $\mathbf{a} = (a^{(j)})_{0 \leq |j| \leq [\kappa]}$ belongs to $\text{Lip}_{\mathbb{R}^n}^\kappa(\mathbb{R}^{n-1} \times \{0\}, T)$ and*

$$(A_s)_{0 \leq s \leq [\kappa]} \longmapsto \mathbf{a} := (a^{(j)})_{0 \leq |j| \leq [\kappa]}$$

defines a continuous operator

$$L_\kappa : \bigoplus_{0 \leq s \leq [\kappa]} \text{Lip}^{\kappa-s}(\mathbb{R}^{n-1}, T) \longrightarrow \text{Lip}_{\mathbb{R}^n}^\kappa(\mathbb{R}^{n-1} \times \{0\}, T).$$

Indeed, since $A_{j_n} \in \text{Lip}^{\kappa-j_n}(\mathbb{R}^{n-1})$ by assumption, we have the estimates

$$\sup_{\mathbb{R}^{n-1} \times \{0\}} \|a^{(j)}\| \leq \sup_{\mathbb{R}^{n-1}} \|\partial^{j'} A_{j_n}\| \leq M_{j_n} := \|A_{j_n}\|_{\text{Lip}^{\kappa-j_n}} \text{ for } |j| \leq [\kappa]. \quad (\text{A.8})$$

On the other hand for any $x', y' \in \mathbb{R}^{n-1}$ and $j = (j', j_n)$ with $|j'| + j_n \leq [\kappa]$ we have

$$\begin{aligned} R_{j,a}^{[\kappa]}((x', 0), (y', 0)) &= a^{(j)}(x', 0) - \sum_{|j+l| \leq [\kappa]} \frac{a^{(j+l)}(y')}{l!} (x' - y', 0)^l \\ &= \partial^{j'} A_{j_n}(x') - \sum_{|j'|+|l'| \leq [\kappa]-j_n} \frac{1}{l'!} \partial^{j'+l'} A_{j_n}(y') (x' - y')^{l'} \\ &= R_{\partial^{j'} A_{j_n}}^{[\kappa]-j_n-|j'|}(x', y'). \end{aligned} \quad (\text{A.9})$$

For the second equality we took into account that $(x' - y', 0)^l = 0$ for all $l = (l', l_n)$ with $l_n > 0$. Since $A_{j_n} \in \text{Lip}^{\kappa-j_n}(\mathbb{R}^{n-1}, T)$, Remark A.1 gives estimates of the form

$$\|R_{\partial^{j'} A_{j_n}}^{[\kappa]-j_n-|j'|}(x', y')\| \leq M_{j'}^{j_n} \|x - y\|^{\kappa-j_n-|j'|}, \quad (\text{A.10})$$

which gives $\|R_{j,a}^{[\kappa]}((x', 0), (y', 0))\| \leq M_{j'}^{j_n} \|(x', 0) - (y', 0)\|^{\kappa-|j|}$. Therefore

$$\|a\|_{\text{Lip}^\kappa} \leq \max\{M_{j'}^s \mid 0 \leq |j'| + s \leq [\kappa]\},$$

which proves the claim.

For the first statement it suffices to put $E_\kappa = \mathcal{E}_\kappa \circ L_\kappa$, where

$$\mathcal{E}_\kappa : \text{Lip}_{\mathbb{R}^n}^\kappa(\mathbb{R}^{n-1} \times \{0\}, T) \longrightarrow \text{Lip}^\kappa(\mathbb{R}^n, T)$$

is Whitney's extension operator given by Theorem A.3.

For the second statement we prove that formula (A.7) for $|j| \leq m$ defines a continuous operator from $\bigoplus_{0 \leq s \leq m} \mathcal{C}^{m-s}(\mathbb{R}^{n-1}, T)$ to the Fréchet space of Whitney jets of class \mathcal{C}^m on $\mathbb{R}^{n-1} \times \{0\}$ (see [8, Section 1.1]), and we use Whitney's original extension theorem for \mathcal{C}^m jets. Replacing $[\kappa]$ by m in (A.8), (A.9), and using (A.3), we obtain for any compact $K \subset \mathbb{R}^{n-1}$ estimates of the form:

$$\begin{aligned} \sup_{K \times \{0\}} \|a^{(j)}\| &\leq \sup_K \|\partial^{j'} A_{j_n}\| \text{ for any } j = (j', j_n) \in \mathbb{N}^n \text{ with } |j| \leq m, \\ q_{m,t,K}(a) &:= \sup \left\{ \left\| \frac{R_{j,a}^m((x', 0), (y', 0))}{\|x' - y'\|^{m-j}} \right\| \mid x', y' \in K, 0 < \|x' - y'\| \leq t, \right. \\ &\quad \left. |j| \leq m \right\} \\ &\leq c \sup \left\{ \left\| \partial^{j'} A_s(x') - \partial^{j'} A_s(y') \right\| \mid x', y' \in K, \|x' - y'\| \leq t, \right. \\ &\quad \left. 0 \leq s \leq m, |j'| = m - s \right\}. \end{aligned}$$

This shows that $\lim_{t \rightarrow 0} q_{m,t,K}(\mathbf{a}) = 0$ and gives estimates for $\sup_{K \times \{0\}} \|a^{(j)}\|$, $\sup_{t > 0} q_{m,t,K}$ in terms of $\sup_K \|\partial^{j'} A_s\|$, $|j'| \leq m - s$. \square

Using Whitney extension theorem for \mathcal{C}^∞ maps [36], we obtain in a similar way:

PROPOSITION A.8. — *For any $(A_s)_{s \in \mathbb{N}} \in \mathcal{C}^\infty(\mathbb{R}^{n-1}, T)^\mathbb{N}$ there exists $A \in \mathcal{C}^\infty(\mathbb{R}^n, T)$ such that*

$$\partial_n^s A(x', 0) = A_s(x') \text{ for } x' \in \mathbb{R}^{n-1}, s \in \mathbb{N}. \quad (\text{A.11})$$

Remark A.9. — Proposition A.8 gives a map $\mathcal{C}^\infty(\mathbb{R}^{n-1}, T)^\mathbb{N} \ni (A_s)_{s \in \mathbb{N}} \mapsto A \in \mathcal{C}^\infty(\mathbb{R}^n, T)$ satisfying (A.6), but such a map can no longer be given by a continuous operator [8].

COROLLARY A.10. — *Let $(a_l)_{l \in \mathbb{N}}$ be a sequence of $\mathcal{C}^\infty(\mathbb{R}^n, T)$ such that for any $l \in \mathbb{N}_{\geq 1}$ and any $s \in \mathbb{N}$ with $s \leq l - 1$ we have $\partial_n^s a_l|_{\mathbb{R}^{n-1} \times \{0\}} = 0$. There exists $a \in \mathcal{C}^\infty(\mathbb{R}^n, T)$ such that for any $m \in \mathbb{N}$ and any $s \in \mathbb{N}$ with $s \leq m$ we have $\partial_n^s(a - \sum_{l=0}^m a_l)|_{\mathbb{R}^{n-1} \times \{0\}} = 0$.*

Proof. — Apply Proposition A.8 to the sequence $(A_s)_{s \in \mathbb{N}} \in \mathcal{C}^\infty(\mathbb{R}^{n-1}, T)^\mathbb{N}$, where

$$A_s(x') := \sum_{l \geq 0} \partial_n^s a_l(x', 0) = \sum_{l=0}^s \partial_n^s a_l(x', 0). \quad \square$$

Propositions A.6, A.7, A.8 can be generalized for sections in vector bundles on manifolds as follows. Let U be an n -dimensional differentiable manifold and E a \mathcal{C}^∞ \mathbb{K} -vector bundle of rank r on U , where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let \mathcal{A}_U be the set of all charts (the maximal atlas) of U and \mathcal{T}_E the set of local trivializations of E . For $\theta : E_V \rightarrow V \times \mathbb{K}^r \in \mathcal{T}_E$ we put $\theta' := p_{\mathbb{K}^r} \circ \theta : V \rightarrow \mathbb{K}^r$.

DEFINITION A.11. — *Let $\kappa \in (0, +\infty) \setminus \mathbb{N}$. We define*

$$\begin{aligned} \mathcal{C}^\kappa(U, T) &:= \left\{ f \in \mathcal{C}^0(U, T) \left| \begin{array}{l} (\chi f|_V) \circ h^{-1} \in \text{Lip}^\kappa(\mathbb{R}^n, T) \\ \text{for any } V \xrightarrow{h} W \in \mathcal{A}_U, \chi \in \mathcal{C}_c^\infty(V, \mathbb{R}) \end{array} \right. \right\}, \\ \Gamma^\kappa(U, E) &:= \left\{ \sigma \in \Gamma^0(U, E) \left| \begin{array}{l} (\chi \theta' \circ \sigma|_V) \circ h^{-1} \in \text{Lip}^\kappa(\mathbb{R}^n, \mathbb{K}^r) \\ \text{for any } V \xrightarrow{h} W \in \mathcal{A}_U, E_V \xrightarrow{\theta} V \times \mathbb{K}^r \in \mathcal{T}_E, \\ \chi \in \mathcal{C}_c^\infty(V, \mathbb{R}) \end{array} \right. \right\}. \end{aligned}$$

Similarly, for a manifold with boundary \bar{U} and a \mathcal{C}^∞ vector bundle E on \bar{U} , the spaces $\mathcal{C}^\kappa(\bar{U}, T)$, $\Gamma^\kappa(\bar{U}, E)$ are defined by the same formulae, but using charts with values in open sets $W \subset \mathbb{R}_+^n$ and the Lipschitz spaces $\text{Lip}^\kappa(\mathbb{R}_+^n, T)$ defined above.

Note that $\mathcal{C}^\kappa(U, T)$, $\mathcal{C}^\kappa(\bar{U}, T)$, $\Gamma^\kappa(U, E)$, $\Gamma^\kappa(\bar{U}, E)$, are naturally Fréchet spaces; they become Banach spaces (in the sense that their topology can be

defined by a single norm) when U , respectively \bar{U} is compact. Definition A.11 is in accordance with Palais' formalism [26, Section 7] and with the definition of the spaces Λ_α for manifolds with boundary [10, Section 14.a]. In particular

Remark A.12. — A section $\sigma \in \Gamma^0(U, E)$ ($\sigma \in \Gamma^0(\bar{U}, E)$) belongs to $\Gamma^\kappa(U, E)$ ($\Gamma^\kappa(\bar{U}, E)$) if and only if for every $x \in U$ ($x \in \bar{U}$) there exists a compact n -dimensional submanifold with boundary $\bar{W} \subset U$ ($\bar{W} \subset \bar{U}$) which is a neighborhood of x in U (in \bar{U}) such that $\sigma|_{\bar{W}} \in \Gamma^\kappa(\bar{W}, E)$.

Let $S \subset U$ be a smooth, closed hypersurface and let $n_S^* \subset T_{U|S}^*$ be the conormal line bundle of S in U ; this line bundle coincides with the annihilator of T_S in the restriction $T_{U|S}^*$ of the contangent bundle T_U^* of U to S .

Let $l, m \in \mathbb{N}$ with $l \leq m$. Let $\sigma \in \Gamma^m(U, E)$. We'll say that order l jet of σ along S vanishes, and we write $j_S^l \sigma = 0$, if the order l jet $j_x^l \sigma$ of σ at x vanishes for any $x \in S$. If this is the case and $l < m$, the intrinsic derivative $D_S^{l+1} \sigma \in \Gamma^{m-l-1}(S, n_S^{*\otimes(l+1)} \otimes E_S)$ or order $l+1$ is defined, and $D_S^{l+1} \sigma = 0$ if and only if $j_S^{l+1} \sigma = 0$ (Section A.6 for details).

COROLLARY A.13.

- (1) Let $\kappa \in [0, +\infty)$ and $m \in \mathbb{N}$ with $m \leq [\kappa]$. There exists a continuous operator

$$E_{S,m}^\kappa : \Gamma^{\kappa-m}(S, n_S^{*\otimes m} \otimes E_S) \longrightarrow \Gamma^\kappa(U, E)$$

such that, for any $b \in \Gamma^{\kappa-m}(S, n_S^{*\otimes m} \otimes E_S)$, putting $\sigma := E_{S,m}^\kappa(b)$, we have

$$j_S^{m-1} \sigma = 0 \text{ (if } m \geq 1), \quad D_S^m \sigma = b, \quad (\text{A.12})$$

and $\sigma|_{U \setminus S} \in \Gamma^\infty(U \setminus S, E)$.

- (2) Let $m \in \mathbb{N}$. For any $b \in \Gamma^\infty(S, n_S^{*\otimes m} \otimes E_S)$ there exists $\sigma \in \Gamma^\infty(U, E)$ such that (A.12) holds.
- (3) Let $(a_l)_{l \in \mathbb{N}}$ be a sequence of $\Gamma^\infty(U, E)$ such that $j_S^{l-1} a_l = 0$ for any $l \geq 1$. There exists $a \in \Gamma^\infty(U, E)$ such that $j_S^m(a - \sum_{l=0}^m a_l) = 0$ for any $m \in \mathbb{N}$.

Proof. — Let us first prove (1). Put $E_m^\kappa := E^\kappa \circ e_m^\kappa$, $F_m^k := F^k \circ f_m^k$, where

$$\begin{aligned} e_m^\kappa : \text{Lip}^{\kappa-m}(\mathbb{R}^{n-1}, T) &\longrightarrow \bigoplus_{0 \leq s \leq [\kappa]} \text{Lip}^{\kappa-s}(\mathbb{R}^{n-1}, T), \\ f_m^k : \mathcal{C}^{k-m}(\mathbb{R}^{n-1}, T) &\longrightarrow \bigoplus_{0 \leq s \leq k} \mathcal{C}^{k-s}(\mathbb{R}^{n-1}, T) \end{aligned} \quad (\text{A.13})$$

are the obvious embeddings. Let $(V_i \xrightarrow{h_i} \mathbb{R}^n)_{i \in I}$ be a system of charts of U and $(E_{V_i} \xrightarrow{\theta_i} V_i \times \mathbb{R}^n)_{i \in I}$ a system of trivializations of E such that

- (1) The family of open sets $(V_i)_{i \in I}$ is locally finite and $\bigcup_{i \in I} V_i \supset S$.
- (2) $\overline{V_i}$ is compact and $h(V_i \cap S) = \mathbb{R}^{n-1} \times \{0\}$ for any $i \in I$.

Via the identifications provided by h_i and θ_i , the operators E_m^κ, F_m^κ give operators

$$E_{S,m,i}^\kappa : \Gamma_c^{\kappa-m}(S \cap V_i, n_S^{*\otimes m} \otimes E_S) \longrightarrow \Gamma^\kappa(V_i, E)$$

satisfying (A.12). The point here is that the intrinsic derivative D_S^m , on sections whose $m-1$ jet along S vanishes, is compatible with vector bundle isomorphisms and diffeomorphic base changes. Let $(\varphi_i : S \rightarrow [0, 1])_{i \in I}$ be a smooth partition of unity on S which is subordinate to the cover $(S \cap V_i)_{i \in I}$ and let, for any $i \in I$, $\chi_i : U \rightarrow [0, 1]$ be a smooth function on U such that $\text{supp}(\chi_i) \subset V_i$ and $\chi \equiv 1$ on a neighborhood of $\text{supp}(\varphi_i)$ (which is compact) in V_i . It suffices to put

$$E_{S,m}^\kappa(b) := \sum_{i \in I} \chi_i E_{S,m,i}^\kappa(\varphi_i b).$$

For (2) and (3) we use Proposition A.8 respectively Corollary A.10 and a similar argument. \square

COROLLARY A.14. — *Let E be a \mathcal{C}^∞ vector bundle on U , $S \subset U$ a separating closed real smooth hypersurface, and $U \setminus S = U^- \cup U^+$ a decomposition of $U \setminus S$ as union of disjoint open subsets such that $\overline{U}^\pm = U^\pm \cup S$. Put $E^\pm := E_{\overline{U}^\pm}$ and let $\kappa \in [0, +\infty]$.*

- (1) *There exists a continuous extension operator $\Gamma^\kappa(\overline{U}^+, E^+) \rightarrow \Gamma^\kappa(U, E)$.*
- (2) *There exists a continuous operator*

$$\mathcal{E}_S : \left\{ (\sigma_-, \sigma_+) \in \Gamma^\kappa(U, E) \times \Gamma^\kappa(U, E) \mid j_S^{[\kappa]}(\sigma_+ - \sigma_-) = 0 \right\} \longrightarrow \Gamma^\kappa(U, E)$$

with the property that, putting $\sigma = \mathcal{E}_S(\sigma_-, \sigma_+)$, we have $\sigma|_{\overline{U}^\pm} = \sigma_\pm|_{\overline{U}^\pm}$.

Proof. — (1) follows from Proposition A.4 using a partition of unity. (2) follows from Proposition A.6 for $\kappa \in (0, +\infty) \setminus \mathbb{N}$ and from a similar gluing principle for \mathcal{C}^m maps if $\kappa = m \in \mathbb{N} \cup \{\infty\}$. \square

A.2. The fiberwise exponential map

Let M be a differentiable manifold, G a Lie group and $p : P \rightarrow M$ a \mathcal{C}^∞ principal G -bundle on M . Let $\iota(\text{Ad})$ be the interior (adjoint) action of G on

itself (on its Lie algebra \mathfrak{g}). Put $\iota(P) := P \times_{\iota} G$, $\text{Ad}(P) := P \times_{\text{Ad}} \mathfrak{g}$. Using Palais's formalism for spaces of sections in locally trivial fiber bundles [26, p. 38], we have:

PROPOSITION A.15. — *Let $\gamma \in [0, \infty]$.*

- (1) *The fiberwise exponential map $\exp : \text{Ad}(P) \rightarrow \iota(P)$ maps $\Gamma^{\gamma}(M, \text{Ad}(P))$ into $\Gamma^{\gamma}(M, \iota(P))$.*
- (2) *There exists an Euclidean structure h on $\text{Ad}(P)$ such that \exp maps diffeomorphically the unit disk bundle $\text{Ad}(P)_0 := \{\xi \in \text{Ad}(P) \mid \|\xi\|_h < 1\}$ with respect to h onto an open neighborhood $\iota(P)_0$ of the identity section id_P in $\iota(P)$. For any such h , the map \exp induces a bijection $\Gamma^{\gamma}(M, \text{Ad}(P)_0) \rightarrow \Gamma^{\gamma}(M, \iota(P)_0)$.*

Proof.

(1). — The map $\exp : \text{Ad}(P) \rightarrow \iota(P)$ is fiber bundle morphism between locally trivial fiber bundles in the sense of [26, Section 10]. The claim follows from [26, Theorem 13.4] taking as base manifold closures $\overline{M}' \subset M$ of relatively compact open submanifolds $M' \subset M$ with smooth boundary.

(2). — The map \exp maps diffeomorphically the zero section $0_{\text{Ad}(P)} \subset \text{Ad}(P)$ onto $\text{id}_P \subset \iota(P)$ and is fiberwise locally invertible at the points of $0_{\text{Ad}(P)}$. By the relative Inverse Function Theorem [13, Exercice 14, Section 1.§8] we obtain an open neighborhood U of $0_{\text{Ad}(P)}$ in $\text{Ad}(P)$ such that $\exp(U)$ is open in $\iota(P)$ and the induced map $U \rightarrow \exp(U)$ is a diffeomorphism. It suffices to choose an Euclidian structure h on $\text{Ad}(P)$ such that the unit disk bundle with respect to h is contained in U .

For the second claim of (2), note that $\exp : \text{Ad}(P)_0 \rightarrow \iota(P)_0$ becomes an isomorphism of \mathcal{C}^{∞} fiber bundles in the sense of [26, Section 10], so the claim follows again by [26, Theorem 13.4]. \square

COROLLARY A.16. — *Let S be a differentiable manifold and P^{\pm} , \mathcal{C}^{∞} principal G -bundles on $S \times \mathbb{R}$. Identify S with $S \times \{0\}$ and let $v : P_S^{-} \rightarrow P_S^{+}$ be a bundle isomorphism of class \mathcal{C}^{γ} . There exists a bundle isomorphism extension $\tilde{v} : P^{-} \rightarrow P^{+}$ of class \mathcal{C}^{γ} of v which is \mathcal{C}^{∞} on $S \times \mathbb{R}^*$.*

Proof. — Let A^{\pm} be a connection of class \mathcal{C}^{∞} on P^{\pm} . Parallel transport with respect to A^{\pm} alongs paths of the form $t \mapsto (u, t)$, $u \in S$, gives \mathcal{C}^{∞} bundle isomorphisms $f^{\pm} : P^{\pm} \xrightarrow{\sim} P_S^{\pm} \times \mathbb{R}$.

The bundles P_S^{-} , P_S^{+} on S are topologically isomorphic, so they are also \mathcal{C}^{∞} isomorphic. Therefore we may suppose $P^{-} = P^{+} = P_S \times \mathbb{R} =: P$ (regarded as bundle on $S \times \mathbb{R}$), where P_S is a \mathcal{C}^{∞} principal G -bundle on S .

The bundle isomorphism v can then be regarded as an element of $\Gamma^\gamma(S, \iota(P_S))$. Let $\iota(P_S)_0$ be an open neighborhood of id in $\iota(P_S)$ as in Proposition A.15. There exists a smooth section $\sigma \in \Gamma^\infty(S \times \mathbb{R}, \iota(P))$ such that $\sigma|_S$ takes values in the disk bundle neighborhood $\iota(P_S)_0 v$ of v . This follows using the density of \mathcal{C}^∞ with respect to the strong \mathcal{C}^0 -topology (see [16, Section 2.1], [16, Theorem 2.6], [16, Exercice 3 p. 56]).

Therefore we have $v = \phi^{-1} \sigma|_S$, where $\phi \in \Gamma^\gamma(S, \iota(P_S)_0)$, because v is of class \mathcal{C}^γ and $\sigma|_S$ of class \mathcal{C}^∞ . Making use of Proposition A.15, let $\psi \in \mathcal{C}^\gamma(S, \text{Ad}(P_S)_0)$ be such that $\phi = \exp(\psi)$. By Corollary A.13 there exists an extension $\tilde{\psi} \in \Gamma^\gamma(S \times \mathbb{R}, \text{Ad}(P))$ of ψ which is \mathcal{C}^∞ on $S \times \mathbb{R}^*$. It suffices to put $\tilde{v} = \exp(-\tilde{\psi})\sigma$. \square

A.3. Gluing bundles along a hypersurface

Let U be a differentiable manifold, $S \subset U$ a separating closed real smooth hypersurface, and $U \setminus S = U^- \cup U^+$ a decomposition of $U \setminus S$ as union of disjoint open subsets such that $\bar{U}^\pm = U^\pm \cup S$. Let P^\pm be a \mathcal{C}^∞ principal G bundle on \bar{U}^\pm , $\gamma \in [0, \infty]$, $v : P_S^- \rightarrow P_S^+$ an isomorphism of class \mathcal{C}^γ , and let $P^v := P^- \amalg_v P^+$ be the topological bundle obtained by gluing P^\pm along S via v . P^v comes with obvious identifications $P^\pm \rightarrow P^v|_{\bar{U}^\pm}$.

DEFINITION A.17. — *A \mathcal{C}^∞ structure \mathfrak{S} on $P^v := P^- \amalg_v P^+$ will be called admissible if, denoting by $P_\mathfrak{S}^v$ the corresponding \mathcal{C}^∞ principal G -bundle, the obvious identifications $P^\pm \rightarrow P_\mathfrak{S}^v|_{\bar{U}^\pm}$ become bundle isomorphisms of class \mathcal{C}^γ on \bar{U}^\pm .*

Let $\text{Aut}^0(P^v) \simeq \Gamma^0(U, P^v \times_\iota G)$ be the gauge group of the topological bundle P^v and $\text{Aut}^0(P^v)_a$ be the subgroup of $\text{Aut}^0(P^v)$ whose elements are the bundle automorphisms $F \in \text{Aut}^0(P^v)$ which induce automorphisms of class \mathcal{C}^γ on P^\pm .

PROPOSITION A.18. — *The set \mathcal{S}_a of admissible \mathcal{C}^∞ structures on P^v is non-empty. The group $\text{Aut}^0(P^v)_a$ acts transitively on \mathcal{S}_a . The stabilizer of an element $\mathfrak{S} \in \mathcal{S}_a$ coincide with the gauge group*

$$\text{Aut}^\infty(P_\mathfrak{S}^v) \simeq \Gamma^\infty(U, P_\mathfrak{S}^v \times_\iota G)$$

of the \mathcal{C}^∞ bundle $P_\mathfrak{S}^v$.

Proof. — Let

$$S \times \mathbb{R} \xrightarrow{\nu \simeq} N \hookrightarrow U$$

be a \mathcal{C}^∞ tubular neighborhood of S in U such that $\nu(S \times \mathbb{R}_\pm) = N \cap \bar{U}^\pm$. Let $q : N \rightarrow S$, $q_\pm : N \cap \bar{U}^\pm \rightarrow S$ be the projections induced by the obvious projections $S \times \mathbb{R} \rightarrow S$, $S \times \mathbb{R}_\pm \rightarrow S$.

Put $\tilde{U}^\pm := U^\pm \cup N$, and let \tilde{P}^\pm be a \mathcal{C}^∞ bundle on \tilde{U}^\pm which extends P^\pm . One obtains easily such an extension by choosing a connection A^\pm of class \mathcal{C}^∞ on P^\pm and noting that parallel transport alongs paths of the form

$$\mathbb{R}_\pm \ni t \mapsto \nu(u, t), \quad u \in S$$

gives \mathcal{C}^∞ bundle isomorphisms $\eta^\pm : q_\pm^*(P_S^\pm) \xrightarrow{\simeq} P_{N \cap \bar{U}^\pm}^\pm$. Therefore it suffices to put $\tilde{P}^\pm := P^\pm \amalg_{\eta^\pm} q^*(P_S^\pm)$.

By Corollary A.16 there exists an extension $\tilde{v} : \tilde{P}_N^- \rightarrow \tilde{P}_N^+$ of class \mathcal{C}^γ of v which is \mathcal{C}^∞ on $N \setminus S$. Put $\tilde{v}^\pm := \tilde{v}|_{N \cap U^\pm}$. We obtain obvious bundle isomorphisms

$$\tilde{P}_{\tilde{U}^-}^- \amalg_{\tilde{v}^+} P_{U^+}^+ \xrightarrow[\simeq]{b} \tilde{P}_{\tilde{U}^-}^- \amalg_{\tilde{v}} \tilde{P}_{\tilde{U}^+}^+ \xrightarrow[\simeq]{a} P^- \amalg_v P^+ = P^v$$

over U , where $\tilde{P}_{\tilde{U}^-}^- \amalg_{\tilde{v}^+} P_{U^+}^+$ is naturally a \mathcal{C}^∞ bundle, $\tilde{P}_{\tilde{U}^-}^- \amalg_{\tilde{v}} \tilde{P}_{\tilde{U}^+}^+$ is naturally a \mathcal{C}^γ bundle, b is a bundle isomorphism of class \mathcal{C}^γ and a is a topological bundle isomorphism. The \mathcal{C}^∞ structure on P^v induced via $a \circ b$ is obviously admissible.

The other statements follow taking into account that $\text{Aut}^0(P^v)$ acts transitively on the set of \mathcal{C}^∞ structures on E^v . \square

A.4. An extension theorem

The following extension result plays a fundamental role in this article. Since I could not find it in standard complex analysis textbooks or articles, I give below a detailed proof based on the regularity of the $\bar{\partial}$ operator. My colleagues Alexandre Boritchev and Karl Oeljeklaus suggested different proofs, which use Morera Theorem (for $\dim(U) = 1$) combined with the well known theorem on separately holomorphic functions (for $\dim(U) > 1$). Another argument, suggested by Christine Laurent-Thiébaud, uses the Hartogs–Bochner extension theorem.

THEOREM A.19. — *Let U, F be complex manifolds and $S \subset U$ a closed, smooth real hypersurface. Let $f : U \rightarrow F$ be a continuous map whose restriction $f|_{U \setminus S}$ is holomorphic. Then f is holomorphic.*

Proof. — It suffices to prove that statement when $F = \mathbb{C}$ and U is open in \mathbb{C}^n , so suppose we are in this case. We will show that $\bar{\partial}f = 0$ in the weak sense around any point $x \in S$; the claim will follow by the regularity property of the $\bar{\partial}$ operator.

Let $B_R \subset \mathbb{R}^{2n}$ be the radius R ball with center $0_{\mathbb{R}^{2n}}$,

$$\bar{B}_R^\pm := \{x \in \bar{B}_R \mid \pm x_{2n} \geq 0\}.$$

For $t \in (-R, R)$, $\varepsilon > 0$ put:

$$\begin{aligned} \bar{B}_R^t &:= \{x \in \bar{B}_R \mid x_{2n} = t\}, \\ \bar{B}_{R,\varepsilon}^\pm &:= \left\{x \in \bar{B}_R^\pm \mid |x_{2n}| \geq \varepsilon\right\}, \\ \bar{B}_{R,\varepsilon} &:= \left\{x \in \bar{B}_R^\pm \mid |x_{2n}| \leq \varepsilon\right\}. \end{aligned}$$

Let $r > 0$ be sufficiently small such that $B(x, r) \subset U$ and there exists a diffeomorphism $\Psi : B(x, r) \rightarrow \mathbb{R}^{2n}$ with $\Psi(x) = 0$ and $\Psi(S \cap B(x, r)) = \mathbb{R}^{2n-1} \times \{0\}$. Let $\phi \in A^{n,n-1}(B(x, r))$ be a type $(n, n-1)$ -form with compact support $K \subset B(x, r)$, and let $R > 0$ be sufficiently large such that $\Psi(K) \subset B_R$. Then

$$\begin{aligned} \int_{B(x,r)} f \bar{\partial} \phi &= \int_{\Psi^{-1}(\bar{B}_R)} f \bar{\partial} \phi \\ &= \lim_{\varepsilon \searrow 0} \int_{\Psi^{-1}(\bar{B}_{R,\varepsilon}^+)} f \bar{\partial} \phi + \lim_{\varepsilon \searrow 0} \int_{\Psi^{-1}(\bar{B}_{R,\varepsilon}^-)} f \bar{\partial} \phi. \end{aligned} \quad (\text{A.14})$$

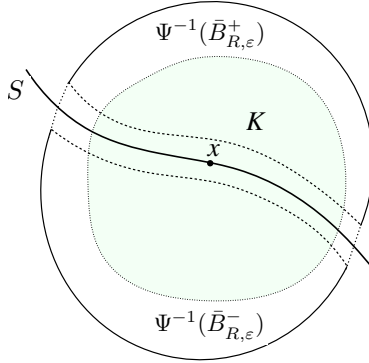


Figure A.1. $\Psi^{-1}(\bar{B}_R)$.

We have used: the measure of $\Psi^{-1}(\bar{B}_{R,\varepsilon})$ (with respect to any Riemannian metric on U) tends to 0 as $\varepsilon \rightarrow 0$. Applying Stokes Theorem to the

form $f\phi$ on the manifold with corners $\Psi^{-1}(\bar{B}_{R,\varepsilon}^\pm)$ (on which f is smooth), we obtain

$$\int_{\Psi^{-1}(\bar{B}_{R,\varepsilon}^\pm)} f\bar{\partial}\phi = - \int_{\Psi^{-1}(\bar{B}_{R,\varepsilon}^\pm)} \bar{\partial}f \wedge \phi + \int_{\partial\Psi^{-1}(\bar{B}_{R,\varepsilon}^\pm)} f\phi = \int_{\Psi^{-1}(\partial\bar{B}_{R,\varepsilon}^\pm)} f\phi,$$

because $\bar{\partial}f = 0$ on $\Psi^{-1}(\bar{B}_{R,\varepsilon}^\pm) \subset U \setminus S$. Endowing B_R^t with the orientation induced from $\mathbb{R}^{2n-1} \times \{0\}$ regarded as boundary of $\mathbb{R}^{2n-1} \times [0, +\infty)$, and noting that ϕ vanishes on $\Psi^{-1}(\partial\bar{B}_R)$, we obtain (see Figure A.1):

$$\lim_{\varepsilon \searrow 0} \int_{\Psi^{-1}(\bar{B}_{R,\varepsilon}^\pm)} f\bar{\partial}\phi = \pm \lim_{\varepsilon \searrow 0} \int_{\Psi^{-1}(\bar{B}_{R,\varepsilon}^\pm)} f\phi = \pm \int_{\Psi^{-1}(\bar{B}_R^0)} f\phi,$$

so, by (A.14), we get $\int_{B(x,r)} f\bar{\partial}\phi = 0$. Therefore $\bar{\partial}f = 0$ around x in distribution sense. \square

COROLLARY A.20. — *Let U, F be complex manifolds and $p : \mathfrak{F} \rightarrow U$ a holomorphic locally trivial fiber bundle with standard fiber F . Let $S \subset U$ be a closed, smooth real hypersurface, and $\mathfrak{f} : U \rightarrow \mathfrak{F}$ a continuous section which is holomorphic on $U \setminus S$. Then \mathfrak{f} is holomorphic.*

Proof. — This follows from Theorem A.19 using local trivializations around the points of S . \square

A.5. Dolbeault operators and bundle almost complex structures

We begin by recalling the well known formalism of Dolbeault operators (semi-connections) on complex vector bundles.

A.5.1. Dolbeault operators on complex vector bundles

Let U be a complex manifold and E a differentiable complex vector bundle of rank r on U . A Dolbeault operator (semi-connection) on E is a first order differential operator

$$\delta : A^0(U, E) \longrightarrow A^{0,1}(U, E)$$

satisfying the Leibniz rule $\delta(f\sigma) = \bar{\partial}f\sigma + f\delta\sigma$ (see for instance [7, Section 2.2.2], [21], [22], [33]). Such an operator has natural extensions $A^{0,q}(U, E) \rightarrow A^{0,q+1}(U, E)$; the square $\delta^2 : A^0(U, E) \rightarrow A^{0,2}(U, E)$ is an order 0 operator, so it corresponds to an endomorphism valued form $F_\delta \in A^{0,2}(U, \text{End}(E))$. By the bundle version of the Nirenberg–Newlander theorem (see Griffiths [11, Proposition p. 419] (see also [2, Theorem 5.1], [19, Proposition I.3.7], [7, Theorem 2.1.53]) the $\text{End}(E)$ -valued $(0,2)$ -form F_δ is

the obstruction to the integrability of δ . More precisely $F_\delta = 0$ if and only if around any point $x \in U$ there exists a local frame $(\theta_1, \dots, \theta_r)$ with $\delta\theta_i = 0$. If this is the case, δ defines a holomorphic structure \mathfrak{h}_δ on E characterized by the condition: a local section s of E is \mathfrak{h}_δ -holomorphic if and only if $\delta s = 0$.

Let now $U^+ \subset U$ be an open set whose closure \bar{U}^+ is a smooth submanifold with boundary, i.e. $\bar{U}^+ = U^+ \cup S$, where S is an oriented real hypersurface of U and $\partial\bar{U}^+ = S$. Put $E^+ := E|_{\bar{U}^+}$, $E_S := E|_S$.

A Dolbeault operator $\delta : A^0(\bar{U}^+, E^+) \rightarrow A^{0,1}(\bar{U}^+, E^+)$ on E^+ and its associated form $F_\delta \in A^{0,2}(\bar{U}^+, \text{End}(E))$ are defined in the same way as for bundles on manifolds without boundary, but, in general, the analogue of the Newlander–Nirenberg Theorem *does not* hold at boundary points [32, Proposition 1.5, Corollary 2.3]. For this reason a Dolbeault operator δ on E^+ satisfying the condition $F_\delta = 0$ will be called *formally integrable* (not integrable). Similarly, for a formally integrable Dolbeault operator δ on E^+ and an open set $V \subset \bar{U}^+$, a section $\sigma \in \Gamma(V, E^+)$ will be called *formally δ -holomorphic* if $\delta\sigma = 0$. This condition implies holomorphy at interior points, but, in general, not at boundary points (not even at boundary points around which a formally δ -holomorphic frame exists).

In this article we make use of a refinement of the above Newlander–Nirenberg for Dolbeault operators with coefficients in \mathcal{C}^κ for $\kappa \in (0, +\infty] \setminus \mathbb{N}$. This result is a special case of the Newlander–Nirenberg theorem for bundle ACS of class \mathcal{C}^κ on principal bundles [33] which will be recalled briefly in the next section.

A.5.2. Bundle almost complex structures on principal bundles

Let G be a complex Lie group, \mathfrak{g} its Lie algebra and $\theta \in A^1(G, \mathfrak{g})$ the canonical left invariant \mathfrak{g} -valued 1-form on G [20, p. 41]. Let $p : P \rightarrow U$ be a principal G -bundle of class \mathcal{C}^∞ on U . Let $\kappa \in [0, +\infty]$.

DEFINITION A.21. — *A bundle almost complex structure (bundle ACS) of class \mathcal{C}^κ on P is an almost complex structure J of class \mathcal{C}^κ on P which makes the G -action $P \times G \rightarrow P$ and the map $p : P \rightarrow U$ pseudo-holomorphic.*

Let \mathcal{J}_P^κ be the space of bundle ACS of class \mathcal{C}^κ on P and let \mathcal{A}_P^κ be the space of sections $\alpha \in \Gamma^\kappa(P, p_*^{-1}(T_U^{0,1})^* \otimes \mathfrak{g}^{1,0})$ satisfying the conditions:

- (Pa) α is invariant with respect to the G action $g \rightarrow {}^tR_{g*} \otimes \text{Ad}_g$ on $p_*^{-1}(T_U^{0,1})^* \otimes \mathfrak{g}^{1,0}$.

$$(Pb) \quad \alpha(a_y^\#) = a^{1,0} \text{ for any } y \in P \text{ and } a \in \mathfrak{g}^\mathbb{C} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}.$$

Here we used the notation $a^\#$ for the vertical vector field associated with a . For any $J \in \mathcal{J}_P^\kappa$ there exists a unique $\alpha_J \in \mathcal{A}_P^\kappa$ such that $T_{P,J}^{0,1} = \ker(\alpha_J)$ and the map

$$\mathcal{J}_P^\kappa \ni J \longmapsto \alpha_J \in \mathcal{A}_P^\kappa$$

is a bijection [33]. Via this bijection \mathcal{J}_P^κ gets the natural structure of an affine space with model space $A_{\text{Ad}}^{0,1}(P, \mathfrak{g}^{1,0})_\kappa$, the space of $\mathfrak{g}^{1,0}$ -valued tensorial forms of type Ad [20, Section II.5], class \mathcal{C}^κ and bidegree $(0,1)$ on P . This space can be further identified [20, p. 76] with the space $A^{0,1}(U, P \times_{\text{Ad}} \mathfrak{g}^{1,0})_\kappa$ of forms of class \mathcal{C}^κ and bidegree $(0,1)$ with values in the associated vector bundle $P \times_{\text{Ad}} \mathfrak{g}^{1,0}$. Identifying $\mathfrak{g}^{1,0}$ with \mathfrak{g} in the standard way, we conclude that \mathcal{J}_P^κ is naturally an affine space with model space $A^{0,1}(U, P \times_{\text{Ad}} \mathfrak{g})_\kappa = A^{0,1}(U, \text{Ad}(P))_\kappa$.

Let $J \in \mathcal{J}_P^\kappa$ with $\kappa \geq 1$. The map

$$\Gamma(P, T_{P,J}^{0,1})^2 \ni (A, B) \xrightarrow{\mathfrak{f}_J} -\alpha_J([A, B])$$

defines a $\mathfrak{g}^{1,0}$ -valued tensorial form of type $(0,2)$ and class $\mathcal{C}^{\kappa-1}$ on P hence an element $\mathfrak{f}_J \in A_{\text{Ad}}^{0,2}(P, \mathfrak{g}^{1,0})_{\kappa-1} = A_{\text{Ad}}^{0,2}(P, \mathfrak{g})_{\kappa-1} = A^{0,2}(U, \text{Ad}(P))_{\kappa-1}$.

The behavior of the map $J \mapsto \mathfrak{f}_J$ with respect to translations in the affine space \mathcal{J}_P^κ is given by the formula

$$\mathfrak{f}_{J+b} = \mathfrak{f}_J + \bar{\mathfrak{t}}_J(b), \tag{A.15}$$

where $\bar{\mathfrak{t}}_J : A_{\text{Ad}}^{0,1}(P, \mathfrak{g}^{1,0})_\kappa \rightarrow A_{\text{Ad}}^{0,2}(P, \mathfrak{g}^{1,0})_{\kappa-1}$ is defined by

$$\bar{\mathfrak{t}}_J(b) = \bar{\partial}_J b + \frac{1}{2}[b \wedge b]$$

(see [33, Proposition 2.9]). Here $\bar{\partial}_J$ stands for the Dolbeault operator on the vector bundle $P \times_{\text{Ad}} \mathfrak{g}^{1,0} \simeq \text{Ad}(P)$ associated with J .

Let $W \subset U$ be an open subset, and $\tau \in \Gamma(W, P)$ be a local section of class \mathcal{C}^∞ of P . Putting

$$\alpha_J^\tau := \tau^*(\alpha) \in A^{0,1}(W, \mathfrak{g})_\kappa$$

and, denoting by $\mathfrak{f}_J^\tau \in A^{0,2}(W, \mathfrak{g})_{\kappa-1}$ the form associated with \mathfrak{f}_J with respect to τ , we have (see [33]):

$$\mathfrak{f}_J^\tau = \bar{\partial} \alpha_J^\tau + \frac{1}{2}[\alpha_J^\tau \wedge \alpha_J^\tau].$$

This formula shows that \mathfrak{f}_J can be also defined for $\kappa \in [0,1)$ as an $\text{Ad}(P)$ -valued form of type $(0,2)$ on U with distribution coefficients.

We refer to [33] for the following principal bundle version of the Nirenberg–Newlander theorem:

THEOREM A.22 (The Nirenberg–Newlander theorem for principal bundles). — *Let G be a complex Lie group and $p : P \rightarrow U$ a differentiable principal G -bundle on U . Let J be a bundle ACS of class \mathcal{C}^κ on P with $\kappa \in (0, +\infty] \setminus \mathbb{N}$. The following conditions are equivalent:*

- (1) $\mathfrak{f}_J = 0$.
- (2) *J is integrable in the following sense: for any point $x \in U$ there exists an open neighborhood W of x and a J -pseudo-holomorphic section $s \in \Gamma^{\kappa+1}(W, P)$.*

If this is the case, J defines a bundle holomorphic reduction \mathfrak{h}_J of the underlying $\mathcal{C}^{\kappa+1}$ -bundle of P . For an open set $W \subset U$, a section $s \in \Gamma^1(W, P)$ is holomorphic with respect to \mathfrak{h}_J if and only if it is J -pseudo-holomorphic; if this is the case then $s \in \Gamma^{\kappa+1}(W, P)$.

For $\kappa \in (0, 1)$ the condition $\mathfrak{f}_J = 0$ is meant in distribution sense. We also refer to [33] for the following regularity result:

COROLLARY A.23. — *Let U be a complex manifold, G a complex Lie group, and P a principal bundle of class \mathcal{C}^∞ on P . Let J be an integrable bundle ACS of class \mathcal{C}^κ on P with $\kappa \in (0, +\infty] \setminus \mathbb{N}$, and let $G \times F \rightarrow F$ be a holomorphic action of G on a complex manifold F . The sheaf of J -holomorphic sections of the associated bundle $P \times_G F$ is contained in the sheaf of sections of class $\mathcal{C}^{\kappa+1}$.*

Let $\iota : G \rightarrow \text{Aut}(G)$ be the morphism which maps any $g \in G$ to the interior automorphism ι_g . An equivariant map $\sigma \in \mathcal{C}_\iota^{\kappa+1}(P, G)$ defines a gauge transformation $\tilde{\sigma} : P \rightarrow P$ of class $\mathcal{C}^{\kappa+1}$ of P and the map

$$\mathcal{C}_\iota^{\kappa+1}(P, G) \ni \sigma \longmapsto \tilde{\sigma} \in \mathcal{G}_P^{\kappa+1}$$

is an isomorphism onto the gauge group $\mathcal{G}_P^{\kappa+1}$ of P . The group $\mathcal{C}_\iota^{\kappa+1}(P, G)$ acts on the space \mathcal{J}_P^κ from the right by the formula

$$J \cdot \sigma := \tilde{\sigma}_*^{-1} \circ J \circ \tilde{\sigma}_*$$

and the corresponding action on \mathcal{A}_P^κ is

$$\alpha \cdot \sigma = \alpha \circ \sigma_*,$$

from which we infer the behavior of the integrability obstruction \mathfrak{f}_J with respect to the gauge symmetry of the space \mathcal{J}_P^κ :

$$\mathfrak{f}_{J \cdot \sigma} = \text{Ad}_{\sigma^{-1}}(\mathfrak{f}_J). \quad (\text{A.16})$$

We have the following formula (see [33, Proposition 2.10]) relating the affine space structure of \mathcal{J}_P^κ to its gauge symmetry:

$$J \cdot \sigma = J + \bar{\mathfrak{l}}_J(\sigma), \quad (\text{A.17})$$

where the map

$$\bar{\mathfrak{l}}_J : \mathcal{C}_l^{\kappa+1}(P, G) \longrightarrow A_{\text{Ad}}^{0,1}(P, \mathfrak{g}^{1,0})_\kappa = A^{0,1}(U, P \times_{\text{Ad}} \mathfrak{g}^{1,0})_\kappa \simeq A^{0,1}(U, \text{Ad}(P))_\kappa$$

is defined by

$$\bar{\mathfrak{l}}_J(\sigma) := \sigma^*(\theta^{1,0})_J^{0,1}.$$

Here $\theta^{1,0}$ is the holomorphic $(1,0)$ -form on G defined as the composition

$$\theta \otimes \text{id}_{\mathbb{C}} : T_G^{\mathbb{C}} \longrightarrow \mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1} \longrightarrow \mathfrak{g}^{1,0}.$$

It is useful to have an explicit formula for $\bar{\mathfrak{l}}_J$ with respect to a local trivialization (or, equivalently, local section) of P . For a local section $\tau \in \Gamma(W, P)$ of class \mathcal{C}^∞ of P put

$$\bar{\mathfrak{l}}_J^\tau(\sigma) := \tau^*(\bar{\mathfrak{l}}_J(\sigma)) \in A^{0,1}(W, \mathfrak{g}^{1,0})_\kappa.$$

We have (see [33, Lemma 2.8]):

$$\bar{\mathfrak{l}}_J^\tau(\sigma) = \sigma_\tau^*(\theta^{1,0})^{0,1} + (\text{Ad}_{\sigma_\tau^{-1}} - \text{id})(\alpha_J^\tau), \quad (\text{A.18})$$

where $\sigma_\tau := \sigma \circ \tau \in \mathcal{C}^{\kappa+1}(W, G)$. Note the following useful formula for the composition $\bar{\mathfrak{f}}_J \circ \bar{\mathfrak{l}}_J$ associated with a bundle ACS J of class \mathcal{C}^1 . For any $\sigma \in \mathcal{C}_l^\kappa(P, G)$ we have [33, Corollary 2.11]:

$$\bar{\mathfrak{f}}_J \circ \bar{\mathfrak{l}}_J(\sigma) = (\text{Ad}_{\sigma^{-1}} - \text{id})(\mathfrak{f}_J). \quad (\text{A.19})$$

Let $J, J' \in \mathcal{J}_P^\kappa$ and $\sigma \in \mathcal{C}_l^{\kappa+1}(P, G)$. We have

$$\alpha_{J' \cdot \sigma} - \alpha_{J \cdot \sigma} = \alpha_{J'} \circ \tilde{\sigma}_* - \alpha_J \circ \tilde{\sigma}_* = (\alpha_{J'} - \alpha_J) \circ \tilde{\sigma}_*.$$

Since $\alpha_{J'} - \alpha_J$ is a tensorial form of type Ad (hence it vanishes on vertical tangent and is Ad -equivariant) we obtain the formula

$$J' \cdot \sigma - J \cdot \sigma = \text{Ad}_{\sigma^{-1}}(J' - J), \quad (\text{A.20})$$

which shows that the group $\mathcal{C}_l^{\kappa+1}(P, G)$ acts on \mathcal{J}_P^κ by affine transformations and the induced linear action on the model vector space $A_{\text{Ad}}^{0,1}(P, \mathfrak{g}^{1,0})_\kappa$ is

$$(\beta, \sigma) \longmapsto \text{Ad}_{\sigma^{-1}}(\beta).$$

We will need:

LEMMA A.24. — *Let J be a bundle ACS of class \mathcal{C}^1 on P . Then*

(1) *For any $\sigma_0, \sigma_1 \in \mathcal{C}_l^1(P, G)$ we have*

$$\bar{\mathfrak{l}}_J(\sigma_1 \sigma_0) = \text{Ad}_{\sigma_0^{-1}}(\bar{\mathfrak{l}}_J(\sigma_1)) + \bar{\mathfrak{l}}_J(\sigma_0). \quad (\text{A.21})$$

(2) *For any $\sigma \in \Gamma^2(U, \iota(P))$ and $\beta \in \Gamma^1(U, \wedge^{0,1} \otimes \text{Ad}(P))$ we have:*

$$\bar{\mathfrak{f}}_J(\text{Ad}_\sigma(\beta - \bar{\mathfrak{l}}_J(\sigma))) = \text{Ad}_\sigma(\bar{\mathfrak{f}}_J(\beta)) + (\text{Ad}_\sigma - \text{id})(\mathfrak{f}_J). \quad (\text{A.22})$$

Proof.

(1). — By (A.17) and (A.20) we have

$$\begin{aligned} \mathfrak{l}_J(\sigma_1\sigma_0) &= J \cdot (\sigma_1\sigma_0) - J = J \cdot (\sigma_1\sigma_0) - J \cdot \sigma_0 + J \cdot \sigma_0 - J \\ &= (J \cdot \sigma_1) \cdot \sigma_0 - J \cdot \sigma_0 + \bar{\mathfrak{l}}_J(\sigma_0) = \text{Ad}_{\sigma_0^{-1}}(J \cdot \sigma_1 - J) + \bar{\mathfrak{l}}_J(\sigma_0) \\ &= \text{Ad}_{\sigma_0^{-1}}(\bar{\mathfrak{l}}_J(\sigma_1)) + \bar{\mathfrak{l}}_J(\sigma_0). \end{aligned}$$

(2). — Using (A.15), (A.16), and (A.20) we obtain:

$$\begin{aligned} \text{Ad}_\sigma(\bar{\mathfrak{f}}_J(\beta)) + (\text{Ad}_\sigma - \text{id})(\mathfrak{f}_J) &= \text{Ad}_\sigma(\bar{\mathfrak{f}}_J(\beta) + \mathfrak{f}_J) - \mathfrak{f}_J = \text{Ad}_\sigma(\mathfrak{f}_{J+\beta}) - \mathfrak{f}_J \\ &= \mathfrak{f}_{(J+\beta) \cdot \sigma^{-1}} - \mathfrak{f}_J = \bar{\mathfrak{f}}_J((J+\beta) \cdot \sigma^{-1} - J) \\ &= \bar{\mathfrak{f}}_J((J+\beta) \cdot \sigma^{-1} - J \cdot \sigma^{-1} + J \cdot \sigma^{-1} - J) \\ &= \bar{\mathfrak{f}}_J(\text{Ad}_\sigma(\beta) + J \cdot \sigma^{-1} - J). \end{aligned} \tag{A.23}$$

On the other hand:

$$J \cdot \sigma^{-1} - J = J \cdot \sigma^{-1} - (J \cdot \sigma) \cdot \sigma^{-1} = -\text{Ad}_\sigma(\bar{\mathfrak{l}}_J\sigma).$$

We used (A.20) with $J' = J \cdot \sigma$ and (A.17). Taking into account (A.23), this completes the proof. \square

A.5.3. The formal integrability condition on manifolds with boundary

The definitions above generalize in an obvious way for a \mathcal{C}^∞ principal G -bundle on a manifold with boundary. The regularity class of a bundle ACS J^+ on a bundle on a manifold with boundary \bar{U}^+ is defined taking into account the regularity class of the associated forms $\alpha_{J^+}^\tau$ in the sense of Definition A.11.

Let now $S \subset U$ be a separating, oriented smooth real hypersurface in U and $U = \bar{U}^- \cup \bar{U}^+$ the corresponding decomposition of U as union of manifolds with boundary. Let P be a principal G -bundle of class \mathcal{C}^∞ on U and let P^\pm, P_S be the restrictions of P to \bar{U}^\pm, S respectively. Let $\kappa \in (0, +\infty] \setminus \mathbb{N}$, $k := [\kappa]$, J be a bundle ACS of class \mathcal{C}^κ on P , and J^\pm be the restriction of J to P^\pm .

Our problem: express the integrability condition on J in terms of its restrictions J^\pm to P^\pm . By the Newlander–Nirenberg theorem for principal bundles (Theorem A.22), the answer is obvious in the case $\kappa > 1$:

Remark A.25. — Suppose $\kappa > 1$. J is integrable if and only if the forms $\mathfrak{f}_{J^\pm} \in A^{0,2}(\bar{U}^\pm, \text{Ad}(P))_{\kappa-1}$ vanish.

The case $\kappa \in (0, 1)$ is more delicate. In this case one can considering the restrictions J^\pm of J to the bundles P_{U^\pm} over the open sets U^\pm and the corresponding distributions \mathfrak{f}_{J^\pm} on U^\pm , but one cannot expect the vanishing of these distributions to imply the integrability of J (i.e. the vanishing of the distribution \mathfrak{f}_J on U). The key observation here is:

Remark A.26. — Let J^+ be a continuous bundle ACS on P^+ . Then the distribution $\mathfrak{f}_{J^+} \in \mathcal{D}'(U^+, \bigwedge_{U^+}^{0,2} \otimes \text{Ad}(P))$ extends as a continuous linear functional on the space $\Gamma_c^1(\bar{U}^+, \bigwedge_{\bar{U}^+}^{n,n-2} \otimes \text{Ad}(P)^*)$ of compactly supported sections of class \mathcal{C}^1 in the indicated vector bundle. If J^+ is of class \mathcal{C}^1 , this extension coincides with the functional associated with the continuous form \mathfrak{f}_{J^+} on \bar{U}^+ .

Proof. — Suppose first that J^+ is of class \mathcal{C}^1 . In this case \mathfrak{f}_{J^+} is a continuous form on \bar{U}^+ , and the associated linear functional on $\Gamma_c^1(\bar{U}^+, \bigwedge_{\bar{U}^+}^{n,n-2} \otimes \text{Ad}(P)^*)$ acts by

$$\langle \mathfrak{f}_{J^+}, \varphi \rangle = \int_{\bar{U}^+} \langle \mathfrak{f}_{J^+} \wedge \varphi \rangle.$$

Let $W \subset^{\text{open}} U$ and $\tau : \bar{W}^+ := W \cap \bar{U}^+ \rightarrow P^+$ be a local section of class \mathcal{C}^2 of P^+ . The associated form $\alpha_{J^+}^\tau$ belongs to $\Gamma^1(\bar{W}^+, \bigwedge_{\bar{W}^+}^{0,1} \otimes \mathfrak{g})$.

For any $\varphi \in \Gamma_c^1(\bar{W}^+, \bigwedge_{\bar{U}^+}^{n,n-2} \otimes \text{Ad}(P)^*)$ let $\varphi^\tau \in \Gamma_c^1(\bar{W}^+, \bigwedge_{\bar{U}^+}^{n,n-2} \otimes \mathfrak{g}^*)$ be the \mathfrak{g}^* -valued form associated with φ with respect to τ . Using Stokes theorem,

$$\begin{aligned} \langle \mathfrak{f}_{J^+}, \varphi \rangle &= \langle \mathfrak{f}_{J^+}^\tau, \varphi^\tau \rangle = \int_{\bar{W}^+} \langle (\bar{\partial} \alpha_{J^+}^\tau + \frac{1}{2} [\alpha_{J^+}^\tau \wedge \alpha_{J^+}^\tau]) \wedge \varphi^\tau \rangle \\ &= \int_{\bar{W}^+} d \langle \alpha_{J^+}^\tau \wedge \varphi^\tau \rangle + \int_{\bar{W}^+} \langle (\alpha_{J^+}^\tau \wedge \bar{\partial} \varphi^\tau) + \frac{1}{2} \langle [\alpha_{J^+}^\tau \wedge \alpha_{J^+}^\tau] \wedge \varphi^\tau \rangle \rangle \\ &= \int_{\partial \bar{W}^+} \langle \alpha_{J^+}^\tau \wedge \varphi^\tau \rangle + \int_{\bar{W}^+} \langle (\alpha_{J^+}^\tau \wedge \bar{\partial} \varphi^\tau) + \frac{1}{2} \langle [\alpha_{J^+}^\tau \wedge \alpha_{J^+}^\tau] \wedge \varphi^\tau \rangle \rangle. \quad (\text{A.24}) \end{aligned}$$

The right hand expression in (A.24) has obviously sense and is continuous with respect to φ^τ (in the \mathcal{C}^1 -topology) even if J^+ is only of class \mathcal{C}^0 and τ is only of class \mathcal{C}^1 , because under these weaker assumptions the form $\alpha_{J^+}^\tau$ remains continuous. Moreover, for J^+ of class \mathcal{C}^0 fixed, this expression gives a well defined (independent of τ) linear functional on $\Gamma_c^1(\bar{W}^+, \bigwedge_{\bar{U}^+}^{n,n-2} \otimes \text{Ad}(P)^*)$. Indeed, we claim that for any $\tau, \tau' \in \Gamma^1(\bar{W}^+, P^+)$ we have

$$\begin{aligned} &\int_{\partial \bar{W}^+} \langle \alpha_{J^+}^\tau \wedge \varphi^\tau \rangle + \int_{\bar{W}^+} \langle (\alpha_{J^+}^\tau \wedge \bar{\partial} \varphi^\tau) + \frac{1}{2} \langle [\alpha_{J^+}^\tau \wedge \alpha_{J^+}^\tau] \wedge \varphi^\tau \rangle \rangle \\ &= \int_{\partial \bar{W}^+} \langle \alpha_{J^+}^{\tau'} \wedge \varphi^{\tau'} \rangle + \int_{\bar{W}^+} \langle (\alpha_{J^+}^{\tau'} \wedge \bar{\partial} \varphi^{\tau'}) + \frac{1}{2} \langle [\alpha_{J^+}^{\tau'} \wedge \alpha_{J^+}^{\tau'}] \wedge \varphi^{\tau'} \rangle \rangle. \quad (\text{A.25}) \end{aligned}$$

By (A.24), this equality is clear when J^+ is of class \mathcal{C}^1 and τ, τ' are of class \mathcal{C}^2 . Fixing φ and writing

$$\begin{aligned} J^+ &= \lim_{n \rightarrow \infty} J_n^+ && (\text{in the } \mathcal{C}^0\text{-topology}), \\ \tau &= \lim_{n \rightarrow \infty} \tau_n, \quad \tau' = \lim_{n \rightarrow \infty} \tau'_n && (\text{in the } \mathcal{C}^1\text{-topology}) \end{aligned}$$

with J_n^+, τ_n, τ'_n of class \mathcal{C}^∞ , we conclude that (A.25) also holds for J^+ of class \mathcal{C}^0 and τ, τ' of class \mathcal{C}^1 . The same formula can be used to show that the linear functionals associated with two sections $\tau \in \Gamma^1(\overline{W}^+, P^+)$, $\tau' \in \Gamma^1(\overline{W}'^+, P^+)$ agree on

$$\Gamma_c^1(\overline{W}^+ \cap \overline{W}'^+, \bigwedge_{\overline{U}^+}^{n, n-2} \otimes \text{Ad}(P)^*),$$

so we obtain a well defined linear functional on $\Gamma_c^1(\overline{U}^+, \bigwedge_{\overline{U}^+}^{n, n-2} \otimes \text{Ad}(P)^*)$, obviously extending the distribution \mathfrak{f}_{J^+} . \square

For a bundle ACS J^+ of class \mathcal{C}^0 on P^+ we will use the notation \mathfrak{f}_{J^+} for the linear functional provided by Remark A.26. Note that \mathfrak{f}_{J^+} can be regarded as an element of the space $\dot{D}'(\overline{U}^+, \bigwedge_{\overline{U}^+}^{0,2} \otimes \text{Ad}(P))$ of $\bigwedge_{\overline{U}^+}^{0,2} \otimes \text{Ad}(P)$ -valued distributions supported by \overline{U}^+ (see [24, Section I.1]). The map $J^+ \mapsto \mathfrak{f}_{J^+}$ is functorial with respect to \mathcal{C}^1 -isomorphisms of principal bundles on \overline{U}^+ , in particular:

Remark A.27. — The equivariance formula (A.16) generalizes to a bundle ACS J^+ of class \mathcal{C}^0 on P^+ and a gauge transformation $\sigma \in \Gamma^1(\overline{U}^+, \iota(P^+))$.

DEFINITION A.28. — *Let P^+ be a principal G -bundle on \overline{U}^+ . A bundle ACS J^+ of class \mathcal{C}^1 on P^+ will be called formally integrable, if $\mathfrak{f}_{J^+} = 0$ in the space of $\text{Ad}(P)$ -valued continuous $(0,2)$ forms on \overline{U}^+ .*

More generally, a bundle ACS J^+ of class \mathcal{C}^0 on P^+ will be called formally integrable, if $\mathfrak{f}_{J^+} = 0$ in the space of $\bigwedge_{\overline{U}^+}^{0,2} \otimes \text{Ad}(P)$ -valued distributions supported by \overline{U}^+ .

With Definition A.28 we have the following generalization of Remark A.25:

PROPOSITION A.29. — *Let J be a bundle ACS of class \mathcal{C}^0 on P and J^\pm its restriction to P^\pm .*

- (1) *If J^\pm are formally integrable, then $\mathfrak{f}_J = 0$ in distribution sense.*
- (2) *Suppose $J \in \mathcal{J}_P^\kappa$ with $\kappa \in (0, +\infty] \setminus \mathbb{N}$. Then J is integrable iff and only if J^\pm are formally integrable.*

Proof.

(1). — Let $\tau \in \Gamma^1(W, P)$ be a local section of P and

$$\varphi \in \Gamma_c^1(W, \bigwedge_U^{n, n-2} \otimes \text{Ad}(P)^*).$$

Put $\overline{W}^\pm := W \cap \overline{U}^\pm$, $\varphi_\pm := \varphi|_{\overline{W}^\pm}$. We have

$$\begin{aligned} & \langle \mathfrak{f}_{J^-}, \varphi_- \rangle + \langle \mathfrak{f}_{J^+}, \varphi_+ \rangle \\ &= \int_{\partial \overline{W}^-} \langle \alpha_J^\tau \wedge \varphi_-^\tau \rangle + \int_{\overline{W}^-} (\langle \alpha_J^\tau \wedge \bar{\partial} \varphi_-^\tau \rangle + \frac{1}{2} \langle [\alpha_J^\tau \wedge \alpha_J^\tau] \wedge \varphi_-^\tau \rangle) \\ &+ \int_{\partial \overline{W}^+} \langle \alpha_J^\tau \wedge \varphi_+^\tau \rangle + \int_{\overline{W}^+} (\langle \alpha_J^\tau \wedge \bar{\partial} \varphi_+^\tau \rangle + \frac{1}{2} \langle [\alpha_J^\tau \wedge \alpha_J^\tau] \wedge \varphi_+^\tau \rangle). \quad (\text{A.26}) \end{aligned}$$

We obviously have $\varphi_-^\tau|_{S \cap W} = \varphi_+^\tau|_{S \cap W} = \varphi^\tau|_{S \cap W}$. Taking into account that the oriented boundaries $\partial \overline{W}^-$, $\partial \overline{W}^+$ coincide with $S \cap W$ endowed with opposite orientations, it follows that $\int_{\partial \overline{W}^-} \langle \alpha_J^\tau \wedge \varphi_-^\tau \rangle + \int_{\partial \overline{W}^+} \langle \alpha_J^\tau \wedge \varphi_+^\tau \rangle = 0$. By (A.26),

$$\langle \mathfrak{f}_{J^-}, \varphi_- \rangle + \langle \mathfrak{f}_{J^+}, \varphi_+ \rangle = \int_W (\langle \alpha_J^\tau \wedge \bar{\partial} \varphi^\tau \rangle + \frac{1}{2} \langle [\alpha_J^\tau \wedge \alpha_J^\tau] \wedge \varphi^\tau \rangle) = \langle \mathfrak{f}_J, \varphi \rangle,$$

so the vanishing of \mathfrak{f}_{J^\pm} as distributions supported by \overline{U}^\pm implies the vanishing of the distribution \mathfrak{f}_J .

(2). — If J^\pm are formally integrable, then $\mathfrak{f}_J = 0$ in distribution sense, so J is integrable by Theorem A.22. Conversely, if J is integrable, then around any point $x \in U$ there exists a local section $\tau : W \rightarrow P$ of class $\mathcal{C}^{\kappa+1}$ which is J -pseudo-holomorphic. Therefore $\alpha_J^\tau = 0$. Put

$$\tau^\pm := \tau|_{W \cap \overline{U}^\pm} : W \cap \overline{U}^\pm \longrightarrow P^\pm.$$

We have $\alpha_J^{\tau^\pm} = \alpha_J^\tau|_{W \cap \overline{U}^\pm} = 0$, so the restriction of \mathfrak{f}_{J^\pm} (regarded as distribution supported by \overline{U}^\pm) to $W \cap \overline{U}^\pm$ vanishes. Therefore $\mathfrak{f}_{J^\pm} = 0$, so J^\pm are formally integrable. \square

Remark A.30. — In the special case $G = \text{GL}(r, \mathbb{C})$ we obtain the formal integrability condition for Dolbeault operators on a vector bundle E^+ on \overline{U}^+ : A Dolbeault operator δ^+ of class \mathcal{C}^1 on E^+ is formally integrable if the continuous $\text{End}(E)$ -valued form F_δ vanishes on \overline{U}^+ . A Dolbeault operator δ^+ of class \mathcal{C}^0 on E^+ is formally integrable if $F_\delta = 0$ in the space of $\bigwedge_{\overline{U}^+}^{0,2} \otimes \text{End}(E^+)$ -valued distributions supported by \overline{U}^+ . With this definition, the analogue for vector bundles of Proposition A.29 holds.

A.6. Intrinsic higher order differentials

Let $k \in \mathbb{N}$ and $l \in \mathbb{Z}$ with $l \leq k$. Let M be a differentiable manifold, F a finite dimensional real vector space, $f \in \mathcal{C}^k(M, F)$, and $x \in M$. The condition

With respect to a local chart around x , all partial derivatives of order $\leq l$ of f at x vanish

is independent of the chosen chart. This follows from the composition formula [23, Section I.6]. If this condition is satisfied, we will say that the order l jet of f at x vanishes, and we shall write $j_x^l f = 0$. For negative l the condition $j_x^l f = 0$ becomes superfluous (satisfied by any $f \in \mathcal{C}^k(M, F)$). Note that, for $l \geq 0$ we have $j_x^l f = 0$ iff and only if $f(x) = 0$ and $j_x^{l-1} df = 0$.

LEMMA A.31. — *Let F_1, F_2, F be finite dimensional real vector spaces, $k \in \mathbb{N}$ and $l, l_1, l_2 \in \mathbb{Z}$, such that $l \leq k, l_1 + l_2 + 1 \leq k$. Let $x \in M$.*

- (1) *Let $b : F_1 \times F_2 \rightarrow F$ be a bilinear map and $f_i \in \mathcal{C}^k(M, F_i)$ with $j_x^{l_i} f_i = 0$. Then $j_x^{l_1+l_2+1} b(f_1, f_2) = 0$. In particular, for $l_1 = -1$, we have the implication*

$$j_x^l f_2 = 0 \Rightarrow j_x^l b(f_1, f_2) = 0.$$

- (2) *Let $f \in \mathcal{C}^k(M, F_1)$ with $j_x^l f = 0$, V_1 an open neighborhood of $\text{im}(f)$ in F_1 and $g \in \mathcal{C}^k(V_1, F_2)$ such that $g(0) = 0$. Then $j_x^l (g \circ f) = 0$.*
- (3) *Let $f_i \in \mathcal{C}^k(M, F_i)$ and $\Phi \in \mathcal{C}^k(F_1 \times F_2, F)$. If $j_{f_1(x)}^l (y_1 \mapsto \Phi(y_1, 0)) = 0$ and $j_x^l f_2 = 0$, then $j_x^l \Phi(f_1, f_2) = 0$.*

Proof. — We may suppose $M = \mathbb{R}^n$, $F_i = \mathbb{R}^{m_i}$, $F = \mathbb{R}^m$.

(1). — The claim follows easily using the Leibniz rule.

(2). — The claim follows by induction using the formula $d(g \circ f)(y) = dg(f(y)) df(y)$ and (1) taking $l_1 = l$ and $l_2 = -1$.

(3). — Induction with respect to l : For $l = 0$, taking into account the assumptions, we have $\Phi(f_1(x), f_2(x)) = \Phi(f_1(x), 0) = 0$. Let $l \geq 1$ and suppose that the statement is true for $l - 1$. For $u \in M = \mathbb{R}^n$ we have:

$$d\Phi(f_1, f_2)(u) = \partial_1 \Phi(f_1(u), f_2(u)) df_1(u) + \partial_2 \Phi(f_1(u), f_2(u)) df_2(u).$$

The assumption $j_{f_1(x)}^l (y_1 \mapsto \Phi(y_1, 0)) = 0$ gives $j_{f_1(x)}^{l-1} (y_1 \mapsto \partial_1 \Phi(y_1, 0)) = 0$. We also have $j_x^{l-1} f_2 = 0$ (because $j_x^l f_2 = 0$), so, the induction hypothesis applied to $(f_1, f_2, \partial_1 \Phi)$ gives $j_x^{l-1} \partial_1 \Phi(f_1, f_2) = 0$. Therefore $j_x^{l-1} (\partial_1 \Phi(f_1, f_2) df_1) = 0$ by (1). On the other hand, the hypothesis $j_x^l f_2 = 0$ implies $j_x^{l-1} df_2 = 0$, so again by (1) we obtain $j_x^{l-1} (\partial_2 \Phi(f_1, f_2) df_2) = 0$. Therefore $j_x^{l-1} (d\Phi(f_1, f_2)) = 0$, so, since $\Phi(f_1(x), f_2(x)) = 0$, we have $j_x^l \Phi(f_1, f_2) = 0$ as claimed. \square

Suppose that $f \in \mathcal{C}^k(M, F)$ with $j_x^l f = 0$ where $0 \leq l < k$. Using the same composition formula cited above it follows that, for tangent vectors $v_1, \dots, v_{l+1} \in T_x M$ the element

$$d_{h(x)}^{l+1}(f \circ h^{-1})(h_*(v_1), \dots, h_*(v_{l+1}))$$

(where $x \in W \xrightarrow{h} W' \subset \mathbb{R}^n$ is a chart around x) of F depends only on v_1, \dots, v_{l+1} , not on h . Therefore, if $j_x^l f = 0$, we obtain a well defined symmetric $(l+1)$ -linear map

$$D_x^{l+1} f : T_{M,x}^{l+1} \longrightarrow F$$

which will be called the *intrinsic differential of order $(l+1)$ of f at x* .

Let now $S \subset M$ be smooth hypersurface. If $j_x^l f = 0$ for any $x \in S$ we'll say that the order l jet of f along S vanishes, and we'll write $j_S^l f = 0$.

Remark A.32. — Suppose that $j_S^l f = 0$, where $0 \leq l < k$, and let $x \in S$. Then $D_x^{l+1} f(v_1, \dots, v_{l+1}) = 0$ if one of the tangent vectors v_i belongs to $T_x S$.

Proof. — We may suppose $M = \mathbb{R}^n$, $S = \mathbb{R}^{n-1} \times \{0\}$. It suffices to prove that $\partial^\alpha f(x', 0)$ for any $x' \in \mathbb{R}^{n-1}$ and any $\alpha \in \mathbb{N}^n$ with $|\alpha| = l+1$ for which there exists $i \in \{1, \dots, n-1\}$ with $\alpha_i > 0$. Let $\alpha \in \mathbb{N}^n$ with $|\alpha| = l+1$ and let $i \in \{1, \dots, n-1\}$ with $\alpha_i > 0$. Denote by (e_1, \dots, e_n) be the canonical basis of \mathbb{R}^n . Putting $\beta := \alpha - e_i$ we have $\beta \in \mathbb{N}^n$, $|\beta| = l$ and

$$\partial^\alpha f(x', 0) = \partial_i(\partial^\beta f)(x', 0).$$

The right hand term vanishes because, since we assumed $j_y^l f = 0$ for any $y \in S$, we have $\partial^\beta f(x' + te_i, 0) = 0$ for any $t \in \mathbb{R}$. \square

Therefore, if $j_S^l f = 0$ and $x \in S$, then $D_x^{l+1} f(v_1, \dots, v_{l+1})$ depends only on the images of v_i in the normal line $n_{S,x} = T_{M,x}/T_{S,x}$, so the family $(D_x^{l+1} f)_{x \in S}$ defines a section

$$D_S^{l+1} f \in \Gamma^0(S, n_S^{*\otimes(l+1)} \otimes F),$$

which will be called the *intrinsic differential of order $(l+1)$ of f along S* .

Let now E be a real vector bundle of rank r and class \mathcal{C}^∞ on M , and let $\sigma \in \Gamma^{l+1}(M, E)$ be a section of class \mathcal{C}^{l+1} of E . Let $x \in M$. For a local trivialization $\theta : E_W \rightarrow W \times \mathbb{R}^r$ put $\sigma^\theta := p_{\mathbb{R}^r} \circ \theta \in \mathcal{C}^{l+1}(W, \mathbb{R}^r)$. The condition

With respect to a local trivialization θ around x we have $j_x^l(\sigma^\theta) = 0$

is independent of θ . If this condition is satisfied, we'll say that the order l jet of σ at x vanishes, and we'll write $j_x^l \sigma = 0$. If this is the case (and $l \geq 0$),

we obtain a well defined intrinsic differential $D_x^{l+1}\sigma : T_{M,x}^{l+1} \rightarrow E_x$ of order $(l+1)$.

If $j_x^l\sigma = 0$ for any $x \in S$, we'll say that the order l jet of σ along S vanishes, we'll write $j_S^l f = 0$, and (if $l \geq 0$) we obtain a well defined intrinsic order $(l+1)$ differential

$$D_S^{l+1}\sigma \in \Gamma^0(S, n_S^{*\otimes(l+1)} \otimes E_S)$$

of σ along S . If E is a complex vector bundle, we can regard $D_S^{l+1}\sigma$ as an element of $\Gamma^0(S, \eta_S^{\otimes(l+1)} \otimes E_S)$, where η_S is the complexified conormal line bundle of S .

LEMMA A.33. — *Let U, V, F be finite dimensional complex vector spaces, $S \subset U$ a smooth real hypersurface, and $f \in \mathcal{C}^k(U, V)$ be such that $j_S^l f = 0$, where $l < k$.*

Put $U_{\mathbb{C}} := U \otimes_{\mathbb{R}} \mathbb{C}$. For $0 \leq s \leq l+1$ regard the order s differential $d^s f$ of f on U as a map $U \rightarrow U_{\mathbb{C}}^{\otimes s} \otimes V$ of class \mathcal{C}^{l+1-s} which takes values in $L_{\text{sym}}^s(U_{\mathbb{C}}, V) \subset U_{\mathbb{C}}^{*\otimes s} \otimes V$.*

Let $\omega \in A^{1,0}(V, F)$ be an F -valued $(1,0)$ form of class \mathcal{C}^∞ on V regarded as element in $\mathcal{C}^\infty(V, \text{Hom}_{\mathbb{C}}(V, F))$ and put

$$\omega^f := \omega \circ f \in \mathcal{C}^k(U, \text{Hom}_{\mathbb{C}}(V, F)).$$

The F -valued forms $f^(\omega)$, $f^*(\omega)^{0,1}$ on U will be regarded as elements of the spaces $\mathcal{C}^{k-1}(U, U_{\mathbb{C}}^* \otimes F)$, $\mathcal{C}^{k-1}(U, U_{\mathbb{C}}^{*0,1} \otimes F)$ respectively.*

Then $j_S^{l-1}(df) = 0$, $j_S^{l-1}(\bar{\partial}f) = 0$, $j_S^{l-1}(f^(\omega)) = 0$, $j_S^{l-1}(f^*(\omega)^{0,1}) = 0$ and the intrinsic order l differentials of df , $\bar{\partial}f$, $f^*(\omega)$ and $f^*(\omega)^{0,1}$ along S are given by the following formulae:*

$$\begin{aligned} D_S^l(df) &= (\text{id}_{\eta_S}^{\otimes l} \otimes (\text{id}_{\eta_S} \otimes \text{id}_V))(D_S^{l+1}f) \\ &\in \Gamma^0(U, \eta_S^{\otimes l} \otimes (\eta_S \otimes V)) \subset \Gamma^0(U, \eta_S^{\otimes l} \otimes (U_{\mathbb{C}}^* \otimes V)), \end{aligned} \quad (\text{A.27})$$

$$\begin{aligned} D_S^l(\bar{\partial}f) &= (\text{id}_{\eta_S}^{\otimes l} \otimes (\psi_S \otimes \text{id}_V))(D_S^{l+1}f) \\ &\in \Gamma^0(U, \eta_S^{\otimes l} \otimes (\eta_S^{0,1} \otimes V)) \subset \Gamma^0(U, \eta_S^{\otimes l} \otimes (U_{\mathbb{C}}^{*0,1} \otimes V)), \end{aligned} \quad (\text{A.28})$$

$$\begin{aligned} D_S^l(f^*(\omega)) &= (\text{id}_{\eta_S}^{\otimes l} \otimes \omega_S^f \cdot (\text{id}_{\eta_S} \otimes \text{id}_V))(D_S^{l+1}f) \\ &\in \Gamma^0(S, \eta_S^{\otimes l} \otimes (\eta_S \otimes F)) \subset \Gamma^0(S, \eta_S^{\otimes l} \otimes (U_{\mathbb{C}}^* \otimes F)), \end{aligned} \quad (\text{A.29})$$

$$\begin{aligned} D_S^l(f^*(\omega)^{0,1}) &= (\text{id}_{\eta_S}^{\otimes l} \otimes \omega_S^f \cdot (\psi_S \otimes \text{id}_V))(D_S^{l+1}f) \\ &\in \Gamma^0(S, \eta_S^{\otimes l} \otimes (\eta_S^{0,1} \otimes F)) \subset \Gamma^0(S, \eta_S^{\otimes l} \otimes (U_{\mathbb{C}}^{*0,1} \otimes F)), \end{aligned} \quad (\text{A.30})$$

where, on the right:

- $\eta_S (\eta_S^{0,1})$ is regarded as a line subbundle of the trivial bundle with fibre $U_{\mathbb{C}}^*$ (respectively $U_{\mathbb{C}}^{*0,1}$) on S ,
- $\omega_S^f \cdot$ denotes the morphism

$$S \times \text{Hom}(U_{\mathbb{C}}, V) \longrightarrow S \times \text{Hom}(U_{\mathbb{C}}, F)$$

of trivial bundles on S defined by pointwise composition with ω^f , and also the induced bundle morphisms on S :

$$\eta_S \otimes V \longrightarrow \eta_S \otimes F, \quad \eta_S^{0,1} \otimes V \longrightarrow \eta_S^{0,1} \otimes F.$$

Proof. — The recursive definition of the higher order differentials gives for $0 \leq s \leq l$

$$d^s(df) = (\text{id}_{U_{\mathbb{C}}^{\otimes s}} \otimes (\text{id}_{U_{\mathbb{C}}^*} \otimes \text{id}_V))(d^{s+1}f) \in \mathcal{C}^0(U, U_{\mathbb{C}}^{*\otimes s} \otimes (U_{\mathbb{C}}^* \otimes V)). \quad (\text{A.31})$$

This implies

$$d^s(\bar{\partial}f) = (\text{id}_{U_{\mathbb{C}}^{\otimes s}} \otimes (p^{0,1} \otimes \text{id}_V))(d^{s+1}f) \in \mathcal{C}^0(U, U_{\mathbb{C}}^{*\otimes s} \otimes (U_{\mathbb{C}}^{*0,1} \otimes V)), \quad (\text{A.32})$$

where $p^{0,1} : U_{\mathbb{C}}^* \rightarrow U_{\mathbb{C}}^{*0,1}$ is the obvious projection, shows that the condition $j_S^l f = 0$ implies $j_S^{l-1}(df) = 0$, $j_S^{l-1}(\bar{\partial}f) = 0$, and proves formulae (A.27), (A.28).

The forms $f^*(\omega)$, $f^*(\omega)^{0,1}$ are given by

$$\begin{aligned} f^*(\omega) &= \omega^f \cdot df \in \mathcal{C}^l(U, U_{\mathbb{C}}^* \otimes F), \\ f^*(\omega)^{0,1} &= \omega^f \cdot \bar{\partial}f \in \mathcal{C}^l(U, U_{\mathbb{C}}^{*0,1} \otimes F), \end{aligned} \quad (\text{A.33})$$

where $\omega^f \cdot$ denotes the morphism

$$U \times \text{Hom}(U_{\mathbb{C}}, V) \longrightarrow U \times \text{Hom}(U_{\mathbb{C}}, F)$$

of trivial bundles on U defined by pointwise composition with ω^f . Since $j_S^{l-1}(df) = 0$, $j_S^{l-1}(\bar{\partial}f) = 0$, we obtain $j_S^{l-1}(f^*(\omega)) = 0$, $j_S^{l-1}(f^*(\omega)^{0,1}) = 0$ and formulae (A.29), (A.30) follow from (A.33) using the Leibniz rule noting that ω_S^f is induced by ω^f . \square

LEMMA A.34.

- (1) Let U , F be complex vector spaces, $S \subset U$ a smooth real hypersurface, and β an F -valued $(0, q)$ form with coefficients \mathcal{C}^{l+1} on U , regarded as element in $\mathcal{C}^{l+1}(U, U_{\mathbb{C}}^{*0,q} \otimes F)$. Suppose that $j_S^l(\beta) = 0$. Then $j_S^{l-1}(\bar{\partial}\beta) = 0$ (if $l \geq 1$) and the intrinsic order l differential of $\bar{\partial}\beta$ along S is given by

$$\begin{aligned} D_S^l(\bar{\partial}\beta) &= (\text{id}_{\eta_S^{\otimes l}} \otimes \wedge (\psi_S \otimes \text{id}_{U^{*0,q} \otimes F}))(D_S^{l+1}\beta) \\ &\in \Gamma^0(U, \eta_S^{\otimes l} \otimes (\eta_S^{0,1} \wedge U_{\mathbb{C}}^{*0,q} \otimes F)) \subset \Gamma^0(U, (\eta_S^{\otimes l} \otimes (U_{\mathbb{C}}^{*0,q+1} \otimes F))), \end{aligned} \quad (\text{A.34})$$

where, on the right, \wedge denotes the bundle morphism

$$\eta_S^{0,1} \otimes (U^{*0,q} \otimes F) \longrightarrow (\eta_S^{0,1} \wedge U^{*0,q}) \otimes F \hookrightarrow S \times (U^{*0,q+1} \otimes F)$$

on S induced by the wedge product $\wedge : U_{\mathbb{C}}^{*0,1} \otimes (U^{*0,q} \otimes F) \rightarrow U^{*0,q+1} \otimes F$.

- (2) More generally, let U be a complex manifold, E a complex vector bundle on U , δ a (not necessarily integrable) Dolbeault operator with coefficients in \mathcal{C}^l on E , and $\beta \in \Gamma^{l+1}(U, \wedge_U^{0,q} \otimes E)$ with $j_S^l(\beta) = 0$. Then $j_S^{l-1}(\delta\beta) = 0$ (if $l \geq 1$), and

$$\begin{aligned} D_S^l(\delta\beta) &= (\text{id}_{\eta_S^{\otimes l}} \otimes \wedge (\psi_S \otimes \text{id}_{\wedge_{U|S}^{0,q} \otimes E_S})) (D_S^{l+1}\beta) \\ &\in \Gamma^0(S, \eta_S^{\otimes l} \otimes (\eta_S^{0,1} \wedge \wedge_{U|S}^{0,q} \otimes E_S)) \subset \Gamma^0(S, \eta_S^{\otimes l} \otimes (\wedge_{U|S}^{0,q+1} \otimes E_S)). \end{aligned} \quad (\text{A.35})$$

Proof.

- (1). — Regard β as an element $\tilde{\beta} \in \mathcal{C}^{l+1}(U, U_{\mathbb{C}}^{*0,q} \otimes F)$. The explicit formula in coordinates for the operator $\bar{\partial}$ on $(0, q)$ forms gives:

$$\begin{aligned} \bar{\partial} \left(\sum_I \beta_I d\bar{z}^I \right) &= \sum_I \bar{\partial} \beta_I \wedge d\bar{z}^I = \wedge \left(\sum_I (d\beta^I)^{0,1} \otimes d\bar{z}^I \right) \\ &= \wedge (p^{0,1} \otimes \text{id}_{U_{\mathbb{C}}^{*0,q} \otimes F}) \left(\sum_I d\beta^I \otimes d\bar{z}^I \right) = \wedge (p^{0,1} \otimes \text{id}_{U_{\mathbb{C}}^{*0,q} \otimes F}) (d\tilde{\beta}), \end{aligned}$$

and (A.34) follows from (A.27) applied to $\tilde{\beta}$.

- (2). — For (A.35) we use the formula of the operator δ with respect to a local trivialization $\tau : E_W \rightarrow W \times \mathbb{C}^r$ of E . Identifying $\Gamma^{l+1}(W, E)$ with $\mathcal{C}^{l+1}(W, \mathbb{C}^r)$ via τ , we have

$$\delta(\beta) = \bar{\partial}\beta + \alpha^\tau \wedge \beta$$

with $\alpha^\tau \in \Gamma^l(W, \wedge_U^{0,1} \otimes \text{End}(E))$. Since we assumed $j_S^l(\beta) = 0$ we have $D_S^l(\alpha^\tau \wedge \beta) = 0$, so (A.35) follows from (A.34). \square

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