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Cyclicity in Besov–Dirichlet spaces from the Corona Theorem (*)

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Abstract. — Tolokonnikov's Corona Theorem is used to obtain two results on cyclicity in Besov–Dirichlet spaces.

Résumé. — Nous utilisons le Théorème de la couronne de Tolokonnikov pour obtenir deux résultats sur la cyclicité dans les espaces de Besov–Dirichlet.

1. Introduction

Let X be a Banach space of analytic functions in the unit disc $\mathbb{D}$ such that the shift operator $S : f(z) \to zf(z)$ is a continuous map of X into itself. The cyclic vectors in X are those functions f such that the polynomial multiples of f are dense in X. Beurling in [6] provided a complete characterization of cyclic vectors in the Hardy space; the cyclic vectors are precisely the outer functions. Cyclic vectors in the Dirichlet space were initially examined by Carleson in [11] and later by Brown and Shields in [9]. In this paper, we focus on studying cyclic vectors in Besov–Dirichlet spaces. Specifically, motivated by the inquiries raised by Brown and Shields [9, Question 3] regarding cyclic vectors in a general Banach space X of analytic functions:

Question. — *If $f, g \in X$, if g is cyclic, and if $|f(z)| \geqslant |g(z)|$ for all $z \in \mathbb{D}$ then must f be cyclic?*

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Article proposé par Frédéric Bayart.

We extend some of Brown and Shields' results on cyclicity to Besov–Dirichlet spaces. We now introduce some notations. For $p \geqslant 1$ and $\alpha > -1$, the Besov space, $\mathcal{D}^p_\alpha$ is the set of holomorphic functions on $\mathbb{D}$ such that

$$\mathcal{D}_{\alpha,p}(f) = \int_{\mathbb{D}} |f'(z)|^p \mathrm{d}A_\alpha(z) < \infty,$$

where $\mathrm{d}A_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha \mathrm{d}A(z)$ and $\mathrm{d}A(z)$ is the normalised Lebesgue measure on the disc. The Besov–Dirichlet space is equipped with the norm

$$\|f\|^p_{\mathcal{D}^p_\alpha} = |f(0)|^p + \mathcal{D}_{\alpha,p}(f).$$

The Besov–Dirichlet space $\mathcal{D}^p_\alpha$ is the set of holomorphic functions f on $\mathbb{D}$ whose derivative f' is a function of the Bergman space $\mathcal{A}^p_\alpha = L^p(\mathbb{D}, \mathrm{d}A_\alpha) \cap \mathrm{Hol}(\mathbb{D})$, where $\mathrm{Hol}(\mathbb{D})$ is the space of holomorphic functions on $\mathbb{D}$. Note that if $p = 2$ and $\alpha = 1$, $\mathcal{D}^2_1$ is the Hardy space H^2 and if $p = 2$ and $\alpha = 0$, then $\mathcal{D}^2_0$ is the classical Dirichlet space $\mathcal{D}$.

Denote by $[f]_{\mathcal{D}^p_\alpha}$ the smallest S-invariant subspace containing f, the vector subspace generated by $\{z^n f,\ n \in \mathbb{N}\}$. We say that $f \in \mathcal{D}^p_\alpha$ is cyclic in $\mathcal{D}^p_\alpha$ if

$$[f]_{\mathcal{D}^p_\alpha} = \mathcal{D}^p_\alpha.$$

The function $f \in H^1$ is called outer function if it is of the form

$$f(z) = \exp \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \varphi(\zeta) |\mathrm{d}\zeta|, \qquad |z| < 1,$$

where φ is nonnegative function in $L^1(\mathbb{T})$ such that $\log \varphi \in L^1(\mathbb{T})$. Note that $|f| = \varphi$ *a.e.* on the unit circle $\mathbb{T} = \partial\mathbb{D}$.

The problem of characterizing the cyclic vectors in the Dirichlet space $\mathcal{D}^2_0$ is much more difficult. In [9], Brown and Shields conjectured that a function f in the Dirichlet space $\mathcal{D}$ is cyclic for the shift operator if and only if f is outer and its boundary zero set is of logarithmic capacity zero. The characterization of cyclic vector of $\mathcal{D}^p_\alpha$ depends on the values of p and α. More precisely our investigation is limited to the case $\alpha + 1 \leqslant p \leqslant \alpha + 2$. Indeed, If $1 < p < \alpha + 1$, then H^p is continuously embedded in $\mathcal{D}^p_\alpha$ see [22], hence every outer function $f \in H^p$ is cyclic for $\mathcal{D}^p_\alpha = \mathcal{A}^p_{\alpha-p}$. On the other hand, if $p > \alpha+2$, then $\mathcal{D}^p_\alpha \subset \mathcal{A}(\mathbb{D}) = \mathrm{Hol}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ becomes Banach algebra see [22], consequently the only cyclic outer functions are the invertible functions, and then any function that vanishes at least at one point is not cyclic in $\mathcal{D}^p_\alpha$. For $f \in \mathcal{A}(\mathbb{D})$, denote

$$\mathcal{Z}(f) = \{\zeta \in \mathbb{T} : f(\zeta) = 0\}$$

the boundary zero set of f. Recall that Brown and Shields conjectured that a function f in the Dirichlet space $\mathcal{D}$ is cyclic for the shift operator if and only if f is outer and its boundary zero set $\mathcal{Z}(f)$ is of logarithmic capacity zero. Here, we will prove the following theorem.

Theorem 1.1. — *Let $p > 1$ such that $\alpha + 1 \leqslant p \leqslant \alpha + 2$. Let $f \in \mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})$ be an outer function such that $\mathcal{Z}(f) = \{1\}$, then f is cyclic in $\mathcal{D}^p_\alpha$.*

The case of the classical Dirichlet space $\mathcal{D}^2_0$ was discovered by Hedenmalm–Shields [20] and generalized by Richter–Sundberg [25]. This result was shown [22] for $\alpha + 1 < p \leqslant \alpha + 2$, the method used for the proof is inspired by that of Hedenmalm and Shields [20]. Note that our result also includes the case where $p = \alpha + 1$. Thanks to [20, Theorem 3], Theorem 1.1 remains true if $\mathcal{Z}(f) = \{1\}$ is replaced by $\mathcal{Z}(f)$ is countable. Our second main result is

Theorem 1.2. — *Let $p > 1$ such that $1 + \alpha \leqslant p \leqslant \alpha + 2$. Let $f, g \in \mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})$ such that*

$$|g(z)| \leqslant |f(z)|, \qquad z \in \mathbb{D}. \tag{1.1}$$

If g is cyclic in $\mathcal{D}^p_\alpha$ then f is cyclic in $\mathcal{D}^p_\alpha$.

This result generalizes that of Brown and Shields [9, Theorem 1], then Aleman [1, Corollary 3.3] for $\mathcal{D}^2_\alpha$ spaces.

The proof of the two theorems is based on the Tolokonnikov Corona Theorem [28]. The idea of using the corona theorem in this context goes back to Roberts for the Bergman space [26], see also [2, 7, 8, 19]. For some results related to cyclic vectors, see [5, 15, 16, 17, 18, 21, 24, 25] and the references therein.

2. Proof of Theorem 1.1 and Theorem 2.5

We recall two results we will need for the proofs. The first is the Corona Theorem of Tolokonnikov [28]

Theorem 2.1. — *Let $1 < p \leqslant \alpha + 2$ and Let $f_1, f_2 \in \mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})$ such that*

$$\inf_{z\in\mathbb{D}} \Big(|f_1(z)| + |f_2(z)|\Big) > \delta > 0.$$

Then there exists $h_1, h_2 \in \mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})$ such that

$$\begin{cases} f_1(z)h_1(z) + f_2(z)h_2(z) = 1, & z \in \mathbb{D} \\ \|h_1\|_{\mathcal{D}^p_\alpha\cap\mathcal{A}(\mathbb{D})} \leqslant \delta^{-\mathbf{A}} \quad \textit{and} \quad \|h_2\|_{\mathcal{D}^p_\alpha\cap\mathcal{A}(\mathbb{D})} \leqslant \delta^{-\mathbf{A}} \end{cases}$$

for some positive constant $\mathbf{A} \geqslant 4$ independent of p and α.

Remark 2.2. — If $p > \alpha + 2$, then $\mathcal{D}^p_\alpha \subset \mathcal{A}(\mathbb{D})$. If $p = 2$ and $\alpha = 0$, $\mathcal{D}^2_0 = H^2$ and we therefore find the classical Carleson–Corona Theorem [12]. In this case, the constant $\mathbf{A} > 2$ instead of $\mathbf{A} \geqslant 4$. If $\alpha = p - 2$, Tolokonnikov [28] showed that $\mathbf{A} = 4$. Nicolau in [23] showed the Corona Theorem but without giving the quantitative version, see also [3, 13].

Let T be a bounded linear operator acting on an infinite dimensional complex Banach space X. The spectrum of T is denoted by $\sigma(T)$. The following corollary is easily obtained by Atzmon's Theorem [4] and Cauchy's inequalities.

Corollary 2.3. — *Let T be an invertible operator on Banach X such that $\sigma(T) = \{1\}$. Suppose that there exist $k \geqslant 0$ and $c > 0$ such that for $\varepsilon > 0$, there exists $c_\varepsilon > 0$*

$$\begin{cases} \|(T - zI)^{-1}\| \leqslant c_\varepsilon \exp \frac{\varepsilon}{1-|z|} & |z| < 1, \\ \|(T - zI)^{-1}\| \leqslant \frac{c}{(|z|-1)^k} & |z| > 1, \end{cases}$$

then $(I - T)^k = 0$.

2.1. Proof of Theorem 1.1

Let $\lambda \in \mathbb{C}$ and put

$$\delta_\lambda := \inf_{z\in\mathbb{D}} \Big(|\lambda - z| + |f(z)|\Big).$$

Since f is an outer function, by [27]

$$\lim_{|z|\to 1-} (1 - |z|) \log 1/|f(z)| = 0.$$

For all $\varepsilon > 0$, there is therefore $c_\varepsilon > 0$ such that

$$|f(z)| \geqslant c_\varepsilon \exp \frac{-\varepsilon}{1 - |z|}, \qquad z \in \mathbb{D}. \tag{2.1}$$

Considering $|\lambda| \neq 1$, we distinguish two cases:

- If $|z - \lambda| \geqslant |1 - |\lambda||/2$, then $\delta_\lambda \geqslant |1 - |\lambda||/2$.
- If $|z - \lambda| \leqslant |1 - |\lambda||/2$, then

$$|1 - |\lambda||/2 \geqslant |z - \lambda| \geqslant |(1 - |\lambda|) - (1 - |z|)| \geqslant |1 - |\lambda|| - |1 - |z||.$$

Thus, we get $1 - |z| \geqslant |1 - |\lambda||/2$ and by (2.1). We then have

$$|f(z)| \geqslant c_\varepsilon \exp \frac{-\varepsilon}{|1 - |\lambda||}.$$

Therefore, we finally get

$$\delta_\lambda \geqslant c_\varepsilon \exp \frac{-\varepsilon}{|1-|\lambda||}.$$

According to the Theorem 2.1, There are $g, h \in \mathcal{D}_\alpha^p \cap \mathcal{A}(\mathbb{D})$ such that

$$\begin{cases} (\lambda - z)g + fh = 1 \\ \|g\|_{\mathcal{D}_\alpha^p \cap \mathcal{A}(\mathbb{D})} \leqslant \delta_\lambda^{-\mathbf{A}} \quad \text{and} \quad \|h\|_{\mathcal{D}_\alpha^p \cap \mathcal{A}(\mathbb{D})} \leqslant \delta_\lambda^{-\mathbf{A}} \end{cases}$$

for some constant $A \geqslant 4$.

Let $[f]_{\mathcal{D}_\alpha^p \cap \mathcal{A}(\mathbb{D})}$ be the closed S-invariant subspace generated by f and let

$$\pi : \mathcal{D}_\alpha^p \cap \mathcal{A}(\mathbb{D}) \to \mathcal{D}_\alpha^p \cap \mathcal{A}(\mathbb{D})/[f]_{\mathcal{D}_\alpha^p \cap \mathcal{A}(\mathbb{D})}$$

be the canonical surjection. We have

$$(\lambda\pi(1) - \pi(z))^{-1} = \pi(g).$$

For $|\lambda| < 1$, we have

$$\begin{aligned} \|(\lambda\pi(1) - \pi(z))^{-1}\| &= \|\pi(g)\| \\ &\leqslant \|g\|_{\mathcal{D}_\alpha^p \cap \mathcal{A}(\mathbb{D})} \\ &\leqslant c_\varepsilon \exp \frac{\varepsilon}{1-|\lambda|}. \end{aligned}$$

For $|\lambda| > 1$, we have

$$\begin{aligned} \|(\lambda\pi(1) - \pi(z))^{-1}\| &\leqslant \|(\lambda - z)^{-1}\|_{\mathcal{D}_\alpha^p \cap \mathcal{A}(\mathbb{D})} \\ &= \frac{1}{|\lambda|-1} + \frac{1}{|\lambda|} + \left(\int_{\mathbb{D}} \frac{\mathrm{d}A_\alpha(z)}{|\lambda - z|^{2p}}\right)^{1/p} \\ &\leqslant \frac{2}{|\lambda|-1} + \frac{1}{(|\lambda|-1)^p}. \end{aligned}$$

The spectrum of π, $\sigma(\pi) = \{1\}$, by Corollary 2.3, $(\pi(1) - \pi(\alpha))^{[p]+1} = 0$, and we get $(1-z)^{[p]+1} \in [f]_{\mathcal{D}_\alpha^p \cap \mathcal{A}(\mathbb{D})}$. Since $(1-z)^{[p]+1}$ is cyclic in $\mathcal{D}_\alpha^p$, f is also cyclic in $\mathcal{D}_\alpha^p$ and the proof is complete.

2.2. Proof of Theorem 1.2

The proof of the Theorem 1.2 is deduced from the following two results.

Lemma 2.4. — *Let $p > 1$ such that $1 + \alpha \leqslant p \leqslant \alpha + 2$. Let $f, g \in \mathcal{D}_\alpha^p \cap \mathcal{A}(\mathbb{D})$, if fg is cyclic, then both f and g are cyclic.*

Proof. — It suffices to show that g is cyclic. Let $\sigma_n(f)$ denote the Fejér means of the partial sums of the power series for f. Since the $\sigma_n(f)$ converges to f in $\mathcal{D}^p_\alpha$, $\sigma_n(f)g$ converge pointwise to fg in $\mathbb{D}$ and

$$\|(\sigma_n(f)g - fg)'\|^p_{\mathcal{A}^p_\alpha} \leqslant \|(\sigma_n(f) - f)\|_\infty \|g'\|^p_{\mathcal{A}^p_\alpha} + \|\sigma_n(f) - f\|^p_{\mathcal{D}^p_\alpha} \|g\|_\infty,$$

we obtain that $\sigma_n(f)g$ converges to fg in $\mathcal{D}^p_\alpha$, which completes the proof. □

The constant N in the following theorem is related to that of the Corona Theorem. If $\alpha = p-2$, we have $N(p-2,p) = 5$.

THEOREM 2.5. — *Let $p > 1$ be such that $\alpha + 1 \leqslant p \leqslant \alpha + 2$, there exists $N = N(\alpha, p)$ which depends only on α and p such that if $f, g \in \mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})$ with*

$$|g(z)| \leqslant |f(z)|, \qquad z \in \mathbb{D},$$

then $[g^N]_{\mathcal{D}^p_\alpha} \subset [f]_{\mathcal{D}^p_\alpha}$.

Proof. — Let $\lambda \in \mathbb{C}$ and set

$$\inf_{z \in \mathbb{D}} \Big\{ |1 - \lambda g(z)| + |f(z)| \Big\} = \delta_\lambda.$$

Considering $\lambda \neq 0$, we have

- If $|g(z)| \leqslant \frac{1}{2|\lambda|}$, then $|1 - \lambda g(z)| \geqslant 1 - |\lambda||g(z)| \geqslant \frac{1}{2}$.
- If $|g(z)| \geqslant \frac{1}{2|\lambda|}$ then $|f(z)| \geqslant \frac{1}{2|\lambda|}$

From this, follows

$$\delta_\lambda \geqslant \frac{1}{2|\lambda|}.$$

According to the Theorem 2.1, there are $F_\lambda, G_\lambda \in \mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})$ such that

$$\begin{cases} (1 - \lambda g)G_\lambda + fF_\lambda = 1, \\ \|F_\lambda\|_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})} \leqslant \delta_\lambda^{-\mathbf{A}} \quad \text{and} \quad \|G_\lambda\|_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})} \leqslant \delta_\lambda^{-\mathbf{A}} \end{cases}$$

for some constant $\mathbf{A} > 4$.

As before, we consider the canonical surjection

$$\pi : \mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D}) \to \left(\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})\right) / [f]_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})}.$$

We have

$$(\pi(1 - \lambda g))^{-1} = \pi(G_\lambda)$$

and

$$\begin{aligned} \|(\pi(1 - \lambda g))^{-1}\|_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})/[f]_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})}} &= \|\pi(G_\lambda)\|_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})/[f]_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})}} \\ &\leqslant \|G_\lambda\|_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})} \\ &\leqslant 2^{\mathbf{A}} |\lambda|^{\mathbf{A}}. \end{aligned}$$

By Liouville's Theorem, $\pi(1-\lambda g)^{-1}$ is polynomial of degree at most $[\mathbf{A}]$. Since $|\lambda g| < 1$, $\pi(1-\lambda g)^{-1} = \sum_{n\geqslant 0} \lambda^n \pi^n(g)$. We obtain $\pi^{[A]+1}(g) = 0$ which means that $g^{[\mathbf{A}]+1} \in [f]_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})}$ and hence $[g^{[\mathbf{A}]+1}]_{\mathcal{D}^p_\alpha} \subset [f]_{\mathcal{D}^p_\alpha}$. □

3. Refinement of the Theorem 1.2

We can improve the estimate (1.1) in Theorem 2.5. The improved estimate we are looking for is given by the following result.

Theorem 3.1. — *Let $p > 1$ such that $\alpha + 1 \leqslant p \leqslant \alpha + 2$. Let $f, g \in \mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})$. Suppose that* $\mathrm{Re}(g) \geqslant 0$ *and there exists $\gamma > 1$ such that.*

$$|g(z)| \leqslant \left(\log \frac{\|f\|_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})}}{|f(z)|}\right)^{-\gamma}, \qquad z \in \mathbb{D} \tag{3.1}$$

then $[g]_{\mathcal{D}^p_\alpha} \subset [f]_{\mathcal{D}^p_\alpha}$.

Proof. — We assume that $\|f\|_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})} = 1$. Let $\lambda \in \mathbb{C}$, we set

$$\inf_{z\in\mathbb{D}} \Big\{|1-\lambda g(z)| + |f(z)|\Big\} = \delta_\lambda.$$

Considering $\lambda \neq 0$, we distinguish two cases:

- If $|g(z)| \leqslant \frac{1}{2|\lambda|}$ then $|1-\lambda g(z)| \geqslant 1 - |\lambda||g(z)| \geqslant \frac{1}{2}$.
- If $|g(z)| \geqslant \frac{1}{2|\lambda|}$, then by (3.1)

$$|f(z)| \geqslant e^{-(2|\lambda|)^{\frac{1}{\gamma}}}.$$

From this follows

$$\delta_\lambda \geqslant e^{-(2|\lambda|)^{\frac{1}{\gamma}}}.$$

By Theorem 2.1, there exists $F_\lambda,\ G_\lambda \in \mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})$ such that

$$\begin{cases} (1-\lambda g)G_\lambda + fF_\lambda = 1, \\ \|F_\lambda\|_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})} \leqslant \delta^{-\mathbf{A}} \quad \text{and} \quad \|G_\lambda\|_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})} \leqslant \delta^{-\mathbf{A}} \end{cases}$$

for some constant $\mathbf{A} \geqslant 4$.

Let π be the canonical surjection

$$\pi : \mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D}) \to (\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D}))\,/[f]_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})}.$$

We have

$$\pi(1-\lambda g)^{-1} = \pi(G_\lambda)$$

and

$$\begin{aligned}\|\pi(1-\lambda g)^{-1}\|_{\mathcal{D}^p_\alpha\cap\mathcal{A}(\mathbb{D})/[f]_{\mathcal{D}^p_\alpha\cap\mathcal{A}(\mathbb{D})}} &= \|\pi(G_\lambda)\|_{\mathcal{D}^p_\alpha\cap\mathcal{A}(\mathbb{D})/[f]_{\mathcal{D}^p_\alpha}}\\ &\leqslant \|G_\lambda\|_{\mathcal{D}^p_\alpha\cap\mathcal{A}(\mathbb{D})}\\ &\leqslant \frac{1}{\delta_\lambda^{\mathbf{A}}} \leqslant e^{\mathbf{A}(2|\lambda|)^{\frac{1}{\gamma}}}.\end{aligned}$$

Let $\ell \in (\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})/[f]_{\mathcal{D}^p_\alpha\cap\mathcal{A}(\mathbb{D})})^*$ with norm $\|\ell\| = 1$ and define φ by

$$\varphi(\lambda) = \langle(\pi(1-\lambda g))^{-1}, \ell\rangle.$$

The function φ is analytic on $\mathbb{C}$ and

$$|\varphi(\lambda)| \leqslant e^{c|\lambda|^{\frac{1}{\gamma}}} \tag{3.2}$$

where $c = 2^{\frac{1}{\gamma}}\mathbf{A}$. Since $\gamma > 1$, there exists θ_γ such that $\frac{\pi}{2}(2-\gamma) < \theta_\gamma < \frac{\pi}{2}\gamma$. We suppose that $\theta_\gamma < \pi$. Consider now the sector

$$\mathcal{S}_{\theta_\gamma} = \{\lambda \in \mathbb{C} : |\arg\lambda| < \theta_\gamma\}.$$

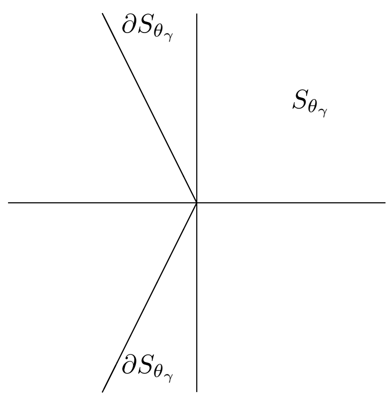

Figure 3.1. The secteur S_{θ_γ}

Let $\lambda \in \partial\mathcal{S}_{\theta_\gamma}$, since $\pi/2 < \theta_\gamma < \pi$, $\mathrm{Re}(\lambda) \leqslant 0$ and $\mathrm{Re}(\frac{1}{\lambda} - g(z)) \leqslant 0$. We obtain

$$\begin{aligned}|1-\lambda g(z)| &= |\lambda|\left|\frac{1}{\lambda} - g(z)\right|\\ &\geqslant |\lambda|\left|\mathrm{Re}\left(\frac{1}{\lambda} - g(z)\right)\right|\\ &\geqslant |\lambda|\frac{|\mathrm{Re}(\lambda)|}{|\lambda|^2} = \frac{|\mathrm{Re}(\lambda)|}{|\lambda|}.\end{aligned}$$

Moreover, $\lambda \in \partial\mathcal{S}_{\theta_\gamma}$, hence

$$\mathrm{Re}(\lambda) = |\lambda|\cos\theta_\gamma.$$

If we set $C_\gamma = |\cos\theta_\gamma|^{-1} \neq 0$, we get

$$\frac{1}{|1-\lambda g(z)|} \leqslant \frac{|\lambda|}{|\mathrm{Re}(\lambda)|} = C_\gamma.$$

Then φ is analytic on $\mathcal{S}_{\theta_\gamma}$, continuous on $\overline{\mathcal{S}_{\theta_\gamma}}$ and satisfies

$$\begin{cases} |\varphi(\lambda)| \leqslant e^{c|\lambda|^{\frac{1}{\gamma}}} & \text{for } \lambda \in \mathcal{S}_{\theta_\gamma} \\ |\varphi(\lambda)| \leqslant C_\gamma & \text{for } \lambda \in \partial\mathcal{S}_{\theta_\gamma}. \end{cases}$$

Since $\frac{1}{\gamma} < \frac{\pi}{2\theta_\gamma}$ with $\theta_\gamma < \frac{\pi}{2}\gamma$, by the Phragmén–Lindelöf principle for a sector $\mathcal{S}_{\theta_\gamma}$, we have

$$|\varphi(\lambda)| \leqslant C_\gamma, \qquad \lambda \in \mathcal{S}_{\theta_\gamma}.$$

The function φ is an entire function and satisfies (3.2) on $\mathbb{C}$. Again using the Phragmén–Lindelöf principle for a sector

$$\mathcal{S} = \mathbb{C} \setminus \mathcal{S}_{\theta_\gamma} = \{\lambda \in \mathbb{C} : \theta_\gamma < \arg(\lambda) < 2\pi - \theta_\gamma\}.$$

Since $2\pi - 2\theta_\gamma$ we get

$$\begin{cases} |\varphi(\lambda)| \leqslant e^{c|\lambda|^{\frac{1}{\gamma}}} & \text{on } \lambda \in \mathcal{S} \\ |\varphi(\lambda)| \leqslant C_\gamma & \text{on } \lambda \in \partial\mathcal{S} \end{cases}$$

Since $\theta_\gamma > \frac{\pi}{2}(2-\gamma)$, $\frac{1}{\gamma} < \frac{\pi}{2\pi - 2\theta_\gamma}$ and

$$|\varphi(\lambda)| \leqslant C_\gamma \qquad \lambda \in \mathcal{S}.$$

Then φ is bounded on $\mathbb{C}$, By Liouville Theorem, φ is a constant function

$$\varphi(\lambda) = \varphi(0) = \langle \pi^{-1}(1), \ell\rangle, \qquad \lambda \in \mathbb{C}.$$

Thus, $\pi^{-1}(1-\lambda g) = \pi^{-1}(1) = \pi(1)$. For $|\lambda g| < 1$, we have $\pi(1) = \pi^{-1}(1 - \lambda g) = \sum_{n\geqslant 0} \lambda^n \pi^n(g)$. Consequently $\pi(g) = 0$ and $g \in [f]_{\mathcal{D}^p_\alpha \cap \mathcal{A}(\mathbb{D})}$, hence $[g]_{\mathcal{D}^p_\alpha} \subset [f]_{\mathcal{D}^p_\alpha}$. □

Remark 3.2. — We will construct two functions f and g that satisfy the condition of the Theorem 3.1, this answers a question of Sasha Borichev. A closed set E of the unit circle is said to be K-set (after Kotochigov), if there exists a positive constant c_E such that for any arc $I \subset \mathbb{T}$

$$\sup_{\zeta\in I} \mathrm{dist}(\zeta, E) \geqslant c_E |I|$$

where I denotes the length of I. K-sets arise as the interpolation sets for Hölder classes. Examples include the generalized cantor set [16, 17], we refer to [10, 14] for more details. Such a set fulfills the following condition

$$\frac{1}{|I|}\int_I \frac{|\zeta|}{\operatorname{dist}(\zeta,E)^{\alpha}} \leqslant |I|^{-\sigma}$$

for

$$\sigma < \left(\log\left(\frac{1}{1-c_E}\right)\right) \Big/ \left(\log\left(\frac{2}{1-c_E}\right)\right).$$

In particular, E has measure zero and $\log\operatorname{dist}(\zeta,E)\in L^1(\mathbb{T})$. Let $p>1$ be such that $\alpha+1\leqslant p\leqslant \alpha+2$. Let us now consider the outer function

$$|g(\zeta)| = \operatorname{dist}(\zeta,E)^{\beta}, \qquad \zeta\in\mathbb{T}.$$

Since E is K-set, by [10], $\operatorname{Re} g(z)>0$,

$$\operatorname{Re} g(z) \asymp |g(z)| \asymp \operatorname{dist}(z,E)^{\beta} \quad \text{and} \quad |g'(z)| \asymp \operatorname{dist}(z,E)^{\beta-1}, \quad z\in\mathbb{D}.$$

If $1/(2+\alpha)<\beta<1$, then

$$\mathcal{D}_{\alpha,p}(g) \asymp \int_{\mathbb{D}} \frac{\mathrm{d}A_{\alpha}(z)}{\operatorname{dist}(z,E)^{p(1-\beta)}} \lesssim \int_0^1 \frac{\mathrm{d}r}{(1-r)^{(\alpha+2)(1-\beta)-\alpha}} < \infty.$$

so $g\in\mathcal{A}(\mathbb{D})\cap\mathcal{D}^p_{\alpha}$. Now let $1/\gamma=\kappa$

$$f(z) = \exp(-1/g^{\kappa}(z)), \qquad z\in\mathbb{D}.$$

We have

$$f'(z) = \kappa\frac{g'(z)}{g(z)^{\kappa+1}}\exp(-1/g^{\kappa}(z)).$$

Thus $|f'(z)|\leqslant |g'(z)|$ and $f\in\mathcal{D}^p_{\alpha}$.

Let us conclude this work with a final remark. Denote by c_0 the logarithmic capacity and by c_{α} the α-capacity for $0<\alpha<1$. The case of Dirichlet spaces $\mathcal{D}^2_{\alpha}$, $0\leqslant\alpha<1$, was studied in [16, 17]. In particular, it was shown in that if $f\in\mathcal{D}^2_{\alpha}\cap\mathcal{A}(\mathbb{D})$, is an outer function such that $\mathcal{Z}(f)$ is a generalized cantor set, then f is cyclic in $\mathcal{D}^2_{\alpha}$ if and only if $c_{\alpha}(\mathcal{Z}(f))=0$. We do not know if this result also holds for $\mathcal{D}^p_{\alpha}\cap\mathcal{A}(\mathbb{D})$.

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Bibliography

[1] A. Aleman, "Hilbert spaces of analytic functions between the Hardy and the Dirichlet space", *Proc. Am. Math. Soc.* **115** (1992), no. 1, p. 97-104.

[2] E. Amar & P. J. Thomas, "Cyclicity of nonvanishing functions in the polydisk and in the ball", *St. Petersbg. Math. J.* **31** (2020), no. 5, p. 751-768.

[3] N. Arcozzi, R. Rochberg, E. T. Sawyer & B. D. Wick, *The Dirichlet space and related function spaces*, Mathematical Surveys and Monographs, vol. 239, American Mathematical Society, 2019, xix+536 pages.

[4] A. Atzmon, "Operators which are annihilated by analytic functions and invariant subspaces", *Acta Math.* **144** (1980), no. 1-2, p. 27-63.

[5] H. Bahajji-El Idrissi & H. El Azhar, "Analytic Besov spaces, approximation, and closed ideals", *Can. Math. Bull.* **66** (2023), no. 1, p. 259-268.

[6] A. Beurling, "On two problems concerning linear transformations in Hilbert space", *Acta Math., Uppsala* **81** (1949), p. 239-255.

[7] A. Bourhim, O. El-Fallah & K. Kellay, "Boundary behaviour of functions of Nevanlinna class", *Indiana Univ. Math. J.* **53** (2004), no. 2, p. 347-395.

[8] B. Bouya, O. El-Fallah & K. Kellay, "Idéaux fermés d'algèbres de Beurling analytiques sur le bidisque", *Bull. Sci. Math.* **134** (2010), no. 6, p. 588-604.

[9] L. Brown & A. L. Shields, "Cyclic vectors in the Dirichlet space", *Trans. Am. Math. Soc.* **285** (1984), p. 269-304.

[10] J. Bruna, "On the peak sets for holomorphic Lipschitz functions", *Indiana Univ. Math. J.* **32** (1983), no. 2, p. 257-272.

[11] L. Carleson, "Sets of uniqueness for functions regular in the unit circle", *Acta Math.* **87** (1952), p. 325-345.

[12] ———, "Interpolations by bounded analytic functions and the corona problem", *Ann. Math.* **76** (1962), p. 547-559.

[13] Ş. Costea, E. T. Sawyer & B. D. Wick, "The Corona Theorem for the Drury–Arveson Hardy space and other holomorphic Besov–Sobolev spaces on the unit ball in $\mathbb{C}^n$", *Anal. PDE* **4** (2011), no. 4, p. 499-550.

[14] E. M. Dyn'kin, "Free interpolation sets for Hölder classes", *Mat. Sb., N. Ser.* **109** (1979), p. 107-128.

[15] O. El-Fallah, K. Kellay, J. Mashreghi & T. Ransford, *A primer on the Dirichlet space*, Cambridge Tracts in Mathematics, vol. 203, Cambridge University Press, 2014, xii+211 pages.

[16] O. El-Fallah, K. Kellay & T. Ransford, "On the Brown-Shields conjecture for cyclicity in the Dirichlet space", *Adv. Math.* **222** (2009), no. 6, p. 2196-2214.

[17] ———, "Cantor sets and cyclicity in weighted Dirichlet spaces", *J. Math. Anal. Appl.* **372** (2010), no. 2, p. 565-573.

[18] ———, "Invariant subspaces of the Dirichlet space", in *Hilbert spaces of analytic functions*, CRM Proceedings & Lecture Notes, vol. 51, American Mathematical Society, 2010, p. 133-141.

[19] O. El-Fallah, K. Kellay & K. Seip, "Cyclicity of singular inner functions from the Corona Theorem", *J. Inst. Math. Jussieu* **22** (2012), no. 4, p. 815-824.

[20] H. Hedenmalm & A. L. Shields, "Invariant subspaces in Banach spaces of analytic functions", *Mich. Math. J.* **37** (1990), no. 1, p. 91-104.

[21] H. B.-E. Idrissi & O. El-Fallah, "Approximation in spaces of analytic functions", *Stud. Math.* **255** (2020), no. 2, p. 209-217.

[22] K. Kellay, F. Le Manach & M. Zarrabi, "Cyclicity in Dirichlet type spaces", in *Complex analysis and spectral theory*, Contemporary Mathematics, vol. 743, American Mathematical Society, 2020, p. 181-193.

[23] A. Nicolau, "The corona property for bounded analytic functions in some Besov spaces", *Proc. Am. Math. Soc.* **110** (1990), no. 1, p. 135-140.
[24] S. Richter & C. Sundberg, "A formula for the local Dirichlet integral", *Mich. Math. J.* **38** (1991), no. 3, p. 355-379.
[25] ———, "Invariant subspaces of the Dirichlet shift and pseudocontinuations", *Trans. Am. Math. Soc.* **341** (1994), no. 2, p. 863-879.
[26] J. W. Roberts, "Cyclic inner functions in the Bergman spaces and weak outer functions in H^p, $0 < p < 1$", *Ill. J. Math.* **29** (1985), p. 25-38.
[27] J. H. Shapiro & A. L. Shields, "Unusual topological properties of the Nevanlinna class", *Am. J. Math.* **97** (1975), no. 4, p. 915-936.
[28] V. A. Tolokonnikov, "The Corona Theorem in algebras of bounded analytic functions", *Transl., Ser. 2, Am. Math. Soc.* **149** (1991), p. 61-93.