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Existence and global Lipschitz estimates for unbounded classical solutions of a Hamilton–Jacobi equation (*)

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Abstract. — The purpose of this article is to prove existence, uniqueness and uniform gradient estimates for unbounded classical solutions of a Hamilton–Jacobi–Bellman equation. Such an equation naturally arises in stochastic control problems. Contrary to the classical literature which handles the case of bounded regular coefficients, we only impose Lipschitz regularity conditions, allowing for a linear growth of coefficients. These Lipschitz assumptions are natural in a probabilistic setting. In principle, these assumptions are compatible with global Lipschitz regularity for the solution. However, to the best of our knowledge, this useful result had not been established before. Our proofs rely on the Ishii–Lions method [36]. We combine several elements from the viscosity solution theory to obtain estimates at the edges of what seems possible.

Résumé. — Cet article établit l'existence et l'unicité de la solution classique non-bornée d'une équation de Hamilton–Jacobi–Bellman, ainsi que des estimées uniformes sur son gradient. Cette équation apparait naturellement en contrôle stochastique. Contrairement aux résultats classiques qui traitent le cas de coefficients bornés et réguliers, nous ne demandons qu'une régularité Lipschitz qui inclut des coefficients non-bornés à croissance linéaire. Ces hypothèses Lipschitz sont naturelles dans un cadre probabiliste. En principe, ces hypothèses permettent d'obtenir une régularité Lipschitz pour la solution. Cependant ce résultat très utile n'avait jamais été établi à notre connaissance. Nos preuves reposent sur la méthode développée par Ishii–Lions [36] pour les solutions de viscosité. Plusieurs éléments classiques sont combinés afin d'obtenir ce résultat à la limite de ce qui semble possible.

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1. Introduction

For any integer $d \geqslant 1$ and $T > 0$, we consider the following Hamilton–Jacobi–Bellman (henceforth, HJB) equation:

$$\begin{cases} \partial_t u(t,x) + cu(t,x) + \frac{1}{2}\operatorname{Tr}[\sigma(t,x)\sigma^\top(t,x)D^2u(t,x)] \\ \qquad\qquad + \inf_{\alpha\in\mathcal{A}}[b(t,x,\alpha)\cdot Du(t,x) + f(t,x,\alpha)] = 0, \\ u(T,x) = g(x), \quad (t,x)\in(0,T)\times\mathbb{R}^d. \end{cases} \tag{1.1}$$

for $\sigma : [0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times d}$, $b : [0,T]\times\mathbb{R}^d\times\mathcal{A} \to \mathbb{R}^d$, $f : [0,T]\times\mathbb{R}^d\times\mathcal{A} \to \mathbb{R}$ and $c \in \mathbb{R}$ is a constant. The control set $\mathcal{A}$ is a possibly unbounded closed domain in $\mathbb{R}^d$. The semi-linear equation (1.1) naturally arises in finite-time horizon stochastic control problems. More specifically, on a filtered probability space $(\Omega, (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathbb{P})$, we consider the controlled dynamics in $\mathbb{R}^d$,

$$\begin{cases} \mathrm{d}X_s^{t,x,\alpha} = b(t, X_s^{t,x,\alpha}, \alpha_s)\mathrm{d}s + \sigma(s, X_s^{t,x,\alpha})\mathrm{d}B_s, \quad t\leqslant s\leqslant T, \\ X_t^{t,x,\alpha} = x, \end{cases} \tag{1.2}$$

where $(B_s)_{0\leqslant s\leqslant T}$ is a $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$-Brownian motion, and the control process $\alpha := (\alpha_s)_{0\leqslant s\leqslant T}$ is any $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$-adapted $\mathcal{A}$-valued stochastic process. The related cost function is

$$V(t,x) := \inf_\alpha \mathbb{E}\int_t^T e^{c(s-t)} f(s, X_s^{t,x,\alpha}, \alpha_s)\mathrm{d}s + g(X_T^{t,x,\alpha}). \tag{1.3}$$

If (1.1) has a $C^{1,2}$ solution u which satisfies suitable bounds, then it is classical that $u = V$ (see e.g. [26, Chapter III.8,Theorem III.8.1]), and it is possible to compute optimal controls for (1.2)–(1.3) by looking for feed-back controls $\overline{\alpha}_s := \overline{\alpha}(s, X_s^{t,x,\overline{\alpha}})$ such that

$$\overline{\alpha}(s,x) \in \operatorname{argmin}_{\alpha\in\mathcal{A}}[b(s,x,\alpha)\cdot Du(s,x) + f(s,x,\alpha)]. \tag{1.4}$$

However, having such a classical solution often requires strong assumptions on the coefficients. A standard reference on this issue is [25, Theorem 6.2], which provides classical solutions for (1.1) when b, f and σ are at least $C_b^{1,2}$ (bounded functions with bounded continuous derivatives) and the control set $\mathcal{A}$ is compact. In the setting of [25], the solution of (1.1) is a $C_b^{1,2}$ function. Our main objective is the following result:

Theorem 1.1. — *If b_2, σ and f_2 are Lipschitz-continuous functions, with σ bounded and $\sigma\sigma^\top$ uniformly elliptic, then the HJB equation*

$$\partial_t u(t,x) + \frac{1}{2}\operatorname{Tr}[\sigma\sigma^\top(t,x)D^2u(t,x)] + b_2(t,x)\cdot Du \\ - \frac{1}{2}|\sigma^\top(t,x)Du(t,x)|^2 + f_2(t,x) = 0, \tag{1.5}$$

with globally Lipschitz terminal data $u(T,x) = g(x)$, has a unique linear growth $C^{1,2}$ solution u, which has a uniformly bounded gradient.

These Lipschitz assumptions on the coefficients are natural in a probabilistic setting. Indeed, from the Itô theorem [48], a natural sufficient assumption for (1.2) to be well-posed is that both b and σ are globally Lipschitz (α must satisfy some moment bound too). In particular, this framework allows for a linear growth of the coefficients, which implies that the solution of (1.5) can have linear growth. Equation (1.5) is a particular case of (1.1) when specifying $b(t,x,\alpha) = \sigma^\top(t,x)\alpha + b_2(t,x)$, $f(t,x,\alpha) = \frac{1}{2}|\alpha|^2 + f_2(t,x)$ together with the unbounded control set $\mathcal{A} = \mathbb{R}^d$. In this case, (1.4) reduces to $\bar{\alpha}(s,x) = -\sigma^\top(s,x)Du(s,x)$. In particular, using the global bound on σ, Theorem 1.1 proves that optimal controls are globally bounded. Theorem 1.1 will be a consequence of Theorem 2.2 below, which is concerned with more general equations of type (1.1). We now summarise the results in the literature that are close to ours.

Well-posedness for (1.5) is related to well-posedness for the linear parabolic equation

$$\partial_t v + b_2 \cdot Dv + \frac{1}{2}\operatorname{Tr}[\sigma\sigma^\top D^2 v] - f_2 v = 0, \tag{1.6}$$

which can be obtained from (1.5) using the Cole–Hopf transform $v = e^{-u}$. Standard references for (1.6) are [28, 31, 41] which provide classical well-posedness and Hölder-regularity estimates when the coefficients b, σ and f_2 are regular enough (at least Hölder-continuous), either in bounded domains, or in $\mathbb{R}^d$ when the coefficients b, σ and f_2 are also globally bounded. Their work also handles Lipschitz non-linear perturbations of (1.6). This latter extension includes (1.1) when the control set $\mathcal{A}$ is compact. Similar results for (1.1) are proved in [39, 40] when f is bounded uniformly in α and the coefficients are twice differentiable.

Well-posedness for linear parabolic equations with unbounded coefficients is a long-standing topic. Historical approaches [8, 9, 38] rely on the parametrix method and provide existence and bounds on a Green function for (1.6) under a Lyapunov-type assumption, but they also require heavy differentiability conditions on the coefficients. See also [23] for a more recent generalisation. On the other side, the solution of (1.6) admits a stochastic representation using the Feynman–Kac formula. This formula can be used in turn to build a classical solution for (1.6), but this approach [29] also requires heavy differentiability assumptions on the coefficients. Another noticeable limitation is the assumption that f_2 is bounded from below, which is necessary to guarantee that solutions do not explode. The removal of this restriction on f_2 can imply the loss of the maximum principle and non-uniqueness situations.

More recently, the works [6, 7, 27, 32, 42]… provide classical well-posedness, existence of Green functions and regularity estimates for linear parabolic equations with unbounded coefficients. These results rely on semi-group methods, under a fairly general Lyapunov-type condition. This framework also extends to Lipschitz non-linear perturbations, see e.g. [1]. However, these results mostly focus on globally bounded solutions, a situation which cannot be expected in our framework for (1.1). In [37], existence of generalised strong solutions with quadratic growth is shown for a class of Hamilton–Jacobi equations whose non-linearity in Du includes the one in (1.5). A uniform bound on Du is proved in [37, Theorem 2.3] in some situations where f_2 is Lipschitz and u is globally bounded, and [37, Theorem 2.7] proves linear growth for Du when f_2 has quadratic growth. However, these results require a strong confinement assumption on the drift b_2 and do not provide classical solutions.

An efficient method to prove global Lipschitz estimates on the solution u of an elliptic or parabolic equation of the kind

$$\partial_t u(t,x) + H(t,x,Du(t,x),D^2u(t,x)) = 0, \tag{1.7}$$

is the classical Bernstein method (see e.g. [31]). By differentiating the equation, this approach looks for a linear equation satisfied by $w := |Du|^2$, and estimates are obtained using standard maximum principle-type arguments. In particular, this method requires extra-regularity properties on u for the differential of (1.7) to make sense, or it needs to approximate (1.7) by a smoothed version and to use continuation methods. A successful application of this approach can be found in [21] under assumptions on H which guarantee that u is $C^{1,2}_b$, namely coercivity and quadratic growth in Du, together with Lipschitz-regularity of $D_x H$ w.r.t. $D^2 u$ and the condition (1.8) below. This method is extended to linear-growth solutions of (1.5) when $\sigma \equiv \mathrm{Id}$ in [22], well-posedness resulting from a fixed-point method. However, [21, 22] cannot handle more general σ, because the Bernstein method they use requires the structural assumption that

$$\forall\, p \in \mathbb{R}^d, \qquad \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |D_x H(t,x,p,0)| \leqslant C[1+|p|], \tag{1.8}$$

for some constant $C > 0$. This assumption is not satisfied by (1.5) in general if $\sigma \neq \mathrm{Id}$.

Another well-posedness approach for (1.1) is the viscosity solution theory [3, 16, 17]. Under suitable structural assumptions like (1.8), existence and comparison principle (hence uniqueness) for bounded uniformly continuous solutions of (1.1) are by now standard results (see e.g. [26, Chapter 6]). Extensions to linear-growth (or more general growth) situations can be found in e.g. [5, 18, 19, 34, 35]. A linear growth situation similar to ours is [30], but still keeping a structural condition of the kind (1.8). A general result that covers existence and uniqueness for viscosity solutions of (1.1) in our Lipschitz setting (and far more general settings) is [20].

The next step is to prove Lipschitz estimates on the obtained viscosity solution. A seminal work to prove regularity estimates is [36]: under uniformly elliptic or parabolic assumptions, the Ishii–Lions method provides Hölder or Lipschitz estimates, see [4] for a summarised presentation. When Lipschitz estimates are available, the Ishii–Lions method even implies semi-concavity estimates. For Lipschitz estimates, the main idea is to prove that

$$\sup_{(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d} u(t,x) - u(t,y) - K|x-y| \leqslant 0$$

for large enough K, by reasoning by contradiction and combining the PDE and the second-order optimality conditions at a maximum point. Similarly, the analogous of the Bernstein method can be developed for viscosity solutions: this is the weak Bernstein method [2]. Contrary to the classical Bernstein method, this method does not require any additional differentiability assumption on the solution, but it requires similar structure conditions on the equation. In [30], Lipschitz estimates and concavity-preserving properties are shown for linear growth viscosity solutions of (1.7). However, their setting does not allow for the linear growth of b in (1.5) and the structure condition (1.8) is required. Lipschitz estimates are also obtained for globally bounded solutions of an equation similar to (1.1) in [43, 44].

After proving existence, uniqueness, and Lipschitz estimates for viscosity solutions, it is possible to obtain semi-concavity and higher-order regularity results using e.g. [13, 14, 33, 49]. This strategy is applied to (1.7) in [21], under (1.8) and assumptions that guarantee the global boundedness of solutions.

On the stochastic side, the representation formula (1.3) suggests a natural way to obtain Lipschitz estimates by coupling the characteristic curves. If $f(s,x,\alpha)$ is Lipschitz in x uniformly in α and $\mathbb{E}[|X_s^{t,x,\alpha} - X_s^{t,y,\alpha}|] \leqslant C|x-y|$ for some $C > 0$, then the estimate easily follows. This method is successfully applied in the reference textbooks [26, 40, 50] and generalised in [12]. In the setting of (1.5), the lack of a priori bounds on α when $\sigma \neq \mathrm{Id}$ does not allow for such a control of $\mathbb{E}[|X_s^{t,x,\alpha} - X_s^{t,y,\alpha}|]$. Nevertheless, the stochastic setting provides a further degree of freedom because the Brownian motion driving $X_s^{t,x,\alpha}$ needs not be the same as the one of $X_s^{t,y,\alpha}$. The reflection coupling [24] exploits this fact to produce uniform in time gradient estimates [46] for (1.6). This method is adapted to HJB equations like (1.5) in [15], but assuming $\sigma \equiv \mathrm{Id}$. A correspondence between reflection coupling and doubling of variables methods is established in [45], recovering and extending the results of [46] in a viscosity solution setting. Another coupling method is developed in [10, 11], exploiting the invariance of the Brownian motion under parabolic re-scaling. This yields global Lipschitz and semi-concavity estimates in (t, x) for (1.5), the main novelty being the Lipschitz-dependence in t. However, their work requires global boundedness of coefficients and Lipschitz-regularity uniformly in α. Their setting can be extended to include linear growth w.r.t. x [10, Remark 2.1] but not w.r.t. α: this does not cover the quadratic term in (1.5).

To the best of our knowledge, the closest results to our setting are the ones in [47], which provide classical well-posedness for polynomial-growth solutions of (1.6) under polynomial-growth conditions on the coefficients. Their proof strategy combines both stochastic representation formulae and classical domain truncation methods. Moreover, this work provides polynomial-growth solutions for (1.1) when the control set $\mathcal{A}$ is compact.

At this point, our strategy for proving Theorem 1.1 is the following one. From [20], comparison, hence uniqueness, already holds for viscosity solutions of (1.5). We then prove a priori Lispchitz estimates for $C^{1,2}$ solutions of (1.5). Existence follows by truncating the non-linearity to make it enter the scope of [47]. To obtain the a priori bounds, we rely on the Ishii–Lions method [34], exploiting the uniform ellipticity in (1.5) by using a strictly concave test function. Since solutions are classical, there is not need for the Jensen–Ishii lemma nor doubling of variables method. No new method is developed, but several tricks from the viscosity solution theory must be combined to obtain this estimate, at the edges of what the setting allows for. Up to our knowledge, this delicate estimate has no equivalent using stochastic approaches: it would be interesting to search for its analogous using coupling methods. Another challenging issue would be to prove Hessian estimates for (1.5) under minor strengthening of our assumptions, similarly to [15, Proposition 3.2].

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2. Assumptions and main result

In the following, the control set $\mathcal{A}$ is a possibly unbounded closed domain in $\mathbb{R}^d$. The notation $|x| := \sqrt{x^\top x}$ refers to the usual Euclidean norm on $\mathbb{R}^d$.

Similarly, we will use the Euclidean norm $|\sigma| := \sqrt{\mathrm{Tr}[\sigma\sigma^\top]}$ on matrices. We write the running cost f as

$$f(t,x,\alpha) = f_1(t,x,\alpha) + f_2(t,x).$$

In the following, we restrict ourselves to

$$b(t,x,\alpha) = b_1(t,x,\alpha) + b_2(t,x), \tag{2.1}$$

the b_1 part being uniformly bounded in x. Although we can handle some more general equations, we emphasise that the main purpose of the following results is to prove Theorem 1.1 in the Introduction. We also handle the case of unbounded diffusion matrices σ. Let us now detail the needed regularity assumptions for our more general results.

Assumption A.1 (Regularity and growth). — *The coefficients b_1, f_1 (resp. b_2, σ and f_2) are continuous functions on $[0,T]\times\mathbb{R}^d\times\mathcal{A}$ (resp. $[0,T]\times\mathbb{R}^d$), which satisfy the following conditions.*

- *Regularity and growth for b_1 and b_2: there exists $L_b > 0$ such that for every (t,x,y,α) in $[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{A}$,*

$$\begin{aligned} |b_1(t,x,\alpha) - b_1(t,y,\alpha)| &\leqslant L_b(1+|\alpha|)|x-y|, \\ |b_2(t,x) - b_2(t,y)| &\leqslant L_b|x-y|, \\ |b_1(t,x,\alpha)| &\leqslant L_b[1+|\alpha|], \\ |b_2(t,x)| &\leqslant L_b[1+|x|]. \end{aligned} \tag{2.2}$$

- *Regularity and coercivity for f_1: there exist $L_{f_1}, c_{f_1}, c'_{f_1}, C_{f_1}, C'_{f_2} > 0$ such that for every (t,x,y,α) in $[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{A}$,*

$$|f_1(t,x,\alpha) - f_1(t,y,\alpha)| \leqslant L_{f_1}|x-y|, \tag{2.3}$$

together with

$$c_{f_1}|\alpha|^2 - c'_{f_1} \leqslant f_1(t,x,\alpha) \leqslant C_{f_1}|\alpha|^2 + C'_{f_1}. \tag{2.4}$$

- *Regularity and growth for f_2: there exists $L_{f_2} \in C([0,T],\mathbb{R}_+)$ such that for every (t,x,y) in $[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$,*

$$|f_2(t,x) - f_2(t,y)| \leqslant L_{f_2}(t)|x-y|, \quad |f_2(t,x)| \leqslant L_{f_2}(t)[1+|x|]. \tag{2.5}$$

- *Regularity and growth for σ and g: there exist L_σ, $L_g > 0$ such that for every (t,x,y) in $[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$,*

$$|\sigma(t,x) - \sigma(t,y)| \leqslant L_\sigma|x-y|^{1/2}, \quad |g(x) - g(y)| \leqslant L_g|x-y|.$$

As explained in Section 1, Lipschitz assumptions on the coefficients are natural in a probabilistic setting to guarantee well-posedness for (1.2). Assumption A.1 is needed to prove Lipschitz estimates on the solution u of (1.1).

Assumption A.2 (Further assumptions on $\sigma\sigma^\top$). —

- *The matrix $\sigma\sigma^\top$ is uniformly elliptic: there exists $\eta_\sigma > 0$ such that for every $(t,x) \in [0,T] \times \mathbb{R}^d$,*

$$\forall\, \xi \in \mathbb{R}^d, \quad \xi^\top \sigma\sigma^\top(t,x)\,\xi \geqslant \eta_\sigma |\xi|^2.$$

- *The matrix $\sigma\sigma^\top$ is uniformly continuous in x: there exists a uniform continuity modulus $m_\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ such that for every (t,x,y) in $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$,*

$$\left|\sigma\sigma^\top(t,x) - \sigma\sigma^\top(t,y)\right| \leqslant m_\sigma(|x-y|).$$

If A.1 is satisfied, uniform continuity for $\sigma\sigma^\top$ always holds if σ is globally bounded (as it is the case in Theorem 1.1). An unbounded σ which satisfies A.1–A.2 is e.g. $\sigma(t,x) = \sqrt{1+|x|}\,\mathrm{Id}$. Assumption A.2 is needed to apply the Ishii–Lions method, and we did not manage to alleviate it. In the presence of degeneracy, existence of classical solutions to (1.1) can be lost.

Definition 2.1 (Local Hölder spaces). — *For integers $n, n' \geqslant 1$ and $\beta, \beta' \in (0,1]$, a function ψ in $C^{n,n'}((0,T) \times \mathbb{R}^d)$ belongs to the local Hölder space $C^{n,n',\beta,\beta'}_{\mathrm{loc}}$ if for every compact set $F \subset (0,T) \times \mathbb{R}^d$, there exists $K_F > 0$ such that ψ and all its derivatives satisfy*

$$\forall\,(t,x),(s,y) \in F, \quad |\psi(t,x) - \psi(s,y)| \leqslant K_F[|t-s|^\beta + |x-y|^{\beta'}]. \tag{2.6}$$

For every differentiable $\psi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, we define the set

$$A[\psi](t,x) := \mathrm{argmin}_{\alpha \in \mathcal{A}} [b_1(t,x,\alpha) \cdot D\psi(t,x) + f_1(t,x,\alpha)]. \tag{2.7}$$

The following assumption is not required to prove Lipschitz estimates. This assumption is copied from [47, Assumption H3-(5)] to obtain existence of classical solutions with Lipschitz non-linearity. It can be seen as a black-box to use results of [47].

Assumption A.3 (Hölder regularity). — *There exists $\beta \in (0,1)$ such that the coefficients b, σ and f belong to $C^{0,0,\beta,1}_{\mathrm{loc}}$. The coefficients b_1 and f_1 are locally Lipschitz in α in the sense of (2.6). Moreover, for every $\beta' \in \{\beta, 1\}$ and every ψ in $C^{0,1,\beta,\beta'}_{\mathrm{loc}}$, $A[\psi]$ is a singleton, defining a function of (t,x) which belongs to $C^{0,0,\beta,\beta'}_{\mathrm{loc}}$.*

Denoting by $B(0,R)$ the centred ball B of $\mathbb{R}^d$ with radius R, we eventually require that the above property on $A[\psi]$ holds when replacing $\mathcal{A}$ by $\mathcal{A} \cap B(0,R)$ for every $R \geqslant R_{\mathcal{A}}$, for a large enough $R_{\mathcal{A}} > 0$.

In practice, regularity properties for $A[\psi]$ are easier to study when assuming convexity properties in α for the coefficients. Once again, we point out that A.1–A.2–A.3 are satisfied in the setting of Theorem 1.1.

Theorem 2.2 (Well-posedness and uniform gradient bound). — *Under A.1–A.2–A.3, Equation* (1.1) *has a unique linear growth solution u in $C([0,T]\times\mathbb{R}^d)\cap C^{1,2,\beta,\beta}_{\mathrm{loc}}$. Moreover, there exists a constant $K>0$ such that*

$$\sup_{(t,x)\in(0,T)\times\mathbb{R}^d}|Du(t,x)|\leqslant K. \tag{2.8}$$

The constant K only depends on the regularity constants introduced in A.1–A.2–A.3, and K only depends on f_2 through $\|L_{f_2}\|_{L^1(0,T)}$.

The dependence on f_2 is emphasised to compare Theorem 2.2 to [22, Theorem 1.1 and Lemma A.2], where a similar dependence is obtained under the additional assumption (1.8). In [22, Lemma A.2], this dependence is obtained using the classical Bernstein method. However, this method does not work without (1.8) and Theorem 2.2 extends the results of [22] to our setting. In particular, following [22], it is possible to extend Theorem 2.2 to measure-valued source terms f_2 that are useful when dealing with constrained stochastic control problems.

Under A.1, for every (t,x,α) in $[0,T]\times\mathbb{R}^d\times\mathcal{A}$, the quantity to minimise on the r.h.s. of (2.7) is bounded from below by

$$F(|\alpha|):=-L_b[1+|\alpha|]|D\psi(t,x)|+c_{f_1}|\alpha|^2-c'_{f_1},$$

and bounded from above by

$$C(\alpha_0):=L_b[1+|\alpha_0|]|D\psi(t,x)|+C_{f_1}|\alpha_0|^2+C'_{f_1},$$

for some (fixed) $\alpha_0\in\mathcal{A}$. As a consequence, minimisers for (2.7) belong to $\{\alpha\in\mathcal{A},\, F(|\alpha|)\leqslant C(\alpha_0)\}$. Using standard computations for second-order polynomials, there exists $L_{\mathcal{A}}>0$ independent of ψ, such that for every (t,x) in $[0,T]\times\mathbb{R}^d$,

$$\forall\,\overline{\alpha}\in A[\psi](t,x),\quad |\overline{\alpha}|\leqslant L_{\mathcal{A}}[1+|D\psi(t,x)|]. \tag{2.9}$$

This property will be crucial in the proofs of Lipschitz estimates. If Du is bounded, then (2.9) guarantees using (1.4) that optimal controls for (1.2)–(1.3) are globally bounded. In the following, all the estimates will only depend on $\mathcal{A}$ through $L_{\mathcal{A}}$.

Remark 2.3 (About the coercivity assumption). — The property (2.9) is the only reason for the decomposition (2.1) and (2.4). The lower bound in (2.4) is a coercivity assumption which is standard in stochastic control. The dependence on $|\alpha|$ for the upper bound is not really restrictive, since this term is only meant to bound the quantity to minimise in (2.7) from above, independently of x. Any other assumption that guarantees (2.9) could replace (2.1)–(2.4).

Remark 2.4 (Adding a linear term). — It would be uneasy to handle a non-constant $c = c(t, x, \alpha)$, because we need cu to be globally-Lipschitz when u is only globally Lipschitz. This is very restrictive, but it would be possible to adapt the result for e.g. $c(t, x, \alpha) = \frac{c_0|\alpha|}{(1+|\alpha|)(1+|x|)}$.

3. Proof of global Lipschitz estimates

As explained in Section 1, we are going to prove a priori global Lipschitz estimates on linear growth $C^{1,2}$ solutions of (1.1). We first start with a short lemma about the growth of such solutions. inherited from [20].

Lemma 3.1. — *Let u in $C([0,T] \times \mathbb{R}^d) \cap C^{1,2}((0,T) \times \mathbb{R}^d)$ be a quadratic growth solution of* (1.1)*, in the sense that*

$$\exists\, C_u > 0 : \forall\, (t, x) \in [0, T] \times \mathbb{R}^d, \quad |u(t, x)| \leqslant C_u[1 + |x|^2].$$

Then under A.1, u has linear growth:

$$\forall\, (t, x) \in [0, T] \times \mathbb{R}^d, \quad |u(t, x)| \leqslant L[1 + |x|],$$

for a positive constant L which does not depend on u and only depends on $(\mathcal{A}, f_2)$ through $(L_{\mathcal{A}}, \|L_{f_2}\|_{L^1})$.

Proof. — The function $z \in \mathbb{R}^d \mapsto (1+|z|^2)^{1/2}$ is C^2 and convex, and thus has non-negative Hessian. From A.1, $(t, x) \mapsto \sqrt{2}(1 + |x|^2)^{1/2}[\int_0^t L_{f_2}(s)\mathrm{d}s + Le^t]$ is a super-solution of (1.1) for large enough $L > 0$. Similarly, $(t, x) \mapsto \sqrt{2}(1+|x|^2)^{1/2}[e^{-\delta t} - \int_0^t L_{f_2}(s)\mathrm{d}s]$ is a sub-solution of (1.1) for large enough $\delta > 0$. The result then follows from the comparison principle proved in [20, Theorem 2.1]. □

Noticeably, the above result allows us to include quadratic growth solutions in the uniqueness result of Theorem 2.2. We now prove Lipschitz estimates by showing that

$$\sup_{t,x,y} u(t, x) - u(t, y) - K|x - y| \leqslant 0 \tag{3.1}$$

for large enough K. The supremum is taken over $t \in [0, T]$ and $x, y \in \mathbb{R}^d$. We reason by contradiction and we aim at combining the PDE and the second-order optimality conditions. The Ishii–Lions method provides a way to obtain a contradiction by making the second-order terms explode at a maximum point $(\bar{x}, \bar{y})$. A key feature for this approach is to prove that $|\bar{x} - \bar{y}| \to 0$ as $K \to +\infty$. Technical difficulties arise at this stage: there is no guarantee that the supremum in (3.1) is finite and this supremum does not need to be realised at some point. This latter difficulty is classically circumvented by adding regularising terms to the supremum and progressively removing them.

However, we need to know a priori that the supremum is finite to ensure that $|\bar{x}-\bar{y}| \to 0$ as $K \to +\infty$. To do so, we reason as in [16, Theorem 5.1] and we use the linear growth to first prove a deteriorated estimate.

Lemma 3.2 (Deteriorated estimate). — *Let u in $C([0,T]\times\mathbb{R}^d) \cap C^{1,2}((0,T)\times\mathbb{R}^d)$ be a solution of* (1.1) *and $L_u > 0$ such that for every $(t,x) \in [0,T]\times\mathbb{R}^d$,*

$$|u(t,x)| \leqslant L_u[1+|x|]. \tag{3.2}$$

Under A.1, there exist constants $\widetilde{K}, \widetilde{M} > 0$ such that

$$\sup_{(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d} u(t,x) - u(t,y) - \widetilde{K}|x-y| \leqslant \widetilde{M}, \tag{3.3}$$

where $(\widetilde{K},\widetilde{M})$ only depends on $(u,\mathcal{A},f_2)$ through $(L_u, L_{\mathcal{A}}, \|L_{f_2}\|_{L^1})$.

Proof. — We proceed in several steps.

Step 1: Regularising parameters. — For any $\gamma > 0$, we notice that $v : (t,x) \mapsto e^{\gamma t}u(t,x)$ is $C^{1,2}$. Moreover, $v(T,x) = e^{\gamma T}g(x)$ and v satisfies

$$\begin{aligned} \partial_t v(t,x) - (\gamma - c)v(t,x) + \frac{1}{2}\operatorname{Tr}[\sigma\sigma^\top(t,x)D^2v(t,x)] \\ + e^{\gamma t}f_2(t,x) + \inf_{\alpha\in\mathcal{A}} L_\alpha[v](t,x) = 0, \end{aligned} \tag{3.4}$$

where for every α in $\mathcal{A}$, we define

$$L_\alpha[v](t,x) := b(t,x,\alpha)\cdot Dv(t,x) + e^{\gamma t}f_1(t,x,\alpha). \tag{3.5}$$

It is equivalent to prove (3.3) for v instead of u. The growth condition (3.2) is satisfied by v with $e^{\gamma T}L_u$ instead of L_u. As a consequence,

$$\begin{aligned} \overline{M} := \sup_{t,x,y} v(t,x) - v(t,y) - \varepsilon t^{-1} \\ - \left[\widetilde{K} - e^{\gamma T}\int_0^t L_{f_2}(s)\mathrm{d}s\right][1+|x-y|^2]^{1/2} - \varepsilon e^{-\delta t}[|x|^2+|y|^2] \end{aligned} \tag{3.6}$$

is finite for every $\varepsilon,\delta,\gamma,\widetilde{K} > 0$, because of the quadratic term. To alleviate notations, we do not emphasise the dependence on γ for v, nor the dependence on $(\varepsilon,\delta,\gamma,\widetilde{K})$ for $\overline{M}$. The L_{f_2}-term within the supremum will allow us to control the f_2-term in (3.4) using only $\|L_{f_2}\|_{L^1}$. In the following, δ, γ and $\widetilde{K}$ are fixed parameters whose values will be chosen later on. On the contrary, ε will be sent to 0, and we restrict ourselves to $\varepsilon \leqslant 1$. Using the continuity and the linear growth of v, the quadratic term ensures that the supremum (3.6) is realised for some $(\bar{t},\bar{x},\bar{y}) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d$. Since $\overline{M}$ is a supremum, it is necessary that $\bar{t} \neq 0$. In the following, we restrict ourselves to $\widetilde{K} \geqslant e^{\gamma T}\|L_{f_2}\|_{L^1}$.

Step 2: Bounds on the optimiser. — If there existed a sequence $(\varepsilon_k)_{k\geqslant 1}$ of positive parameters converging to 0 and such that the related sequence $(\bar{t}_k)_{k\geqslant 1}$ had a constant sub-sequence equal to T, then for every $t \in [0,T]$ and $x, y \in \mathbb{R}^d$, we would have (up to re-labelling the sequence)

$$\begin{aligned}
&v(t,x) - v(t,y) - \varepsilon_k t^{-1} \\
&\qquad - \left[\widetilde{K} - e^{\gamma T}\int_0^t L_{f_2}(s)\mathrm{d}s\right][1+|x-y|^2]^{1/2} - \varepsilon_k e^{-\delta t}[|x|^2+|y|^2] \\
&\leqslant e^{\gamma T} g(\bar{x}) - e^{\gamma T} g(\bar{y}) - \varepsilon_k T^{-1} \\
&\qquad - \left[\widetilde{K} - e^{\gamma T}\int_0^T L_{f_2}(s)\mathrm{d}s\right][1+|\bar{x}-\bar{y}|^2]^{1/2} - \varepsilon_k e^{-\delta T}[|\bar{x}|^2+|\bar{y}|^2],
\end{aligned}$$

and using the assumption on g in A.1, the r.h.s. is negative as soon as $\widetilde{K} \geqslant e^{\gamma T}(L_g + \|L_{f_2}\|_{L^1})$. Similarly, if there existed $(\varepsilon_k)_{k\geqslant 1}$ that converges to 0 while $\overline{M} \leqslant 0$ for every k, then for every $t \in [0,T]$ and $x, y \in \mathbb{R}^d$, we would have

$$\begin{aligned}
&v(t,x) - v(t,y) - \varepsilon_k t^{-1} - \varepsilon_k e^{-\delta t}[|x|^2+|y|^2] \\
&\qquad\qquad - \left[\widetilde{K} - e^{\gamma T}\int_0^t L_{f_2}(s)\mathrm{d}s\right][1+|x-y|^2]^{1/2} \leqslant 0.
\end{aligned}$$

In both cases, taking the limit would yield

$$\sup_{t,x,y} v(t,x) - v(t,y) - \left[\widetilde{K} - e^{\gamma T}\int_0^t L_{f_2}(s)\mathrm{d}s\right][1+|x-y|^2]^{1/2} \leqslant 0.$$

In particular, this would imply the desired result.

We thus restrict ourselves to $\widetilde{K} \geqslant e^{\gamma T}(L_g + \|L_{f_2}\|_{L^1})$, and we can assume without loss of generality that $\bar{t} \neq T$ and $\overline{M} > 0$ for small enough ε. From $\overline{M} > 0$ and the linear growth of v,

$$\begin{gathered}
[\widetilde{K} - e^{\gamma T}\|L_{f_2}\|_{L^1}]|\bar{x}-\bar{y}| \leqslant v(\bar{t},\bar{x}) - v(\bar{t},\bar{y}), \\
\varepsilon e^{-\delta t}[|\bar{x}|^2+|\bar{y}|^2] \leqslant v(t,\bar{x}) - v(t,\bar{y}) \leqslant 2L_u e^{\gamma T}[2+|\bar{x}|+|\bar{y}|].
\end{gathered} \tag{3.7}$$

Up to changing L_u in $\max(1, L_u)$, the second equation implies the bound

$$\varepsilon|\bar{x}| + \varepsilon|\bar{y}| \leqslant 4e^{(\delta+\gamma)T} L_u, \tag{3.8}$$

which prevents $\varepsilon|\bar{x}|$ and $\varepsilon|\bar{y}|$ from exploding.

Step 3: Optimality conditions. — Let us now define

$$\bar{p} := \left[\widetilde{K} - e^{\gamma T}\int_0^t L_{f_2}(s)\mathrm{d}s\right] D_z[1+|z|^2]^{1/2}\Big|_{z=\bar{x}-\bar{y}}.$$

We notice that $|\bar{p}| \leqslant \widetilde{K} - e^{\gamma T}\|L_{f_2}\|_{L^1} \leqslant \widetilde{K}$. Since $\bar{t} \notin \{0, T\}$, the first-order optimality conditions in (3.6) provide

$$\begin{cases} Dv(\bar{t},\bar{x}) = \bar{p} + 2\varepsilon e^{-\delta\bar{t}}\bar{x}, \\ Dv(\bar{t},\bar{y}) = \bar{p} - 2\varepsilon e^{-\delta\bar{t}}\bar{y}, \\ \partial_t v(\bar{t},\bar{x}) - \partial_t v(\bar{t},\bar{y}) = -e^{\gamma T} L_{f_2}(\bar{t})[1+|\bar{x}-\bar{y}|^2]^{1/2} \\ \qquad\qquad\qquad -\varepsilon\bar{t}^{-2} - \varepsilon\delta e^{-\delta\bar{t}}[|\bar{x}|^2 + |\bar{y}|^2]. \end{cases} \tag{3.9}$$

We now write the second-order optimality conditions in (x, y) for (3.6):

$$\begin{pmatrix} D^2v(\bar{t},\bar{x}) & 0 \\ 0 & -D^2v(\bar{t},\bar{y}) \end{pmatrix} \leqslant \left[\widetilde{K} - e^{\gamma T}\int_0^{\bar{t}} L_{f_2}(s)\mathrm{d}s\right] \begin{pmatrix} \bar{A} & -\bar{A} \\ -\bar{A} & \bar{A} \end{pmatrix} + 2\varepsilon e^{-\delta t}\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \tag{3.10}$$

where $\bar{A} := D_z^2[1+|z|^2]^{1/2}\big|_{z=\bar{x}-\bar{y}}$, the inequality being understood in the sense of quadratic forms. The matrix $\bar{A}$ is bounded by a constant C_0 independent of $\bar{x}-\bar{y}$. We multiply the l.h.s. of (3.10) by $\begin{pmatrix} \sigma^\top(\bar{t},\bar{x}) & \sigma^\top(\bar{t},\bar{y}) \end{pmatrix}$ and the r.h.s. by the transpose of this matrix:

$$\begin{aligned} &\sigma^\top(\bar{t},\bar{x})D^2v(\bar{t},\bar{x})\sigma(\bar{t},\bar{x}) - \sigma^\top(\bar{t},\bar{y})D^2v(\bar{t},\bar{y})\sigma(\bar{t},\bar{y}) \\ &\leqslant 2\varepsilon e^{-\delta\bar{t}}[\sigma\sigma^\top(\bar{t},\bar{x}) + \sigma\sigma^\top(\bar{t},\bar{y})] \\ &\quad + \left[\widetilde{K} - e^{\gamma T}\int_0^{\bar{t}} L_{f_2}(s)\mathrm{d}s\right][\sigma(\bar{t},\bar{x}) - \sigma(\bar{t},\bar{y})]^\top \bar{A}[\sigma(\bar{t},\bar{x}) - \sigma(\bar{t},\bar{y})]. \end{aligned} \tag{3.11}$$

We now take the trace and we use the assumption on σ in A.1. Since $L_{f_2} \geqslant 0$, $\sigma\sigma^\top$ has linear growth and $\varepsilon[|\bar{x}| + |\bar{x}|]$ is bounded from (3.8), the symmetry property of the trace yields

$$\frac{1}{2}\operatorname{Tr}[\sigma\sigma^\top D^2v(\bar{t},\bar{x})] - \frac{1}{2}\operatorname{Tr}[\sigma\sigma^\top D^2v(\bar{t},\bar{y})] \leqslant C_0 L_\sigma^2 \widetilde{K}|\bar{x}-\bar{y}] + C_1, \tag{3.12}$$

for constants C_0, C_1 which do not depend on $(\varepsilon, \widetilde{K}, \mathcal{A})$, and which only depends on (u, f_2) through $(L_u, e^{\gamma T}\|L_{f_2}\|_{L^1})$. Moreover, C_0 does not depend on γ.

Step 4: Gathering the PDE. — We now write the PDE (3.4) at $(\bar{t},\bar{x})$ and $(\bar{t},\bar{y})$. Let $\bar{\alpha}$ be a minimiser in α of $L_\alpha[v](\bar{t},\bar{x})$, defined in (3.5). Subtracting both PDEs, we get using (3.12) that

$$\begin{aligned} &(\gamma - c)[v(\bar{t},\bar{x}) - v(\bar{t},\bar{y})] \\ &\qquad \leqslant \partial_t v(\bar{t},\bar{x}) - \partial_t v(\bar{t},\bar{y}) + L_{f_2}(\bar{t})e^{\gamma\bar{t}}[f_2(\bar{t},\bar{x}) - f_2(\bar{t},\bar{y})] \\ &\qquad\qquad + L_{\bar{\alpha}}[v](\bar{t},\bar{x}) - L_{\bar{\alpha}}[v](\bar{t},\bar{y}) + C_0 L_\sigma^2\widetilde{K}|\bar{x}-\bar{y}] + C_1. \end{aligned}$$

We then replace $\partial_t v(\bar t,\bar x)-\partial_t v(\bar t,\bar y)$ by its computed value in (3.9). Assumption A.1 ensures that the L_{f_2}-term in (3.9) is greater than $L_{f_2}(\bar t)e^{\gamma \bar t}[f_2(\bar t,\bar x)-f_2(\bar t,\bar y)]$, so that

$$\begin{aligned}(\gamma-c)[v(\bar t,\bar x)-v(\bar t,\bar y)]&\\ \leqslant -\varepsilon \bar t^{-2}-\varepsilon\delta e^{-\delta\bar t}[|\bar x|^2+|\bar y|^2]&\\ +L_{\bar\alpha}[v](\bar t,\bar x)-L_{\bar\alpha}[v](\bar t,\bar y)+C_0L_\sigma^2\widetilde K|\bar x-\bar y|+C_1.&\quad (3.13)\end{aligned}$$

Since $-\varepsilon\bar t^{-2}\leqslant 0$, we can get rid of this term. Within (2.9), changing f_2 in $e^{\gamma T}f_2$ amounts to replacing L_b by $e^{-\gamma T}L_b$. Since $e^{-\gamma T}\leqslant 1$, the bound $|\bar\alpha|\leqslant L_{\mathcal A}[1+|Dv(\bar t,\bar y)|]$ still holds. Moreover, from (3.8)–(3.9),

$$|Dv(\bar t,\bar y)|\leqslant \widetilde K+8e^{(\delta+\gamma)T}L_u \quad\text{and}\quad |Dv(\bar t,\bar x)-Dv(\bar t,\bar y)|\leqslant 2\varepsilon e^{-\delta\bar t}[|\bar x|+|\bar y|].$$

We then bound each term of $L_{\bar\alpha}[v](\bar t,\bar x)-L_{\bar\alpha}[v](\bar t,\bar y)$. For the b-term, we use the Lipschitz assumption (2.2) on b_2 and we simply bound b_1:

$$\begin{aligned}&b(\bar t,\bar x,\bar\alpha)\cdot Dv(\bar t,\bar x)-b(\bar t,\bar y,\bar\alpha)\cdot Dv(\bar t,\bar y)\\ &\quad\leqslant |Dv(\bar t,\bar y)||b(\bar t,\bar x,\bar\alpha)-b(\bar t,\bar y,\bar\alpha)|+|b(\bar t,\bar x,\bar\alpha)||Dv(\bar t,\bar x)-Dv(\bar t,\bar y)|\\ &\quad\leqslant L_b[\widetilde K+8e^{(\delta+\gamma)T}L_u][2+2|\bar\alpha|+|\bar x-\bar y|]\\ &\qquad\quad +2\varepsilon e^{-\delta\bar t}L_b[2+|\bar x|+|\bar\alpha|][|\bar x|+|\bar y|],\end{aligned}$$

where $\varepsilon[|\bar x|+|\bar y|]$ is bounded independently of ε from (3.8). For the last term:

$$|e^{\gamma t}f_1(\bar t,\bar x,\bar\alpha)-e^{\gamma t}f_1(\bar t,\bar y,\bar\alpha)|\leqslant e^{\gamma T}L_{f_2}|\bar x-\bar y|.$$

We recall that $\varepsilon\leqslant 1$. Gathering everything, (3.13) becomes

$$\begin{aligned}&(\gamma-c)[v(\bar t,\bar x)-v(\bar t,\bar y)]\\ &\leqslant -\varepsilon\delta e^{-\delta\bar t}[|\bar x|^2+|\bar y|^2]+2\varepsilon\delta L_b e^{-\delta\bar t}|\bar x|[|\bar x|+|\bar y|]\\ &\quad +C_2(\widetilde K+1)(|\bar\alpha|+1)+|\bar x-\bar y|[\widetilde K(L_b+C_0L_\sigma^2)+L_{f_1}e^{\gamma T}+C_3],\quad (3.14)\end{aligned}$$

for constants C_2, C_3 which do not depend on $(\varepsilon,\widetilde K,\mathcal A)$, and which only depend on (u,f_2) through $(L_u,\|L_{f_2}\|_{L^1})$. Using the bound (2.9) on $|\bar\alpha|$, we eventually get that

$$\begin{aligned}(\gamma-c)[v(\bar t,\bar x)-v(\bar t,\bar y)]\leqslant \varepsilon[-\delta+2L_b]e^{-\delta\bar t}[|\bar x|^2+|\bar y|^2]+C_4&\\ +|\bar x-\bar y|[\widetilde K(L_b+C_0L_\sigma^2)+C_5],&\end{aligned}$$

for constants C_4,C_5 which do not depend on ε, and which only depend on $(u,\mathcal A,f_2)$ through $(L_u,L_{\mathcal A},\|L_{f_2}\|_{L^1})$.

Step 5: Choice of parameters. — Let us fix the values of parameters:

$$\delta:=2L_b,\quad \gamma:=c+L_b+C_0L_\sigma^2+2,$$

$$\widetilde{K} := \max(e^{\gamma T} L_g + e^{\gamma T} \|L_{f_2}\|_{L^1}, C_5 + e^{\gamma T} \|L_{f_2}\|_{L^1}),$$

where we recall that C_0 was not depending on anything. This yields

$$(\gamma - c)[v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y})] \leqslant [\gamma - c - 1][\widetilde{K} - e^{\gamma T} \|L_{f_2}\|_{L^1}]|\bar{x} - \bar{y}| + C_4,$$

so that, using (3.7),

$$\overline{M} \leqslant v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) \leqslant C_4.$$

This uniform bound on $\overline{M}$ allows us to send ε to 0 to get that

$$\sup_{t,x,y} v(t,x) - v(t,y) - \left[\widetilde{K} - e^{\gamma T} \int_0^t L_{f_2}(s)\mathrm{d}s\right][1 + |x-y|^2]^{1/2} \leqslant C_4.$$

We conclude the proof by choosing $\widetilde{M} := \widetilde{K} + C_4$. □

We are now ready to prove the Lipschitz estimate using the Ishii–Lions method. To exploit the uniform ellipticity of $\sigma\sigma^\top$, we are going to use a strictly concave function ψ as a test function.

Proposition 3.3 (Lipschitz estimate). — *Let u in $C([0,T] \times \mathbb{R}^d) \cap C^{1,2}((0,T) \times \mathbb{R}^d)$ be a solution of (1.1) and $L_u > 0$ such that for every $(t,x) \in [0,T] \times \mathbb{R}^d$,*

$$|u(t,x)| \leqslant L_u[1 + |x|].$$

Under A.1–A.2, there exists a positive constant $K > 0$ such that

$$\forall\, t \in [0,T], \forall\, (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad |u(t,x) - u(t,y)| \leqslant K|x-y|, \qquad (3.15)$$

where K only depends on $(u, \mathcal{A}, f_2)$ through $(L_u, L_{\mathcal{A}}, \|L_{f_2}\|_{L^1})$.

Proof. — As previously, we look for $K > 0$ such that

$$\sup_{t,x,y} u(t,x) - v(t,y) - K|x-y| \leqslant 0.$$

We first introduce regularising parameters.

Step 1: Regularising parameters. — As in the proof of Lemma 3.2, for any $\gamma > 0$, we introduce the $C^{1,2}$ solution $v : (t,x) \mapsto e^{\gamma t} u(t,x)$ of (3.4). It is equivalent to prove (3) for v instead of u, and v has linear growth with constant $e^{\gamma T} L_u$. To make use of uniform ellipticity, we consider a smooth function $\rho : \mathbb{R}_+ \to [0,1]$ such that

$$\begin{cases} \rho(z) = \frac{1}{3} z^{3/2} & \text{for } z \in [0,1], \\ \rho(z) = 0 & \text{for } z \geqslant 2, \end{cases}$$

and we define the non-negative function $\psi(z) := z - \rho(z)$. As previously, we introduce regularising parameters $\varepsilon, \delta, \varepsilon > 0$, before considering the supremum

$$\overline{M} := \sup_{t,x,y} v(t,x) - v(t,y) - \varepsilon t^{-1} - \varepsilon e^{-\delta t}[|x|^2 + |y|^2] - \left[K - e^{\gamma T} \int_0^t L_{f_2}(s) \mathrm{d}s \right] \psi(|x-y|). \quad (3.16)$$

As previously, γ, δ and K are fixed parameters whose values will be chosen later on. On the contrary, ε will be sent to 0, and we restrict ourselves to $\varepsilon \leqslant 1$. The L_{f_2}-term within the supremum will allow us to control the f_2-term in (3.4) using only $\|L_{f_2}\|_{L^1}$. In the following, we restrict ourselves to $K \geqslant e^{\gamma T} \|L_{f_2}\|_{L^1}$.

The quadratic term ensures that the supremum (3.16) is realised for some $(\bar{t}, \bar{x}, \bar{y}) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$. Moreover, $\bar{t} \neq 0$ because the supremum is finite. If there existed a sequence $(\varepsilon_k)_{k \geqslant 1}$ of positive parameters which converges to 0 such that the related sequence $(\bar{t}_k)_{k \geqslant 1}$ had a constant sub-sequence equal to T, we would get the desired result as soon as $K \geqslant e^{\gamma T} L_g + e^{\gamma T} \|L_{f_2}\|_{L^1}$, reasoning as in Step 2 in the proof of Lemma 3.2. Similarly, the result would be direct if there existed $(\varepsilon_k)_{k \geqslant 1}$ which converges to 0 while $\overline{M} \leqslant 0$ for every k.

Step 2: Bounds on the optimiser. — We thus restrict ourselves to $K \geqslant e^{\gamma T} L_g + e^{\gamma T} \|L_{f_2}\|_{L^1}$ and we reason by contradiction, assuming that $\bar{t} \neq T$ and $\overline{M} > 0$ for ε small enough: we are going to show that this cannot happen if K is larger than a certain threshold which does not depend on ε. From Lemma 3.2, we deduce that there exist large enough constant $\widetilde{K}, \widetilde{M} > 0$ such that

$$\sup_{t,x,y} v(t,x) - v(t,y) - \widetilde{K}|x-y| \leqslant \widetilde{M},$$

where $(\widetilde{K}, \widetilde{M})$ only depends on $(u, \mathcal{A}, f_2)$ through $(L_u, L_{\mathcal{A}}, \|L_{f_2}\|_{L^1})$. From $\overline{M} > 0$, we get that

$$\left[K - e^{\gamma T} \int_0^t L_{f_2}(s) \mathrm{d}s - \widetilde{K} \right] [|\bar{x} - \bar{y}| - \rho(|\bar{x} - \bar{y}|)] \leqslant v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) - \widetilde{K}[|\bar{x} - \bar{y}| - \rho(|\bar{x} - \bar{y}|)].$$

For every $z \geqslant 0$, we have $\frac{1}{2} z - \rho(z) \geqslant 0$ and $\rho(z) \leqslant 1$, hence

$$\frac{1}{2} \left[K - e^{\gamma T} \int_0^t L_{f_2}(s) \mathrm{d}s - \widetilde{K} \right] |\bar{x} - \bar{y}| \leqslant \widetilde{K} + \sup_{t,x,y} v(t,x) - v(t,y) - \widetilde{K}|x-y| \leqslant \widetilde{K} + \widetilde{M}.$$

Up to increasing the value of $\widetilde{K}$, we can thus assume that for $K \geqslant \widetilde{K}$,

$$|\bar{x} - \bar{y}| \leqslant C_0 K^{-1}, \tag{3.17}$$

for some $C_0 > 0$ which only depends on $(e^{\gamma T}, \widetilde{M}, \widetilde{K}, \|L_{f_2}\|_{L^1})$. Up to increasing $\widetilde{K}$ again, we can then assume that $|\bar{x} - \bar{y}| \leqslant 1$ for $K \geqslant \widetilde{K}$. Using the linear growth of v and $\overline{M} > 0$,

$$\varepsilon e^{-\delta \bar{t}}[|\bar{x}|^2 + |\bar{y}|^2] \leqslant v(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) \leqslant L_u e^{\gamma T}[2 + |\bar{x}| + |\bar{y}|].$$

Up to changing L_u in $\max(1, L_u)$, this implies that

$$\varepsilon|\bar{x}| + \varepsilon|\bar{y}| \leqslant 4e^{(\delta+\gamma)T} L_u, \tag{3.18}$$

preventing $\varepsilon|\bar{x}|$ and $\varepsilon|\bar{y}|$ from exploding.

Step 3: Optimality conditions. — To ease computations, we introduce the function $\chi : z \in \mathbb{R}^d \mapsto |z|$. From $\overline{M} > 0$, necessarily $\bar{x} \neq \bar{y}$. Since $\bar{t} \notin \{0, T\}$, the first-order optimality conditions in (3.16) provide

$$\begin{cases} Dv(\bar{t}, \bar{x}) = \left[K - e^{\gamma T} \int_0^t L_{f_2}(s)\mathrm{d}s\right] \psi'(|\bar{x} - \bar{y}|) D\chi(\bar{x} - \bar{y}) + 2\varepsilon e^{-\delta \bar{t}} \bar{x}, \\ Dv(\bar{t}, \bar{y}) = \left[K - e^{\gamma T} \int_0^t L_{f_2}(s)\mathrm{d}s\right] \psi'(|\bar{x} - \bar{y}|) D\chi(\bar{x} - \bar{y}) - 2\varepsilon e^{-\delta \bar{t}} \bar{y}, \\ \partial_t v(\bar{t}, \bar{x}) - \partial_t v(\bar{t}, \bar{y}) \\ \qquad\qquad = -e^{\gamma T} L_{f_2}(\bar{t})[1 + |\bar{x} - \bar{y}|^2]^{1/2} - \varepsilon \bar{t}^{-2} - \varepsilon \delta e^{-\delta \bar{t}}[|\bar{x}|^2 + |\bar{y}|^2]. \end{cases} \tag{3.19}$$

Since $|\bar{x} - \bar{y}| \leqslant 1$, we notice that $0 \leqslant \psi'(|\bar{x} - \bar{y}|) \leqslant 1$. We then write the second-order optimality conditions in (x, y) for (3.16):

$$\begin{pmatrix} D^2 v(\bar{t}, \bar{x}) & 0 \\ 0 & -D^2 v(\bar{t}, \bar{y}) \end{pmatrix} \leqslant \left[K - e^{\gamma T} \int_0^{\bar{t}} L_{f_2}(s)\mathrm{d}s\right] \begin{pmatrix} \bar{A} & -\bar{A} \\ -\bar{A} & \bar{A} \end{pmatrix} + 2\varepsilon e^{-\delta \bar{t}} \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \tag{3.20}$$

where

$$\bar{A} := \psi''(|\bar{x} - \bar{y}|) D\chi(\bar{x} - \bar{y}) \otimes D\chi(\bar{x} - \bar{y}) + \psi'(|\bar{x} - \bar{y}|) D^2\chi(\bar{x} - \bar{y}).$$

For every $(p, q) \in \mathbb{R}^d \times \mathbb{R}^d$, applying (3.20) to the vector $\begin{pmatrix} p & q \end{pmatrix}$ and using $L_{f_2} \geqslant 0$, we get that

$$p^\top D^2 v(\bar{t}, \bar{x}) p - q^\top D^2 v(\bar{t}, \bar{y}) q \\ \leqslant \left[K - \int_0^{\bar{t}} L_{f_2(s)}\mathrm{d}s\right] (p - q)^\top \bar{A} (p - q) + 2\varepsilon e^{-\delta \bar{t}}[|p|^2 + |q|^2]. \tag{3.21}$$

Since $|D\chi(z)|^2 = 1$ for every z, we get that $D^2\chi(z)D\chi(z) = 0$. Using (3.21) with $p = -q = D\chi(\bar{x} - \bar{y})$ thus yields

$$D\chi(\bar{x} - \bar{y})^\top [D^2 v(\bar{t}, \bar{x}) - D^2 v(\bar{t}, \bar{y})] D\chi(\bar{x} - \bar{y}) \leqslant 4\left[K - e^{\gamma T} \int_0^{\bar{t}} L_{f_2}(s) \mathrm{d}s\right] \psi''(|\bar{x} - \bar{y}|) + 4\varepsilon e^{-\delta \bar{t}}.$$

Similarly, taking $p = q$ in (3.21) gives that

$$\forall\, p \in \mathbb{R}^d, \quad p^\top [D^2 v(\bar{t}, \bar{x}) - D^2 v(\bar{t}, \bar{y})] p \leqslant 4\varepsilon e^{-\delta \bar{t}} |p|^2. \tag{3.22}$$

Completing $D\chi(\bar{x} - \bar{y})$ in an orthornomal basis $(D\chi(\bar{x} - \bar{y}), p_2, \ldots, p_d)$ of $\mathbb{R}^d$, this proves that

$$\begin{aligned}\mathrm{Tr}[D^2 v(\bar{t}, \bar{x}) - D^2 v(\bar{t}, \bar{y})] &= D\chi(\bar{x} - \bar{y})^\top [D^2 v(\bar{t}, \bar{x}) - D^2 v(\bar{t}, \bar{y})] D\chi(\bar{x} - \bar{y}) \\ &\quad + \sum_{i=2}^{d} p_i^\top [D^2 v(\bar{t}, \bar{x}) - D^2 v(\bar{t}, \bar{y})] p_i \\ &\leqslant 4\left[K - e^{\gamma T} \int_0^{\bar{t}} L_{f_2}(s) \mathrm{d}s\right] \psi''(|\bar{x} - \bar{y}|) + 4\varepsilon d e^{-\delta \bar{t}}. \end{aligned} \tag{3.23}$$

As $K \to +\infty$, $|\bar{x} - \bar{y}| \to 0$ from (3.17), and our specific choice of ψ makes the r.h.s. of (3.23) go to $-\infty$: this is the main trick of the Ishii–Lions method.

Step 4: Control of the second-order terms. — Let us introduce the matrix $a := \sigma\sigma^\top$. From A.2, $(t, x) \mapsto a(t, x)$ is uniformly continuous with modulus m_σ. We decompose:

$$\begin{aligned}\mathrm{Tr}[a(\bar{t}, \bar{x}) D^2 v(\bar{t}, \bar{x}) - a(\bar{t}, \bar{y}) D^2 v(\bar{t}, \bar{y})] &= \frac{1}{2} \mathrm{Tr}\big\{a(\bar{t}, \bar{x})[D^2 v(\bar{t}, \bar{x}) - D^2 v(\bar{t}, \bar{y})]\big\} \\ &\quad + \frac{1}{2} \mathrm{Tr}\big\{a(\bar{t}, \bar{y})[D^2 v(\bar{t}, \bar{x}) - D^2 v(\bar{t}, \bar{y})]\big\} \\ &\quad + \frac{1}{2} \mathrm{Tr}\big\{[a(\bar{t}, \bar{x}) - a(\bar{t}, \bar{y})][D^2 v(\bar{t}, \bar{x}) + D^2 v(\bar{t}, \bar{y})]\big\}. \end{aligned} \tag{3.24}$$

We now apply Lemma A.1 in Appendix with $A = a(\bar{t}, \bar{x})$, $B = D^2 v(\bar{t}, \bar{x}) - D^2 v(\bar{t}, \bar{y})$, $m = \eta_\sigma / 2$ and $M = 4\varepsilon e^{-\delta \bar{t}}$. The inequality $A \geqslant m\,\mathrm{Id}$ stems from A.2, while $B \leqslant M\,\mathrm{Id}$ results from (3.22). This gives that

$$\mathrm{Tr}\big\{a(\bar{t}, \bar{x})[D^2 v(\bar{t}, \bar{x}) - D^2 v(\bar{t}, \bar{y})]\big\} \leqslant \frac{\eta_\sigma}{2} \mathrm{Tr}[D^2 v(\bar{t}, \bar{x}) - D^2 v(\bar{t}, \bar{y})] + 4\varepsilon e^{-\delta \bar{t}} \mathrm{Tr}[a(\bar{t}, \bar{x})],$$

and $\mathrm{Tr}[a(\bar{t},\bar{x})]$ has linear growth from A.1. Using (3.18), $4\varepsilon e^{-\delta\bar{t}}\,\mathrm{Tr}[a(\bar{t},\bar{x})]$ is bounded by a constant C_1 which only depends on $(L_\sigma, T, \delta, L_u)$. The same bound works for $\mathrm{Tr}\{a(\bar{t},\bar{y})[D^2v(\bar{t},\bar{x}) - D^2v(\bar{t},\bar{y})]\}$. We then use that $A \mapsto \sup_{|\xi|=1}\xi^\top A\xi$ is a sub-multiplicative norm on symmetric matrices and that all norms are equivalent in finite dimension. By A.1 and the continuity of the trace, there exists a constant $C_2 > 0$ which only depends on the dimension d such that

$$\begin{aligned}\mathrm{Tr}\big\{[a(\bar{t},\bar{x}) - a(\bar{t},\bar{y})][D^2v(\bar{t},\bar{x}) + D^2v(\bar{t},\bar{y})]\big\}&\\ \leqslant C_2 m_\sigma(|\bar{x}-\bar{y}|)\sup_{|\xi|=1}\xi^\top[D^2v(\bar{t},\bar{x}) + D^2v(\bar{t},\bar{y})]\xi.&\quad (3.25)\end{aligned}$$

Using (3.20), we now apply Lemma A.2 with $A = \overline{A}$, $X = D^2v(\bar{t},\bar{x})$, $Y = -D^2v(\bar{t},\bar{x})$, $m = 2\varepsilon e^{-\delta\bar{t}}$ and $C = K - e^{\gamma T}\int_0^t L_{f_2}(s)\mathrm{d}s$. As a consequence, the r.h.s. of (3.25) is bounded by

$$\begin{aligned}\sqrt{2}C_2 m_\sigma(|\bar{x}-\bar{y}|)\bigg\{4\varepsilon e^{-\delta\bar{t}} + 2\bigg[K - e^{\gamma T}\int_0^t L_{f_2}(s)\mathrm{d}s\bigg]&\\ + \sup_{|\xi|=1}\xi^\top[D^2v(\bar{t},\bar{x}) - D^2v(\bar{t},\bar{y})]\xi\bigg\}.&\end{aligned}$$

The supremum that appears within the above quantity equals the spectral radius of the symmetric matrix $D^2u(\bar{t},\bar{x}) - D^2u(\bar{t},\bar{y})$. From (3.22), any eigenvalue of $D^2v(\bar{t},\bar{x}) - D^2v(\bar{t},\bar{y})$ is bounded from above by $4\varepsilon e^{-\delta\bar{t}}$. As a consequence, any eigenvalue of $D^2v(\bar{t},\bar{x}) - D^2v(\bar{t},\bar{y})$ is bounded from below by

$$\mathrm{Tr}[D^2v(\bar{t},\bar{x}) - D^2v(\bar{t},\bar{y})] - 4(d-1)\varepsilon e^{-\delta\bar{t}},$$

so that

$$\sup_{|\xi|=1}\xi^\top[D^2v(\bar{t},\bar{x}) - D^2v(\bar{t},\bar{y})]\xi \leqslant |\mathrm{Tr}[D^2v(\bar{t},\bar{x}) - D^2v(\bar{t},\bar{y})]| + 4d\varepsilon e^{-\delta\bar{t}}.$$

At the end of the day, gathering everything from (3.24) yields

$$\begin{aligned}&\mathrm{Tr}[a(\bar{t},\bar{x})D^2v(\bar{t},\bar{x}) - a(\bar{t},\bar{y})D^2v(\bar{t},\bar{y})]\\ &\quad\leqslant \frac{\eta_\sigma}{2}\,\mathrm{Tr}[D^2v(\bar{t},\bar{x}) - D^2v(\bar{t},\bar{y})] + C_1\\ &\qquad + \sqrt{2}C_2 m_\sigma(|\bar{x}-\bar{y}|)\bigg[2\varepsilon e^{-\delta\bar{t}} + K + 2d\varepsilon e^{-\delta\bar{t}}\\ &\qquad\qquad + \frac{1}{2}|\mathrm{Tr}[D^2v(\bar{t},\bar{x}) - D^2v(\bar{t},\bar{y})]|\bigg].\quad (3.26)\end{aligned}$$

Using (3.23), we see that the r.h.s. of (3.26) goes to $-\infty$ as $K \to +\infty$.

Step 5: Gathering the PDE. — We now write the PDE (3.4) at $(\bar t,\bar x)$ and $(\bar t,\bar y)$. Let $\bar\alpha$ be a minimiser in α of $L_\alpha[v](\bar t,\bar x)$, defined in (3.5). Subtracting both PDEs, we get that

$$\begin{aligned}&(\gamma-c)[v(\bar t,\bar x)-v(\bar t,\bar y)]\\&=\partial_t v(\bar t,\bar x)-\partial_t v(\bar t,\bar y)+L_{f_2}(\bar t)e^{\gamma\bar t}[f_2(\bar t,\bar x)-f_2(\bar t,\bar y)]\\&\quad+L_{\bar\alpha}[v](\bar t,\bar x)-L_{\bar\alpha}[v](\bar t,\bar y)+\frac12\operatorname{Tr}[a(\bar t,\bar x)D^2v(\bar t,\bar x)-a(\bar t,\bar y)D^2v(\bar t,\bar y)].\end{aligned}$$

Using (3.18)–(3.19), we now reason as in Step 4 in the proof of Lemma 3.2: we replace $\partial_t v(\bar t,\bar x)-\partial_t v(\bar t,\bar y)$ by its computed value in (3.19), we use it to get rid of $L_{f_2}(\bar t)e^{\gamma\bar t}[f_2(\bar t,\bar x)-f_2(\bar t,\bar y)]$, we get rid of the $-\varepsilon\bar t^{-2}$ term, and then we control the $L_{\bar\alpha}[v](\bar t,\bar x)-L_{\bar\alpha}[v](\bar t,\bar y)$ term. For the b-term, we now use the Lipschitz assumption (2.2) on b_1 and b_2:

$$\begin{aligned}&b(\bar t,\bar x,\bar\alpha)\cdot Dv(\bar t,\bar x)-b(\bar t,\bar y,\bar\alpha)\cdot Dv(\bar t,\bar y)\\&\quad\leqslant|Dv(\bar t,\bar y)||b(\bar t,\bar x,\bar\alpha)-b(\bar t,\bar y,\bar\alpha)|+|b(\bar t,\bar x,\bar\alpha)||Dv(\bar t,\bar x)-Dv(\bar t,\bar y)|\\&\quad\leqslant L_b[\widetilde K+8e^{(\delta+\gamma)T}L_u](1+|\bar\alpha|)|\bar x-\bar y|+2\varepsilon e^{-\delta\bar t}L_b[1+|\bar x|+|\bar\alpha|][|\bar x|+|\bar y|],\end{aligned}$$

where from (3.18), $\varepsilon[|\bar x|+|\bar y|]$ is bounded independently of ε. Similarly,

$$|e^{\gamma t}f_1(\bar t,\bar x,\bar\alpha)-f_1(\bar t,\bar y,\bar\alpha)|\leqslant e^{\gamma T}L_{f_2}|\bar x-\bar y|.$$

As in Step 4 in the proof of Lemma 3.2, we obtain using (2.9) that $|\bar\alpha|\leqslant L_{\mathcal A}[1+|Dv(\bar t,\bar y)|]$. Gathering everything, the analogous of (3.14) now reads

$$\begin{aligned}&(\gamma-c)[v(\bar t,\bar x)-v(\bar t,\bar y)]\\&\qquad\leqslant\varepsilon[2L_b-\delta]\delta e^{-\delta\bar t}[|\bar x|^2+|\bar y|^2]\\&\qquad\quad+\frac12\operatorname{Tr}[a(\bar t,\bar x)D^2v(\bar t,\bar x)-a(\bar t,\bar y)D^2v(\bar t,\bar y)]\\&\qquad\quad+L_b[\widetilde K+8e^{(\delta+\gamma)T}L_u](1+K)|\bar x-\bar y|+C_3(1+K+|\bar x-\bar y|),\end{aligned}$$

for a constant C_3 which does not depend on (ε,K), and which only depends on $(u,\mathcal A,f_2)$ through $(L_u,L_{\mathcal A},\|L_{f_2}\|_{L^1})$.

Step 6: Contradiction. — We now fix the values $\gamma:=c$ and $\delta:=2L_b$. Using (3.17) and imposing that $K\geqslant1$, we get that

$$\begin{aligned}0\leqslant&\frac12\operatorname{Tr}[a(\bar t,\bar x)D^2v(\bar t,\bar x)-a(\bar t,\bar y)D^2v(\bar t,\bar y)]\\&+2C_0L_b[K+8e^{(\delta+\gamma)T}L_u]+2C_3(1+K+C_0).\end{aligned}\tag{3.27}$$

Similarly, (3.26) becomes

$$\begin{aligned}
&\mathrm{Tr}[a(\bar{t},\bar{x})D^2v(\bar{t},\bar{x}) - a(\bar{t},\bar{y})D^2v(\bar{t},\bar{y})] \\
&\leqslant \frac{\eta_\sigma}{2}\,\mathrm{Tr}[D^2v(\bar{t},\bar{x}) - D^2v(\bar{t},\bar{y})] + C_1 \\
&\quad + \sqrt{2}C_2 m_\sigma(|\bar{x}-\bar{y}|)\big[2 + K + 2d + \frac{1}{2}|\mathrm{Tr}[D^2v(\bar{t},\bar{x}) - D^2v(\bar{t},\bar{y})]|\big], \quad (3.28)
\end{aligned}$$

and $m_\sigma(|\bar{x}-\bar{y}|) \to 0$ as $K \to +\infty$. From (3.17)–(3.23),

$$\mathrm{Tr}[D^2v(\bar{t},\bar{x}) - D^2v(\bar{t},\bar{y})] \leqslant -3\sqrt{C_0}K^{3/2} + 4d,$$

proving that $\mathrm{Tr}[D^2v(\bar{t},\bar{x}) - D^2v(\bar{t},\bar{y})]$ goes to $-\infty$ as $K \to +\infty$ uniformly in ε. From (3.28), $\mathrm{Tr}[a(\bar{t},\bar{x})D^2v(\bar{t},\bar{x}) - a(\bar{t},\bar{y})D^2v(\bar{t},\bar{y})]$ goes to $-\infty$ at least as fast as $-K^{3/2}$ as $K \to +\infty$, uniformly in ε. From this, there exists a finite value $K' \geqslant \max(1, \widetilde{K}, e^{\gamma T}(L_g + \|L_{f_2}\|_{L^1}))$ independent of ε such that the r.h.s. of (3.27) is always negative for $K \geqslant K'$. This gives the desired contradiction. We moreover notice that K' only depends on $(u, \mathcal{A}, f_2)$ through $(L_u, L_{\mathcal{A}}, \|L_{f_2}\|_{L^1})$. □

Proof of Theorem 2.2. — From Lemma 3.1, there exists a growth constant L which is valid for any $C^{1,2}$ linear growth solution of (1.1), and L only depends on $\mathcal{A}$ through $L_{\mathcal{A}}$. Let K be the value given by Proposition 3.3 for $L_u = L$: K only depends on $\mathcal{A}$ through $L_{\mathcal{A}}$. K can thus be chosen as an increasing function of $L_{\mathcal{A}}$. Up to choosing a larger $L_{\mathcal{A}}$ in (2.9), we can assume that $L_{\mathcal{A}}[1+K] \geqslant R_{\mathcal{A}}$, where $R_{\mathcal{A}}$ is the constant given by A.3. Let us define the compact sub-set of $\mathcal{A}$:

$$\mathcal{A}' := \mathcal{A} \cap B(0, L_{\mathcal{A}}[1 + L + K]).$$

We now introduce a truncated version of (1.1): we truncate the non-linearity by using the compact control set $\mathcal{A}'$ instead of $\mathcal{A}$. From [47, Theorem 3.1], the resulting truncated equation has a unique linear growth solution $\widetilde{u}$ in $C([0,T]\times\mathbb{R}^d) \cap C^{1,2,\beta,\beta}_{\mathrm{loc}}$. From the definition of $\mathcal{A}'$, (2.9) is still valid when replacing $\mathcal{A}$ by $\mathcal{A}'$, with the same constant $L_{\mathcal{A}}$. Since the Lipschitz estimate of Proposition 3.3 only depends on $\mathcal{A}$ through $L_{\mathcal{A}}$, $\widetilde{u}$ is K-Lipschitz continuous with the same constant K. This proves that $\widetilde{u}$ does not feel the truncation, hence $\widetilde{u}$ is a $C^{1,2}$ solution of (1.1).

Reciprocally, any linear growth solution in $C([0,T]\times\mathbb{R}^d)\cap C^{1,2,\beta,\beta}_{\mathrm{loc}}$ of (1.1) is K-Lipschitz continuous from Lemma 3.1 and Proposition 3.3, hence it is a solution of the truncated equation. Since uniqueness holds for the truncated equation, this concludes the proof. □

Remark 3.4 (Possible improvements of Lipschitz hypotheses). — The Lipschitz assumptions on the coefficients are two-fold. For instance, (2.3) provides a control of $|f_1(t,x,\alpha) - f_1(t,y,\alpha)|$ when $|x-y|$ is large, together with local regularity when $|x-y|$ is small. Both these properties are needed in the proofs, but not simultaneously. The proof of Lemma 3.2 only uses this control when $|x-y|$ is large, whereas the proof of Proposition 3.3 uses it when $|x-y|$ is small. This suggests a weakening of the assumption by separating these properties: we could require that $r_0 > 0$ exists such that

$$\begin{cases} |f_1(t,x,\alpha) - f_1(t,y,\alpha)| \leqslant 1 + L_{f_1}|x-y| + h(|\alpha|), & \text{if } |x-y| > r_0, \\ |f_1(t,x,\alpha) - f_1(t,y,\alpha)| \leqslant L_{f_1}(1+|\alpha|^2)|x-y|^\mu, & \text{if } |x-y| \leqslant r_0, \end{cases}$$

for any non-negative $h : \mathcal{A} \to \mathbb{R}$ and $\mu \in (0,1]$. Using this assumption instead of (2.3), the proof of Lemma 3.2 would still work without any change. The proof of Proposition 3.3 should then be modified by considering $\rho(z) = z^{1+\nu}$ with $\nu < \mu$, instead of $\rho(z) = z^{3/2}$. We could split the condition (2.2) similarly. For the sake of clarity, we refrained ourselves from adding these subtleties in A.1.

Appendix A. Linear algebra results

This appendix gathers two linear algebra results that were needed for Step 4 in the proof of Proposition 3.3. These lemmas correspond to intermediary results which can be respectively found within the proofs of [36, Proposition III.1] and [36, Lemma III.1]. We provide the proofs for the sake of completeness.

Lemma A.1. — *Let A and B be two symmetric matrices in $\mathbb{R}^d$ with $A \geqslant m\,\mathrm{Id}$ and $B \leqslant M\,\mathrm{Id}$ for real numbers $m, M \geqslant 0$. We have that*

$$\mathrm{Tr}[AB] \leqslant m\,\mathrm{Tr}[B] + M[\mathrm{Tr}[A] - dm].$$

Proof. — We first handle the case $M = 0$. Let us write $A = P^\top D P$ for some orthogonal matrix P and $D := \mathrm{diag}(d_1, \dots, d_d)$. The d_i are the eigenvalues of A and all satisfy $d_i \geqslant m$. Moreover, using the symmetry property of the trace,

$$\mathrm{Tr}[AB] = \mathrm{Tr}[D(PBP^\top)] = \sum_{i=1}^d d_i e_i^\top (PBP^\top) e_i,$$

where $(e_1, \dots, e_d)$ is an orthonormal basis of $\mathbb{R}^d$. From the non-positiveness assumption $M \leqslant 0$ on B, we have $e_i^\top (PBP^\top) e_i \leqslant 0$, so that

$$\mathrm{Tr}[AB] \leqslant m \sum_{i=1}^d e_i^\top (PBP^\top) e_i = m\,\mathrm{Tr}[PBP^\top] = m\,\mathrm{Tr}[B].$$

In the general case B does not satisfy the non-positiveness condition, but $B - M\,\mathrm{Id}$ does. The above result applied to $B - M\,\mathrm{Id}$ then yields

$$\mathrm{Tr}[AB] - M\,\mathrm{Tr}[A] \leqslant m\,\mathrm{Tr}[B] - mM\,\mathrm{Tr}[\mathrm{Id}] = m\,\mathrm{Tr}[B] - dmM,$$

concluding the proof. □

Lemma A.2. — *Let A, X and Y be symmetric matrices in $\mathbb{R}^d$ such that*

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leqslant C \begin{pmatrix} A & -A \\ -A & A \end{pmatrix} + m \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \tag{A.1}$$

for real numbers $C, m > 0$. Then

$$\sup_{|\xi|=1} \xi^\top (X - Y)\xi \leqslant \sqrt{2}\left[2m + 2C + \sup_{|\xi|=1} \xi^\top (X+Y)\xi\right].$$

Proof. — The inequality (A.1) is unchanged if we multiply each side on the left and on the right by the symmetric matrix $\left(\begin{smallmatrix} \mathrm{Id} & \mathrm{Id} \\ \mathrm{Id} & -\mathrm{Id} \end{smallmatrix}\right)$. This yields

$$\begin{pmatrix} X+Y & X-Y \\ X-Y & X+Y \end{pmatrix} \leqslant 4C \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} + 2m \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}.$$

For any (t, ξ) in $\mathbb{R} \times \mathbb{R}^d$, we apply this to the vector $\begin{pmatrix} t\xi & \xi \end{pmatrix}$, so that

$$t^2 \xi^\top (X+Y)\xi + 2t\xi^\top (X-Y)\xi + \xi^\top (X+Y)\xi \leqslant 4C\xi^\top A\xi + 2m(t^2+1)|\xi|^2.$$

Since this holds for every t in $\mathbb{R}$, we get that

$$\begin{aligned}
[\xi^\top (X-Y)\xi]^2 &\leqslant [\xi^\top (X+Y)\xi - 2m|\xi|^2][\xi^\top (X+Y)\xi - 4C\xi^\top A\xi - 2m|\xi|^2] \\
&\leqslant \frac{1}{2}[\xi^\top (X+Y)\xi - 2m|\xi|^2]^2 \\
&\qquad + \frac{1}{2}[\xi^\top (X+Y)\xi - 4C\xi^\top A\xi - 2m|\xi|^2]^2.
\end{aligned}$$

To conclude, we apply this to any normalised vector ξ and we take the square-root of each side. □

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