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The 10-web by conics on the quartic del Pezzo surface ^(*)

LUC PIRIO ⁽¹⁾

ABSTRACT. — We study and compare the webs $\mathcal{W}_{\mathrm{dP}_d}$ defined by the conic fibrations on a given smooth del Pezzo surface dP_d of degree d for $d = 4$ and $d = 5$. In a previous paper, we proved that for any positive integer $d \leq 6$, the web $\mathcal{W}_{\mathrm{dP}_d}$ carries a particular abelian relation $\mathbf{HLog}_d$, whose components all are weight $7 - d$ anti-symmetric hyperlogarithms. The web $\mathcal{W}_{\mathrm{dP}_5}$ is a geometric model of the exceptional Bol’s web and the relation $\mathbf{HLog}_5$ corresponds to the famous “Abel’s identity” $(\mathcal{Ab})$ of the dilogarithm. Bol’s web together with the relation $(\mathcal{Ab})$ enjoy several remarkable properties of different kinds. We show that almost all of them admit natural generalizations to the pair $(\mathcal{W}_{\mathrm{dP}_4}, \mathbf{HLog}_4)$.

RÉSUMÉ. — Nous étudions et comparons les tissus $\mathcal{W}_{\mathrm{dP}_d}$ définis par les fibrations coniques sur une surface de del Pezzo lisse dP_d de degré d , pour $d = 4$ et $d = 5$. Dans un article précédent, nous avons prouvé que pour tout entier positif $d \leq 6$, le tissu en coniques $\mathcal{W}_{\mathrm{dP}_d}$ sur une surface de del Pezzo dP_d de degré d porte une relation abélienne particulière $\mathbf{HLog}_d$, dont toutes les composantes sont des hyperlogarithmes antisymétriques de poids $7 - d$. Le tissu $\mathcal{W}_{\mathrm{dP}_5}$ est un modèle géométrique du tissu de Bol et la relation $\mathbf{HLog}_5$ correspond à la fameuse « identité d’Abel » $(\mathcal{Ab})$ du dilogarithme. Le tissu de Bol ainsi que la relation $(\mathcal{Ab})$ jouissent de plusieurs propriétés remarquables de différentes natures. Nous montrons que presque toutes ces propriétés admettent des généralisations naturelles à la paire $(\mathcal{W}_{\mathrm{dP}_4}, \mathbf{HLog}_4)$.

In the whole paper, we work over the field $\mathbb{C}$ of complex numbers, in the analytic or algebraic category depending on the context. We will often work with the affine coordinates x, y on the complex projective plane $\mathbb{P}^2$ which correspond to the embedding $\mathbb{C}^2 \hookrightarrow \mathbb{P}^2, (x, y) \mapsto [x : y : 1]$.

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Article proposé par Jérémy Blanc.

1. Introduction

“Cauchy’s identity”

$$\mathrm{Log}(x) + \mathrm{Log}(y) - \mathrm{Log}(xy) = 0 \quad (1.1)$$

satisfied by the logarithm $\mathrm{Log}(\cdot)$ is certainly one of the most important functional identities in mathematics. It admits a “weight 2” generalization, the so-called “Abel’s” or “five-term identity”

$$R(x) - R(y) - R\left(\frac{x}{y}\right) - R\left(\frac{1-y}{1-x}\right) + R\left(\frac{x(1-y)}{y(1-x)}\right) = 0 \quad (\mathbf{Ab})$$

which is satisfied for any $(x, y) \in \mathbb{R}^2$ such that $0 < x < y < 1$, by *Rogers’ dilogarithm* that is the function defined by

$$R(x) = \mathbf{Li}_2(x) + \frac{1}{2} \mathrm{Log}(x) \mathrm{Log}(1-x) - \frac{\pi^2}{6}$$

for $x \in]0, 1[$.⁽¹⁾ Abel’s dilogarithmic identity $(\mathbf{Ab})$ appears in several domains of mathematics hence connecting them.⁽²⁾

In [12]⁽³⁾, we described a series of hyperlogarithmic functional identities $(\mathbf{HLog}_d)$ for d ranging from 1 to 6, constructed from del Pezzo surfaces in a uniform way and such that $(\mathbf{HLog}_6)$ and $(\mathbf{HLog}_5)$ precisely coincide with the classical identities (1.1) and $(\mathbf{Ab})$ respectively. For any $d \in \{1, \dots, 6\}$, there is one identity $(\mathbf{HLog}_d)$ for each smooth degree d del Pezzo surface dP_d , and the former functional equation can be described geometrically as follows: up to post-compositions by projective automorphisms, there exists only a finite number κ_d of “conic fibrations” $U_i : \mathrm{dP}_d \rightarrow \mathbb{P}^1$ ($i = 1, \dots, \kappa_d$). For each U_i , the set Σ_i of values $\sigma \in \mathbb{P}^1$ such that the fiber $U_i^{-1}(\sigma)$ decomposes into the sum of two “lines” has cardinality $8 - d$. To each Σ_i is canonically associated a certain “complete antisymmetric hyperlogarithm AH_{Σ_i} ” on $\mathbb{P}^1$, of weight $7 - d$, which is well-defined up to sign. In [12], we proved that for a suitable choice of the AH_{Σ_i} ’s⁽⁴⁾, the following identity holds true at the generic point of the considered del Pezzo surface dP_d :

$$\sum_{i=1}^{\kappa_d} AH_{\Sigma_i}(U_i) \equiv 0. \quad (\mathbf{HLog}_d)$$

⁽¹⁾ Here $\mathbf{Li}_2$ stands for the classical bilogarithm, defined as the sum of the series $\mathbf{Li}_2(z) = \sum_{n=1}^{+\infty} z^n/n^2$ for $|z| < 1$.

⁽²⁾ To cite a few: hyperbolic geometry, K -theory of number fields, web geometry, theory of cluster algebras, mirror symmetry of log CY manifolds, scattering amplitudes, etc.

⁽³⁾ See also the longer preliminary version [46].

⁽⁴⁾ In [12], we give an effective construction of the suitable hyperlogarithms AH_{Σ_i} ’s such that the functional identity $\sum_{i=1}^{\kappa_d} AH_{\Sigma_i}(U_i) = 0$ be indeed satisfied. Moreover, we give a conceptual interpretation of this construction in terms of representations of the Weyl group associated to the considered del Pezzo surface.

Considering the importance of the functional identities (1.1) $\simeq \mathbf{HLog}_6$ and $(\mathbf{Ab}) \simeq \mathbf{HLog}_5$, it is natural to wonder about the case of the other $(\mathbf{HLog}_d)$'s. In this paper, we deal with $(\mathbf{HLog}_4)$ which we investigate through the prism of web geometry.

Web geometry is the study of webs which are geometric objects formed (locally) by the configuration of a finite number of foliations whose leaves intersect transversally. It proved to be an interesting tool to study geometrically functional identities such as the $(\mathbf{HLog}_d)$'s. Let $V_1, \dots, V_k$ be holomorphic submersions on a complex manifold M , connected and of dimension 2 say, such that $dV_i \wedge dV_j$ does not vanish identically for any $i \neq j$. The level sets of the V_i 's define a k -web denoted by

$$\mathcal{W} = \mathcal{W}(V_1, \dots, V_k).$$

Via projective duality, one classically associates a linear web $\mathcal{W}_C$ to any reduced algebraic curve C , giving rise to the important notions of “*algebraic and algebraizable webs*”. It is relevant to try mimicking for a general web, all the constructions of classical algebraic geometry which apply to algebraic curves. This leads to the notion of “*abelian relation*” (ab. AR) for an arbitrary web such as the k -web $\mathcal{W}$ above: it is a k -tuple $(F_i)_{i=1}^k$ of (germs of) holomorphic functions such that the following functional identity is satisfied:

$$\sum_{i=1}^k F_i(V_i) = 0. \quad (1.2)$$

The abelian relations of $\mathcal{W}$ form a vector space⁽⁵⁾ $\mathbf{AR}(\mathcal{W})$ whose dimension is known as the *rank* of the web and denoted by $\mathrm{rk}(\mathcal{W}) = \dim \mathbf{AR}(\mathcal{W})$. By a result of Bol, we always have $\mathrm{rk}(\mathcal{W}) \leq (k-1)(k-2)/2$. For an algebraic web $\mathcal{W}_C$, the space $\mathbf{AR}(\mathcal{W}_C)$ linearly identifies with $\mathbf{H}^0(C, \omega_C^1)$ hence the rank of $\mathcal{W}_C$ is equal to the arithmetic genus of C : one has

$$\mathrm{rk}(\mathcal{W}_C) = p_a(C) = \frac{1}{2}(d-1)(d-2)$$

with $d = \deg(C)$ therefore Bol's bound actually is an equality in this case.

The identity $(\mathbf{HLog}_d)$ above precisely is of the form (1.2) which naturally leads to consider the associated web, namely the web

$$\mathcal{W}_{\mathrm{dP}_d} = \mathcal{W}(U_1, \dots, U_{\kappa_d})$$

on dP_d , defined by all the fibrations in conics on it. For $d = 6$, taking dP_6 as the blow-up of $\mathbb{P}^2$ at the vertices p_1, p_2, p_3 of the standard projective frame relative to the choice of the affine embedding $\mathbb{C}^2 \subset \mathbb{P}^2$ given by

⁽⁵⁾ Actually, globally the ARs form a local system but we will not elaborate further on this here.

$(x, y) \mapsto [x : y : 1]$ in coordinates, we obtain that

$$\mathcal{W}_{\mathrm{dP}_6} = \mathcal{W}(x, y, x/y)$$

which is one of the two classical normal forms of the so-called *hexagonal 3-web*.⁽⁶⁾

The case of $\mathcal{W}_{\mathrm{dP}_5}$ is much more interesting: viewing dP_5 as the blow up of the plane at the points p_i for $i = 1, \dots, 4$ with p_1, p_2, p_3 the same as the points defined above and $p_4 = [1 : 1 : 1]$, we obtain that with respect to the standard affine coordinates (x, y) , one has

$$\mathcal{W}_{\mathrm{dP}_5} \simeq \mathcal{W}\left(x, y, \frac{x}{y}, \frac{1-y}{1-x}, \frac{x(1-y)}{y(1-x)}\right).$$

The five rational functions appearing as first integrals for $\mathcal{W}_{\mathrm{dP}_5}$ in the chosen coordinates are precisely those appearing as arguments of Rogers' dilogarithm in Abel's identity (**Ab**). We recognize the five first integrals of the famous Bol's web **B**. This web is quite famous in web geometry first, since it is the first example of an exceptional web ever discovered and second, because it enjoys several remarkable properties that we are going to discuss below. Considering this, it is natural to investigate the webs $\mathcal{W}_{\mathrm{dP}_d}$ for $d = 4, 3, 2, 1$.

Each web $\mathcal{W}_{\mathrm{dP}_d}$ carries the hyperlogarithmic abelian relation corresponding to $(\mathbf{HLog}_d)$ hence it is interesting to look at the whole space of the ARs and more generally to its properties. In what follows, we first discuss several properties of Bol's web then discuss in the same way the case of $\mathcal{W}_{\mathrm{dP}_4}$ which is the main object of study of this text.

Notation. In what follows, we will denote by $\mathbf{HLog}^{7-d}$ the identity $(\mathbf{HLog}_d)$ (for any $d = 1, \dots, 6$), this in order to emphasize the weight $7 - d$ of the hyperlogarithms involved in it.

Warning. Some of the statements in the next two subsections may look a bit cryptic at first sight, even for people familiar with web geometry. Everything will be carefully explained further.

⁽⁶⁾ The other normal form we are referring to here being $\mathcal{W}(x, y, x + y)$.

1.1. Many remarkable properties of Bol's web

Abel's identity ($\mathcal{Ab}$) gives rise to an abelian relation for $\mathcal{B}$, which we denote by $\mathcal{Ab}$.

Given a planar web $\mathcal{W}$, one sets $\mathcal{AR}(\mathcal{W})$ for the space of its abelian relations. Inspired by the terminology introduced by Damiano in [17], we will say that an AR of $\mathcal{W}$ is “combinatorial” if it is of minimal length, that is if only 3 of its components are not trivial, and we will denote by $\mathcal{AR}_C(\mathcal{W})$ the subspace of $\mathcal{AR}(\mathcal{W})$ spanned by the ARs of this kind.

It has been known for a long time that Bol's web is very particular as a planar web since it enjoys all the following properties:

- (1) [Geometric definition]. $\mathcal{B} \simeq \mathcal{W}_{\text{dP}_5}$: *Bol's web is equivalent to the web by conics on dP_5 .*
- (2) [Non linearizability]. *Bol's web is not linearizable hence not algebraizable.*
- (3) [Abelian relations].

(a) *All the ARs of $\mathcal{B}$ are polylogarithmic, of weight 1 or 2:*

- *the subspace $\mathcal{AR}_{\log}(\mathcal{B})$ of logarithmic ARs of $\mathcal{B}$ has dimension 5. It coincides with the space $\mathcal{AR}_C(\mathcal{B})$ of combinatorial ARs of $\mathcal{B}$;*
- *the subspace $\mathcal{AR}^2(\mathcal{B})$ of dilogarithmic ARs of $\mathcal{B}$ is spanned by $\mathcal{Ab}$; and there is a decomposition in direct sum*

$$\mathcal{AR}(\mathcal{B}) = \mathcal{AR}_{\log}(\mathcal{B}) \oplus \langle \mathcal{Ab} \rangle. \quad (\star)$$

- (b) *Abel's AR $\mathcal{Ab}$ spans the whole space $\mathcal{AR}_{\log}(\mathcal{B})$ by residues/monodromy.*
- (c) *Conversely, one can reconstruct Abel's identity $\mathcal{Ab}$ from a basis of $\mathcal{AR}_{\log}(\mathcal{B})$.*

- (4) [Rank]. *Bol's web has maximal rank hence is exceptional.*
- (5) [“Algebraization”]. *Let $\mathcal{W}$ be a germ of 5-web at the origin of $\mathbb{C}^2$, equivalent to Bol's web. There are two distinct canonical ways to recover $(\text{dP}_5, \mathcal{B})$ from $\mathcal{W}$:*
 - *the first is from the space $\mathcal{AR}_C(\mathcal{W})$ of combinatorial ARs of $\mathcal{W}$;*
 - *the second approach is via the canonical map $\varphi_{\mathcal{W}} : (\mathbb{C}^2, 0) \rightarrow \mathcal{M}_{0,5}$ associated to $\mathcal{W}$.*

- (6) [Weyl group action].

(a) *There is a natural action of the Weyl group $W(A_4) = \mathfrak{S}_5$ on the space of abelian relations $\mathcal{AR}(\mathcal{B})$ for which $(\star)$ is the decomposition into irreducible $\mathfrak{S}_5$ -representations.*

- (b) *In particular, $\langle \mathbf{Ab} \rangle$ is $\mathfrak{S}_5$ -stable and isomorphic to the signature as a $\mathfrak{S}_5$ -representation.*
- (7) [Hexagonality & Characterization]. *Bol's web is hexagonal and is essentially characterized by this property since "for $k \geq 3$, a hexagonal planar k -web either is linearizable and equivalent to a web formed by k pencils of lines or $k = 5$ and it is equivalent to $\mathbf{B}$ ".*
- (8) [Construction à la GM]. *$\mathbf{B}$ can be constructed following the approach of Gelfand and MacPherson [27]: it can be seen as the quotient, under the action of the Cartan torus $H_4 \subset \mathrm{SL}_5(\mathbb{C})$, of a natural H_4 -equivariant web defined on the grassmannian $G_2(\mathbb{C}^5)$.*
- (9) [Modularity]. *Bol's web is modular: it is equivalent to the web $\mathbf{W}_{0,5}$ on $\bar{\mathcal{M}}_{0,5}$ defined by the five forgetful maps $\bar{\mathcal{M}}_{0,5} \rightarrow \bar{\mathcal{M}}_{0,4}$.*
- (10) [Cluster web]. *Bol's web is of cluster type: up to equivalence, it can be defined by means of the $\mathcal{X}$ -cluster variables of the finite type cluster algebra of type A_2 .*

Some of the statements above are obvious/tautological (e.g. the first), some others are very classical ((2), (3) and (7) for instance, see [6]) or have been proved rather recently (such as (6), (8) or (9), cf. [17] and [27]). The fifth statement is new (and actually provocative without further explanations, as it seems to contradict the fourth!). In any case, all of these are well-known or easy to prove and it is the fact that all are satisfied by $(\mathbf{B}, \mathbf{Ab})$ which makes this pair such an interesting object.

Although rather long, the list above is not complete since an interesting feature of $(\mathbf{Ab})$, related to (8), is missing in it. Namely that the real identity $(\mathbf{Ab})$ can be obtained geometrically within Gelfand–MacPherson formalism [27], by means of the invariant representative of the first Pontryagin class of the tautological bundle on the real grassmannian $G_2(\mathbb{R}^5)$. Since this is a phenomenon taking place in a real setting and because we do not have any counterpart for $\mathbf{HLog}^3$ yet, this result has not been included in the list above (although it truly deserves to be part of it). We will discuss this matter a bit further in Section 5.9.

*

Since $\mathbf{B}$ is equivalent to $\mathbf{W}_{\mathrm{dP}_5}$ and because what can be considered as its most important feature, namely the fact it carries the dilogarithmic abelian relation $\mathbf{Ab}$, generalizes to $\mathbf{W}_{\mathrm{dP}_4}$, one can wonder whether the properties listed above admit analogs for this 10-web or not. It turns out that it is indeed the case as we are going to show now.

1.2. Many remarkable properties of $\mathcal{W}_{\text{dP}_4}$

The main aim of this text is to prove that the following facts hold true, where dP_4 stands for an arbitrary (fixed) smooth del Pezzo quartic surface.

- (1) [Geometric definition]. $\mathcal{W}_{\text{dP}_4}$ is the web defined by the ten pencils of conics on dP_4 .
- (2) [Non linearizability]. $\mathcal{W}_{\text{dP}_4}$ is not linearizable hence not equivalent to an algebraic web.
- (3) [Abelian relations].
 - (a) All the abelian relations of $\mathcal{W}_{\text{dP}_4}$ are hyperlogarithmic, of weight 1, 2 or 3:

- the subspace $\mathbf{AR}^1(\mathcal{W}_{\text{dP}_4})$ of logarithmic ARs of $\mathcal{W}_{\text{dP}_4}$ has dimension 20. It coincides with the space $\mathbf{AR}_C(\mathcal{W}_{\text{dP}_4})$ of combinatorial abelian relations of $\mathcal{W}_{\text{dP}_4}$;
- the subspace $\mathbf{AR}^2(\mathcal{W}_{\text{dP}_4})$ of weight 2 hyperlogarithmic ARs has dimension 15: it is the direct sum of two subspaces, a first one, $\mathbf{AR}_{\text{sym}}^2$, of “symmetric” ARs, of dimension 5 and another one denoted by $\mathbf{AR}_{\text{asym}}^2$ of “antisymmetric” ARs, of dimension 10;
- the subspace $\mathbf{AR}^3(\mathcal{W}_{\text{dP}_4})$ of weight 3 hyperlogarithmic ARs of $\mathcal{W}_{\text{dP}_4}$ is spanned by $\mathbf{HLog}^3$;
- there is a decomposition in direct sum

$$\mathbf{AR}(\mathcal{W}_{\text{dP}_4}) = \mathbf{AR}^1(\mathcal{W}_{\text{dP}_4}) \oplus (\mathbf{AR}_{\text{sym}}^2 \oplus \mathbf{AR}_{\text{asym}}^2) \oplus \langle \mathbf{HLog}^3 \rangle. \quad (\clubsuit)$$

- (b) $\mathbf{HLog}^3$ spans the whole space $\mathbf{AR}_{\text{asym}}^2$ of antisymmetric ARs of $\mathcal{W}_{\text{dP}_4}$ by residues/monodromy.
 - (c) Conversely, one can reconstruct the weight 3 hyperlogarithmic identity $\mathbf{HLog}^3$ from a basis of $\mathbf{AR}_{\text{asym}}^2$.
- (4) [Rank]. The web $\mathcal{W}_{\text{dP}_4}$ has maximal rank hence is exceptional.
- (5) [“Algebraization”]. Let $\mathcal{W}$ be a germ of 10-web at the origin of $\mathbb{C}^2$, equivalent to $\mathcal{W}_{\text{dP}_4}$. There are two distinct canonical ways to recover the latter (and the quartic del Pezzo surface dP_4) from $\mathcal{W}$:
 - the first is from the space $\mathbf{AR}_C(\mathcal{W})$ of combinatorial ARs of $\mathcal{W}$;
 - the second approach is via the canonical map $\varphi_{\mathcal{H}\mathcal{W}_5} : (\mathbb{C}^2, 0) \rightarrow \mathcal{M}_{0,5}$ associated to any linearizable hexagonal 5-subweb $\mathcal{H}\mathcal{W}_5$ of $\mathcal{W}$.
- (6) [Weyl group action].
 - (a) The Weyl group $W = W(D_5)$ acts naturally on $\mathbf{AR}(\mathcal{W}_{\text{dP}_4})$ and $(\clubsuit)$ actually is its decomposition into irreducible W -modules.

- (b) *In particular, $\langle \mathbf{HLog}^3 \rangle$ is W -stable and isomorphic to the signature as a W -representation.*
- (7) [Hexagonality & Characterization]. $\mathcal{W}_{\mathrm{dP}_4}$ *is characterized by its hexagonal 3-subwebs.*
- (8) [Construction à la GM]. $\mathcal{W}_{\mathrm{dP}_4}$ *can be seen as the quotient, under the action of the Cartan torus $H_5 \subset \mathrm{Spin}_{10}(\mathbb{C})$, of a natural H_5 -equivariant web defined on the spinor variety $\mathbb{S}_5$.*
- (9) [Modularity]. *The web $\mathcal{W}_{\mathrm{dP}_4}$ is modular: it can be obtained as the pull-back under a natural map $\mathrm{dP}_4 \dashrightarrow \mathrm{Conf}_6(\mathbb{P}^2)$ of the web induced by the 30 rational maps $\mathrm{Conf}_6(\mathbb{P}^2) \dashrightarrow \mathrm{Conf}_4(\mathbb{P}^1) = \mathcal{M}_{0,4}$ on the space of projective configurations of six points on $\mathbb{P}^2$.*
- (10) [Cluster web]. *The web $\mathcal{W}_{\mathrm{dP}_4}$ is of cluster type: it can be obtained by means of some of the $\mathcal{X}$ -cluster variables of the finite type cluster algebra of type D_4 .*

A few remarks about these statements are in order.

- As in the case of Bol’s web in Section 1.1, all the assertions above do not have the same status. For instance, the first is tautological and the second is almost obvious. The fourth point was known before (see just below), (6b) has been established in [46]. All the other statements (namely (3), (5), (6a), and from (7) to (10)) are new.
- Classically in web geometry, a maximal rank web is said to be exceptional precisely when it is not algebraizable: for such a web, one of these two adjectives means exactly the opposite of the other. Since both $\mathcal{W}_{\mathrm{dP}_5} \simeq \mathcal{B}$ and $\mathcal{W}_{\mathrm{dP}_4}$ are exceptional in the classical sense, the use of the term “algebraization” about them could appear completely inconsistent at first reading. The terminology that we introduce here is a bit provocative and has to be understood as follows: if these two webs are not linearizable, each of them has as a natural model a web formed by the pencils of conics on a certain rational surface. The term “algebraization” used here in relation to them refers to the process of constructing an equivalence to this “natural algebraic model” when starting from any other web analytically equivalent to it.
- Birationally equivalent models of $\mathcal{W}_{\mathrm{dP}_4}$ were considered in the author’s PhD thesis [44], in the unpublished text [49] and also later in the article [41]. That this web is exceptional (which is equivalent to the points (2) and (4) together) is mentioned in [44] and [49] and

more thoroughly investigated by Pereira who computed the hyperlogarithmic ranks of a birational model of $\mathcal{W}_{\mathrm{dP}_4}$ (cf. the proof of Theorem 5.1 in [41]).

- Many of the results established in the present text were previously announced without proof in our preprint [46] (in particular, in Section 4.2 therein).

What one has to have in mind is that, considering that almost all (if not all) the remarkable properties of the pair Abel’s identity $(\mathcal{A}b) \simeq \mathbf{HLog}_5$ / Bol’s web $\mathcal{B} \simeq \mathcal{W}_{\mathrm{dP}_5}$ admit formally very similar counterparts for the pair $\mathbf{HLog}_4$ / $\mathcal{W}_{\mathrm{dP}_4}$, then it is not unreasonable to see the latter as the most natural weight 3 generalization of the former.

The quest of natural generalizations to any weight of Abel’s dilogarithmic identity is a long standing one which has received many contributions by several authors. But except until very recently and for very few cases, all these contributions were sticking to the case of polylogarithms or more precisely to iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ constructed from words in the alphabet with the two logarithmic forms $\mathrm{dlog}(u) = du/u$ and $\mathrm{dlog}(u-1) = du/(u-1)$ as letters. The discovery of the identities $(\mathbf{HLog}_d)$ for $d = 4, 3, 2, 1$ as well as the striking fact that $\mathcal{B} \simeq \mathcal{W}_{\mathrm{dP}_5}$ and $\mathcal{W}_{\mathrm{dP}_4}$ look so similar regarding the numerous remarkable properties they satisfy may be an indication that natural generalizations to higher weights of the five terms relation of the dilogarithm have to be looked for in the more general setting of hyperlogarithms.

However, what makes the functional equations of polylogarithms so interesting is the role that these identities are playing in other areas of mathematics, such as in hyperbolic geometry (volume of hyperbolic polytopes) or in the K-theory of number fields (regarding to regulators and Zagier’s polylogarithmic conjecture). For the moment, we are not aware of any occurrence of the identity $\mathbf{HLog}^3$ within other fields of mathematics, which would be necessary for it to be fully recognized as the genuine weight 3 generalization of Abel’s identity.

1.3. Structure of the paper

The rest of the paper is organized as follows.

In Section 2, we introduce the material about hyperlogarithms, webs and del Pezzo surfaces which will be used in the rest of the text. Everything here is well-known or straightforward, hence no proof has been included but only some references when they are needed. Section 3 is where we discuss all the properties of Bol’s web cited in Section 1.1 above with more details. That

$\mathcal{W}_{\text{dP}_4}$ satisfies all the properties listed in Section 1.2 is proved in Section 4, which constitutes the main part of the paper. In the last Section 5, we discuss some questions left open by the results obtained here that we find interesting, such as that of constructing several other versions of the identity $\mathbf{HLog}^3$.

*

To help the reader to navigate throughout the text, we offer the table below which indicates where in the text are discussed the different properties of the webs under consideration.

Property	$\mathcal{B} \simeq \mathcal{W}_{\text{dP}_5}$	$\mathcal{W}_{\text{dP}_4}$
Description of the space of ARs as a W -module	Sections 3.3 & 3.4	Section 4.2
Canonical algebraizations	Section 3.5	Section 4.4
Combinatorial characterization	Section 2.2.4	Section 4.3
Description as a modular web	page 1043	Section 4.6
Description à la Gelfand–MacPherson	Section 3.6	Section 4.5
Description as a cluster web	Section 3.2	Section 4.7

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2. Hyperlogarithms, webs and del Pezzo surfaces in a nutshell

Here we first give a quick account of the classical theory of hyperlogarithms then we discuss some points of web geometry. After short reminders on webs, we discuss some new (although rather elementary) notions concerning abelian relations, related to hyperlogarithms. We also introduce a simple formalism which will prove to be convenient to investigate the rich combinatorial structure of the spaces of ARs of the del Pezzo webs we will study in the next sections. Then we recall some well-known facts about the geometry of del Pezzo surfaces and the Weyl groups actions on their Picard lattices.

Almost no examples are given in this section. The two cases of $\mathcal{W}_{\text{dP}_5}$ and $\mathcal{W}_{\text{dP}_4}$ studied in detail in the following sections will serve to show, among other things, the relevance of the notions introduced here.

2.1. Hyperlogarithms

Hyperlogarithms are special functions considered first by Poincaré (1884) and Lappo–Danilevski (1928) in the realm of complex ordinary differential equations. Some modern references are [10], [59] and [23]. For more specific references on hyperlogarithms in relation with the notion of abelian relation in web geometry, see [45, §1.2], [46, §2.1] or [12].

Let Y be a connected complex manifold. We fix a finite dimensional subspace $\mathbf{H} \subset \mathbf{H}^0(Y, \Omega_Y^1)$ we are going to work with. We also assume that $\mathbf{H}$ has a basis $(\omega_1, \dots, \omega_m)$ whose elements satisfy the following assumptions: these forms are closed and their wedge products all vanish, i.e.

$$d\omega_i = 0 \quad \text{and} \quad \omega_i \wedge \omega_j = 0 \quad (2.1)$$

for all $i, j = 1, \dots, m$.

Given a base point $y \in Y$, we consider continuous paths $\gamma^z : [0, 1] \rightarrow Y$ joining y to another point $z \in Y$, all supported in a small open ball B_y around y in Y . For any “weight” $w \geq 1$, we identify any pure tensor $\underline{\omega} = \omega_{i_1} \otimes \dots \otimes \omega_{i_w} \in \mathbf{H}^{\otimes w}$ with the word $\omega_{i_1} \dots \omega_{i_w}$. Setting inductively

$$\begin{aligned} L_{\omega_{i_1}} : z &\mapsto \int_{\gamma^z} \omega_{i_1}, \\ L_{\omega_{i_1}\omega_{i_2}} : z &\mapsto \int_{\gamma^z} L_{\omega_{i_1}} \omega_{i_2} \\ \text{and } L_{\omega_{i_1}\dots\omega_{i_w}} : z &\mapsto \int_{\gamma^z} L_{\omega_{i_2}\dots\omega_{i_w}} \omega_{i_1}, \end{aligned} \quad (2.2)$$

we obtain iterated integrals which, thanks to the conditions (2.1), do not depend on the path γ^z but only on its endpoint hence give rise to holomorphic germs $L_{\underline{\omega}} \in \mathcal{O}_{Y,y}$. Extending $\underline{\omega} \mapsto L_{\underline{\omega}}$ linearly to the whole tensorial power $\mathbf{H}^{\otimes w}$, we define a realization map

$$\Pi_y^w : \mathbf{H}^{\otimes w} \longrightarrow \mathcal{O}_{Y,y} \quad (2.3)$$

which is obviously $\mathbb{C}$ -linear and turns out to be injective. The shuffle product $\sqcup$ on words in the letters $\omega_1, \dots, \omega_m$ gives rise to another product on the full tensorial algebra $\mathbf{H}^{\otimes \bullet} = \bigoplus_{w \geq 1} \mathbf{H}^{\otimes w}$, again denoted by $\sqcup$ and which enjoys the nice property of making of

$$\Pi_y = \bigoplus_{w \geq 1} \Pi_y^w : (\mathbf{H}^{\otimes \bullet}, \sqcup) \longrightarrow \mathcal{O}_{Y,y}$$

an injective map of complex algebras.⁽⁷⁾ For $L \in \text{Im}(\mathbf{II}_y)$, we define its “*weight*” as the minimum $w(L)$ of the set of w ’s such that $L \in \mathbf{II}_y(\bigoplus_{w' \leq w} \mathbf{H}^{\otimes w'})$ and the “*symbol of L* ” (at y a priori) is the element $\mathcal{S}_y(L) \in \mathbf{H}^{\otimes w(L)}$ such that the weight of $L - L_{\mathcal{S}_y(L)}$ is strictly less than that of L .

Using the formulas (2.2), one sees that any iterated integral $L \in \text{Im}(\mathbf{II}_y)$ extends holomorphically along any path in Y and gives rise to a holomorphic multivalued function on the whole Y , again denoted by L . If L^γ stands for the analytic continuation along a path $\gamma : [0, 1] \rightarrow Y$ joining y to the other extremity $y' = \gamma(1) \in Y$, then $L^\gamma \in \text{Im}(\mathbf{II}_{y'}) \subset \mathcal{O}_{Y, y'}$ and moreover $w(L) = w(L^\gamma)$ and $\mathcal{S}_y(L) = \mathcal{S}_{y'}(L^\gamma)$. This shows that the notions of weight and of symbol of an iterated integral constructed from $\mathbf{H}$ do not depend on the chosen base point but only on its analytic extension as a multivalued function on the whole Y .

One difficulty when working with multivalued holomorphic functions is that it is often painful to be rigorous and precise about which branch of them has to be considered so that a certain property be satisfied. This can be avoided for iterated integrals by using their symbols which offer an efficient way to deal with them algebraically.

*

The hyperlogarithms which are involved in this text (in relation with the abelian relations of planar webs) are among the simplest, and are those obtained when Y is the complement of a finite set Σ in $\mathbb{P}^1$ and when one takes $\mathbf{H} = \mathbf{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\text{Log } \Sigma))$. In this case, if $\sigma_1, \dots, \sigma_{m+1}$ stand for the elements of Σ (assumed to be pairwise distinct), one takes an affine coordinate z such that $\sigma_{m+1} = \infty$ and the basis of $\mathbf{H}$ we are going to work with is that whose elements are the following logarithmic 1-forms, for $i = 1, \dots, m$:

$$\omega_i = d\text{Log}(z - \sigma_i) = \frac{dz}{z - \sigma_i}.$$

Hyperlogarithms on $\mathbb{P}^1$ have been considered by many authors, ancient or contemporary. The properties of these functions, for instance their monodromy, are well-known: as explained in [10] (cf. Corollary 6.5 therein), given an element ϖ of $\mathbf{H}^{w-1}$ (for some weight $w \geq 1$), then modulo $\mathbf{H}\text{Log}^{\leq w}$, for any $i \in \{1, \dots, k\}$, one has

$$(\mathcal{M}_{\gamma_k} - \text{Id})(L_{\omega_i \varpi}) \equiv \begin{cases} 0 & \text{if } k \neq i \\ 2i\pi L_{\varpi} & \text{if } k = i \end{cases} \quad (2.4)$$

⁽⁷⁾ The product on $\mathcal{O}_{Y, y}$ is of course the one induced by the usual (pointwise) product of holomorphic functions.

where for each k , γ_k stands for a continuous path in $\mathbb{P}^1 \setminus \Sigma$, whose index with respect to σ_i is 1 if $i = k$ and 0 otherwise.

2.2. Webs

As references on webs, we mention the books [6] and [42]. The first part of our memoir [45] may be useful too.

2.2.1.

In this text, we only deal with a very restrictive class of planar webs, our definition will be restricted to them. Here a d -web $\mathcal{W}$ for $d \geq 1$ on a (connected and smooth) algebraic surface X will designate a d -tuple $(\mathcal{F}_{U_i})_{i=1}^d$ of foliations formed by the level sets of non constant rational functions $U_i : X \dashrightarrow \mathbb{P}^1$ which are such that $dU_i \wedge dU_j \neq 0$ at the generic point of X , for any $i, j = 1, \dots, d$ distinct. One can and will assume that each of the U_i 's is non composite (or equivalently, the generic fibers of the U_i 's are connected). In this case we use the following notation to denote $\mathcal{W}$:

$$\mathcal{W} = \mathcal{W}(U_1, \dots, U_d).$$

Any non constant map $\tilde{U}_i$ defined on an open subset of X such that $dU_i \wedge d\tilde{U}_i = 0$ is a first integral (for the i -th foliation) of $\mathcal{W}$. Another d -web $\mathcal{W}'$ defined on another surface X' is said to be “*equivalent*” to $\mathcal{W}$ if there exists a (local) biholomorphism φ between two open subsets in X' and X such that the pull-backs $\varphi^*(U_i) = U_i \circ \varphi$ for $i = 1, \dots, d$ form a complete set of first integrals for $\mathcal{W}'$ (possibly up to a relabelling of the foliations).

2.2.2.

An “*abelian relation*” (ab. AR) for $\mathcal{W}(U_1, \dots, U_d)$ is a tuple of exact holomorphic 1-forms $(dF_i)_{i=1}^d$ such that

$$\sum_{i=1}^d U_i^*(dF_i) = 0$$

or equivalently, such that $\sum_{i=1}^d F_i(U_i) = 0$ holds true identically up to the constants (on a suitable open domain in X). Any AR extends holomorphically along any path supported in the regular set $Y = Y_{\mathcal{W}}$ of $\mathcal{W}$, defined as

the biggest open subset in X on which all the foliations of the web are regular and intersect transversally. For that reason, the abelian relations form a local system $\mathbf{AR}(\mathcal{W})$ on Y rather than a vector space. However we will use, a bit abusively, the same notation to designate the restriction of $\mathbf{AR}(\mathcal{W})$ on any open domain in Y , restriction which canonically has the structure of a vector space.

2.2.3.

For any $i = 1, \dots, d$, let $\mathfrak{R}_i$ be the finite subset of $\mathbb{P}^1$ whose elements are the λ 's such that there exists an irreducible component of $U_i^{-1}(\lambda)$ which is invariant by another foliation $\mathcal{F}_{U_j}$ ($j \neq i$) of the web. Then the open subset

$$Z = Z_{\mathcal{W}} = X \setminus \bigcup_{i=1}^d U_i^{-1}(\mathfrak{R}_i)$$

contains Y and one verifies that any germ of AR $(F_i(U_i))_{i=1}^d$ at some point $y \in Y$ extends holomorphically along any continuous path supported in Z . Equivalently, for any i , setting $y_i = U_i(y)$, the germ $F_i \in \mathcal{O}_{\mathbb{P}^1, y_i}$ extends to a global but a priori multivalued holomorphic function on $\mathbb{P}^1 \setminus \mathfrak{R}_i$.

2.2.4. Combinatorics of the space of abelian relations

Let us introduce general notions which we believe are relevant for studying webs. Let $\mathcal{W}$ be a planar web, formed by d distinct foliations $\mathcal{F}_1, \dots, \mathcal{F}_d$ on its definition domain in $\mathbb{C}^2$.

For any $k = 3, \dots, d$, one sets/defines

- $\text{SubW}_k(\mathcal{W})$ is the set of k -subwebs of $\mathcal{W}$;
- $\text{SubW}(\mathcal{W})$ is the set of subwebs of $\mathcal{W}$ (i.e. $\text{SubW}(\mathcal{W}) = \bigcup_{k=3}^d \text{SubW}_k(\mathcal{W})$);
- $\mathbf{AR}_3(\mathcal{W})$ denotes the subspace of $\mathbf{AR}(\mathcal{W})$ spanned by the ARs of the 3-subwebs of $\mathcal{W}$;
- $r_3(\mathcal{W})$ stands for the dimension of $\mathbf{AR}_3(\mathcal{W})$;
- $r_{\mathcal{W}} : \text{SubW}(\mathcal{W}) \rightarrow \mathbb{N}$ is the function associating $r_3(\mathcal{W})$ to any subweb $\mathcal{W}$ of $\mathcal{W}$;
- $r_{\mathcal{W},k} : \text{SubW}_k(\mathcal{W}) \rightarrow \mathbb{N}$ is the restriction of $r_{\mathcal{W}}$ to $\text{SubW}_k(\mathcal{W})$.

The functions $r_{\mathcal{W},k}$'s are combinatorial objects invariantly attached to $\mathcal{W}$. They allow to state in a nice and concise way some properties of webs. For instance, let us consider the interesting case of Bol's web $\mathcal{B} \simeq \mathcal{W}_{\text{dP}_5}$: since it is hexagonal, one has identically $r_{\mathcal{B},3} = 1$ and $r_{\mathcal{B},4} = 3$ and one verifies that $r_{\mathcal{B},5} \equiv 5$. On the other hand, if $\mathcal{L}$ is a planar web formed by five pencils of lines, then it is a straightforward exercise to determine the triple $(r_{\mathcal{L},k})_{k=3,4,5}$ depending on the combinatorial type of the vertices of the five pencils of lines of $\mathcal{L}$.⁽⁸⁾ From this, one deduces the following combinatorial characterization of $\mathcal{B}$:

THEOREM (Bol [7]). — *Bol's web is characterized by $r_{\mathcal{B}}$. More precisely, any 5-web $\mathcal{W}$ such that $r_{\mathcal{W},3} \equiv 1$, $r_{\mathcal{W},4} \equiv 3$ and $r_{\mathcal{W}} = r_{\mathcal{W},5} \equiv 5$ is equivalent to Bol's web.*

This result shows that the $r_{\mathcal{W},k}$'s can be useful to study webs up to equivalence. Given two planar d -webs $\mathcal{W}'$ and $\mathcal{W}''$ as above, let us say that the functions $r_{\mathcal{W}'}$ and $r_{\mathcal{W}''}$ (or $r_{\mathcal{W}',k}$ and $r_{\mathcal{W}'',k}$ for any $k = 3, \dots, d$) are equivalent, denoted by $r_{\mathcal{W}'} \sim r_{\mathcal{W}''}$ (and similarly for $r_{\mathcal{W}',k}$ and $r_{\mathcal{W}'',k}$) if both coincide, possibly up to a relabelling of the foliations composing one of them.

In [11], Burau first determines $r_{\mathcal{W}_{\text{dP}_3},3}$ and then proves the following characterization result:

THEOREM (Burau [11]). — *The class of 27-webs $\mathcal{W}_{\text{dP}_3}$'s is characterized by $r_{\mathcal{W}_{\text{dP}_3},3}$: any 27-web $\mathcal{W}$ such that $r_{\mathcal{W},3} \sim r_{\mathcal{W}_{\text{dP}_3},3}$ is equivalent to the 27-web in conics on a cubic surface in $\mathbb{P}^3$.*

In Section 4.3, we will show that there is an analogous result for the web $\mathcal{W}_{\text{dP}_4}$, this for any smooth quartic del Pezzo surface dP_4 .

2.2.5. Hyperlogarithmic abelian relations

As we are going to prove, all the abelian relations of $\mathcal{W}_{\text{dP}_4}$ are hyperlogarithmic hence all can be found using the “symbolic” approach we are going to describe. This method is elementary and amounts to elementary linear algebra computations in tensorial spaces. It has been used to find some ARs of webs first by Robert [49], then by Pereira [41]. But the use of a symbolic

⁽⁸⁾ For instance, when $\mathcal{L}$ is *parallelizable*, that is formed by five pencils of lines whose vertices are aligned, then one has $r_{\mathcal{L},3} \equiv 1$ (hexagonality), $r_{\mathcal{L},4} \equiv 2$ and $r_{\mathcal{L},5} \equiv 3$. In the case when exactly four pencils of $\mathcal{L}$ have their vertices on a same line, one has $r_{\mathcal{L},3} \equiv 1$, $\text{Im}(r_{\mathcal{L},4}) = \{2, 3\}$ and $r_{\mathcal{L},5} \equiv 5$, etc.

approach for solving a functional equation can even be found before, for instance in [30, §4].

We continue to use the notation used above. For any i , let $\mathcal{H}_i$ stand for the space of rational 1-forms on $\mathbb{P}^1$, regular on $\mathbb{P}^1 \setminus \mathfrak{R}_i$ and having at most logarithmic poles at each point of $\mathfrak{R}_i$, i.e.

$$\mathcal{H}_i = \mathbf{H}^0\left(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\mathrm{Log} \mathfrak{R}_i)\right).$$

We set $\mathfrak{R}_i = \{\mathfrak{r}_{i,s}\}_{s=0}^{m_i}$ with the $\mathfrak{r}_{i,s}$'s pairwise distinct and $\mathfrak{r}_{i,0} = \infty$. Then the 1-forms $\eta_{i,s} = d\mathrm{Log}(z - \mathfrak{r}_{i,s}) = dz/(z - \mathfrak{r}_{i,s})$ for $s = 1, \dots, m_i$, form a basis of $\mathcal{H}_i$ hence their pull-backs $\Theta_{i,s} = U_i^*(\eta_{i,s}) = d\mathrm{Log}(U_i - \mathfrak{r}_{i,s}) = dU_i/(U_i - \mathfrak{r}_{i,s})$ form a basis of the subspace $\mathbf{H}_i = U_i^*(\mathcal{H}_i)$ of the space $\Omega_{\mathbb{C}(X)}^1 \cap \mathbf{H}^0(Y, \Omega_Y^1)$ of rational 1-forms on X whose restrictions on Y are holomorphic. We set

$$\mathbf{H} = \mathbf{H}_{\mathcal{W}} = \sum_{i=1}^d \mathbf{H}_i \subset \Omega_{\mathbb{C}(X)}^1 \cap \mathbf{H}^0(Y, \Omega_Y^1).$$

Then for any weight $w \geq 1$, we define a vector space of “(symbolic) weight w hyperlogarithmic abelian relations”, denoted by $\mathbf{HLogAR}^w = \mathbf{HLogAR}^w(\mathcal{W})$, by requiring that the following sequence of vector spaces be exact:

$$0 \longrightarrow \mathbf{HLogAR}^w \longrightarrow \bigoplus_{i=1}^{10} \mathbf{H}_i^{\otimes w} \xrightarrow{\Phi^w} \mathbf{H}^{\otimes w}, \quad (2.5)$$

where the map Φ^w is the one defined as follows: if $\Omega = (\Omega_i)_{i=1}^d$ with $\Omega_i \in \mathbf{H}_i^{\otimes w}$ for $i \leq d$, then $\Phi^w(\Omega) = \sum_{i=1}^d \iota_i^w(\Omega_i)$, where for any i , ι_i^w stands for the monomorphism $\mathbf{H}_i^{\otimes w} \hookrightarrow \mathbf{H}^{\otimes w}$ naturally induced by the inclusion $\mathbf{H}_i \subset \mathbf{H}$.

We fix an arbitrary base point $y \in Y$ and we set $y_i = U_i(y) \in \mathbb{P}^1 \setminus \mathfrak{R}_i$ for $i = 1, \dots, d$. For any i , since $\mathbf{H}_i = U_i^*(\mathcal{H}_i)$ and because this is obvious for $\mathcal{H}_i$, the $\Theta_{i,s}$ satisfy the conditions (2.1), hence there are well-defined iterated integration morphisms

$$\mathbf{II}_{Y,y}^w : \mathbf{H}_i^{\otimes w} \longrightarrow \mathcal{O}_{Y,y} \quad \text{and} \quad \mathbf{II}_{\mathbb{P}^1,y_i}^w : \mathcal{H}_i^{\otimes w} \longrightarrow \mathcal{O}_{\mathbb{P}^1,y_i}$$

which are easily seen to satisfy $\mathbf{II}_{Y,y}^w = U_i^* \circ \mathbf{II}_{\mathbb{P}^1,y_i}^w$ where U_i^* stands here for the natural map $\mathcal{H}_i^{\otimes w} \simeq \mathbf{H}_i^{\otimes w}$ induced by the pull-back map $U_i^* : \mathcal{H}_i \simeq \mathbf{H}_i$. In other terms, for any $\Theta = \Theta_{i,s_1} \otimes \dots \otimes \Theta_{i,s_w} = U_i^*(\eta)$ with $\eta = \eta_{i,s_1} \otimes \dots \otimes \eta_{i,s_w}$, one has

$$L_{\Theta} = \mathbf{II}_{Y,y}^w(\Theta) = \mathbf{II}_{\mathbb{P}^1,y_i}^w(\eta) = L_{\eta} \circ U_i$$

as holomorphic germs at y and this extends by linearity to any element of $\mathbf{H}_i^{\otimes w}$.

From the elementary facts above, it follows that for any element $\Omega = (\Omega_i)_{i=1}^d \in \mathbf{HLogAR}^w$, then there exists $(\omega_i)_{i=1}^d \in \bigoplus_{i=1}^d \mathcal{H}_i^{\otimes w}$ such that one has $\Omega_i = U_i^*(\omega_i)$ for any $i = 1, \dots, d$. Thus the d -tuple of hyperlogarithms $(L_{\omega_i})_{i=1}^d$ satisfies

$$\sum_{i=1}^d L_{\omega_i}(U_i) = 0$$

(equality in $\mathcal{O}_{Y,y}$) hence $L_{\Omega} = (L_{\omega_i}(U_i))_{i=1}^d$ is a hyperlogarithmic AR of weight w for $\mathcal{W}$ at y . Denoting by $\mathbf{AR}(\mathcal{W})$ the space of ARs of $\mathcal{W}$ on any fixed but arbitrary open domain containing y , we thus obtain a linear map $\mathbf{II}^w : \mathbf{HLogAR}^w \rightarrow \mathbf{AR}(\mathcal{W})$ whose injectivity follows at once from that of (2.3). Using the symbol, we obtain that all the spaces $\mathbf{II}^w(\mathbf{HLogAR}^w)$'s for $w \geq 1$ are in direct sum in $\mathbf{AR}(\mathcal{W})$ from which we obtain a linear injective map

$$\mathbf{II} = \mathbf{II}_{\mathcal{W}} : \mathbf{HLogAR} = \bigoplus_{w \geq 1} \mathbf{HLogAR}^w \longrightarrow \mathbf{AR}(\mathcal{W}). \quad (2.6)$$

For each $w \geq 1$, we set

$$\mathbf{AR}^w(\mathcal{W}) = \mathbf{II}^w(\mathbf{HLogAR}^w)$$

and call its elements the weight w hyperlogarithmic ARs of $\mathcal{W}$ (at y). A global perspective requires a bit of additional formalism: the $\mathbf{AR}^w(\mathcal{W})$'s defined locally above actually do not give rise to global local systems on Y , but the direct sums

$$\mathbf{AR}^{\leq w}(\mathcal{W}) = \bigoplus_{w' \leq w} \mathbf{AR}^{w'}(\mathcal{W})$$

do (this follows from the fact that hyperlogarithms have unipotent monodromies). They are the pieces of what we call the “*hyperlogarithmic filtration*” $\mathbf{HF}_{\mathcal{W}} = \mathbf{AR}^{\leq \bullet}(\mathcal{W})$ of $\mathcal{W}$. As one verifies easily, for any weight w , one recovers the symbolic space $\mathbf{HLogAR}^w$ as the w -th piece of the graded space associated to $\mathbf{HF}_{\mathcal{W}}$:

$$\mathrm{Gr}^w(\mathbf{HF}_{\mathcal{W}}) = \mathbf{HLogAR}^w.$$

Since they are algebraic objects, it is much more convenient to work with the symbolic spaces $\mathbf{HLogAR}^w$ for $w \geq 1$, than with the associated genuine (multivalued) hyperlogarithmic ARs, because of their multivaluedness. This is what we will do with the del Pezzo webs $\mathcal{W}_{\mathrm{dP}_5}$ and $\mathcal{W}_{\mathrm{dP}_4}$, to be examined in more detail later.

We define the “*weight w (resp. the total) hyperlogarithmic rank* $\mathrm{HLrk}^w(\mathcal{W})$ ” (resp. $\mathrm{HLrk}(\mathcal{W})$) of the web $\mathcal{W}$ as the dimension of $\mathbf{HLogAR}^w$

(resp. of $\mathbf{HLogAR}$): one has

$$\begin{aligned} \mathrm{HLrk}^w(\mathcal{W}) &= \dim \mathbf{HLogAR}^w \\ \text{and } \mathrm{HLrk}(\mathcal{W}) &= \dim \mathbf{HLogAR} = \sum_{w \geq 1} \mathrm{HLrk}^w(\mathcal{W}). \end{aligned}$$

We will say that $\mathcal{W}$ has “*all its ARs hyperlogarithmic*” when the map (2.6) is an isomorphism or equivalently when one has $\mathrm{HLrk}(\mathcal{W}) = \mathrm{rk}(\mathcal{W})$.

For $w \geq 1$ fixed, the symmetric group $\mathfrak{S}_w$ acts naturally on the tensor spaces $\mathbf{H}_i^{\otimes w}$ ’s and $\mathbf{H}^{\otimes w}$ and the map Φ^w is equivariant for these actions. It follows that $\mathfrak{S}_w$ acts naturally on $\mathbf{HLogAR}^w$ and that (2.5) actually can be seen as a short exact sequence in the category of $\mathfrak{S}_w$ -modules. As a consequence, it follows that there is a decomposition of $\mathbf{HLogAR}^w$ as a direct sum of irreducible $\mathfrak{S}_w$ -modules. These being encoded by the partitions λ of w , one obtains a $\mathfrak{S}_w$ -equivariant decomposition

$$\mathbf{HLogAR}^w = \bigoplus_{\lambda \vdash w} \mathbf{HLogAR}_{\lambda}^w$$

where for any partition $\lambda \vdash w$, $\mathbf{HLogAR}_{\lambda}^w$ stands for the sum of all the submodules of $\mathbf{HLogAR}^w$ which are isomorphic to “the” irreducible $\mathfrak{S}_w$ -module V_{λ} associated with the partition λ .

There is nothing new for $w = 1$, the first interesting case is when $w = 2$. There are only two partitions to consider, (2) and (1, 1) and the corresponding decomposition can be written

$$\mathbf{HLogAR}^2 = \mathbf{HLogAR}_{\mathrm{asym}}^2 \oplus \mathbf{HLogAR}_{\mathrm{sym}}^2$$

where $\mathbf{HLogAR}_{\mathrm{sym}}^2$ (resp. $\mathbf{HLogAR}_{\mathrm{asym}}^2$) stands for the space of d -tuples $(\Omega_i)_{i=1}^d \in \bigoplus_{i=1}^d \mathbf{H}_i^{\otimes 2}$ such that $\sum_i \Omega_i = 0$ and where each symbol Ω_i is a linear combination of symmetrized elements $\eta_{i,s} \circ \eta_{i,s'} = (\eta_{i,s} \otimes \eta_{i,s'} + \eta_{i,s'} \otimes \eta_{i,s})/2$ (resp. antisymmetrized elements $\eta_{i,s} \wedge \eta_{i,s'} = (\eta_{i,s} \otimes \eta_{i,s'} - \eta_{i,s'} \otimes \eta_{i,s})/2$) with $s, s' = 1, \dots, m_i$. Accordingly, the corresponding decomposition of $\mathbf{AR}^2(\mathcal{W})$ will be written

$$\mathbf{AR}^2(\mathcal{W}) = \mathbf{AR}_{\mathrm{asym}}^2(\mathcal{W}) \oplus \mathbf{AR}_{\mathrm{sym}}^2(\mathcal{W}).$$

2.2.6. Monodromies and residues

Here we describe two ways to get new ARs from a given one. The first is topological: it relies on analytic continuation and applies to any abelian relation. The second is algebraic and only concerns hyperlogarithmic abelian relations.

We continue to use the notation considered above. Let y be in Y and let $\mathbf{A}$ be a germ of abelian relation at this point. As explained above, $\mathbf{A}$ extends holomorphically along any path γ in Y originating from y . If γ is a loop based at y , the analytic continuation $\mathcal{M}_{[\gamma]}(\mathbf{A}) = \mathbf{A}^\gamma$ of $\mathbf{A}$ along γ is an AR of $\mathcal{W}$ at y which only depends on the homotopy class $[\gamma]$ of γ as a based loop. We thus define a “*monodromy representation*”

$$\begin{aligned} \pi_1(Y, y) &\longrightarrow \mathrm{GL}(\mathbf{AR}(\mathcal{W}))^{(9)} \\ [\gamma] &\longmapsto \mathcal{M}_{[\gamma]} : \mathbf{A} \mapsto \mathbf{A}^\gamma \end{aligned} \quad (2.7)$$

whose orbits it could be interesting to study. For instance, given a certain subspace $E \subset \mathbf{AR}(\mathcal{W})$, one will say that “*the abelian relation $\mathbf{A}$ spans E by monodromy*” whenever one has

$$\mathrm{Monod}(\mathbf{A}) \stackrel{\mathrm{def}}{=} \left\langle \pi_1(Y, y) \cdot \mathbf{A} \right\rangle = E.$$

In the extremal case when $\mathrm{Monod}(\mathbf{A}) = \mathbf{AR}(\mathcal{W})$, then $\mathbf{A}$ has to be considered as “the” fundamental AR for the web $\mathcal{W}$ since all the others can be obtained from it by analytic continuation (up to linear combinations). Abel’s AR of Bol’s web is an example of such a very specific AR.

If one limits oneself to considering only hyperlogarithmic ARs, there are even more structures associated with the action of monodromy. Let $\mathbf{HLogAR}^{\leq \bullet}$ be the trivial filtration defined by $\mathbf{HLogAR}^{\leq w} = \bigoplus_{w' \leq w} \mathbf{HLogAR}^{w'}$ for any $w \geq 1$. Because hyperlogarithms have unipotent monodromy, the choice of any base point $y \in Y$ gives rise to a structure of unipotent $\pi_1(Y, y)$ -module on the filtered space $\mathbf{HLogAR}^{\leq \bullet}$ given by

$$[\gamma] \cdot \Omega = \Pi_y^{-1}(\gamma \cdot \Pi_y(\Omega)) \quad (2.8)$$

for any $[\gamma] \in \pi_1(Y, y)$ and any $\Omega \in \mathbf{HLogAR}^w$. This structure of $\pi_1(Y, y)$ -module defined by (2.8) depends on the base point y but in a non essential way that we let the reader to make precise. To (2.8) is associated a representation

$$\rho_{Y, y} : \pi_1(Y, y) \longrightarrow \mathrm{Aut}(\mathbf{HLogAR}^{\leq \bullet})$$

which depends on y . We let the reader verify that the image $\Pi_1 = \mathrm{Im}(\rho_{Y, y})$ is a subgroup of the automorphism group of the filtration $\mathbf{HLogAR}^{\leq \bullet}$ which is well-defined up to conjugation.

For hyperlogarithmic ARs, there is another, more algebraic, way to get new ARs from a given one. Assume that all the elements of the space $\mathbf{H} = \sum_i \mathbf{H}_i$ have logarithmic poles along any irreducible component of $Z = X \setminus Y$.

⁽⁹⁾ More rigorously, the monodromy representation has values in $\mathrm{GL}(\mathbf{AR}(\mathcal{W})_y)$ where $\mathbf{AR}(\mathcal{W})_y$ stands for the vector space of “*germs of ARs of $\mathcal{W}$ at y* ”. We take the liberty here and in what follows of not being very rigorous with the notation.

For such a component D , there is a residue map $\text{Res}_D : \mathbf{H} \mapsto \mathbb{C}$.⁽¹⁰⁾ For any pure tensor $h_1 \otimes \cdots \otimes h_w \in \mathbf{H}^{\otimes w}$, one sets

$$\text{Res}_D(h_1 \otimes \cdots \otimes h_w) = \text{Res}_D(h_1)(h_2 \otimes \cdots \otimes h_w) \quad (2.9)$$

and extending it by linearity, we get a linear map $\text{Res}_D : \mathbf{H}^{\otimes w} \rightarrow \mathbf{H}^{\otimes(w-1)}$. We define similarly $\text{Res}_{i,D} : \mathbf{H}_i^{\otimes w} \rightarrow \mathbf{H}_i^{\otimes(w-1)}$ for each i and one verifies that the residue maps commute, which gives us a residue map at the level of symbolic hyperlogarithmic abelian relations:

$$\begin{aligned} \text{Res}_D : \mathbf{HLogAR}^w &\longrightarrow \mathbf{HLogAR}^{w-1} \\ (\Omega_i)_{i=1}^d &\longmapsto (\text{Res}_{i,D}(\Omega_i))_{i=1}^d. \end{aligned}$$

Given a hyperlogarithmic abelian relation $\Omega \in \mathbf{HLogAR}^w$ and an irreducible component D as above, the iterates

$$(\text{Res}_D)^{\circ k}(\Omega) \in \mathbf{HLogAR}^{w-k}, \quad k = 0, 1, \dots, w-1$$

generate the smallest Res_D -stable subspace of $\mathbf{HLogAR}$ containing Ω : by definition, it is the “*space spanned by Ω by taking residues along D* ”. When taking the span of these spaces for all the irreducible components D of Z , we obtain the “*space obtained from Ω by residues*”, denoted by

$$\text{Res}(\Omega) = \left\langle (\text{Res}_D)^{\circ k}(\Omega) \mid k \geq 0, D \subset Z \text{ irred} \right\rangle.$$

For hyperlogarithmic ARs, the operations to get new ARs for a given one by monodromy or by “residues” are related, at least “locally”. Indeed, let y_D be a generic point of an irreducible component D of Z . We fix a small transversal to D at y_D , namely a holomorphic embedding $T_D : \Delta \rightarrow X$ of the unit disk $\Delta \subset \mathbb{C}$ such that $T_D(\Delta)$ is transverse to D at $T_D(0) = y_D$. We fix $z \in \Delta^*$, we set $y = T_D(z)$ and we consider the loop $\gamma_D : [0, 1] \rightarrow \Delta, t \mapsto ze^{2i\pi t}$ whose homotopy class spans $\pi_1(\Delta^*, z) \simeq \mathbb{Z}$. Then $\tilde{\gamma}_D = T_D \circ \gamma_D$ spans the “*local fundamental group along D (at y_D)*” and the monodromy map $\mathcal{M}_{[\tilde{\gamma}_D]}$ (cf. (2.7)) induces an endomorphism of $\mathbf{F}^w \mathbf{HLogAR} = \bigoplus_{w' \leq w} \mathbf{HLogAR}^{w'}$ for any weight $w \geq 1$, that we call the “*local monodromy operator around D at y_D* ” and denote by $\mathcal{M}_{D, y_D}^{\text{loc}}$.

Considering the monodromy of hyperlogarithms which is well-known (see (2.4)) and in particular unipotent, one obtains first that the “*local variation around D at y_D* ”, which by definition is the linear operator

$$V_{[\tilde{\gamma}_D]} = \mathcal{M}_{[\tilde{\gamma}_D]} - \text{Id},$$

⁽¹⁰⁾ A priori, the image of an element of $\mathbf{H}$ by Res_D is a holomorphic function on D minus a finite subset of points.

sends $\mathbf{HLogAR}^w$ into $\mathbf{HLogAR}^{w-1}$. A bit more work gives us that, as linear maps from $\mathbf{HLogAR}^w$ to $\mathbf{HLogAR}^{w-1}$, the following equality holds true:

$$V_{[\gamma_D]} = 2i\pi \operatorname{Res}_D.$$

For $\Omega \in \mathbf{HLogAR}^w$ and $y \in Y$ arbitrary, the space $\langle \pi_1(Y, y) \cdot \Omega \rangle$ spanned by the global monodromy of course contains those spanned by the “local monodromies” around all the irreducible components of Z .⁽¹¹⁾ It follows that

$$\operatorname{Res}(\Omega) \subset \operatorname{Monod}(\Omega).$$

When the local monodromies $\mathcal{M}_D^{\text{loc}}$ around the irreducible divisors D of Z span the global monodromy in (2.7),⁽¹²⁾ constructing new ARs from a hyperlogarithmic abelian relation by residues is just an algebraic counterpart to the topological method by monodromy and in particular one has

$$\operatorname{Res}(\Omega) = \operatorname{Monod}(\Omega) \tag{2.10}$$

for any $\Omega \in \mathbf{HLogAR}$. The case when Y can be taken as the complement of a curve in $\mathbb{P}^2$ is interesting because it covers the cases of del Pezzo surfaces as we will see further (cf. (4.35)). Given a reduced algebraic curve $C \subset \mathbb{P}^2$ of degree d , there is a classical description of the set of generators of the fundamental group of the complement $\mathbb{P}^2 \setminus C$ due to Van Kampen. Let $L \subset \mathbb{P}^2$ be a line intersecting C transversally in d distinct points $q_1, \dots, q_d$. One sets $L^* = L \setminus C$, one chooses a base point $y \in L^*$ and for all $i = 1, \dots, d$, one takes a loop γ_i supported in L^* , starting from y and going close to q_i , making a small circle around this point in the direct order, and then returning to y by traveling along the same path as at the beginning but in the opposite direction. Moreover, one assumes that the supports of the loops γ_i do not meet, except at y . Then the homotopy classes of the γ_i ’s generate $\pi_1(L^*, y)$ hence generate $\pi_1(\mathbb{P}^2 \setminus C, y)$ up to the natural embedding of the former fundamental group into the latter. It follows that, in this situation, equality (2.10) holds true.

An interesting and specific case is when $\mathcal{W}$ has maximal rank with a peculiar abelian relation $\mathcal{A}$ such that

$$\operatorname{Res}(\mathcal{A}) = \operatorname{Monod}(\mathcal{A}) = \mathbf{AR}(\mathcal{W}).$$

In such a case, $\mathcal{A}$ is essentially uniquely defined and has to be considered as “the” most fundamental abelian relation of the considered web $\mathcal{W}$.

⁽¹¹⁾ These claims are not fully correct as currently stated. We let the reader make this precise and fully rigorous.

⁽¹²⁾ Details must be added to make this sentence valid. This is left to the reader.

2.3. Del Pezzo surfaces and their webs by conics

We first recall some well-known material about del Pezzo surfaces then we discuss the webs induced by the fibrations in conics on them. We end by remarking that the Weyl group naturally acting on the Picard lattice of such a surface acts also on the space of (symbolic) hyperlogarithms we have to work with in order to handle the hyperlogarithmic ARs of the del Pezzo webs. We will be fairly succinct in this subsection, referring to Chapter 4 of Manin's book [36] and to Chapter 8 of Dolgachev's book [21] for the basics of the theory of del Pezzo surfaces.

2.3.1. Del Pezzo surfaces

Here we denote by r an integer in $\{3, 4, 5, 6, 7, 8\}$ and d stands for $9 - r \in \{1, 2, 3, 4, 5, 6\}$. A del Pezzo surface is the total space of the blow-up $b : \text{dP}_d \rightarrow \mathbb{P}^2$ of the projective plane at r points $p_1, \dots, p_r$ in general position. To indicate the number of points in the blow up, we use the notation X_r as well. The anticanonical sheaf $-K_{X_r}$ is ample (even very ample if $r \leq 6$) with anticanonical degree $(-K_r)^2$ equal to d .

The Picard lattice $\mathbf{Pic}(X_r)$ is freely spanned over $\mathbb{Z}$ by the class $\mathbf{h}$ of the pull-back under b of a general line in $\mathbb{P}^2$ together with the classes ℓ_i of the exceptional divisor $\ell_i = b^{-1}(p_i)$ for $i = 1, \dots, r$: one has

$$\mathbf{Pic}(X_r) = \mathbb{Z}\mathbf{h} \oplus \bigoplus_{i=1}^r \mathbb{Z}\ell_i \simeq \mathbb{Z}^{r+1}.$$

Moreover, the intersection pairing $(\cdot, \cdot) : \mathbf{Pic}(X_r)^2 \rightarrow \mathbb{Z}$ is non degenerate with signature $(1, r)$. The canonical class of X_r is given by

$$K_r = K_{X_r} = -3\mathbf{h} + \sum_{i=1}^r \ell_i$$

and is such that $K_r^2 = d$. Its orthogonal

$$K_r^\perp = \{ \boldsymbol{\rho} \in \mathbf{Pic}(X_r) \mid (\boldsymbol{\rho} \cdot K_r) = 0 \}$$

is free of rank r and spanned by the r classes

$$\begin{aligned} \boldsymbol{\rho}_i &= \ell_i - \ell_{i+1} \quad (i = 1, \dots, r-1) \\ \text{and} \quad \boldsymbol{\rho}_r &= \mathbf{h} - \ell_1 - \ell_2 - \ell_3. \end{aligned} \tag{2.11}$$

Each of the class $\boldsymbol{\rho}_i$ is such that $(\boldsymbol{\rho}_i, K_r) = 0$ and $\boldsymbol{\rho}_i^2 = -2$. Together with the positive definite symmetric form $-(\cdot, \cdot)|_{K_r^\perp}$, the $\boldsymbol{\rho}_i$'s define a root

system of type E_r , with the convention that $E_4 = A_4$ and $E_5 = D_5$. The corresponding set of roots is

$$\mathcal{R}_r = \{\boldsymbol{\rho} \mid (\boldsymbol{\rho}, K_r) = 0 \text{ and } \boldsymbol{\rho}^2 = -2\}$$

which is contained in the associated root space

$$R_r = K_r^\perp \otimes_{\mathbb{Z}} \mathbb{R} = \bigoplus_{i=1}^r \mathbb{R} \boldsymbol{\rho}_i.$$

Endowed with (the restriction of) $-(\cdot, \cdot)$, the latter is a Euclidean vector space. For each root $\boldsymbol{\rho}$, the map

$$\begin{aligned} s_{\boldsymbol{\rho}} : R_r &\longrightarrow R_r, \\ d &\longmapsto d + (d, \boldsymbol{\rho}) \boldsymbol{\rho} \end{aligned} \tag{2.12}$$

is a hyperplane reflection of R_r , admitting $\boldsymbol{\rho}$ as (-1) -eigenvector hence with invariant hyperplane $\boldsymbol{\rho}^\perp \subset R_r$. The subgroup of the group of orthogonal transformations of R_r spanned by the $s_i = s_{\boldsymbol{\rho}_i}$ for $i = 1, \dots, r$ is a Coxeter group (that is it is finite). It is the so-called “Weyl group of type E_r ” and will be denoted by $W(E_r)$ or shortly by W_r .

In the two cases we are interested in this paper, namely when $r = 4, 5$, there are isomorphisms

$$W_4 = W(A_4) \simeq \mathfrak{S}_5 \quad \text{and} \quad W_5 = W(D_5) \simeq (\mathbb{Z}/2\mathbb{Z})^4 \ltimes \mathfrak{S}_5$$

which can be described explicitly in terms of the generators s_i of the two Weyl groups (this is left to the reader).

2.3.2. Lines and conics

We now discuss several notions, objects and facts related to the lines and to the conics contained in del Pezzo surfaces.

By definition, a “line” is a class $\ell \in \mathbf{Pic}(X_r)$ such that $(K_r, \ell) = \ell^2 = -1$. This terminology makes sense for the following reason: from $\ell^2 = -1$, one deduces first that the associated linear system $|\ell|$ is a singleton which is a smooth rational curve embedded in X_r , denoted by ℓ (ie. by the same but unbolded notation). Second, when $-K$ is very ample (that is when $r \leq 6$), the condition $(K_r, \ell) = -1$ means that $\varphi_{|-K|}(\ell)$ is a projective line in the target projective space $|-K_r|^\vee \simeq \mathbb{P}^d$.

The set $\mathcal{L}_r$ of lines included in X_r is finite and its elements all can be explicitly given. In the two cases we will consider in this paper, these sets are

$$\begin{aligned} \mathcal{L}_4 &= \{\ell_i, \mathbf{h} - \ell_j - \ell_k\} \\ \text{and} \quad \mathcal{L}_5 &= \{\ell_i, \mathbf{h} - \ell_j - \ell_k, 2\mathbf{h} - \ell_{tot}\}, \end{aligned}$$

where for $r = 4, 5$, the indices i, j, k are such that $i = 1, \dots, r$ and $1 \leq j < k \leq r$ and where we use the notation $\ell_{tot} = \sum_{i=1}^r \ell_i$. It follows that one has $|\mathcal{L}_4| = 10$ and $|\mathcal{L}_5| = 16$.

A “conic class” on X_r is an element $\mathbf{c} \in \mathbf{Pic}(X_r)$ such that $(-K_r, \mathbf{c}) = 2$ and $\mathbf{c}^2 = 0$. From the second condition, one deduces that for any such class, the linear system $|\mathbf{c}|$ has dimension 1 and gives rise to a fibration $\varphi_{\mathbf{c}} : X_r \rightarrow |\mathbf{c}|^{\vee} \simeq \mathbb{P}^1$ whose fibers are conics (i.e. rational curves with anticanonical degree 2). The description of conic classes is classical and not difficult: for instance, see Table 2 in [12] for the case $r = 8$ (that is, the case of del Pezzo surface of degree 1), from which the corresponding descriptions for any r can be deduced easily. In the two cases $r = 4$ and $r = 5$ of interest in this paper, we recall that $\ell_{tot} = \sum_{i=1}^r \ell_i$ and setting $\ell_i = \ell_{tot} - \ell_i = \sum_{j \neq i} \ell_j$ for $i = 1, \dots, r$, one has:

$$\begin{aligned} \mathcal{K}_4 &= \{\mathbf{h} - \ell_i, 2\mathbf{h} - \ell_{tot} \mid i = 1, \dots, 4\} \\ \text{and } \mathcal{K}_5 &= \{\mathbf{h} - \ell_i, 2\mathbf{h} - \ell_i \mid i = 1, \dots, 5\}. \end{aligned} \tag{2.13}$$

The Weyl group W_r acts transitively on the set of lines $\mathcal{L}_r$ which gives rise to a structure of W_r -module on $\mathbf{R}^{\mathcal{L}_r}$ for any ring $\mathbf{R}$ (such as the ring of integers $\mathbb{Z}$ or the field $\mathbb{C}$). Since the action of each generator s_i of W_r on the set of lines $\mathcal{L}_r$ can be described explicitly, determining the decomposition of $\mathbb{C}^{\mathcal{L}_r}$ into irreducible W_r -modules can be done by straightforward computations within the theory of characters of W_r (see Section 3.2 and the Appendix in our preprint [46]):

PROPOSITION 2.1. — *For $r = 4, \dots, 8$, one has the following decompositions of $\mathbb{C}^{\mathcal{L}_r}$ into irreducible W_r -modules:*⁽¹³⁾

$$\begin{aligned} \mathbb{C}^{\mathcal{L}_4} &= \mathbf{1} \oplus V_{[41]}^4 \oplus V_{[32]}^5 \\ \mathbb{C}^{\mathcal{L}_5} &= \mathbf{1} \oplus V_{[4,1]}^5 \oplus V_{[3,2]}^{10} \\ \mathbb{C}^{\mathcal{L}_6} &= \mathbf{1} \oplus V^{6,1} \oplus V^{20,2} \\ \mathbb{C}^{\mathcal{L}_7} &= \mathbf{1} \oplus V^{7,1} \oplus V^{21,3} \oplus V^{27,2} \\ \mathbb{C}^{\mathcal{L}_8} &= \mathbf{1} \oplus V^{8,1} \oplus V^{35,2} \oplus V^{84,4} \oplus V^{112,3}. \end{aligned} \tag{2.14}$$

By its very definition, the set of lines $\mathcal{L}_r$ is included in the Picard group of X_r hence there is a natural $\mathbb{Z}$ -linear map

$$\mathbb{Z}^{\mathcal{L}_r} \longrightarrow \mathbf{Pic}(X_r) \tag{2.15}$$

which is easily seen to be W_r -equivariant and surjective. Because W_r acts by permutations on $\mathcal{L}_r$, it lets invariant the total sum $\sum_{\ell \in \mathcal{L}_r} \ell$ hence it

⁽¹³⁾ See [46, Remark 3.3] for some explanations about the notations used for the W_r -irreducibles.

follows that the trivial W_r -representation $\mathbf{1}$ is a subrepresentation of both spaces in (2.15). Because $K^\perp$ is a subrepresentation of $\mathbf{Pic}(X_r)$ which is the fundamental reflection representation $V^{r,1}$ of W_r (with $V^{4,1} = V_{[41]}^4$ and $V^{5,1} = V_{[4,1]}^5$ for $r = 4, 5$ respectively), we get that for $r = 4, \dots, 8$, as a W_r -module, one has

$$\mathbf{Pic}(X_r) = \mathbf{1} \oplus V^{r,1}.$$

For any r , one denotes by $\mathbf{K}_r$ the kernel of (2.15) in the category of W_r -representations.

COROLLARY 2.2. — *As W_r -representations, one has*

$$\begin{aligned} \mathbf{K}_4 &= V_{[32]}^5 & \mathbf{K}_5 &= V_{[3,2]}^{10} \\ \mathbf{K}_6 &= V^{20,2} & \mathbf{K}_7 &= V^{21,3} \oplus V^{27,2} \\ \mathbf{K}_8 &= V^{35,2} \oplus V^{84,4} \oplus V^{112,3}. \end{aligned} \quad (2.16)$$

(We will give a natural geometric interpretation of the W_r -module $\mathbf{K}_r$ below in Section 2.3.3).

For each conic fibration $\varphi_{\mathbf{c}} : X_r \rightarrow \mathbb{P}^1$, we denote by $\Sigma_{\mathbf{c}}$ its “*spectrum*” which by definition is the subset formed by the points σ of the target projective line such that the preimage $\varphi_{\mathbf{c}}^{-1}(\sigma)$ is a non irreducible conic, namely the sum of two lines of X_r : one has

$$\Sigma_{\mathbf{c}} = \{\sigma \in \mathbb{P}^1 \mid \exists \ell_{\mathbf{c}}, \ell'_{\mathbf{c}} \in \mathcal{L}_r \text{ such that } \varphi_{\mathbf{c}}^{-1}(\sigma) = \ell_{\mathbf{c}} + \ell'_{\mathbf{c}}\}.$$

It is known that $\Sigma_{\mathbf{c}}$ admits exactly $r - 1$ elements, which can also be characterized as the singular values of $\varphi_{\mathbf{c}}$. One verifies easily that any tangency point between two distinct conic fibrations on X_r necessarily lies on a line which is invariant by both fibrations. In particular, if one denotes by $U_r = X_r \setminus L_r$ the complement of the union of lines $L_r = \bigcup_{\ell \in \mathcal{L}_r} \ell \subset X_r$, then

$$\forall \mathbf{c}, \mathbf{c}' \in \mathcal{K}_r : \mathbf{c} \neq \mathbf{c}' \implies d\varphi_{\mathbf{c}} \wedge d\varphi_{\mathbf{c}'} \text{ does not vanish on } U_r. \quad (2.17)$$

The objects and quantities just considered above regarding lines and conic classes are given in explicit form in the following table:

r	3	4	5	6	7	8
E_r	$A_2 \times A_1$	A_4	D_5	E_6	E_7	E_8
$W_r = W(E_r)$	$\mathfrak{S}_3 \times \mathfrak{S}_2$	$\mathfrak{S}_5$	$(\mathbb{Z}/2\mathbb{Z})^4 \ltimes \mathfrak{S}_5$	$W(E_6)$	$W(E_7)$	$W(E_8)$
$\omega_r = W_r $	12	5!	$2^4 \cdot 5!$	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
$l_r = \mathcal{L}_r $	6	10	16	27	56	240
$\kappa_r = \mathcal{K}_r $	3	5	10	27	126	2160

2.3.3. Some other properties of del Pezzo surfaces

We briefly discuss some properties of del Pezzo surfaces which we will use at some points further on.

Let $X_r = \text{dP}_d$ be a del Pezzo surface as above (with $d = 9 - r$). By combining Riemann–Roch theorem and Kodaira vanishing, it follows that for any ample divisor A on X_r , one has $h^0(\mathcal{O}_{X_r}(A)) = \chi(\mathcal{O}_{X_r}(A)) = (1/2)A^2 - (A, K) + 1$. Since $-K$ is ample, one gets (for any $m \in \mathbb{N}^*$):

$$h^0(\mathcal{O}_{X_r}(-mK)) = \frac{1}{2}(m^2 + m)K^2 + 1 = \frac{1}{2}(m^2 + m)d + 1.$$

Let us now discuss some differential 1-forms with logarithmic poles along lines on X_r . First, remark that there is a short exact sequence of sheaves

$$0 \longrightarrow \Omega_{X_r}^1 \longrightarrow \Omega_{X_r}^1(\text{Log } L_r) \xrightarrow{\text{Res}} \bigoplus_{\ell \in \mathcal{L}_r} \mathcal{O}_\ell \longrightarrow 0$$

where for any line ℓ , the structure sheaf $\mathcal{O}_\ell$ is identified with its push-forward $i_*^\ell(\mathcal{O}_\ell)$ on X_r by the inclusion map $i^\ell : \ell \hookrightarrow X_r$. The first terms of the associated long exact sequence of cohomology groups are

$$0 \longrightarrow \mathbf{H}^0(X, \Omega_{X_r}^1) \longrightarrow \mathbf{H}^0(X_r, \Omega_{X_r}^1(\text{Log } L_r)) \xrightarrow{\text{Res}} \mathbb{C}^{\mathcal{L}_r} \longrightarrow \mathbf{H}^1(X_r, \Omega_{X_r}^1) \quad (2.18)$$

where the map Res is the direct sum of the Res_ℓ for $\ell \in \mathcal{L}_r$, each map Res_ℓ consisting in taking the residue along the line $\ell \subset X_r$.

Since X_r is rational, the space $\mathbf{H}^0(X_r, \Omega_{X_r}^1)$ is trivial hence the space of logarithmic 1-forms

$$\mathbf{H}_{X_r} = \mathbf{H}^0(X_r, \Omega_{X_r}^1(\text{Log } L_r))$$

identifies with the kernel of the map $\mathbb{C}^{\mathcal{L}_r} \rightarrow \mathbf{H}^1(X_r, \Omega_{X_r}^1)$. Because of the identifications $\mathbf{H}^1(X_r, \Omega_{X_r}^1) \simeq \mathbf{H}^{1,1}(X_r, \mathbb{C}) = \mathbf{H}^2(X_r, \mathbb{C}) = \mathbf{Pic}(X_r)_{\mathbb{C}} = \mathbf{Pic}(X_r) \otimes_{\mathbb{Z}} \mathbb{C}$, this map can be interpreted as the complexification of the surjective cycle map $\mathbb{Z}^{\mathcal{L}_r} \rightarrow \mathbf{Pic}(X_r)$, $\ell \mapsto \ell$ considered above in (2.15). Hence $\mathbf{H}_{X_r}$ identifies naturally with the kernel $\mathbf{K}_r$ discussed above. We deduce the following corollary for any $r = 4, \dots, 8$:

COROLLARY 2.3. — *By means of the map Res in (2.18), the space of logarithmic forms $\mathbf{H}_{X_r}$ is naturally a subrepresentation of $\mathbb{C}^{\mathcal{L}_r}$. And as such, $\mathbf{H}_{X_r}$ is isomorphic to the W_r -module $\mathbf{K}_r$ in (2.16).*

As far we are aware of, the description of $\mathbf{H}_{X_r}$ as a W_r -module and its decomposition in irreducibles does not appear in the existing literature, except in the case when $r = 4$ which is the subject of [22, Lem. 2.1].

Remark 2.4. —

- (1) The group $\text{Aut}(X_r)$ of automorphisms of X_r naturally acts on $\mathbf{H}_{X_r}$ as well. One verifies that this action is compatible with the W_r -action defined just above up to the natural monomorphism of groups $\text{Aut}(X_r) \hookrightarrow W_r$.

When $r = 4$, one has $X_4 = \text{dP}_5 \simeq \overline{\mathcal{M}}_{0,5}$ with $\text{Aut}(X_4) = \text{Aut}(\overline{\mathcal{M}}_{0,5}) = \text{Aut}(\mathcal{M}_{0,5}) = \mathfrak{S}_5$ and in this case $\text{Aut}(X_4) \hookrightarrow W_4$ happens to be an isomorphism. This does not longer holds true for $r > 4$. For instance when $r = 5$, one has $\text{Aut}(X_5) \simeq (\mathbb{Z}/2\mathbb{Z})^4$ for a generic del Pezzo quartic surface $X_5 = \text{dP}_4$ (cf. [21, §8.6.4]) and up to identifying W_5 with $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathfrak{S}_5$, the morphism $\text{Aut}(X_5) \hookrightarrow W_5$ corresponds to the natural inclusion $(\mathbb{Z}/2\mathbb{Z})^4 \hookrightarrow (\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathfrak{S}_5$ of the first factor.

- (2) Because the map $\mathbb{Z}^{\mathcal{L}_r} \rightarrow \mathbf{Pic}(X)$ is defined over $\mathbb{Z}$, (2.18) can be lifted to a short exact sequence of W_r -representations $0 \rightarrow \mathbf{H}_{X_r}^{\mathbb{Z}} \rightarrow \mathbb{Z}^{\mathcal{L}} \rightarrow \mathbf{Pic}(X) \rightarrow 0$ defined over the ring of integers, where $\mathbf{H}_{X_r}^{\mathbb{Z}}$ stands for the free $\mathbb{Z}$ -module of elements in $\mathbf{H}_{X_r} = \mathbf{H}^0(X_r, \Omega_{X_r}^1(\text{Log } L_r))$ whose residues along the lines in X_r all are integers. We will not use this arithmetical fact in what follows.

2.3.4. The del Pezzo web $\mathcal{W}_{\text{dP}_d}$

Here we just give the general definition of the webs $\mathcal{W}_{\text{dP}_d}$ before focusing on the two cases we are interested in this paper, namely $\mathcal{W}_{\text{dP}_5}$ and $\mathcal{W}_{\text{dP}_4}$ that we make explicit in some affine coordinates.

By definition, the “*del Pezzo web*” of the considered del Pezzo surface $X_r = \text{dP}_d$, denoted by $\mathcal{W}_{X_r}$ or $\mathcal{W}_{\text{dP}_d}$, is the web formed by all the pencils of conics on the surface: one has

$$\mathcal{W}_{\text{dP}_d} = \mathcal{W}_{X_r} = \mathcal{W}(\varphi_{\mathbf{c}} \mid \mathbf{c} \in \mathcal{K}_r).$$

From (2.17), it follows that the web $\mathcal{W}_{\text{dP}_d}$ is regular on the complement $U_r \subset X_r$ of the union of all lines contained in the considered del Pezzo surface. In the case when $r = 4$, U_4 corresponds to $\mathcal{M}_{0,5}$ up to the natural identification $X_4 = \text{dP}_5 \simeq \overline{\mathcal{M}}_{0,5}$.

In the two cases $r = 4$ and $r = 5$ of interest in this paper, the sets $\mathcal{K}_r$ are described explicitly in (2.13) which allows to give simple birational models

for the two del Pezzo webs $\mathcal{W}_{\mathrm{dP}_5}$ and $\mathcal{W}_{\mathrm{dP}_4}$. Let $p_1, \dots, p_4$ and p_5 be the points (in general position) in $\mathbb{P}^2$ such that X_r is the total space of the blow-up $b = b_r : X_r \rightarrow \mathbb{P}^2$. We assume that the first four points p_i are the vertices of the fundamental tetrahedron in $\mathbb{P}^2$ (that is $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, $p_3 = [0 : 0 : 1]$, $p_4 = [1 : 1 : 1]$ and $p_5 = [a : b : 1]$ with affine coordinate $(a, b) \in \mathbb{C}^2$ such that $ab(a-1)(b-1)(a-b) \neq 0$). Then there is no difficulty to see that the following holds true:

PROPOSITION 2.5. —

- (1) *The direct image of $\mathcal{W}_{\mathrm{dP}_5}$ by b is the 5-web on $\mathbb{P}^2$ formed by the four pencils of lines with vertices $p_1, \dots, p_4$ plus the pencil of conics passing through these four points. Analytically, one has*

$$b_*(\mathcal{W}_{\mathrm{dP}_5}) = \mathcal{W}\left(x, y, \frac{x}{y}, \frac{1-y}{1-x}, \frac{x(1-y)}{y(1-x)}\right).$$

- (2) *The direct image of $\mathcal{W}_{\mathrm{dP}_4}$ by b is the 10-web on $\mathbb{P}^2$ formed by the five pencils of lines with vertices $p_1, \dots, p_5$ plus the five pencils of conics passing through four points among these five.*

Analytically, one has $b_(\mathcal{W}_{\mathrm{dP}_4}) = \mathcal{W}(U_1, \dots, U_{10})$ where the U_i 's are the following ten rational functions (with P standing for the polynomial $(1-b)x - (1-a)y - (a-b)$):*

$$\begin{array}{lll} U_1 = x & U_2 = \frac{1}{y} & U_3 = \frac{y}{x} \\ U_4 = \frac{x-y}{x-1} & U_5 = \frac{b(a-x)}{ay-bx} & U_6 = \frac{P}{(x-1)(y-b)} \\ U_7 = \frac{(x-y)(y-b)}{yP} & U_8 = \frac{xP}{(x-y)(x-a)} & U_9 = \frac{y(x-a)}{x(y-b)} \\ U_{10} = \frac{x(y-1)}{y(x-1)}. \end{array}$$

2.3.5. The Weyl group action on the hyperlogarithmic ARs of a del Pezzo' web

Since the Weyl group $W_4 \simeq \mathfrak{S}_5$ identifies with the group of automorphisms of $U_4 \simeq \mathcal{M}_{0,5}$, it is not surprising (and it has been known for a long time) that it acts linearly on the space of abelian relations of $\mathcal{W}_{\mathrm{dP}_5}$. Since the inclusion $\mathrm{Aut}(U_r) \hookrightarrow W_r$ is strict when $r > 4$ (see Remark 2.4.1), that a similar fact happens for all the other del Pezzo webs is not obvious. This is what we discuss below.

Given a conic class $\mathbf{c} \in \mathcal{K}_r$, we denote by $\mathbf{c}^{red}$ the set of $r - 1$ non irreducible conics in the linear system $|\mathbf{c}| \subset \mathbb{P}\mathbf{H}^0(X_r, -2K_r)$. As a 1-cycle, each non irreducible conic $c \in \mathbf{c}^{red}$ can be written $c = l_c + l'_c$ for two distinct lines $l_c, l'_c \in \mathcal{L}_r$. Then we can define a natural injection $\mathbf{c}^{red} \hookrightarrow \mathbb{C}\mathcal{L}_r$ which in turn allows us to consider $\mathbb{C}\mathbf{c}^{red}$ as a subspace of $\mathbb{C}\mathcal{L}_r$. The stabilizer $W_{\mathbf{c}}$ of $\mathbf{c}$ in W_r acts by permutations on $\mathbf{c}^{red}$ from which one gets a $W_{\mathbf{c}}$ -action on $\mathbb{C}\mathbf{c}^{red}$. Considering the action of the latter group on $\mathcal{L}_r$ induced by restriction of the action of W_r , one obtains that the linear injection $\mathbb{C}\mathbf{c}^{red} \hookrightarrow \mathbb{C}\mathcal{L}_r$ is a morphism of $W_{\mathbf{c}}$ -representations. Since $W_{\mathbf{c}}$ acts by permutations on $\mathbf{c}^{red}$, the element $1_{\mathbf{c}} = \sum_{c \in \mathbf{c}^{red}} c \in \mathbb{C}\mathbf{c}^{red}$ is $W_{\mathbf{c}}$ -invariant and admits as $W_{\mathbf{c}}$ -invariant supplementary the subspace $\mathcal{U}_{\mathbf{c}}$ spanned over $\mathbb{C}$ by the elements $c - c' \in \mathbb{C}\mathbf{c}^{red}$ for all $c, c' \in \mathbf{c}^{red}$. We denote by $1_{\mathbf{c}} = \langle 1_{\mathbf{c}} \rangle$ the trivial representation spanned by $1_{\mathbf{c}}$.

LEMMA 2.6. —

- (1) As a group, $W_{\mathbf{c}}$ is isomorphic to $W(D_{r-1})$.
- (2) The decomposition $\mathbb{C}\mathbf{c}^{red} = 1_{\mathbf{c}} \oplus \mathcal{U}_{\mathbf{c}}$ actually is a direct sum of irreducible $W_{\mathbf{c}}$ -modules.
- (3) As $W_{\mathbf{c}}$ -representations, one has $\mathcal{U}_{\mathbf{c}} \simeq V_{[(r-2)1]}$ for $r = 4, 6$ and $\mathcal{U}_{\mathbf{c}} \simeq V_{[2,2]+}$ for $r = 5$.⁽¹⁴⁾

Proof. — Since W_r acts transitively on $\mathcal{K}_r$, it suffices to prove the lemma when $\mathbf{c} = \mathbf{h} - \mathbf{l}_1$. In this case, one verifies that $W_{\mathbf{c}_1}$ is isomorphic to the subgroup of W_r generated by the s_i 's for $i = 2, \dots, r$. Thus $W_{\mathbf{c}_1} \simeq W(D_{r-1})$ (with the convention that $D_3 = A_3$).

That $\mathbb{C}\mathbf{c}^{red} = 1_{\mathbf{c}} \oplus \mathcal{U}_{\mathbf{c}}$ is a decomposition into $W_{\mathbf{c}_1}$ -modules is clear. To determine $\mathcal{U}_{\mathbf{c}}$, we notice that the elements of $\mathbf{c}^{red}$ are the conics $c_j = [\mathbf{h} - \mathbf{l}_1 - \mathbf{l}_j] + [\mathbf{l}_j]$'s for $j = 2, \dots, r$, where each element between brackets is an element of $\mathcal{L}_r$ and the notation means that c_j has the two lines $\mathbf{h} - \mathbf{l}_1 - \mathbf{l}_j$ and $\mathbf{l}_j$ (viewed here as curves contained in X_r) as irreducible components. Direct computations give that as permutations of $\mathbf{c}^{red} = \{c_2, \dots, c_r\}$: s_j is the transposition exchanging c_j and c_{j+1} for $j = 2, \dots, r - 1$ and s_r is the transposition exchanging c_2 and c_3 . By elementary character theory of $W(D_{r-1})$ (see the Appendix), one gets (3). $\square$

A point which will be relevant for our purpose, is that there is a nice geometric interpretation of $\mathcal{U}_{\mathbf{c}}$. We denote by $\varphi_{\mathbf{c}} : X_r \rightarrow |\mathbf{c}| \simeq \mathbb{P}^1$ the

⁽¹⁴⁾ We recall that the irreducible representations of $W(D_n)$ are encoded by bipartitions of n . This is a 1-1 correspondence when n is odd. When n is even, given a partition η of $n/2$, there are two non isomorphic $W(D_n)$ -irreps associated to the bipartition (η, η) which are denoted by $[\eta, \eta]^+$ and $[\eta, \eta]^-$. For more details, see [26, §5.6].

fibration in conics associated to $\mathbf{c}$. It is known that the “*spectrum*” $\Sigma_{\mathbf{c}} = \{\sigma \in \mathbb{P}^1 \mid \varphi_{\mathbf{c}}^{-1}(\sigma) = \ell_{\mathbf{c}} + \ell'_{\mathbf{c}} \text{ with } \ell_{\mathbf{c}}, \ell'_{\mathbf{c}} \in \mathcal{L}_r\}$ of $\varphi_{\mathbf{c}}$ has cardinality $r - 1$ from which it follows that the space of logarithmic holomorphic 1-forms

$$\mathcal{H}_{\mathbf{c}} = \mathbf{H}^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(\text{Log } \Sigma_{\mathbf{c}}))$$

has dimension $r - 2$. If z stands for an affine coordinate on $\mathbb{P}^1$ with respect to which ∞ belongs to $\Sigma_{\mathbf{c}}$, then denoting by $\sigma_{\mathbf{c}}^1, \dots, \sigma_{\mathbf{c}}^{r-2}$ the elements of $\Sigma_{\mathbf{c}} \cap \mathbb{C}$, one has that the logarithmic 1-forms $dz/(z - \sigma_{\mathbf{c}}^i)$, $i = 1, \dots, r - 2$, form a basis of $\mathcal{H}_{\mathbf{c}}$. Since $\varphi_{\mathbf{c}}$ is a morphism, it follows that

$$\mathbf{H}_{\mathbf{c}} = \varphi_{\mathbf{c}}^*(\mathcal{H}_{\mathbf{c}}) = \left\langle \frac{d\varphi_{\mathbf{c}}}{\varphi_{\mathbf{c}} - \sigma_{\mathbf{c}}^i} \right\rangle_{i=1}^{r-2} \simeq \mathbb{C}^{r-2}$$

is naturally a subspace of $\mathbf{H}_{X_r} = \mathbf{H}^0(X_r, \Omega_{X_r}^1(\text{Log } L_r))$ hence of $\mathbb{C}^{\mathcal{L}_r}$ (as it follows from the injectivity of the map Res in (2.18)). For any $\mathbf{c}$, we denote the linear inclusion of $\mathbf{H}_{\mathbf{c}}$ into $\mathbb{C}^{\mathcal{L}_r}$ by

$$\iota_{\mathbf{c}} : \mathbf{H}_{\mathbf{c}} \hookrightarrow \mathbb{C}^{\mathcal{L}_r}. \quad (2.19)$$

LEMMA 2.7. — *As subspaces of $\mathbb{C}^{\mathcal{L}_r}$, $\mathbf{H}_{\mathbf{c}}$ and $\mathcal{U}_{\mathbf{c}}$ coincide. In particular, $\mathbf{H}_{\mathbf{c}}$ is $W_{\mathbf{c}}$ -stable and is an irreducible representation.*

Proof. — That $\mathbf{H}_{\mathbf{c}}$ is included in $\mathcal{U}_{\mathbf{c}}$ (when both spaces are seen as subspaces of $\mathbb{C}^{\mathcal{L}_r}$) immediately follows from the residues theorem applied to the 1-forms $dz/(z - \sigma_{\mathbf{c}}^i)$ which form a basis of $\mathcal{H}_{\mathbf{c}}$. One concludes by noticing that both $\mathbf{H}_{\mathbf{c}}$ and $\mathcal{U}_{\mathbf{c}}$ have dimension $r - 2$. $\square$

From this lemma together with what has been said above about $\mathcal{U}_{\mathbf{c}}$ in Lemma 2.6, we get that $\mathbf{H}_{\mathbf{c}}$ naturally is an irreducible representation of $W_{\mathbf{c}}$. For any w in this group, we denote by $w \cdot \omega$ the action of w on any $\omega \in \mathbf{H}_{\mathbf{c}}$ when this space is viewed as a $W_{\mathbf{c}}$ -representation. We will use the same notation, with a dot $\cdot$, for the action of the whole Weyl group W_r on $\mathbf{H}_{X_r}$ or $\mathbb{C}^{\mathcal{L}_r}$. Thanks to Lemma 2.7, all these notations are compatible.

From now on we fix a base conic class in $\mathcal{K}$ that we denote by $\mathbf{c}_0$. Since W_r acts transitively on $\mathcal{K}$, for each element $\mathbf{c}$ in it, there exists $w_{\mathbf{c}_0}^{\mathbf{c}}$ (with $w_{\mathbf{c}_0}^{\mathbf{c}_0} = 1$) such that $w_{\mathbf{c}_0}^{\mathbf{c}} \cdot \mathbf{c}_0 = \mathbf{c}$. Since $W_r/W_{\mathbf{c}_0} \simeq \mathcal{K}_r$, it follows that

$$W_r = \bigsqcup_{\mathbf{c} \in \mathcal{K}} w_{\mathbf{c}_0}^{\mathbf{c}} W_{\mathbf{c}_0}. \quad (2.20)$$

Remark that for any $\mathbf{c} \in \mathcal{K}_r$, one has $w_{\mathbf{c}_0}^{\mathbf{c}} W_{\mathbf{c}_0} = \{w \in W_r \mid w \cdot \mathbf{c}_0 = \mathbf{c}\}$.

Given a fixed weight $\nu > 0$, the inclusions (2.19) actually land in $\mathbf{H}_{X_r}$ hence give rise to inclusions between the ν -th tensorial powers $\mathbf{H}_{\mathbf{c}}^{\otimes \nu} \hookrightarrow \mathbf{H}_{X_r}^{\otimes \nu}$, again denoted by $\iota_{\mathbf{c}}$. For any $w \in W_r$ and any $\eta \in \mathbf{H}_{\mathbf{c}}^{\otimes \nu}$, letting w act on

$\iota_{\mathbf{c}}(\eta)$ gives an element of $\text{Im}(\iota_{w\mathbf{c}}) \subset \mathbf{H}_{X_r}^{\otimes \nu}$ hence one can set

$$w \bullet \eta = \iota_{w\mathbf{c}}^{-1}(w \cdot \iota_{\mathbf{c}}(\eta)) \in \mathbf{H}_{w\mathbf{c}}^{\otimes \nu}, \quad (2.21)$$

a definition which gives rise to a map

$$\bullet : W_r \times \left(\bigoplus_{\mathbf{c} \in \mathcal{K}} \mathbf{H}_{\mathbf{c}}^{\otimes \nu} \right) \longrightarrow \bigoplus_{\mathbf{c} \in \mathcal{K}} \mathbf{H}_{\mathbf{c}}^{\otimes \nu}. \quad (2.22)$$

From now on, we no longer write the inclusions $\iota_{\mathbf{c}}$ to simplify formulas. For any $\mathbf{c} \in \mathcal{K}_r$ we have $w_{\mathbf{c}_0}^{\mathbf{c}} \mathbf{H}_{\mathbf{c}_0}^{\otimes \nu} = \mathbf{H}_{\mathbf{c}}^{\otimes \nu}$ and for any $w \in W$, setting $\mathbf{c}' = w\mathbf{c}$, we have $(ww_{\mathbf{c}_0}^{\mathbf{c}})\mathbf{c}_0 = \mathbf{c}'$ hence $ww_{\mathbf{c}_0}^{\mathbf{c}} \in w_{\mathbf{c}_0}^{\mathbf{c}'} W_{\mathbf{c}_0}$ thus there exists $w' \in W_{\mathbf{c}_0}$ such that $ww_{\mathbf{c}_0}^{\mathbf{c}} = w_{\mathbf{c}_0}^{\mathbf{c}'} w'$ in W_r . From this, one gets that given an element $\eta_{\mathbf{c}}$ of $\mathbf{H}_{\mathbf{c}}^{\otimes \nu}$, first one has $\eta_{\mathbf{c}} = w_{\mathbf{c}_0}^{\mathbf{c}} \eta_{\mathbf{c}_0}$ for some $\eta_{\mathbf{c}_0} \in \mathbf{H}_{\mathbf{c}_0}^{\otimes \nu}$ hence

$$w \bullet \eta_{\mathbf{c}} = w \cdot w_{\mathbf{c}_0}^{\mathbf{c}} \eta_{\mathbf{c}_0} = (ww_{\mathbf{c}_0}^{\mathbf{c}}) \eta_{\mathbf{c}_0} = (w_{\mathbf{c}_0}^{\mathbf{c}'} w') \eta_{\mathbf{c}_0} = w_{\mathbf{c}_0}^{\mathbf{c}'} (w' \eta_{\mathbf{c}_0}).$$

As explained in [25, §3.3], this defines a W_r -action on $\bigoplus_{\mathbf{c} \in \mathcal{K}} \mathbf{H}_{\mathbf{c}}^{\otimes \nu}$, which is isomorphic to the one induced by the $W_{\mathbf{c}_0}$ -action on $\mathbf{H}_{\mathbf{c}_0}^{\otimes \nu}$. This gives us the following

LEMMA 2.8. —

- (1) *The map $\bullet$ defined in (2.22) makes of the direct sum $\bigoplus_{\mathbf{c} \in \mathcal{K}} \mathbf{H}_{\mathbf{c}}^{\otimes \nu}$ a W_r -representation isomorphic to the induced representation $\text{Ind}_{W_{\mathbf{c}_0}}^{W_r}(\mathbf{H}_{\mathbf{c}_0}^{\otimes \nu})$.*
- (2) *The linear map*

$$\begin{aligned} \Phi_r^{\nu} = \Phi_{X_r}^{\nu} : \bigoplus_{\mathbf{c} \in \mathcal{K}} \mathbf{H}_{\mathbf{c}}^{\otimes \nu} &\longrightarrow \mathbf{H}_{X_r}^{\otimes \nu} \\ (\eta_{\mathbf{c}})_{\mathbf{c} \in \mathcal{K}} &\longmapsto \sum_{\mathbf{c} \in \mathcal{K}} \eta_{\mathbf{c}} \end{aligned} \quad (2.23)$$

is W_r -equivariant therefore requiring that

$$0 \longrightarrow \mathbf{ARHLog}^{\nu} \longrightarrow \bigoplus_{\mathbf{c} \in \mathcal{K}} \mathbf{H}_{\mathbf{c}}^{\otimes \nu} \xrightarrow{\Phi_r^{\nu}} \mathbf{H}_{X_r}^{\otimes \nu} \quad (2.24)$$

be an exact sequence of W_r -representations naturally defines a W_r -action on the kernel $\mathbf{ARHLog}^{\nu}$ which identifies with the space of (symbolic) weight ν hyperlogarithmic abelian relations of the web $\mathcal{W}_{\text{dP}_d}$.

For any $\nu \geq 1$, let h_r^{ν} stand for the dimension of the image of $\Phi_r^{\nu} : \bigoplus_{\mathbf{c} \in \mathcal{K}} \mathbf{H}_{\mathbf{c}}^{\otimes \nu} \rightarrow \mathbf{H}_{X_r}^{\otimes \nu}$. Obviously, $\mathbf{ARHLog}^{\nu}$ is of dimension $\kappa_r \cdot (r-2)^{\nu} - h_r^{\nu}$. Thanks to the preceding lemma, one can easily compute the dimension h_r^{ν} in the simplest cases: because Φ_r^{ν} is W_r -equivariant and non trivial (since

any $\mathbf{H}_{\mathbf{c}}^{\otimes \nu} \rightarrow \mathbf{H}_{X_r}^{\otimes \nu}$ is an embedding hence non trivial), its image is a non zero W_r -submodule of $\mathbf{H}_{X_r}^{\otimes \nu}$. If the latter tensor product is irreducible as a W_r -module then necessarily $\nu = 1$ and Φ_r^1 is surjective. This implies that $h_r^1 = \dim_{\mathbb{C}} \mathbf{H}_{X_r} = l_r - r - 1$ (with $l_r = |\mathcal{L}_r|$). Then from Corollary 2.3 and (2.16) we deduce the

LEMMA 2.9. — *For $r = 4, 5, 6$, one has*

$$\mathrm{HLrk}_r^1 = \dim_{\mathbb{C}}(\mathbf{ARHLog}^1) = \kappa_r \cdot (r - 2) - l_r + r + 1.$$

The explicit values of $l_r = |\mathcal{L}_r|$, $\kappa_r = |\mathcal{K}_r|$ and HLrk_r^1 for $r = 4, 5, 6$ are given in the table below.

r	4	5	6
l_r	10	16	27
κ_r	5	10	27
HLrk_r^1	5	20	88

2.3.6. Some remarks about the space of hyperlogarithmic ARs of del Pezzo webs

Let $\mathcal{W}$ be a web as in Section 2.2 and in Section 2.2.5 whose notation we use here. One assumes that this web carries many hyperlogarithmic abelian relations, so that its spaces $\mathbf{HLogAR}^w$ of symbolic hyperlogarithmic ARs are not all trivial. From the preceding subsections, it comes that the following facts hold true (for any $y \in Y$):

- (1) each weighted piece $\mathbf{HLogAR}^w$ is naturally a $\mathfrak{S}_w$ -module;
- (2) the filtered space $\mathbf{HLogAR}^{\leq \bullet}$ naturally is a (unipotent) $\pi_1(Y, y)$ -module;
- (3) there is a well-defined residue map

$$\mathrm{Res}_D : \mathbf{HLogAR}^w \longrightarrow \mathbf{HLogAR}^{w-1}$$

for any irreducible component $D \subset Z$.

The case of a del Pezzo web $\mathcal{W}_{\mathrm{dPd}}$ (with $d = 9 - r$) is particularly interesting since in addition to the above, the following holds true as well:

- (4) the Weyl group $W(E_r)$ acts on each space $\mathbf{HLogAR}^w$.

In [12], we have proved that any del Pezzo web $\mathcal{W}_{\text{dP}_d}$ for $d = 1, \dots, 6$ carries a hyperlogarithmic abelian relation $\mathbf{HLog}^{7-d}$ of weight $7 - d$. Considering the above four points, it follows that this web is likely to carry many other hyperlogarithmic ARs, possibly of smaller weights. In other words, it is natural to expect that the spaces $\mathbf{HLogAR}^w$ for $w \leq 7 - d$ are 'rather big'. Having spaces of a priori high dimension compatible with all the algebraic structures listed in the fourth points above is not at all trivial and makes of these spaces interesting objects of study, at least for us. We hope to come back to this in a future work.

2.4. Cluster algebras and cluster webs

In this subsection, we introduce and discuss some elements of the theory of cluster algebras and some interesting webs one can derive from some of them. Our treatment below is very succinct, we refer to Section 4.7 below for more specific details and to our long memoir [45] for a more panoramic perspective.

We start with some basic definitions of the theory of cluster algebras. Let n be a fixed positive integer. A (*labeled*) *seed* of rank n is a triple $\mathcal{S} = (\mathbf{A}, \mathbf{x}, B)$ where $\mathbf{A} = (a_1, \dots, a_n)$ is a n -tuple of indeterminates, $\mathbf{x} = (x_1, \dots, x_n)$ is another one (independent of the former) and $B = (b_{ij})_{i,j=1}^n$ is a skew-symmetrizable $n \times n$ integer matrix, called the *exchange matrix* of the seed. The associated (*labeled*) $\mathcal{A}$ -seed (resp. $\mathcal{X}$ -seed) is the pair $(\mathbf{A}, B)$ (resp. $(\mathbf{x}, B)$). The n -tuples $\mathbf{A}$ and $\mathbf{x}$ are called " $\mathcal{A}$ – (resp. $\mathcal{X}$ –) clusters" and their elements a_i and x_i " $\mathcal{A}$ – (resp. $\mathcal{X}$ –) cluster variables" respectively. Note that all these objects are a priori associated to the considered seed.

By definition, for $k \in \{1, \dots, n\}$, the (*cluster*) *mutation* in the k -th direction of the seed $\mathcal{S}$ is the new seed

$$\mathcal{S}' = (\mathbf{A}', \mathbf{x}', B') = \left(\mu_k(\mathbf{A}, B), \mu_k(\mathbf{x}, B), \mu_k(B) \right) = \mu_k(\mathcal{S})$$

with $B' = (b'_{ij})_{i,j=1}^n = \mu_k(B)$ being the *matrix mutation* of B in direction k , whose coefficients b'_{ij} are given by the following formulas:

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } k \in \{i, j\}; \\ b_{ij} & \text{if } k \notin \{i, j\} \text{ and } b_{ik}b_{kj} \leq 0; \\ b_{ij} + |b_{ik}|b_{kj} & \text{if } k \notin \{i, j\} \text{ and } b_{ik}b_{kj} > 0, \end{cases}$$

and where the new clusters $\mathbf{A}' = (a'_1, \dots, a'_n) = \mu_k(\mathbf{A}, B)$ and $\mathbf{x}' = (x'_1, \dots, x'_n) = \mu_k(\mathbf{x}, B)$ are defined by the following formulas for the corresponding cluster variables a'_j 's and x'_j 's:

$$\begin{aligned} a'_j &= \begin{cases} a_j & \text{if } j \neq k \\ a'_k = \frac{1}{a_k} \left(\prod_{b_{\ell k} > 0} a_\ell^{b_{\ell k}} + \prod_{b_{\ell k} < 0} a_\ell^{-b_{\ell k}} \right) & \text{if } j = k \end{cases} \\ x'_j &= \begin{cases} x_j^{-1} & \text{if } j = k \\ x'_j = x_j \left(1 + x_k^{\max(0, -b_{kj})} \right)^{-b_{kj}} & \text{if } j \neq k. \end{cases} \end{aligned} \quad (2.25)$$

Taking $\mathbf{S} = (\mathbf{A}, \mathbf{x}, B)$ as initial seed, by considering all the seeds obtained by successive mutations from it, one can construct the associated “*cluster exchange pattern*”, which is the countable family of seeds $\mathbf{S}_t = (\mathbf{A}^t, \mathbf{x}^t, B^t)$ indexed by the vertices t of the n -regular tree $\mathbb{T}^n$, with the initial seed $\mathbf{S} = \mathbf{S}_{t_0}$ associated to the root t_0 of this tree, the edges of $\mathbb{T}^n$ being labeled by elements of $\{1, \dots, n\}$ in the natural consistent way: two vertices are linked as follows $t \xrightarrow{k} t'$ in $\mathbb{T}^n$ if and only if the two corresponding seeds satisfy $\mathbf{S}_{t'} = \mu_k(\mathbf{S}_t)$ (or equivalently $\mathbf{S}_t = \mu_k(\mathbf{S}_{t'})$).

For $\mathcal{Z}$ standing for $\mathcal{A}$ or $\mathcal{X}$ and accordingly $\mathbf{z} = (z_i)_{i=1}^s$ standing for $\mathbf{A}$ or $\mathbf{x}$, the “ *$\mathcal{Z}$ -cluster algebra $\mathbf{Z}_{\mathbf{S}}$* ” with initial seed $\mathbf{S}$ is the subalgebra of rational functions in the initial cluster coordinates z_i 's spanned by all the $\mathcal{Z}$ -cluster variables of seeds obtained from $\mathbf{S}$ by a finite sequence of successive mutations: one has $\mathbf{Z}_{\mathbf{S}} = \langle z_i^t \mid t \in \mathbb{T}^n, i = 1, \dots, n \rangle \subset \mathbb{Q}(\mathbf{z}) = \mathbb{Q}(z_1, \dots, z_n)$.

To each vertex t of $\mathbb{T}^n$, one sets $\mathbf{ZT}^t = \text{Spec}(\mathbb{Q}[z_1^t, \dots, z_n^t, 1/z_1^t, \dots, 1/z_n^t])$ for $\mathcal{Z} \in \{\mathcal{A}, \mathcal{X}\}$ and one considers the monomial map $p^t : \mathbf{AT}^t \rightarrow \mathbf{XT}^t$ characterized by the relations $(p^t)^*(x_i^t) = \prod_{j=1}^n (a_i^t)^{b_{ij}^t}$ for $i = 1, \dots, n$. The cluster tori $\mathbf{ZT}^t$ can be glued together by means of the corresponding (sequences of) mutations and give rise to the so-called *$\mathcal{Z}$ -cluster variety*:

$$\mathbf{Z}_{\mathbf{S}} = \left(\bigsqcup_{t \in \mathbb{T}^n} \mathbf{ZT}^t \right) / \mathbf{Z}\text{-mut} \quad (2.26)$$

which has the structure of a scheme (not necessarily of finite type nor separated⁽¹⁵⁾).

The monomial maps p^t commute with mutations hence can be seen as the restrictions of a global well-defined cluster map

$$p : \mathbf{AS} \longrightarrow \mathbf{XS}$$

⁽¹⁵⁾ The $\mathcal{A}$ -cluster variety is always separated but in general not the associated $\mathcal{X}$ -cluster variety.

between the two corresponding cluster varieties. The triple $(\mathcal{A}_{\mathcal{S}}, \mathcal{X}_{\mathcal{S}}, p)$ is the “*cluster ensemble*” associated to the initial seed $\mathcal{S}$. To simplify the writing and since it still makes sense (despite not being fully rigorous), one often replaces the subscript $\mathcal{S}$ by B in the notations, and even sometimes just removes it.

Actually, there are more structures associated to the cluster ensemble $(\mathcal{A}_B, \mathcal{X}_B, p)$ since, as explained in [24, §2]:

- (1) the map $p : \mathcal{A}_B \rightarrow \mathcal{X}_B$ corresponds to the quotient map under the $H_{\mathcal{A}}$ -action of a certain algebraic torus $H_{\mathcal{A}}$ acting on the $\mathcal{A}$ -cluster variety;
- (2) there is an “exact sequence” of cluster varieties/maps

$$\mathcal{A}_B \xrightarrow{p} \mathcal{X}_B \xrightarrow{\lambda} H_{\mathcal{X}} \longrightarrow 1 \quad (2.27)$$

where $H_{\mathcal{X}}$ is a torus and λ is a map with a monomial expression in each $\mathcal{X}$ -cluster torus $\mathcal{X}\mathbf{T}^t$. That (2.27) be “exact” means that one has $\mathcal{U}_B = \text{Im}(p) = \lambda^{-1}(1)$ as subvarieties of $\mathcal{X}_B$. The subvariety $\mathcal{U}_B$ is called the “*secondary cluster variety*” by some authors and has been proved to be relevant regarding web geometry and the theory of functional of polylogarithms (eg. see [45, Thm. 0.7]).

We will consider the cluster variables as rational functions on the initial cluster tori. They enjoy several remarkable properties (separation formula, sign-coherence, positivity, Laurent phenomenon for the $\mathcal{A}$ -cluster variables) that we will not review here. Given a finite set Σ of cluster variables, one can consider the “*cluster web*” $\mathcal{W}_{\Sigma}$ formed by the foliations admitting the elements of Σ as first integrals. A nice feature of the theory of cluster algebras is that it comes with several ways to get finite sets Σ of cluster variables giving rise to webs carrying polylogarithmic ARs. The first way is given by the so called cluster algebras of “*finite type*”, which are by definition those admitting only a finite number of clusters. An early fundamental result of Fomin and Zelevinsky shows that the classification of such algebras is parallel to that of Dynkin diagrams. Thus given a Dynkin diagram Δ of rank $n \geq 2$, the set Σ_{Δ} of all the $\mathcal{X}$ -cluster variables of the corresponding cluster algebra is finite, which allows us to define the “*cluster web of Dynkin type Δ* ” as the web

$$\mathcal{X}\mathcal{W}_{\Delta} = \mathcal{W}_{\Sigma_{\Delta}}.$$

It is a (generalized) web of codimension 1 in n -variables.

More elaborated constructions of cluster webs can be considered, such as the “*secondary cluster web*” of Dynkin type Δ , which by definition is the

trace of $\mathcal{X}\mathcal{W}_\Delta$ along the secondary cluster manifold $\mathcal{U} \subset \mathcal{X}$:

$$\mathcal{U}\mathcal{W}_\Delta = (\mathcal{X}\mathcal{W}_\Delta)|_{\mathcal{U}}.$$

Note that $\mathcal{U}\mathcal{W}_\Delta$ is distinct from $\mathcal{X}\mathcal{W}_\Delta$ if and only if $\mathcal{U}$ is a proper subvariety of $\mathcal{X}$, that is if and only if the initial exchange matrix B has rank strictly less than n , in which case one has $\dim(\mathcal{U}) = \text{rk}(B)$. The relevance of the notion of cluster web is illustrated by several webs associated to classical functional identities satisfied by some polylogarithms (such as Spence–Kummer identity of the trilogarithm, or Kummer’s one of the tetralogarithm) which can be proved to be webs of cluster type (see [45, §5.2]). But for the purpose of this paper, we need to go further and define a more general notion of cluster web.

Let Σ be a finite set of $\mathcal{X}$ -cluster variables defining a cluster web $\mathcal{X}\mathcal{W}_\Sigma$. Assume that the map p in (2.27) has rank $m < n$, or equivalently $H_{\mathcal{X}}$ has positive dimension $n - m$. Since λ is dominant, it has rank $n - m$ hence for any $\tau \in H_{\mathcal{X}}$, the fiber $\mathcal{X}_{B,\tau} = \lambda^{-1}(\tau)$ is a m -dimensional subvariety of $\mathcal{X}_B$. One defines a more general kind of cluster web by considering the trace of $\mathcal{X}\mathcal{W}_\Sigma$ along $\mathcal{X}_{B,\tau}$: one sets

$$\mathcal{X}\mathcal{W}_{\Sigma,\tau} = (\mathcal{X}\mathcal{W}_\Sigma)|_{\lambda=\tau}.$$

In Section 4.7, we will prove that any del Pezzo’s web $\mathcal{W}_{\text{dP}_4}$ is a cluster web of the form just above.

*

In the next two sections, we make as explicit as possible the general material of the current section for the two del Pezzo webs $\mathcal{W}_{\text{dP}_5}$ and $\mathcal{W}_{\text{dP}_4}$.

3. Bol’s web

There is no moduli for quintic del Pezzo surface: for dP_5 , we take the total space of the blow-up $b : \text{dP}_5 \rightarrow \mathbb{P}^2$ at the four points $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, $p_3 = [0 : 0 : 1]$ and $p_4 = [1 : 1 : 1]$. Instead of dealing with $\mathcal{W}_{\text{dP}_5}$ on dP_5 , we are going to work on $\mathbb{P}^2$ with the push-forward web $b_*(\mathcal{W}_{\text{dP}_5})$, which is entirely equivalent (cf. Proposition 2.5). This push-forward is classically known as Bol’s web, denoted by $\mathcal{B}$ which, in the coordinates x, y corresponding to the affine embedding $\mathbb{C}^2 \rightarrow \mathbb{P}^2$, $(x, y) \mapsto [x : y : 1]$, is the web given by the arguments of Rogers’ dilogarithm in ($\mathcal{A}b$):

$$\mathcal{B} = \mathcal{W}\left(x, y, \frac{x}{y}, \frac{y-1}{x-1}, \frac{x(y-1)}{y(x-1)}\right).$$

Recall that L_4 stands for the union of the ten lines contained in $X_4 = \mathrm{dP}_5$. Bol's web is regular on $U = b(\mathrm{dP}_5 \setminus L_4) \subset \mathbb{P}^2$ which coincides with the complement in $\mathbb{C}^2$ of the affine arrangement of five lines $\mathcal{A} \subset \mathbb{C}^2$ cut out by

$$xy(x-1)(y-1)(x-y) = 0.$$

The map $(x, y) \mapsto [0, 1, x, y, \infty]$, where the latter expression stands for the projective equivalence class of the configuration of 5 points $(0, 1, x, y, \infty)$ on the projective line, induces an affine biholomorphism $\varphi : U \xrightarrow{\sim} \mathcal{M}_{0,5}$ which extends to an isomorphism $\mathrm{dP}_5 \simeq \overline{\mathcal{M}}_{0,5}$. The web $\varphi_*(\mathcal{B})$ on $\mathcal{M}_{0,5}$ is the 5-web denoted by $\mathcal{W}_{\mathcal{M}_{0,5}}$ whose first integrals are the five forgetful maps $\mathcal{M}_{0,5} \rightarrow \mathcal{M}_{0,4}$.

*

Our goal below is to establish all the properties of this web stated in Section 1.1. Some of them are classical and well-known, we will be quite succinct about them.

3.1. Non linearizability

For completeness, we recall the standard argument for proving the non linearizability of Bol's web (see [42, §6.1] for more details): in the coordinates x, y we are working with, Bol's web is seen to be formed by four pencils of lines plus a pencil of conics. A classical result says that any (local) linearization of a linear planar 4-web is necessarily induced by a projective transformation of $\mathbb{P}^2$. Since such a map cannot send any smooth conic onto a line, there is no chance that it could linearize the pencil of conics of $\mathcal{B}$ which is therefore non linearizable (and consequently non algebraizable).

3.2. Cluster web of type A_2

To deal with the ARs of $\mathcal{B}$, it is convenient to perform a change of coordinates in order to get its cluster model. As an unordered 5-web, Bol's web is obviously equivalent to

$$\mathcal{W}\left(\frac{1}{x-1}, \frac{y-x}{x-1}, y-1, \frac{y(x-1)}{y-x}, \frac{x}{y-x}\right)$$

and using the birational change of variables $\Psi : (u_1, u_2) \mapsto (x, y)$ defined by

$$x = \frac{1+u_1}{u_1} \quad \text{and} \quad y = \frac{1+u_1+u_2}{u_1}$$

(cf. the “zig-zag map” of [45, §6.3.2]), one obtains that $\mathcal{B}$ is equivalent to the web defined by the following five rational first integrals u_1 , u_2 , $(1 + u_2)/u_1$, $(1 + u_1 + u_2)/(u_1 u_2)$ and $(1 + u_1)/u_2$. This web is the $\mathcal{X}$ -cluster web of type A_2 (see [45, §3.2.1], especially Figure 10 there): one has

$$\Psi^*(\mathcal{B}) = \mathcal{W}\left(u_1, u_2, \frac{1 + u_2}{u_1}, \frac{1 + u_1 + u_2}{u_1 u_2}, \frac{1 + u_1}{u_2}\right) = \mathcal{XW}_{A_2}. \quad (3.1)$$

3.3. The abelian relations of $\mathcal{XW}_{A_2}$

Dealing with the cluster model of Bol’s web above is quite convenient to write down its ARs in a nice form. The presentation below is essentially taken from [45, §5.1.1].

3.3.1.

Denoting by X_ℓ for $\ell = 1, \dots, 5$ the first integrals of $\mathcal{XW}_{A_2}$ given in (3.1), that is

$$\begin{aligned} X_1 &= u_1, & X_2 &= u_2, & X_3 &= \frac{1 + u_2}{u_1}, \\ X_4 &= \frac{1 + u_1 + u_2}{u_1 u_2}, & X_5 &= \frac{1 + u_1}{u_2} \end{aligned} \quad (3.2)$$

and setting $X_0 = X_5 = (1 + u_1)/u_2$ as well as $X_6 = X_1 = u_1$, one verifies that the identity

$$X_{\ell-1} X_{\ell+1} = 1 + X_\ell$$

is satisfied for any $\ell = 1, \dots, 5$ (these are the identities defining the “*Y-system of type A_2* ”, see [45, §3.3.1.3]). For any such ℓ , taking the logarithm of the previous relation gives

$$\mathrm{Log}(X_{\ell-1}) - \mathrm{Log}(1 + X_\ell) + \mathrm{Log}(X_{\ell+1}) = 0, \quad (3.3)$$

a functional identity which corresponds to a logarithmic and combinatorial AR for $\mathcal{XW}_{A_2}$, that we will denote by ARLog_ℓ . The ℓ -th component of ARLog_ℓ is $-\mathrm{Log}(1 + X_\ell)$ for each ℓ . Since the $\mathrm{Log}(1 + X_\ell)$ ’s (for $\ell = 1, \dots, 5$) are linearly independent, it follows that the ARLog_ℓ ’s form a free family of elements of $\mathbf{AR}_{\log}(\mathcal{XW}_{A_2})$ which actually is a basis of this space. One has

$$\mathbf{AR}_{\log}(\mathcal{XW}_{A_2}) = \mathbf{AR}_C(\mathcal{XW}_{A_2}) = \left\langle \mathrm{ARLog}_i \right\rangle_{i=1}^5 \simeq \mathbb{C}^5.$$

The “*cluster dilogarithm R*” is the function defined by the integral formula

$$\mathrm{R}(x) = \frac{1}{2} \int_0^x \left(\frac{\mathrm{Log}(1 + v)}{v} - \frac{\mathrm{Log}(v)}{1 + v} \right) \mathrm{d}v$$

for any $x \geq 0$. It is well-known that it satisfies the functional identity

$$R(u_1) + R(u_2) + R\left(\frac{1+u_2}{u_1}\right) + R\left(\frac{1+u_1+u_2}{u_1 u_2}\right) + R\left(\frac{1+u_1}{u_2}\right) = \frac{\pi^2}{2} \quad (3.4)$$

for any $u_1, u_2 > 0$, which is nothing else but Abel's 5-term identity written in cluster form. One denotes by $\mathbf{Ab}$ the weight 2 polylogarithmic “cluster abelian relation” associated to (3.4).

One thus has

$$\mathbf{AR}(\mathcal{XW}_{A_2}) = \langle \mathbf{ARLog}_1, \dots, \mathbf{ARLog}_5 \rangle \oplus \langle \mathbf{Ab} \rangle, \quad (3.5)$$

from which one recovers the well-known fact (due to Bol) that $\mathcal{XW}_{A_2} \simeq \mathbf{B}$ has maximal rank, with all its ARs being polylogarithmic of weight 1 or 2.

Another nice feature of choosing the cluster variables X_ℓ as first integrals for the $\mathcal{X}$ -cluster web of type A_2 is that one can easily construct the dilogarithmic identity (3.4) from the five logarithmic abelian relations $\mathbf{LogAR}_\ell$'s. Indeed, considering the differential of $\mathbf{LogAR}_\ell$, it follows that

$$\frac{dX_\ell}{1+X_\ell} = \frac{dX_{\ell-1}}{X_{\ell-1}} + \frac{dX_{\ell+1}}{X_{\ell+1}}$$

for any ℓ . Then summing for ℓ in $\mathbb{Z}/5\mathbb{Z}$ and setting $\Theta = \sum_\ell \mathbf{LogAR}_\ell \frac{dX_\ell}{X_\ell}$, one gets

$$\begin{aligned} \Theta &= \sum_\ell \left(-\mathbf{Log}(X_{\ell-1}) + \mathbf{Log}(1+X_\ell) - \mathbf{Log}(X_{\ell+1}) \right) \frac{dX_\ell}{X_\ell} \\ &= \sum_\ell \mathbf{Log}(1+X_\ell) \frac{dX_\ell}{X_\ell} - \sum_\ell \mathbf{Log}(X_\ell) \left(\frac{dX_{\ell-1}}{X_{\ell-1}} + \frac{dX_{\ell+1}}{X_{\ell+1}} \right) \\ &= \sum_\ell \left(\frac{\mathbf{Log}(1+X_\ell)}{X_\ell} - \frac{\mathbf{Log}(X_\ell)}{1+X_\ell} \right) dX_\ell \\ &= \sum_\ell 2R'(X_\ell) dX_\ell = 2d \left(\sum_\ell R(X_\ell) \right). \end{aligned} \quad (3.6)$$

Since all the $\mathbf{LogAR}_\ell$'s vanish identically, the same holds true for the total derivative of $\sum_{\ell=1}^5 R(u_\ell)$ hence this sum is identically equal to a constant. We thus have recovered very symmetrically the A_2 -cluster dilogarithmic identity (3.4) from the logarithmic identities $\mathbf{LogAR}_\ell$'s.

Using (3.2) and the identity

$$2d \left(\sum_\ell R(X_\ell) \right) = - \sum_\ell \mathbf{LogAR}_\ell \cdot dX_\ell / X_\ell,$$

one can compute easily the residues of the abelian relation $\mathbf{Ab}$ associated to (3.4). Given P one of the F -polynomials $u_1, u_2, 1+u_1, 1+u_2$ and $1+u_1+u_2$, one denotes by $\text{Res}_P(\mathbf{Ab})$ the residue of $\mathbf{Ab}$ along the divisor cut out by $P = 0$ in $\mathbb{C}^2$. Then easy computations show that, up to non zero multiplicative constants, one has:

$$\begin{aligned}\text{Res}_{u_1}(\mathbf{Ab}) &= \text{LogAR}_1 - \text{LogAR}_3 - \text{LogAR}_4 \\ \text{Res}_{u_2}(\mathbf{Ab}) &= \text{LogAR}_2 - \text{LogAR}_4 - \text{LogAR}_5 \\ \text{Res}_{1+u_1}(\mathbf{Ab}) &= \text{LogAR}_5 \\ \text{Res}_{1+u_2}(\mathbf{Ab}) &= \text{LogAR}_3 \\ \text{and } \text{Res}_{1+u_1+u_2}(\mathbf{Ab}) &= \text{LogAR}_4.\end{aligned}$$

Clearly, the five residue abelian relations above form another basis of the space $\mathbf{AR}_{\log}(\mathcal{XW}_{A_2})$.

3.3.2. The symbolic abelian relations of $\mathcal{XW}_{A_2}$

It is interesting to discuss succinctly the structure of the space of symbolic ARs of $\mathcal{XW}_{A_2}$ (if only to prepare the reader for what is to come, since this is the approach we will use to describe the ARs of $\mathcal{W}_{\text{dP}_4}$ in the next section).

We set

$$\begin{aligned}F_1 &= u_1, & F_2 &= u_2, & F_3 &= 1 + u_1, \\ F_4 &= 1 + u_2, & \text{and } F_5 &= 1 + u_1 + u_2.\end{aligned}$$

Then the rational differential 1-forms

$$\kappa_\ell = \text{dLog}(F_\ell) = \text{d}F_\ell / F_\ell \tag{3.7}$$

for $\ell = 1, \dots, 5$ are such that their pull-backs under the blow-up map $b : X_4 = \text{dP}_5 \rightarrow \mathbb{P}^2$ form a basis of the space $\mathbf{H} = \mathbf{H}_{X_4}$ of rational 1-forms on dP_5 with logarithmic poles along the divisor of lines $L_4 \subset X_4$ and holomorphic on $U_4 = X_4 \setminus L_4$.

One sets $\Sigma = \{0, -1, \infty\} \subset \mathbb{P}^1$. The logarithmic 1-forms $\text{dLog}(z) = \text{d}z/z$ and $\text{dLog}(z+1) = \text{d}z/(z+1)$ form a basis of $\mathbf{H}^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\text{Log}(\Sigma)))$. Hence for any $\ell = 1, \dots, 5$, the two 1-forms

$$\nu_{\ell,1} = X_\ell^* \left(\frac{\text{d}z}{z} \right) = \frac{\text{d}X_\ell}{X_\ell} \quad \text{and} \quad \nu_{\ell,2} = X_\ell^* \left(\frac{\text{d}z}{z+1} \right) = \frac{\text{d}X_\ell}{X_\ell + 1}$$

form a basis of $\mathbf{H}_\ell = X_\ell^*(\mathbf{H}^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\text{Log}(\Sigma))))$ which is a subspace of the 5-dimensional vector space $\mathbf{H}$. Hence any $\nu_{\ell,s}$ ($s = 1, 2$) admits a unique

expression as a linear combination in the κ_i 's. These decompositions are the following (with $\nu_\ell = (\nu_{\ell,1}, \nu_{\ell,2})$ for any ℓ):

$$\begin{aligned}\nu_1 &= (\kappa_1, \kappa_3) \\ \nu_2 &= (\kappa_2, \kappa_4) \\ \nu_3 &= (\kappa_4 - \kappa_1, \kappa_5 - \kappa_1) \\ \nu_4 &= (\kappa_5 - \kappa_1 - \kappa_2, \kappa_3 + \kappa_4 - \kappa_1 - \kappa_2) \\ \text{and } \nu_5 &= (\kappa_3 - \kappa_2, \kappa_5 - \kappa_2).\end{aligned}\tag{3.8}$$

We set $\nu_0 = \nu_5$ and $\nu_6 = \nu_1$. For any $\ell = 1, \dots, 5$, the symbolic relation corresponding to the identity (3.3) is

$$\nu_{\ell-1,1} - \nu_{\ell,2} + \nu_{\ell+1,1} = 0\tag{3.9}$$

and the weight 2 antisymmetric symbolic relation corresponding to the functional identity (3.4) is

$$\sum_{\ell=1}^5 \nu_{\ell,1} \wedge \nu_{\ell,2} = 0,\tag{3.10}$$

an identity in $\wedge^2 \mathbf{H}$ which can be shown to be satisfied by an easy direct computation.

3.4. Birational symmetries and the Weyl group action

The two following maps

$$\begin{aligned}\Phi : (u_1, u_2) &\longmapsto \left(\frac{1+u_2}{u_1}, \frac{1+u_1+u_2}{u_1 u_2} \right) \\ \text{and } \Psi : (u_1, u_2) &\longmapsto \left(\frac{1+u_1}{u_2}, \frac{1+u_1+u_2}{u_1 u_2} \right),\end{aligned}$$

are birational symmetries of $\mathcal{XW}_{A_2}$ (which can be qualified as “cluster symmetries” since their components are cluster coordinates). The former map has order 5 whereas the second is an involution. Their pull-backs on a certain affine chart $\mathbb{C}^2$ of $\mathcal{M}_{0,5}$ under the map $(x, y) \mapsto [\infty, 0, 1, x, y] \mapsto (u_1, u_2) = (1/(x-1), (x-y)/(1-x))$ (cf. the map $\mathbf{U}_2$ in [45, §6.3.2]) correspond on $\mathcal{M}_{0,5}$ to the cyclic shift $\phi : [p_1, \dots, p_5] \mapsto [p_5, p_1, \dots, p_4]$ and to the dihedral reflection $\psi : [p_1, \dots, p_5] \mapsto [p_2, p_1, p_5, p_4, p_3]$ respectively. It follows that Φ and Ψ generate a group of cluster symmetries of $\mathcal{XW}_{A_2}$ isomorphic to the dihedral group of the pentagon (note that, instead of Ψ , one could have considered the simpler involution $\Gamma : (u_1, u_2) \mapsto (u_2, u_1)$ since one has $\langle \Phi, \Psi \rangle = \langle \Phi, \Gamma \rangle$ as groups of Cremona transformations).

The involution exchanging the fourth and the fifth points of any configuration in $\mathcal{M}_{0,5}$ is written $(x, y) \mapsto (y, x)$ in the affine coordinates x, y and as follows in the coordinates u_1, u_2 :

$$J : (u_1, u_2) \mapsto \left(\frac{u_1}{1 + u_2}, -\frac{u_2}{1 + u_2} \right).$$

The 5-cycle $(12 \cdots 5)$ together with the transposition (45) generate the whole permutation group $\mathfrak{S}_5$ which identifies naturally to $\text{Aut}(\mathcal{M}_{0,5})$. To this group corresponds, in the coordinates u_1, u_2 , the one generated by Φ and J which is the group of birational symmetries of $\mathcal{XW}_{A_2}$: one has natural identifications

$$\mathfrak{S}_5 \simeq \text{Aut}(\mathcal{M}_{0,5}) \simeq \langle \Phi, J \rangle.$$

It is straightforward to verify that the space $\mathbf{H}$ spanned by the κ_ℓ 's is stable by pull-backs under the maps Φ and J and that the two corresponding linear actions are characterized by the following relations:

$$\begin{aligned} \Phi^*(\kappa_1) &= \kappa_4 - \kappa_1 & J^*(\kappa_1) &= \kappa_1 - \kappa_4 \\ \Phi^*(\kappa_2) &= \kappa_5 - \kappa_1 - \kappa_2 & J^*(\kappa_2) &= \kappa_2 - \kappa_4 \\ \Phi^*(\kappa_3) &= \kappa_5 - \kappa_1 - \kappa_2 & J^*(\kappa_3) &= \kappa_5 - \kappa_4 \\ \Phi^*(\kappa_4) &= \kappa_3 + \kappa_4 - \kappa_1 - \kappa_2 & J^*(\kappa_4) &= -\kappa_4 \\ \Phi^*(\kappa_5) &= \kappa_4 + \kappa_5 - \kappa_1 - \kappa_2 & J^*(\kappa_5) &= \kappa_3 - \kappa_4. \end{aligned} \tag{3.11}$$

The space $\mathbf{H}$ is easily seen to be an irreducible $\mathfrak{S}_5$ -module of dimension 5. Moreover one immediately gets that the trace of the matrix of J^* in the basis $(\kappa_\ell)_{\ell=1}^5$ is 1. From the character table of $\mathfrak{S}_5$, one gets that, as a $\mathfrak{S}_5$ -representation, $\mathbf{H}$ is isomorphic to the one with Young symbol [32].

From the explicit formulas (3.8), (3.9), and (3.11), it is just a matter of computations in linear algebra to determine the action of Φ^* and J^* on the space $\mathbf{ARHLog}^1$ (isomorphic to $\mathbf{AR}_{\log}(\mathcal{XW}_{A_2})$) which is freely spanned as a complex vector space by the formal ARs corresponding to the algebraic relations (3.9) for $\ell = 1, \dots, 5$. We obtain that, as a $\mathfrak{S}_5$ -representation, $\mathbf{ARHLog}^1$ is irreducible and isomorphic to the irreducible representation with Young symbol [221]. Finally, from the formulas (3.11) above, one also gets $J^*(\nu_{1,1}) = -J^*(\nu_{3,1})$ and $J^*(\nu_{1,2}) = J^*(\nu_{3,2})$ hence $J^*(\nu_{1,1} \wedge \nu_{1,2}) = -\nu_{3,1} \wedge \nu_{3,2}$. It follows immediately from this that the weight 2 antisymmetric AR corresponding to (3.10) is a non-trivial 1-dimensional representation of $\mathfrak{S}_5 \simeq \langle \Phi, J \rangle$: we obtain that $\langle \mathbf{Ab} \rangle$ is isomorphic to the signature representation of $\mathfrak{S}_5$.

Remark 3.1. — The isomorphisms of $\mathfrak{S}_5$ -representations

$$\mathbf{ARHLog}^1 \simeq V_{[221]}^5 \quad \text{and} \quad \langle \mathbf{Ab} \rangle \simeq \mathbf{sign} = V_{[1^5]}^1$$

have been stated explicitly first by Damiano (see [17] or the introduction of [47]).

3.5. Canonical algebraizations of Bol's web

For planar webs with maximal rank, being exceptional is precisely the opposite of being algebraizable in the classical terminology. Hence one has to explain the meaning of the heading of this subsection.

Here we take the standpoint that Bol's web is (naturally equivalent to) the web $\mathcal{W}_{\mathcal{M}_{0,5}}$ which is defined by natural rational (hence algebraic) first integrals on the algebraic surface $\bar{\mathcal{M}}_{0,5} \simeq \text{dP}_5$. What we mean by the notion of a “*canonical algebraization of Bol's web*” is a canonical way, given a germ of 5-web $\mathcal{W}$ (at the origin of $\mathbb{C}^2$ say) to first recognize whether $\mathcal{W}$ is equivalent to $\mathcal{B}$, then, if it is indeed the case, to construct in a canonical way a local biholomorphism $\psi_{\mathcal{W}} : (\mathbb{C}^2, 0) \rightarrow \mathcal{M}_{0,5}$ such that the push-forward of $\mathcal{W}$ by $\psi_{\mathcal{W}}$ coincides with $\mathcal{W}_{\mathcal{M}_{0,5}} \simeq \mathcal{B}$ (possibly up to reindexing the foliations).

We describe below two constructions of such canonical maps. These two constructions are a priori distinct but both are canonical, hence both coincide for Bol's web. The first construction is the most direct and elementary but seems to be new. The second approach is more in the classical spirit of web geometry, still is new as well. It consists in constructing an algebraization map by means of (some of) the ARs of the web under consideration.

In the rest of this subsection, $\mathcal{W}$ stands for a fixed 5-web defined on an open domain $U \subset \mathbb{C}^2$.

3.5.1. Algebraization via the combinatorial abelian relations

Let $\mathbf{AR}_C(\mathcal{W})$ be the space of combinatorial ARs of $\mathcal{W}$ (on U) which by definition is the subspace of $\mathbf{AR}(\mathcal{W})$ spanned by the ARs carried by the 3-subwebs of $\mathcal{W}$. We define $\mathbf{ar}_C(\mathcal{W})$ as the subspace of $\Omega^1(U)$ spanned by the components of all the elements of $\mathbf{AR}_C(\mathcal{W})$. Note that $\mathbf{ar}_C(\mathcal{W})$ is invariantly attached to $\mathcal{W}$: for any biholomorphism $F : \tilde{U} \rightarrow U$, one has $F^*(\mathbf{AR}_C(\mathcal{W})) = \mathbf{AR}_C(F^*(\mathcal{W}))$ hence

$$F^*(\mathbf{ar}_C(\mathcal{W})) = \mathbf{ar}_C(F^*(\mathcal{W})). \quad (3.12)$$

We can call the dimension of $\mathbf{ar}_C(\mathcal{W})$ the “*combinatorial A-rank*” of $\mathcal{W}$. Remark that the dimension of $\mathbf{ar}_C(\mathcal{W})$ is positive if and only if $\mathbf{AR}_C(\mathcal{W})$ is non-trivial, which is often verified by the webs we consider in practice. In what follows, we set

$$\mathfrak{r} = \dim(\mathbf{ar}_C(\mathcal{W})).$$

LEMMA 3.2. — *If $\mathfrak{r}$ is positive then necessarily $\mathfrak{r} \geq 2$ and the evaluation map $\text{ev}_u : \mathbf{ar}_C(\mathcal{W}) \rightarrow \Omega_{U,u}^1 \simeq \mathbb{C}^2$ is a surjective linear map of $\mathbb{C}$ -vector spaces for any $u \in U$.*

Proof. — This follows immediately from the fact that the valuation of a non-trivial abelian relation with 3 terms is zero at any point. $\square$

We assume that $\mathfrak{r} > 0$ from now on. In this case, it follows from the preceding lemma that $\text{Ker}(\text{ev}_u)$ is a subspace of codimension 2 of $\mathbf{ar}_C(\mathcal{W})$ hence its annihilator, denoted by An_u , is a 2-plane in the dual space $\mathbf{ar}_C(\mathcal{W})^\vee$. Composing the map $u \mapsto An_u$ from U into the grassmannian $G_2(\mathbf{ar}_C(\mathcal{W})^\vee)$ with the Plücker embedding ρ of this grassmannian variety into the second wedge product of $\mathbf{ar}_C(\mathcal{W})^\vee$, we obtain a morphism

$$\begin{aligned} \Psi_{\mathcal{W}} : U &\longrightarrow \mathbb{P}\left(\wedge^2 \mathbf{ar}_C(\mathcal{W})^\vee\right) \simeq \mathbb{P}^{\binom{\mathfrak{r}}{2}-1} \\ u &\longmapsto \wedge^2 An_u. \end{aligned}$$

We let the reader verify that $\Psi_{\mathcal{W}}$ is invariantly attached to $\mathcal{W}$ as well in the sense that, for any biholomorphism F as above, one has

$$\Psi_{\mathcal{W}} \circ F = \Psi_{F^*(\mathcal{W})}$$

up to the natural identification between the target spaces induced by (3.12).

Given local coordinates x, y and a basis $(\eta_i)_{i=1}^{\mathfrak{r}}$ of $\mathbf{ar}_C(\mathcal{W})$, the map $\Psi_{\mathcal{W}}$ can be described explicitly as follows: for any $i = 1, \dots, \mathfrak{r}$, let $\eta_i = \eta_i^1 dx + \eta_i^2 dy$ be the decomposition of η_i in the basis (dx, dy) and let us set $\eta^s = (\eta_1^s, \dots, \eta_{\mathfrak{r}}^s)$ for $s = 1, 2$. Then up to the linear identification between $\mathbf{ar}_C(\mathcal{W})$ and its dual induced by the choice of the basis $(\eta_i)_{i=1}^{\mathfrak{r}}$, one has that $\eta^1 \wedge \eta^2 \in \wedge^2 \mathbb{C}^{\mathfrak{r}}$ identifies with $\wedge^2 An_u$ from which it follows that $\Psi_{\mathcal{W}}$ is the projectivization of the map with components the 2×2 minors of the $2 \times \mathfrak{r}$ matrix whose two lines are η^1 and η^2 : one has

$$\Psi_{\mathcal{W}} = \left[\eta_i^1 \eta_j^2 - \eta_i^2 \eta_j^1 \right]_{1 \leq i < j \leq \mathfrak{r}} : U \longrightarrow \mathbb{P}^{\binom{\mathfrak{r}}{2}-1}.$$

When $\Psi_{\mathcal{W}}$ is étale at some point, then $S_{\mathcal{W}} = \text{Im}(\Psi_{\mathcal{W}})$ is a (a priori analytic) surface geometrically attached to $\mathcal{W}$ (up to a projective automorphism of the target projective space) hence the push forward web $(\Psi_{\mathcal{W}})_*(\mathcal{W})$ is a canonical model of $\mathcal{W}$.

Remark 3.3. — Note that when $\mathcal{W}$ is a maximal rank 5-web, another canonical model of it has been previously considered, as the push-forward of $\mathcal{W}$ by the associated Poincaré–Blaschke map $P_{\mathcal{W}}$ (cf. [42, §5.2 and §5.3] for more details). When $\mathcal{W}$ is such that both push-forwards $(\Psi_{\mathcal{W}})_*(\mathcal{W})$ and $(P_{\mathcal{W}})_*(\mathcal{W})$ are well-defined, it would be interesting to compare these “two canonical models”. Since the combinatorial ARs are the less transcendent

ones among the ARs, one can expect the surface $S_{\mathcal{W}} = \text{Im}(\Psi_{\mathcal{W}})$ to be less transcendental/more algebraic (in a vague sense here but which it would be advisable to make more precise) than Poincaré–Blaschke surface $\Sigma_{\mathcal{W}} = \text{Im}(P_{\mathcal{W}})$.

*

We now specialize the material introduced above in the specific case of Bol’s web. For this aim, as a model for this web, it is more convenient to work with the web $\mathcal{W}_{\mathcal{M}_{0,5}}$ induced by the five forgetting maps on $\mathcal{M}_{0,5}$. For this web, one has

PROPOSITION 3.4. —

- (1) *The space $\text{ar}_C(\mathcal{W}_{\mathcal{M}_{0,5}})$ coincides with the space of holomorphic 1-forms on $\mathcal{M}_{0,5}$ with logarithmic poles along $\partial\mathcal{M}_{0,5}$, i.e. as subspaces of $\Omega^1(\mathcal{M}_{0,5})$, one has*

$$\text{ar}_C(\mathcal{W}_{\mathcal{M}_{0,5}}) = \mathbf{H}^0\left(\overline{\mathcal{M}}_{0,5}, \Omega_{\overline{\mathcal{M}}_{0,5}}^1\left(\text{Log}(\partial\mathcal{M}_{0,5})\right)\right).$$

In particular $r = 5$ and $\text{ar}_C(\mathcal{W}_{\mathcal{M}_{0,5}})$ is well-defined on the whole $\mathcal{M}_{0,5}$.

- (2) *The map $\Psi_{\mathcal{W}_{\mathcal{M}_{0,5}}}$ takes its values into a 5-dimensional linear subspace $\mathbb{P}\mathcal{W} \simeq \mathbb{P}^5$ of $\mathbb{P}(\wedge^2 \text{ar}_C(\mathcal{W}_{\mathcal{M}_{0,5}})^\vee) \simeq \mathbb{P}^9$. Moreover the corresponding map $\mathcal{M}_{0,5} \rightarrow \mathbb{P}\mathcal{W} \simeq \mathbb{P}^5$ is the restriction to $\mathcal{M}_{0,5}$ of the anticanonical embedding of $\overline{\mathcal{M}}_{0,5}$ into $\mathbb{P}^5$.*

Proof. — In the case of $\mathcal{X}\mathcal{W}_{A_2}$, let U be an open domain containing the positive real orthant $(\mathbb{R}_{>0})^2$. Then it follows from (3.5) that $\text{ar}_C(\mathcal{X}\mathcal{W}_{A_2})$ is the vector space spanned by the five 1-forms κ_ℓ ’s of (3.7) hence in particular one has $r = 5$. According to [45, Prop. 6.21], the affine rational map $\mathbb{C}^2 \dashrightarrow \mathcal{M}_{0,5}$, $(x, y) \mapsto [\infty, 0, 1, x, y]$ induces an affine isomorphism between $\mathbb{C}^2 \setminus A$ where A is the arrangement of lines cut out by $xy(x-1)(y-1)(x-y) = 0$ and the complement of the cluster arrangement cut out by $u_1u_2(1+u_1) \times (1+u_2)(1+u_1+u_2) = 0$ in the “cluster affine chart” $\mathbb{C}_{u_1, u_2}^2 = \text{Spec}(\mathbb{C}[u_1, u_2])$. What corresponds to $\text{ar}_C(\mathcal{X}\mathcal{W}_{A_2})$ in the coordinates x, y is the vector space with basis

$$\frac{dx}{x}, \quad \frac{dy}{y}, \quad \frac{dx}{x-1}, \quad \frac{dy}{y-1} \quad \text{and} \quad \frac{dx-dy}{x-y}. \quad (3.13)$$

If $b : \mathbf{Bl}_{p_1, \dots, p_5}(\mathbb{P}^2) \rightarrow \mathbb{P}^2$ stands the blow up at the points $p_1 = [1, 0, 0]$, $p_2 = [0, 1, 0]$, $p_3 = [0, 0, 1]$ and $p_4 = [1, 1, 1]$, then up to the identification $\mathbf{Bl}_{p_1, \dots, p_5}(\mathbb{P}^2) \simeq \overline{\mathcal{M}}_{0,5}$, one easily verifies that the pull-backs under b of the 1-forms (3.13) form a basis of the space of 1-forms on $\mathcal{M}_{0,5}$ with logarithmic poles along $\partial\overline{\mathcal{M}}_{0,5}$.

Once the first point has been established, the second follows from [22, §2.4]. $\square$

Because $\mathbf{ar}_C(\mathcal{W})$ and $\Psi_{\mathcal{W}}$ are attached to $\mathcal{W}$ in an invariant way (see above), the previous proposition gives us immediately the following result which provides an effective tool to verify whether a given 5-web is equivalent to Bol's web or not.

Recall that following C. Segre [52] (see also [42, §1.4.4]), any surface $S \subset \mathbb{P}^5$ with non-degenerate second-order osculation at a generic point either is included in the Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ or carries a 5-web $\mathcal{W}_S$ which is projectively attached to S . Moreover, in the last case (which is the generic one), given any étale map $\varphi : U \rightarrow \mathbb{C}^6$ inducing a local parametrization of S when projectivized, there is an explicit formula for a global symmetric form $W_\varphi \in \text{Sym}^5(T_U^*)$ defining the pull-back web $\varphi^*(\mathcal{W}_S)$ on U .

COROLLARY 3.5. — *A 5-web $\mathcal{W}$ on a domain $U \subset \mathbb{C}^2$ is equivalent to Bol's web if and only if all the three following statements hold true:*

- (1) *one has $\dim(\mathbf{ar}_C(\mathcal{W}_{\mathcal{M}_{0,5}})) = 5$;*
- (2) *the map $\Psi_{\mathcal{W}}$ has values in a linear subspace $\mathbb{P}W \subset \mathbb{P}^9$ of dimension 5 and $\text{Im}(\Psi_{\mathcal{W}}) = \Psi_{\mathcal{W}}(U)$ is included in a quintic del Pezzo surface anticanonically embedded in $\mathbb{P}W$;*
- (3) *the initial web $\mathcal{W}$ and the pull-back $\Psi_{\mathcal{W}}^*(\mathcal{W}_\Sigma)$ coincide.*

This result has to be compared to the following one:

PROPOSITION 3.6 (Bol's theorem [7]). — *Let $\mathcal{W}$ be a hexagonal d -web. Then either it is linearizable hence equivalent to d pencils of lines, or $d = 5$ and $\mathcal{W}$ is equivalent to Bol's web.*

Since a hexagonal web $\mathcal{W}$ has maximal rank, in order to verify if such a web is linearizable, it suffices to verify whether it is compatible with a projective connexion or not (see [6, §30] or Section 6.1.5 in the modern reference [42]) which is equivalent to the fact that the slopes of the foliations of $\mathcal{W}$ satisfy some differential algebraic identities (see [42, Prop. 6.1.10]). Because hexagonality can be verified by means of formal computations (of differential algebras), Bol's theorem above provides an effective way to verify if a given 5-web is equivalent to Bol's web or not. However, the criterion evoked is non constructive in the sense that, when $\mathcal{W}$ is equivalent to Bol's web, it does not indicate a natural (or even better, a canonical) way to construct a map φ from the definition domain of $\mathcal{W}$ to $\mathcal{M}_{0,5}$ such that $\mathcal{W} = \varphi^*(\mathcal{W}_{\mathcal{M}_{0,5}})$.

Our Corollary 3.5 for its part, is constructive and gives a canonical way of constructing such a map. However, it has its own disadvantage which is of relying on the explicit determination of the space $\mathbf{ar}_C(\mathcal{W})$ which requires to determine all the combinatorial ARs of the web under scrutiny. From a computational perspective, determining the abelian relation of a hexagonal 3-web is a bit more involved than just verifying that this 3-web is indeed flat since it requires one additional step which consists in solving a homogeneous linear ODE of the first order. Solving such a differential equation can be done by performing one integration but this is a supplementary task which does not occur when using the criterion for characterizing Bol's web described in the preceding paragraph.

The new material introduced above has also the interesting feature that it makes sense not only for hexagonal webs but more generally for webs with sufficiently many combinatorial ARs. Obviously, one has

$$\mathfrak{r} = \dim(\mathbf{ar}_C(\mathcal{W})) \leq \dim(\mathbf{AR}_C(\mathcal{W})) \leq \mathrm{rk}(\mathcal{W})$$

and there are several examples of webs with maximal rank for which $\mathfrak{r}$ is significantly lower than the rank. The map $\Psi_{\mathcal{W}}$ could turn out to be an interesting tool to study webs of maximal rank and with sufficiently many combinatorial ARs.

*

But it turns out that, given a 5-web $\mathcal{W}$, there exists a canonical and more direct effective criterion for first ascertaining whether it is equivalent to Bol's web and second, when it is the case, to build in a canonical way a local biholomorphism φ for which one has $\mathcal{W} = \varphi^*(\mathcal{W}_{\mathcal{M}_{0,5}})$. This is elementary but has not been remarked before and it is the purpose of the next subsection.

3.5.2. Algebraization via the canonical map $\Phi_{\mathcal{W}}$

We first introduce a general elementary construction (namely the “canonical map $\Phi_{\mathcal{W}}$ ” associated to a given web $\mathcal{W}$) which is the main ingredient which is considered below and will be used further on in Section 4.4.2 when dealing with $\mathcal{W}_{\mathrm{dP}_4}$.

Given a holomorphic submersion X on a domain of $\mathbb{C}^2$, its “*slope function* ζ_X ” is the meromorphic function defined by the formula

$$\zeta_X = \partial_{u_1}(X)/\partial_{u_2}(X),$$

with the convention that ζ_X is constant and identically equal to $\infty \in \mathbb{P}^1$ in the case when $\partial_{u_2}(X)$ vanishes identically.

Now let d be bigger than or equal to 3 and let $\mathcal{W}$ be a d -web defined on an open domain $\Omega \subset \mathbb{C}^2$, with no singularity. Then there exist holomorphic submersions $U_i : \Omega \rightarrow \mathbb{C}$ for $i = 1, \dots, d$, such that $\mathcal{W}$ is formed by the foliations defined by the U_i 's. Because $\mathcal{W}$ has been assumed to have no singular points, the slope function ζ_{U_i} can be seen as a holomorphic morphism $\zeta_{U_i} : \Omega \rightarrow \mathbb{P}^1$ for any i and the ζ_{U_i} 's take pairwise distinct values at any point of Ω . Hence the map

$$\Phi_{\mathcal{W}} := [\zeta_{U_i}]_{i=1}^d : \Omega \longrightarrow \mathcal{M}_{0,d} \quad (3.14)$$

is easily verified to be a well-defined holomorphic morphism which is independent of the choice of any local coordinates and is invariantly attached to the web, i.e. for any local biholomorphism F taking values into Ω , as holomorphic maps from $F^{-1}(\Omega)$ to $\mathcal{M}_{0,d}$, one has

$$\Phi_{F*(\mathcal{W})} = \Phi_{\mathcal{W}} \circ F.$$

Now recall that $\mathcal{M}_{0,d}$ naturally carries a $\binom{d}{4}$ -web of codimension 1, denoted by $\mathcal{W}_{\mathcal{M}_{0,d}}$, which is the one admitting as first integrals the forgetful maps $f_I : \mathcal{M}_{0,d} \rightarrow \mathcal{M}_{0,4} \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $[p_1, \dots, p_d] \mapsto [p_{i_1}, \dots, p_{i_4}]$ for all 4-tuples $I = (i_1, \dots, i_4)$ such that $1 \leq i_1 < i_2 < i_3 < i_4 \leq d$.

When $\Phi_{\mathcal{W}}$ has rank 2 at some point, the pull-back $\Phi_{\mathcal{W}}^*(\mathcal{W}_{\mathcal{M}_{0,d}})$ is a web on Ω , possibly with singularities and possibly formed by strictly less than $\binom{d}{4}$ foliations, but which is invariantly attached to $\mathcal{W}$. It is then natural to try to better understand the initial web $\mathcal{W}$ via its canonical map $\Phi_{\mathcal{W}}$, in terms of some properties of $\mathcal{W}_{\mathcal{M}_{0,d}}$.

*

We now look at the previous construction when specialized to Bol's web. To perform the computations, it is more convenient to work with $\mathcal{X}\mathcal{W}_{A_2}$ and its cluster first integrals X_ℓ given in (3.2). The 5-tuple of slopes associated to the cluster first integrals of $\mathcal{X}\mathcal{W}_{A_2}$ is

$$\zeta_{A_2} = \left(\infty, 0, \frac{-(1+u_2)}{u_1}, \frac{u_2(1+u_2)}{u_1(1+u_1)}, \frac{-u_2}{1+u_1} \right).$$

At any point $(u_1, u_2) \in (\mathbb{R}_{>0})^2$, the coordinates of ζ_{A_2} are pairwise distinct points of $\mathbb{P}^1$ and the map $\Phi_{\mathcal{X}\mathcal{W}_{A_2}}$ has rank 2. Thus the same holds true at any point of a sufficiently small open domain of $\mathbb{C}^2$ containing this positive quarter of plane. We fix such an open domain that we denote by Ω . The map $\Phi_{\mathcal{X}\mathcal{W}_{A_2}} : \Omega \rightarrow \mathcal{M}_{0,5}$ being étale, the pull-back of $\mathcal{W}_{\mathcal{M}_{0,5}}$ under it, that we will denote by $\mathcal{X}\mathcal{W}'_{A_2}$, is a genuine regular 5-web on Ω .

It is not difficult to make $\mathcal{X}\mathcal{W}'_{A_2}$ explicit in the coordinates u_1, u_2 . For $i = 1, \dots, 5$, one denotes by $\zeta_{A_2}(i)$ the 4-tuple obtained from ζ_{A_2} by removing

its i -th coordinate. As a (cluster) cross-ratio, we choose the one defined by

$$\kappa(a_1, a_2, a_3, a_4) = -\frac{(a_1 - a_4)(a_2 - a_3)}{(a_1 - a_3)(a_2 - a_4)}$$

for any 4-tuple $(a_1, \dots, a_4)$ of pairwise distinct points of $\mathbb{P}^1$. The first integrals of $\mathcal{X}\mathcal{W}'_{A_2}$ are the five rational maps $\kappa(\zeta_{A_2}(i))$ for $i = 1, \dots, 5$ and through elementary computations, one obtains that as an ordered 5-web on Ω , one has

$$\mathcal{X}\mathcal{W}'_{A_2} = \mathcal{W}\left(u_1, u_2, \frac{1+u_2}{u_1}, \frac{1+u_1+u_2}{u_1u_2}, \frac{1+u_1}{u_2}\right).$$

The map $\kappa(\zeta_{A_2}(\ell))$ coincides with X_ℓ for ℓ odd, and with $-(1+X_\ell)$ for ℓ even. We thus have that $\mathcal{X}\mathcal{W}_{A_2}$ and $\mathcal{X}\mathcal{W}'_{A_2}$ are the same (moreover as ordered 5-webs). This observation immediately gives us the

PROPOSITION 3.7. — *A 5-web $\mathcal{W}$ is equivalent to Bol's web if and only if $\mathcal{W} = \Phi_{\mathcal{W}}^*(\mathcal{W}_{\mathcal{M}_{0,5}})$.*

It should be noted that the condition that $\Phi_{\mathcal{W}}^*(\mathcal{W}_{\mathcal{M}_{0,5}})$ is a 5-web requires that $\Phi_{\mathcal{W}}$ is generically of rank 2, a condition that we assume to be implicitly satisfied here.

Proposition 3.7 may look tautological at first sight, but actually it is not and even better, an explicit and effective criterion for characterizing Bol's web can be extracted from it. Assuming that $\mathcal{W} = \mathcal{W}(U_1, \dots, U_5)$ for some first integrals $U_1, \dots, U_5$, one sets

$$X_i = (\partial U_i / \partial y) \partial_x - (\partial U_i / \partial x) \partial_y$$

for $i = 1, \dots, 5$ and for any j, k such that $1 \leq j < k \leq 5$, one denotes by $\mathcal{J}_{jk}$ the jacobian determinant of the map (U_j, U_k) in the coordinates x, y , i.e.

$$\mathcal{J}_{jk} = \begin{vmatrix} \partial U_j / \partial x & \partial U_j / \partial y \\ \partial U_k / \partial x & \partial U_k / \partial y \end{vmatrix}.$$

Then the condition $\mathcal{W} = \Phi_{\mathcal{W}}^*(\mathcal{W}_{\mathcal{M}_{0,5}})$ in Proposition 3.7 is equivalent to the fact that for any i, j, k, l, m such that $\{i, j, k, l, m\} = \{1, \dots, 5\}$, the following relation is identically satisfied

$$X_i \left(\frac{\mathcal{J}_{jm} \mathcal{J}_{kl}}{\mathcal{J}_{jl} \mathcal{J}_{km}} \right) = 0.$$

Verifying that these relations hold true is straightforward and only requires to compute rational expressions in the U_i 's and their partial derivatives up to order 2. We thus get a new criterion for characterizing 5-webs equivalent to Bol's web, which, from our point of view, is better than the two considered in the preceding subsection: it requires less computations and

furthermore for any 5-web $\mathcal{W} \simeq \mathcal{B}$, it furnishes the “algebraization map” essentially for free, by means of rational expressions in the first order partial derivatives of the first integrals of the web $\mathcal{W}$ under scrutiny.

3.6. The Gelfand–MacPherson construction

We now discuss the nice geometric construction of Bol’s web as the quotient of an equivariant web given by Gelfand and MacPherson in their paper [27]. We first use the formalism they used (in terms of grassmannians, etc.) then recast most things from another perspective (the key notion being that of “Cox variety”) which is the one which will generalize and will be used to construct geometrically $\mathcal{W}_{\text{dP}_4}$ further in Section 4.5.

3.6.1.

Let $n \in \mathbb{N}^*$ be arbitrary, set $N = n + 3 \geq 4$ and denote by $G_2(V)$ the grassmannian variety of 2-planes in $V = \mathbf{K}^N$ where $\mathbf{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$. Let $(e_i)_{i=1}^N$ be the canonical basis of V and $(x_i)_{i=1}^N$ be the corresponding linear coordinates. Denoting by

$$P : G_2(V) \hookrightarrow \mathbb{P}(\wedge^2 V) : \xi \mapsto [\Delta_{ij}(\xi)]_{1 \leq i < j \leq N}$$

the corresponding Plücker embedding, we define $G_2^*(V)$ as the Zariski open subset of $G_2(V)$ formed by “*generic*” 2-planes, namely the 2-planes whose all Plücker coordinates do not vanish:

$$G_2^*(V) = P^{-1} \left(\wedge^2 V \setminus \left(\bigcup_{1 \leq i < j \leq N} \{\Delta_{ij} = 0\} \right) \right).$$

The special linear group $\text{SL}_N(\mathbf{K})$ acts on the grassmannian and its Cartan torus $H_N = H_{A_{n+2}}$ formed by diagonal matrices lets $G_2^*(V)$ invariant and acts freely on it in such a way that $G_2^*(V)/H_N$ is a smooth manifold with the quotient map

$$\chi_N = \chi_{H_{A_{n+2}}} : G_2^*(V) \longrightarrow G_2^*(V)/H_N$$

being a $\mathbf{K}$ -algebraic submersion. According to Gelfand and MacPherson, there is a nice geometric interpretation of the image $\chi_N(P)$ in the quotient of any element $P \in G_2^*(V)$: being generic, such a 2-plane intersects transversally each of the coordinate hyperplanes $H_i = \{x_i = 0\}$ hence the intersections $P \cap H_i$ are N lines in P , which moreover are pairwise distinct. Setting $\ell_i(P)$ for

the line $P \cap H_i$ ($i = 1, \dots, N$) viewed as an element of $\mathbb{P}(P)$, and identifying the latter with a fixed projective line $\mathbb{P}^1$, one constructs a well-defined map

$$\begin{aligned} \mu_N : G_2^*(V) &\longrightarrow \text{Conf}_N(\mathbb{P}^1) = \mathcal{M}_{0,N} \\ P &\longmapsto [\ell_i(P)]_{i=1}^N \end{aligned}$$

which can be verified to be not only H_N -invariant, but to provide a geometric model for the quotient map χ_N : there is a natural isomorphism $G_2^*(V)/H_N \simeq \mathcal{M}_{0,N}$ such that the following diagram commutes:

$$\begin{array}{ccc} G_2^*(V) & \xrightarrow{\chi_N} & G_2^*(V)/H_N \\ & \searrow \mu_N & \downarrow \wr \\ & & \mathcal{M}_{0,N}. \end{array}$$

For $i = 1, \dots, N$, we set $V_i = V/\langle e_i \rangle \simeq \mathbf{K}^{N-1}$ and we denote by $\Psi_i : V \rightarrow V_i$ the corresponding linear projection. Since any generic 2-plane P intersects $\langle e_i \rangle$ transversally, the image $\Psi_i(P)$ is a 2-plane in V_i which can be verified to be generic as well. We thus have a well-defined morphism $G_2^*(V) \rightarrow G_2^*(V_i)$, $P \mapsto \Psi_i(P)$, again denoted by Ψ_i , which is such that the following diagram also is commutative (where $\psi_i : \mathcal{M}_{0,N} \rightarrow \mathcal{M}_{0,N-1}$ stands for the i -th forgetful map):

$$\begin{array}{ccc} G_2^*(V) & \xrightarrow{\Psi_i} & G_2^*(V_i) \\ \mu_N \downarrow & & \downarrow \mu_{N-1} \\ \mathcal{M}_{0,N} & \xrightarrow{\psi_i} & \mathcal{M}_{0,N-1}. \end{array} \quad (3.15)$$

Each map Ψ_i is a surjective $\mathbf{K}$ -algebraic regular submersion inducing a foliation by surfaces on $G_2^*(V)$. Taking these foliations together gives us what we call the “*Gelfand–MacPherson web*”

$$\mathcal{W}_{G_2(V)}^{GM} = \mathcal{W}(\Psi_1, \dots, \Psi_{n+3}),$$

which is a $(n+3)$ -web in a generalized sense on the grassmannian $G_2(V)$ (without singularities on its generic part $G_2^*(V)$). One easily checks that the foliations of $\mathcal{W}_{G_2(V)}^{GM}$ are H_N -equivariant (actually the same can be said about the maps Ψ_i ’s themselves, up to some technical details left to the reader). It follows that the direct image/quotient $(\mu_N)_*(\mathcal{W}_{G_2(V)}^{GM}) = \mathcal{W}_{G_2(V)/H_N}^{GM}$ exists as a web on $G_2^*(V)/H_N \simeq \mathcal{M}_{0,n+3}$. Considering the diagrams (3.15) for $i = 1, \dots, n+3$, it follows that

$$\mathcal{W}_{G_2(V)/H_{n+2}}^{GM} = \mathcal{W}_{\mathcal{M}_{0,n+3}}$$

where the right hand side stands for Burau–Damiano’s curvilinear web on $\mathcal{M}_{0,n+3}$, which is the web by rational curves induced by the forgetful maps $\psi_i : \mathcal{M}_{0,n+3} \rightarrow \mathcal{M}_{0,n+2}$ for $i = 1, \dots, n+3$.⁽¹⁶⁾

From the preceding discussion, we deduce that

Gelfand–MacPherson’s web $\mathcal{W}_{G_2(V)}^{GM}$ is H_{n+2} -equivariant and its torus quotient naturally identifies with Burau–Damiano’s curvilinear web $\mathcal{W}_{\mathcal{M}_{0,n+3}}$ on $\mathcal{M}_{0,n+3}$. (3.16)

For $n = 2$, we have $\mathcal{W}_{\mathcal{M}_{0,5}} \simeq \mathcal{B}$ hence the above statement applies and gives a description of Bol’s web $\mathcal{B}$ as an equivariant quotient by a torus action of a natural web defined by linear maps on the grassmannian of 2-planes in $\mathbf{K}^5$.

Remark 3.8. — The above geometric way to get Bol’s web is particularly interesting if we consider the major result obtained by Gelfand and MacPherson which is to construct geometrically (the real version of) Abel’s dilogarithmic identity, by integrating along the fibers of the actions of the different tori involved invariant representatives of some compatible characteristic classes on the grassmannians of 2-planes in $\mathbb{R}^5$ and in $\mathbb{R}_i^4 = \mathbb{R}^5 / \langle e_i \rangle$ for $i = 1, \dots, 5$ (see [27, §0.3] or Section 5.9 further in this text).

3.6.2.

In order to generalize the above geometric construction of Bol’s web $\mathcal{B} \simeq \mathcal{W}_{\mathrm{dP}_5}$ to $\mathcal{W}_{\mathrm{dP}_4}$, it is interesting to recast it from the bottom, that is in terms of del Pezzo’s quintic surface $\mathrm{dP}_5 \simeq \mathcal{M}_{0,5}$. Our main references for the material below are [54] (see also [55, §3]), [4], [53] and [19]. We identify the quintic del Pezzo surface to the total space, denoted by X_4 below, of the blow-up of $\mathbb{P}^2$ at four points $p_1, \dots, p_4$ in general position: $\mathrm{dP}_5 \simeq X_4 = \mathbf{Bl}_{p_1+\dots+p_4}(\mathbb{P}^2)$. As a $\mathbb{Z}$ -basis for the Picard lattice $\mathbf{Pic}_{\mathbb{Z}}(X_4) \simeq \mathbb{Z}^5$ of X_4 , we take $\mathcal{E} = (\ell_i)_{i=0}^4$ where $\ell_0 = \mathbf{h}$ stands for the pull-back of the class of a generic line in $\mathbb{P}^2$ and where ℓ_i is the class of the exceptional divisor in X_4 associated to p_i for $i = 1, \dots, 4$.

For $\ell \in \{\mathbf{h}, \ell_1, \dots, \ell_4\}$, let $\mathcal{L}_{\ell}^{\circ}$ stand for the total space of the line bundle $\mathcal{O}_{X_4}(\ell)$ with the image of the zero section removed and let us set

$$\mathcal{T}_{X_4} = \mathcal{L}_{\ell_0}^{\circ} \times_{X_4} \cdots \times_{X_4} \mathcal{L}_{\ell_4}^{\circ}.$$

⁽¹⁶⁾ We refer the interested reader to our recent paper [47] for much more about the webs $\mathcal{W}_{\mathcal{M}_{0,n+3}}$, for $n \geq 2$.

It is a torus bundle over X_4 on which naturally acts the *Néron–Severi torus* of X_4 , defined by

$$T_{NS}(X_4) = \mathrm{Hom}(\mathbf{Pic}_{\mathbb{Z}}(X_4), \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\mathcal{E}}.$$

The associated quotient map relative to this action identifies with the bundle map $\mathcal{T}_{X_4} \rightarrow X_4$. Taking the intersection with the anticanonical class defines a linear form $(-K, \cdot)$ on $\mathbf{Pic}_{\mathbb{Z}}(X_4)$ which gives rise to a 1-parameter subgroup $\tau_{\kappa} = \mathbb{C}^*(-K, \cdot) \subset T_{NS}(X_4)$. Setting

$$\mathrm{T}_{X_4} = T_{NS}(X_4)/\tau_{\kappa} \simeq (\mathbb{C}^*)^4,$$

we obtain that $\tau : \mathcal{T}_{X_4} = \mathcal{T}_{X_4}/\tau_{\kappa} \rightarrow X_4$ is a T_{X_4} -torsor over X_4 . Then we set

$$\mathcal{T}_{X_4}^* = \tau^{-1}(X_4^*) \subset \mathcal{T}_{X_4}$$

where X_4^* is the complement of the line divisor $L_4 = \bigcup_{\ell \in \mathcal{L}_4} \ell$ in X_4 , that is $X_4^* = X_4 \setminus L_4$.

For each $\ell \in \mathcal{L}_4$, we fix a nonzero section $x_{\ell} \in \mathbf{H}^0(X_4, \mathcal{O}(\ell))$ which is determined up to multiplication by a nonzero scalar since $\mathbf{H}^0(X_4, \mathcal{O}(\ell))$ has dimension 1. Since x_{ℓ} does not vanish on $\mathcal{T}_{X_4}^*$ for any ℓ , there exists a morphism

$$\begin{aligned} F : \mathcal{T}_{X_4}^* &\longrightarrow \mathbb{P}(\mathbb{C}^{\mathcal{L}_4}) \simeq \mathbb{P}^{l_4-1} = \mathbb{P}^9 \\ t &\longmapsto [x_{\ell}(t)]_{\ell \in \mathcal{L}_4} \end{aligned} \tag{3.17}$$

which can be proved to be an embedding. Note that F is canonically defined up to post-composition by an element of the torus formed by the linear automorphisms of $\mathbb{P}(\mathbb{C}^{\mathcal{L}_4})$ with a diagonal representative with respect to the basis $\mathcal{L}_4$. In order to describe the image of F in $\mathbb{P}^9$, let $(X_{\ell})_{\ell \in \mathcal{L}_4}$ be the homogeneous coordinates corresponding to the x_{ℓ} 's considered above and for any ℓ , let H_{ℓ} stand for the coordinate hyperplane cut out by the equation $X_{\ell} = 0$. Then the Zariski closure $\mathbf{G}_{2,5}$ of $F(\mathcal{T}_{X_4}^*)$ in $\mathbb{P}(\mathbb{C}^{\mathcal{L}_4})$ is isomorphic to the image of a Plücker embedding $P : G_2(\mathbb{C}^5) \hookrightarrow \mathbb{P}^9$ and moreover, one has

$$\begin{aligned} \mathcal{T}_{X_4}^* &\xrightarrow{\sim} F(\mathcal{T}_{X_4}^*) = \mathbf{G}_{2,5}^* = P(G_2^*(\mathbb{C}^5)) \\ &= \mathbf{G}_{2,5} \setminus \left(\bigcup_{\ell \in \mathcal{L}_4} H_{\ell} \right) \subset \mathbf{G}_{2,5} \subset \mathbb{P}^9. \end{aligned}$$

The grassmannian $\mathbf{G}_{2,5}$ is homogeneous under the action of a subgroup G of $\mathrm{PGL}(\mathbb{C}^{\mathcal{L}_4})$ isomorphic to $\mathrm{PGL}_5(\mathbb{C})$. Moreover, there exists an embedding $\mathrm{T}_{X_4} \hookrightarrow G$ whose image will be denoted by $\mathbf{T}_{X_4}$, making of $F : \mathcal{T}_{X_4}^* \rightarrow \mathbf{G}_{2,5}^*$ a $(\mathrm{T}_{X_4}, \mathbf{T}_{X_4})$ -equivariant isomorphism. Because the action of $\mathbf{T}_{X_4}$ on $\mathbf{G}_{2,5}^*$ is isomorphic to the one of H_4 on $G_2^*(\mathbb{C}^5)$ discussed above, this gives an intrinsic

way to obtain the geometric framework of Gelfand and MacPherson allowing to construct Bol's web geometrically as an equivariant quotient.

Actually, not only the fibration $G_2^*(\mathbb{C}^5) \rightarrow X_4^*$ can be recovered that way but its extension over the whole del Pezzo surface X_4 can be as well. The main ingredient for that is the *Cox ring* of X_4 (with respect to $\mathcal{E}$) which by definition is the commutative $\mathbb{C}$ -algebra

$$\mathrm{Cox}(X_4) = \bigoplus_{m_0, \dots, m_4 \in \mathbb{Z}} \mathbf{H}^0\left(X_4, \mathcal{O}(m_0H + m_1E_1 + \dots + m_4E_4)\right)$$

where H is a fixed generic element of $|\mathbf{h}|$ and E_i is the exceptional divisor associated to p_i for $i = 1, \dots, 4$, with the multiplication being induced by the pointwise multiplication of sections. It is naturally $\mathbb{Z}_{\geq 0}$ -graded (by taking the intersection with the anticanonical class $-K_{X_4}$) and it is known (see e.g. [4]) to be generated by degree 1 elements which are global nonzero elements of $\mathbf{H}^0(X_4, \mathcal{O}(\ell))$ for ℓ ranging in the set of lines included in X_4 , with relations which all are homogeneous of degree 2 with respect to the considered grading. From this, it can be deduced that

$$\mathbb{P}(X_4) = \mathrm{Proj}\left(\mathrm{Cox}(X_4)\right)$$

is a projective variety, which contains $\mathcal{T}_{X_4}^*$ as a Zariski open subset and is such that (3.17) extends to a canonical embedding $\bar{F} : \mathbb{P}(X_4) \hookrightarrow \mathbb{P}(\mathbb{C}^{\mathcal{L}_4})$ whose image is the grassmannian $\mathbf{G}_{2,5}$. Moreover, the T_{X_4} -action on $\mathcal{T}_{X_4}^*$ extends to $\mathbb{P}(X_4)$ and the map $\bar{F}$ is (T_{X_4}, T_{X_4}) -equivariant, as is its restriction to $\mathcal{T}_{X_4}^*$. The set $\mathbf{G}_{2,5}^s$ of stable points for the T_{X_4} -action coincides with that of semi-stable points (cf. [54]) and contains $\mathcal{T}_{X_4}^*$ as a Zariski open subset. One thus recovers the whole del Pezzo surface as the GIT quotient of $\mathbf{G}_{2,5}^s$ by T_{X_4} : one has $X_4 = \mathbf{G}_{2,5}^s // T_{X_4}$.

From the discussion above, one deduces that

Gelfand–MacPherson's geometric description of Bol's web (3.16)
can be recovered in an intrinsic and almost canonical way, by means (3.18)
of the Cox ring theoretic material discussed just above.

The point is that this Cox-ring theoretic material can be generalized to all the del Pezzo surfaces of degree $d \in \{2, \dots, 5\}$ which opens the door for a perspective à la Gelfand and MacPherson on the corresponding del Pezzo's webs $\mathcal{W}_{\mathrm{dP}_d}$. We will discuss the case when $d = 4$ further below in Section 4.5.

4. The web $\mathcal{W}_{\mathrm{dP}_4}$

This section is the main core of this article. In it, we prove that the web of conics of a quartic del Pezzo surface satisfies all the properties listed in Section 1.2 above.

4.1. The web $\mathcal{W}_{\mathrm{dP}_4}$ and the identity $\mathbf{HLog}^3$ in explicit form

It is interesting to make the web $\mathcal{W}_{\mathrm{dP}_4}$ and the hyperlogarithmic identity $\mathbf{HLog}^3$ as explicit as possible.

Let dP_4 be a fixed smooth del Pezzo quartic surface, that we see as the total space blow-up $b : \mathrm{dP}_4 \rightarrow \mathbb{P}^2$ at the following five points:

$$\begin{aligned} p_1 &= [1 : 0 : 0], & p_2 &= [0 : 1 : 0], & p_3 &= [0 : 0 : 1] \\ p_4 &= [1 : 1 : 1] & \text{and} & & p_5 &= [\pi : \gamma : 1] \end{aligned} \quad (4.1)$$

for some fixed parameters $\pi, \gamma \in \mathbb{C}$ such that these five points are in general position. This latter condition is equivalent to the fact that

$$\pi\gamma(\pi - 1)(\gamma - 1)(\pi - \gamma) \neq 0, \quad (4.2)$$

a condition that we assume to be satisfied in what follows. Let x, y be the affine coordinates corresponding to the affine embedding $\mu : (x, y) \mapsto [x : y : 1]$. One verifies easily that the pull-back of $\mathcal{W}_{\mathrm{dP}_4}$ under the birational map $b^{-1} \circ \mu$ is the web $\mathcal{W}(U_1, \dots, U_{10})$ defined by the ten following rational functions:

$$\begin{aligned} U_1 &= x & U_6 &= \frac{(1-x)\gamma + x + (\pi-1)y - \pi}{(x-1)(y-\gamma)} \\ U_2 &= \frac{1}{y} & U_7 &= \frac{(x-y)(y-\gamma)}{y(\pi y - \gamma x - \pi + \gamma + x - y)} \\ U_3 &= \frac{y}{x} & U_8 &= \frac{-x(x(\gamma-1) + (1-y)\pi - \gamma + y)}{(x-y)(x-\pi)} \\ U_4 &= \frac{x-y}{x-1} & U_9 &= \frac{y(x-\pi)}{x(y-\gamma)} \\ U_5 &= \frac{\gamma(\pi-x)}{\pi y - \gamma x} & U_{10} &= \frac{x(y-1)}{y(x-1)}. \end{aligned} \quad (4.3)$$

The rational curves in $\mathbb{P}^2$ corresponding to the 11 lines in dP_4 distinct from the exceptional divisors $\ell_i = b^{-1}(p_i)$ ($i = 1, \dots, 5$) are the lines at infinity plus the closures in $\mathbb{P}^2$ of the affine curves with equation $\mathcal{L}_i = 0$,

where the $\mathcal{L}'_i$'s are the components of the following 10-tuple of polynomials in x, y :

$$\begin{aligned} \mathcal{L} = (\mathcal{L}_i)_{i=1}^{10} = & \left(x, y, y - \gamma, x - 1, x - \pi, x - y, y - 1, \right. \\ & \gamma((x - y)\pi + x(y - 1)) - \pi y(x - 1), \\ & \left. \gamma(x - 1) - \pi(y - 1) + y - x, \gamma x - \pi y \right). \end{aligned}$$

We denote by $A_{\mathcal{L}}$ the union in $\mathbb{C}^2$ of the curves cut out by the equations $\mathcal{L}_i = 0$ with $i = 1, \dots, 10$. Then $\mathcal{W}(U_1, \dots, U_{10})$ is a non singular web on the Zariski open set $b(\mathrm{dP}_4 \setminus L_5) = \mathbb{C}^2 \setminus A_{\mathcal{L}}$.

One considers the associated logarithmic forms (for $i = 1, \dots, 10$):

$$h_i = \mathrm{dlog}(\mathcal{L}_i) = \mathrm{d}\mathcal{L}_i / \mathcal{L}_i. \quad (4.4)$$

Then for each i , the spectrum⁽¹⁷⁾ of the fibration in conics $U_i \circ b : \mathrm{dP}_4 \rightarrow \mathbb{P}^1$ is the support of the following normalized 4-tuple

$$\mathfrak{R}_i = (0, 1, r_i, \infty)$$

where for each i , r_i is the i -th component of the following 10-tuple of complex numbers:

$$\begin{aligned} \mathfrak{r} = (r_i)_{i=1}^{10} = & \left(\pi, \frac{1}{\gamma}, \frac{\gamma}{\pi}, \frac{\pi - \gamma}{\pi - 1}, \frac{\gamma(\pi - 1)}{\pi - \gamma}, \frac{\gamma - \pi}{\gamma}, \right. \\ & \left. \frac{1}{1 - \pi}, 1 - \gamma, \frac{\pi - 1}{\gamma - 1}, \frac{\pi(\gamma - 1)}{\gamma(\pi - 1)} \right). \end{aligned}$$

Remark that since (4.2) is assumed to hold true:

- (i) all the $\mathcal{L}_i$'s are linearly independent as affine equations in x and y ; hence
- (ii) the same holds true for the h_i 's (as logarithmic forms in the same variables); and
- (iii) none of the r_i 's coincides with an element of $\{0, 1, \infty\} \subset \mathbb{P}^1$ hence each $\mathfrak{R}_i$ indeed is a 4-tuple of pairwise distinct elements of $\mathbb{P}^1$.

The h_i 's form a basis of a subspace denoted by $\mathbf{H}$, of the space of rational 1-forms on $\mathbb{P}^2$. For every i , one sets $\eta_i = (\eta_{i,s})_{s=1}^3$ with $\eta_{i,s} = \mathrm{dLog}(U_i - \mathfrak{R}_{i,s})$ for $s = 1, 2, 3$, that is

$$\eta_{i,1} = \frac{\mathrm{d}U_i}{U_i}, \quad \eta_{i,2} = \frac{\mathrm{d}U_i}{U_i - 1} \quad \text{and} \quad \eta_{i,3} = \frac{\mathrm{d}U_i}{U_i - r_i}.$$

⁽¹⁷⁾ By definition, the “spectrum” of a map $f : \mathrm{dP}_4 \dashrightarrow \mathbb{P}^1$ is the set of values $\lambda \in \mathbb{P}^1$ such that $f^{-1}(\lambda)$ is not irreducible.

Any $\eta_{i,s}$ is an element of $\mathbf{H}$ hence admits a unique expression as a linear combination in the h_j 's (see below for a conceptual explanation of this fact) and all of these decompositions can be obtained in explicit form by means of straightforward computations. One gets the following formulas:

$$\begin{aligned}
 \eta_1 &= (h_1, h_4, h_5) \\
 \eta_2 &= (-h_2, h_7 - h_2, h_3 - h_2) \\
 \eta_3 &= (-h_1 + h_2, -h_1 + h_6, -h_1 + h_{10}) \\
 \eta_4 &= (-h_4 + h_6, h_7 - h_4, h_9 - h_4) \\
 \eta_5 &= (-h_{10} + h_5, h_3 - h_{10}, h_9 - h_{10}) \\
 \eta_6 &= (-h_3 + h_9 - h_4, h_7 - h_3 - h_4 + h_5, -h_3 - h_4 + h_8) \\
 \eta_7 &= (h_3 - h_9 + h_6 - h_2, h_7 - h_9 + h_{10} - h_2, -h_9 - h_2 + h_8) \\
 \eta_8 &= (h_9 + h_1 - h_5 - h_6, h_4 - h_5 - h_6 + h_{10}, -h_5 - h_6 + h_8) \\
 \eta_9 &= (-h_3 - h_1 + h_5 + h_2, -h_3 - h_1 + h_{10}, -h_3 - h_1 + h_8) \\
 \text{and } \eta_{10} &= (h_7 + h_1 - h_4 - h_2, -h_4 + h_6 - h_2, -h_4 - h_2 + h_8).
 \end{aligned} \tag{4.5}$$

For a triple (a, b, c) of pairwise distinct points on $\mathbb{C}$ and given a base point $\zeta \in \mathbb{C} \setminus \{a, b, c\}$, we consider the weight 3 hyperlogarithm $H_{a,b,c}^\zeta$ defined by

$$H_{a,b,c}^\zeta(z) = \int_\zeta^z \left(\int_\zeta^{u_3} \left(\int_\zeta^{u_2} \frac{du_1}{u_1 - c} \right) \frac{du_2}{u_2 - b} \right) \frac{du_3}{u_3 - a}$$

for any z sufficiently close to ζ , and we denote by $AH_{a,b,c}^\zeta$ its antisymmetrization:

$$AH_{a,b,c}^\zeta = \frac{1}{6} \left(H_{a,b,c}^\zeta - H_{a,c,b}^\zeta - H_{b,a,c}^\zeta + H_{b,c,a}^\zeta + H_{c,a,b}^\zeta - H_{c,b,a}^\zeta \right).$$

We now fix a base point $\xi \in \mathbb{C}^2 \setminus A_{\mathcal{L}}$ and for $i = 1, \dots, 10$, we set $\xi_i = U_i(\xi) \in \mathbb{C} \setminus \{0, 1, r_i\}$ and

$$AH_i^3 = AH_{0,1,r_i}^{\xi_i}.$$

For any i , the symbol $\mathcal{S}_i = \mathcal{S}(AH_i^3(U_i))$ of $AH_i^3(U_i) = U_i^*(AH_i^3)$ is the antisymmetric tensor

$$\mathcal{S}_i = \frac{dU_i}{U_i} \wedge \frac{dU_i}{U_i - 1} \wedge \frac{dU_i}{U_i - r_i} \in \wedge^3 \mathbf{H}.$$

A basis of the space of weight 3 tensors $\wedge^3 \mathbf{H}$ is given by the $h_i \wedge h_j \wedge h_k$'s for all triples (i, j, k) such that $1 \leq i < j < k \leq 10$. From the formulas in (4.5),

it is just an elementary computational matter to express each symbol $\mathcal{S}_i$ as a linear combination of the $h_i \wedge h_j \wedge h_k$'s. For instance, one has

$$\mathcal{S}_1 = h_1 \wedge h_4 \wedge h_5$$

$$\mathcal{S}_2 = -h_2 \wedge h_3 \wedge h_7$$

$$\mathcal{S}_3 = h_1 \wedge h_2 \wedge h_{10} - h_1 \wedge h_2 \wedge h_6 - h_1 \wedge h_6 \wedge h_{10} + h_2 \wedge h_6 \wedge h_{10}, \text{ etc.}$$

With the explicit expressions of the symbols $\mathcal{S}_i$'s at hand, another elementary computation gives us that

$$\sum_{i=1}^{10} \mathcal{S}_i = \sum_{i=1}^{10} \left(\frac{dU_i}{U_i} \wedge \frac{dU_i}{U_i - 1} \wedge \frac{dU_i}{U_i - r_i} \right) = 0$$

in $\wedge^3 \mathbf{H}$, from which one immediately deduces that the identity $\mathbf{HLog}^3$ is written in explicit form

$$\sum_{i=1}^{10} AH_i^3(U_i) = 0, \quad (4.6)$$

a functional relation which holds true identically on any sufficiently small neighbourhood of ζ .

Another thing which can be obtained quite easily from the explicit formulas (4.5) is the determination of the weight 2 antisymmetric ARs one can obtain from $\mathbf{HLog}^3$ by taking residues. For any $i = 1, \dots, 10$, let ℓ_i be the "line" corresponding to the equation $\mathcal{L}_i = 0$ and denote by $\mathcal{R}_i$ the weight 2 abelian relation obtained by taking the residue of $\mathbf{HLog}^3$ along it, i.e.

$$\mathcal{R}_i = \text{Res}_{\ell_i}(\mathbf{HLog}^3) = \left(\text{Res}_{\ell_i}(\mathcal{S}_j) \right)_{j=1}^{10}.$$

Let $\chi_1, \dots, \chi_{10}$ be the elements of the formal dual basis of the basis $(h_1, \dots, h_{10})$ of $\mathbf{H}$: the χ_i 's are uniquely determined by the relations $h_j(\chi_i) = \delta_{ij}$ for all $i, j = 1, \dots, 10$. For each i , one denotes by $\mathbf{H}_i$ the subspace of $\mathbf{H}$ with basis dU_i/U_i , $dU_i/(U_i - 1)$ and $dU_i/(U_i - r_i)$.

LEMMA 4.1. — *For any $i, j = 1, \dots, 10$, the following hold true:*

- *the residue $\text{Res}_{\ell_j}(\mathcal{S}_i)$ is non-zero if and only if ℓ_j is contracted by U_i ;*
- *when ℓ_j is contracted by U_i , then $z_{ij} = U_i(\ell_j)$ is one of the elements of the 4-tuple $\mathfrak{R}_i = (0, 1, r_i, \infty)$, the k -th say. Then $\text{Res}_{\ell_j}(\mathcal{S}_i)$ coincides with $(-1)^{k-1}/3$ times the pull-back under U_i of the weight two symbol of $AH_{\mathfrak{R}_{ik}}^2$ where $\mathfrak{R}_{ik} = (\mathfrak{R}_{i,s})_{s=1, s \neq k}^4$. The explicit formula for $\text{Res}_{\ell_j}(\mathcal{S}_i)$ according to the value of z_{ij} is given in the following*

table:

z_{ij}	$3 \operatorname{Res}_{\ell_j}(\mathcal{S}_i)$
0	$U_i^* \left(\frac{dz}{z-1} \wedge \frac{dz}{z-r_i} \right)$
1	$-U_i^* \left(\frac{dz}{z} \wedge \frac{dz}{z-r_i} \right)$
r_i	$U_i^* \left(\frac{dz}{z} \wedge \frac{dz}{z-1} \right)$
∞	$-U_i^* \left(\left(\frac{dz}{z} - \frac{dz}{z-r_i} \right) \wedge \left(\frac{dz}{z-1} - \frac{dz}{z-r_i} \right) \right)$

For $a, b \in \mathbb{C}$ and $c \in \mathbb{P}^1$, one considers the formal expression

$$\mathfrak{R}_{a,b,c}^V = \left(\frac{dV}{V-a} - \frac{dV}{V-c} \right) \wedge \left(\frac{dV}{V-b} - \frac{dV}{V-c} \right)$$

with the convention that $\frac{dV}{V-c} = 0$ when $c = \infty$ (hence the elements of the second row in the table above may have been written $\mathfrak{R}_{1,r_i,\infty}^{U_i}$, $-\mathfrak{R}_{0,r_i,\infty}^{U_i}$, $\mathfrak{R}_{0,1,\infty}^{U_i}$ and $-\mathfrak{R}_{0,1,r_i}^{U_i}$, in this order).

With the preceding lemma at hand, it is straightforward to compute the weight 2 abelian relations obtained by taking residues of $\mathbf{HLog}^3$ along the ℓ_i 's. One has:

$$\begin{aligned} \operatorname{Res}_{\ell_1}(\mathbf{HLog}^3) &= \left(\mathfrak{R}_{1,r_1,\infty}^{U_1}, 0, -\mathfrak{R}_{0,1,r_3}^{U_3}, 0, 0, 0, 0, \mathfrak{R}_{1,r_8,\infty}^{U_8}, -\mathfrak{R}_{0,1,r_9}^{U_9}, \mathfrak{R}_{1,r_{10},\infty}^{U_{10}} \right) \\ \operatorname{Res}_{\ell_2}(\mathbf{HLog}^3) &= \left(0, -\mathfrak{R}_{0,1,r_2}^{U_2}, \mathfrak{R}_{1,r_3,\infty}^{U_3}, 0, 0, 0, -\mathfrak{R}_{0,1,r_7}^{U_7}, 0, \mathfrak{R}_{1,r_9,\infty}^{U_9}, -\mathfrak{R}_{0,1,r_{10}}^{U_{10}} \right) \\ \operatorname{Res}_{\ell_3}(\mathbf{HLog}^3) &= \left(0, \mathfrak{R}_{0,1,\infty}^{U_2}, 0, 0, -\mathfrak{R}_{0,r_5,\infty}^{U_5}, -\mathfrak{R}_{0,1,r_6}^{U_6}, \mathfrak{R}_{1,r_7,\infty}^{U_7}, 0, -\mathfrak{R}_{0,1,r_9}^{U_9}, 0 \right) \\ \operatorname{Res}_{\ell_4}(\mathbf{HLog}^3) &= \left(-\mathfrak{R}_{0,r_1,\infty}^{U_1}, 0, 0, -\mathfrak{R}_{0,1,r_4}^{U_4}, 0, -\mathfrak{R}_{0,1,r_6}^{U_6}, 0, -\mathfrak{R}_{0,r_8,\infty}^{U_8}, 0, -\mathfrak{R}_{0,1,r_{10}}^{U_{10}} \right) \\ \operatorname{Res}_{\ell_5}(\mathbf{HLog}^3) &= \left(\mathfrak{R}_{0,1,\infty}^{U_1}, 0, 0, 0, \mathfrak{R}_{1,r_5,\infty}^{U_5}, -\mathfrak{R}_{0,r_6,\infty}^{U_6}, 0, -\mathfrak{R}_{0,1,r_8}^{U_8}, \mathfrak{R}_{1,r_9,\infty}^{U_9}, 0 \right) \\ \operatorname{Res}_{\ell_6}(\mathbf{HLog}^3) &= \left(0, 0, -\mathfrak{R}_{0,r_3,\infty}^{U_3}, \mathfrak{R}_{1,r_4,\infty}^{U_4}, 0, 0, \mathfrak{R}_{1,r_7,\infty}^{U_7}, -\mathfrak{R}_{0,1,r_8}^{U_8}, 0, -\mathfrak{R}_{0,r_{10},\infty}^{U_{10}} \right) \\ \operatorname{Res}_{\ell_7}(\mathbf{HLog}^3) &= \left(0, -\mathfrak{R}_{0,r_2,\infty}^{U_2}, 0, -\mathfrak{R}_{0,r_4,\infty}^{U_4}, 0, -\mathfrak{R}_{0,r_6,\infty}^{U_6}, -\mathfrak{R}_{0,r_7,\infty}^{U_7}, 0, 0, \mathfrak{R}_{1,r_{10},\infty}^{U_{10}} \right) \\ \operatorname{Res}_{\ell_8}(\mathbf{HLog}^3) &= \left(0, 0, 0, 0, \mathfrak{R}_{0,1,\infty}^{U_6}, \mathfrak{R}_{0,1,\infty}^{U_7}, \mathfrak{R}_{0,1,\infty}^{U_8}, \mathfrak{R}_{0,1,\infty}^{U_9}, \mathfrak{R}_{0,1,\infty}^{U_{10}} \right) \\ \operatorname{Res}_{\ell_9}(\mathbf{HLog}^3) &= \left(0, 0, 0, \mathfrak{R}_{0,1,\infty}^{U_4}, \mathfrak{R}_{0,1,\infty}^{U_5}, \mathfrak{R}_{1,r_6,\infty}^{U_6}, -\mathfrak{R}_{0,1,r_7}^{U_7}, \mathfrak{R}_{1,r_8,\infty}^{U_8}, 0, 0 \right) \\ \operatorname{Res}_{\ell_{10}}(\mathbf{HLog}^3) &= \left(0, 0, \mathfrak{R}_{0,1,\infty}^{U_3}, 0, -\mathfrak{R}_{0,1,r_5}^{U_5}, 0, -\mathfrak{R}_{0,r_7,\infty}^{U_7}, -\mathfrak{R}_{0,r_8,\infty}^{U_8}, -\mathfrak{R}_{0,r_9,\infty}^{U_9}, 0 \right). \end{aligned}$$

For any $i \leq 10$, there exist exactly 3 residues $\text{Res}_{\ell_j}(\mathbf{HLog}^3)$ the i -th components of which are non zero. Moreover, these components form a basis of $\wedge^2 \mathbf{H}_i$. For instance, for $i = 1$, $\text{Res}_{\ell_j}(\mathbf{HLog}^3)$ has a non zero first component only for $j = 1, 4, 5$, and these components are $\mathfrak{R}_{1,r_1,\infty}^{U_1}$, $-\mathfrak{R}_{0,r_1,\infty}^{U_1}$ and $\mathfrak{R}_{0,1,\infty}^{U_1}$ respectively, that is are the pull-backs under U_i of the three 2-wedge products

$$\frac{du}{u-1} \wedge \frac{du}{u-r_1}, \quad -\frac{du}{u} \wedge \frac{du}{u-r_1} \quad \text{and} \quad \frac{du}{u} \wedge \frac{du}{u-1}$$

which obviously form a basis of $\wedge^2 \mathbf{H}^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\text{Log } \mathfrak{R}_1))$. From this, one immediately deduces that the ten residues $\text{Res}_{\ell_i}(\mathbf{HLog}^3)$'s are linearly independent. Since $\text{Res}_{\ell_i}(\mathbf{HLog}^3)$ is an element of $\mathbf{AR}_{\text{asym}}^2(\mathcal{W}_{\text{dP}_4})$ and because this space is of dimension 10 (see the next subsection), it follows that

$$\text{Res}(\mathbf{HLog}^3) = \langle \text{Res}_{\ell_i}(\mathbf{HLog}^3) \rangle_{i=1}^{10} = \mathbf{AR}_{\text{asym}}^2(\mathcal{W}_{\text{dP}_4}). \quad (4.7)$$

4.2. The symbolic abelian relations of $\mathcal{W}_{\text{dP}_4}$

Here we apply the approach of Section 2.3.5 to the web under scrutiny: $\mathbf{H}$ stands for the space of rational 1-forms, regular on $\mathbb{C}^2 \setminus A$ and with logarithmic singularities along the components of A spanned by the $h_i = \text{dLog}(\mathcal{L}_i)$ for $i = 1, \dots, 10$. These are linearly independent hence $\dim \mathbf{H} = 10$. For any i , let $\mathbf{H}_i$ be the subspace of $\mathbf{H}$ with basis the 1-form $\eta_{i,s} = \text{dLog}(U_i - \mathfrak{r}_{i,s})$ for $s = 1, 2, 3$.

For any weight $w \geq 1$, one defines a vector space of “*symbolic weight w hyperlogarithmic abelian relations*”, denoted by $\mathbf{HLogAR}^w = \mathbf{HLogAR}^w(\mathcal{W}_{\text{dP}_4})$, by requiring that the following sequence of vector spaces be exact:

$$0 \longrightarrow \mathbf{HLogAR}^w \longrightarrow \bigoplus_{i=1}^{10} \mathbf{H}_i^{\otimes w} \longrightarrow \mathbf{H}^{\otimes w}. \quad (4.8)$$

The symmetric group $\mathfrak{S}_w$ naturally acts linearly on $\mathbf{H}^{\otimes w}$ by permuting the components of the pure tensors: one has $\sigma \cdot k_1 \otimes \dots \otimes k_w = k_{\sigma(1)} \otimes \dots \otimes k_{\sigma(w)}$ for any $\sigma \in \mathfrak{S}_w$ and any $k_1 \otimes \dots \otimes k_w \in \mathbf{H}^{\otimes w}$ and this action is extended linearly to the whole space $\mathbf{H}^{\otimes w}$. Each of the inclusions $\mathbf{H}_i \hookrightarrow \mathbf{H}$ is obviously $\mathfrak{S}_w$ -equivariant, hence (4.8) is a sequence of $\mathfrak{S}_w$ -representations, with the map $\mathbf{HLogAR}^w \rightarrow \bigoplus_{i=1}^{10} \mathbf{H}_i^{\otimes w}$ being injective by the very definition of $\mathbf{HLogAR}^w$ for any $w \geq 1$. In particular for $w = 2$, one has a decomposition in direct sum

$$\mathbf{HLogAR}^2 = \mathbf{HLogAR}_{\text{sym}}^2 \oplus \mathbf{HLogAR}_{\text{asym}}^2$$

where $\mathbf{HLogAR}_{\text{sym}}^2$ (resp. $\mathbf{HLogAR}_{\text{asym}}^2$) stands for the symbolic weight 2 hyperlogarithmic ARs which are symmetric (resp. antisymmetric) with respect to the action of $\mathfrak{S}_2 \simeq \{\pm 1\}$.

Using the explicit decomposition of the $\eta_{i,s}$ in the basis formed by the h_j with $j = 1, \dots, 10$, it is just a matter of linear algebra computations to express any tensor $\eta_{i,s_1} \otimes \dots \otimes \eta_{i,s_w} \in \mathbf{H}_i^{\otimes w}$ as a linear combination in the $h_{i_1 \dots i_w} = h_{i_1} \otimes \dots \otimes h_{i_w} \in \mathbf{H}^{\otimes w}$ for all tuples $(i_1, \dots, i_w)$ such that $1 \leq i_1 \leq \dots \leq i_w \leq 10$. By direct elementary computations, we obtain the following points:

- the space $\mathbf{HLogAR}^1$ has dimension 20 (coherently with Lemma 2.9). This space admits a basis formed by combinatorial ARs hence the subspace of combinatorial abelian relations $\mathbf{AR}_C(\mathcal{W}_{\text{dP}_4})$ coincides with that of logarithmic ones: one has

$$\mathbf{AR}_{\log}(\mathcal{W}_{\text{dP}_4}) \simeq \mathbf{HLogAR}^1;$$

- the spaces $\mathbf{HLogAR}_{\text{sym}}^2$ and $\mathbf{HLogAR}_{\text{asym}}^2$ have dimension 5 and 10 respectively, with $\mathbf{HLogAR}_{\text{asym}}^2$ spanned by the residues of $\mathbf{HLog}^3$ (see (4.7) above);
- the space $\mathbf{HLogAR}^3$ is spanned by $\mathbf{HLog}^3 = (\eta_{i,1} \wedge \eta_{i,2} \wedge \eta_{i,3})_{i=1}^{10}$ hence has dimension 1;
- we have $\text{rk}(\mathcal{W}_{\text{dP}_4}) = \sum_{k=1}^3 \dim(\mathbf{HLogAR}^k) = 20 + (10 + 5) + 1 = 36$ hence $\mathcal{W}_{\text{dP}_4}$ has maximal rank with all its ARs being hyperlogarithmic of weight less than or equal to 3.

As explained in Section 2.3.5, all the subspaces of ARs considered in the first three points above are naturally acted upon by the Weyl group $W = W(D_5)$. It is interesting to describe more precisely the structures of these spaces as W -representations. It is the purpose of the next subsection.

4.2.1. Subspaces of ARs as W -modules

We recall some notations and introduce new ones which we will use here, in relation with the representations of the Weyl group W and in particular its actions on the set of lines included in the fixed del Pezzo quartic surface we are working with:

- we use the notation of Section 2.3: for instance, $\mathcal{K}$ stands for the set of conic classes in the Picard lattice of the considered del Pezzo quartic, etc. We set $\mathbf{c}_1 = \mathbf{h} - \ell_1$: it is the class of the privileged conic

fibration we will consider (which is the one such that $b_*|\mathbf{c}_1|$ is the linear system of lines on the projective plane passing through the point p_1);

- then $\mathbf{c}_1^{red}$ denotes the set of non irreducible conics in the pencil $|\mathbf{c}_0|$: a bit abusively, one can write

$$\mathbf{c}_1^{red} = \{(\mathbf{h} - \ell_1 - \ell_i) + (\ell_i) \mid i = 2, \dots, 5\},$$

with the convention here that $(\mathbf{h} - \ell_1 - \ell_i) + (\ell_i)$ stands for the non irreducible conic whose two irreducible components are the two lines $h - \ell_1 - \ell_i$ and ℓ_i , this for any $i = 2, \dots, 5$

- we set $W_{\mathbf{c}_1} = \text{Fix}_W(\mathbf{c}_1) = \langle s_2, \dots, s_5 \rangle$. This subgroup of $W = W(D_5) = \langle s_1, \dots, s_5 \rangle$ is naturally isomorphic to $W(D_4)$ and naturally acts (by permutations) on $\mathbf{c}_1^{red}$;
- we use the classical labelling of irreducible representation of $W = W(D_5)$ or $W_{\mathbf{c}_1} = W(D_4)$ by means of bipartitions of 5 and 4 respectively (e.g. see [26, §5.6]).
- we will denote by $\mathbb{1}$ the trivial representation of the considered Weyl group. For instance, we have $\mathbb{1} = V_{[.5]}^1$ when dealing with $W \simeq W(D_5)$;
- we denote by $\mathbb{Z}^{\mathbf{c}_1^{red}}$ the free $\mathbb{Z}$ -module spanned by the elements of $\mathbf{c}_1^{red}$ and by $\mathbf{H}_{\mathbf{c}_1}$ the submodule of elements with vanishing trace. Since $W_{\mathbf{c}_1}$ acts by permutations on $\mathbf{c}_1^{red}$, $\mathbb{Z}^{\mathbf{c}_1^{red}}$ naturally is a $W_{\mathbf{c}_1}$ -representation of which $\mathbf{H}_{\mathbf{c}_1}$ is a direct factor. More precisely, one has $\mathbb{Z}^{\mathbf{c}_1^{red}} = \mathbb{1} \oplus \mathbf{H}_{\mathbf{c}_1}$ as $W_{\mathbf{c}_1}$ -modules (cf. Lemma 2.6).

Weight 1 abelian relations. A description of the W -module structure of $\mathbf{HLogAR}^1$ will follow from the

LEMMA 4.2. —

- (1) As $W_{\mathbf{c}_1}$ -representations, one has

$$\mathbb{Z}^{\mathbf{c}_1^{red}} = \mathbb{1} \oplus \mathbf{H}_{\mathbf{c}_1} \quad \text{with} \quad \mathbf{H}_{\mathbf{c}_1} \simeq V_{[2,3]}^3.$$

- (2) As W -representations, one has

$$\bigoplus_{\mathbf{c} \in \mathcal{K}} \mathbf{H}_{\mathbf{c}} \simeq \text{Ind}_{W_{\mathbf{c}_1}}^W(\mathbf{H}_{\mathbf{c}_1}) \simeq V_{[2,3]}^{10} \oplus V_{[2,21]}^{20}.$$

Proof. — This is obtained by explicit computations in GAP3 (detailed in the Appendix). $\square$

Since $\bigoplus_{\mathbf{c} \in \mathcal{K}} \mathbf{H}_{\mathbf{c}} \rightarrow \mathbf{H}_{\text{dP}_4}$ is non zero and because the target is an irreducible W -representation, it follows that this map is surjective. Hence for

$w = 1$, (4.8) is an exact sequence and because we have vertical isomorphisms as follows

$$\begin{array}{ccccccc} 0 \longrightarrow \mathbf{HLogAR}^1 & \longrightarrow & \bigoplus_{c \in \mathcal{K}} \mathbf{H}_c & \longrightarrow & \mathbf{H}_{dP_4} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & V_{[2,21]}^{20} \oplus V_{[2,3]}^{10} & \longrightarrow & V_{[2,3]}^{10} & \longrightarrow & 0, \end{array} \quad (4.9)$$

we deduce that as a W -representation, $\mathbf{HLogAR}^1$ is irreducible and isomorphic to $V_{[2,21]}^{20}$.

Weight 2 abelian relations. Aiming to decompose into W -irreducibles the space of weight 2 hyperlogarithmic ARs $\mathbf{HLogAR}^2$, we first determine those of the two W -modules $\bigoplus_{c \in \mathcal{K}} \mathbf{H}_c^{\otimes 2}$ and $\mathbf{H}_{dP_4}^{\otimes 2}$. Since $M^{\otimes 2} = M^{\wedge 2} \oplus M^{\odot}$ for any W -module M (with $M^2 = \text{Asym}(M) = \wedge^2 M$ and $M^{\odot 2} = \text{Sym}(M)$), one can deal with symmetric and antisymmetric weight 2 tensors separately.

We deal with the case of antisymmetric ARs first. From straightforward computations with GAP3 (cf. the Appendix), we easily obtain the following decompositions into W -irreducible representations: one has

$$\begin{aligned} \wedge^2 \mathbf{H}_{c_1} &\simeq \wedge^2 V_{[2,2]}^3 - \simeq V_{[11]}^3 - \\ \text{and } \wedge^2 \mathbf{H}_{dP_4} &\simeq \wedge^2 V_{[2,3]}^{10} = V_{[11,21]}^{20} \oplus V_{[11,3]}^{10} \oplus V_{[1,31]}^{15}. \end{aligned} \quad (4.10)$$

On the other hand, remarking that $\bigoplus_{c \in \mathcal{K}} \wedge^2 \mathbf{H}_c \simeq \text{Ind}_{W_{c_1}}^W (\wedge^2 \mathbf{H}_{c_1})$ and computing with GAP3 again, one gets

$$\begin{aligned} \bigoplus_{c \in \mathcal{K}} \wedge^2 \mathbf{H}_c &\simeq \text{Ind}_{W_{c_1}}^W (\wedge^2 \mathbf{H}_{c_1}) \\ &\simeq \text{Ind}_{W_{c_1}}^W (V_{[11]}^3 -) = V_{[11,111]}^{10} \oplus V_{[11,21]}^{20}. \end{aligned} \quad (4.11)$$

Putting (4.10) and (4.11) together into the following diagram of W -representations

$$\begin{array}{ccccc} 0 \longrightarrow \mathbf{HLogAR}_{\text{asym}}^2 & \longrightarrow & \bigoplus_{c \in \mathcal{K}} \wedge^2 \mathbf{H}_c & \longrightarrow & \wedge^2 \mathbf{H}_{dP_4} \\ & & \downarrow & & \downarrow \\ & & V_{[11,111]}^{10} \oplus V_{[11,21]}^{20} & \longrightarrow & V_{[11,21]}^{20} \oplus V_{[11,3]}^{10} \oplus V_{[1,31]}^{15}, \end{array}$$

we deduce that $\mathbf{HLogAR}_{\text{asym}}^2$ is an irreducible W -module isomorphic to $V_{[11,111]}^{10}$.

We proceed similarly regarding the symmetric weight 2 hyperlogarithmic ARs. By computations with GAP3 (see the Appendix), one gets

$$\begin{aligned} (\mathbf{H}_{\mathfrak{c}_1})^{\odot 2} &\simeq V_{[2,2]-}^3 \oplus V_{[.22]}^2 \oplus \mathbf{1} \\ \text{and } (\mathbf{H}_{\mathrm{dP}_4})^{\odot 2} &\simeq V_{[2,21]}^{20} \oplus V_{[1,22]}^{10} \oplus V_{[2,3]}^{10} \oplus V_{[.32]}^5 \oplus V_{[1,4]}^5 \oplus V_{[.41]}^4 \oplus \mathbf{1}. \end{aligned}$$

On the other hand, since

$$\begin{aligned} \mathrm{Ind}_{W_{\mathfrak{c}_1}}^W (V_{[2,2]-}^3) &= V_{[2,3]}^{10} \oplus V_{[2,21]}^{20} \\ \mathrm{Ind}_{W_{\mathfrak{c}_1}}^W (V_{[.22]}^2) &= V_{[.32]}^5 \oplus V_{[.221]}^5 \oplus V_{[1,22]}^{10} \\ \mathrm{Ind}_{W_{\mathfrak{c}_1}}^W (\mathbf{1}) &= V_{[1,4]}^5 \oplus V_{[.14]}^4 \oplus V_{[.5]}^1 \end{aligned} \tag{4.12}$$

(as it follows from Table A.3), one deduces that $\mathbf{HLogAR}_{\mathrm{sym}}^2 \simeq V_{[.221]}^5$ as a W -module.

The weight 3 abelian relation. In [46] (see also [12]), we proved that $\mathbf{HLog}^3$ can be identified with an explicit element $(\omega_{\mathfrak{c}})_{\mathfrak{c} \in \mathcal{K}}$ of $\bigoplus_{\mathfrak{c} \in \mathcal{K}} \wedge^3 \mathbf{H}_{\mathfrak{c}}$ which is W -stable and spans a 1-dimensional subrepresentation $\langle (\omega_{\mathfrak{c}})_{\mathfrak{c} \in \mathcal{K}} \rangle$ isomorphic to the signature representation $\mathbf{sign}_{D_5}$ of $W = W(D_5)$.

*

We gather all the results of the above considerations in the following

PROPOSITION 4.3. —

- (1) *The Weyl group $W(D_5)$ naturally acts on the spaces of hyperlogarithmic ARs*

$$\begin{aligned} &\mathbf{HLogAR}^1(\mathcal{W}_{\mathrm{dP}_4}), \quad \mathbf{HLogAR}_{\mathrm{sym}}^2(\mathcal{W}_{\mathrm{dP}_4}), \\ &\mathbf{HLogAR}_{\mathrm{asym}}^2(\mathcal{W}_{\mathrm{dP}_4}) \quad \text{and} \quad \mathbf{HLogAR}^3(\mathcal{W}_{\mathrm{dP}_4}). \end{aligned}$$

- (2) *As $W(D_5)$ -modules, the following isomorphisms hold true:*

$$\begin{aligned} [\text{weight } 1] \quad &\mathbf{HLogAR}^1 \simeq V_{[2,21]}^{20}; \\ [\text{weight } 2] \quad &\mathbf{HLogAR}_{\mathrm{asym}}^2 \simeq V_{[11,111]}^{10} \quad \text{and} \quad \mathbf{HLogAR}_{\mathrm{sym}}^2 \simeq V_{[.221]}^5; \\ [\text{weight } 3] \quad &\mathbf{HLogAR}^3 = \langle \mathbf{HLog}^3 \rangle \simeq V_{[.15]}^1 = \mathbf{sign}_{D_5}. \end{aligned}$$

In particular, all these subspaces of ARs are irreducible $W(D_5)$ -representations.

4.3. Subwebs and combinatorial characterization of $\mathcal{W}_{\text{dP}_4}$

The purpose of the present subsection is to give a description of the remarkable subwebs of $\mathcal{W}_{\text{dP}_4}$ and to deduce from this a combinatorial characterization for this web similar to those stated about Bol's web and Burau's one $\mathcal{W}_{\text{dP}_3}$ in Section 2.2.4.

Because our considerations below are invariant up to (local analytic) equivalence, we are going to work with a simple and quite concrete model for $\mathcal{W}_{\text{dP}_4}$, namely its birational model on the projective plane obtained as its direct image $b_*(\mathcal{W}_{\text{dP}_4})$ by the blow-up $b : \text{dP}_4 \rightarrow \mathbb{P}^2$ at five points in general position. We denote by $p_1, \dots, p_5$ these points and by P their set: $P = \{p_1, \dots, p_5\}$.

Then for $i = 1, \dots, 5$, we set

- $\mathcal{L}_i$ for the pencil of lines of the plane passing through p_i ; and
- $\mathcal{C}_i$ for the pencil of conics passing through all the p_k 's except p_i .

Then we have

$$b_*(\mathcal{W}_{\text{dP}_4}) = \mathcal{W}_P = \mathcal{W}(\mathcal{L}_1, \dots, \mathcal{L}_5, \mathcal{C}_1, \dots, \mathcal{C}_5)$$

and this latter web is the planar model for $\mathcal{W}_{\text{dP}_4}$ we are going to work with in this subsection.

To describe in an efficient way some subwebs of $\mathcal{W}_P$, it will be useful to use the following alternative notations for its foliations: for any $i = 1, \dots, 5$, we set

$$\mathcal{F}_i^+ = \mathcal{L}_i \quad \text{and} \quad \mathcal{F}_i^- = \mathcal{C}_i.$$

It will be also convenient to use the following notation: for $\underline{\epsilon} = (\epsilon_i)_{i=1}^5 \in \{\pm 1\}^5$, we will denote by $p(\underline{\epsilon}) \in \{0, \dots, 5\}$ the number of indices i such that $\epsilon_i = +1$; and I will stand for $\{1, \dots, 5\}$.

By direct (but lengthy) computations, it is not difficult to establish the following list of remarkable subwebs of $\mathcal{W}_P$:

- [Hexagonal 3-subwebs of $\mathcal{W}_P$]: the subwebs of $\mathcal{W}_P$ of this kind are exactly those of the form

$$\mathcal{W}(\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon, \mathcal{F}_k^\epsilon) \quad \text{and} \quad \mathcal{W}(\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon, \mathcal{F}_k^{-\epsilon})$$

where i, j, k stand for pairwise distinct elements of I and $\epsilon = \pm$. There are $2\binom{5}{3} = 20$ subwebs of the first type and $2\left(\binom{5}{2} \times 3\right) = 60$ subwebs of the second kind. Thus there are as many associated abelian relations $\text{LogAR}_{ijk}^\epsilon$ and $\text{LogAR}_{ijk}^{\epsilon,-}$. By elementary linear algebra, one gets that there are precisely 60 linear relations between

these 80 ARs, hence it follows that the length 3 logarithmic ARs of $\mathcal{W}_P$ span a subspace of $\mathbf{AR}^1(\mathcal{W}_P)$, of dimension 20. Since it is precisely the dimension of both $\mathbf{AR}^1(\mathcal{W}_{\mathrm{dP}_4})$ and $\mathbf{AR}^1(\mathcal{W}_P)$, these spaces coincide hence one has

$$\begin{aligned} \left\langle \mathrm{LogAR}_{ijk}^\epsilon, \mathrm{LogAR}_{ijk}^{\epsilon, -} \right\rangle &= \mathbf{AR}_C(\mathcal{W}_P) \\ &= \mathbf{HLogAR}^1(\mathcal{W}_P). \end{aligned}$$

The description above of the hexagonal 3-subwebs of $\mathcal{W}_P \simeq \mathcal{W}_{\mathrm{dP}_4}$ fully characterizes the map $r_{\mathcal{W}_P, 3} = r_{\mathcal{W}_{\mathrm{dP}_4}, 3}$.

- [Hexagonal 4-subwebs of $\mathcal{W}_P$]: the subwebs of $\mathcal{W}_P$ of this type are exactly those of the form

$$\mathcal{W}(\mathcal{F}_{i_1}^{\epsilon_1}, \dots, \mathcal{F}_{i_4}^{\epsilon_4})$$

with $1 \leq i_1 < \dots < i_4 \leq 5$ and $\underline{\epsilon} = (\epsilon_i)_{i=1}^4 \in \{\pm 1\}^4$.

- [Hexagonal 5-subwebs of $\mathcal{W}_P$]: the subwebs of this type are exactly those of the form

$$\mathcal{W}(\mathcal{F}_1^{\epsilon_1}, \dots, \mathcal{F}_5^{\epsilon_5})$$

with $\underline{\epsilon} = (\epsilon_i)_{i=1}^5 \in \{\pm 1\}^5$. The ones which are hexagonal and linearizable hence equivalent to five pencils of lines, exactly are the ones with $p(\underline{\epsilon})$ even, all the others (with $p(\underline{\epsilon})$ odd) being equivalent to Bol's web.

From above, one immediately deduces the following fact which we will use later: for any hexagonal 5-subweb $\mathcal{W}(\mathcal{F}_i^{\epsilon_i})_{i=1}^5$, the complementary subweb $\mathcal{W}(\mathcal{F}_i^{-\epsilon_i})_{i=1}^5$ is hexagonal too, but of the opposite type: if the former web is linearizable, then the latter is equivalent to Bol's web, and vice versa.

- Finally, it can be verified that $\mathcal{W}_P$ does not admit any hexagonal k -subweb for $k \geq 6$.

In view of giving a combinatorial characterization of $\mathcal{W}_{\mathrm{dP}_4}$, let us introduce the following terminology:

- given a smooth quartic del Pezzo surface dP_4 , we denote by $\langle \mathrm{dP}_4 \rangle$ the set of all unordered projective configurations $[\{p_1, \dots, p_5\}] \in \mathrm{Conf}_5(\mathbb{P}^2)/\mathfrak{S}_5$ such that $\mathrm{dP}_4 \simeq \mathbf{Bl}_{p_1, \dots, p_5}(\mathbb{P}^2)$. It is (classically?) known that $\langle \mathrm{dP}_4 \rangle$ actually is a singleton (for a recent reference, see (2) of [32, Thm. 2.1] for instance).⁽¹⁸⁾

⁽¹⁸⁾ We think it is worth mentioning that there is another way of obtaining $\langle \mathrm{dP}_4 \rangle$: the anticanonical image of dP_4 in $\mathbb{P}^4$ is cut out by a pencil of quadrics P_{dP_4} among which

- given a 10-web $\mathcal{W}$, one will say that $r_{\mathcal{W},3}$ is of del Pezzo's type if, possibly up to a relabelling of its foliations, one has $r_{\mathcal{W},3} = r_{\mathcal{W}_{\text{dP}_4},3}$.

THEOREM 4.4. —

- (1) Let $\mathcal{W}$ be a planar 10-web such that $r_{\mathcal{W},3}$ be of del Pezzo's type. Then there exists a quartic del Pezzo surface dP_4 such that $\mathcal{W} \simeq \mathcal{W}_{\text{dP}_4}$.
- (2) Given another smooth quartic del Pezzo surface dP'_4 , the following assertions are equivalent:
 - the two webs $\mathcal{W}_{\text{dP}_4}$ and $\mathcal{W}_{\text{dP}'_4}$ are analytically equivalent (as unordered webs);
 - the two projective configurations $\langle \text{dP}_4 \rangle$ and $\langle \text{dP}'_4 \rangle$ coincide: one has $\langle \text{dP}_4 \rangle = \langle \text{dP}'_4 \rangle$;
 - the two considered del Pezzo quartic surfaces are isomorphic: one has $\text{dP}_4 \simeq \text{dP}'_4$.

Actually, the following stronger result holds true: any local analytic equivalence between the del Pezzo webs $\mathcal{W}_{\text{dP}_4}$ and $\mathcal{W}_{\text{dP}'_4}$ is the restriction of a global isomorphism between dP_4 and dP'_4 .

Proof. — The proof is by a case analysis, our two main tools being first the fact that for any $d \geq 4$, a linearizable d -web admits only one linearization up to composition by projective transforms (see [42, Cor. 6.1.9]); and second, Bol's theorem giving a description of hexagonal webs (cf. page 1021 above).

Let $\mathcal{W} = (\mathcal{F}_1, \dots, \mathcal{F}_{10})$ be as in the first point of the statement. One can and will moreover assume that the two combinatorial functions $r_{\mathcal{W},3}$ and $r_{\mathcal{W}_{\text{dP}_4},3}$ coincide. In particular, the 5-subweb $\mathcal{H}_5 = (\mathcal{F}_1, \dots, \mathcal{F}_5)$ is hexagonal hence according to Bol's theorem, there are two possibilities: either $\mathcal{H}_5$ is equivalent to a web formed by 5 pencils of lines or $\mathcal{H}_5$ is equivalent to Bol's web.

Let us first assume that the second alternative occurs. Thus one can find and one fixes (a priori local) holomorphic coordinates x and y such that $\mathcal{H}_5 = \mathcal{W}(U_1, \dots, U_5)$ with $U_1 = x$, $U_2 = y$, $U_3 = x/y$, $U_4 = (y-1)/(x-1)$ and $U_5 = x(y-1)/(y(x-1))$. One denotes by $V_1, \dots, V_5$ five other first integrals such that $\mathcal{W} = \mathcal{W}(U_1, \dots, U_5, V_1, \dots, V_5)$ and with the corresponding function $r_{\mathcal{W},3}$ not only of del Pezzo's type but coinciding exactly with $r_{\mathcal{W}_{\text{dP}_4},3}$. Considering the description of the hexagonal 5-subwebs of $\mathcal{W}_P \simeq \mathcal{W}_{\text{dP}_4}$ given above, it follows that the subweb $\mathcal{W}(U_1, \dots, U_4, V_5) = \mathcal{W}(x, y, x/y, (y-1)/(x-1), V_5)$ is hexagonal too. Because $\mathcal{W}(x, y, x/y,$

exactly five are singular. The non-ordered projective configuration formed by these five singular quadrics as elements of $P_{\text{dP}_4} \simeq \mathbb{P}^1$ coincides with $\langle \text{dP}_4 \rangle$ (see [3, §5]).

$(y-1)/(x-1)$ is linear, the foliation defined by V_5 is necessarily a pencil of lines when considered in the coordinates x, y . Similarly, the subweb $\mathcal{W}(U_1, U_2, U_3, U_5, V_4) = \mathcal{W}(x, y, x/y, x(y-1)/y(x-1), V_4)$ is hexagonal. In the coordinates X, Y defined by $X = 1/x$ and $Y = 1/y$, its first four foliations become the pencils of lines whose some first integrals are the functions $X, Y, X/Y$ and $(X-1)/(Y-1)$. Since the foliations $\mathcal{F}_4$ and $\mathcal{F}_9$ defined by U_4 and V_4 are distinct, the latter is a pencil of lines in the coordinates X, Y , from which it follows that $\mathcal{F}_{V_4}$ is a pencil of conics when considered in the initial coordinates x, y . Arguing similarly, one gets that in these coordinates, one has

$$\mathcal{W}\left(x, y, \frac{x}{y}, \frac{y-1}{x-1}, \frac{x(y-1)}{y(x-1)}, V_1, \dots, V_5\right)$$

with $\mathcal{F}_{V_k}$ being a pencil of conics for $k = 1, \dots, 4$ and $\mathcal{F}_{V_5}$ a pencil of lines.

From the description of hexagonal 3-subwebs of $\mathcal{W}$ which corresponds to the one for $\mathcal{W}_P$ given above, it follows that for any triple of indices (i, j, k) such that $1 \leq i < j < k \leq 5$, the 3-subwebs $\mathcal{W}(U_i, U_j, V_k)$, $\mathcal{W}(U_i, V_j, V_k)$ and $\mathcal{W}(V_i, V_j, V_k)$ are hexagonal or equivalently, their Blaschke–Dubourdieu’s curvatures vanish identically. These vanishings give several PDEs which have to be satisfied by the V_i ’s which, in turn, give rise to many polynomial relations satisfied by the coefficients of the pencils of conics/lines that the V_i ’s are. It is straightforward to solve the corresponding polynomial system (using a computer algebra system) to get that the web under scrutiny is of the form

$$\mathcal{W}\left(x, y, \frac{x}{y}, \frac{y-1}{x-1}, \frac{x(y-1)}{y(x-1)}, \frac{(x-y)(y-\beta)}{(y-1)(\beta x - \alpha y)}, \frac{(x-y)(x-\alpha)}{(x-1)(\beta x - \alpha y)}, \frac{(y-1)(x-\alpha)}{(x-1)(y-\beta)}, \frac{y(x-\alpha)}{x(y-\beta)}, \frac{x-\alpha}{y-\beta}\right)$$

for some complex parameters α and β . In other terms, one has $\mathcal{W} = \mathcal{W}_P$ for $P = \{p_1, \dots, p_5\}$ with $p_1 = [1 : 0 : 0], \dots, p_4 = [1 : 1 : 1]$ and $p_5 = [\alpha : \beta : 1]$. This proves the first point of the theorem when the initial hexagonal 5-subweb considered is assumed not to be linearizable.

Let us now consider the case when $\mathcal{W}$ does not admit any 5-subweb equivalent to Bol’s web (otherwise the analysis just above would apply and give that $\mathcal{W} = \mathcal{W}_P$ for some set P of 5 points on $\mathbb{P}^2$). Then our initially considered hexagonal subweb $\mathcal{H}_5$ as well as its complement $\mathcal{H}_5^c$, which by definition is the 5-web formed by the foliations of $\mathcal{W}$ which are not in $\mathcal{H}_5$, both are equivalent to 5 pencils of lines. Arguing as above, one could endeavour to show that there exist affine coordinates with respect to which both $\mathcal{H}_5$ and $\mathcal{H}_5^c$ are formed by pencils of lines. But this is not possible: it

would imply that $\mathcal{W} = \mathcal{H}_5 \boxtimes \mathcal{H}_5^c$ is hexagonal or equivalently that one has $r_{\mathcal{W},3} \equiv 1$, contradicting the hypothesis $r_{\mathcal{W},3} = r_{\mathcal{W}_{\mathrm{dP}_4},3}$ since $r_{\mathcal{W}_{\mathrm{dP}_4},3} \not\equiv 1$.

Given a linearizable hexagonal 5-subweb $\mathcal{H}_5$ of $\mathcal{W}_{\mathrm{dP}_4}$, it admits a linear model formed by the pencils of lines through five points $q_{\mathcal{H}_5,1}, \dots, q_{\mathcal{H}_5,5}$ in general position in $\mathbb{P}^2$. It is not difficult to verify that the unordered configuration

$$\{q_{\mathcal{H}_5,i}\}_{i=1}^5 \in \mathrm{Conf}_5(\mathbb{P}^2)/\mathfrak{S}_5$$

coincides with $\langle \mathrm{dP}_4 \rangle$ which immediately implies the first part of (2). That any local analytic equivalence between $\mathcal{W}_{\mathrm{dP}_4}$ and $\mathcal{W}_{\mathrm{dP}'_4}$ is the restriction of a global isomorphism between dP_4 and dP'_4 follows from the fact that any linearizable 5-web admits only one linearization up to composition with a projective transformation (details are left to the reader). This ends the proof of the theorem. $\square$

4.4. Algebraization of $\mathcal{W}_{\mathrm{dP}_4}$

In this subsection we explain how the algebraization results of Section 3.5 generalize to the web $\mathcal{W}_{\mathrm{dP}_4}$.

4.4.1. Algebraization of $\mathcal{W}_{\mathrm{dP}_4}$ via its combinatorial abelian relations

We now specialize the material of Section 3.5.1 to the case of $\mathcal{W}_{\mathrm{dP}_4}$. By direct computations, one can determine a basis of $\mathbf{AR}_C(\mathcal{W}_{\mathrm{dP}_4})$ from which one can deduce that the logarithmic 1-forms h_i 's defined in (4.4) (with $i = 1, \dots, 10$) form a basis of $\mathbf{ar}_C(\mathcal{W}_{\mathrm{dP}_4})$. It is then straightforward, by direct computations again, to get the following

PROPOSITION 4.5. — *A 10-web $\mathcal{W}$ defined on a domain of $\mathbb{C}^2$ is equivalent to a del Pezzo's web $\mathcal{W}_{\mathrm{dP}_4}$ if and only if all the three following statements hold true:*

- (1) *one has $r = \dim(\mathbf{ar}_C(\mathcal{W}_{\mathrm{dP}_4})) = 10$;*
- (2) *the map $\Psi_{\mathcal{W}}$ has values in a linear subspace $\mathbb{P}W \subset \mathbb{P}^{\binom{10}{2}-1}$ of dimension 24 and $\mathrm{Im}(\Psi_{\mathcal{W}}) = \Psi_{\mathcal{W}}(U)$ is included in the image of the triple anticanonical embedding of a quartic del Pezzo surface dP_4 in $\mathbb{P}^{24}$: one has $\Psi_{\mathcal{W}}(U) \subset \mathrm{Im}(\Phi_{|-3K_{\mathrm{dP}_4}|}) \subset \mathbb{P}^{24}$;*

(3) *the two push-forward webs*

$$(\Psi_{\mathcal{W}})_*(\mathcal{W}) \quad \text{and} \quad (\Phi_{|-3K_{\mathrm{dP}_4}|})_*(\mathcal{W}_{\mathrm{dP}_4})$$

coincide.

Rough sketch of the proof. — It suffices to verify that the three points of the Proposition indeed hold true when the 10-web under scrutiny is equivalent to $\mathcal{W}_{\mathrm{dP}_4}$. For such a web, we take the web $\mathcal{W}$ on $\mathbb{C}^2 \setminus A_{\mathcal{L}}$ defined by the 10 rational functions in (4.3). Then it is just a matter of computation to verify that the logarithmic 1-forms h_i defined in (4.4) (with $i = 1, \dots, 10$) form a basis of $\mathbf{ar}_C(\mathcal{W})$, from which one gets that the map $\Psi_{\mathcal{W}}$ is rational and induced by the linear system $\mathcal{L}$ of plane curves of degree 9 having a point of multiplicity 3 at each of the points p_i (cf. 4.1). The Proposition follows from the fact that $\mathcal{L}$ precisely is the push-forward of $|-3K_{\mathrm{dP}_4}|$ by the blow-up map $b : \mathrm{dP}_4 \rightarrow \mathbb{P}^2$ at the p_i 's. $\square$

4.4.2. Algebraization of $\mathcal{W}_{\mathrm{dP}_4}$ via a canonical map

In order to generalize Proposition 3.7 to $\mathcal{W}_{\mathrm{dP}_4}$, the most direct attempt would be to consider the associated full canonical map $\Phi_{\mathcal{W}_{\mathrm{dP}_4}}$ which takes values into the 7-dimensional moduli space $\mathcal{M}_{10}$. But it can be verified that the pull-back of the 210-web $\mathcal{W}_{\mathcal{M}_{0,10}}$ under $\Phi_{\mathcal{W}_{\mathrm{dP}_4}}$ is a 80-web, hence is a web formed by a relatively important number of foliations. Therefore, it is possibly not the most convenient web to consider in view of stating a criterion characterizing $\mathcal{W}_{\mathrm{dP}_4}$ as a 10-web.

Instead of dealing with $\Phi_{\mathcal{W}_{\mathrm{dP}_4}}$, we are going to consider the canonical map $\Phi_{\mathcal{H}_5}$ associated to a fixed (but otherwise arbitrary) hexagonal 5-subweb $\mathcal{H}_5$ of $\mathcal{W}_{\mathrm{dP}_4}$. By elementary (but lengthy) computations, one verifies that $\mathcal{W}_{\mathrm{dP}_4}$ is the 10-web obtained by taking the juxtaposition of $\mathcal{H}_5$ with the pull-back of $\mathcal{W}_{\mathcal{M}_{0,5}}$ under $\Phi_{\mathcal{H}_5}$: one has

$$\mathcal{W}_{\mathrm{dP}_4} = \mathcal{H}_5 \boxtimes \Phi_{\mathcal{H}_5}^*(\mathcal{W}_{\mathcal{M}_{0,5}}).$$

This immediately gives us the following result which has to be seen as an invariant criterion characterizing del Pezzo's web $\mathcal{W}_{\mathrm{dP}_4}$:

PROPOSITION 4.6. — *Let $\mathcal{W}$ be a planar 10-web. The three following properties are equivalent:*

- (1) *the web $\mathcal{W}$ is equivalent to a del Pezzo's web $\mathcal{W}_{\mathrm{dP}_4}$;*
- (2) *there exists a hexagonal 5-subweb $\mathcal{H}_5 \subset \mathcal{W}$ such that*

$$\mathcal{W} = \mathcal{H}_5 \boxtimes \Phi_{\mathcal{H}_5}^*(\mathcal{W}_{\mathcal{M}_{0,5}}); \tag{4.13}$$

- (3) the web $\mathcal{W}$ admits a hexagonal 5-subweb and (4.13) holds true for any such subweb $\mathcal{H}_5$.

Proof. — As mentioned before, that (1) $\Rightarrow$ (3) can be verified by straightforward computations. Since (3) $\Rightarrow$ (1) is obvious, it suffices to prove that (2) $\Rightarrow$ (1). But this follows from the following two facts, which can be established by elementary computations as well:

- (1) a hexagonal 5-web $\mathcal{H}$ is such that $\mathcal{H} \boxtimes \Phi_{\mathcal{H}}^*(\mathcal{W}_{\mathcal{M}_{0,5}})$ is a 10-web if and only if there exists a local biholomorphism φ such that $\mathbf{H} = \varphi_*(\mathcal{H})$ is the linear web formed by five pencils of lines whose vertices $v_1, \dots, v_5$ are five points in general position in $\mathbb{P}^2$;
- (2) in this case $\mathbf{H} \boxtimes \Phi_{\mathbf{H}}^*(\mathcal{W}_{\mathcal{M}_{0,5}})$ is the 10-web formed by $\mathbf{H}$ together with the five pencils of conics passing through all the v_i 's except one, that is $\mathbf{H} \boxtimes \Phi_{\mathbf{H}}^*(\mathcal{W}_{\mathcal{M}_{0,5}})$ is the direct image by the blow-up map of the del Pezzo's web on the del Pezzo quartic surface $\mathrm{dP}_4 = \mathbf{Bl}_{v_1, \dots, v_5}(\mathbb{P}^2)$. $\square$

As in the case of Bol's web (cf. the end of Section 3.5.2), an effective criterion can be deduced from the above proposition for a web $\mathcal{W}(U_1, \dots, U_{10})$ to be equivalent to a del Pezzo web $\mathcal{W}_{\mathrm{dP}_4}$ (the explicit statement of this criterion being a bit involved, it is left to the interested reader).

To end this section, we mention that several preliminary experiments led us to realize that one can obtain many webs carrying interesting ARs (in particular exceptional webs) under the form

$$\mathcal{H}_k \boxtimes \Phi_{\mathcal{H}_k}^*(\mathcal{W}_{\mathcal{M}_{0,k}})$$

where $\mathcal{H}_k$ is ranging the space of planar hexagonal k -webs with $k \geq 4$. See Section 5.12 at the end where some interesting cases are considered. We believe that there is much more to be understood regarding the notion of canonical map $\Phi_{\mathcal{W}}$ of a web $\mathcal{W}$, especially when the latter is hexagonal. We hope to investigate this further in a future work.

4.5. Geometric construction of $\mathcal{W}_{\mathrm{dP}_4}$ à la Gelfand–MacPherson

This subsection is devoted to establishing that there is a geometric construction of $\mathcal{W}_{\mathrm{dP}_4}$ which is similar to Gelfand–MacPherson's one of Bol's web $\mathcal{B} \simeq \mathcal{W}_{\mathrm{dP}_5}$ recalled above in Section 3.6. However the case of $\mathcal{W}_{\mathrm{dP}_4}$ is more involved than the latter one and we need to introduce some material and recall some results about it first.

In the first subsection below, building on some work by Gelfand and some of its collaborators, for any simple Lie group G , we sketch a general notion of Gelfand–MacPherson web on any rational homogeneous space G/P equivariantly embedded in the projectivization $\mathbb{P}V_\omega$ of an irreducible G -representation V_ω . Since we will only consider a specific case in what follows, we are very sketchy and do not give many details (nor proofs).

In the next subsection, after having recalled some results about the embedding of the Cox variety of del Pezzo surfaces into certain homogeneous spaces, we explain how the web $\mathcal{W}_{\mathrm{dP}_4}$ can be obtained from Gelfand–MacPherson’s web on the tenfold spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$.

4.5.1. Webs associated to moment polytopes of projective homogeneous spaces

The material presented below is taken from [28]. The case of type A (which corresponds to that of usual grassmannian manifolds) has been considered before by Gelfand and MacPherson in [27]. We will also use freely some results of [2] and [60, §3]. The presentation below is quite succinct, more details and proofs will be provided in the paper in preparation [43].

Let D be a connected Dynkin diagram of rank r and $G = G(D)$ a simple Lie group of type D . We fix a Cartan torus H with Lie algebra $\mathfrak{h}$ and set of roots $\Pi \subset \mathfrak{h}_{\mathbb{R}}^*$ and choose a subset of positive roots Π^+ which in turn defines the associated subset of simple roots $\Phi = \{\alpha_1, \dots, \alpha_r\} \subset \Pi^+$ which is naturally in bijection with the nodes of D . We denote by $\omega_1, \dots, \omega_r$ the fundamental weights of the considered root system. We fix a standard parabolic subgroup $P \subset G$ and we set $I(P)$ for the subset of $\{1, \dots, r\}$ whose elements are the indices i such that α_i is also a root of P . Then $\omega = \omega(P) = \sum_{i \in I(P)} \omega_i$ is a dominant weight and we denote by $V = V_\omega$ the G -module such that the representation $G \rightarrow \mathrm{GL}(V_\omega)$ is irreducible with highest weight ω . One has $\mathrm{Stab}_G(\mathbb{C}e_\omega) = P$ for a highest weight vector $e_\omega \in V_\omega$, hence we get a G -equivariant embedding $\rho = \rho_\omega : G/P \hookrightarrow \mathbb{P}V_\omega$. The image $X = \rho(G/P)$ is a G -homogeneous projective variety which is the projectivization of the G -orbit of the highest-weight vector e_ω . Let $\mathfrak{W}$ be the set of weights in V , their multiplicities being taken into account, and let $(e_w)_{w \in \mathfrak{W}}$ be a weight basis of V . Then for $x \in G/P$, there exists a $\mathfrak{W}$ -tuple of complex numbers $(p^w(x))_{w \in \mathfrak{W}} \in \mathbb{C}^{\mathfrak{W}}$, unique up to a non-zero rescaling, such that

$$\rho(x) = \left[\sum_{w \in \mathfrak{W}} p^w(x) e_w \right] \in \mathbb{P}V$$

where $[\cdot] : V \setminus \{0\} \rightarrow \mathbb{P}V$ stands for the projectivization map. The $p^w(x)$'s for $w \in \mathfrak{W}$ are called the “*generalized Plücker coordinates*” of x .

Let $W = N_G(H)/H$ be the associated Weyl group, which acts orthogonally on $\mathfrak{h}_{\mathbb{R}}^*$ leaving invariant the considered root lattice. The W -orbit $W \cdot \omega$ of the highest weight is included in $\mathfrak{W}$ and the points of $W \cdot \omega$ are the vertices of the so-called *weight or moment polytope*

$$\Delta = \Delta_{D,\omega} \subset \mathfrak{h}_{\mathbb{R}}^* \simeq \mathbb{R}^r.$$

The elements of $\mathfrak{W} \setminus W \cdot \omega$ all lie inside Δ . In the two cases we are interested in in this paper, the Weyl group W acts transitively on $\mathfrak{W}$, that is the representation V_{ω} is “*minuscule*”. From now on, essentially to simplify the exposition below, we assume that V_{ω} is of this kind. This implies in particular that the considered highest weight $\omega = \omega(P)$ is one of the fundamental weights ω_i hence corresponds to one of the nodes of the Dynkin diagram D . Equivalently, P is one of the r maximal standard parabolic subgroups of G . To exclude trivial situations, we also require that the dimension of H is strictly less than the one of G/P , this in order that quotienting the latter space by the action of H gives rise to a positive dimensional quotient (in some sense to be made precise). This amounts to leaving aside the minuscule pairs of type (A_n, ω_1) and (A_n, ω_n) for which one has $X = G/P \simeq \mathbb{P}^n$ which is such that X/H is a point.

By definition, the associated *moment map* $\mu = \mu_X = \mu_{D,\omega}$ is the map

$$\begin{aligned} \mu : X &\longrightarrow \mathfrak{h}_{\mathbb{R}}^*, \\ x &\longmapsto \frac{\sum_{w \in \mathfrak{W}} |p^w(x)|^2 w}{\sum_{w \in \mathfrak{W}} |p^w(x)|^2}. \end{aligned}$$

It is a real-analytic map which satisfies nice convexity properties. Let $H_{>0} \simeq (\mathbb{R}_{>0})^r$ stand for the connected component of the identity of the split real form $H(\mathbb{R})$ of the Cartan torus H . Then the following hold:

- one has $\mu(X) = \Delta$;
- $X^* = \mu^{-1}(\mathring{\Delta})$ is a Zariski-open subset of X on which the moment map is regular (a real-analytic submersion of rank r at any $x^* \in X^*$);
- for any $x^* \in X^*$, μ induces a real-analytic isomorphism between the real orbit $H_{>0} \cdot x^*$ and the interior $\mathring{\Delta}$ of the weight polytope, which extends to an isomorphism of real analytic varieties with corners to the closures: $\mu : \overline{H_{>0} \cdot x^*} \xrightarrow{\sim} \Delta$;
- for $\sigma \in N_G(H)$, let $[\sigma]$ be the corresponding element in $N(H)/H \simeq W$. Then the vertex $[\sigma] \cdot \omega$ of Δ is the image by μ of the point of X corresponding to the coset $\sigma \cdot P$ viewed as a point of G/P

- the H -action on X^* is sufficiently nice for $\mathcal{Y}^* = X^*/H$ to be a smooth quasi-projective manifold with the quotient map $\chi_H : X^* \rightarrow \mathcal{Y}^*$ being a H -torsor. Moreover, the quotient $\mathcal{Y}^*$ is rational.⁽¹⁹⁾

Remark 4.7. — The quotient $\mathcal{Y}^* = X^*/H$ is not the most natural one to be considered from the point of view of geometric invariant theory. Indeed, in [53], the authors show that in some cases (which include in particular the “minuscule cases” we are interested in), the set X^s of H -stable points of X contains all the $x \in X$ for which there exists at most one weight w such that $p^w(x) = 0$. Setting $X^{sf} = X^s \cap X^f$ where X^f stands for the Zariski open subset of points $x \in X$ such that $\text{Stab}_H(x) = Z(G)$, one has $\text{codim}(X \setminus X^{sf}) \geq 2$ and X^* is strictly contained in X^{sf} . The quotient $\mathcal{Y} = X^{sf}/H$ is a smooth quasi projective variety containing $\mathcal{Y}^*$ as a Zariski open subset and enjoying nice properties⁽²⁰⁾ which are a priori nicer than those satisfied by $\mathcal{Y}^*$. Depending on what one is aiming for, it might be more interesting to work with $\mathcal{Y}$ instead of its dense open subset $\mathcal{Y}^*$, but it will not be the case in this text.

The main objects we are going to work with are the (closed) faces of the weight polytope Δ , especially its “*facets*” that is its faces of codimension 1. We now recall some facts/results of [60, §3] which will be relevant regarding our purpose. The Weyl group W acts on the set $\mathfrak{F}(\Delta)$ of faces of Δ . Given such a face F , let W_F be the subgroup of $\text{Stab}_W(F)$ generated by the transformations $w \in W$ leaving F invariant and such that the mirror hyperplane $\ker(w) \subset \mathfrak{h}_{\mathbb{R}}^*$ passes through the center of mass m_F of F but does not contain the whole face (equivalently: m_F belongs to $\ker(w)$ which is orthogonal to F). Then W_F acts on the affine span $\langle F \rangle$ with m_F as unique fixed point. Making of $\langle F \rangle$ a vector space by taking m_F as origin, one obtains that $\Pi_F = \Pi \cap \langle F \rangle$ is a root system of rank $\dim(F)$ and W_F identifies as its Weyl group. Moreover, W_F acts transitively on the set of vertices of F .

We will be essentially interested in the facets of the weight polytope Δ . These faces are weight polytopes of rank $r-1$ whose types can be described as follows: let $C \subset \mathfrak{h}_{\mathbb{R}}^*$ be the dominant Weyl chamber associated to the chosen set of simple roots Φ . Given a facet F , there exists $w \in W$ such that wF is a “*dominant facet*” (in Vinberg’s terminology), that is a 1-codimensional face of Δ such that its intersection with C has dimension $r-1$. Hence in order to describe the different facets of Δ one can restrict to study the dominant ones.

⁽¹⁹⁾ This has been explained to us by N. Perrin.

⁽²⁰⁾ For instance: the Weyl group W embeds into the automorphisms group of $\mathcal{Y}$ (see [56, Thm. 2.2]) and the Picard lattice of $\mathcal{Y}$ is naturally isomorphic to that of a del Pezzo surface obtained as the blow-up of $\mathbb{P}^2$ at r points in general position, cf. [53, p. 418].

One proceeds as follows to get a complete list of the Dynkin types of the dominant facets of Δ : let D_ω be the Dynkin diagram D with the vertex corresponding to ω in black, all the other vertices being white.⁽²¹⁾ Then the marked Dynkin diagrams associated to the dominant facets are those, denoted by $D_{\omega, \omega'}$ obtained by removing an extremal vertex $\omega' \neq \omega$ from D_ω (see the figure below for the case relevant to describing $\mathcal{W}_{\text{dP}_4}$ à la Gelfand–MacPherson further on). Given an arbitrary facet F , let $w \in W$ be such that wF be dominant. One sets $D_F = D_{\omega, \omega'}$ where the latter is the rank $r - 1$ Dynkin diagram with a blacked vertex associated to wF , according to the procedure described just above and one denotes by ω_F the marked vertex of D_F . One verifies that the pair (D_F, ω_F) is well-defined, that is does not depend on the considered element w of the Weyl group such that wF be dominant.

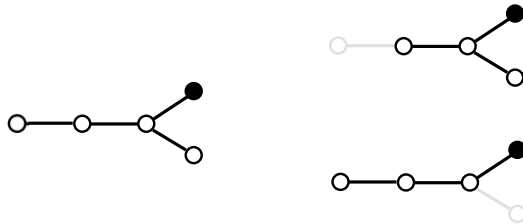


Figure 4.1. The marked Dynkin diagram of type (D_5, ω_4) on the left and the two possible types of rank four 1-marked Dynkin diagrams for the facets on the right: of type (D_4, ω_3) (up) and (A_4, ω_4) (bottom).

We are now going to associate geometric objects (homogeneous varieties, rational maps, etc) to the faces of Δ . Actually, since one is mainly interested in facets in this paper, we will restrict to this case below, even if most things could be stated in much more generality.

Given a 1-codimensional face F of Δ , always assumed to be topologically closed, we set/define:

- $V_F = \bigoplus_{w \in F \cap \mathfrak{W}} \mathbb{C}e_w$ and $V^F = \bigoplus_{w \in \mathfrak{W} \setminus F} \mathbb{C}e_w$ so that $V = V_F \oplus V^F$;
- $\Pi_F : V = V_F \oplus V^F \rightarrow V_F$ is the linear projection with kernel V^F ;
- For Γ one of the groups G , P or H , one sets
$$\Gamma^F = \text{Stab}_\Gamma(V_F \oplus V^F) = \{ \gamma \in \Gamma \mid \gamma(V_F) = V_F \text{ and } \gamma(V^F) = V^F \}.$$

⁽²¹⁾ Since the representation under consideration is minuscule, the corresponding dominant weight ω is fundamental hence naturally identifies to one of the nodes of the Dynkin diagram.

Note that since H is the Cartan torus of G and because $\mathfrak{W}$ is a weight basis, one has $H^F = H$;

- $K^F = \{g \in G^F \mid \exists \lambda \in \mathbb{C}^* \text{ s.t. } g|_{V_F} = \lambda \text{Id}_{V_F}\}$. There is a sequence of inclusions of subgroups $K^F \subset P^F \subset G^F$;
- $\Gamma_F = \Gamma^F / (K^F \cap \Gamma^F)$ for Γ being G , P or H . More explicitly, one has $G_F = G^F / K^F$, $H_F = H / (H \cap K^F)$ and $P_F = P^F / K^F$;
- $X_F = X \cap \mathbb{P}V_F$ and $X_F^* = \mu^{-1}(\mathring{F})$ where $\mathring{F}$ stands for the relative interior of F .

With these notations, one can state the

PROPOSITION 4.8. —

- (1) *The group G_F is a simple complex Lie group of type D_F , H_F is a Cartan subgroup and P_F is the minimal standard parabolic subgroup of G_F associated to the fundamental root ω_F .*
- (2) *As a G_F -representation, V_F is irreducible and is the irrep of highest weight ω_F .*
- (3) *The variety X_F is homogeneous under the natural action of G^F and this action factorizes through the quotient map $G^F \rightarrow G_F$. Moreover, one has*

$$X_F \simeq G^F / P^F = G_F / P_F.$$

- (4) *There is a natural affine isomorphism between the affine span of F in $\mathfrak{h}_{\mathbb{R}}^*$ and the dual of the real Lie algebra of the torus H_F such that the restriction of the moment map on X_F identifies with that on X_F associated to the triple (G_F, P_F, V_F) , i.e. one has $\mu|_{X_F} = \mu_{D_F, \omega_F}$. In particular, F identifies with the moment polytope Δ_{D_F, ω_F} of X_F .*

Both the projectivization $\mathbb{P}V \dashrightarrow \mathbb{P}V_F$ of the linear projection Π_F and its restriction on $X = G/P \subset \mathbb{P}V$ will be again denoted the same in what follows.

PROPOSITION 4.9. —

- (1) *The image of the projection $\Pi_F : X \dashrightarrow \mathbb{P}V_F$ coincides with X_F . In short:*

$$X_F \stackrel{\text{def}}{=} X \cap \mathbb{P}V_F = \Pi_F(X) = G_F / P_F.$$

Moreover, in terms of the moment map, one has

$$X_F = \mu^{-1}(F) = \overline{X_F^*}.$$

- (2) *The induced dominant rational map $\Pi_F : X \dashrightarrow X_F$ is (G^F, G_F) -equivariant.*

- (3) *The map Π_F is defined at any point of X^* , one has $\Pi_F(X^*) = X_F^*$ and the induced surjective morphism $\Pi_F : X^* \rightarrow X_F^*$ is (H, H_F) -equivariant.*

We postpone the proofs of the two above propositions to the paper to come [43]. Here we make only the following general commentaries: most of the points of the proposition 4.8 follow quite directly from the results of [60, §3], the one requiring the most effort being the third point.

As for Proposition 4.9, if F is a facet of the weight polytope, its affine span in $\mathfrak{h}_{\mathbb{R}}^*$ admits an equation of the form $\varphi_F = \alpha_F$ where φ_F is a non zero linear form and α_F a certain real number. Because the dual of $\mathfrak{h}_{\mathbb{R}}^*$ identifies with $\mathfrak{h}_{\mathbb{R}}$, there exists $\zeta_F \in \mathfrak{h}_{\mathbb{R}}$ such that $\varphi_F(\cdot) = (\zeta_F, \cdot)$ as linear forms on $\mathfrak{h}_{\mathbb{R}}^*$. Since $\zeta_F \neq 0$, there is a one-parameter subgroup $h_F : \mathbb{C}^* \rightarrow H, t \mapsto \exp(t\zeta_F)$ associated to it. Then most of Proposition 4.9 follows from Białynicki-Birula theory for the $\mathbb{C}^*$ -action on X provided by h_F . For instance, X_F is an irreducible component of the subvariety X^{h_F} of X formed by its h_F -invariant points, one has $\Pi_F(x) = \lim_{t \rightarrow 0} h_F(t) \cdot x$ for any $x \in X^*$, etc. More details will be given in a forthcoming text.

From the last point of the preceding proposition one immediately gets the

COROLLARY 4.10. — *For any facet F of the moment polytope, there exists a well-defined surjective morphism $\pi_F : \mathcal{Y}^* \rightarrow \mathcal{Y}_F^* = X_F^*/H_F$ such that the following diagram commutes:*

$$\begin{array}{ccc} X^* & \xrightarrow{\Pi_F} & X_F^* \\ \chi_H \downarrow & & \downarrow \chi_{H_F} \\ \mathcal{Y}^* & \xrightarrow{\pi_F} & \mathcal{Y}_F^* \end{array}$$

Remark 4.11. — The face maps Π_F and π_F considered above are specific cases of more general mappings associated with projective varieties acted upon by a torus already considered in the literature, e.g. in [29] or in [9].

The map $\pi_F : \mathcal{Y}^* \rightarrow \mathcal{Y}_F^*$ is constant hence defines a trivial foliation on $\mathcal{Y}^*$ if and only if $\mathcal{Y}_F^*$ reduces to a point. Such a case is not interesting for our purpose and a facet of this kind will be said “**W**-irrelevant”. Accordingly, facets not of this type will be said to be “**W**-relevant”.

Example 4.12. — The case of the pair (A_4, ω_2) is considered in depth in [27] (see also Section 4.5 above). In this case, one has $X = G/P = G_2(\mathbb{C}^5)$ and the image $\mu(X) \subset \mathbb{R}^5$ of the associated moment map is the hypersimplex $\Delta_{5,2}$ formed by the 5-tuples of elements in $[0, 1]$ summing up to 2: one has $\Delta_{5,2} = \{(t_i)_{i=1}^5 \in [0, 1]^5 \mid \sum_{i=1}^5 t_i = 2\}$. This polytope has 10 facets, which

are the intersections $\Delta_{5,2}(i, \epsilon) = \Delta_{5,2} \cap \{t_i = \epsilon\}$ for $i = 1, \dots, 5$ and $\epsilon = 0, 1$. For any i , the linear projection from $\mathbb{R}^5$ to $\mathbb{R}^4$ given by disregarding the i th coordinate induces an isomorphism from the facet $\Delta_{5,2}(i, 1)$ onto a 3-simplex in $\mathbb{R}^4$, which reflects the fact that $X_{\Delta_{5,2}(i,1)} \simeq \mathbb{P}^3$. Moreover the action of $H_{\Delta_{5,2}(i,1)} \simeq (\mathbb{C}^*)^3$ on it identifies with the usual toric action hence it follows that $\mathcal{Y}_{\Delta_{5,2}(i,1)}^*$ is a point. This shows that the facets $\Delta_{5,2}(i, 1)$ ($i = 1, \dots, 5$) all are $\mathcal{W}$ -irrelevant. Each facet $\Delta_{5,2}(i, 0)$ is isomorphic to the hypersimplex $\Delta_{4,2} = \{(\tau_j)_{j=1}^4 \in [0, 1]^4 \mid \sum_{j=1}^4 \tau_j = 2\}$ which is the moment polytope of the $\mathrm{PGL}_4(\mathbb{C})$ -homogeneous variety $X_{\Delta_{5,2}(i,0)}$ which is isomorphic to the grassmannian $G_2(\mathbb{C}^4)$. One has $\mathcal{Y}^* = G_2(\mathbb{C}^5)^*/H_4 \simeq \mathcal{M}_{0,5}$ and for each facet $\Delta_{5,2}(i, 0)$, $\mathcal{Y}_{\Delta_{5,2}(i,0)}^* = G_2(\mathbb{C}^4)^*/H_3 \simeq \mathcal{M}_{0,4} \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and up to the above identifications, the map $\pi_{\Delta_{5,2}(i,0)} : \mathcal{Y}^* \rightarrow \mathcal{Y}_{\Delta_{5,2}(i,0)}^*$ of Corollary 4.10 corresponds to the i th forgetful map $\mathcal{M}_{0,5} \rightarrow \mathcal{M}_{0,4}$. In particular, all five facets $\Delta_{5,2}(i, 0)$ are $\mathcal{W}$ -relevant.

From the preceding example, it follows that Gelfand–MacPherson’s web on $G_2(\mathbb{C}^5)$ (resp. its equivariant quotient by the H_4 -action) previously considered in Section 3.6.1 is the web on $G_2(\mathbb{C}^5)^*$ (resp. on $\mathcal{Y}^* \simeq \mathcal{M}_{0,5}$) whose first integrals exactly are the maps $\Pi_F : G_2(\mathbb{C}^5)^* \rightarrow G_2(\mathbb{C}^4)^*$ (resp. π_F) for F ranging in the set of $\mathcal{W}$ -relevant facets of the associated moment polytope. This suggests the following definition (in which we continue to use the notations introduced above and where we denote by $\mathfrak{F}_{\mathcal{W}}(\Delta)$ the set of $\mathcal{W}$ -relevant facets of the considered moment polytope Δ):

DEFINITION 4.13. — *Given a minuscule pair (D, ω) as above, the “Gelfand–MacPherson web” associated to it, denoted by $\mathcal{W}_X^{GM}$, is the “web” on X^* induced by the facet maps $\Pi_F : X^* \rightarrow X_F^*$ for all $\mathcal{W}$ -relevant facets F : one has*

$$\mathcal{W}_X^{GM} = \mathcal{W}(\Pi_F \mid \forall F \in \mathfrak{F}_{\mathcal{W}}(\Delta)).$$

This web is equivariant under the action of H hence descends to the quotient $\mathcal{Y}^$ and gives rise to a web denoted by $\mathcal{W}_{\mathcal{Y}}^{GM}$, which will be also called the “Gelfand–MacPherson web” associated to (D, ω) (but on $\mathcal{Y}^*$): one has*

$$\mathcal{W}_{\mathcal{Y}}^{GM} = (\mathcal{W}_X^{GM})/_H = \mathcal{W}(\pi_F \mid \forall F \in \mathfrak{F}_{\mathcal{W}}(\Delta)). \quad (4.14)$$

A remark is in order about the use of the term “web” in this definition: it does not refer to the classical notion of web but to the more general one introduced in [45] under the name of “*generalized web*”, which is just a collection of (possibly singular and/or of different codimensions) foliations on a given space, with the unique requirement that two distinct foliations intersect transversally.

Example 4.14. —

- (1) As it follows from Example 4.12 above, for the pair (A_4, ω_2) , one recovers Bol’s web since in this case, one has

$$\mathcal{W}_{\mathcal{Y}}^{GM} = \mathcal{W}_{\mathcal{M}_{0,5}} \simeq \mathcal{W}_{\text{dP}_5} \simeq \mathcal{B}.$$

- (2) From [27], one gets a simple general description of any Gelfand–MacPherson web associated to a pair (A_n, ω_k) with k such that $1 < k \leq n/2$ (which corresponds to $X = G/P = G_k(\mathbb{C}^{n+1})$). Assume that both n and g are greater than or equal to 3 (this just in order to give a simple uniform description for the associated Gelfand–MacPherson web). Then $\mathcal{Y}^*$ can be identified with a Zariski open subset of $\text{Conf}_{n+k}(\mathbb{P}^{k-1})$, that is the space of projective equivalence classes of $(n+k)$ -tuples of points in general position in $\mathbb{P}^{k-1}$. The facet maps all are $\mathcal{W}$ -relevant in this case and can be described as follows as rational maps defined on the birational model $\text{Conf}_{n+k}(\mathbb{P}^{k-1})$ of $\mathcal{Y}^*$: there are $2(n+k)$ facets hence as many facet maps, the half of which are the forgetful maps $\text{Conf}_{n+k}(\mathbb{P}^{k-1}) \rightarrow \text{Conf}_{n+k-1}(\mathbb{P}^{k-1})$, the $n+k$ others being the maps $\text{Conf}_{n+k}(\mathbb{P}^{k-1}) \rightarrow \text{Conf}_{n+k-1}(\mathbb{P}^{k-2})$ induced by the linear projection from a point of the configuration. The forgetful maps and those induced by projections from a point define foliations of dimension $k-1$ and $n-k$ respectively. Hence except when $n = 2k-1$, the Gelfand–MacPherson web as defined above is a $2(n+k)$ -web of mixed codimensions.

The case which we will prove to be the relevant one for describing del Pezzo’s web $\mathcal{W}_{\text{dP}_4}$ à la Gelfand–MacPherson is the one associated to the pair (D_5, ω_4) or equivalently, to the spinor tenfold $\mathbb{S}_5$. In this case, $\mathcal{Y}^*$ is 5-dimensional and the Gelfand–MacPherson web $\mathcal{W}_{\mathcal{Y}}^{GM}$ is a 10-web of codimension 2. In the next subsection, we describe quite explicitly this web (cf. Proposition 4.24) and then just after, we recall some results about the Cox variety of a del Pezzo surface and explain how one can deduce from them a way to geometrically construct $\mathcal{W}_{\text{dP}_4}$ from Gelfand–MacPherson’s web $\mathcal{W}_{\mathbb{S}_5}^{GM}$ (see Proposition 4.30).

4.5.2. The GM-web of the spinor tenfold $\mathbb{S}_5$

Our goal here is to specialize the theory sketched above in Section 4.5.1 in the case to be used for describing $\mathcal{W}_{\text{dP}_4}$, namely the one associated to the pair (D_5, ω_4) . In this case $G = \text{Spin}_{10}(\mathbb{C})$ and $X = G/P$ is the spinor

tenfold $\mathbb{S}_5$ and our ultimate goal here is to describe as explicitly as possible the associated Gelfand–MacPherson webs.

We start by stating some general facts about spinor varieties, especially the spinor tenfold we are interested in. Everything here is classical hence we do not give any proof. Among several recent references on spinor varieties, we mention [37] and [16].

Notation and generalities. All what is discussed in this preliminary paragraph actually holds true for all spinor varieties $\mathbb{S}_n$ with $n \geq 5$. We start below by stating everything in the general case $n \geq 5$, with the exception of some details that we make explicit in the case $n = 5$ which is the one relevant for our purpose.

For any $n \geq 5$, we will use the following notations:

- one sets $i^* = i + n$ for $i = 1, \dots, n$;
- we set $I = \{1, \dots, n\}$, $I^* = \{1^*, \dots, n^*\}$ and $J = I \sqcup I^*$;
- setting $(i^*)^* = i$ for $i \in I$, the map $j \mapsto j^*$ defines an involution of J exchanging I and I^* ;
- we consider a complex vector space E of dimension $2n$, with a basis $(e_j)_{j \in J}$ indexed by J . By means of this basis, we identify E with $\mathbb{C}^{2n}$;
- for $1 \leq i_1 < \dots < i_k \leq n$, we write $e_{i_1 \dots i_k}$ for $e_{i_1} \wedge \dots \wedge e_{i_k}$;
- we denote by x_i, x_{i^*} for $i = 1, \dots, n$ the linear coordinates on E dual to the e_i, e_{i^*} ;
- we endow/equip E with the quadratic form $Q(x) = \sum_{i=1}^n x_i x_{i^*}$;
- a n -dimensional subspace $U \subset E$ is totally isotropic if Q vanishes identically on U ;
- we set $V = \bigoplus_{i=1}^n \mathbb{C} e_i$ and $W = \bigoplus_{i=1}^n \mathbb{C} e_{i^*}$. Both are totally isotropic n -planes. Moreover each of them is the dual of the other (where the duality is the one induced on E by Q);
- the orthogonal grassmannian $OG_n(\mathbb{C}^{2n})$ is the subset of $G_n(\mathbb{C}^{2n})$ formed by the totally isotropic n -dimensional subspaces of $\mathbb{C}^{2n}$;
- $\mathrm{SO}(\mathbb{C}^{2n}, Q)$ (or just $\mathrm{SO}(\mathbb{C}^{2n})$ to simplify) denotes the algebraic subgroup of $\mathrm{SL}(\mathbb{C}^{2n})$ formed by the linear transformations of E with determinant 1 and letting Q invariant. The fundamental group of $\mathrm{SO}(\mathbb{C}^{2n})$ is $\mathbb{Z}/2\mathbb{Z}$ and its universal covering is the (complex) *spin group* $\mathrm{Spin}_{2n} = \mathrm{Spin}(\mathbb{C}^{2n})$;

- for $i = 1, \dots, 5$, we set
 - $\pi_i : E \rightarrow E/\langle e_i \rangle$ and $\pi_{i^*} : E \rightarrow E/\langle e_{i^*} \rangle$;
 - $H_i = \{x_i = 0\} = (\bigoplus_{j \neq i} \mathbb{C}e_j) \oplus W$ and $H_{i^*} = \{x_{i^*} = 0\} = V \oplus (\bigoplus_{j^* \neq i^*} \mathbb{C}e_{j^*})$;

Spinor varieties, half-spin representations and the Wick embeddings. Let $\xi = [A, B]$ be a $n \times 2n$ matrix by blocks, with $A, B \in M_n(\mathbb{C})$ such that $\text{rk}(\xi) = 2n$. Then one denotes by $\langle \xi \rangle \in G_n(\mathbb{C}^{2n})$ the subspace of $\mathbb{C}^{2n}$ spanned by the n row vectors of ξ . When A (resp. B) is invertible, one obviously has $\langle [A, B] \rangle = \langle [\text{Id}_n, A^{-1}B] \rangle$ (resp. $\langle [A, B] \rangle = \langle [B^{-1}A, \text{Id}_n] \rangle$). As is well-known, $\langle \xi \rangle = \langle [A, B] \rangle$ is a totally isotropic subspace hence belongs to the orthogonal grassmannian $OG_n(\mathbb{C}^{2n})$ if and only if the matrix $A^t B$ is antisymmetric.

Another well-known fact is that the orthogonal grassmannian has two isomorphic disjoint connected components:

$$OG_n(\mathbb{C}^{2n}) = OG_n^+(\mathbb{C}^{2n}) \sqcup OG_n^-(\mathbb{C}^{2n}).$$

Let $\xi_1 = [A_1, B_1]$ and $\xi_2 = [A_2, B_2]$ be two $n \times 2n$ -matrices defining two totally isotropic n -planes in $\mathbb{C}^{2n}$. These two n -planes belong to the same connected component $OG_n^\pm(\mathbb{C}^{2n})$ of $OG_n(\mathbb{C}^{2n})$ if and only if $\dim(\langle \xi_1 \rangle + \langle \xi_2 \rangle) = \text{rk}(\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix})$ has the same parity as n .

There is a decomposition in direct sum $\wedge^n E = \wedge^n E^+ \oplus \wedge^n E^-$ such that for $\varepsilon = \pm$, one has $\mathbb{P}\wedge^n E^\varepsilon = \langle OG_n^\varepsilon(E) \rangle$ and $OG_n^\varepsilon(E) = G_n(E) \cap \mathbb{P}\wedge^n E^\varepsilon$. Both embeddings $OG_n^\varepsilon(E) \subset \mathbb{P}\wedge^n E^\varepsilon$ are (non canonically) isomorphic. Each is not minimal but is given by the second quadratic Veronese map of a minimal embedding of $OG_n^\pm(E)$ into the projectivization of an irreducible representation $S^\pm = S_n^\pm$ of dimension 2^{n-1} of the spin group Spin_{2n} , called the “*half-spin representation*”. The corresponding variety in $\mathbb{P}S^\pm$ is denoted by $\mathbb{S}_n^\pm$ and is called the n -th spinor variety. It is the Spin_{2n} -orbit of a highest weight vector in $S^\pm$ hence one has $\mathbb{S}^\pm = \text{Spin}_{2n}/P$ for a certain parabolic subgroup P . The group Spin_{2n} is a simple complex Lie group of type D_n and the parabolic subgroup P just mentioned is the maximal one associated to the $(n-1)$ -th node of the associated Dynkin diagram (in black in the figure below):

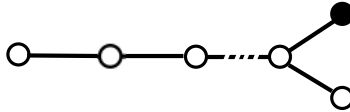


Figure 4.2. The marked Dynkin diagram associated to the spinor variety $\mathbb{S}_n^+ = \text{Spin}_{2n}/P_{n-1}$.

The two half-spin representations $S^\pm$ are abstractly indistinguishable from each other but for conveniency we will use the notations of [37] by identifying $S^\pm$ with $\wedge^{\pm\varrho}V$ where $\varrho = \varrho(n) \in \{\text{odd}, \text{even}\}$ stands for the parity of n (and $-\varrho$ for the opposite) and where

$$\wedge^{\text{even}}V = \bigoplus_{k=0}^{\lfloor n/2 \rfloor} \wedge^{2k}V \quad \text{and} \quad \wedge^{\text{odd}}V = \bigoplus_{k=0}^{\lfloor n/2 \rfloor} \wedge^{2k+1}V.$$

It is interesting to describe a classical affine parametrization of (a Zariski open subset of) $\mathbb{S}^+$ in terms of antisymmetric matrices. We denote by $\text{Asym}_n(\mathbb{C})$ the vector space of antisymmetric matrices of size n . Given a subset $K \subset \llbracket n \rrbracket$ and a matrix $M = (m_{ij})_{i,j=1}^n \in \text{Asym}_n(\mathbb{C})$, we set

- (1) $\ell = \ell(K) = \text{Card}(K) \in \{0, \dots, n\}$;
- (2) $k_1, \dots, k_\ell$ for the elements of K labeled by increasing order: $1 \leq k_1 < \dots < k_\ell \leq n$;
- (3) $e_K = e_{k_1} \wedge \dots \wedge e_{k_\ell} \in \wedge^\ell V$;
- (4) ${}^cK = \llbracket n \rrbracket \setminus K$;
- (5) M_K for the antisymmetric submatrix $(m_{k_u k_v})_{u,v=1}^\ell \in \text{Asym}_\ell(\mathbb{C})$;
- (6) $\text{Pf}(M)$ (resp. $\text{Pf}_K(M)$) for the Pfaffian of M (resp. of M_K).

Then the “*Wick parametrization*” of $\mathbb{S}_n^+$ is the morphism defined as the projectivisation of the affine embedding

$$\begin{aligned} W_n^+ : \text{Asym}_n(\mathbb{C}) &\longmapsto S_n^+ = \wedge^\varrho V \\ M &\longmapsto \sum_K \text{Pf}(M_K) e_{K^c} \end{aligned} \tag{4.15}$$

where in the sum K ranges in the set of subsets of $\llbracket n \rrbracket$ of even cardinality. We will denote the Wick parametrization $\text{Asym}_n(\mathbb{C}) \rightarrow \mathbb{P}S_n^+$ by W_n^+ as well. It is known that W_n^+ embeds $\text{Asym}_n(\mathbb{C}) \simeq \mathbb{C}^{n(n-1)/2}$ into $\mathbb{P}S_n^+$ and parametrizes a Zariski open subset of $\mathbb{S}_n^+$. One verifies that with the notation introduced above, the totally isotropic n -plane $W_n(A)$ associated to a given antisymmetric matrix $A \in \text{Asym}_n(\mathbb{C})$ is represented by the $n \times 2n$ -matrix $[\text{Id}_n, A]$.

Because this case is relevant for our purpose, let us make the fifth Wick embedding W_5^+ more explicit. We set $e_{klm} = e_k \wedge e_l \wedge e_m \in \wedge^3 V$ for k, l, m such that $1 \leq k < l < m \leq 5$, $e_{1\dots 5} = e_1 \wedge e_2 \wedge \dots \wedge e_5 \in \wedge^5 V \simeq \mathbb{C}$ and $\Sigma_5 = \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i < j \leq 5\}$. For $x = (x_{ij})_{(i,j) \in \Sigma_5} \in \mathbb{C}^{\Sigma_5}$, we denote by $A(x)$ the antisymmetric matrix whose (i, j) -th coefficient is x_{ij} for any

$(i, j) \in \Sigma_5$. Then we have

$$\begin{aligned} \mathbb{C}^{\Sigma_5} \simeq \text{Asym}_5(\mathbb{C}) &\longrightarrow \mathbb{P}(\wedge^{\text{odd}} V) \simeq \mathbb{P}^{15} \\ (x_{ij})_{(i,j) \in \Sigma_5} = x &\longmapsto A(x) \longmapsto e_{1\dots 5} + \sum_{1 \leq i < j \leq 5} x_{ij} e_{klm} \\ &+ \sum_{i=1, \dots, 5} \text{Pf}(A_i) e_i, \end{aligned} \quad (4.16)$$

where for $(i, j) \in \Sigma_5$, (k, l, m) stands for the triple of increasing integers such that $\{i, j, k, l, m\} = \{1, \dots, 5\}$ and where for any i , A_i stands for the 4×4 antisymmetric matrix obtained by removing the i th row and the i th column to $A(x)$.

The Cartan torus and the weights. The Cartan torus H we are dealing with is the rank n torus formed of diagonal matrices

$$D_h = \begin{bmatrix} h & 0 \\ 0 & h^{-1} \end{bmatrix} \in \text{SO}(\mathbb{C}^{2n}) \quad (4.17)$$

with $h = \text{Diag}(h_1, \dots, h_n) \in \text{GL}_{2n}(\mathbb{C})$. The action of $D(h)$ on the n -plane associated to $\xi = [A, B]$ (with A, B such that $A^t B \in \text{Asym}_n(\mathbb{C})$) is given by $\langle \xi \rangle \cdot D_h = \langle [A, B] \cdot D_h \rangle$ with $[A, B] \cdot D_h = [Ah, Bh^{-1}]$. In the case when $A = \text{Id}_n$ (with B antisymmetric), one has

$$\langle [\text{Id}_n, B] \rangle \cdot D_h = \langle [\text{Id}_n, h^{-1} B h^{-1}] \rangle.$$

For any i , we set $\nu_i \in \mathfrak{h}_{\mathbb{R}}^*$ for the map associating h_i to D_h with $h = \text{Diag}(h_1, \dots, h_n)$. Then the ν_i 's form a basis of $\mathfrak{h}_{\mathbb{R}}^*$ allowing to identify it with $\mathbb{R}^n$.

From the property of the pfaffian that $\text{Pf}(MCM^t) = \det(M) \text{Pf}(C)$ for any antisymmetric matrix C and any arbitrary square matrix M (both of the same size), one deduces that for any $D_h \in H$, any $A \in \text{Asym}_n(\mathbb{C})$ and any $K \subset \llbracket n \rrbracket$ of even cardinality, one has

$$\text{Pf}_K(h^{-1} A h^{-1}) = \text{Pf}_K(A) / h^K \quad (4.18)$$

with $h^K = \prod_{l=1}^{\ell} h_{\kappa_l}$ if $K = \{\kappa_1, \dots, \kappa_{\ell}\}$. It follows that for any such subset K , the line $\mathbb{C} e_{K^c} \subset \wedge^{\ell} V$ is stable under the action of the Cartan torus, hence $\{e_{K^c}\}$ for K ranging in the subsets of $\llbracket n \rrbracket$ with even cardinal not only forms a vector basis of $\wedge^{\ell} V = S^+$, but actually is a weight basis of it as a Spin_{2n} -representation. For any such K , let $w(K)$ be the corresponding weight of e_{K^c} . It is an element of $\mathfrak{h}_{\mathbb{R}}^* = \bigoplus_{i=1}^n \mathbb{R} \nu_i \simeq \mathbb{R}^n$ which can be determined explicitly in a very down to earth and elementary way from (4.18). Combined with the fact that $e_{1\dots n} = e_1 \wedge \dots \wedge e_n$ is known as being a vector with the highest

weight which is $w(\emptyset) = \frac{1}{2} \sum_{i=1}^n \gamma_i \in \mathfrak{h}_{\mathbb{R}}^*$, it follows that for any K with even cardinal, one has

$$w(K) = w(\emptyset) - \sum_{k \in K} \gamma_k = \frac{1}{2} \sum_{l \in K^c} \gamma_l - \frac{1}{2} \sum_{k \in K} \gamma_k.$$

Up to the identification $\mathfrak{h}_{\mathbb{R}}^* \simeq \mathbb{R}^n$ provided by the basis $(\gamma_i)_{i=1}^n$, we obtain the following explicit description of the set $\mathfrak{W}_n^+$ of weights of the half-spin representation S_n^+ :

$$\mathfrak{W}_n^+ = \left\{ \frac{1}{2} (\epsilon_i)_{i=1}^n \in \{\pm 1\}^n \mid \epsilon_1 \cdots \epsilon_n = 1 \right\}$$

and for any $w = (w_1, \dots, w_n)$ in this set, an associated weight vector is given by $e_w = \bigwedge_{\{k | w_k > 0\}} e_k$. In this case, $(\frac{1}{2}, \dots, \frac{1}{2})$ is the highest weight and $e_{1\dots n}$ is a highest weight vector.

Since it is the case relevant for our purpose in this text, the suitable subsets K together with the associated weights $w(K)$, the weight vector e_{K^c} as well as the corresponding coordinates $p_K(x)$ in the Wick embedding (with $x \in \mathbb{C}^{\Sigma_5}$) are given in the following table (where the following conventions are used: the subsets K are written as even tuples of increasing integers in $\llbracket 5 \rrbracket$ and i, j, k, l, m stand for the elements of $\llbracket 5 \rrbracket$ with $i < j$ and $k < l < m$):

Table 4.1. The weight vectors, the associated Wick coordinates and the corresponding weights.

K	$\emptyset$	(i, j)	(j, k, l, m)
e_{K^c}	$e_{1\dots 5}$	e_{klm}	e_i
$p_K(x)$	1	x_{ij}	$\text{Pf}(A_i)$
$w(K)$	$\frac{1}{2}(\nu_1 + \dots + \nu_5)$	$\frac{1}{2}(\nu_k + \nu_l + \nu_m - \nu_i - \nu_j)$	$\frac{1}{2}(\nu_i - \nu_j - \nu_k - \nu_l - \nu_m)$

We will use a similar description of the weights and of the associated weight vectors for the other half-spin representation S_n^+ : the set of weights is

$$\mathfrak{W}_n^- = \left\{ \frac{1}{2} (\epsilon_i)_{i=1}^n \mid \forall i : \epsilon_i \in \{\pm 1\}, \epsilon_1 \cdots \epsilon_n = -1 \right\}$$

and for any $w = (w_1, \dots, w_n)$ in this set, an associated weight vector is given by $e_w = \bigwedge_{\{k | w_k > 0\}} e_k$. In this case, $(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$ is the highest weight and $e_1 \wedge \dots \wedge e_{n-1}$ is a highest weight vector.

Isomorphisms. Here we describe some isomorphisms between the half-spin representations under consideration which we will use further on. In what follows, ε stands for $+$ or $-$.

For $i, j \in \llbracket n \rrbracket$ distinct, let g_{ij} be the linear involution of E defined, in the considered basis $(e_1, \dots, e_n, e_{1^*}, \dots, e_{n^*})$ by exchanging e_i and e_j as well as e_{i^*} and e_{j^*} and letting all the others e_k and e_{k^*} unchanged. One has $g_{ij} \in \mathrm{SO}(E)$ hence interior conjugation by g_{ij} gives rise to another representation that we will denote by $S_{n,ij}^\varepsilon$ to differentiate it from the initial S_n^ε . The underlying vector spaces are the same, namely $\wedge^{\varepsilon\varrho} V$ and the isomorphism $\wedge^{\varepsilon\varrho} V \xrightarrow{\sim} \wedge^{\varepsilon\varrho} V$, again denoted by g_{ij} , is the natural map $e_L \mapsto e_{(ij)L}$ where $(ij)L$ stands for image of the subset $L \subset \llbracket n \rrbracket$ by the transposition (ij) . If one denotes by $\mathbb{S}_{n,ij}^\varepsilon$ the projectivisation of the closed orbit in $\wedge^{\varepsilon\varrho} V$ when this space is viewed as the representation $S_{n,ij}^\varepsilon$, then the restriction of the map $g_{ij} \in \mathrm{GL}(\wedge^{\varepsilon\varrho} V)$ gives rise to an isomorphism from $\mathbb{S}_n^\varepsilon$ onto $\mathbb{S}_{n,ij}^\varepsilon$. We remark that the line spanned by a highest weight vector in S_n^ε is left invariant by g_{ij} if and only if $\varepsilon = +$ of $\varepsilon = -$ and $n \notin \{i, j\}$. Consequently in these cases, and in these cases only, one has $\mathbb{S}_{n,ij}^\varepsilon = \mathbb{S}_n^\varepsilon$ as subvarieties of $\mathbb{P}(\wedge^{\varepsilon\varrho} V)$.

The two half-spin representations S_n^+ and S_n^- are isomorphic but not in a canonical way. We will need below to consider some isomorphisms between them that we are going to describe now.

We fix $i \in \llbracket n \rrbracket$ and for any subset $L \subset \llbracket n \rrbracket$, we set

$$L^i = \begin{cases} L \setminus \{i\} & \text{if } i \in L; \\ L \cup \{i\} & \text{if } i \notin L. \end{cases}$$

The map $L \mapsto L^i$ induces a bijection from the set of subsets of $\llbracket n \rrbracket$ whose cardinal has the same parity as n onto the one of subsets with the opposite parity. Then associating e_{L^i} to e_L gives rise to a linear isomorphism

$$\Gamma_i : \wedge^{\pm\varrho} V \xrightarrow{\sim} \wedge^{\mp\varrho} V \quad (4.19)$$

which can be shown to induce an isomorphism of representations from $S_n^\pm$ onto $S_n^\mp$. The induced linear isomorphism of $\mathfrak{h}_{\mathbb{R}}^*$, denoted by $\bar{\Gamma}_i$, is characterized by the relations $\bar{\Gamma}_i(\nu_j) = \nu_j$ if $j \neq i$ and $\bar{\Gamma}_i(\nu_i) = -\nu_i$. The corresponding bijection from $\mathfrak{W}_n^\pm$ onto $\mathfrak{W}_n^\mp$ simply involves replacing the i -th coordinate of the weight by its opposite, leaving all other coordinates unchanged.

Wick parametrizations of $\mathbb{S}_n^-$. If the Wick parametrization (4.15) is a privileged birational parametrization of the spinor variety $\mathbb{S}_n^+$, we do not have anything similar at disposal regarding the other spinor manifold $\mathbb{S}_n^-$.

However one gets several parametrizations of (a Zariski open subset of) $\mathbb{S}_n^-$ by composing W_n^+ with a projective isomorphism G from S_n^+ onto S_n^- . For such a G , one can for instance take a composition $\Gamma_n \circ g_{ij}$ for any distinct $i, j \in \llbracket n \rrbracket$. Any such map $\mathrm{Asym}_n(\mathbb{C}) \rightarrow \mathbb{S}_n^-$ will be called a “Wick parametrization” of $\mathbb{S}_n^-$. This generalizes in a straightforward way to any spinor avatars $\mathbb{S}_{n,ij}^-$.

The weight polytope and its facets. For $\varepsilon \in \{+, -\}$, the set of weights of S_n^ε is the set $\mathfrak{W}_n^\varepsilon$ of n -tuples $\frac{1}{2}(\epsilon_1, \dots, \epsilon_n) \in \mathfrak{h}^* \simeq \mathbb{R}^n$ with $\epsilon_i \in \{\pm 1\}$ for all indices $i \in \llbracket n \rrbracket$ and such that $\epsilon_1 \cdots \epsilon_n = \varepsilon 1$ (i.e. the number of ϵ_i 's equal to -1 is even if $\varepsilon = +$ and is odd when $\varepsilon = -$).

It is known that the moment polytope Δ_n^ε of $S_n^\varepsilon \subset \mathbb{P}S_n^\varepsilon$ coincides with the weight polytope: one has $\Delta_n^\varepsilon = \text{Conv}(\mathfrak{W}_n^\varepsilon) \subset \mathfrak{h}_\mathbb{R}^*$. From the description of $\mathfrak{W}_n^\varepsilon$ recalled just above, it follows that this polytope is a half-hypercube of dimension n whose combinatorial properties are well-known. In particular, its facets are known: these are 2^{n-1} hypersimplices and $2n$ are half-hypercubes. The former facets are easily seen to be of type (A_{n-1}, ω_{n-1}) (see Figure 4.1) hence all are $\mathcal{W}$ -irrelevant. For this reason, we will not consider them further. On the other hand, the $2n$ half-hypercubic facets all are $\mathcal{W}$ -relevant and are easy to describe: they come in pairs $(\Delta_{n,i}^{\varepsilon,+}, \Delta_{n,i}^{\varepsilon,-})$ for $i = 1, \dots, n$, with

$$\Delta_{n,i}^{\varepsilon,\pm} = \Delta_n^\varepsilon \cap \{\lambda_i = \pm 1/2\} = \text{Conv}(\mathfrak{W}_{n,i}^{\varepsilon,\pm}) \quad (4.20)$$

where $\mathfrak{W}_{n,i}^{\varepsilon,\pm}$ stands for the subset of $\mathfrak{W}_n^\varepsilon$ formed by its elements whose i -th coordinates are $\pm \frac{1}{2}$. Obviously, if $p_i : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ stands for the linear projection consisting in forgetting the i -th coordinate, one has

$$p_i(\mathfrak{W}_{n,i}^{\varepsilon,\pm}) = \mathfrak{W}_{n-1}^{\pm\varepsilon} \quad \text{and} \quad p_i(\Delta_{n,i}^{\varepsilon,\pm}) = \Delta_{n-1}^{\pm\varepsilon}.$$

The face maps. Our goal here is to describe the face maps associated to the $2n$ $\mathcal{W}$ -relevant hypercubic facets of Δ_n^+ . We fix $i \in \llbracket n \rrbracket$ in what follows.

For $\varepsilon = \pm$, we denote by $S_{n,i}^{+,\varepsilon}$ the subspace of S_n^+ spanned by the weight vectors with weight in $\mathfrak{W}_{n,i}^{+,\varepsilon}$: $S_{n,i}^{+,+}$ (resp. $S_{n,i}^{+,-}$) admits as a basis the set of weight vectors e_L for all $L \subset \llbracket n \rrbracket$ with $L^c = \llbracket n \rrbracket \setminus L$ of even cardinal and with $i \in L$ (resp. with $i \notin L$): one has

$$S_{n,i}^{+,+} = \bigoplus_{\substack{L \subset \llbracket n \rrbracket, i \in L \\ \#L^c \text{ even}}} \mathbb{C}e_L \quad \text{and} \quad S_{n,i}^{+,-} = \bigoplus_{\substack{L \subset \llbracket n \rrbracket, i \notin L \\ \#L^c \text{ even}}} \mathbb{C}e_L.$$

The subgroup $\text{Spin}_{2n,i}$ of Spin_{2n} which lets invariant the decomposition in direct sum

$$S_n^+ = S_{n,i}^{+,+} \oplus S_{n,i}^{+,-} \quad (4.21)$$

is naturally isomorphic to $\text{Spin}_{2(n-1)}$. Moreover, up to the identification $\text{Spin}_{2n,i} \simeq \text{Spin}_{2(n-1)}$, $S_{n,i}^{+,+}$ and $S_{n,i}^{+,-}$ identify to the two half-spin representations S_{n-1}^+ and S_{n-1}^- respectively: this follows easily from the well-known fact that for any $n \geq 5$, one has $S_n^+ \downarrow_{\text{Spin}_{2(n-1)}} = S_{n-1}^+ \oplus S_{n-1}^-$ (cf. [18, Prop. 5.1] for instance).

For $\varepsilon \in \{-, +\}$, one denotes by $\Pi_{n,i}^\varepsilon$ the linear projection from S_n^+ onto $S_{n,i}^{+, \varepsilon}$ relatively to the decomposition in direct sum (4.21):

$$\begin{aligned} \Pi_{n,i}^\varepsilon : S_n^+ &= S_{n,i}^{+,+} \oplus S_{n,i}^{+,-} \longrightarrow S_{n,i}^{+, \varepsilon} \\ (x^+, x^-) &\longmapsto x^\varepsilon. \end{aligned}$$

We will use the same notation for the linear projection onto the corresponding projective spaces $\mathbb{P}S_n^+ \simeq \mathbb{P}^{2^n-1}$ and $\mathbb{P}S_{n,i}^{+, \varepsilon} \simeq \mathbb{P}^{2^{n-2}-1}$.

In this setting, Proposition 4.9 takes the following form:

PROPOSITION 4.15. — *Up to the natural identification $S_{n,i}^{+, \varepsilon} \simeq S_{n-1}^\varepsilon$ mentioned above:*

- *the intersection $\mathbb{S}_{n,i}^\varepsilon \stackrel{\text{def}}{=} S_n^+ \cap \mathbb{P}S_{n,i}^{+, \varepsilon}$ identifies with $\mathbb{S}_{n-1}^\varepsilon$;*
- *by restriction, the projection $\Pi_{n,i}^\varepsilon$ induces a dominant rational map $\Phi_{n,i}^\varepsilon : \mathbb{S}_n^+ \dashrightarrow \mathbb{S}_{n-1}^\varepsilon$.*

It is easy to give an explicit expression for the map $\Phi_{n,i}^+ : \mathbb{S}_n^+ \dashrightarrow \mathbb{S}_{n-1}^+$ in terms of “Wick coordinates”. Let $\iota : \llbracket n \rrbracket \setminus \{i\} \xrightarrow{\sim} \llbracket n-1 \rrbracket$ stand for the bijection preserving the usual orders of the source and of the target.⁽²²⁾ Given a subset $K \subset \llbracket n \rrbracket$ of even cardinal and such that $i \notin K$, $\iota(K)$ is a subset of $\llbracket n-1 \rrbracket \setminus \{i\}$ with even cardinal too. Hence the map which associates $e_{\iota(K)^c}$ to e_{K^c} for any such K gives rise to a linear isomorphism of vector spaces $\iota : S_{n,i}^{+,+} \rightarrow S_{n-1}^+$ whose projectivisation $\mathbb{P}S_{n,i}^{+,+} \rightarrow \mathbb{P}S_{n-1}^+$ will be denoted the same. On the other hand, the components of the composition

$$\Pi_{n,i}^\varepsilon \circ W_n^+ : \text{Asym}_n(\mathbb{C}) \longrightarrow \mathbb{P}S_{n,i}^{+,+}$$

are immediately seen to be the pfaffians $\text{Pf}(A_K)$ for all subsets $K \subset \llbracket n \rrbracket \setminus \{i\}$ with even cardinal. It is then easy to get the following result giving a nice expression in birational coordinates for $\Phi_{n,i}^+$:

PROPOSITION 4.16. — *The following diagram of rational maps is commutative*

$$\begin{array}{ccccc} \text{Asym}_n(\mathbb{C}) & \xhookrightarrow{W_n^+} & \mathbb{S}_n^+ & \xhookrightarrow{\quad} & \mathbb{P}S_n^+ \xrightarrow{\Pi_{n,i}^+} \mathbb{P}S_{n,i}^{+,+} \\ \downarrow \tilde{\Phi}_{n,i}^+ & & \downarrow \Phi_{n,i}^+ & & \downarrow \iota \\ \text{Asym}_{n-1}(\mathbb{C}) & \xhookrightarrow{W_{n-1}^+} & \mathbb{S}_{n-1}^+ & \xhookrightarrow{\quad} & \mathbb{P}S_{n-1}^+ \end{array}$$

⁽²²⁾ More explicitly, for any $j \in \llbracket n \rrbracket \setminus \{i\}$, one has $\iota(j) = j$ if $j < i$ and $\iota(j) = j-1$ if $i < j$.

where $\tilde{\Phi}_{n,i}^+$ stands for the map associating to any antisymmetric matrix $A \in \text{Asym}_n(\mathbb{C})$ the submatrix $A_i \in \text{Asym}_{n-1}(\mathbb{C})$ obtained by removing from it its i -th line and i -th column.

Even if we will not use this, it is interesting to give a more geometric description of the map $\Phi_{n,i}^+ : \mathbb{S}_n^+ \dashrightarrow \mathbb{S}_{n-1}^+$. To this aim, we need to identify $\mathbb{S}_n^+$ with one of the components of the orthogonal grassmannian $OG_n(\mathbb{C}^{2n})$, which we are going to do as follows: always assuming that $V = \bigoplus_{i=1}^n \mathbb{C}e_i$ is in the component of $OG_n(\mathbb{C}^{2n})$ identified with $\mathbb{S}_n^+$, we see that the latter identifies with the set of totally isotrope subspaces $U \in OG_n(\mathbb{C}^{2n})$ such that the parity of the codimension in $\mathbb{C}^{2n}$ of the space spanned by U and V is the same as that of n :

$$\mathbb{S}_n^+ \simeq \{U \in OG_n(\mathbb{C}^{2n}) \mid n + \text{codim}(\langle U, V \rangle) \text{ is even}\}.$$

PROPOSITION 4.17. — *Up to the identifications above for $\mathbb{S}_n^+$ and $\mathbb{S}_{n-1}^+$, the face map $\Phi_{n,i}^+ : \mathbb{S}_n^+ \dashrightarrow \mathbb{S}_{n-1}^+$ is the rational application associating to any generic $\Pi \in \mathbb{S}_n^+$ the totally isotropic $(n-1)$ -plane $\pi_{i^*}(\Pi \cap H_i) = (\Pi \cap H_i) / \langle e_{i^*} \rangle$ of $E_{i,i^*} = H_i / \langle e_{i^*} \rangle$.*

We now turn to the description of the face maps associated to the facets $\Delta_{n,i}^{+,-}$ of Δ_n^+ . Rigorously speaking, such a map is a rational map

$$\Phi_{n,i}^- : \mathbb{S}_n^+ \dashrightarrow \mathbb{S}_{n-1}^-.$$

The preceding geometric description of the maps $\Phi_{n,i}^+ : \mathbb{S}_n^+ \dashrightarrow \mathbb{S}_{n-1}^+$ associated to the facets $\Delta_{n,i}^{+,+}$ has the peculiarity of suggesting the corresponding statement for the other facets we are now considering.

PROPOSITION 4.18. — *The face map $\Phi_{n,i}^- : \mathbb{S}_n^+ \dashrightarrow \mathbb{S}_{n-1}^-$ is the rational application associating to any generic $\Pi \in \mathbb{S}_n^+$ the totally isotropic $(n-1)$ -plane $\pi_i(\Pi \cap H_{i^*}) = (\Pi \cap H_{i^*}) / \langle e_i \rangle$ of $E_{i^*,i} = H_{i^*} / \langle e_i \rangle$.*

Although the geometric descriptions of the face maps $\Phi_{n,i}^+$ and $\Phi_{n,i}^-$ given above are completely similar, the description of the maps $\Phi_{n,i}^-$ in “Wick coordinates” is not as simple as the one for the maps $\Phi_{n,i}^+$ given in Proposition 4.16, the main reason behind this being that this requires an identification between $\mathbb{S}_{n-1}^-$ and $\mathbb{S}_{n-1}^+$ and there is no canonical such identification. Actually, given i , for any $j \in \llbracket n-1 \rrbracket \setminus \{i\}$, using the identification (4.19) (but with V replaced by V_i), we can construct a birational model $\tilde{\Phi}_{n,i}^{-,j} : \text{Asym}_n(\mathbb{C}) \dashrightarrow \text{Asym}_{n-1}(\mathbb{C})$ of $\Phi_{n,i}^-$. However, since all these birational models are obtained in the same way and to avoid writing down too many formulas, we only treat below the case of $\tilde{\Phi}_{n,n}^{-,n-1}$ (ie. $i = n$ and

$j = n - 1$) that we make entirely explicit in Wick coordinates. The corresponding description of any other $\Phi_{n,i}^{-,j}$, for any distinct i and j , will follow immediately from that (see Corollary 4.20 below where all the $\Phi_{5,i}^{-,j}$ are explicitly given).

We are going to use the following notations and conventions:

- $x = (x_{ij})_{1 \leq i < j \leq n}$ stands for an element of $\mathbb{C}^{\Sigma_n}$ and $A = A(x)$ denotes the antisymmetric matrix whose (i, j) -th coefficient is x_{ij} for any $(i, j) \in \Sigma_n$;
- ϱ stands for the parity of n ;
- one sets $V_k^+ = \bigoplus_{i=1, i \neq k}^n \mathbb{C}e_i$ for any $k = 1, \dots, n$;

The inclusion $\llbracket n-1 \rrbracket \subset \llbracket n \rrbracket$ gives rise to a canonical isomorphism $S_{n,n}^{+,-} \simeq S_{n-1}^-$ that we will denote by Ξ_n . Then for $j \in \llbracket n-1 \rrbracket$, one gets a linear isomorphism $\Gamma_j \circ \theta_n : S_n^+ \rightarrow S_{n-1}^+$. Setting $\Lambda_{n,j} = \Gamma_j \circ \theta_n \circ \Pi_{n,n}^- : S_n^+ \rightarrow S_{n-1}^+$ and considering the composition $\Lambda_{n,j} \circ W_n^+ : \text{Asym}_n(\mathbb{C}) \rightarrow S_{n-1}^+$, we get an affine map which gives rise to an affine model for $\Phi_{n,n}^- \circ W_n^+ : \text{Asym}_n(\mathbb{C}) \rightarrow S_{n-1}^-$ but takes values into S_{n-1}^+ . Explicit formulas of the maps involved in the previous considerations are made explicit in the diagram page 1096.

We consider now that $j = n - 1$ and we denote by $\Psi_n = \Lambda_{n,j} \circ W_n^+$ the composition of the linear maps above. Our goal is to find a map $\text{Asym}_n(\mathbb{C}) \dashrightarrow \text{Asym}_{n-1}(\mathbb{C})$, $A \mapsto \tilde{A}$ such that for a generic antisymmetric matrix, $\Psi(A)$ and $W_{n-1}^+(\tilde{A})$ coincide as elements of S_{n-1}^+ , possibly up to multiplication by a non zero scalar. To this end, we will proceed by synthesis analysis.

Among the elements of the basis of S_{n-1}^+ we are dealing with, the one of highest weight is $e_{\llbracket n-1 \rrbracket} = e_1 \wedge \dots \wedge e_{n-1}$ which is e_{cM} for M the empty set. For $A = (x_{ij})_{i,j=1}^n \in \text{Asym}_n(\mathbb{C})$, the $e_{\llbracket n-1 \rrbracket}$ -coefficient of $\Psi(A)$ is $x_{n-1,n}$ hence in view of our purpose, assuming that this coefficient of A does not vanish, it is relevant to consider $\Psi(A)/x_{n-1,n}$ which is to be written as

$$\frac{1}{x_{n-1,n}} \Psi(A) = W_{n-1}^+(\tilde{A}) \quad (4.22)$$

for a certain $\tilde{A} \in \text{Asym}_{n-1}(\mathbb{C})$. Denoting by $\tilde{x}_{ij}$ the coefficients of $\tilde{A}$ with $i, j = 1, \dots, n-1$, we find that in the decomposition of $W_{n-1}^+(\tilde{A})$ in the weight basis:

- for $i < n-1$, $\tilde{x}_{i,n-1}$ is the coefficient of e_{cM} for $M = \{i, n-1\}$. In this case, one has $M^{n-1} \cup \{n\} = \{i, n\}$ hence the corresponding coefficient of $\Psi(A)/x_{n-1,n}$ is $\text{Pf}(A_{\{i,n\}})/x_{n-1,n} = x_{i,n}/x_{n-1,n}$;

$$\mathrm{Asym}_n(\mathbb{C}) \xrightarrow{W_n^+} S_n^+ = \wedge^e V \xrightarrow{\Pi_{n,n}^+} S_{n,n}^- \xrightarrow{\Xi_n} S_{n-1}^- \xrightarrow{\Gamma_j} \wedge^{-e} V_j \simeq S_{n-1}^+$$

$\Lambda_{n,j}$

$$A \longmapsto \sum_{\substack{K \subset \llbracket n \rrbracket \\ \#K=0[2]}} \mathrm{Pf}(A_K) e_{cK} \longmapsto \sum_{\substack{K \subset \llbracket n \rrbracket \\ \#K=0[2] \\ n \in K}} \mathrm{Pf}(A_K) e_{cK} \longmapsto \sum_{\substack{L \subset \llbracket n-1 \rrbracket \\ \#L=1[2]}} \mathrm{Pf}(A_{L \cup \{n\}}) e_{cL} \longmapsto \sum_{\substack{M \subset \llbracket n-1 \rrbracket \\ \#M=0[2]}} \mathrm{Pf}(A_{M^j \cup \{n\}}) e_{cM}$$

- for i, j with $1 \leq i < j < n - 1$, $\tilde{x}_{i,j}$ is the coefficient of e_{c_M} for $M = \{i, j\}$. In this case, one has $M^{n-1} \cup \{n\} = \{i, j, n-1, n\}$ hence the corresponding coefficient of $\Psi(A)/x_{n-1,n}$ is the quotient of the pfaffian of the 4×4 -antisymmetric matrix $A_{\{i,j,n-1,n\}}$ by $x_{n-1,n}$.

This short analysis shows that given $A \in \text{Asym}_n(\mathbb{C})$ generic, if there exists $\tilde{A} \in \text{Asym}_{n-1}(\mathbb{C})$ such that (4.22) holds true, then it is unique and its coefficients $\tilde{x}_{i,j}$ are given by

$$\begin{aligned} \tilde{x}_{k,n-1} &= \frac{x_{k,n}}{x_{n-1,n}} & \text{for } k = 1, \dots, n-2 \\ \text{and } \tilde{x}_{i,j} &= \frac{\text{Pf}(A_{\{i,j,n-1,n\}})}{x_{n-1,n}} & \text{for } (i, j) \in \Sigma_{n-2}. \end{aligned} \quad (4.23)$$

Assume that $\tilde{A}$ indeed stands for the antisymmetric matrix whose coefficients are given by the above formulas. That the equality (4.22) holds true in S_{n-1}^+ is equivalent to the fact that the following rational identities in the coefficients x_{ij} of A are satisfied:

$$\frac{\text{Pf}(A_{M^{n-1} \cup \{n\}})}{x_{n-1,n}} = \text{Pf}(\tilde{A}_M), \quad (4.24)$$

for all subsets $M \subset \llbracket n-1 \rrbracket$ with even cardinal.

These identities are known to hold true: they are particular cases of a more general family of identities between “overlapping pfaffians” established by Knuth (see [40, Prop. 2.1]).

PROPOSITION 4.19. — *The following diagram of rational maps is commutative*

$$\begin{array}{ccccc} \text{Asym}_n(\mathbb{C}) & \xhookrightarrow{W_n^+} & S_n^+ & \hookrightarrow & \mathbb{P}S_n^+ \\ \tilde{\Phi}_{n,n-1}^- \downarrow & & \downarrow \Phi_{n,n-1}^- & & \downarrow \Lambda_{n,n-1} \\ \text{Asym}_{n-1}(\mathbb{C}) & \xhookrightarrow{W_{n-1}^+} & S_{n-1}^+ & \hookrightarrow & \mathbb{P}S_{n-1}^+, \end{array}$$

where $\tilde{\Phi}_{n,n}^-$ stands for the rational map associating to an antisymmetric matrix $A \in \text{Asym}_n(\mathbb{C})$ the antisymmetric $n \times n$ -matrix whose coefficients are given by formulas (4.23).

It is only for convenience that this proposition has been stated for the specific case of the face map with respect to $\Delta_{n,n}^-$, when the considered linear projection $\mathbb{P}S_n^+ \dashrightarrow \mathbb{P}S_{n-1}^+$ is induced by Λ_n^{n-1} . Up to an explicit bijection for any $j \in \llbracket n-1 \rrbracket$, it is not difficult to give a similar formula in the case when

the linear projection $\mathbb{P}S_n^+ \dashrightarrow \mathbb{P}S_{n-1}^+$ has been chosen to be the one induced by Λ_n^j . And more generally, for any $i \in \llbracket n-1 \rrbracket$, one can give similar explicit formulas for several models $\tilde{\Phi}_{n,i}^{-,j} : \text{Asym}_n(\mathbb{C}) \dashrightarrow \text{Asym}_{n-1}(\mathbb{C})$ for the face map with respect to the facet $\Delta_{n,i}^-$, one model for each $j \in \llbracket n-1 \rrbracket \setminus \{i\}$. The details and the (somewhat tedious!) task of making the $\tilde{\Phi}_{n,i}^{-,j}$ explicit in the general case are left to the reader.

However, as an example, let us consider the case when $n = 5$ which moreover is the one relevant considering the purpose of this paper. From easy computations, one gets that $\tilde{\Phi}_{5,5}^- = \tilde{\Phi}_{5,5}^{-,4} : \text{Asym}_5(\mathbb{C}) \dashrightarrow \text{Asym}_4(\mathbb{C})$ is the rational map

$$[x_{i,j}]_{i,j=1}^5 \mapsto \begin{bmatrix} 0 & \frac{x_{1,2}x_{4,5} - x_{1,4}x_{2,5} + x_{1,5}x_{2,4}}{x_{4,5}} & \frac{x_{1,3}x_{4,5} - x_{1,4}x_{3,5} + x_{1,5}x_{3,4}}{x_{4,5}} & \frac{x_{1,5}}{x_{4,5}} \\ -\frac{x_{1,2}x_{4,5} - x_{1,4}x_{2,5} + x_{1,5}x_{2,4}}{x_{4,5}} & 0 & \frac{x_{2,3}x_{4,5} - x_{2,4}x_{3,5} + x_{2,5}x_{3,4}}{x_{4,5}} & \frac{x_{2,5}}{x_{4,5}} \\ -\frac{x_{1,3}x_{4,5} - x_{1,4}x_{3,5} + x_{1,5}x_{3,4}}{x_{4,5}} & -\frac{x_{2,3}x_{4,5} - x_{2,4}x_{3,5} + x_{2,5}x_{3,4}}{x_{4,5}} & 0 & \frac{x_{3,5}}{x_{4,5}} \\ -\frac{x_{1,5}}{x_{4,5}} & -\frac{x_{2,5}}{x_{4,5}} & -\frac{x_{3,5}}{x_{4,5}} & 0 \end{bmatrix}.$$

In order to describe the other face maps $\tilde{\Phi}_{5,i}^{-,j}$ for $i, j \in \llbracket 5 \rrbracket$ distinct:

- we denote by x_{ij} the rational map associating its (i, j) -th coefficient to a matrix;
- let k_1, k_2 and k_3 be such that $\llbracket 5 \rrbracket \setminus \{i, j\} = \{k_1, k_2, k_3\}$ and $1 \leq k_1 < k_2 < k_3 \leq 5$;
- we consider the rational map

$$\begin{aligned} \Psi_{i,j} : \text{Asym}_5(\mathbb{C}) &\dashrightarrow \text{Asym}_{\llbracket 5 \rrbracket \setminus \{i\}}(\mathbb{C}) \\ (x_{ij})_{i,j \in \llbracket 5 \rrbracket} &\mapsto (\tilde{x}_{u,v})_{u,v \in \llbracket 5 \rrbracket \setminus \{i\}} \end{aligned}$$

whose coefficients $\tilde{x}_{u,v}$ are given by

- $\tilde{x}_{k,j} = x_{ki}/x_{ij}$ for $k \in \{1, \dots, i-1\}$ with $k \neq j$;
- $\tilde{x}_{k,j} = -x_{ik}/x_{ij}$ for $k \in \{i+1, \dots, 5\}$ with $k \neq j$;
- $\tilde{x}_{k_a, k_b} = \text{Pf}(A_{\{k_a, k_b, i, j\}})/x_{ij}$ for all a, b such that $1 \leq a < b \leq 3$.

COROLLARY 4.20. — *For any $i, j \in \llbracket 5 \rrbracket$ distinct, $\Psi_{i,j}$ is an expression in some Wick's coordinates for the face map $\Phi_{5,i}^- : \mathbb{S}_5^+ \dashrightarrow \mathbb{S}_4$ associated to the facet $\Delta_{5,i}^-$ of the moment polytope Δ_5 .*

The Gelfand–MacPherson web of the spinor variety. Writing $\mathbb{S}_n$ for $\mathbb{S}_n^+$, we can gather all the results above in the following statement which gives a geometric and explicit description of the Gelfand–MacPherson web of $\mathbb{S}_n$:

PROPOSITION 4.21. —

- (1) *As a web defined by rational first integrals on $\mathbb{S}_n$, one has*

$$\mathcal{W}_{\mathbb{S}_n}^{GM} = \mathcal{W}\left(\Phi_{n,i}^+, \Phi_{n,i}^- \mid i = 1, \dots, n\right).$$

It is a $2n$ -web of dimension $n - 1$ on the n -th spinor variety (which is of dimension $n(n - 1)/2$).

- (2) *Geometric descriptions of the face maps $\Phi_{n,i}^+$ and $\Phi_{n,i}^-$ are given above in Proposition 4.17 and Proposition 4.18 respectively.*
- (3) *Up to the birational identifications $\text{Asym}_m(\mathbb{C}) \simeq \mathbb{S}_m^+$ provided by Wick's maps for $m = n, n - 1$, a corresponding birational model of the face map $\Phi_{n,i}^+ : \mathbb{S}_n^+ \dashrightarrow \mathbb{S}_{n-1}^+$ is given by*

$$\begin{aligned} \tilde{\Phi}_{n,i}^+ : \text{Asym}_n(\mathbb{C}) &\dashrightarrow \text{Asym}_{n-1}(\mathbb{C}) \\ A &\longrightarrow A_i. \end{aligned}$$

As for the n other face maps $\Phi_{n,i}^- : \mathbb{S}_n^+ \dashrightarrow \mathbb{S}_{n-1}^-$, for each $j \in \llbracket n \rrbracket \setminus \{i\}$ and up to some explicit identifications, it admits a birational model

$$\begin{aligned} \tilde{\Phi}_{n,i}^{-,j} : \text{Asym}_n(\mathbb{C}) &\dashrightarrow \text{Asym}_{n-1}(\mathbb{C}) \\ A &\longrightarrow \tilde{A}_{ij} \end{aligned}$$

where up to sign, the non trivial coefficients of $\tilde{A}_{ij}$ are of the form $x_{k,i}/x_{i,j}$ or $\text{Pf}(A_{\{k,l,i,j\}})/x_{i,j}$ for $k, l \in \llbracket n \rrbracket \setminus \{i, j\}$.

Birational models for the quotients of $\mathbb{S}_5$ and $\mathbb{S}_4$ by the Cartan tori.

For any $n \geq 4$, we recall that $\mathbb{S}_n^*$ stands for the complement in $\mathbb{S}_n \subset \mathbb{P}S_n^+$ of the union of the hyperplanes of coordinates (relatively to the weight basis $\{e_{c_L} \mid L \subset \llbracket n \rrbracket \text{ even}\}$ of $S_n^+ = \wedge^e V$). The stabilizer of any $\xi \in \mathbb{S}_n^*$ in H is $\{\pm \text{Id}_E\} \simeq \mathbb{Z}/2\mathbb{Z}$ thus $H' = H/\langle \pm \text{Id}_E \rangle$ acts freely on $\mathbb{S}_n^*$ hence the quotient $\mathcal{Y}_n^* = \mathbb{S}_n^*/H'$ exists as a smooth quasi-projective variety. However, even in the cases when $n = 5$ and $n = 4$ which are the two cases relevant for describing $\mathcal{W}_{\text{dP}_4}$ in terms of a Gelfand–MacPherson web, describing precisely $\mathcal{Y}_n^*$ is a bit delicate. Since it will suffice for what we are aiming for, we will instead consider and work with simple birational models of $\mathcal{Y}_5^*$ and $\mathcal{Y}_4^*$ that we are going to describe now. Note that everything below can be easily generalized to n arbitrary but we will not elaborate on this as it is of no interest for our purpose.

We deal in detail with the case when $n = 5$ and are more succinct about the case when $n = 4$, since it is treated in a very similar way. Let $A_5(x) =$

$(x_{i,j})_{i,j=1}^5 \in \text{Asym}_5(\mathbb{C})$ be a 5×5 antisymmetric matrix such that any of its non diagonal coefficients $x_{i,j}$ (with $i \neq j$) does not vanish. One sets

$$H(x) = \begin{bmatrix} h(x) & 0 \\ 0 & h(x)^{-1} \end{bmatrix}$$

where $h(x)$ stands for the diagonal matrix $\text{Diag}(\sqrt{h_1}, \dots, \sqrt{h_5})$ whose diagonal coefficients are given by

$$\begin{aligned} \sqrt{h_1} &= \frac{x_{3,4}x_{1,5}x_{1,2}}{x_{2,3}x_{4,5}} & \sqrt{h_2} &= \frac{x_{1,2}x_{2,3}x_{4,5}}{x_{3,4}x_{1,5}} \\ \sqrt{h_3} &= \frac{x_{2,3}x_{3,4}x_{1,5}}{x_{1,2}x_{4,5}} & \sqrt{h_4} &= \frac{x_{3,4}x_{1,2}x_{4,5}}{x_{2,3}x_{1,5}} & \sqrt{h_5} &= \frac{x_{1,5}x_{2,3}x_{4,5}}{x_{3,4}x_{1,2}}. \end{aligned} \quad (4.25)$$

Actually, the presence of square roots in this definition induces a certain ambiguity with the consequence that $H(x)$ is not well-defined as an element of the diagonal torus in $\text{GL}(E)$. However, this ambiguity can be resolved arguing as follows: the set of diagonal matrices D_h for $h \in (\mathbb{C}^*)^5$ described above (in (4.17)) is the image of the Cartan torus H_{D_5} of Spin_{10} in $\text{GL}(E)$. The map $(\mathbb{C}^*)^5 \rightarrow \mathbb{C}^*$, $(h_s)_{s=1}^5 \mapsto h_i$, denoted abusively by h_i from now on, is a character on H which admits a square root when pulled-back on H_{D_5} , which will be denoted by $\sqrt{h_i}$. With this notation, the coordinate ring of H_{D_5} is $\mathbb{C}[H_{D_5}] = \mathbb{C}[h_1^{\pm 1/2}, \dots, h_5^{\pm 1/2}]$ and now the formulas (4.25) unambiguously define a rational map $\mathbf{H} : \text{Asym}_5(\mathbb{C}) \dashrightarrow H_{D_5}$. For x as above, $\mathbf{H}$ is defined at x and given an arbitrary matrix $A(y) = (y_{i,j})_{i,j=1}^5 \in \text{Asym}_5(\mathbb{C})$, one has

$$\begin{aligned} A(y) \circ \mathbf{H}(x) &= h(x)^{-1} \cdot \left[\text{Id}_5, A(y) \right] \cdot H(x) \\ &= \begin{bmatrix} 0 & \frac{y_{1,2}}{\sqrt{h_1}\sqrt{h_2}} & \frac{y_{1,3}}{\sqrt{h_1}\sqrt{h_3}} & \frac{y_{1,4}}{\sqrt{h_1}\sqrt{h_4}} & \frac{y_{1,5}}{\sqrt{h_1}\sqrt{h_5}} \\ -\frac{y_{1,2}}{\sqrt{h_1}\sqrt{h_2}} & 0 & \frac{y_{2,3}}{\sqrt{h_2}\sqrt{h_3}} & \frac{y_{2,4}}{\sqrt{h_2}\sqrt{h_4}} & \frac{y_{2,5}}{\sqrt{h_2}\sqrt{h_5}} \\ -\frac{y_{1,3}}{\sqrt{h_1}\sqrt{h_3}} & -\frac{y_{2,3}}{\sqrt{h_2}\sqrt{h_3}} & 0 & \frac{y_{3,4}}{\sqrt{h_3}\sqrt{h_4}} & \frac{y_{3,5}}{\sqrt{h_3}\sqrt{h_5}} \\ -\frac{y_{1,4}}{\sqrt{h_1}\sqrt{h_4}} & -\frac{y_{2,4}}{\sqrt{h_2}\sqrt{h_4}} & -\frac{y_{3,4}}{\sqrt{h_3}\sqrt{h_4}} & 0 & \frac{y_{4,5}}{\sqrt{h_4}\sqrt{h_5}} \\ -\frac{y_{1,5}}{\sqrt{h_1}\sqrt{h_5}} & -\frac{y_{2,5}}{\sqrt{h_2}\sqrt{h_5}} & -\frac{y_{3,5}}{\sqrt{h_3}\sqrt{h_5}} & -\frac{y_{4,5}}{\sqrt{h_4}\sqrt{h_5}} & 0 \end{bmatrix} \end{aligned}$$

where $\circ$ stands here for the action of $\mathbf{H}(x)$ on the point of $\mathbb{S}_5$ corresponding to $A(y)$ and where the dots $\cdot$ in the expression after the first equal sign refer to matricial products. Taking $y = x$ in the above identity, an elementary

computation gives us that one has $A(x) \circ \mathbf{H}(x) = A'(x)$ with

$$A'(x) = \begin{bmatrix} 0 & 1 & \frac{x_{4,5}x_{1,3}}{x_{1,5}x_{3,4}} & \frac{x_{2,3}x_{1,4}}{x_{3,4}x_{1,2}} & 1 \\ -1 & 0 & 1 & \frac{x_{1,5}x_{2,4}}{x_{1,2}x_{4,5}} & \frac{x_{3,4}x_{2,5}}{x_{2,3}x_{4,5}} \\ -\frac{x_{4,5}x_{1,3}}{x_{1,5}x_{3,4}} & -1 & 0 & 1 & \frac{x_{1,2}x_{3,5}}{x_{1,5}x_{2,3}} \\ -\frac{x_{2,3}x_{1,4}}{x_{3,4}x_{1,2}} & -\frac{x_{1,5}x_{2,4}}{x_{1,2}x_{4,5}} & -1 & 0 & 1 \\ -1 & -\frac{x_{3,4}x_{2,5}}{x_{2,3}x_{4,5}} & -\frac{x_{1,2}x_{3,5}}{x_{1,5}x_{2,3}} & -1 & 0 \end{bmatrix}.$$

Conversely, one verifies that $A(x) = A'(x) \circ \mathbf{H}(x)^{-1}$. Then denoting by $u_{a,b}$ the coefficients of $A'(x)$ over the diagonal which are distinct from 1 (ie. $u_{1,3} = x_{4,5}x_{1,3}/(x_{1,5}x_{3,4})$, $u_{2,3} = x_{2,3}x_{1,4}/(x_{3,4}x_{1,2})$ etc), using $\sqrt{h_i}$ ($i = 1, \dots, 5$) for the quantities defined in (4.25) and considering all these quantities as rational functions in the x_{ij} 's, we get a rational map

$$\begin{aligned} \text{Asym}_5(\mathbb{C}) &\dashrightarrow \mathbb{C}_u^5 \times H_{D_5} \\ A(x) &\longmapsto \left((u_{1,3}, \dots, u_{3,5}), \left(\sqrt{h_1}, \dots, \sqrt{h_5} \right) \right) \end{aligned} \quad (4.26)$$

which is easily shown to provide a birational model for the quotient of $\mathbb{S}_5$ by H_{D_5} :

PROPOSITION 4.22. — *The map (4.26) is birational. The corresponding isomorphism of fields of rational functions $\mathbb{C}(\mathbb{S}_5) \simeq \mathbb{C}(\text{Asym}_5(\mathbb{C})) \simeq \mathbb{C}(u) \otimes_{\mathbb{C}} \mathbb{C}(H_{D_5})$ is such that*

$$\mathbb{C}(\mathbb{S}_5)^{H_{D_5}} \simeq \mathbb{C}(\mathbb{C}_u^5 \times H_{D_5})^{H_{D_5}} \simeq \mathbb{C}(u).$$

In particular, the rational map $\mathcal{P}_5 : \text{Asym}_5(\mathbb{C}) \dashrightarrow \mathbb{C}^5$ associating to an antisymmetric matrix $[x_{i,j}]_{i,j=1}^5$ the following 5-tuple

$$\left(\frac{x_{4,5}x_{1,3}}{x_{1,5}x_{3,4}}, \frac{x_{1,5}x_{2,4}}{x_{1,2}x_{4,5}}, \frac{x_{1,2}x_{3,5}}{x_{1,5}x_{2,3}}, \frac{x_{2,3}x_{1,4}}{x_{3,4}x_{1,2}}, \frac{x_{3,4}x_{2,5}}{x_{2,3}x_{4,5}} \right)$$

is a birational model in Wick's coordinates, for the quotient map $\chi_5 : \mathbb{S}_5 \rightarrow \mathcal{Y}_5 = \mathbb{S}_5/H_{D_5}$.

This proposition can be generalized for $n \geq 4$ arbitrary. However, this requires two similar but formally distinct treatments according to the parity of n . We will not elaborate further on this but only give the corresponding statement in the case when $n = 4$ which we are going to use in the next paragraph.

PROPOSITION 4.23. — *The rational map*

$$\mathcal{P}_4 : \text{Asym}_4(\mathbb{C}) \dashrightarrow \mathbb{C}^2$$

$$\begin{bmatrix} 0 & x_{1,2} & x_{1,3} & x_{1,4} \\ -x_{1,2} & 0 & x_{2,3} & x_{2,4} \\ -x_{1,3} & -x_{2,3} & 0 & x_{3,4} \\ -x_{1,4} & -x_{2,4} & -x_{3,4} & 0 \end{bmatrix} \mapsto \left(\frac{x_{1,4}x_{2,3}}{x_{1,2}x_{3,4}}, \frac{x_{1,3}x_{2,4}}{x_{1,2}x_{3,4}} \right)$$

is a birational model for the quotient map

$$\chi_4 : \mathbb{S}_4 \dashrightarrow \mathcal{Y}_4 = \mathbb{S}_4/H_{D_4}.$$

Birational models for the equivariant quotient of the Gelfand–MacPherson web $\mathcal{W}_{\mathbb{S}_5}^{GM}$. Our aim here is to give an explicit birational model for the H_{D_5} -equivariant quotient of Gelfand–MacPherson web as defined in (4.14).

For $n = 4, 5$, we denote by $\mathbf{Y}_n$ the affine complex space which is the target space of the map $\mathcal{P}_n$ of the two preceding propositions (thus $\mathbf{Y}_5 = \mathbb{C}^5$ and $\mathbf{Y}_4 = \mathbb{C}^2$). From the two previous propositions and Corollary 4.10, it follows that there exist birational maps $\omega_n : \mathbf{Y}_n \xrightarrow{\sim} \mathcal{Y}_n$ for $n = 5, 4$; and dominant rational maps $\varphi_{5,i}^\varepsilon : \mathcal{Y}_5 \dashrightarrow \mathcal{Y}_4$ and $\psi_i^\varepsilon : \mathbb{C}^5 = \mathbf{Y}_5 \dashrightarrow \mathbf{Y}_4 = \mathbb{C}^2$ for $i = 1, \dots, 5$ and $\varepsilon = \pm$, making the following diagram commutative:

$$\begin{array}{ccccccc} \text{Asym}_5(\mathbb{C}) & \xrightarrow{W_5} & \mathbb{S}_5 & \xrightarrow{\Phi_{5,i}^\varepsilon} & \mathbb{S}_4 & \xleftarrow{W_4} & \text{Asym}_4(\mathbb{C}) \\ \downarrow \mathcal{P}_5 & & \downarrow \pi_5 & & \downarrow \pi_4 & & \downarrow \mathcal{P}_4 \\ \mathbb{C}^5 & \xrightarrow{\omega_5} & \mathcal{Y}_5 & \xrightarrow{\varphi_{5,i}^\varepsilon} & \mathcal{Y}_4 & \xleftarrow{\omega_4} & \mathbb{C}^2 \\ & & & \searrow \psi_{5,i}^\varepsilon & & & \\ & & & & & & \end{array} \quad (4.27)$$

A green dotted arrow labeled σ points from $\mathbb{C}^5$ to $\text{Asym}_5(\mathbb{C})$.

Remarks. —

- (i) This is a diagram of rational maps hence all the arrows in it should be dashed according to our typographic convention. They are not just for aesthetic reasons;
- (ii) The green map σ with the dotted arrow on the left is defined a few paragraphs below, cf. (4.28)).

Since $\mathcal{P}_5$ and $\mathcal{P}_4$ are birational models for the torus quotients of $\mathbb{S}_5$ and $\mathbb{S}_4$ respectively, it follows that the pull-back under ω_5 of the equivariant quotient of $\mathcal{W}_{\mathbb{S}_5}^{GM}$ by H_{D_5} is the web on $\mathbb{C}^5$ defined by the rational maps $\psi_{5,i}^\varepsilon$: one has

$$\omega_5^* \left(\left(\mathcal{W}_{\mathbb{S}_5}^{GM} \right) /_{H_{D_5}} \right) = \mathcal{W} \left(\psi_{5,i}^\varepsilon \mid i = 1, \dots, 5, \varepsilon = \pm \right).$$

Because we have explicit rational expressions for the maps $\Phi_{5,i}^\varepsilon$ (in Wick's coordinates) as well as for $\mathcal{P}_5$ and $\mathcal{P}_4$, one can easily get expressions of the same kind for the maps $\psi_{5,i}^\varepsilon$'s hence give an explicit birational model for $\mathcal{W}_{\mathcal{Y}_5}^{GM} = \mathcal{W}_{\mathbb{S}_5}^{GM} / H_{D_5}$.

Let us start by making explicit the maps $\psi_i^+ = \psi_{5,i}^+$'s (for which there is no ambiguity). Setting

$$y = (y_{1,3}, y_{1,4}, y_{2,4}, y_{2,5}, y_{3,5}),$$

we see that the rational map $\sigma : \mathbb{C}^5 \dashrightarrow \text{Asym}_5(\mathbb{C})$ given by

$$y \mapsto \begin{bmatrix} 0 & 1 & y_{1,3} & y_{1,4} & 1 \\ -1 & 0 & 1 & y_{2,4} & y_{2,5} \\ -y_{1,3} & -1 & 0 & 1 & y_{3,5} \\ -y_{1,4} & -y_{2,4} & -1 & 0 & 1 \\ -1 & -y_{2,5} & -y_{3,5} & -1 & 0 \end{bmatrix} \quad (4.28)$$

is a rational section of $\mathcal{P}_5$. It follows that for $i = 1, \dots, 5$, one has

$$\psi_i^+ = \mathcal{P}_4 \circ W_4^{-1} \circ \Phi_{5,i}^+ \circ W_5 \circ \sigma : \mathbb{C}^5 \dashrightarrow \mathbb{C}^2.$$

Since $\tilde{\Phi}_i^+ = W_4^{-1} \circ \Phi_{5,i}^+ \circ W_5$ is known (see Proposition 4.21.3), it is just a matter of elementary computations to get the following explicit and quite simple formulas for the $\psi_{5,i}^+$'s:

$$\begin{aligned} \psi_1^+(y) &= (y_{2,5}, y_{2,4}y_{3,5}) & \psi_2^+(y) &= \left(\frac{1}{y_{1,3}}, \frac{y_{1,4}y_{3,5}}{y_{1,3}} \right) \\ \psi_3^+(y) &= (y_{2,4}, y_{1,4}y_{2,5}) & \psi_4^+(y) &= \left(\frac{1}{y_{3,5}}, \frac{y_{1,3}y_{2,5}}{y_{3,5}} \right) \\ \psi_5^+(y) &= (y_{1,4}, y_{1,3}y_{2,4}). \end{aligned} \quad (4.29)$$

We now turn to the face maps $\psi_i^- = \psi_{5,i}^-$ for $i = 1, \dots, 5$. As explained above, for $i \in \llbracket 5 \rrbracket$ given, there is no natural choice for an expression in Wick's coordinates for the map $\Phi_{5,i}^-$ and one has to choose and work with one of the maps $\tilde{\Phi}_{5,i}^{-,j}$ with $j \neq i$. But once such a map has been chosen, possibly in an arbitrary way, one can compute an explicit rational expression for a model of ψ_i^- with respect to some Wick's coordinates, which we will denote by $\psi_{i,j}^-$.

For example, the map $\widetilde{\Phi}_{5,1}^{-,5}$ is the rational map which associates to a generic antisymmetric matrix $(x_{ij})_{i,j=1}^5 \in \text{Asym}_5(\mathbb{C})$ the following one

$$\begin{bmatrix} 0 & \frac{x_{1,2}x_{3,5}-x_{1,3}x_{2,5}+x_{1,5}x_{2,3}}{x_{1,5}} & \frac{x_{1,2}x_{4,5}-x_{1,4}x_{2,5}+x_{1,5}x_{2,4}}{x_{1,5}} & \frac{x_{1,2}}{x_{1,5}} \\ \dots & 0 & \frac{x_{1,3}x_{4,5}-x_{1,4}x_{3,5}+x_{1,5}x_{3,4}}{x_{1,5}} & \frac{x_{1,3}}{x_{1,5}} \\ \vdots & \ddots & 0 & \frac{x_{1,4}}{x_{1,5}} \\ \dots & \dots & \dots & 0 \end{bmatrix}$$

From this, one easily deduces that the rational map $\psi_{1,5}^- : \mathbb{C}^5 \dashrightarrow \mathbb{C}^2$ is given by

$$\psi_{1,5}^-(y) = \left(\frac{y_{1,4}y_{3,5} - y_{1,3} - 1}{y_{1,4}(y_{1,3}y_{2,5} - y_{3,5} - 1)}, \frac{y_{1,3}(y_{1,4}y_{2,5} - y_{2,4} - 1)}{y_{1,4}(y_{1,3}y_{2,5} - y_{3,5} - 1)} \right) \quad (4.30)$$

for $y = (y_{1,3}, y_{1,4}, y_{2,4}, y_{2,5}, y_{3,5}) \in \mathbb{C}^5$. Similar explicit formulas can be given for any of the $\psi_{i,j}^-$'s: for instance, straightforward but a bit lengthy (hence not reproduced here) computations give us that

$$\begin{aligned} \psi_{2,5}^-(y) &= \left(\frac{y_{2,4}y_{3,5} - y_{2,5} - 1}{y_{2,4}(y_{1,3}y_{2,5} - y_{3,5} - 1)}, \frac{y_{1,4}y_{2,5} - y_{2,4} - 1}{y_{2,4}(y_{1,3}y_{2,5} - y_{3,5} - 1)} \right) \\ \psi_{3,5}^-(y) &= \left(\frac{y_{1,3}(y_{2,4}y_{3,5} - y_{2,5} - 1)}{y_{1,3}y_{2,5} - y_{3,5} - 1}, \frac{y_{1,4}y_{3,5} - y_{1,3} - 1}{y_{1,3}y_{2,5} - y_{3,5} - 1} \right) \\ \psi_{4,5}^-(y) &= \left(\frac{y_{1,4}(y_{2,4}y_{3,5} - y_{2,5} - 1)}{y_{1,4}y_{2,5} - y_{2,4} - 1}, \frac{y_{2,4}(y_{1,4}y_{3,5} - y_{1,3} - 1)}{y_{1,4}y_{2,5} - y_{2,4} - 1} \right) \\ \psi_{5,4}^-(y) &= \left(\frac{y_{2,4}y_{3,5} - y_{2,5} - 1}{y_{3,5}(y_{1,4}y_{2,5} - y_{2,4} - 1)}, \frac{y_{2,5}(y_{1,4}y_{3,5} - y_{1,3} - 1)}{y_{3,5}(y_{1,4}y_{2,5} - y_{2,4} - 1)} \right). \end{aligned} \quad (4.31)$$

To simplify the writing, we set $\psi_i^- = \psi_{i,5}^-$ for $i = 1, \dots, 4$ and $\psi_5^- = \psi_{5,4}^-$.

PROPOSITION 4.24. — *In the system of rational coordinates $(y_{1,3}, y_{1,4}, y_{2,4}, y_{2,5}, y_{3,5})$ on $\mathbb{C}^5 \simeq \mathcal{Y}_5$, Gelfand–MacPherson’s web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ is defined by the ten rational first integrals ψ_i^ε for $i = 1, \dots, 5$ and $\varepsilon = \pm$.*

Serganova–Skorobogatov’s embedding and a description of $\mathcal{W}_{\text{dP}_4}$ à la Gelfand–MacPherson. Our goal in this paragraph is to explain how to construct $\mathcal{W}_{\text{dP}_4}$ from $\mathcal{W}_{\mathcal{Y}_5}^{GM}$. For this sake, we need to recall first a result (due to Popov in its Diplomarbeit [48], see also [53, 19]) about a natural embedding of the Cox variety of a given del Pezzo quartic surface dP_4 into the spinor tenfold.

The construction sketched in Section 3.6.2 for the del Pezzo quintic surface can be generalized to any del Pezzo surface dP_d for $d \in \{2, \dots, 6\}$. Given a del Pezzo surface of this degree, there exists a blow-up $b : X_r = \text{Bl}_{p_1, \dots, p_r}(\mathbb{P}^2) \rightarrow \mathbb{P}^2$ of $r = 9 - d$ points $p_1, \dots, p_r$ in general position in $\mathbb{P}^2$

such that the considered del Pezzo surface identifies with X_r . Such a description of dP_d is not unique⁽²³⁾ but we fix one and denote the surface by X_r from now on to indicate this. We will denote X_r^* the affine Zariski-open subset of X_r which is the complement of the union of all lines contained in X_r .

For any $i = 1, \dots, r$, we set $E_i = b^{-1}(p_i)$ and we denote by $\mathbf{e}_i$ the class of this exceptional divisor in the Picard lattice $\mathbf{Pic}(X_r)$ of X_r . As mentioned above, $\mathbf{Pic}(X_r)$ is freely spanned over $\mathbb{Z}$ by the $\mathbf{e}_i$'s ($i = 1, \dots, r$) and by the class, denoted by $\mathbf{h}$, of the preimage $H = b^{-1}(L)$ of a line $L \subset \mathbb{P}^2 \setminus \{p_1, \dots, p_r\}$. By definition, the “Cox ring” associated to the divisors $H, E_1, \dots, E_r$ whose classes are generators of the Picard lattice is

$$\mathrm{Cox}(X_r) = \bigoplus_{m_0, \dots, m_r \in \mathbb{Z}} \mathbf{H}^0\left(X_r, \mathcal{O}_{X_r}\left(m_0 H + \sum_{i=1}^r m_i E_i\right)\right),$$

the ring structure being defined by the usual multiplication of sections.

One defines the “affine Cox variety” $\mathbf{A}(X_r)$ as the spectrum of the Cox ring:

$$\mathbf{A}(X_r) = \mathbf{Spec}(\mathrm{Cox}(X_r)).$$

The Cox ring is naturally graded by the $\mathbb{Z}_{\geq 0}$ -monoid of effective classes. From the latter, we get a $\mathbb{Z}_{\geq 0}$ -grading defined by $(-K, \cdot)$ which allows us to define the “Projective Cox variety”

$$\mathbb{P}(X_r) = \mathbf{Proj}(\mathrm{Cox}(X_r)).$$

The “Néron-Severi torus” of X_r , which by definition is

$$T_{NS} = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{Pic}(X_r), \mathbb{C}^*) \simeq (\mathbb{C}^*)^{\{\mathbf{h}, \mathbf{e}_1, \dots, \mathbf{e}_r\}},$$

naturally acts on the Cox ring by $t \cdot \sigma_D = t_{\mathbf{h}}^{n_0} t_{\mathbf{e}_1}^{n_1} \dots t_{\mathbf{e}_r}^{n_r} \sigma_D$ for any $t = (t_{\mathbf{h}}, t_{\mathbf{e}_1}, \dots, t_{\mathbf{e}_r}) \in T_{NS}$ and any section $\sigma_D \in \mathbf{H}^0(X_r, \mathcal{O}_{X_r}(n_0 H - \sum_{i=1}^r n_i E_i))$ with $D = n_0 \mathbf{h} - \sum_{i=1}^r n_i \mathbf{e}_i \in \mathrm{Pic}_{\mathbb{Z}}(X_r)$. Let τ_K be the 1-parameter subgroup of T_{NS} spanned by the $\mathbb{Z}$ -linear form $(-K, \cdot) : \mathbf{Pic}_{\mathbb{Z}}(X_r) \rightarrow \mathbb{Z}$. Then quotienting by τ_K gives rise to a surjective quotient map $\mathbf{A}(X_r) \setminus \{0\} \rightarrow \mathbb{P}(X_r)$ from which one gets that the quotient torus $T_r = T_{NS}/\tau_K \simeq (\mathbb{C}^*)^r$ naturally acts on $\mathbb{P}(X_r)$.

Here are some basic and fundamental results about the (affine and projective) Cox varieties of X_r and the torus actions on them considered above:

⁽²³⁾ The set of descriptions of dP_d as such a blow-up is in one-to-one correspondence with the set of sets of r pairwise disjoint lines on dP_d . The Weyl group W_r acts transitively on the latter set which allows to compute its cardinal explicitly (cf. [34]).

- the Cox ring $\text{Cox}(X_r)$ is generated by the (classes of the) lines contained in X_r . Denoting by σ^ℓ a nonzero section in $\mathbf{H}^0(X_r, \mathcal{O}_{X_r}(\ell))$ for any $\ell \in \mathcal{L}_r$, there is an isomorphism $\text{Cox}(X_r) \simeq \mathbb{C}[\sigma_\ell \mid \ell \in \mathcal{L}_r] / J_r$ for a certain ideal J_r which is spanned by elements of $(-K)$ -degree 2;
- for any divisor Z on X_r , we denote $\mathcal{L}_Z^\circ$ the total space of the line bundle associated to the sheaf $\mathcal{O}_{X_r}(Z)$ with the zero section removed. Then the fiber product over X_r of the $\mathbb{C}^*$ -bundles $\mathcal{L}_Z^\circ$ for $Z \in \{H, E_1, \dots, E_r\}$, denoted by $\mathcal{T}_{X_r}$, embeds as an open-Zariski subset into $\mathbf{A}(X_r)$: one has

$$\mathcal{T}_{X_r} = \mathcal{L}_H^\circ \times_{X_r} \mathcal{L}_{E_1}^\circ \times_{X_r} \cdots \times_{X_r} \mathcal{L}_{E_r}^\circ \subset \mathbf{A}(X_r). \quad (4.32)$$

The Néron–Severi torus naturally acts on $\mathcal{T}_{X_r}$ making of the projection $\nu : \mathcal{T}_{X_r} \rightarrow X_r$ a T_{NS} -bundle in a natural way. We denote by $\mathcal{T}_{X_r}^*$ the preimage of X_r^* by ν : $\mathcal{T}_{X_r}^* = \nu^{-1}(X_r^*)$.

- the set of lines $\mathcal{L}_r$ identifies with the set of weights of a minuscule representation of the Lie group G_r of type E_r . Identifying G_r with its image in $\text{GL}(\mathbb{C}^{\mathcal{L}_r})$, we get by restriction a group monomorphism from the Cartan torus H_r of G_r into the torus of matrices which are diagonal with respect to the natural basis (namely $\mathcal{L}_r$) of $\mathbb{C}^{\mathcal{L}_r}$. The stabilizer P_r of the line spanned by the highest weight ℓ_h is a maximal parabolic subgroup of G_r and the orbit

$$\mathcal{G}_r = G_r \cdot \langle \ell_h \rangle \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r})$$

is closed and identifies with the projective homogeneous space G_r/P_r . Finally, let T_r be the subtorus of $\text{GL}(\mathbb{C}^{\mathcal{L}_r})$ spanned by H_r and the 1-dimensional subtorus formed by scalar matrices;

- seeing each global section $\sigma_\ell \in \mathbf{H}^0(X_r, \mathcal{O}_{X_r}(\ell))$ as a non constant regular rational function on $\mathbf{A}(X_r)$, one constructs an embedding

$$\widehat{\varrho}_r = (\sigma_\ell)_{\ell \in \mathcal{L}_r} : \mathbf{A}(X_r) \hookrightarrow \mathbb{C}^{\mathcal{L}_r}$$

which is (T_{NS}, T_r) -equivariant and which, after projectivization, induces a H_r -equivariant embedding $\varrho_r : \mathbb{P}(X_r) \hookrightarrow \mathbb{P}(\mathbb{C}^{\mathcal{L}_r})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{A}(X_r) \setminus \{0\} & \xhookrightarrow{\widehat{\varrho}_r} & \mathbb{C}^{\mathcal{L}_r} \setminus \{0\} \\ \downarrow & & \downarrow \\ \mathbb{P}(X_r) & \xhookrightarrow{\varrho_r} & \mathbb{P}(\mathbb{C}^{\mathcal{L}_r}), \end{array}$$

where the (surjective) vertical maps are given by quotienting by τ_K and $\mathbb{C}^* \text{Id}_{\mathbb{C}^{\mathcal{L}_r}}$ respectively. We denote by $\mathcal{P}_r$ the image of ϱ_r : one has

$\mathcal{P}_r = \varrho(\mathbb{P}(X_r)) \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r})$. Its is an irreducible closed subvariety of $\mathbb{P}(\mathbb{C}^{\mathcal{L}_r})$, of dimension $r + 2$;

- an important result in this area is that for suitable choices of the sections σ_ℓ for ℓ ranging in $\mathcal{L}_r$, one can assume that ϱ_r lands into $\mathcal{G}_r \simeq G_r/P_r$: one has

$$\mathbb{P}(X_r) \hookrightarrow \mathcal{P}_r \subset \mathcal{G}_r \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r}); \quad (4.33)$$

- let $\mathcal{G}_r^{sf}$ be the subset of $\mathcal{G}_r$ formed by its points whose stabilizer in H_r coincides with the center of G_r , i.e. $\mathcal{G}_r^{sf} = \{x \in \mathcal{G}_r = G_r/P_r \mid \text{Stab}_{H_r}(x) = Z(G_r)\}$. Of course $\mathcal{G}_r^{sf}$ is H_r -invariant and from general and classical results of geometric invariant theory, it follows that $\mathcal{Y}_r = \mathcal{G}_r^{sf}/H_r$ is a smooth quasiprojective variety and that the quotient mapping $\chi : \mathcal{G}_r^{sf} \rightarrow \mathcal{Y}_r$ is a geometric quotient of $\mathcal{G}_r^{sf}$ by H_r ;
- for $\ell \in \mathcal{L}_r$, one sets $\mathcal{H}_\ell$ for the hyperplane in $\mathbb{P}(\mathbb{C}^{\mathcal{L}_r})$ defined by the vanishing of the ℓ -th coordinate and we denote by $\mathcal{H}_{\mathcal{L}_r}$ the union of these hyperplanes: $\mathcal{H}_{\mathcal{L}_r} = \bigcup_{\ell \in \mathcal{L}_r} \mathcal{H}_\ell$. Then for any subset $\Gamma \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r})$ (resp. $Y \subset \mathcal{Y}_r$), one sets $\Gamma^* = \Gamma \setminus \mathcal{H}_{\mathcal{L}_r}$ (resp. $Y^* = \chi(\chi^{-1}(Y)^*)$);
- denote by $\mathbb{P}(X_r)^\circ$ the image of the T_{NS} -torsor $\mathcal{T}_{X_r}$ (cf. (4.32)) by the projection $\mathbf{A}(X_r) \setminus \{0\} \rightarrow \mathbb{P}(X_r)$. Then the map ϱ_r embeds $\mathbb{P}(X_r)^\circ$ into $\mathcal{G}_r^{sf}$ and there exists an embedding $f_{SS} : X_r \hookrightarrow \mathcal{Y}_r$, named after Serganova and Skorobogatov, which makes the following diagram commutative (where we set $\mathcal{P}_r^\circ = \varrho_r(\mathbb{P}(X_r)^\circ)$):

$$\begin{array}{ccc} \mathbb{P}(X_r)^\circ & \xrightarrow{\varrho_r} & \mathcal{P}_r^\circ \subset \mathcal{G}_r^{sf} \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r}) \\ \downarrow & & \downarrow \gamma \\ X_r & \xrightarrow{f_{SS}} & \mathcal{Y}_r. \end{array} \quad (4.34)$$

Moreover, identifying X_r with its image by f_{SS} in $\mathcal{Y}_r$, the intersection $X_r \cap \mathcal{Y}_r^*$ coincides with the complement X_r^* of the union of all the lines contained in it: coherently with the notation, one has $X_r^* = X_r \cap \mathcal{Y}_r^*$;

- the initial del Pezzo surface X_r can be recovered from its projective Cox variety $\mathbb{P}(X_r) \simeq \mathcal{P}_r$ as the GIT quotient of the latter by the torus H_r : one has $X_r = \mathbb{P}(X_r)//H_r$;
- let $N_r = N_{G_r}(H_r)$ be the normalizer of the torus H_r in G_r . The homogeneous space $\mathcal{G}_r$ and its open subset $\mathcal{G}_r^{sf}$ both can be seen to be N_r -invariant hence one deduces an action by automorphisms of

the Weyl group $W_r = N_r/H_r$ on the quotient $\mathbf{Y}_r = \mathcal{G}_r^{sf}/H_r$. On the other hand, W_r acts by permutations on the set of lines $\mathcal{L}_r$ hence gives rise to a linear action of W_r in $\mathbb{C}^{\mathcal{L}_r}$ which leaves $\mathcal{G}_r^{sf}$ invariant. The map $\gamma : \mathcal{G}_r^{sf} \rightarrow \mathbf{Y}_r$ can be proved to be W_r -equivariant with respect to these two actions;

- for $w \in W_r$, let ψ_w be the automorphism of $\mathbf{Y}_r$ given by the group monomorphism $W \hookrightarrow \text{Aut}(\mathbf{Y}_r)$ mentioned above. Any such ψ_w lets $\mathbf{Y}_r^*$ invariant. As a consequence, it follows that the map $f_{SS} : X_r \hookrightarrow \mathbf{Y}_r$ is not unique: another such map can be obtained by post-composing f_{SS} with an automorphism ψ_w , for any element w of the Weyl group.

Remark 4.25. —

- (1) For $r = 4$, all the material above can be found in [54]. Actually, many things simplify in this case, since one has $\mathcal{P}_r = \mathcal{G}_r = G_2(\mathbb{C}^5)$.
- (2) In the case when $r = 5$, which is the one we are interested in this paper, the first occurrence in the literature of a map as in (4.33) we are aware of is the one at the top of [48, p. 35].

We set $\mathcal{L}' = \mathcal{L}_r \setminus \{E_1, \dots, E_r\}$. For $\ell \in \mathcal{L}'$, its direct image $\bar{\ell} = b(\ell)$ by b is an irreducible rational curve in $\mathbb{P}^2$, of degree $\delta_\ell \in \{1, 2, 3\}$. We fix one of the lines ℓ_∞ by viewing $\bar{\ell}_\infty$ at infinity in the projective plane, with respect to the choice of some fixed affine coordinates x, y on $\mathbb{C}^2 = \mathbb{P}^2 \setminus \bar{\ell}_\infty$. For any $\ell \in \mathcal{L}_r \setminus \{E_1, \dots, E_r, \ell_\infty\}$, let $F_\ell \in \mathbb{C}[x, y]$ be an irreducible polynomial of degree δ_ℓ such that $\bar{\ell}$ is the closure of the affine curve in $\mathbb{C}^2$ cut out by $F_\ell(x, y) = 0$. We set also $F_{E_i} = 1$ for any $i = 1, \dots, r$ and $F_{\ell_\infty} = 1$. We denote by $U_{\mathcal{L}_r}$ the open subset of $\mathbb{C}^2$ whose points are the pairs (x, y) such that $F_\ell(x, y)$ does not vanish for any line ℓ : in other terms, one has

$$U_{\mathcal{L}_r} = \mathbb{P}^2 \setminus \left(\bigcup_{\ell \in \mathcal{L}'} \bar{\ell} \right) = b(X_r^*) \simeq X_r^*. \quad (4.35)$$

PROPOSITION 4.26. — *Up to renormalizing each F_ℓ by a suitable non zero multiplicative constant, the rational morphism*

$$\begin{aligned} F_{\mathcal{L}_r} &= [F_\ell]_{\ell \in \mathcal{L}_r} : U_{\mathcal{L}_r} \longrightarrow \mathbb{P}(\mathbb{C}^{\mathcal{L}_r}) \\ (x, y) &\longmapsto [F_\ell(x, y)]_{\ell \in \mathcal{L}_r} \end{aligned} \quad (4.36)$$

can be made taking its values into $\mathcal{G}_r^{sf} \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}})$ and such that the following diagram commutes:

$$\begin{array}{ccc}
 & \xrightarrow{F_{\mathcal{L}_r}} & \mathcal{G}_r^{sf} \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r}) \\
 \mathbb{C}^2 \supset U_{\mathcal{L}_r} \simeq X_r^* \subset X_r & \xrightarrow{f_{SS}} & \downarrow \gamma \\
 & & \mathcal{Y}_r.
 \end{array} \tag{4.37}$$

Proof. — We first recall a well-known fact: let $\mu : \mathcal{L} \rightarrow X_r$ be the total space of a line bundle L over X_r . Given a non zero section $\sigma \in \mathbf{H}^0(X_r, L)$ with associated divisor $D_\sigma = \{\sigma = 0\} \subset X_r$, one gets a regular morphism $\tilde{\sigma}^{-1} : \mu^{-1}(X_r \setminus D) \rightarrow \mathbb{C}$, which associates to $\xi \in \mathcal{L}_x = \mu^{-1}(x)$ the quotient $\xi/\sigma(x)$. It extends to a global rational function $\tilde{\sigma}^{-1} : \mathcal{L} \dashrightarrow \mathbb{P}^1$ which is such that the rational function $\tilde{\sigma}^{-1} \circ \sigma : X \dashrightarrow \mathbb{P}^1$ is constant equal to 1.

We fix non zero sections $\sigma_{E_i} \in \mathbf{H}^0(X_r, \mathcal{O}_{X_r}(E_i))$ ($i = 1, \dots, 5$) and $\sigma_\infty \in \mathbf{H}^0(x_r, \mathcal{O}_{X_r}(H))$, the latter section being such that $b(\{\sigma_\infty = 0\}) = \tilde{\ell}_\infty$ (in other terms, if i and j are the two indices such that p_i and p_j lie on $\tilde{\ell}_\infty$, one has $\sigma_\infty = \sigma_{\ell_\infty} \cdot \sigma_{E_i} \cdot \sigma_{E_j}$ with $\sigma_{\ell_\infty} \in \mathbf{H}^0(X_r, \mathcal{O}_{X_r}(\ell_\infty)) \setminus \{0\}$). Then $(\sigma_\infty, \sigma_{E_1}, \dots, \sigma_{E_r})$ gives a global section of the T_{NS} -bundle $\mathcal{T}_{X_r}^* \rightarrow X_r^*$ defined as the restriction of $\mathcal{T}_{X_r} \rightarrow X_r$ over X_r^* . Viewing $\mathcal{T}_{X_r}^*$ as a Zariski open subset of $\mathbf{A}(X_r)$, one can consider

$$\hat{\varrho}_r \circ (\sigma_\infty, \sigma_{E_1}, \dots, \sigma_{E_r}) \circ b^{-1} : U_{\mathcal{L}_r} \simeq X_r^* \longrightarrow (\mathbb{C}^*)^{\mathcal{L}_r}.$$

Obviously, the projectivization of this map is the map (4.36) and the commutativity of (4.37) follows immediately from that of (4.34). $\square$

Remark 4.27. — The fact that ϱ_r can be assumed to land into $G_r/P_r \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r})$ and the existence of an embedding $f_{SS} : X_r \hookrightarrow \mathcal{Y}_r$ such that the diagram (4.34) commutes is proved by a recurrence on r in [53]. We believe that these results could be obtained in a more direct and explicit way by considering the map (4.36).

The idea of considering the mapping (4.36) has been suggested to us by A.-M. Castravet.

In the case under consideration, since the Wick map (4.16) is birational, an explicit way to construct a rational map $F_{\mathcal{L}_r}$ landing into the spinor tenfold and making (4.37) commutative is by giving its factorization $\tilde{F}_{\mathcal{L}_r} : \mathbb{C}^2 \dashrightarrow \text{Asym}_5(\mathbb{C})$ by W_5^+ (ie. one has $W_5^+ \circ \tilde{F}_{\mathcal{L}_r} = F_{\mathcal{L}_r}$ as rational maps).

In order to relate our notations with the standard ones of [8, Planche IV], we introduce the basis $(\mathbf{F}_i)_{i=1}^5$ of the root space $R_5 = (-K)^\perp$ defined by

the following relations

$$\begin{aligned} \mathbf{F}_i - \mathbf{F}_{i+1} &= \boldsymbol{\rho}_i = \mathbf{e}_i - \mathbf{e}_{i+1} & (\text{for } i = 1, \dots, 4) \\ \text{and } \mathbf{F}_4 + \mathbf{F}_5 &= \boldsymbol{\rho}_5 = \mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3. \end{aligned}$$

Explicitly, one has:

$$\begin{aligned} \mathbf{F}_1 &= \frac{1}{2} \left(\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 + \mathbf{h} \right) \\ \mathbf{F}_2 &= \frac{1}{2} \left(-\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 + \mathbf{h} \right) \\ \mathbf{F}_3 &= \frac{1}{2} \left(-\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 + \mathbf{h} \right) \\ \mathbf{F}_4 &= \frac{1}{2} \left(-\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 + \mathbf{e}_4 - \mathbf{e}_5 + \mathbf{h} \right) \\ \text{and } \mathbf{F}_5 &= \frac{1}{2} \left(-\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{h} \right). \end{aligned}$$

The weight w_ℓ of a line $\ell \in \mathcal{L}_r$ is given by its orthogonal projection onto the root space R_5 , perpendicularly to the canonical class $K = -3\mathbf{h} + \sum_{i=1}^5 \mathbf{e}_i$. Straightforward computations give us the following formulas for the w_ℓ 's expressed in the basis $(\mathbf{F}_i)_{i=1}^5$ of R_5 defined just above:

$$\begin{aligned} w_{\mathbf{e}_1} &= \left(\frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2} \right) & w_{\mathbf{h}-\mathbf{e}_1-\mathbf{e}_5} &= \left(\frac{-1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2} \right) \\ w_{\mathbf{e}_2} &= \left(\frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2} \right) & w_{\mathbf{h}-\mathbf{e}_2-\mathbf{e}_3} &= \left(\frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{1}{2} \right) \\ w_{\mathbf{e}_3} &= \left(\frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2} \right) & w_{\mathbf{h}-\mathbf{e}_2-\mathbf{e}_4} &= \left(\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2} \right) \\ w_{\mathbf{e}_4} &= \left(\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2} \right) & w_{\mathbf{h}-\mathbf{e}_2-\mathbf{e}_5} &= \left(\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2} \right) \\ w_{\mathbf{e}_5} &= \left(\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2} \right) & w_{\mathbf{h}-\mathbf{e}_3-\mathbf{e}_4} &= \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2} \right) \\ w_{\mathbf{h}-\mathbf{e}_1-\mathbf{e}_2} &= \left(\frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) & w_{\mathbf{h}-\mathbf{e}_3-\mathbf{e}_5} &= \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2} \right) \\ w_{\mathbf{h}-\mathbf{e}_1-\mathbf{e}_3} &= \left(\frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{1}{2} \right) & w_{\mathbf{h}-\mathbf{e}_4-\mathbf{e}_5} &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2} \right) \\ w_{\mathbf{h}-\mathbf{e}_1-\mathbf{e}_4} &= \left(\frac{-1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2} \right) & w_{2\mathbf{h}-\sum_{i=1}^5 \mathbf{e}_i} &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right). \end{aligned}$$

We assume that the considered del Pezzo quartic surface X_5 is the total space of the blow-up of the projective plane at the five points in general

position $p_1, \dots, p_5$ which are defined as the projectivization of the following five points of $\mathbb{C}^3 \setminus \{0\}$:

$$\begin{aligned}\widehat{p}_1 &= (1, 0, 0), & \widehat{p}_2 &= (0, 1, 0), & \widehat{p}_3 &= (0, 0, 1), \\ \widehat{p}_4 &= (1, 1, 1), & \widehat{p}_5 &= (a, b, 1).\end{aligned}$$

We set $\widehat{p}_{xy} = (x, y, 1) \in \mathbb{C}[x, y]^3$ and for any i, j such that $1 \leq i < j \leq 5$, we introduce the following equation of the line through p_i and p_j in the affine coordinates x, y :

$$P_{ij} = \det(\widehat{p}_i, \widehat{p}_j, \widehat{p}_{xy}) = 0$$

(hence one has $P_{12} = 1$, $P_{13} = -y$, $P_{14} = 1 - y$, etc.). As an affine equation for the conic passing through the five points p_i 's, we take

$$\mathcal{C}_{ab} = (a - b)xy + b(1 - a)x + a(b - 1)y.$$

From the comparison between the weights of the lines w_ℓ 's given above and those associated to the Wick coordinates as indicated in Table 4.1, it follows easily that the map $\widetilde{F}_{\mathcal{L}} : \mathbb{C}^2 \dashrightarrow \text{Asym}_5(\mathbb{C})$ discussed above is necessarily of the form $(x, y) \dashrightarrow (\lambda_{ij} P_{ij})_{i,j=1}^5$, where the λ_{ij} 's are suitable scalar constants symmetric in i and j . Let us consider the rational map $\mathbf{F} : \mathbb{C}^2 \dashrightarrow \text{Asym}_5(\mathbb{C})$ defined by

$$\mathbf{F}(x, y) = \frac{1}{\mathcal{C}_{ab}} \begin{bmatrix} 0 & -a - b & \cdots & \cdots & -b + y \\ a + b & 0 & \ddots & & \vdots \\ (a + 1)y & x(1 + b) & 0 & \ddots & \vdots \\ (y - 1)a & (x - 1)b & x - y & 0 & -\mu \\ b - y & a - x & ay - bx & \mu & 0 \end{bmatrix}$$

with $\mu = (1 - b)x + (a - 1)y + b - a$.

For any $i = 1, \dots, 5$, we denote by $\mathbf{F}(x, y)_i$ the 4×4 antisymmetric matrix obtained from $\mathbf{F}(x, y)$ by removing its i -th line and its i -th column. By direct computations, we get the

LEMMA 4.28. — *For $i = 1, \dots, 5$, one has:*

$$\text{Pf}(\mathbf{F}(x, y)_i) = \frac{1}{\mathcal{C}_{ab}}.$$

From above, it comes that the relations $\text{Pf}(\mathbf{F}(x, y)_i) = 1/\mathcal{C}_{ab}$ for $i = 1, \dots, 5$ are precisely those ensuring that the following holds true:

COROLLARY 4.29. — *The map $W_5 \circ \mathbf{F} : \mathbb{C}^2 \dashrightarrow \mathbb{S}_5 \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}^r})$ is of the form (4.36). Consequently, $\mathcal{P}_5 \circ \mathbf{F} : \mathbb{C}^2 \dashrightarrow \mathbb{C}^5$ is a birational model for Serganova–Skorobogatov's embedding $f_{SS} : X_r \hookrightarrow \mathcal{Y}_r$.*

Up to some conventions distinct from ours⁽²⁴⁾, an explicit formula for a map $\mathcal{A}_{SV} : \mathbb{C}^2 \dashrightarrow \text{Asym}_5(\mathbb{C})$ such that $W_5 \circ \mathcal{A}_{SV}$ be of the form (4.36) can be found in [57, §4] (hence the subscript “SV” which is for “Sturmfels–Velasco”). This map is given by

$$\mathcal{A}_{SV}(x, y) = \begin{bmatrix} 0 & \frac{(y-x)(s_2-s_1)}{(b-a)} & \frac{(1-x)(s_3-s_1)}{(1-a)} & -\frac{xs_1}{a} & 1 \\ \vdots & 0 & \frac{(1-y)(s_3-s_2)}{(1-b)} & -\frac{ys_2}{b} & 1 \\ \vdots & \ddots & 0 & -s_3 & 1 \\ \vdots & & \ddots & 0 & 1 \\ \vdots & \dots & \dots & \dots & 0 \end{bmatrix}$$

where the s_i ’s stand for some auxiliary generic (but fixed) scalar parameters.

Compared to our map $\mathbf{F}$ and with regard to the variables x and y , Sturmfels–Velasco’s map $\mathcal{A}_{SV}$ has two nice features: first it is a polynomial map while the coefficients of $\mathbf{F}(x, y)$ are genuine rational functions; secondly, the formula above for $\mathcal{A}_{SV}$ is just the specialization in the case under consideration of a more general formula (related to a natural embedding into the n -th spinor variety $\mathbb{S}_n$, of the Cox variety of the blow-up of $\mathbb{P}^n$ in $n+3$ points in general position, this for any $n \geq 2$). Since it leads to simpler formulas, we prefer to work with Sturmfels–Velasco map $\mathcal{A}_{SV}$ below, although it involves some auxiliary generic scalar parameters (the s_k ’s for $k = 1, 2, 3$).

It follows from Proposition 4.22 and Proposition 4.26 that the map $\mathcal{P}_5 \circ \mathcal{A}_{SV} : \mathbb{C}^2 \dashrightarrow \mathbb{C}^5$ is a birational model for $\chi_5 \circ W_5^+ \circ \mathcal{A}_{SV} = f_{SS}$. A straightforward computation gives us that, in the rational coordinates $(y_{1,3}, y_{1,4}, y_{2,4}, y_{2,5}, y_{3,5})$ on the target space, $\mathcal{P}_5 \circ \mathcal{A}_{SV}$ is given explicitly by

$$(x, y) \mapsto \left(\frac{(x-1)s_{13}}{(a-1)s_3}, \frac{x(y-1)(a-b)s_{123}}{(x-y)a(b-1)s_3s_{12}}, \frac{y(a-b)s_2}{(x-y)bs_{12}}, \frac{(b-1)s_3}{(y-1)s_{23}}, \frac{(x-y)(b-1)s_{12}}{(y-1)(a-b)s_{23}} \right)$$

where $s_{ij} = s_i - s_j$ for $i, j = 1, \dots, 3$.

For any $i = 1, \dots, 5$, let $\mathcal{F}_i^+$ be the foliation on $\mathbb{C}^2$ with first integral

$$\psi_i^+ \circ \mathcal{P}_5 \circ \mathcal{A}_{SV} : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$$

⁽²⁴⁾ The main difference between the convention of [57] and ours is that, in the case under scrutiny, Sturmfels and Velasco take $\wedge^{\text{even}} V$ for S^+ , and not $\wedge^{\text{odd}} V$ as we do in this text.

(cf. (4.29) for the maps ψ_i^+). A priori $\mathcal{F}_i^+$ might be a trivial foliation (that is a foliation by points, which would occur exactly when $\psi_i^+ \circ \mathcal{P}_5 \circ \mathcal{A}_{SV}$ is generically of rank 2) but it turns out that this is precisely not the case. Indeed, all the maps $\psi_i^+ \circ \mathcal{P}_5 \circ \mathcal{A}_{SV}$ are generically of rank 1, all the $\mathcal{F}_i^+$'s are foliations by rational curves and direct easy computations give that these are the following:

$$\begin{aligned} \mathcal{F}_1^+ &= \mathcal{F}_y, & \mathcal{F}_2^+ &= \mathcal{F}_x, & \mathcal{F}_3^+ &= \mathcal{F}_{\frac{x}{y}}, \\ \mathcal{F}_4^+ &= \mathcal{F}_{\frac{y-1}{x-1}} & \text{and} & & \mathcal{F}_5^+ &= \mathcal{F}_{\frac{x(y-1)}{y(x-1)}}. \end{aligned}$$

For any $i = 1, \dots, 5$, let $\mathcal{F}_i^-$ be the foliation on $\mathbb{C}^2$ whose first integral is any one of the maps $\psi_i^{-,j} \circ \mathcal{P}_5 \circ \mathcal{A}_{SV} : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$ with $j \neq i$ (cf. the formulas (4.30) and (4.31)). As for the $\mathcal{F}_i^+$'s, a priori $\mathcal{F}_i^-$ might be a trivial foliation but, as before, it turns out that this is precisely not the case. Indeed, the $\mathcal{F}_i^-$'s are foliations by rational curves and direct easy computations give that these are the following five:

$$\begin{aligned} \mathcal{F}_1^- &= \mathcal{F}_{\frac{(x-y)(a-x)}{x((y-1)a+(1-x)b-y+x)}} & \mathcal{F}_2^- &= \mathcal{F}_{\frac{(x-y)(b-y)}{(ay-bx-a+b+x-y)y}} \\ \mathcal{F}_3^- &= \mathcal{F}_{\frac{(x-1)(b-y)}{(-1+y)a+(1-x)b-y+x}} & \mathcal{F}_4^- &= \mathcal{F}_{\frac{(b-y)x}{ay-bx}} & \mathcal{F}_5^- &= \mathcal{F}_{\frac{b-y}{ay-bx}}. \end{aligned}$$

In more geometric terms, recalling that for any i , one denotes by $\mathcal{L}_{p_i}$ the pencil of lines through p_i and by $\mathcal{C}_{\widehat{p_i}}$ the pencil of conics through the p_j 's for $j \in \{1, \dots, 5\} \setminus \{i\}$, one has

$$\begin{aligned} \mathcal{F}_1^+ &= \mathcal{L}_{p_1} & \mathcal{F}_2^+ &= \mathcal{L}_{p_2} & \mathcal{F}_3^+ &= \mathcal{L}_{p_3} & \mathcal{F}_4^+ &= \mathcal{L}_{p_4} & \mathcal{F}_5^+ &= \mathcal{C}_{\widehat{p_5}} \\ \mathcal{F}_1^- &= \mathcal{C}_{\widehat{p_1}} & \mathcal{F}_2^- &= \mathcal{C}_{\widehat{p_2}} & \mathcal{F}_3^- &= \mathcal{C}_{\widehat{p_3}} & \mathcal{F}_4^- &= \mathcal{C}_{\widehat{p_4}} & \mathcal{F}_5^- &= \mathcal{L}_{p_5}. \end{aligned}$$

These foliations are affine birational models of the ten fibrations in conics on the quartic del Pezzo surface dP_4 under consideration. We thus get the

PROPOSITION 4.30. — *As ordered 5-webs, one has*

$$\mathcal{W}(\mathcal{L}_{p_1}, \dots, \mathcal{L}_{p_4}, \mathcal{C}_{\widehat{p_5}}) = f_{SS}^* \left(\mathcal{W}(\psi_i^+)_{i=1}^5 \right) = \mathcal{A}_{SV}^* \left(\mathcal{W}(\Phi_i^+)_{i=1}^5 \right)$$

and

$$\mathcal{W}(\mathcal{C}_{\widehat{p_1}}, \dots, \mathcal{C}_{\widehat{p_4}}, \mathcal{L}_{p_5}) = f_{SS}^* \left(\mathcal{W}(\psi_i^-)_{i=1}^5 \right) = \mathcal{A}_{SV}^* \left(\mathcal{W}(\Phi_i^-)_{i=1}^5 \right).$$

Consequently, del Pezzo's web $\mathcal{W}_{dP_4}$ is the pull-back, under Serganova–Skorobogatov embedding $f_{SS} : dP_4^* \hookrightarrow \mathcal{Y}_5^*$, of the quotient by the action of the Cartan torus H_{D_5} of Spin_{10} , of Gelfand–MacPherson web $\mathcal{W}_{S_5}^{GM}$ on the tenfold spinor variety S_5 . In mathematical terms, one has

$$\mathcal{W}_{dP_4} = f_{SS}^* \left(\mathcal{W}_{\mathcal{Y}_5}^{GM} \right) = \mathcal{A}_{SV}^* \left(\mathcal{W}_{S_5}^{GM} \right).$$

We believe that this description *à la Gelfand–MacPherson* of $\mathcal{W}_{\mathrm{dP}_4}$ is quite interesting and deserves further investigations. We make a few comments on this in the subsections Section 5.9 and Section 5.10 further.

4.6. The web $\mathcal{W}_{\mathrm{dP}_4}$ is modular

In this subsection, we show that a del Pezzo $\mathcal{W}_{\mathrm{dP}_4}$ can be obtained in a natural way from a web defined by modular rational maps between modular spaces. The general set-up is that of [45, §1.2.7.2] (see also §1.2.7.3 therein) and our arguments below rely on explicit computations which we will not detail.

Let $\mathrm{Conf}_6(\mathbb{P}^2)$ be the space of configurations of six points in general position on $\mathbb{P}^2$: if $Z \subset (\mathbb{P}^2)^6$ stands for the algebraic subset of 6-tuples of points not in linear general position, one has

$$\mathrm{Conf}_6(\mathbb{P}^2) = \left((\mathbb{P}^2)^6 - Z \right) / \mathrm{PGL}_3(\mathbb{C}).$$

To any pair (i, J) where i is an element of $\{1, \dots, 6\}$ and J a subset of $\{1, \dots, 6\} \setminus \{i\}$ of cardinality 4 is associated a rational morphism

$$\pi_{i,J} : \mathrm{Conf}_6(\mathbb{P}^2) \longrightarrow \mathcal{M}_{0,4}, [p_1, \dots, p_6] \longmapsto [\pi_i(p_j)]_{j \in J}$$

where $\pi_i : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ is the linear projection from the i th point of the configuration. There are $6 \times \binom{5}{4} = 30$ such maps and any two of these define two distinct foliations of codimension 1 on $\mathrm{Conf}_6(\mathbb{P}^2)$. We thus get a natural (modular) 30-web by hypersurfaces⁽²⁵⁾ on this space:

$$\mathcal{W}_{\mathrm{Conf}_6(\mathbb{P}^2)} = \mathcal{W}(\pi_{i,J} \mid i = 1, \dots, 6, J \subset \{1, \dots, 6\} \setminus \{i\}, |J| = 4).$$

Given a quartic del Pezzo surface dP_4 , let $q_1, \dots, q_5$ be 5 points in general position in $\mathbb{P}^2$ such that dP_4 identifies with the total space of the blow-up at the q_i 's. Denoting by $\beta : \mathrm{dP}_4 \rightarrow \mathbb{P}^2$ the corresponding morphism, we get a well-defined rational map by setting

$$B : \mathrm{dP}_4 \dashrightarrow \mathrm{Conf}_6(\mathbb{P}^2), x \longmapsto [q_1, \dots, q_5, \beta(x)]. \quad (4.38)$$

⁽²⁵⁾ The term “*web*” refers here to a more general notion than the one classically referred to. Indeed, the foliations of $\mathcal{W}_{\mathrm{Conf}_6(\mathbb{P}^2)}$ do not satisfy the “general position property” usually required in web geometry but the weaker property that two of its foliations intersect transversally (note that this is somehow tautological here since this only means that these two foliations of codimension 1 are distinct which is indeed the case!). For a discussion about the more general notion of “web” considered here, see [45, §1.1.1].

PROPOSITION 4.31. — *The web $\mathcal{W}_{\mathrm{dP}_4}$ coincides with the pull-back of $\mathcal{W}_{\mathrm{Conf}_6(\mathbb{P}^2)}$ under the map B :*

$$\mathcal{W}_{\mathrm{dP}_4} = B^*(\mathcal{W}_{\mathrm{Conf}_6(\mathbb{P}^2)}).$$

Note that the map B in (4.38) is not canonical since there are several non equivalent ways to describe the considered del Pezzo quartic surface dP_4 as the total space of a blow-up of $\mathbb{P}^2$ at five points in general position. However the pull-back of $\mathcal{W}_{\mathrm{Conf}_6(\mathbb{P}^2)}$ under any such map is independent of it and is $\mathcal{W}_{\mathrm{dP}_4}$.

4.7. The web $\mathcal{W}_{\mathrm{dP}_4}$ as a cluster web

In this subsection, we show that a del Pezzo web $\mathcal{W}_{\mathrm{dP}_4}$ admits a birational model which is a cluster web (of type D_4). Our arguments below rely on explicit computations that we omit.

4.8. The cluster algebra setting

As the initial exchange matrix of type D_4 that we will consider, we take

$$B_{D_4} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The $\mathcal{X}$ -cluster web of type D_4 associated to B_{D_4} is the web denoted by $\mathcal{XW}_{D_4}$, whose first integrals are the $\mathcal{X}$ -cluster variables obtained from the initial seed

$$\mathcal{S}_0 = ((a_i)_{i=1}^4, (x_i)_{i=1}^4, B_{D_4}).$$

It is a 52-web in four variables which has been studied in [45] (see Section 7.3.1.2 and more specifically Section 7.3.2.1 therein). Since the cluster variables can be constructed explicitly (using a computer algebra system), this web can be made entirely explicit but there is no point in giving explicit expressions for its 52 cluster first integrals here.

Let $\mathcal{W}_{\mathrm{dP}_4}$ be a given fixed del Pezzo's web. The purpose of this section is to explain how one can obtain a cluster description of it from $\mathcal{XW}_{D_4}$ by specializing a general construction the broad outlines of which were described in [45, §8.4.2] to the case under consideration.

To this aim, we have to deal with the associated “cluster ensemble” introduced by Fock and Goncharov in [24]. In the case we are interested in, it

is the triple $(\mathcal{A}_{D_4}, \mathcal{X}_{D_4}, p)$ where $\mathcal{A}_{D_4}$ and $\mathcal{X}_{D_4}$ stand respectively for the $\mathcal{A}$ - and $\mathcal{X}$ -cluster variety constructed from $\mathcal{S}_0$ and $p : \mathcal{A}_{D_4} \rightarrow \mathcal{X}_{D_4}$ is the cluster map

Actually, there are more structures associated to the cluster ensemble $(\mathcal{A}_{D_4}, \mathcal{X}_{D_4}, p)$ since, as explained in [24, §2]:

- (1) the map $p : \mathcal{A}_{D_4} \rightarrow \mathcal{X}_{D_4}$ corresponds to the quotient map under the $H_{\mathcal{A}}$ -action of a certain algebraic torus $H_{\mathcal{A}}$ acting on the $\mathcal{A}$ -cluster variety;
- (2) there is an “exact sequence” of cluster varieties/maps

$$\mathcal{A}_{D_4} \xrightarrow{p} \mathcal{X}_{D_4} \xrightarrow{\lambda} H_{\mathcal{X}} \longrightarrow 1 \quad (4.39)$$

where $H_{\mathcal{X}}$ is a torus and λ is a map with a monomial expression in each $\mathcal{X}$ -cluster chart. That (4.39) be “exact” means that one has $\mathcal{U} = \text{Im}(p) = \lambda^{-1}(1)$ as subvarieties of $\mathcal{X}_{D_4}$.

Since B_{D_4} is a 4×4 matrix of rank 2, the map p has rank 2 as well hence both $H_{\mathcal{A}}$ and $H_{\mathcal{X}}$ are tori of dimension 2. In the initial cluster coordinates $a_1, \dots, a_4$ and $x_1, \dots, x_4$, the two cluster maps p and λ are written:

$$\begin{aligned} p : (a_s)_{s=1}^4 &\longmapsto (a_2^{-1}, a_1 a_3 a_4, a_2^{-1}, a_2^{-1}) \\ \text{and } \lambda : (x_i)_{i=1}^4 &\longmapsto \left(\frac{x_1}{x_3}, \frac{x_1}{x_4} \right). \end{aligned}$$

For $\tau \in (\mathbb{C}^*)^2$, one defines $\mathcal{XW}_{D_4, \tau}$ as the restriction of $\mathcal{XW}_{D_4}$ along $\mathcal{X}_{\tau} = \lambda^{-1}(\tau) \subset \mathcal{X}_{D_4}$:

$$\mathcal{XW}_{D_4, \tau} = (\mathcal{XW}_{D_4})|_{\lambda=\tau}.$$

Here are some facts/remarks which can be made about the cluster webs $\mathcal{XW}_{D_4, \tau}$:

- For $\tau \in H_{\mathcal{X}}$ generic, $\mathcal{XW}_{D_4, \tau}$ is a 38-web (see below for explicit expressions for the cluster first integrals of this web).
- For $\tau = (a, b)$ with $(a - 1)(b - 1) = 0$ but $(a, b) \neq (1, 1)$, $\mathcal{XW}_{D_4, \tau}$ is a 28-web.
- The case when $\tau = \mathbf{1} = (1, 1)$ is particularly interesting. Indeed, $\mathcal{XW}_{D_4, \mathbf{1}}$ is a 18-web which is equivalent to the web associated to Kummer’s identity of the tetralogarithm (see [45, §5.2]). In particular, $\mathcal{XW}_{D_4, \mathbf{1}}$ carries polylogarithmic ARs of weight 4.

In view of giving a cluster description of $\mathcal{W}_{\text{dP}_4}$ (up to equivalence), we are going to consider a certain 10-subweb of the 38-web $\mathcal{XW}_{D_4, \tau}$ for $\tau = (a, b) \in$

$H_{\mathcal{X}}$ generic (that is such that a and b are such that $ab(a-1)(b-1) \neq 0$), which will be denoted by $\mathcal{XW}_{D_4, \tau}^{10}$.

For $\tau = (a, b)$ as above, the first integrals for $\mathcal{XW}_{D_4, \tau}$ are given by considering the pull-backs of the 52 $\mathcal{X}$ -cluster variables obtained from the seed $\mathcal{S}_0$ under the affine parametrization

$$(u, v) \mapsto (x_1, \dots, x_4) = (u^{-1}, v, u^{-1}, u^{-1})$$

of $\mathcal{U}$.⁽²⁶⁾ There are some redundancies among these pull-backs from which one can extract a list of explicit expressions for the 38 first integrals of the web $\mathcal{XW}_{D_4, \tau}$ (see page 1118).

The ten cluster variables in red in this list are those which are the first integrals of the subweb we are interested in, which is defined as follows as an ordered 10-web:

$$\begin{aligned} \mathcal{XW}_{D_4, \tau}^{10} = \mathcal{W} \Big(& \frac{1+v}{au}, \frac{1+u+v}{bu(1+u)}, \frac{1+bu+v}{u(1+bu)}, \frac{1}{u}, \frac{1+au+v}{u(1+au)}, \\ & \frac{abu^3 + (a(b+1)+b)u^2 + u(a+b+1) + v + 1}{uv}, \\ & \frac{abu^2 + (a+b)u + v + 1}{buv}, \frac{au^2 + au + u + v + 1}{uv}, \\ & \frac{abu^3 + (a(b+1)+b)u^2 + (v+1)(a+b+1)u + (v+1)^2}{au^2v}, \\ & \left. \frac{bu^2 + bu + u + v + 1}{uv} \right). \end{aligned}$$

We consider the affine birational change of variables $\Phi : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$, $(x, y) \mapsto (u, v)$ given by

$$\begin{aligned} u &= \frac{(b-1)x - by + 1}{b(1-y)} \\ \text{and } v &= (b-1) \frac{bx - (b-1)y - xy}{b(1-y)^2}. \end{aligned} \tag{4.40}$$

By straightforward computations, one obtains that as a 10-tuple of rational functions, one has

$$\Phi^*(\mathcal{XW}_{D_4, \tau}^{10}) = (V_i)_{i=1}^{10}$$

⁽²⁶⁾ We use u^{-1} instead of u in this parametrization in order to get “nicer” formulas for the cluster first integrals of $\mathcal{XW}_{D_4, \tau}$.

$$\begin{aligned}
& \frac{1}{u}, \frac{1}{v}, \frac{au+1}{v}, \frac{u+1}{v}, \frac{bu+1}{v}, \frac{v+1}{au}, \frac{(u+1)(au+1)}{v}, \\
& \frac{(u+1)(bu+1)}{v}, \frac{v+1+u}{bu(u+1)}, \frac{(bu+1)(au+1)}{v}, \frac{v+1+u}{uv}, \\
& \frac{bu+v+1}{u(bu+1)}, \frac{bu+v+1}{buv}, \frac{au+v+1}{u(au+1)}, \frac{au+v+1}{auv}, \\
& \frac{abu^3 + (a(b+1)+b)u^2 + u(a+b+1) + v+1}{uv}, \\
& \frac{bu^2 + bu + u + v + 1}{uv}, \frac{au^2 + au + u + v + 1}{uv}, \\
& \frac{abu^2 + (a+b)u + v + 1}{buv}, \frac{(u+1)(bu+1)(au+1)}{v}, \\
& \frac{abu^3 + (a(b+1)+b)u^2 + (v+1)(a+b+1)u + (v+1)^2}{au^2v}, \\
& \frac{(v+1+u)(bu+v+1)}{(bu^2+bu+u+v+1)au}, \frac{(v+1+u)(au+v+1)}{(au^2+au+u+v+1)bu}, \\
& \frac{(bu+v+1)(au+v+1)}{u(abu^2+(a+b)u+v+1)}, \frac{bu^2+bu+u+v+1}{au(u+1)(bu+1)}, \\
& \frac{au^2+au+u+v+1}{bu(u+1)(au+1)}, \frac{abu^2+(a+b)u+v+1}{u(bu+1)(au+1)}, \\
& \frac{(bu^2+bu+u+v+1)(au^2+au+u+v+1)}{uv(v+1+u)}, \\
& \frac{(bu^2+u(b+1)+v+1)(abu^2+(a+b)u+v+1)}{vbu(bu+v+1)}, \\
& \frac{(au^2+(a+1)u+v+1)(abu^2+(a+b)u+v+1)}{vau(au+v+1)}, \\
& \frac{(v+1+u)(bu+v+1)}{u^2vb}, \frac{(v+1+u)(au+v+1)}{u^2va}, \\
& \frac{(bu^2+u(b+1)+v+1)(au^2+(a+1)u+v+1)(abu^2+(a+b)u+v+1)}{au^3bv^2}, \\
& \frac{(bu+v+1)(au+v+1)}{vbu^2a}, \frac{(v+1+u)(bu+v+1)(au+v+1)}{u^3vba}, \\
& \frac{(bu^2+bu+u+v+1)(au^2+au+u+v+1)}{vbu^2a(u+1)}, \\
& \frac{(bu^2+u(b+1)+v+1)(abu^2+(a+b)u+v+1)}{u^2va(bu+1)}, \\
& \frac{(au^2+(a+1)u+v+1)(abu^2+(a+b)u+v+1)}{u^2vb(au+1)}.
\end{aligned}$$

where the V_i 's are the following elements of $\mathbb{Q}[a, b](x, y)$:

$$\begin{aligned}
 V_1 &= \frac{b-y}{a(y-1)} \\
 V_2 &= \frac{1}{x-1} \\
 V_3 &= \frac{y}{x-y} \\
 V_4 &= \frac{b(y-1)}{b(x-y)-x+1} \\
 V_5 &= \frac{b((a-1)y+b-a)}{a((b-1)x-by+1)+b(y-1)} \\
 V_6 &= \frac{a(b-1)x^2 + ((y-1)b - (y+1)(b-1)a)x + a(b-1)y}{(b-y)x - y(b-1)} \\
 V_7 &= \frac{((1-a)y+ax)(y-1)}{(-b-x+1)y+bx} \\
 V_8 &= \frac{(a(x-1)+b)(y-1)}{(-b-x+1)y+bx} \\
 V_9 &= \frac{((ax+b-1)y-ax)b}{a((-b-x+1)y+bx)} \\
 V_{10} &= \frac{xb(y-1)}{(-b-x+1)y+bx}.
 \end{aligned}$$

Then one sets

$$(\pi, \gamma) = \left(\frac{a-b}{a}, \frac{a-b}{a-1} \right) \quad (4.41)$$

(which is equivalent to $(a, b) = (\gamma/(\gamma-\pi), (1-\pi)\gamma/(\gamma-\pi))$ and

$$\begin{aligned}
 q_1 &= [1 : 0 : 0], & q_2 &= [0 : 1 : 0], & q_3 &= [1 : 0 : 0], \\
 q_4 &= [1 : -1 : 1], & q_5 &= [\pi : \gamma : 1].
 \end{aligned}$$

From (4.41), one has $ab(a-1)(b-1) \neq 0$ if and only if $\pi\gamma(\pi-1)(\gamma-1)(\pi-\gamma) \neq 0$. Since the former condition has been assumed, the same holds for the second which geometrically means that the five points q_i are in general position in $\mathbb{P}^2$. One easily checks that for $i = 1, \dots, 5$, the rational function V_i is a primitive first integral for the pencil of lines through q_i , whereas V_{5+i} is a primitive first integral for the pencil of conics through all the q_k 's except q_i . Denoting here by

$$\beta_{\pi, \gamma} : \text{dP}_4(\pi, \gamma) = \mathbf{Bl}_{q_1+\dots+q_5}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$$

the blow-up of the plane at the q_i 's, one obtains the

PROPOSITION 4.32. — *For any parameters $\pi, \gamma \in \mathbb{C}$ such that $\pi\gamma(\pi - 1)(\gamma - 1)(\pi - \gamma) \neq 0$, one has*

$$\mathcal{W}_{\mathrm{dP}_4(\pi, \gamma)} = (\Phi \circ \beta_{\pi, \gamma})^*(\mathcal{XW}_{D_4, \tau})$$

where $\tau = (a, b)$ is related to (π, γ) via the relation (4.41). In particular, $\mathcal{W}_{\mathrm{dP}_4(\pi, \gamma)}$ is cluster.

This proposition is established by means of simple arguments or calculations that are explicit but do not say much about either the conceptual context underlying the proof or the numerous calculations we have performed to obtain it. We think it is worth saying a few words about the process that led us to establish that any del Pezzo web associated with a quartic del Pezzo surface is a cluster web. There were three main points to be investigated for establishing this:

- (1) first, given a del Pezzo quartic dP_4 , we had the intuition to look for $\mathcal{W}_{\mathrm{dP}_4}$ as a subweb of a certain cluster 38-web $\mathcal{XW}_{D_4, \tau}$;
- (2) then given a web $\mathcal{XW}_{D_4, \tau}$, the task was to find one of its 10-subwebs likely to be equivalent to $\mathcal{W}_{\mathrm{dP}_4}$;
- (3) finally, once such a 10-subweb $\mathcal{XW}_{D_4, \tau}^{10}$ was identified, the problem was to find an analytic equivalence of the latter with $\mathcal{W}_{\mathrm{dP}_4}$.

We dealt with these three points as follows:

- regarding the first, we recall that Bol’s web $\mathcal{W}_{\mathrm{dP}_5}$, which lives on dP_5 , is also (equivalent to) the $\mathcal{X}$ -cluster web of type A_2 hence is naturally defined on the cluster variety $\mathcal{X}_{A_2}$. The latter essentially⁽²⁷⁾ is the Looijenga interior $\mathrm{dP}_5 \setminus P$ where P is an anticanonical pentagon in dP_5 .⁽²⁸⁾ This suggests to consider the case of the Looijenga interior $U = \mathrm{dP}_4 \setminus P_4$ where P_4 now stands for an anticanonical square in $|-K_{\mathrm{dP}_4}|$. Using [35, §2.4] and especially Theorem 2.13 therein, we come to the conclusion that U can be described as a fiber of the map λ for the cluster ensemble of type D_4 ;
- having identified the cluster algebra set-up which might be the right one, it is straightforward to get the cluster variables defining $\mathcal{XW}_{D_4, \tau}$. The problem we are facing at this step is that the latter web is a 38-web and it is not at all clear, assuming that it is possible

⁽²⁷⁾ “Essentially” means here “up to a birational map inducing an isomorphism in codimension 1”.

⁽²⁸⁾ An “anticanonical pentagon” is an element of the anticanonical linear system formed by 5 lines cyclically labeled with the i -th intersecting only lines with labels $i - 1$ and $i + 1$ in two distinct points.

(what we proved to be the case), how to distinguish 10 of the 38 cluster variables on $\mathcal{X}_{D_4, \tau}$ which could define a web equivalent to $\mathcal{W}_{\text{dP}_4}$. We succeeded in this task by brute force computations of some arithmetic and combinatorial invariants of the 10-subwebs of $\mathcal{X}\mathcal{W}_{D_4, \tau}$. Heavy and lengthy computations led us to consider more closely the web $\mathcal{X}\mathcal{W}_{D_4, \tau}^{10}$ defined by the ten red cluster variables of page 1118, which share the same invariants with $\mathcal{W}_{\text{dP}_4}$.

- Finally, we deduce the explicit expression (4.40) for the components of an analytic equivalence between $\mathcal{X}\mathcal{W}_{D_4, \tau}^{10}$ and $\mathcal{W}_{\text{dP}_4}$ by means of a more thorough investigation/comparison of the invariants of these two webs.

5. Questions and perspectives

In this section we discuss some questions and perspectives we find interesting considering the results obtained above in this text together with others that we briefly recall.

5.1. Del Pezzo's web on singular del Pezzo's quartic surfaces.

The singular quartic surfaces of $\mathbb{P}^4$ which can be obtained by blowing-up up five points on $\mathbb{P}^2$ no longer assumed to be in general position have been classified by ancient geometers and all are explicitly known (e.g. see [51], [58], [31, p. 286], etc. For modern references, see [15, §4] or [20, §3.4]).

Since the del Pezzo webs of the smooth quartic del Pezzo surfaces all satisfy so many nice features, it is natural to wonder about the properties of the webs formed by the fibrations in conics on the singular quartic del Pezzo surfaces. By means of direct computations, we have studied the webs by conics on the del Pezzo quartic surfaces with isolated singular points, mainly with regard to their rank and their abelian relations. We have found that way interesting webs, many of maximal rank, that we are going to describe succinctly below.

We consider only quartic surfaces in $\mathbb{P}^4$ with only finitely many singular points. There are 15 possibilities for the type of singularities for such a surface (cf. [15, Prop. 5.6] or [20, Table 6]), and for each of them there are specific numbers for the lines and for the conic fibrations on the corresponding singular quartic surfaces. Since we are interested in exceptional webs, one has to restrict oneself to singular quartic surfaces $S \subset \mathbb{P}^4$ carrying at

least 5 pencils of conics, so that the most singular types of such surfaces can be left aside and we just have to deal with the type of singularities, denoted by Σ , being one of the following ones: $A_1, 2A_1, A_2, 3A_1, A_1A_2, A_3$.⁽²⁹⁾

We use the following notations:

- Σ denotes an element of $\{\emptyset, A_1, 2A_1, A_2, 3A_1, A_1A_2, A_3\}$;
- $S = S(\Sigma)$ stands for a del Pezzo quartic surface in $\mathbb{P}^4$ with Σ as type of singularities;
- any singular del Pezzo quartic $S(\Sigma)$ admits a desingularization $\nu : \tilde{S}(\Sigma) \rightarrow S(\Sigma)$ the composition of which with the inclusion $S(\Sigma) \subset \mathbb{P}^4$ is given by the anticanonical linear system $|-K_{\tilde{S}(\Sigma)}|$. Moreover, as for smooth del Pezzo quartic surfaces, $\tilde{S}(\Sigma)$ is obtained as the blow-up $\beta_\Sigma : \tilde{S}(\Sigma) \rightarrow \mathbb{P}^2$ in five points, but these points are no longer in general position (they even can be infinitely near). The rational map $\phi_{|-K_{\tilde{S}(\Sigma)}|} \circ \beta_\Sigma^{-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ is given by a linear system of cubic curves whose base locus scheme will be denoted by B_Σ ;
- any quartic del Pezzo $S(\Sigma) \subset \mathbb{P}^4$ is cut out by two linearly independent quadratic equations, each being obtained from a symmetric 5×5 matrix. Denoting by A and B these matrices, one can always assume that A is regular and define the *Segre symbol* $\mathcal{S}_\Sigma$ of S , as an uplet of integers encoding the decomposition into Jordan blocks of $A^{-1}B \in \text{Sym}_5(\mathbb{C})$;
- μ_Σ is the dimension of the moduli space of del Pezzo quartic surfaces $S(\Sigma)$;
- ℓ_Σ denotes the number of lines contained in $S(\Sigma)$;
- κ_Σ stands for the number of pencils of conics on $S(\Sigma)$;
- $\mathcal{W}_{S(\Sigma)}$ denotes the κ_Σ -web defined by all conic fibrations on $S(\Sigma)$.

Depending on the type of singularities Σ , most of the elements of the list above are given in Table 5.1 below. In the last column, we give an explicit expression for the birational model of $\mathcal{W}_{S(\Sigma)}$ obtained by taking the pull-back of this web under the map $\phi_{|-K_{\tilde{S}(\Sigma)}|} \circ \beta_\Sigma^{-1}$. The base loci B_Σ are pictured using the convention that genuine points on the base $\mathbb{P}^2$ are in black, whereas the infinitely near points are pictured white.

⁽²⁹⁾ Here the notation $3A_1$ (resp. A_1A_2) refers to quartic surfaces whose singular set is formed by three singular points of type A_1 (resp. by two singular points, one of type A_1 , the other of type A_2), etc.

Using several effective tools, we have studied all the webs explicated in this table, especially with regard to their abelian relations and their rank. From this, we get Table 5.1

PROPOSITION 5.1. — *The del Pezzo's webs in Table 5.1 all have maximal rank hence are exceptional.*

Generally speaking, the webs by conics on del Pezzo quartic surfaces with finitely many singularities all are of maximal rank and this seems to be verified as well for many cubic surfaces of $\mathbb{P}^3$ with the same type of singularities (however, recall that $\mathcal{W}_{\text{dP}_3}$ does not have maximal rank for a generic smooth del Pezzo cubic dP_3). Given that being of maximum rank (and exceptional when non-linearizable) is a rather exceptional phenomenon for a web, we believe that the fact that many del Pezzo webs on smooth or singular del Pezzo surfaces are of maximum rank cannot be a mere coincidence. This leads us to ask the

QUESTION 5.2. — *Does it exist (and if yes, which is it) a general conceptual reason explaining why so many del Pezzo's webs (that is, webs formed by considering all the conic fibrations) of del Pezzo surfaces without or with only finitely many singular points are of maximal rank?*

5.2. Behavior of the $\mathcal{W}_{\text{dP}_d}$'s in families and degeneracies

For $d \in \{1, 2, 3, 4\}$ there is a $2(5 - d)$ -dimensional algebraic family $d\mathcal{P}_d$ of degree d del Pezzo surfaces giving rise to a family of webs $\mathcal{W}_{\text{dP}_d} \rightarrow d\mathcal{P}_d$ over it. First, it is interesting to ask how the webs $\mathcal{W}_{\text{dP}_d}$'s behave when dP_d varies in $d\mathcal{P}_d$.

Another kind of natural questions is how the $\mathcal{W}_{\text{dP}_d}$'s may degenerate: there are modular compactifications $\overline{d\mathcal{P}_d}$ of the moduli space $d\mathcal{P}_d$ and it is natural to wonder first whether $\mathcal{W}_{\text{dP}_d} \rightarrow d\mathcal{P}_d$ can be extended over $\overline{d\mathcal{P}_d}$, then if it is the case, what are the properties (in particular those concerning its ARs) of a web $\mathcal{W}_{\text{dP}'_d}$ on a degenerate del Pezzo surface $\text{dP}'_d \in \overline{d\mathcal{P}_d} \setminus d\mathcal{P}_d$. Since the singular del Pezzo surfaces considered in Section 5.1 are such degenerations, questions asked here are generalizations of those raised in the above subsection.

In view of considering a very concrete example, let us consider the webs on $\mathbb{P}^2$ defined by the 10 rational first integrals (4.3), denoted by $\mathcal{W}_{\text{dP}_4(\gamma, \pi)}$. Assuming that (4.2) is satisfied, each such web is a planar model of the 10-web by conics on a smooth del Pezzo quartic surface which will be denoted by $\text{dP}_4(\gamma, \pi)$. Let us consider the naive specialization $\mathcal{W}_{\text{dP}_4, -1} =$

Table 5.1. Webs by conics on quartic del Pezzo surfaces with finitely many singular points, for all types of singularities Σ such that $\kappa_\Sigma \geq 5$. (Base points on the base $\mathbb{P}^2$ are in black, whereas those in white are infinitely near the other points they are touching).

Σ	B_Σ	$\mathcal{S}_\Sigma$	μ_Σ	ℓ_Σ	κ_Σ	Birational model for the κ_Σ -web $\mathcal{W}_\Sigma(\Sigma)$
$\emptyset$		[11111]	2	16	10	$\mathcal{W}_{\text{dP}_4} = \mathcal{W}\left(x, y, \frac{x}{y}, \frac{y-1}{x-1}, \dots, \frac{(y-1)(x-\alpha)}{(x-1)(y-\beta)}, \frac{y(x-\alpha)}{x(y-\beta)}, \frac{x-\alpha}{y-\beta}\right)$
A_1		[2111]	1	12	8	$\mathcal{W}\left(x, y, x - ay, \frac{x}{y}, \frac{y-1}{x-1}, \frac{x(y-1)}{y(x-ay+a-1)}, \frac{(x-ay)(y-1)}{y(x-ay+a-1)}, \frac{(x-1)(x-ay)}{x(x-ay+a-1)}\right)$
$2A_1$		[221]	0	9	6	$\mathcal{W}\left(x, y, x - y, \frac{x}{y}, \frac{y-1}{x-1}, \frac{x(y-1)}{y(x-1)}\right)$
		[(11)111]	1	8	7	$\mathcal{W}\left(x, y, x - y, \frac{x}{y}, \frac{x(x-y)}{x-ay}, \frac{y(x-y)}{x-ay}, \frac{xy}{x-ay}\right)$
$3A_1$		[(11)21]	0	6	5	$\mathcal{W}\left(x, y, \frac{x}{y}, \frac{xy}{x+y}, \frac{y^2}{x+y}\right)$
A_2		[311]	0	8	6	$\mathcal{W}\left(x, y, \frac{x}{y}, \frac{y-1}{x-1}, \frac{x(y-1)}{y(x-1)}, \frac{x-y}{xy}\right)$
A_1A_2		[32]	0	6	4	$\mathcal{W}\left(x, y, x - y, \frac{x}{y}, \frac{xy}{x-y}\right)$
A_3		[(21)11]	0	4	5	$\mathcal{W}\left(y, \frac{x}{y}, \frac{1+x}{y}, \frac{y^2+x}{xy}, \frac{x^2+y^2+x}{xy}\right)$

$\lim_{\pi, \gamma \rightarrow -1} \mathcal{W}_{\mathrm{dP}_4(\gamma, \pi)}$ of the webs $\mathcal{W}_{\mathrm{dP}_4(\gamma, \pi)}$'s: it is the web whose first integrals are the limits of all the U_i 's in (4.3) when both parameters π and γ go to -1 . It is no longer a 10-web but the following 8-web

$$\mathcal{W}_{\mathrm{dP}_4, -1} = \mathcal{W}\left(x, \frac{1}{y}, \frac{y}{x}, \frac{x-y}{x-1}, \frac{x+1}{x-y}, \frac{x-y}{(x-1)(y+1)}, \frac{y(x+1)}{x(y+1)}, \frac{x(y-1)}{y(x-1)}\right).$$

One verifies that this web has maximal rank 21 with all its ARs being generalized hyperlogarithms of weight 1 or 2.⁽³⁰⁾ Denote by $\mathbf{HLog}_{\gamma, \pi}^3$ the weight 3 hyperlogarithmic functional identity carried by $\mathcal{W}_{\mathrm{dP}_4(\gamma, \pi)}$ for γ and π generic (that is, satisfying (4.2)). From the description of the ARs of $\mathcal{W}_{\mathrm{dP}_4, -1}$ just mentioned, it comes that $\mathbf{HLog}_{\gamma, \pi}^3$ somehow disappears under the specialization $(\gamma, \pi) \rightarrow (-1, -1)$. It would be interesting to understand this phenomenon better and more generally to see how the ARs of the generic web $\mathcal{W}_{\mathrm{dP}_4(\gamma, \pi)}$ and those of the specialization $\mathcal{W}_{\mathrm{dP}_4, -1}$ are related.

5.3. Global single-valued version of $\mathbf{HLog}^3$

An interesting well-known feature of Abel's identity (**Ab**) is that there is a global single-valued real analytic version of it. More precisely, let D stand for the “*Bloch–Wigner dilogarithm*”, which is defined for any $z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ by

$$D(z) = \mathrm{Im}\left(\mathbf{Li}_2(z)\right) + \mathrm{Arg}(1-z) \cdot \mathrm{Log}|z|, \quad (5.1)$$

where $\mathrm{Arg} : \mathbb{C}^* \rightarrow]-\pi, \pi]$ denotes the main branch of the complex argument. Bloch–Wigner's dilogarithm D is real-analytic and extends continuously to the whole Riemann sphere. Denoting again by D this extension, one has $D(0) = D(1) = D(\infty) = 0$ and the most remarkable of its features is that it satisfies the following global version of (**Ab**): one has

$$D(x) - D(y) - D\left(\frac{x}{y}\right) - D\left(\frac{1-y}{1-x}\right) + D\left(\frac{x(1-y)}{y(1-x)}\right) = 0$$

for any $x, y \in \mathbb{P}^1$ such that none of the five arguments of D in this identity be indeterminate.

The hyperlogarithmic identity $\mathbf{HLog}^3$ is holomorphic, because the functions AH_i^3 's involved in it are holomorphic. On the other hand this functional identity is only locally satisfied, because the AH^3 's extend to global but multivalued functions on the whole projective line. Remark that the function AH^2 involved in $\mathbf{HLog}^2 \simeq \mathbf{Ab}$ is a holomorphic version of Rogers dilogarithm R which admits D as a single-valued global cousin. The point

⁽³⁰⁾ Here, the adjective phrase “*generalized hyperlogarithmic*” refers to the notion of “*generalized iterated integrals*” considered in [45, §1.5.3].

is that it has been proved in [10, §7] (see also [59, §2.5] or [13, §3]) that the most general hyperlogarithm HL admits a single-valued global version HL^{sing} . Specializing this to the case of the weight 3 antisymmetric hyperlogarithms AH_i^3 involved in $\mathbf{HLog}^3$ gives global single-valued functions $AH_i^{3,sing}$ for $i = 1, \dots, 10$, regarding which the following questions immediately arise:

QUESTION 5.3. —

- (1) *Can one give an explicit formula for the $AH_i^{3,sing}$'s?*
- (2) *Do the $AH_i^{3,sing}$'s satisfy the global univalued version of $\mathbf{HLog}^3$, i.e. using the same notations than in the first page of this text, do we have identically $\sum_{i=1}^{10} AH_i^{3,sing}(U_i) = 0$ on the whole considered del Pezzo quartic surface dP_4 ?*

We believe that the answer to (2) is positive, possibly up to considering suitable single-valued versions of the antisymmetric hyperlogarithms AH_i^3 . An approach for answering this question might be to use Theorem 1.1 of [13, §3] but this would require first to build a “motivic lift” of the identity $\mathbf{HLog}^3$.

5.4. Motivic lift

In [13], the authors discuss motivic avatars of hyperlogarithms, some single-valued (still motivic) versions of these, classical realisations of them, as well as a motivic approach of the functional identities satisfied by the hyperlogarithms. Thanks to its algebraic nature, we expect that $\mathbf{HLog}^3$ indeed can be lifted into the “motivic world”. It would be interesting to investigate this more rigourously.

5.5. Supersymmetric version of $\mathbf{HLog}^3$

Several authors have undertaken to build a theory of cluster superalgebras. In the preprint [33], the authors explore physical applications of this notion. In particular, in the fourth section, they discuss a “super cluster polylogarithm identity” which can be seen as a supersymmetric version of Abel’s identity $\mathcal{Ab} \simeq \mathbf{HLog}^2$. Because $\mathbf{HLog}^3$ is cluster as well (as shown above in Section 4.7), it is natural to ask the

QUESTION 5.4. — *Does it exist a cluster supersymmetric version of $\mathbf{HLog}^3$?*

5.6. Non-archimedean version of $\mathbf{HLog}^3$

Analytic p -adic versions of classical polylogarithms have been constructed by Coleman in [14] where he moreover proved that the p -adic version of Rogers dilogarithm satisfies a (or “the”?) p -adic analog of the 5 terms identity (cf. [14] or [61]). Since Coleman’s work (where another approach was used), a theory of non-archimedean iterated integrals has become available (see [5]) which led us to think that the antisymmetric hyperlogarithms AH_i^3 involved in $\mathbf{HLog}^3$ should have non-archimedean avatars. If this is so, it is then natural to ask the

QUESTION 5.5. — *Does it exist a non-archimedean version of $\mathbf{HLog}^3$?*

5.7. Quantum version of $\mathbf{HLog}^3$

The identity $\mathcal{A}b \simeq \mathbf{HLog}^2$ can be seen as a classical limit of a quantum pentagon dilogarithmic identity $q\mathcal{A}b$ (see [45, §8.6.1] and the references given there). Does this generalize to $\mathbf{HLog}^3$? More precisely:

QUESTION 5.6. — *Does it exist a quantum version $q\mathbf{HLog}^3$ of the identity $\mathbf{HLog}^3$ such that the latter could be seen as a classical limit of the former?*

5.8. Scattering diagram interpretation of $\mathbf{HLog}^3$

There is now a conceptual interpretation of why the 5-terms dilogarithmic identity (or even its quantum version $q\mathcal{A}b$) holds true in terms of a certain “scattering diagram”. Indeed, that $(\mathcal{A}b)$ (or $q\mathcal{A}b$) holds reflects the fact that a certain (quantum) “scattering diagram” is consistent (see [38, 39] by Nakanishi, or his book “Cluster Algebras and Scattering Diagrams”).

On another hand, there is a scattering diagram SC_{dP_4} associated to any del Pezzo surface dP_4 which has been made explicit by Argüz [1]. Considering this and what has been said in the preceding paragraph, it is natural to wonder whether something similar does occur for $\mathbf{HLog}^3$ with respect to SC_{dP_4} :

QUESTION 5.7. — *Can the identity $\mathbf{HLog}^3$ be more conceptually interpreted in terms of some (algebraic, combinatorial or topological) properties of the scattering diagram SC_{dP_4} ?*

5.9. Construction of $\mathbf{HLog}^3$ à la Gelfand–MacPherson

One of the most exciting results obtained by Gelfand and MacPherson in [27] is the geometric construction, henceforth providing a conceptual explanation for it, which they give of Abel’s identity (**Ab**) of the dilogarithm. In concise terms, in a real setting:

- first they identify, for any $n \geq 1$, the quotient of a certain Zariski open subset $G_2^\times(\mathbb{R}^{n+3})$ formed of “generic” 2-planes in $\mathbb{R}^{n+3}$ by the action of the positive part $H_{>0} = H(\mathbb{R}_{>0}) \simeq (\mathbb{R}_{>0})^{n+m-1}$ of the Cartan torus H of $\mathrm{SL}_{m+n}(\mathbb{R})$, with the real moduli space $\mathcal{M}_{0,n+3}(\mathbb{R})$ of projective configurations of $n+3$ points on the real projective line;
- then they define a general notion of “*polylogarithmic differential form*” on $\mathcal{M}_{0,n+3}(\mathbb{R})$, which is any form $\omega(P)_{n+3}$ obtained by integrating along the fibers of the $H_{>0}$ -action on $G_2^\times(\mathbb{R}^{n+3})$, a $\mathrm{SO}_{n+3}(\mathbb{R})$ -invariant closed differential form $\Omega(P)$ whose cohomology class coincides with a previously given stable characteristic class $P \in \mathbf{H}^*(G_2(\mathbb{R}^{n+3}), \mathbb{R})$;
- via an isomorphism induced by the moment map, the (closure in the grassmannian of the) $H_{>0}$ -orbit of any generic 2-plane can be seen to have the combinatorial structure of the hypersimplex $\Delta_2^{n+3} = \{(t_i)_{i=1}^{n+3} \in [0, 1]^{n+3} \mid \sum_i t_i = 2\}$. This polytope has $2(n+3)$ -facets (1-codimensional faces), which are the intersections $\Delta_2^{n+3} = \Delta_{2,i,\nu}^{n+3} \cap \{t_i = \nu\}$ for $i = 1, \dots, n+3$ and $\nu \in \{0, 1\}$. Up to isomorphisms induced by natural linear inclusions $\mathbb{R}^{n+2} \hookrightarrow \mathbb{R}^{n+3}$, the $\Delta_{2,i,0}^{n+3}$ ’s are $(n+1)$ -simplices while the $\Delta_{2,i,1}^{n+3}$ ’s are hypersimplices Δ_2^{n+2} ;
- for any i , let $\mathbb{R}_i^{n+2}$ be the coordinate hyperplane in $\mathbb{R}^{n+3}$ defined by the vanishing of the i -th coordinate. Then $G_2(\mathbb{R}_i^{n+2})$ is naturally contained in $G_2(\mathbb{R}^{n+3})$ and is $H_{>0}$ -stable. Moreover, there exists a $H_{>0}$ -equivariant rational map $F_i : G_2(\mathbb{R}^{n+3}) \dashrightarrow G_2(\mathbb{R}_i^{n+2})$ the $H_{>0}$ -reduction of which is the i -th forgetful map $f_i : \mathcal{M}_{0,n+3}(\mathbb{R}) \rightarrow \mathcal{M}_{0,n+2}(\mathbb{R})$;
- when $P \in \mathbf{H}^4(G_2(\mathbb{R}^{n+2}), \mathbb{R})$ stands for the first Pontryagin class of the rank 2 tautological bundle on $G_2(\mathbb{R}^{n+2})$, Gelfand and MacPherson prove that the following facts hold true:
 - (1) the polylogarithmic differential forms $\omega(P)_{n+3}$ for $n = 1, 2$ satisfy

$$d\omega(P)_5 = \sum_{i=1}^5 f_i^*(\omega(P)_4), \quad (5.2)$$

a differential relation in $\Omega^1(\mathcal{M}_{0,5}(\mathbb{R}))$ which follows from Stokes' theorem for integration along the fibers of the $H_{>0}$ -actions on $G_2(\mathbb{R}^5)$ and $G_2(\mathbb{R}^4)$;

- (2) the function $\omega(P)_5$ vanishes identically whereas up to the identification $\mathcal{M}_{0,4}(\mathbb{R}) \simeq \mathbb{R} \setminus \{0, 1\}$ provided by the cross-ratio, the differential 1-form $\omega(P)_4$ coincides with the total derivative of Rogers dilogarithm R : one has

$$\omega(P)_5 = 0 \quad \text{and} \quad \omega(P)_4 = dR; \quad (5.3)$$

- from (5.2) and (5.3) together, it follows immediately that $0 = \sum_{i=1}^5 f_i^*(dR)$, which is nothing else but the differential form of Abel's identity (**Ab**).

Because $\mathcal{W}_{\text{dP}_4}$ can be obtained from Gelfand–MacPherson web $\mathcal{W}_{\mathbf{y}_5}^{GM} = (\mathcal{W}_{\mathbb{S}_5}^{GM})/H_5$ and because of the many similarities between $\mathbf{HLog}^2 \simeq \mathbf{Ab}$ and $\mathbf{HLog}^3$, we are naturally led to ask the following

QUESTION 5.8. — *Can the functional identity $\mathbf{HLog}^3$ be obtained à la Gelfand–MacPherson by means of an invariant differential form $\Omega(P)$ on a real form $\mathbf{S}_5$ of the spinor tenfold $\mathbb{S}_5$ which represents a certain characteristic class $P \in \mathbf{H}^*(\mathbf{S}_5, \mathbb{R})$?*

We believe that the proper answer to this question is “no” but this is because it is not the relevant question to ask, as we explain in the next subsection.

5.10. Study of the Gelfand–MacPherson web $\mathcal{W}_{\mathbf{y}_5}^{GM}$ and of its abelian relations

The so many similarities between the webs $\mathcal{W}_{\text{dP}_5}$ and $\mathcal{W}_{\text{dP}_4}$ may let us think that the del Pezzo's webs $\mathcal{W}_{\text{dP}_d}$ are the most natural generalizations of $\mathcal{W}_{\text{dP}_5} \simeq \mathcal{B}$ but this expectation is too naive, as it appears as soon as one considers the next web $\mathcal{W}_{\text{dP}_3}$ since, by a direct computation, one verifies that the Blaschke–Dubourdieu's curvature of this web is nonzero, which implies that it does not have maximal rank contrarily to $\mathcal{W}_{\text{dP}_5}$ and $\mathcal{W}_{\text{dP}_4}$. Another point which can be opposed to the claim that the $\mathcal{W}_{\text{dP}_d}$'s are the most natural generalization of $\mathcal{W}_{\text{dP}_5}$ is that starting from $d = 4$ included, there is actually a positive dimensional moduli of such webs, which is not the case for the previous two webs, namely $\mathcal{W}_{\text{dP}_5}$ and also $\mathcal{W}_{\text{dP}_6} = \mathcal{W}(x, y, x/y)$ which are unique.

Since the $\mathcal{W}_{\text{dP}_4}$'s can all be obtained from Gelfand–MacPherson web $\mathcal{W}_{\mathbf{y}_5}^{GM}$, it is not unreasonable to ask whether this latter web cannot be seen

as a more natural generalization of $\mathcal{W}_{\mathrm{dP}_5} \simeq \mathcal{W}_{\mathcal{Y}_4}^{GM}$ than the $\mathcal{W}_{\mathrm{dP}_4}$'s. Preliminary results indicate that it is indeed the case. The web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ is a well-defined 10-web on the 5-dimensional rational variety $\mathcal{Y}_5$, defined by 10 rational first integrals $\psi_i^\epsilon : \mathcal{Y}_5 \dashrightarrow \mathbb{S}_4^\times / H_{D_4} \simeq \mathbb{P}^2$ with $i = 1, \dots, 5$ and $\epsilon = \pm$. What seems to be the relevant abelian relations to be considered for this web are the 2-ARs, namely the tuples of 2-forms $(\eta_i^\epsilon)_{i,\epsilon}$ such that $\sum_{i,\epsilon} (\psi_i^\epsilon)^* (\eta_i^\epsilon) = 0$ (possibly just locally) on $\mathcal{Y}_5$. They form a vector space which we denote by $\mathbf{AR}^2(\mathcal{W}_{\mathcal{Y}_5}^{GM})$. Following the same approach as the one described in [45, §1.3.4.], one can define the “virtual 2-rank” $\rho^{(2)}(\mathcal{W})$ of any subweb $\mathcal{W}$ of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$. If ψ stands for one of the first integrals ψ_i^ϵ of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$, we denote by

- $\mathbb{C}(\psi)$ the subalgebra of rational functions on $\mathcal{Y}_5$ formed by compositions $f \circ \psi$ with $f \in \mathbb{C}(\mathbb{S}_4 / H_{D_4}) \simeq \mathbb{C}(\mathbb{P}^2)$;
- $\mathrm{Log} \mathbb{C}(\psi)$ the family of multivalued functions on $\mathcal{Y}_5$ of the form $\mathrm{Log}(f \circ \psi)$ with $f \in \mathbb{C}(\mathbb{S}_4 / H_{D_4}) \simeq \mathbb{C}(\mathbb{P}^2)$;
- $\mathrm{dLog} \mathbb{C}(\psi)$ the space of ψ -logarithmic differential 1-forms, that is of rational 1-forms on $\mathcal{Y}_5$ of the form $\mathrm{dLog}(f \circ \psi) = \mathrm{d}(f \circ \psi) / (f \circ \psi)$ with $f \in \mathbb{C}(\mathbb{S}_4 / H_{D_4}) \simeq \mathbb{C}(\mathbb{P}^2)$.

By means of explicit computations in Maple, we have established the

THEOREM 5.9. —

- (1) One has $\rho^{(2)}(\mathcal{W}_{\mathcal{Y}_5}^{GM}) = 11$ and $\rho^{(2)}(\mathcal{W}) \leq 1$ for every 5-subweb $\mathcal{W}$ of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$.
- (2) Among all the 5-subwebs of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$, exactly 16 have virtual 2-rank equal to 1. These are the subwebs

$$\mathcal{W}^\epsilon = \mathcal{W}(\psi_1^{\epsilon_1}, \dots, \psi_5^{\epsilon_5})$$

for the sixteen 5-tuples $\epsilon = (\epsilon_i)_{i=1}^5 \in \{\pm 1\}^5$ such that $p(\epsilon) = \#\{i \mid \epsilon_i = 1\}$ is odd.⁽³¹⁾ Each such subweb $\mathcal{W}^\epsilon$ actually has maximal rank 1, with $\mathbf{AR}^2(\mathcal{W}^\epsilon)$ spanned by a 2-AR $\mathbf{LogAR}^\epsilon$ which is complete, irreducible and logarithmic, in the sense that the $\psi_i^{\epsilon_i}$ -th component of $\mathbf{LogAR}^\epsilon$ belongs to $\mathrm{dLog} \mathbb{C}(\psi_i^{\epsilon_i}) \wedge \mathrm{dLog} \mathbb{C}(\psi_i^{\epsilon_i})$ for every $i = 1, \dots, 5$.

- (3) The $\mathbf{LogAR}^\epsilon$'s for all odd 5-tuples ϵ 's span a vector space denoted by $\mathbf{AR}_C^2(\mathcal{W}_{\mathcal{Y}_5}^{GM})$ and called the space of “combinatorial ARs” of

⁽³¹⁾ This has to be compared with the description of Bol's subwebs of $\mathcal{W}_{\mathrm{dP}_4}$ given page 1072.

$\mathcal{W}_{\mathcal{Y}_5}^{GM}$. Moreover, one has

$$\mathrm{rk}_C^{(2)}(\mathcal{W}_{\mathcal{Y}_5}^{GM}) = \dim \mathbf{AR}_C^2(\mathcal{W}_{\mathcal{Y}_5}^{GM}) = \rho^{(2)}(\mathcal{W}_{\mathcal{Y}_5}^{GM}) - 1 = 10.$$

- (4) There exists an AR of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$, denoted by $\mathbf{DilogAR}^{(2)}$, which is complete, irreducible and whose components are dilogarithmic, in the sense that for any first integral ψ_i^ϵ of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$, the ψ_i^ϵ -th component of $\mathbf{DilogAR}^{(2)}$ belongs to $\mathrm{Log} \mathbb{C}(\psi_i^\epsilon) \mathrm{dLog} \mathbb{C}(\psi_i^\epsilon) \wedge \mathrm{dLog} \mathbb{C}(\psi_i^\epsilon)$. Moreover, one has

$$\mathbf{AR}^2(\mathcal{W}_{\mathcal{Y}_5}^{GM}) = \mathbf{AR}_C^2(\mathcal{W}_{\mathcal{Y}_5}^{GM}) \oplus \langle \mathbf{DilogAR}^{(2)} \rangle \quad (5.4)$$

which implies $\mathrm{rk}^{(2)}(\mathcal{W}_{\mathcal{Y}_5}^{GM}) = \rho^{(2)}(\mathcal{W}_{\mathcal{Y}_5}^{GM}) = 11$: the web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ has maximal 2-rank.⁽³²⁾

- (5) One can associate a divisor $\mathcal{D}_w$ of $\mathcal{Y}_5$ to each weight w of the minuscule half-spin representation S_5^+ . Then the 2-ARs of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ are regular on the complement in $\mathcal{Y}_5$ of the union of all the $\mathcal{D}_w$'s which coincides with $\mathcal{Y}_5^\times$. Moreover, the sixteen ARs $\mathbf{LogAR}^\epsilon$'s are exactly the logarithmic ARs obtained by considering the residues of $\mathbf{DilogAR}^{(2)}$ along the $\mathcal{D}_w$'s: for any odd 5-tuple ϵ , there exists a uniquely defined weight $w(\epsilon)$ such that, up to multiplication by a non-zero constant, one has $\mathrm{Res}_{\mathcal{D}_{w(\epsilon)}}(\mathbf{DilogAR}^{(2)}) = \mathbf{LogAR}^\epsilon$.
- (6) The action of the Weyl group $W(D_5)$ by automorphisms on $\mathcal{Y}_5$ (cf. [56, Thm. 2.2]) gives rise to a linear $W(D_5)$ -action on $\mathbf{AR}^2(\mathcal{W}_{\mathcal{Y}_5}^{GM})$ which lets invariant the decomposition in direct sum (5.4). This decomposition actually is the one in $W(D_5)$ -irreducibles: the irrep associated to the 1-dimensional component $\langle \mathbf{DilogAR}^{(2)} \rangle$ is the signature representation whereas $\mathbf{AR}_C^2(\mathcal{W}_{\mathcal{Y}_5}^{GM})$ is the $W(D_5)$ -irreducible module $V_{[11,1^3]}^{10}$.⁽³³⁾
- (7) For any del Pezzo surface dP_4 , using Serganova–Skorobogatov embedding $f_{SS} : \mathrm{dP}_4 \hookrightarrow \mathcal{Y}_5$ (cf. (4.34) above), the weight 3 hyperlogarithmic abelian relation $\mathbf{HLog}^3$ of $\mathcal{W}_{\mathrm{dP}_4}$ (resp. the sixteen elements of $\mathbf{HLogAR}_{\mathrm{asym}}^2$ equivalent to $\mathbf{HLog}^2$) can be obtained in a natural way from $\mathbf{DilogAR}^{(2)}$ (resp. from the sixteen logarithmic 2-abelian relations $\mathbf{LogAR}^\epsilon \in \mathbf{AR}_C^2(\mathcal{W}_{\mathcal{Y}_5}^{GM})$).

⁽³²⁾ It would be more rigorous to state this as “the 2-rank of $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ is AMP”, using the terminology introduced in [45, §1.3.5].

⁽³³⁾ Seeing it as a decomposition in $W(D_5)$ -irreducibles, (5.4), must be compared with some results given in Section 4.2.1: in some sense which could be made precise, $\mathbf{AR}_C^2(\mathcal{W}_{\mathcal{Y}_5}^{GM})$ corresponds to $\mathbf{HLogAR}_{\mathrm{asym}}^2$ and $\mathbf{DilogAR}^{(2)}$ to $\mathbf{HLog}^3$.

The similarities between the statements above with the corresponding ones for Bol's web $\mathcal{B} \simeq \mathcal{W}_{\mathrm{dP}_5}$ in Section 1.1 are even more striking than those between the latter statements and those in Section 1.2 about $\mathcal{W}_{\mathrm{dP}_4}$. For this reason, and also because the theorem above can be generalized to all the Gelfand–MacPherson webs $\mathcal{W}_{\mathcal{Y}_r}^{GM}$ for $r = 4, \dots, 8$ (see the next subsection below), we believe that these latter webs are those which must really be considered as the most direct/fundamental generalizations of $\mathcal{B} \simeq \mathcal{W}_{\mathrm{dP}_5} \simeq \mathcal{W}_{\mathcal{Y}_4}^{GM}$, and not really the del Pezzo's webs $\mathcal{W}_{\mathrm{dP}_{9-r}}$ which actually are 2-dimensional slices of the Gelfand–MacPherson webs. All these considerations make it natural to ask the following question, which we believe is the one to be considered instead of Question 5.8:

QUESTION 5.10. — *Can the dilogarithmic 2-abelian relation $\mathbf{DilogAR}^{(2)}$ of the web $\mathcal{W}_{\mathcal{Y}_5}^{GM}$ be obtained following the geometric approach of Gelfand–MacPherson, by integrating along the orbits of the Cartan torus H_{D_5} , of an invariant differential form $\Omega(P)$ on a real form $\mathbf{S}_5$ of the spinor tenfold $\mathbb{S}_5$ which represents a certain characteristic class $P \in \mathbf{H}^*(\mathbf{S}_5, \mathbb{R})$?*

We plan to dedicate a future paper to the theorem and the question above (see [43]).

5.11. Generalizations of the questions above to the identities $\mathbf{HLog}^{r-2}$ for $r = 4, \dots, 8$

The questions asked for $\mathbf{HLog}^3$ until Section 5.7 (included) actually generalize straightforwardly to the identities $\mathbf{HLog}^{r-2}$ for any $r = 4, \dots, 8$.

It is not clear for the moment whether the webs $\mathcal{W}_{\mathrm{dP}_d}$ are of cluster type for $d \in \{3, 2, 1\}$ but since they are Fano surfaces, there exists in each case a d -gone with rational components $D \subset \mathrm{dP}_d$ such that (dP_d, D) be a log Calabi–Yau pair (i.e. $K_{\mathrm{dP}_d} + D$ is trivial).

In a work in progress, we have established that:

- all the del Pezzo's webs $\mathcal{W}_{\mathrm{dP}_d}$'s can be obtained from Gelfand–MacPherson web $\mathcal{W}_{\mathcal{Y}_{9-d}}^{GM}$ or, in other terms: there are versions of Proposition 4.30 for $\mathcal{W}_{\mathrm{dP}_d}$ for any $d \in \{4, \dots, 8\}$;
- Theorem 5.9 generalizes to any Gelfand–MacPherson web $\mathcal{W}_{\mathcal{Y}_{9-d}}^{GM}$ for $d = 4, \dots, 7$.⁽³⁴⁾

⁽³⁴⁾ We believe that there exists a version of Theorem 5.9 for the 2160-web $\mathcal{W}_{\mathcal{Y}_8}^{GM}$ but this is not proved yet.

These preliminary but already interesting results make it possible to ask a suitable version of Question 5.10 for any Gelfand–MacPherson web $\mathcal{W}_{\mathcal{Y}_{9-d}}^{GM}$ for $d = 4, \dots, 7$ (and possibly for $d = 8$ as well). We will study all this more in depth in our coming work [43].

5.12. Playing with canonical maps

In addition to its “naturalness”, the notion of canonical map of a planar web introduced before in this text (see Section 3.5.2) seems to be relevant with respect to the question of constructing planar webs carrying many abelian relations. Since hexagonal webs are of maximal rank, it is a natural and rather easy task to look at some new webs which can be constructed from hexagonal webs and their canonical maps. We discuss this for some interesting examples below.

For $k \geq 4$, let $\mathcal{H}$ be a hexagonal planar k -web: there exist k pairwise distinct points $p_1, \dots, p_k$ in $\mathbb{P}^2$, not necessarily in general position, such that $\mathcal{H} = \mathcal{W}(\mathcal{L}_{p_1}, \dots, \mathcal{L}_{p_k})$, where $\mathcal{L}_{p_i}$ stands for the pencil of lines through p_i for any $i = 1, \dots, k$. We define the “canonical extension” of $\mathcal{H}$ as the planar web denoted by $\overline{\mathcal{H}}^{can}$, obtained by juxtaposing to it the preimage of $\mathcal{W}_{\mathcal{M}_{0,k}}$ under its canonical map $\Phi_{\mathcal{H}} : \mathbb{P}^2 \dashrightarrow \mathcal{M}_{0,k}$: in mathematical symbols, one has

$$\overline{\mathcal{H}}^{can} = \mathcal{H} \boxtimes \Phi_{\mathcal{H}}^*(\mathcal{W}_{\mathcal{M}_{0,k}}).$$

The first two examples to have in mind are those with the p_i ’s in general position for $k = 4$ and $k = 5$: the corresponding canonical extensions are (models of) the two del Pezzo’s webs studied in this text, namely $\mathcal{W}_{dP_5}$ and $\mathcal{W}_{dP_4}$ respectively. Since these two webs carry interesting hyperlogarithmic ARs, it is natural to ask about the general case:

QUESTION 5.11. — *Given k points $p_1, \dots, p_k \in \mathbb{P}^2$, pairwise distinct but not necessarily in general position, what can be said about the canonical extension of the hexagonal web $\mathcal{H} = \mathcal{W}(\mathcal{L}_{p_1}, \dots, \mathcal{L}_{p_k})$, in particular regarding its abelian relations ?*

If our main interest lies in the ARs of $\overline{\mathcal{H}}^{can}$, even determining its basic invariants already is interesting. For instance, even determining the degree $\bar{k}^{can}$ of $\overline{\mathcal{H}}^{can}$ does not seem obvious: one expects that $\bar{k}^{can}$ only depends on the combinatorial type of the projective configuration $[p_i]_{i=1}^k \in \mathcal{M}_{0,k}$ but what might be a closed formula for $\bar{k}^{can}$ is not clear (at least to the author of these lines).

Here is a sample of a few explicit examples we have considered for some degenerated configurations of k points in the plane for $k = 5, 6$. One checks that $\Phi_{\mathcal{H}}$ is constant (hence $\mathcal{H} = \overline{\mathcal{H}}^{can}$) when all the vertices p_i 's are aligned hence we will not consider this case further.

Five points. There are four possible combinatorial types for the p_i 's, we consider each of them separately:

- when four of the p_i 's are on a same line but not the fifth point, then again $\mathcal{H} = \overline{\mathcal{H}}^{can}$ hence there is not much interesting to add.
- When there are two distinct sets of 3 aligned points among the p_i 's, then a normal form for the associated hexagonal 5-web is $\mathcal{H} = \mathcal{W}(x, y, x - y, \frac{x}{y}, \frac{y-1}{x-1})$. From elementary computations, we get that $\overline{\mathcal{H}}^{can} = \mathcal{H} \boxtimes \mathcal{F}_{\frac{x(y-1)}{y(x-1)}} = \mathcal{W}(x, y, x - y, \frac{x}{y}, \frac{y-1}{x-1}, \frac{x(y-1)}{y(x-1)})$ and that this 6-web has maximal rank (hence is exceptional), with the following invariants:

$$\begin{aligned} \text{Hex}_{\bullet}(\overline{\mathcal{H}}^{can}) &= (16, 9, 2, 0) \\ \text{and } \text{Flat}_{\bullet}(\overline{\mathcal{H}}^{can}) &= (16, 11, 2, 1). \end{aligned}$$

- If exactly three of the p_i 's are aligned, then one can assume that $\mathcal{H} = \mathcal{W}(x, y, x - y, \frac{x}{y}, \frac{y-b}{x-a})$ with $ab(a-b) \neq 0$. Elementary computations give us that $\overline{\mathcal{H}}^{can} = \mathcal{H} \boxtimes \mathcal{W}(\frac{y(y-x+a-b)}{(y-b)(x-y)}, \frac{x(y-x+a-b)}{(x-a)(x-y)}, \frac{x(y-b)}{y(x-a)})$ and that this 8-web is of maximal rank (hence is exceptional) and has the following numerical invariants:

$$\begin{aligned} \text{Hex}_{\bullet}(\overline{\mathcal{H}}^{can}) &= (38, 28, 8, 0, 0, 0) \\ \text{and } \text{Flat}_{\bullet}(\overline{\mathcal{H}}^{can}) &= (38, 34, 8, 4, 0, 1). \end{aligned}$$

- Finally, when the p_i 's are in general position, then one can take $\mathcal{H} = \mathcal{W}(x, y, \frac{x}{y}, \frac{y-1}{x-1}, \frac{y-b}{x-a})$ with $ab(a-1)(b-1)(a-b) \neq 0$. Hence one has $\overline{\mathcal{H}}^{can} = \mathcal{W}_{\text{dP}_4}$ where dP_4 is the del Pezzo quartic surface obtained as the blow-up of $\mathbb{P}^2$ at the p_i 's. Moreover, one has

$$\begin{aligned} \text{Hex}_{\bullet}(\mathcal{W}_{\text{dP}_4}) &= (80, 80, 32, 0, 0, 0, 0, 0) \\ \text{and } \text{Flat}_{\bullet}(\mathcal{W}_{\text{dP}_4}) &= (80, 90, 32, 10, 0, 5, 0, 1). \end{aligned}$$

Six points. There are eight possible combinatorial types for the p_i 's (see [28, p. 141]). Several computations lead us to believe that in essentially any case, the canonical extension is not flat (hence not of maximal rank), except when the combinatorial type of the p_i 's is of a specific type.

- Assuming that the p_i 's are the vertices of the non Fano arrangement (e.g. see Figure 2 in [46]), one can assume that $\mathcal{H} = \mathcal{W}(x, y, \frac{x}{y}, \frac{y-1}{x-1}, \frac{y-1}{x}, \frac{y}{x-1})$ in which case one gets that $\overline{\mathcal{H}}^{can}$ coincides with

$$\mathcal{H} \boxtimes \mathcal{W}\left(\frac{x(y-1)}{y(x-1)}, \frac{(x-1)(y-1)}{xy}, \frac{y(y-1)}{x(x-1)}\right).$$

It can be verified (cf. [46, §4.3]) that the 9-web $\overline{\mathcal{H}}^{can}$ is a model of the well-known trilogarithmic Spence–Kummer's web $\mathcal{W}_{SK}$ which is exceptional and carries two linearly independent trilogarithmic abelian relations (see [45, §2.2.3.1] for more details).

The consideration of the preceding examples indicates (in our opinion) that the study of the configurations of points in the plane giving hexagonal webs $\mathcal{H}$ such that $\overline{\mathcal{H}}^{can}$ be of maximal rank is interesting and deserves a further study.

Appendix. Some representation-theoretic computations in GAP

In this appendix we briefly explain the computations we used to establish some representation-theoretic facts stated above in the text. These computations were mainly performed using the software GAP3 [50] which has the advantage of being available online.⁽³⁵⁾

The set-up

First, to define the Weyl group we will work with and return the corresponding Dynkin diagram within GAP, we run the following commands:

```
gap> WD5:=CoxeterGroup("D",5);
gap> PrintDiagram( WD5 );
```

The last command returns the following diagram:

$$\begin{array}{c} 1 \\ \searrow \\ 3 - 4 - 5. \\ \nearrow \\ 2 \end{array} \quad (\text{A.1})$$

To define and get the corresponding characters table, we type

⁽³⁵⁾ One can run GAP3 by typing the command `/ext/bin/gap3` in a CoCalc terminal window.

```
gap> TableWD5:=CharTable(WD5);
gap> Display(TableWD5);
```

From the outcome of the last command, one can extract the character table A.1 (in which the dot $\cdot$ is used instead of 0 to make its reading easier).

Table A.1. Character table of the Weyl group of type D_5 .

	(1 ⁵ .)	(1 ³ .1 ²)	(1.1 ⁴)	(21 ³ .)	(1 ² .21)	(21.1 ²)	(.21 ³)	(221.)	(1.22)	(2.21)	(311.)	(1.31)	(3.11)	(32.)	(.32)	(41.)	(.41)	(5.)
[1 ² .1 ³]	10	-2	2	-4	2	.	-2	2	-2	.	1	-1	1	-1	1	.	.	.
[1.1 ⁴]	5	1	-3	-3	-1	1	3	1	1	-1	2	.	-2	.	.	-1	1	.
[.1 ⁵]	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1
[1 ³ .2]	10	-2	2	-2	.	2	-4	-2	2	.	1	-1	1	1	-1	.	.	.
[1 ² .21]	20	-4	4	-2	2	-2	2	.	.	.	-1	1	-1	1	-1	.	.	.
[1.21 ²]	15	3	-9	-3	-1	1	3	-1	-1	1	.	.	.	.	.	1	-1	.
[.21 ³]	4	4	4	-2	-2	-2	-2	.	.	.	1	1	1	1	1	.	.	-1
[1.2 ²]	10	2	-6	.	.	.	.	2	2	-2	-2	.	2	.	.	.	.	.
[2.21]	20	-4	4	2	-2	2	-2	.	.	.	-1	1	-1	-1	1	.	.	.
[.2 ² 1]	5	5	5	-1	-1	-1	-1	1	1	1	-1	-1	-1	-1	-1	1	1	.
[1 ² .3]	10	-2	2	2	.	-2	4	-2	2	.	1	-1	1	-1	1	.	.	.
[1.31]	15	3	-9	3	1	-1	-3	-1	-1	1	.	.	.	.	.	-1	1	.
[.31 ²]	6	6	6	.	.	.	.	-2	-2	-2	.	.	.	.	.	.	.	1
[2.3]	10	-2	2	4	-2	.	2	2	-2	.	1	-1	1	1	-1	.	.	.
[.32]	5	5	5	1	1	1	1	1	1	1	-1	-1	-1	1	1	-1	-1	.
[1.4]	5	1	-3	3	1	-1	-3	1	1	-1	2	.	-2	.	.	1	-1	.
[.41]	4	4	4	2	2	2	2	.	.	.	1	1	1	-1	-1	.	.	-1
[.5]	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

A few explanations are in order about this table. As is well-known (e.g. see [26, §5.6]), for any odd integer $n \geq 4$,⁽³⁶⁾ the characters as well as the conjugacy classes of the Weyl group of type D_n can be labelled by means of the “bipartitions of n ”, that is pairs (λ, λ') of partitions such that $|\lambda| + |\lambda'| = n$. In Table A.1, we use $[\lambda, \lambda']$ for labelling the line associated to the corresponding character while (λ, λ') is used to label the column associated to the corresponding conjugacy class. When λ is the empty partition, we just write $[\cdot, \lambda']$ and $(\cdot, \lambda')$ respectively and similarly when $\lambda' = \emptyset$.

The roots of the root space of type D_5 associated to a del Pezzo quartic surface we considered in Section 2.3.1 are the ρ_i ’s for $i = 1, \dots, 5$, defined as follows: one has $\rho_i = e_i - e_{i+1}$ for $i = 1, \dots, 4$ and $\rho_5 = h - e_1 - e_2 - e_3$. One deduces that the associated Dynkin diagram is the following

$$\begin{array}{c} \rho_1 - \rho_2 - \rho_3 \begin{array}{l} \nearrow \rho_4 \\ \searrow \rho_5 \end{array} \end{array} \quad (\text{A.2})$$

⁽³⁶⁾ The case when n is even is a bit more involved but fully understood as well.

Let $\sigma_1, \dots, \sigma_5$ be the generators of WD5 as encoded in GAP3, with σ_i corresponding to the vertex labeled by i in (A.1). We recall that the generators of $W = \langle s_1, \dots, s_5 \rangle$ are $s_1, \dots, s_5$ where $s_i = s_{\rho_i}$ for any i (cf. Section 2.3.1). Comparing (A.2) with (A.1), it comes that the isomorphism between WD5 and W that we are considering is induced by the following relations between the σ_i 's and the s_j 's:

$$\sigma_1 = s_4, \quad \sigma_2 = s_5, \quad \sigma_3 = s_3, \quad \sigma_4 = s_2, \quad \sigma_5 = s_1. \quad (\text{A.3})$$

For $\mathbf{c}_1 = \mathbf{h} - \mathbf{e}_1 \in \mathcal{K}$, we label as follows the elements of $\mathbf{c}_1^{\text{red}}$:

$$\begin{aligned} \mathbf{c}_{1,2} &= \mathbf{e}_2 + (\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2) & \mathbf{c}_{1,3} &= \mathbf{e}_3 + (\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_3) \\ \mathbf{c}_{1,4} &= \mathbf{e}_4 + (\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_4) & \mathbf{c}_{1,5} &= \mathbf{e}_5 + (\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_5) \end{aligned}$$

(recall our convention: here $\mathbf{e}_i + (\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_i)$ stands for the non irreducible conic on the (fixed) del Pezzo quartic surface we are working with, whose two irreducible components are the two lines $\mathbf{e}_i$ and $\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_i$, this for any $i = 2, \dots, 5$).

The stabilizer $W_{\mathbf{c}_1}$ of $\mathbf{c}_1$ in W is the subgroup spanned by the s_j 's for $j = 2, \dots, 5$. It is a Weyl group of type D_4 :

$$W_{\mathbf{c}_1} = \langle s_2, \dots, s_5 \rangle \simeq W(D_4).$$

From (A.3), it comes that the corresponding subgroup in GAP can be defined by means of the following command

```
gap> Wc1:= ReflectionSubgroup( WD5, [1,2,3,4] );
```

and we get the characters table of $W_{\mathbf{c}_1}$ (see Table A.2 below) using the GAP command

```
gap> TableWc1:=CharTable(Wc1); Display(TableWc1);
```

Table A.2. Character table of the Weyl subgroup $W_{\mathbf{c}_1}$ (of type D_4).

	1111.	11.11	.1111	211.	1.21	2.11	22.+	22.-	.22	31.	.31	4.+	4.-
11 ⁺	3	-1	3	-1	1	-1	3	-1	-1	0	0	-1	1
11 ⁻	3	-1	3	-1	1	-1	-1	3	-1	0	0	1	-1
1.111	4	0	-4	-2	0	2	0	0	0	1	-1	0	0
.1111	1	1	1	-1	-1	-1	1	1	1	1	1	-1	-1
11.2	6	-2	6	0	0	0	-2	-2	2	0	0	0	0
1.21	8	0	-8	0	0	0	0	0	0	-1	1	0	0
.211	3	3	3	-1	-1	-1	-1	-1	-1	0	0	1	1
2 ⁺	3	-1	3	1	-1	1	3	-1	-1	0	0	1	-1
2 ⁻	3	-1	3	1	-1	1	-1	3	-1	0	0	-1	1
.22	2	2	2	0	0	0	2	2	2	-1	-1	0	0
1.3	4	0	-4	2	0	-2	0	0	0	1	-1	0	0
.31	3	3	3	1	1	1	-1	-1	-1	0	0	-1	-1
.4	1	1	1	1	1	1	1	1	1	1	1	1	1

Our goal is now to determine the character, denoted by $\chi_{\mathbf{c}_1}$, of the $W_{\mathbf{c}_1}$ -representation $\mathbb{Z}^{\mathbf{c}_1^{red}}$. To this end, one first computes the permutation matrix M_i of s_i viewed as an endomorphism of $\mathbb{Z}^{\mathbf{c}_1^{red}}$ when expressed in the basis $(\mathbf{c}_{1,i})_{i=2}^5$. Straightforward computations give us the following formulas:

$$M_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad M_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In view of determining $\chi_{\mathbf{c}_1} = \chi_{\mathbb{C}^{\mathbf{c}_1^{red}}}$, we need to construct explicit representatives of the conjugacy classes in $W_{\mathbf{c}_1}$. This can be achieved in GAP3 by means of the following command:

```
gap> List(ConjugacyClasses(Wc1),
         x->CoxeterWord(Wc1,Representative(x)));
```

which returns the following string of 13 sequences:

```
[ [], [1, 2], [1, 2, 3, 1, 2, 3, 4, 3, 1, 2, 3, 4],
  [1], [1, 2, 3], [1, 2, 4], [1, 4], [2, 4],
  [1, 3, 1, 2, 3, 4], [1, 3], [1, 2, 3, 4],
  [1, 4, 3], [2, 4, 3] ]
```

The k -th element of the above list corresponds to an explicit representative of the conjugacy class associated to the k -th column of Table A.1 in terms of the generators of $W_{\mathbf{c}_1}$ the software GAP is dealing with. More precisely, if this string is $[i_1, \dots, i_m]$, the corresponding representative is $\sigma_{i_1} \cdots \sigma_{i_m}$, where $\sigma_1, \dots, \sigma_5$ stand for the generators of $WD5$ implicitly defined when the latter group was defined. More explicitly, let ν be the permutation such that the relations (A.3) are equivalent to $\sigma_i = s_{\nu(i)}$ for $i = 1, \dots, 5$ (namely one has $\nu = (1425)$). For $[i_1, \dots, i_m]$ in the list above, the value taken by the character $\chi_{\mathbf{c}_1}$ when evaluated on the corresponding conjugacy class is given by the trace

$$\text{Tr}(M_{\nu(i_1)} \cdots M_{\nu(i_m)}) \in \mathbb{Z}.$$

From the material above, it is then just a (straightforward but lengthy) computational matter to determine $\chi_{\mathbf{c}_1}$ explicitly: one gets $\chi_{\mathbf{c}_1} = \mathbb{1} + \chi_{\mathbf{H}_{\mathbf{c}_1}}$ where $\mathbb{1}$ denotes the trivial character and where $\chi_{\mathbf{H}_{\mathbf{c}_1}}$ is given by

$$\chi_{\mathbf{H}_{\mathbf{c}_1}} = (3, -1, 3, 1, -1, 1, -1, 3, -1, 0, 0, -1, 1) .$$

Then using Table A.2, one finally gets that as $W_{\mathbf{c}_1}$ -representation, one has

$$\mathbb{Z}^{\mathbf{c}_1^{\text{red}}} = \mathbf{1} \oplus \mathbf{H}_{\mathbf{c}_1} \quad \text{with} \quad \mathbf{H}_{\mathbf{c}_1} \simeq V_{[2,2]^-}^3. \quad (\text{A.4})$$

Because one obviously has $\bigoplus_{\mathbf{c} \in \mathcal{K}} \mathbf{H}_{\mathbf{c}} \simeq \text{Ind}_{W_{\mathbf{c}_1}}^W(\mathbf{H}_{\mathbf{c}_1})$ as W -representations, one can use (A.4) to determine the decomposition of $\bigoplus_{\mathbf{c} \in \mathcal{K}} \mathbf{H}_{\mathbf{c}}$ into irreducibles. For that purpose, one only needs the induction table from $W_{\mathbf{c}_1}$ to W which can be obtained via the following GAP command:

```
gap> Display( InductionTable( Wc1, WD5) );
```

We get the table below which has to be used as follows: each column corresponds to the W -induction of the $W_{\mathbf{c}_1}$ -irrep labeled by the bipartition of 4 given in the top entry of the column. Denoting by $[\lambda, \lambda']$ this bipartition, the coefficients of the column correspond to the multiplicity of the W -irreducibles appearing in the decomposition of the induction of $V_{[\lambda, \lambda']}$ from $W_{\mathbf{c}_1}$ to W .

Table A.3. Induction table from $W_{\mathbf{c}_1}$ to W (of type D_4 and D_5 respectively).

	$[11]^+$	$[11]^-$	$[1,1^3]$	$[1,111]$	$[11,2]$	$[1,21]$	$[2,11]$	$[2,2]^+$	$[2,2]^-$	$[2,22]$	$[1,3]$	$[3,1]$	$[4]$
$[1^2,1^3]$	1	1	1										
$[1,1^4]$			1	1									
$[1,1^5]$				1									
$[1^3,2]$			1		1								
$[1^2,21]$	1	1			1	1							
$[1,21^2]$			1			1	1						
$[2,1^3]$				1			1						
$[1,2^2]$						1				1			
$[2,21]$					1	1		1	1				
$[2,2^2]$							1			1			
$[1^2,3]$					1						1		
$[1,31]$						1					1	1	
$[3,1^2]$							1					1	
$[2,3]$								1	1		1		
$[3,2]$										1		1	
$[1,4]$											1		1
$[4,1]$												1	1
$[5]$													1

Because $H_{\mathbf{c}_1}$ is isomorphic to the $W_{\mathbf{c}_1}$ -representation $V_{[2,2]^-}$, the above table gives us that

$$\bigoplus_{\mathbf{c} \in \mathcal{K}} \mathbf{H}_{\mathbf{c}} \simeq \text{Ind}_{W_{\mathbf{c}_1}}^W(V_{[2,2]^-}^3) \simeq V_{[2,21]}^{20} \oplus V_{[2,3]}^{10}.$$

The weight 2 hyperlogarithmic ARs of $\mathcal{W}_{\mathrm{dP}_4}$

One has $\mathbf{H}_{\mathrm{dP}_4} \simeq V_{[2,3]}^{10}$ (cf. (4.9)) hence the associated W -character $\chi_{\mathbf{H}_{\mathrm{dP}_4}}$ as well as $\wedge^2 \chi_{\mathbf{H}_{\mathrm{dP}_4}}$ can be defined and the latter can be determined using the following commands in GAP3:

```
gap> chi_HdP4:=TableWD5.irreducibles[14];
gap> Wedge2_chi_HdP4:=AntiSymmetricParts(TableWD5,
                                         [chi_HdP4],2)[1];
```

which return

```
[ 45, -3, -3, 3, 3, -5, 3, -3, 1, 1,
                                0, 0, 0, 0, 0, -1, 1, 0 ]
```

The multiplicities appearing in the decomposition of $\wedge^2 \mathbf{H}_{\mathrm{dP}_4}$ into irreducibles are obtained by typing

```
gap> List(TableWD5.irreducibles,t->ScalarProduct
          (TableWD5,t,Wedge2_chi_HdP4[1]));
```

which returns the string

```
[ 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0 ]
```

from which one deduces that, as W -modules, one has

$$\wedge^2 \mathbf{H}_{\mathrm{dP}_4} \simeq V_{[11,21]}^{20} \oplus V_{[11,3]}^{10} \oplus V_{[1,31]}^{15}. \quad (\text{A.5})$$

Working similarly, one gets $\wedge^2 \mathbf{H}_{\mathbf{e}_1} \simeq V_{[1,1]}^3$ as $W_{\mathbf{e}_1}$ -modules. Using Table A.3, one deduces that as W -modules, one has

$$\bigoplus_{\mathbf{e} \in \mathcal{K}} \wedge^2 \mathbf{H}_{\mathbf{e}} \simeq \mathrm{Ind}_{W_{\mathbf{e}_1}}^W (\wedge^2 \mathbf{H}_{\mathbf{e}_1}) \simeq V_{[11,111]}^{10} \oplus V_{[11,21]}^{20}. \quad (\text{A.6})$$

Because the map $\bigoplus_{\mathbf{e} \in \mathcal{K}} \wedge^2 \mathbf{H}_{\mathbf{e}} \rightarrow \wedge^2 \mathbf{H}_{\mathrm{dP}_4}$ is non zero, it follows from (A.5) and (A.6) that as a W -representation, one has necessarily

$$\mathbf{HLogAR}_{\mathrm{asym}}^2 \simeq V_{[11,111]}^{10}.$$

We now consider the representations involved in the description of $\mathbf{HLogAR}_{\mathrm{sym}}^2$ as a W -module. First we construct the character of the second symmetric product of $\mathbf{H}_{\mathrm{dP}_4}$, then determine its decomposition into irreducible W -representations by typing

```
gap> Sym2_chi_HdP4:=SymmetricParts(TableWD5,
                                   [chi_HdP4],2)[1];
gap> ScalarProduct(TableWD5,t,Sym2_chi_HdP4));
```

The last command returns the following string

```
[0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1]
```

from which we deduce (using Schur orthogonality relations) the decomposition we are looking for:

$$(\mathbf{H}_{\text{dP}_4})^{\odot 2} \simeq \mathbb{1} \oplus V_{[.41]}^4 \oplus V_{[1.4]}^5 \oplus V_{[.32]}^5 \oplus V_{[2.3]}^{10} \oplus V_{[1.22]}^{10} \oplus V_{[2.21]}^{20}.$$

We proceed similarly as in the antisymmetric case for determining the decomposition of $\bigoplus_{\mathbf{c} \in \mathbf{K}} (\mathbf{H}_{\mathbf{c}})^{\odot 2}$. First we get the decomposition $(\mathbf{H}_{\mathbf{c}_1})^{\odot 2}$ as a $W_{\mathbf{c}_1}$ -representation by means of the following commands:

```
gap> chi_Hc1:=TableWc1.irreducibles[9];
gap> Sym2_chi_Hc1:=SymmetricParts(TableWc1,
                                [chi_Hc1],2)[1]:
gap> List(TableWc1.irreducibles,t->
          ScalarProduct(TableWc1,t,Sym2_chi_Hc1));
```

which return the string

```
[0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 1]
```

Hence we obtain that $(\mathbf{H}_{\mathbf{c}_1})^{\odot 2} \simeq V_{[2.2]-}^3 \oplus V_{[.22]}^2 \oplus \mathbb{1}$ as a $W_{\mathbf{c}_1}$ -representation. Then as explained above in (4.12), one deduces that $\mathbf{HLogAR}_{\text{sym}}^2 \simeq V_{[.221]}^5$ as a W -representation.

The $W_{\mathbf{c}_1}$ -representation $\mathbf{H}_{\mathbf{c}_1}$ when $r = 6$

First, in order to define the Weyl group we will work with in this case and return the corresponding Dynkin diagram within GAP, we run the following commands:

```
gap> WE6:=CoxeterGroup("E",6);
gap> PrintDiagram( WE6 );
```

The last command returns the following diagram:

$$\begin{array}{ccccccc} & & 2 & & & & \\ & & | & & & & \\ 1 & - & 3 & - & 4 & - & 5 - 6 \end{array} \quad (\text{A.7})$$

We define and get the corresponding characters table by typing

```
gap> TableWE6:=CharTable(WE6);
gap> Display(TableWE6);
```

The following diagram illustrates the labelling of the fundamental roots considered when $r = 6$ (see (2.11)):

$$\begin{array}{c} \rho_6 \\ | \\ \rho_1 - \rho_2 - \rho_3 - \rho_4 - \rho_5 \end{array} \quad (\text{A.8})$$

Let us denote by $\sigma_1, \dots, \sigma_6$ the generators of the group **WE6** in GAP. Comparing (A.7) and (A.8), one deduces the following relations between the generators $s_i = s_{\rho_i}$ ($i = 1, \dots, r$) of the Weyl group considered in Section 2.3.1 and the σ_i 's implemented in GAP by means of the command **WE6:=CoxeterGroup("E",6)**:

$$\begin{array}{lll} \sigma_1 = s_1, & \sigma_3 = s_2, & \sigma_4 = s_3 \\ \sigma_5 = s_4, & \sigma_6 = s_5, & \sigma_2 = s_6. \end{array} \quad (\text{A.9})$$

The stabilizer $W_{\mathbf{c}_1}$ of $\mathbf{c}_1$ in W is the subgroup spanned by the s_j 's for $j = 2, \dots, 6$. It is a Weyl group of type D_5 :

$$W_{\mathbf{c}_1} = \langle s_2, \dots, s_6 \rangle \simeq W(D_5). \quad (\text{A.10})$$

The elements of the set $\mathbf{c}_1^{\text{red}}$ of reducible conics in the pencil $|\mathbf{c}_1|$ are the $\mathbf{c}_{1,i} = (\mathbf{e}_i) + (\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_i)$'s for $i = 2, \dots, 6$. The action of the s_k 's on $\mathbf{c}_1^{\text{red}}$ for $k = 2, \dots, 6$ is as follows: for $k = 2, \dots, 5$, s_k acts as the transposition exchanging $\mathbf{c}_{1,k}$ and $\mathbf{c}_{1,k+1}$ while s_6 acts as s_2 . The corresponding matrices in the basis $\mathfrak{C}_1 = (\mathbf{c}_{1,i})_{i=2}^6$ of $\mathbb{C}^{\mathbf{c}_1^{\text{red}}} \simeq \mathbb{C}^5$ are the corresponding transposition matrices and will be denoted by

$$M_k = \text{Mat}_{\mathfrak{C}_1}(s_k) \in \text{GL}_5(\mathbb{Z}) \quad k = 2, \dots, 6.$$

From (A.9), the matrices $\mathcal{M}_k = \text{Mat}_{\mathfrak{C}_1}(\sigma_k)$ for $k = 2, \dots, 6$ are the following ones:

$$\begin{array}{lll} \mathcal{M}_1 = M_1, & \mathcal{M}_2 = M_6, & \mathcal{M}_3 = M_2, \\ \mathcal{M}_4 = M_3, & \mathcal{M}_5 = M_4, & \mathcal{M}_6 = M_5. \end{array} \quad (\text{A.11})$$

From (A.9) and (A.10), it comes that the group **Wc1** corresponding to $W_{\mathbf{c}_1}$ as well as its characters table are defined as follows in GAP:

```
gap> Wc1:=ReflectionSubgroup(WE6,[2,3,4,5,6]);
gap> TableWc1:=CharTable(Wc1);Display(TableWc1);
```

In view of determining the character $\chi_{\mathbf{c}_1}$ of the $W_{\mathbf{c}_1}$ -representation $H_{\mathbf{c}_1}$, we need to construct explicit representatives of the conjugacy classes in $W_{\mathbf{c}_1}$. This can be achieved in GAP3 by means of the following command:

```
gap> List(ConjugacyClasses(Wc1),
x->CoxeterWord(Wc1,Representative(x)));
```

which returns the following collection of strings:

[[], [2, 3], [2, 3, 4, 2, 3, 4, 5, 4, 2, 3, 4, 5],
 [2], [2, 3, 4], [2, 3, 5],
 [2, 3, 4, 2, 3, 4, 5, 4, 2, 3, 4, 5, 6],
 [2, 5], [2, 4, 2, 3, 4, 5], [2, 3, 4, 6],
 [2, 4], [2, 3, 4, 5], [2, 3, 5, 6],
 [2, 4, 6], [2, 4, 2, 3, 4, 5, 6], [2, 5, 4],
 [2, 3, 4, 5, 6], [2, 5, 4, 6]]

It is then just a matter of elementary computations with the matrices $\mathcal{M}_2, \dots, \mathcal{M}_6$ to get the character of $\mathbb{C}^{\mathbf{c}_1^{red}}$ as a $W_{\mathbf{c}_1}$ -representation. For instance, the value of this character evaluated on the conjugacy class encoded by the string [2, 3, 5] is the trace of the matricial product $\mathcal{M}_2\mathcal{M}_3\mathcal{M}_5$, etc. Proceeding in this way, one easily obtains that

$$\chi_{\mathbb{C}^{\mathbf{c}_1^{red}}} = (5, 5, 5, 3, 3, 3, 3, 1, 1, 1, 2, 2, 2, 0, 0, 1, 1, 0).$$

This character can be decomposed as the sum of the trivial character with the one associated with the $W_{\mathbf{c}_1}$ -representation $V_{[.41]}^4$. It follows that

$$\mathbf{H}_{\mathbf{c}_1} = V_{[.41]}^4. \quad (\text{A.12})$$

In fact, if one assumes that $\mathbf{H}_{\mathbf{c}_1}$ is irreducible, there is another way for establishing (A.12). Indeed, there are only two irreducible representations of $W_{\mathbf{c}_1} \simeq W(D_5)$ of degree 4, the two associated with the bipartitions [.2111] and [.41] respectively. Moreover, we know that $\text{Ind}_{W_{\mathbf{c}_1}}^{W(E_6)}(\mathbf{H}_{\mathbf{c}_1})$ must contain $\mathbf{H}_{X_6} \simeq V^{20,2}$ as an irreducible $W(E_6)$ -subrepresentation (see Section 2.3.5). One obtains the induction table from $W_{\mathbf{c}_1}$ to $W(E_6)$ by typing the following command in GAP3:

```
gap> Display(InductionTable(Wc1, WE6));
```

We obtain easily that

$$\begin{aligned} \text{Ind}_{W_{\mathbf{c}_1}}^{W_6} \left(V_{[.41]}^4 \right) &= V^{20,2} \oplus V^{24,6} \oplus V^{64,4} \\ \text{while } \text{Ind}_{W_{\mathbf{c}_1}}^{W_6} \left(V_{[.2111]}^4 \right) &= V^{20,20} \oplus V^{24,12} \oplus V^{64,13}. \end{aligned}$$

Because $\mathbf{H}_{X_6} = \mathbf{H}^0(X_6, \Omega_{X_6}^1(\mathbf{Log} L_6))$ is $V^{20,2}$ according to Corollary 2.3, only the first case can occur hence necessarily (A.12) holds true.

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