Annales de la Faculté des Sciences de Toulouse

MATHÉMATIQUES

BLAKE J. BOUDREAUX

T-Polynomial Convexity and Holomorphic Convexity

Tome XXXIV, nº 4 (2025), p. 1147–1157.

https://doi.org/10.5802/afst.1828

Publication membre du centre
Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2258-7519

Annales de la faculté des sciences de Toulouse Volume XXXIV, n° 4, 2025
pp. 1147-1157

T-Polynomial Convexity and Holomorphic Convexity [(*)]

Blake J. Boudreaux [(1)]

Abstract. — We compare the T-polynomial convexity of Guedj with holomorphic convexity away from the support of T. In particular, we prove an Oka–Weil theorem for T-polynomial convexity. We also show a sufficient condition for when the notions of T-polynomial convexity and holomorphic convexity of $X \setminus \operatorname{Supp} T$ coincide in the class of complex projective algebraic manifolds.

Résumé. — Nous comparons la convexité T-polynomiale de Guedj avec la convexité holomorphe sur le complément du support de T. En particulier, nous démontrons un théorème d'Oka–Weil pour la convexité T-polynomiale. Nous montrons également une condition suffisante pour que les notions de convexité T-polynomiale et de convexité holomorphe de $X \setminus \operatorname{Supp} T$ coïncident dans la classe des variétés algébriques projectives complexes.

1. Introduction and Preliminaries

A compact set $K \subset \mathbb{C}^n$ is said to be polynomially convex if it agrees with its hull

$$\widehat{K}_{\mathbb{C}^n} = \left\{ z \in X : |P(z)| \leqslant \sup_K |P| \text{ for all holomorphic polynomials } P \right\}. \quad (1.1)$$

Polynomial convexity can be generalized to Stein manifolds in a straightforward manner by replacing polynomials with entire functions in the above definition and retains many of the same properties. This is appropriately dubbed holomorphic convexity [4]. On the other hand, there is no intrinsic

(*) Reçu le 18 mars 2024, accepté le 4 octobre 2024.

Keywords: holomorphic convexity, polynomial convexity, Stein manifolds, complex projective manifolds.

2020 *Mathematics Subject Classification:* 32E05, 32U40.

(1) Department of Mathematics, Middlesex College, The University of Western Ontario, London, Ontario, Canada, N6A 5B7 — *Current address:* Department of Mathematical Sciences, 850 West Dickson Street, SCEN Room 309 Fayetteville, AR, United States, 72701 — bb225@uark.edu

Article proposé par Jasmin Raissy.

definition of polynomial convexity on complex projective manifolds. Indeed, the compact set $K = \{[1 : e^{i\theta} : e^{-i\theta}] \in \mathbb{CP}^2 : 0 \leqslant \theta \leqslant 2\pi\}$ is polynomially convex when viewed as a subset of $\mathbb{CP}^2 \setminus \{z_0 = 0\} \cong \mathbb{C}^2$, but it is not polynomially convex when viewed as a subset of $\mathbb{CP}^2 \setminus \{z_1 = 0\} \cong \mathbb{C}^2$.

However, viewing $\mathbb{C}^n$ as a subset of complex projective space $\mathbb{CP}^n$, polynomials in $\mathbb{C}^n$ are simply rational functions in $\mathbb{CP}^n$ having poles in the complex hyperplane at infinity. This suggests a generalization of polynomial convexity to a complex projective manifold X by replacing the family of polynomials in (1.1) with the family of meromorphic functions on X having poles in a fixed hypersurface $Z \subset X$.

Since the hypersurface Z corresponds to a closed positive current of integration $[Z]$ having bidegree $(1,1)$, this motivates a further extension of these ideas, due to Guedj [7]. Suppose a positive current T of bidegree $(1,1)$ on a complex projective manifold X has *integral cohomology class*; i.e., $[T] \in H^2_{\mathrm{dR}}(X,\mathbb{R}) \cong H^2(X,\mathbb{R})$ lies in the image of the morphism $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{R})$ induced by the containment $\mathbb{Z} \hookrightarrow \mathbb{R}$. Then there exists a holomorphic line bundle L on X and a (singular) metric φ of L for which $\mathrm{dd}^c \varphi = T$ [10, Theorem 5]. We consider compact sets $K \subset X$ for which the hull

$$p_T(K) = \left\{ z \in X \ : \begin{array}{r} |\sigma|e^{-k\varphi}(z) \leqslant \sup\limits_K |\sigma|e^{-k\varphi} \\ \text{for all } \sigma \in \Gamma(X,L) \text{ and } k \in \mathbb{N} \end{array} \right\} \tag{1.2}$$

and K agree.(1) Note that if T is a current of integration corresponding to a complex hypersurface Z, then the objects with respect to which the hull is being taken in (1.2) take the form $|\sigma/s^k|$, where s is a holomorphic section of some line bundle on X whose zero divisor coincides with Z; that is, they become precisely the family of meromorphic functions on X with poles on Z.

There is also the following weaker notion, known as T-polynomial convexity.(2) Note that it does not require T to have integral cohomology class.

Definition 1.1. — *Let T be a positive closed current of bidegree $(1,1)$ on a complex projective manifold X and let K be a compact subset of X. We define the T-polynomially convex hull of K by*

$$\widehat{K}^T = \left\{ z \in X \ : \ f(z) \leqslant \sup_K f \ \textit{for all } f \in \mathcal{C}_T(X) \ \textit{such that } \mathrm{dd}^c f \geqslant -T \right\},$$

(1) Here and throughout we make the slight abuse of notation that $|\sigma|e^{-k\varphi}(z)$ stands for $|\sigma|_{k\varphi}(z)$, the action of the metric $k\varphi$ on the section σ at the point $z \in X$.

(2) It should be noted that, while the notions of convexity corresponding to $p_T(K)$ and $\widehat{K}^T$ are analogous to convexity with respect to polynomial and plurisubharmonic functions, respectively, we call the latter *T-polynomial convexity* to be consistent with the literature.

where $\mathcal{C}_T(X)$ denotes the set of functions $f \in L^1(X)$ such that $\exp(f+\varphi)$ is continuous whenever φ is a local dd^c-potential of T. Note in particular that any f in $\mathcal{C}_T(X)$ is lower semicontinuous.

Polynomial convexity is of interest, in particular, in view of the Oka–Weil theorem [12]. In its simplest formulation, it states that a holomorphic function defined on a neighbourhood of a polynomially convex compact set in $\mathbb{C}^n$ is the uniform limit on K of a sequence of polynomials. This has been extended to T-polynomial convexity when T is a current of integration corresponding to a positive divisor [7, Theorem 3.10]. Our first result is to extend this further. We need an additional assumption on T, called condition (C). This means that $\widehat{K}^T \subset\subset X \setminus \operatorname{Supp} T$ whenever $K \subset\subset X \setminus \operatorname{Supp} T$. Here $c_1 : \operatorname{Pic}(X) \to H^2(X,\mathbb{Z})$ denotes the first Chern class homomorphism.

Theorem 1.2. — *Let T be a positive closed current of bidegree $(1,1)$ satisfying condition (C) on a projective algebraic manifold X. Assume $[T] = c_1(L)$ for some positive holomorphic line bundle L on X. Suppose that the compact set $K \subset X \setminus \operatorname{Supp} T$ is T-polynomially convex and that $f \in \mathcal{O}(K)$. Then there exist $u_j, v_j \in \Gamma(X, L^{N_j})$, $j \in \mathbb{N}$, so that the meromorphic function u_j/v_j approximates f uniformly on K as $j \to \infty$. Furthermore,*

(i) $\frac{1}{N_j}\log|v_j| - \varphi \xrightarrow{j\to\infty} 0$ *uniformly on compact subsets of $X \setminus \operatorname{Supp} T$, where φ is a (singular) metric of L with $\mathrm{dd}^c\varphi = T$;*

(ii) $\frac{1}{N_j}\left[v_j^{-1}(0)\right] \xrightarrow{j\to\infty} T$ *in the weak sense of currents;*

(iii) $v_j^{-1}(0) \xrightarrow{j\to\infty} \operatorname{Supp} T$ *in the Hausdorff metric;*

(iv) $\nu\Big(\frac{1}{N_j}\left[v_j^{-1}(0)\right], z\Big) \xrightarrow{j\to\infty} \nu(T,z)$ *for all $z \in X$.*

Remark. — The assumptions are T are not surprising, since the tools available to us require the environment in which we are working to be as "Stein-like" as possible. Condition (C) and the positivity of L are analogous to the conditions of holomorphic convexity and separability in the definition of a Stein manifold, respectively.

In the spirit of the previous remark, our next result shows that, in this context, the notion of T-polynomial convexity in $X \setminus \operatorname{Supp} T$ is equivalent to the notion of holomorphic convexity on $X \setminus \operatorname{Supp} T$.

Theorem 1.3. — *Let T be a positive closed current of bidegree $(1,1)$ satisfying condition (C) on a projective algebraic manifold X. Assume $[T] = c_1(L)$ for some positive holomorphic line bundle L. Then $\widehat{K}^T = \widehat{K}_{X\setminus \operatorname{Supp} T}$ for all compact subsets $K \subset X \setminus \operatorname{Supp} T$, where $\widehat{K}_{X\setminus\operatorname{Supp} T}$ denotes the holomorphically convex hull of K in the Stein manifold $X \setminus \operatorname{Supp} T$.*

Guedj showed that $X \setminus \operatorname{Supp} T$ is Stein on a compact Kähler manifold X whenever T satisfies condition (C) and T is cohomologous to a Kähler form [7, Theorem 3.8]. If X is a *homogeneous* projective manifold, then every (1,1)-current of integral cohomology class satisfies condition (C), so in such a setting the assumption of condition (C) in the previous statement is redundant. We end the introduction with a natural question.

Question. — *Let T be a positive current of bidegree $(1,1)$ on a complex projective manifold X with $c_1(L) = [T]$ for some positive line bundle L. Does T satisfy condition (C)?*

2. T-polynomial convexity and holomorphic convexity of $X \setminus \operatorname{Supp} T$

It is natural to ask under what conditions $p_T(K) = \widehat{K}^T$ holds for all compact subsets K. It has been shown that $p_T(K) \subseteq \widehat{K}^T$ with equality whenever L is positive and X is homogeneous [7, Proposition 3.2]. Our first observation is that the assumption of homogeneity on X is not necessary for equality. This follows from a somewhat recent extension result, quoted below.

Theorem 2.1 ([1, Theorem B′]). — *Let X be a subvariety of a projective manifold V and L be an ample line bundle on V. Then any (singular) positive metric of $L|_X$ is the restriction of a (singular) positive metric of L on V.*

Corollary 2.2. — *Let X be a projective algebraic manifold and T be a positive closed current of bidegree $(1,1)$ on X. Assume that $[T] = c_1(L)$ for some positive line bundle L, and let φ be a (singular) metric on L with $\mathrm{dd}^c\,\varphi = T$. Then $p_T(K) = \widehat{K}^T$ for all compact subsets $K \subset X$.*

Proof. — The inclusion $\widehat{K}^T \subseteq p_T(K)$ is straightforward. Indeed, if $z \notin p_T(K)$, then there exists a k and a $\sigma \in \Gamma(X, L^k)$ with $|\sigma| e^{-k\varphi}(z) > \sup_K |\sigma| e^{-k\varphi}$. Therefore

$$\frac{1}{k}\log|\sigma(z)| - \varphi(z) > \sup_K \left(\frac{1}{k}\log|\sigma| - \varphi\right)$$

and so $z \notin \widehat{K}^T$ since $\frac{1}{k}\log|\sigma|(z) - \varphi(z)$ is a member of $\mathcal{C}_T(X)$.

For the reverse inclusion, fix $z \notin \widehat{K}^T$. In view of the Kodaira embedding theorem we can assume $X \subset \mathbb{CP}^N$ with $\mathcal{O}(1)|_X = L^k$ for some k. Then there exists a positive metric ψ of L on X with e^ψ continuous and $(\psi - \varphi)(z) > \sup_K(\psi - \varphi)$ [7, Proposition 3.2]. Extend $k\psi$ and $k\varphi$ to metrics $\widetilde{\psi}$ and $\widetilde{\varphi}$, respectively, on $\mathbb{CP}^N$ by Theorem 2.1 and set $\widetilde{T} := \mathrm{dd}^c\,\widetilde{\varphi}$. Now $z \notin \widehat{K}^{\widetilde{T}}$, and so we have $z \notin p_{\widetilde{T}}(K)$, since $\mathbb{CP}^N$ is homogeneous. Thus there exist an integer M and a section $s \in \Gamma(\mathbb{CP}^N, \mathcal{O}(M))$ with

$$|s|e^{-M\widetilde{\varphi}}(z) > \sup_K |s|e^{-M\widetilde{\varphi}}.$$

The restriction of s to X yields a section of L^{Mk} satisfying the above inequality. We conclude that $z \notin p_T(K)$. □

The proof Theorem 1.2 requires a result following from the work of Guedj, which will be included for completeness. Owing to its technical nature, the proof will be postponed to the next section.

Theorem 2.3 (cf. [7, Theorem 4.1]). — *Let T be a positive closed current of bidegree $(1,1)$ satisfying condition (C) on a projective algebraic manifold X. Assume $[T] = c_1(L)$ for some positive holomorphic line bundle. Then there exist $N_j \in \mathbb{N}$ and $s_j \in \Gamma(X, L^{N_j})$, $j \in \mathbb{N}$, so that*

(i) $\frac{1}{N_j}\log|s_j| - \varphi \xrightarrow{j\to\infty} 0$ *uniformly on compact subsets of* $X \setminus \operatorname{Supp} T$, *where φ is a metric of L with $\mathrm{dd}^c\varphi = T$;*
(ii) $T_j = \frac{1}{N_j}\left[s_j^{-1}(0)\right] \xrightarrow{j\to\infty} T$ *in the weak sense of currents;*
(iii) $s_j^{-1}(0) \xrightarrow{j\to\infty} \operatorname{Supp} T$ *in the Hausdorff metric;*
(iv) $\nu\left(\frac{1}{N_j}\left[s_j^{-1}(0)\right], z\right) \xrightarrow{j\to\infty} \nu(T, z)$ *for any $z \in X$.*

Proof of Theorem 1.2. — Let $U \subset\subset X \setminus \operatorname{Supp} T$ be a neighbourhood of K on which f is defined. For each $a \in \mathrm{b}U \setminus K$, Corollary 2.2 provides a positive integer k and a $u \in \Gamma(X, L^k)$ so that

$$|u|e^{-k\varphi}(a) > \sup_K |u|e^{-k\varphi}.$$

Let s_j be the sections granted by Theorem 2.3, and set $T_j = \frac{1}{N_j}[s_j^{-1}(0)]$. By (i), we have

$$|u|e^{-\frac{k}{N_j}\log|s_j|}(a) > \sup_K |u|e^{-\frac{k}{N_j}\log|s_j|}$$

and hence

$$\left|\frac{u^{N_j}}{s_j^k}\right|(a) > \sup_K \left|\frac{u^{N_j}}{s_j^k}\right|$$

for large j. This inequality holds in a neighbourhood of K, so there is a neighbourhood of a that is disjoint from $\widehat{K}^{T_j}$. By compactness of $\mathrm{b}U$ we can repeat this process finitely many times to see that $\widehat{K}^{T_j} \subset\subset U$ for large j.

The Kodaira embedding theorem then shows that for every sufficiently large j there exists a sequence of meromorphic functions $\{h_\ell/s_j^{P_{j,\ell}}\}_{\ell=1}^{\infty}$ approximating f uniformly on K [7, Theorem 3.10]. Here $P_{j,\ell} \in \mathbb{N}$ and $h_\ell \in \Gamma(X, L^{P_{j,\ell}})$. A diagonalization argument then shows that for every $\varepsilon > 0$ there exists a j_0 so that

$$\sup_K \left(\frac{h_j}{s_j^{P_{j,j}}} - f \right) < \varepsilon$$

whenever $j \geqslant j_0$. □

The proof of Theorem 1.3 follows from Theorem 1.2.

Proof of Theorem 1.3. — Let K be a compact subset of $X \setminus \operatorname{Supp} T$.

First suppose $z \notin \widehat{K}^T$. Then there exists a $f \in \mathcal{C}_T(X)$ with $\mathrm{dd}^c f \geqslant -T$ and

$$f(z) > \sup_K f. \tag{2.1}$$

If $z \in \operatorname{Supp} T$, then clearly $z \notin \widehat{K}_{X \setminus \operatorname{Supp} T}$, so we can assume that $z \notin \operatorname{Supp} T$. Note that f is plurisubharmonic on $X \setminus \operatorname{Supp} T$, so the inequality (2.1) indicates that z does not belong to the convex hull of K with respect to plurisubharmonic functions on $X \setminus \operatorname{Supp} T$. Since $X \setminus \operatorname{Supp} T$ is Stein [7, Theorem 0.4], this hull coincides with $\widehat{K}_{X \setminus \operatorname{Supp} T}$ (e.g. see Forstnerič [6, Corollary 2.5.3]). Therefore we have $\widehat{K}_{X \setminus \operatorname{Supp} T} \subseteq \widehat{K}^T$.

For the reverse inclusion, suppose $z \notin \widehat{K}_{X \setminus \operatorname{Supp} T}$. Condition (C) is satisfied by T, so $z \notin \widehat{K}^T$ whenever $z \in \operatorname{Supp} T$ and we can thus assume $z \notin \operatorname{Supp} T$. Then there exists a $f \in \mathcal{O}(X \setminus \operatorname{Supp} T)$ with

$$|f(z)| > \sup_K |f|. \tag{2.2}$$

Again, by condition (C), the T-polynomially convex hull of $K \cup \{z\}$ is contained in $X \setminus \operatorname{Supp} T$, and an application of Theorem 1.2 yields sequences $u_j, v_j \in \Gamma(X, L^{N_j})$, $N_j \in \mathbb{N}$, so that the meromorphic functions $\{u_j/v_j\}_{j=1}^{\infty}$ approximate f uniformly on $K \cup \{z\}$ and v_j satisfies conditions (i)–(iv) of the theorem. Since inequality (2.2) is strict, we have $|u_j/v_j|(z) > \sup_K |u_j/v_j|$, or

$$|u_j| e^{-\log|v_j|}(z) > \sup_K \Big(|u_j| e^{-\log|v_j|} \Big)$$

for large j. Furthermore, condition (i) of Theorem 1.2 asserts that $\frac{1}{N_j} \log|v_j| - \varphi \xrightarrow{j \to \infty} 0$ uniformly on $K \cup \{z\}$, which in turn implies

$$|u_j| e^{-N_j \varphi}(z) > \sup_K \big(|u_j| e^{-N_j \varphi} \big)$$

for large j. □

3. Proof of Theorem 2.3

We require the following proposition. Here $E_c(T) = \{z \in X : \nu(T, z) \geqslant c\}$ and $E^+(T) = \{z \in X : \nu(T, z) \geqslant 0\}$.

Proposition 3.1 (cf. [7, Proposition 4.2]). — *Let T be a positive closed current of bidegree $(1,1)$ satisfying condition (C) on a projective algebraic manifold X such that $[T] = c_1(L)$ for some positive holomorphic line bundle L. Suppose φ is a (singular) metric on L with $\mathrm{dd}^c\,\varphi = T$, K is a compact subset of $X \setminus \operatorname{Supp} T$ with $\widehat{K}^T = K$, and fix a Kähler form ω on X. Then for every open set V with $K \subset V \subset\subset X \setminus \operatorname{Supp} T$ and every $\delta > 0$, we can find $M \in \mathbb{N}$ and construct a positive (singular) metric ψ of L^M on X and a section $h \in \Gamma(V, L^M)$ such that*

(i) $K \subset \{a \in V : |h|_\psi \geqslant 1\} = \{a \in V : |h|_\psi \equiv 1\} \subset\subset V$
(ii) $\left\|\frac{\psi}{M} - \varphi\right\|_{L^\infty(\overline{V})} \leqslant \delta$ *and* $\left\|\frac{\psi}{M} - \varphi\right\|_{L^1(X)} \leqslant \delta$,
(iii) $\sup_X \left|\nu\big(\frac{1}{M}\mathrm{dd}^c\,(\psi), \cdot\big) - \nu(\mathrm{dd}^c\,\varphi, \cdot)\right| \leqslant \delta$,
(iv) $\mathrm{dd}^c\,\psi \geqslant \varepsilon\omega$ *in a neighbourhood of* $\operatorname{Supp} T$ *for some constant* $\varepsilon > 0$,
(v) ψ *is continuous in* $X \setminus \operatorname{Supp} T$ *and smooth on a dense subset of* $X \setminus E_{c_0}(T)$ *for some* $c_0 > 0$ *which can be made arbitrarily small.*

Proof of Theorem 2.3. — Since T satisfies condition (C), we can find a sequence $\{K_n\}_{n=1}^\infty$ of T-polynomially convex compact subsets of $X \setminus \operatorname{Supp} T$ which exhaust $X \setminus \operatorname{Supp} T$. Fix a sequence $\{\delta_n\}_{n=1}^\infty$ of positive numbers tending to zero, and open neighbourhoods $V_n \subset\subset X \setminus \operatorname{Supp} T$ of K_n. Via Proposition 3.1 we construct M_n, positive metrics ψ_n of L^{M_n} and holomorphic sections h_n of L^{M_n} in V_n with the properties (i)–(v).

Fix a sequence of points $\{a_j\}_{j=1}^\infty$, dense in $\operatorname{Supp} T$, such that for all $n \in \mathbb{N}$ we have $a_1, \dots, a_n \in \operatorname{Supp} T \setminus E_{c_n}(T)$, where $\{c_n\}_{n=1}^\infty$ is a sequence of positive numbers converging to zero with ψ_n smooth and $\mathrm{dd}^c\,\psi_n > 0$ at the points $a_1, \dots, a_n$. Further, let $\{F_n\}_{n=1}^\infty$ be a sequence of compact subsets of $X \setminus E^+(T)$ with $\bigcup_{j=1}^n F_n = X \setminus E^+(T)$ and $K_n \cup \{a_1, \dots, a_n\} \subset F_n \subset\subset X \setminus E_{c_n}(T)$.

For the following we will treat n as fixed and consequently omit subscripts of "n" to simplify notation. Fix an open covering $\{U_\alpha\}_{\alpha \in A}$ of X by trivializations of L fine enough so that for each a_j there is a $\alpha_j \in A$ so that $a_j \in U_{\alpha_j}$ and U_{α_j} contains no other elements of the discrete set $\{a_1, \dots, a_n\}$.

Since $\mathrm{dd}^c\,\psi > 0$ on $\operatorname{Supp} T$, there are holomorphic polynomials P_j so that $\mathrm{Re}(P_j)(a_j) = \psi_{\alpha_j}(a_j)$ and $\psi_{\alpha_j}(z) - \mathrm{Re}(P_j)(z) \geqslant c_j|z - a_j|^2$ for some $c_j > 0$ in a local coordinate patch W_j of a_j with $W_j \subset U_{\alpha_j}$; we can further choose the W_j small enough so that $W_j \cap U_\beta = \varnothing$ for all $\beta \in A \setminus \{\alpha_j\}$.

Let $\chi_j \in \mathcal{C}_0^\infty(W_j)$, $0 \leqslant \chi_j \leqslant 1$, be a bump function that is identically equal to one in a neighbourhood of a_j and define smooth sections $f_j = \{f_j^\alpha\}_{\alpha \in A}$ of L^{NM} by

$$f_j^\alpha = \begin{cases} \chi_j e^{NP_j}, & \text{whenever } \alpha = \alpha_j \\ 0, & \text{whenever } \alpha \neq \alpha_j, \end{cases}$$

where N is a large constant that will be chosen momentarily. Let $\xi : X \to [0,1]$ be another smooth bump function which is identically equal to one on a neighbourhood of $K' = \left\{z \in X \,:\, |h|e^{-\psi}(z) \geqslant 1\right\}$ and has support disjoint from $\operatorname{Supp} T$ and the supports of $\chi_1, \ldots, \chi_n$.

Define $u = \xi h^N + \sum_{j=1}^n f_j$. This is a smooth global section of L^{NM} on X with the properties:

$$|u|_{N\psi} = 1 \qquad \text{on } K \cup \{a_1, \ldots, a_n\}, \text{ and} \tag{3.1}$$

$$|u|_{N\psi} \leqslant 1 \qquad \text{on } X \text{ and strict outside a neighbourhood of } K \cup \{a_1, \ldots, a_n\} \tag{3.2}$$

The smooth $\bar\partial$-closed form $\bar\partial u$ is a $(0,1)$-form with values in L^{NM}, or alternatively, a smooth $\bar\partial$-closed $(\dim X, 1)$-form with values in $L^{NM} \otimes K_X^*$, where K_X denotes the canonical bundle on X.

Set $N = N_1 + N_2$, where N_2 is chosen so that $L^{N_2 M} \otimes K_X^*$ is positive, and N_1 is a large positive integer to be determined soon. Fix a Kähler form ω on X, $\varepsilon > 0$, and a smooth metric G of $L^{N_2 M} \otimes K_X^*$ with $\mathrm{dd}^c G \geqslant \varepsilon\omega$. We solve a $\bar\partial$-problem with L^2-estimates associated to the metric $\theta = N_1\psi + G$ [2, Theorem 3.1] to find a smooth $(\dim X, 0)$-form v with values in $L^{NM} \otimes K_X^*$ satisfying $\bar\partial v = \bar\partial u$ and

$$\int_X |v|^2 e^{-2\theta} \mathrm{d}V_\omega \leqslant \frac{1}{\varepsilon} \int_X |\bar\partial u|^2 e^{-2\theta} \mathrm{d}V_\omega,$$

where $\mathrm{d}V_\omega$ denotes the Kähler volume element $\frac{1}{\dim(X)!}\omega^{\dim(X)}$. Since

$$\begin{aligned} &\operatorname{Supp}(\bar\partial\xi) \subset \{z \in X \,:\, |h|_\psi(z) < 1\} \\ \text{and} \quad &\operatorname{Supp}(\bar\partial\chi_j) \subset \left\{z \in W_j \,:\, |e^{MP_j}|_{-\psi} < 1\right\}, \end{aligned}$$

we can find a number $a < 1$ so that $|\bar\partial u|^2 e^{-2N\psi} \lesssim a^{2N_1}$, where the implied constant is independent of N_1. Then

$$\int_X |v|^2 e^{-2\theta} \mathrm{d}V_\omega \lesssim a^{2N_1}. \tag{3.3}$$

Now since ψ is continuous on $X \setminus E_c(T)$, it is uniformly continuous on a continuous on a compact neighbourhood of F which is compact in $X \setminus E_c(T)$. For a fixed $z \in F$, choose an $r > 0$ small enough so that $e^\eta a < 1$, where

$\eta = \sup_{w\in B(z,r)} |\psi(w) - \psi(z)|$ represents the uniform oscillation of ψ on $B(z,r)$. A lemma of Hörmander–Wermer [8, Lemma 4.4] yields

$$\begin{aligned}
&|v(z)|^2 \\
&\lesssim r^2 \sup_{w\in B(z,r)} |\bar\partial v(w)|^2 + r^{-2\dim(X)} \|v\|^2_{L^2(B(z,r))} \\
&= r^2 \sup_{w\in B(z,r)} \left(e^{2N\psi(w)} |\bar\partial v(w)|^2 e^{-2N\psi(w)}\right) + r^{-2\dim(X)} \|v\|^2_{L^2(B(z,r))} \\
&\leqslant r^2 e^{2N\psi(z)} e^{2N\eta} \sup_{w\in B(z,r)} \left(|\bar\partial v(w)|^2 e^{-2N\psi(w)}\right) + r^{-2\dim(X)} \|v\|^2_{L^2(B(z,r))} \\
&\lesssim e^{2N\psi(z)} e^{2N_1\eta} \left(\sup_{B(z,r)} \left(|\bar\partial v|^2 e^{-2N\psi}\right) + \int_{B(z,r)} |v|^2 e^{-2\theta} \mathrm{d}V_\omega \right) \\
&\lesssim (e^\eta a)^{2N_1} e^{2N\psi(z)}, \qquad (3.4)
\end{aligned}$$

where again the implied constants are independent of N_1. Here $B(z,r)$ stands for the pullback of a Euclidean ball via a coordinate chart.

Choose r so that $e^\eta a < 1$ and N_1 so that $|v|e^{-N\psi} \leqslant \frac{1}{n-1}$ on F_n. Set $S = \frac{n-1}{n}(u - v)$. Then

$$|S|e^{-N\psi} \leqslant \frac{n-1}{n}|u|e^{-N\psi} + \frac{n-1}{n}|v|e^{-N\psi} \leqslant 1 \qquad \text{on } F_n \qquad (3.5)$$

and likewise

$$|S|e^{-N\psi} \geqslant \frac{n-1}{n}|u|e^{-N\psi} - \frac{n-1}{n}|v|e^{-N\psi} \geqslant \frac{n-2}{n} \quad \text{on } K_n \cup \{a_1, \dots, a_n\} \qquad (3.6)$$

by (3.1) and (3.2) above.

Given $\varepsilon > 0$ and a compact subset $A \subset X \setminus \operatorname{Supp} T$, by (3.5) and (3.6) we have $\|\frac{1}{N_n} \log|S_n| - \psi_n\|_{L^\infty(A)} < \varepsilon/2$. By choice of n large enough so that both δ_n is smaller than $\varepsilon/2$ and $A \subset K_n \cap F_n$, one also has $\|\frac{1}{M_n}\psi - \varphi\|_{L^\infty(A)} < \varepsilon/2$ by property (ii) of Proposition 3.1. The triangle inequality then shows conclusion (i).

For conclusions (ii)–(iv), we will show

$$\nu\left(\frac{1}{N_n}\mathrm{dd}^c(\log|S_n|), z\right) \geqslant \left(1 - \frac{1}{\sqrt{N_n}}\right)\nu(\mathrm{dd}^c\psi_n, z) - \frac{1}{N_n} \qquad \text{for all } z \in E_{c_n}(\mathrm{dd}^c\psi_n) \qquad (3.7)$$

and

$$\int_X |S_n|e^{-N_n\psi_n} \leqslant C, \qquad \text{for } C > 0 \text{ independent of } n. \qquad (3.8)$$

Note that u is identically zero in a neighbourhood of any point $z \in E_{c_n}(\mathrm{dd}^c\,\psi_n)$, so v is holomorphic there and the finiteness of the integral $\int_X |v|^2 e^{-2\theta}\mathrm{d}V_\omega$ forces v to vanish to an order greater than or equal to $N_1\nu(\mathrm{dd}^c\,\psi_n, x) - 1$ at z (cf. Kiselman [9, Theorem 3.2]). We thus have

$$\begin{aligned}\nu\left(\frac{1}{N}\mathrm{dd}^c\,(\log|S|), z\right) &\geqslant \frac{N_1}{N}\nu(\mathrm{dd}^c\,\psi_n, z) - \frac{1}{N}\\ &\geqslant \left(1 - \frac{1}{\sqrt{N}}\right)\nu(\mathrm{dd}^c\,\psi_n, z) - \frac{1}{N},\end{aligned}$$

for N_1 large enough, since $N = N_1 + N_2$ and N_2 is fixed. This establishes the inequality (3.7). To see the inequality (3.8), note that we have $\int_X |v|^2 e^{-2\theta}\mathrm{d}V_\omega \lesssim 1$ from (3.3), where the implied constant is independent of n; from this it follows that $\int_X |v|^2 e^{-2N\psi}\mathrm{d}V_\omega \lesssim 1$ as well. From this, we obtain

$$\int_X |S| e^{-N\psi}\mathrm{d}V_\omega \lesssim \int_X |S|^2 e^{-2N\psi}\mathrm{d}V_\omega \lesssim 1.$$

To see (ii), the inequalities (3.5) and (3.6), along with the Lelong–Poincaré equation and (ii) of Proposition 3.1 implies that $T_n = \mathrm{dd}^c\left(\frac{1}{N_n M_n}\log|S_n|\right)$ converges weakly towards T in $X \setminus E^+(T)$. Since the decomposition theorem of Siu [11] gives $T = \sum_i \lambda_i[Z_i] + R$, where λ_i are positive constants, Z_i are hypersurfaces, and R is a residual current with $E_c(R) \geqslant 2$ for all $c > 0$, the Hausdorff dimension of $E^+(R) = E^+(T) \setminus \bigcup_i Z_i$ is less than or equal than $2\dim(X) - 4$, hence T_n actually converges towards T on $X \setminus \bigcup_i Z_i$ (e.g. [5]).

Also, note that $\limsup_{n\to\infty} \nu(T_n, x) \leqslant \nu(T, x)$ for all $x \notin \bigcup_i Z_i$ since T_n converges to T in the weak sense of currents there (cf. Demailly [3, Proposition III.5.12]); hence the inequality (3.7) and condition (iii) of Proposition 3.1 gives that $\nu(T_n, x) \to \nu(T, x)$ for all $x \in E^+(T) \setminus \bigcup_i Z_i$ and hence $\nu(T_n, x) \to \nu(T, x)$ for all $x \notin \bigcup_i Z_i$. Now, the sequence $\{T_n\}_n$ is bounded in the sense of currents by the inequality (3.8) and (ii) of Proposition 3.1 (see Demailly [3, p. 19]), so it admits a subsequence weakly converging to a positive closed current T' of bidegree $(1,1)$ on X. Since $T' = T$ on $X \setminus \bigcup_i Z_i$ and $T' \geqslant T$ on $\bigcup_i Z_i$ by the inequality (3.7), it follows that $T = T'$ on all of X because $c_1(L) = [T] = [T']$ and the fact that X it is compact and Kähler. Therefore T_n converges weakly towards T on X and $\nu(T_n, x) \to \nu(T, x)$ for all $x \in X$.

Lastly, since $|S_n| > 0$ on K_n and $T_n \to T$, the varieties $\{S_n = 0\}$ converge towards $\operatorname{Supp} T$ in the Hausdorff metric. $\square$

Remark. — It should be noted that, while the focus of this note is on complex projective manifolds, T-polynomial convexity can be defined on Stein manifolds. In fact, on a Stein manifold, every line bundle is positive and every positive closed (1,1)-current satisfies condition (C), so Theorems 1.2 and 1.3 can be stated in that setting with some hypotheses removed. The methods of proof are identical, with one exception: Stein manifolds are not compact. In place of compactness of the complex projective manifold, one instead takes advantage of the compactness of a large sublevel set of some strictly plurisubharmonic exhaustion function, along with an appropriate patching argument.

Bibliography

[1] D. Coman, V. Guedj & A. Zeriahi, "Extension of plurisubharmonic functions with growth control", *J. Reine Angew. Math.* **676** (2013), p. 33-49.

[2] J.-P. Demailly, "Singular Hermitian metrics on positive line bundles", in *Complex Algebraic Varieties: Proceedings of a Conference held in Bayreuth, Germany, April 2–6, 1990*, Lecture Notes in Mathematics, vol. 1507, Springer, 1992, p. 87-104.

[3] ———, *Complex analytic and differential geometry*, Université de Grenoble I, 2012.

[4] J. E. Fornæss, F. Forstnerič & E. F. Wold, "Holomorphic approximation: the legacy of Weierstrass, Runge, Oka–Weil, and Mergelyan", in *Advancements in complex analysis. From theory to practice*, Springer, 2020, p. 133-192.

[5] J. E. Fornæss & N. Sibony, "Oka's inequality for currents and applications", *Math. Ann.* **301** (1995), no. 3, p. 399-419.

[6] F. Forstnerič, *Stein manifolds and holomorphic mappings. The homotopy principle in complex analysis*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 56, Springer, 2017, xiv+562 pages.

[7] V. Guedj, "Approximation of currents on complex manifolds", *Math. Ann.* **313** (1999), no. 3, p. 437-474.

[8] L. Hörmander & J. Wermer, "Uniform approximation on compact sets in C^n", *Math. Scand.* **23** (1968), p. 5-21.

[9] C. O. Kiselman, "Attenuating the singularities of plurisubharmonic functions", *Ann. Pol. Math.* **60** (1994), no. 2, p. 173-197.

[10] B. Shiffman, *Singular metrics and singular differential forms in complex geometry (notes by M. Reimpell)*, Schriftenreihe des Graduiertenkollegs Geometrie und Mathematische Physik, vol. 12, Ruhr-Universität Bochum, 1995.

[11] Y.-T. Siu, "Analyticity of sets associated to Lelong numbers and the extension of closed positive currents", *Invent. Math.* **27** (1974), p. 53-156.

[12] E. L. Stout, *Polynomial convexity*, Progress in Mathematics, vol. 261, Birkhäuser, 2007.