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On K-moduli of Fano threefolds with degree 28 and Picard rank 4 [(*)]

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Abstract. — We analyse the local structure of the K-moduli space of Fano varieties at a toric singular K-polystable Fano 3-fold, which deforms to smooth Fano 3-folds with anticanonical volume 28 and Picard rank 4. In particular, by constructing an algebraic deformation of this toric singular Fano, we show that the irreducible component of K-moduli parametrising these smooth Fano 3-folds is a rational surface.

Résumé. — On étudie la structure locale de l'espace de K-modules des variétés de Fano autour d'une variété torique singulière K-polystable, qui se déforme en une variété de Fano lisse de dimension 3, volume anticanonique 28 et rang de Picard 4. En particulier, par la construction d'une déformation algébrique de cette variété torique singulière on montre que la composante irréductible des K-modules correspondante est une surface rationnelle.

1. Introduction

1.1. Context

We work over $\mathbb{C}$. In the study of Fano varieties *K-stability* [29, 54] has become a fundamental topic in recent years, for two main reasons: it is the algebraic counterpart of the existence of Kähler–Einstein metrics [21, 55], and it has allowed to construct projective moduli spaces for Fano varieties.

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More precisely, by [3, 11, 13, 14, 33, 37, 41, 56, 58], for every positive integer n and for every positive rational number v, there exists a projective scheme $M^{\mathrm{Kps}}_{n,v}$ over $\mathbb{C}$, whose closed points are in natural one-to-one correspondence with K-polystable Fano varieties X of dimension n and anticanonical volume v. This is called a *K-moduli space*. We refer the reader to [57] for a survey on this topic.

An important problem is to decide if a given Fano variety is K-stable, K-polystable or K-semistable: this is the *Calabi problem*. For smooth del Pezzo surfaces this has been completely solved in [53]. For the general member of the 105 deformation families of smooth Fano 3-folds, the Calabi problem has been solved in [6]. For all (not necessarily general) smooth Fano 3-folds there is much recent and ongoing work, e.g. [9, 17, 18, 20, 28, 39]. Other smooth Fano varieties are studied in [2, 59].

The Calabi problem for toric Fano manifolds has been translated into an easy combinatorial condition via differential geometry methods: a toric Fano manifold admits a Kähler–Einstein metric if and only if the moment polytope of its toric boundary has barycentre at the origin [8, 42]. Blum and Jonsson [12] found a completely algebraic proof of the fact that the same condition is equivalent to the K-polystability of a (possibly singular) toric Fano variety. K-stability of Fano varieties with the action of certain algebraic groups (e.g. T-varieties or spherical varieties) has been investigated in [27, 32, 52].

It is natural to investigate the geometry of K-moduli spaces of Fano varieties. Only few cases are known, e.g. smooth del Pezzo surfaces [43, 47], cubic 3-folds [40], quartic 3-folds [1] and cubic 4-folds [38].

Here we present results about a specific family of smooth Fano 3-folds by using toric geometry and deformation theory techniques which originate in mirror symmetry [22].

1.2. The family no. 4.2

There is only one deformation family of *smooth* Fano 3-folds with Picard rank 4 and anticanonical volume 28: this is the 2nd entry in the Mori–Mukai list [44, 45] (and also in [48]) of families of smooth Fano 3-folds with Picard rank 4, thus it is denoted by no. 4.2 in [6]. This family is denoted by MM_{4-3} in [23, Section 87] because the authors have reordered the families of smooth Fano 3-folds of Picard rank 4 so that the anticanonical volume increases along the list. The other invariants of this family are $h^{1,2}(X) = 1$ and $\chi(T_X) = -1$.

Each member of the family no. 4.2 is the blow-up of the cone over a smooth quadric surface $S \subset \mathbb{P}^3$ with centre the disjoint union of the vertex and an elliptic curve on S. Each member of the family no. 4.2 is K-polystable by [6, Section 4.6], hence gives a point in $M^{\mathrm{Kps}}_{3,28}$, which is the K-moduli space of 3-dimensional K-polystable Fano varieties with anticanonical volume 28.

We prove the following:

Theorem 1.1. — *Let $M = M^{\mathrm{Kps}}_{3,28}$ be the K-moduli space of K-polystable Fano 3-folds with anticanonical volume 28. Let $M^{\mathrm{sm}}_{\rho=4} \subset M$ be the locus which parametrises the smooth K-polystable Fano 3-folds with anticanonical volume 28 and Picard rank 4.*

Then there exists an open subscheme U of M such that U is a smooth rational surface, $U \cap M^{\mathrm{sm}}_{\rho=4} \neq \varnothing$ and $U \smallsetminus M^{\mathrm{sm}}_{\rho=4}$ is a smooth rational curve.

We therefore conclude that the irreducible component of $M^{\mathrm{Kps}}_{3,28}$ containing $M^{\mathrm{sm}}_{\rho=4}$ is a rational surface.

1.3. Idea of the proof

We pick a singular toric Fano 3-fold X whose singular locus consists of 4 ordinary double points: this has reflexive ID 735 in the Graded Ring Data Base [16]. This variety X is K-polystable because the barycentre of the moment polytope of its toric boundary is the origin. It has been shown to deform to the members of the family no. 4.2 by Galkin in his PhD thesis [30]. Moreover, by [34, Section 5.1] the base space of the miniversal deformation of X is a polydisc of dimension 4. However, no information about the local structure, in the Zariski topology, of the K-moduli space can be derived from this infinitesimal study.

Here, we construct an explicit algebraic model of the miniversal deformation of X over a Zariski neighbourhood of the origin in the affine space $\mathbb{A}^4$, i.e. we exhibit a flat algebraic deformation of X over $\mathbb{A}^4$ such that its formal completion at the origin is the miniversal deformation of X.

To achieve this we use the Laurent inversion method introduced in [24] and used to systematically construct (not necessarily smooth) Fano 3-folds in [31, 51]. Indeed, we find a toric 4-fold F such that X is a divisor in F and we are able to smooth X by deforming it inside the linear system $|\mathcal{O}_F(X)|$. We then show that this algebraic deformation induces the miniversal one.

We deduce that the K-moduli space is unirational near the point $[X]$ corresponding to X. Moreover, by using deformation-theoretic techniques in Proposition 2.1 (8) we show that the K-moduli space is smooth of dimension 2 at $[X]$. This proves the rationality in the statement.

Remark 1.2. — In this paper we describe geometric properties of a 2-dimensional component of K-moduli of smoothable Fano 3-folds, by analysing the deformation theory of a singular toric Fano 3-fold. In later work, Cheltsov, Duarte Guerreiro, Fujita, Krylov and Martinez-Garcia [19, Corollary 1.14] express this rational surface as a GIT quotient. Their methods are independent of ours, and rely on the birational description of smooth members of the family no. 4.2.

Remark 1.3. — Let X' be the Gorenstein toric Fano 3-fold which appears in [16] with reflexive ID 1518 and canonical ID 61936. We expect that X' deforms to two different families of smooth Fano 3-folds, namely no. 4.2 (which is the family studied in this paper) and no. 2.21 (which is the family consisting of the blowups of the smooth quadric 3-fold at a twisted quartic curve). Both families have anticanonical volume 28, but they have different Betti numbers. The general member of the family no. 2.21 is K-polystable by [6].

By [19] the two irreducible components of $M^{\mathrm{Kps}}_{3,28}$ generically parametrising K-polystable members of the families no. 2.21 and no. 4.2 belong to distinct connected components of $M^{\mathrm{Kps}}_{3,28}$. This is not a contradiction with the expected properties of X' described above because X' is not K-semistable, hence it does not give a point in K-moduli.

Notation and conventions

The set of non-negative (resp. positive) integers is denoted by $\mathbb{N}$ (resp. $\mathbb{N}^+$). We work over an algebraically closed field of characteristic zero, denoted by $\mathbb{C}$. Every toric variety or toric singularity is assumed to be normal. A Fano variety is a normal projective variety whose anticanonical divisor is $\mathbb{Q}$-Cartier and ample. A del Pezzo surface is a Fano variety of dimension 2.

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We use the Magma computational algebra software [15] in Section A.

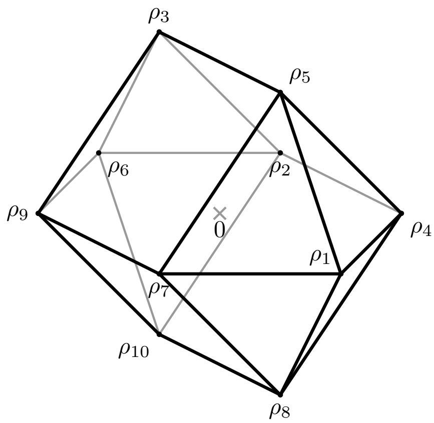

Figure 2.1. The polytope P in Proposition 2.1

2. A toric variety

Here we study a specific toric Fano 3-fold, which was already studied in [34, Section 5.1]. Its associated polytope first appeared in the classification of reflexive polytopes [36] by Kreuzer and Skarke and is part of the Graded Ring Database "Toric canonical Fano 3-folds" list [16, 35], appearing with canonical ID 674679 and reflexive ID 735.

Proposition 2.1. — *In the lattice $N = \mathbb{Z}^3$ consider the polytope P whose vertices are:*

$$\begin{aligned} &\rho_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \rho_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \rho_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \quad \rho_4 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \quad \rho_5 = \begin{pmatrix}1\\0\\1\end{pmatrix}, \\ &\rho_6 = -\begin{pmatrix}1\\0\\0\end{pmatrix}, \ \rho_7 = -\begin{pmatrix}0\\1\\0\end{pmatrix}, \ \rho_8 = -\begin{pmatrix}0\\0\\1\end{pmatrix}, \ \rho_9 = -\begin{pmatrix}1\\1\\0\end{pmatrix}, \ \rho_{10} = -\begin{pmatrix}1\\0\\1\end{pmatrix}. \end{aligned} \tag{2.1}$$

Let X be the toric 3-fold associated to the face fan of P. Then the following assertions hold.

1. *X is a K-polystable Fano 3-fold with anticanonical volume 28.*
2. *The singular locus of X consists of 4 ordinary double points.*
3. *The base of the miniversal deformation of X is smooth of dimension 4, i.e. Def_X is isomorphic to the formal spectrum of the power series ring $\mathbb{C}[\![t_\alpha, t_\beta, t_\gamma, t_\delta]\!]$.*

(4) *X deforms to the 2nd entry in the Mori–Mukai list* [44, 45] *of smooth Fano 3-folds with Picard rank 4 (the other invariants are $(-K)^3 = 28$ and $h^{1,2} = 1$).*
(5) *The automorphism group $\mathrm{Aut}(P) \subset \mathrm{GL}(N) \simeq \mathrm{GL}_3(\mathbb{Z})$ of the polytope P is isomorphic to the semidirect product $D_8 \rtimes C_2$, where C_2 is the cyclic group of order 2 and D_8 is the dihedral group of order 8; more precisely, $\mathrm{Aut}(P)$ is generated by the following matrices:*

$$g_1 = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ of order } 2,$$
$$g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ of order } 2,$$
$$g_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \text{ of order } 4.$$

(6) *The automorphism group $\mathrm{Aut}(X)$ of X is isomorphic to $T_N \rtimes \mathrm{Aut}(P)$, where $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ is the 3-dimensional algebraic torus whose cocharacter lattice is N.*
(7) *The formal action of $\mathrm{Aut}(X)$ on $\mathrm{Def}_X \simeq \mathrm{Spf}\, \mathbb{C}[\![t_\alpha, t_\beta, t_\gamma, t_\delta]\!]$ is given by:*
   - *the torus T_N acts linearly and diagonally on $t_\alpha, t_\beta, t_\gamma, t_\delta$ with weights $(0,1,1)$, $(0,1,-1)$, $(0,-1,-1)$ $(0,-1,1)$ in M, respectively,*
   - *the involution g_2 leaves t_α and t_γ fixed and swaps t_β and t_δ,*
   - *the involution g_1 fixes everything,*
   - *g_3 acts as $t_\alpha \mapsto -t_\beta$, $t_\beta \mapsto t_\gamma$, $t_\gamma \mapsto t_\delta$, $t_\delta \mapsto -t_\alpha$.*

(8) *The scheme $M^{\mathrm{Kps}}_{3,28}$ is smooth of dimension 2 in a neighbourhood of the point $[X]$ corresponding to X. More precisely, $t_\alpha t_\beta t_\gamma t_\delta$ and $t_\alpha t_\gamma - t_\beta t_\delta$ are regular formal parameters of $M^{\mathrm{Kps}}_{3,28}$ at $[X]$.*
(9) *In a neighbourhood of $[X]$ in $M^{\mathrm{Kps}}_{3,28}$ the locus of non-smooth K-polystable Fano varieties is a smooth curve passing through $[X]$.*

The proof of (1)–(4) is contained in [34, Section 5.1]; here we recap the argument briefly.

Proof. — The polytope P has 20 edges (all of them have lattice length 1) and has 12 2-dimensional faces (8 of them are triangles and 4 of them are quadrilaterals). It is depicted in Figure 2.1. Let Σ_X be the face fan of P.

The 3-dimensional cones of Σ_X are

$$\begin{gathered}\sigma_{145},\ \sigma_{157},\ \sigma_{148},\ \sigma_{178},\\ \sigma_{6910},\ \sigma_{2610},\ \sigma_{369},\ \sigma_{236},\\ \sigma_{2345},\ \sigma_{24810},\ \sigma_{78910},\ \sigma_{3579}.\end{gathered}\tag{2.2}$$

Here σ_{ijk} (resp. σ_{ijkl}) is the 3-dimensional cone in N whose rays are ρ_i, ρ_j, ρ_k (resp. ρ_i, ρ_j, ρ_k, ρ_l). X is the toric variety associated to Σ_X.

(1). — X is Fano, because it is defined by the face fan of a Fano polytope. Let P° be the polar of P: it is the moment polytope of the toric boundary of X. Since P° has normalised volume 28, we have $(-K_X)^3 = 28$. Since $P = -P$ we have that the barycentre of P° is the origin, therefore X is K-polystable by [10, Corollary 1.2] and [12, Corollary 7.17].

(2). — The simplicial cones of Σ_X are smooth. The non-simplicial cones of Σ_X are $\sigma_\alpha = \sigma_{2345}$, $\sigma_\beta = \sigma_{24810}$, $\sigma_\gamma = \sigma_{78910}$, $\sigma_\delta = \sigma_{3579}$. Let U_α (resp. U_β, U_γ, U_δ) be the toric open affine subscheme of X associated to the cone σ_α (resp. σ_β, σ_γ, σ_δ), i.e. $U_\alpha = \operatorname{Spec}\mathbb{C}[\sigma_\alpha^\vee \cap M]$.

Each of the cones σ_α, σ_β, σ_γ, σ_δ is $\mathrm{GL}_3(\mathbb{Z})$-equivalent to the cone over a unit square placed at height 1, i.e. to the cone spanned by e_3, e_1+e_3, e_2+e_3, $e_1+e_2+e_3$ in $\mathbb{Z}^3$ where e_1, e_2, e_3 is the standard basis of $\mathbb{Z}^3$. Therefore U_α, U_β, U_γ, U_δ are all isomorphic to $\operatorname{Spec}\mathbb{C}[x,y,z,w]/(xy - zw)$, which has an ordinary double point. Hence the singular locus of X is made up of 4 ordinary double points.

More precisely, we fix an isomorphism between U_α (resp. U_β, U_γ, U_δ) and $\operatorname{Spec}\mathbb{C}[x,y,z,w]/(xy - zw)$ by giving names to the minimal generators of the monoid $\sigma_\alpha^\vee \cap M$ (resp. $\sigma_\beta^\vee \cap M$, $\sigma_\gamma^\vee \cap M$, $\sigma_\delta^\vee \cap M$):

$$\begin{array}{llll} x_\alpha = (-1,1,1) & y_\alpha = (1,0,0) & z_\alpha = (0,0,1) & w_\alpha = (0,1,0),\\ x_\beta = (0,0,-1) & y_\beta = (0,1,0) & z_\beta = (-1,1,0) & w_\beta = (1,0,-1),\\ x_\gamma = (0,-1,0) & y_\gamma = (0,0,-1) & z_\gamma = (-1,0,0) & w_\gamma = (1,-1,-1),\\ x_\delta = (0,0,1) & y_\delta = (0,-1,0) & z_\delta = (-1,0,1) & w_\delta = (1,-1,0). \end{array}$$

(3). — Since $\mathrm{H}^1(T_X) = 0$ and $\mathrm{H}^2(T_X) = 0$ [49, Lemma 4.4] and X has isolated singularities, the deformations of X can be identified with the deformations of the singularities of X (see for instance [49, Section 14.2.1]). More precisely, the product of the restriction maps

$$\mathrm{Def}_X \longrightarrow \mathrm{Def}_{U_\alpha} \times \mathrm{Def}_{U_\beta} \times \mathrm{Def}_{U_\gamma} \times \mathrm{Def}_{U_\delta}\tag{2.3}$$

is smooth and induces an isomorphism on tangent spaces. Since an ordinary double point has a smooth 1-dimensional miniversal deformation space, we conclude.

For future use we need to specify the meaning of t_α, t_β, t_γ, t_δ. We will always use the smooth functor

$$\operatorname{Spf}\mathbb{C}[\![t_\alpha]\!] \longrightarrow \operatorname{Def}_{U_\alpha},$$

which induces an isomorphism on tangent spaces and is induced by the formal deformation of U_α over $\operatorname{Spf}\mathbb{C}[\![t_\alpha]\!]$ given by the equation

$$x_\alpha y_\alpha - z_\alpha w_\alpha + t_\alpha = 0, \tag{2.4}$$

where the meaning of x_α, y_α, z_α, w_α is specified in the proof of (2) above. We make the similar conventions for t_β, t_γ and t_δ and we will always stick to these conventions.

(4). — By [34, Proposition 2.5] the 2nd Betti number of X is 4. Since the smoothing of a 3-fold ordinary double point has a Milnor fibre homotopically equivalent to the 3-sphere, the 2nd Betti number of the general smoothing of X is 4. Moreover, in smoothing X, also the anticanonical degree is preserved. In the Mori–Mukai list [44, 45] there is only one family of smooth Fano 3-folds with Picard rank 4 and anticanonical degree 28.

(5). — We now compute the automorphism group of the polytope P. Any automorphism of P should preserve the ρ_1-ρ_6 axis. There is only one reflection that switches ρ_1 and ρ_6, and we choose it as the generator of a C_2-subgroup of $\operatorname{Aut}(P)$. As a matrix in $\mathrm{GL}(N) = \mathrm{GL}_3(\mathbb{Z})$ the generator of the C_2 above is

$$g_1 = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

All other automorphisms of P fix ρ_1. They must also preserve the square obtained by intersecting H with P, where $H = \mathbb{R}\frac{\rho_2+\rho_4}{2} + \mathbb{R}\frac{\rho_3+\rho_5}{2}$. This square has symmetry group D_8 and all of its symmetries lift uniquely to symmetries of P that fix ρ_1. Moreover, these do not commute with the generator of C_2, thus $\operatorname{Aut}(P) = D_8 \rtimes C_2$. The matrix representations of the two generators of D_8 are:

$$g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad g_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

(6). — This follows from [34, Proposition 2.8] because no facet of the polar polytope P° has interior lattice points.

(7). — The vector space $\mathbb{T}^1_X$, which is the tangent space of Def_X, is a 4-dimensional representation of $\operatorname{Aut}(X) = T_N \rtimes \operatorname{Aut}(P)$. Using (2.3) we identify $\mathbb{T}^1_X$ with the tangent space to the deformations of the 4 singular points of X.

The vector $(0,1,1) \in M$ is the only one such that by pairing with the primitive generators of the cone σ_α, namely ρ_2, ρ_3, ρ_4, ρ_5, we obtain 1. In other words, $-(0,1,1) \in M$ is the vertex of P° which is dual to the face of P whose vertices are ρ_2, ρ_3, ρ_4, ρ_5. By [5] $\mathbb{T}^1_{U_\alpha}$ is the 1-dimensional representation of the torus $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ with weight $(0,-1,-1)$. This implies that the coordinate t_α has degree $(0,-1,-1) \in M$ with respect to T_N.

In similar ways one proves that t_β, t_γ, t_δ have degrees $(0,-1,1)$, $(0,1,1)$, $(0,1,-1)$ respectively. We now focus on the action of the 3 generators g_1, g_2, g_3 of $\pi_0(\mathrm{Aut}(X))$ given in (5).

The involution g_1 acts on the vertices of P by $\rho_1 \leftrightarrow \rho_6$, $\rho_2 \leftrightarrow \rho_4$, $\rho_3 \leftrightarrow \rho_5$, $\rho_7 \leftrightarrow \rho_9$, $\rho_8 \leftrightarrow \rho_{10}$. In particular, by remembering that the torus orbits on X correspond bijectively to the cones of the face fan of P, we can see that U_α is g_1-invariant and g_1 acts via pull-back on the regular functions of U_α. For example

$$g_1(x_\alpha) = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = y_\alpha$$

and

$$g_1(z_\alpha) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = w_\alpha.$$

This shows that g_1 maps the miniversal deformation (2.4) of U_α to itself. Hence g_1 maps t_α to itself. In a similar way, one can show that U_β, U_γ and U_δ are g_1-invariant and g_1 fixes t_β, t_γ and t_δ too.

Let us now focus on g_2. On the vertices of P this involution acts as $\rho_2 \leftrightarrow \rho_3$, $\rho_4 \leftrightarrow \rho_5$, $\rho_7 \leftrightarrow \rho_8$, $\rho_9 \leftrightarrow \rho_{10}$ and fixes ρ_1 and ρ_6. One can see that U_α is g_2-invariant and $g_2(x_\alpha) = x_\alpha$, $g_2(y_\alpha) = y_\alpha$, $g_2(z_\alpha) = w_\alpha$. Therefore g_2 maps the miniversal deformation (2.4) of U_α to itself, hence t_α is fixed by g_2.

One sees that $g_2(U_\beta) = U_\delta$. So, via pull-back, g_2 maps a regular function on U_δ to a regular function on U_β. A similar calculation as above gives $g_2(y_\delta) = x_\beta$, $g_2(x_\delta) = y_\beta$, $g_2(z_\delta) = z_\beta$, $g_2(w_\delta) = w_\beta$. So the miniversal deformation $x_\delta y_\delta - z_\delta w_\delta + t_\delta = 0$ of U_δ is mapped by g_2 to the deformation $x_\beta y_\beta - z_\beta w_\beta + t_\delta = 0$ of U_β. This latter deformation is induced by the chosen miniversal deformation $x_\beta y_\beta - z_\beta w_\beta + t_\beta = 0$ of U_β by setting $t_\delta = t_\beta$. Therefore g_2 swaps t_δ and t_β.

In a similar way, one can show that g_2 leaves t_γ fixed.

Let us now focus on g_3. The order 4 matrix g_3 acts as follows on the vertices of P: ρ_1 and ρ_6 are fixed and there are two orbits of cardinality 4, namely $\rho_2 \mapsto \rho_{10} \mapsto \rho_9 \mapsto \rho_3 \mapsto \rho_2$ and $\rho_5 \mapsto \rho_4 \mapsto \rho_8 \mapsto \rho_7 \mapsto \rho_5$.

One sees $g_3(U_\alpha) = U_\beta$. Via pull-back along g_3, we get $x_\beta \mapsto w_\alpha$, $y_\beta \mapsto z_\alpha$, $z_\beta \mapsto x_\alpha$, $w_\beta \mapsto y_\alpha$. So the miniversal deformation $x_\beta y_\beta - z_\beta w_\beta + t_\beta = 0$ of U_β is mapped via g_3 to the deformation $w_\alpha z_\alpha - x_\alpha y_\alpha + t_\beta = 0$ of U_α. This shows that g_3 maps t_α to $-t_\beta$.

In a similar way, one obtains $g_3(U_\beta) = U_\gamma$, $g_3(U_\gamma) = U_\delta$, $g_3(U_\delta) = U_\alpha$ and that g_3 acts as $t_\beta \mapsto t_\gamma \mapsto t_\delta \mapsto -t_\alpha$.

(8). — By (7) the invariant subring of the formal action of the torus $T_N \simeq (\mathbb{C}^*)^3$ on $\mathbb{C}[\![t_\alpha, t_\beta, t_\gamma, t_\delta]\!]$ is $\mathbb{C}[\![t_\alpha t_\gamma, t_\beta t_\delta]\!]$, which is a power series ring in 2 indeterminates. The involutions g_1 and g_2 act trivially on $\mathbb{C}[\![t_\alpha t_\gamma, t_\beta t_\delta]\!]$, whereas g_3 acts on $\mathbb{C}[\![t_\alpha t_\gamma, t_\beta t_\delta]\!]$ with order 2 and maps $t_\alpha t_\gamma$ to $-t_\beta t_\delta$. In conclusion, the subring of $\mathbb{C}[\![t_\alpha, t_\beta, t_\gamma, t_\delta]\!]$ of the invariants elements under the action of $\operatorname{Aut}(X)$ is $\mathbb{C}[\![t_\alpha t_\beta t_\gamma t_\delta, t_\alpha t_\gamma - t_\beta t_\delta]\!]$, which is a power series ring in 2 indeterminates.

By the Luna étale slice theorem for algebraic stacks [4] the local structure of $\mathcal{M}^{\mathrm{Kss}}_{3,28} \to M^{\mathrm{Kps}}_{3,28}$ at the point $[X]$ is given by the commutative square

$$\begin{tikzcd}
{[\operatorname{Spf} \mathbb{C}[\![t_\alpha, t_\beta, t_\gamma, t_\delta]\!] \;/\; \operatorname{Aut}(X)]} \arrow[r] \arrow[d] & \mathcal{M}^{\mathrm{Kss}}_{3,28} \arrow[d] \\
\operatorname{Spf} \mathbb{C}[\![t_\alpha t_\beta t_\gamma t_\delta, t_\alpha t_\gamma - t_\beta t_\delta]\!] \arrow[r] & M^{\mathrm{Kps}}_{3,28}
\end{tikzcd}$$

where the horizontal maps are formally étale and maps the closed point to $[X]$. This implies that $M^{\mathrm{Kps}}_{3,28}$ is smooth of dimension 2 near $[X]$ and that $t_\alpha t_\beta t_\gamma t_\delta$, $t_\alpha t_\gamma - t_\beta t_\delta$ are local analytic coordinates.

(9). — Here, to simplify, we work in the analytic category. Since $t_\alpha, t_\beta, t_\gamma, t_\delta$ are the smoothing parameters of an ordinary double point, it is clear that in the miniversal deformation of X, which now we think over a small polydisc Δ^4 of dimension 4 (with local analytic coordinates $t_\alpha, t_\beta, t_\gamma, t_\delta$), the smooth fibres are exactly those where $t_\alpha \neq 0$, $t_\beta \neq 0$, $t_\gamma \neq 0$, $t_\delta \neq 0$. Let D be the discriminant locus, i.e. the locus in the base space Δ^4 of the miniversal deformation of X where the fibres are singular; hence $D = \{t_\alpha t_\beta t_\gamma t_\delta = 0\} \subset \Delta^4$ is the union of 4 hyperplanes.

Let Δ^2 be a small 2-dimensional polydisc which gives the local structure of $M^{\mathrm{Kps}}_{3,28}$ at $[X]$. The holomorphic map $\Delta^4 \to \Delta^2$ is given by

$$(t_\alpha, t_\beta, t_\gamma, t_\delta) \longmapsto (t_\alpha t_\beta t_\gamma t_\delta, t_\alpha t_\gamma - t_\beta t_\delta).$$

The image of D via this map is the locus in Δ^2 where the first coordinate is zero, therefore it is a germ of a smooth curve passing through the origin. The image of D is exactly the locus in Δ^2 of singular K-polystable Fanos close to $[X]$. $\square$

An immediate consequence of Proposition 2.1 is:

Corollary 2.2. — *Let $M = M^{\mathrm{Kps}}_{3,28}$ be the K-moduli space of K-polystable Fano 3-folds with anticanonical volume 28. Let $M^{\mathrm{sm}}_{\rho=4} \subset M$ be the locus which parametrises the smooth K-polystable Fano 3-folds with anticanonical volume 28 and Picard rank 4. Then there exists an open subscheme U of M such that U is a smooth surface and $U \smallsetminus M^{\mathrm{sm}}_{\rho=4}$ is a smooth curve.*

This is a portion of Theorem 1.1; the only things which are missing are the statements about the rationality of U and of $U \smallsetminus M^{\mathrm{sm}}_{\rho=4}$.

3. Embeddings in toric varieties

Now we realise the toric Fano 3-fold X of Proposition 2.1 inside a toric Fano 4-fold.

Proposition 3.1. — *In the lattice $N_F = \mathbb{Z}^4$ consider the lattice polytope P_F whose vertices are the columns $r_1, \dots, r_7 \in N_F$ of the matrix*

$$R_F = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 & 1 \end{bmatrix}.$$

Let F be the toric variety associated to the face fan of P_F. Then:

(1) *F is a smooth Fano 4-fold and is the GIT quotient $\mathbb{A}^7 /\!\!/ (\mathbb{C}^*)^3$ given by the weight matrix*

u_1	u_2	u_3	u_4	u_5	u_6	u_7
1	0	1	0	0	−1	0
0	1	0	1	0	−1	0
0	0	0	0	1	1	1

and by the stability condition $(1,1,1)$;

(2) *if X is the toric Fano 3-fold considered in Proposition 2.1, then the linear map $N_X = \mathbb{Z}^3 \to N_F = \mathbb{Z}^4$ induced by the matrix*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

induces a toric morphism $X \to F$ which is a closed embedding and identifies X with the divisor in F defined by the equation

$$u_5u_7 - u_6^2u_1u_2u_3u_4 = 0$$

in the Cox coordinates of F.

Remark 3.2. — The matrix A gives an injective homomorphism of algebraic tori $(\mathbb{C}^*)^3 \hookrightarrow (\mathbb{C}^*)^4$ and X is the closure in F of the orbit of $(1,1,1,1)$ under the action of $(\mathbb{C}^*)^3$ induced by A.

Remark 3.3. — The toric variety F in Proposition 3.1 is a smooth Fano 4-fold with ID 97 in [46].

Remark 3.4. — The weight matrix of F and the line bundle

$$L = \mathcal{O}_F \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

of which the equation of X is a section of have previously appeared in [23, Section 87]. We refer to the analysis there justifying why the Picard rank of X is 4, whereas the Picard rank of its ambient space F is 3.

Proof of Proposition 3.1. —

(1). — The divisor sequence [26][Theorem 4.1.3] of F is

$$0 \longrightarrow M_F \simeq \mathbb{Z}^4 \xrightarrow{(R_F)^T} \mathrm{Div}_T(F) \simeq \mathbb{Z}^7 \xrightarrow{D_F} \mathrm{Pic}(F) \simeq \mathbb{Z}^3 \longrightarrow 0$$

where D_F is the 3×7 matrix in the statement (1).

Thus D_F is the weight matrix for F; in other words, the action of the torus $(\mathbb{C}^*)^3$ on $\mathbb{A}^7$ with weights given by D_F is such that F is the corresponding GIT quotient $\mathbb{A}^7 /\!\!/ (\mathbb{C}^*)^3$ with respect to a certain stability condition. Since F is Fano and $-K_F$ is the $(\mathbb{C}^*)^3$-linearised line bundle on the quotient with weights $(1,1,3) \in \mathrm{Pic}(F) \simeq \mathbb{Z}^3$, the stability condition is given by the chamber which contains $(1,1,3)$. This chamber also contains $(1,1,1)$, as it is the positive orthant.

It is immediate to check that F is smooth as all relevant 3×3 minors of R_F are equal to ± 1.

(2). — Let $\rho_1, \ldots, \rho_{10}$ be the vectors in (2.1), i.e. the primitive generators of the rays of the fan defining X. The map induced by A sends the ρ_i's onto the hyperplane with normal vector $m_X := (1,1,-1,-1)$:

$$A\left(\ \rho_1 \mid \cdots \mid \rho_{10}\ \right) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ -1 & 1 & 1 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & -1 & -1 \end{bmatrix}.$$

One can check that each cone of Σ_X is sent via A to a cone in Σ_F. Furthermore, the image of the fan Σ_X equals the intersection of the hyperplane $m_X^\perp$ with Σ_F. Therefore the image of the toric morphism $X \to F$ is a prime divisor on F. We want to determine its equation in the Cox coordinates of F.

Each lattice element $A(\rho_i)$ belongs to exactly one 2-dimensional cone of F. Let us write it as a linear combination, with positive coefficients, of the primitive generators of the fan Σ_F. For instance

$$A(\rho_1) = (0,0,-1,1)^T = (0,0,-1,-1)^T + 2(0,0,0,1)^T = r_6 + 2r_7.$$

The matrix which express these combinations is

	r_1	r_2	r_3	r_4	r_5	r_6	r_7
$A(\rho_1)$	0	0	0	0	0	1	2
$A(\rho_2)$	1	0	0	0	1	0	0
$A(\rho_3)$	0	1	0	0	1	0	0
$A(\rho_4)$	1	0	0	0	0	0	1
$A(\rho_5)$	0	1	0	0	0	0	1
$A(\rho_6)$	0	0	0	0	2	1	0
$A(\rho_7)$	0	0	1	0	0	0	1
$A(\rho_8)$	0	0	0	1	0	0	1
$A(\rho_9)$	0	0	1	0	1	0	0
$A(\rho_{10})$	0	0	0	1	1	0	0

Let B be the 7×10 matrix given by transposing this matrix. We have a commutative diagram with exact rows

$$\begin{tikzcd}
 & & & \mathbb{Z} & \\
0 \arrow[r] & (\mathrm{Pic}(F))^\vee \arrow[r] & (\mathrm{Div}_T F)^\vee \arrow[r, "R_F"] & N_F \arrow[r] \arrow[u, two heads, "m_X"] & 0 \\
0 \arrow[r] & (\mathrm{Cl}(X))^\vee \arrow[r] \arrow[u] & (\mathrm{Div}_T X)^\vee \arrow[r, "R_X"] \arrow[u, "B^T"] & N_X \arrow[r] \arrow[u, "A"] & 0
\end{tikzcd}$$

which dualises to

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[d, "m_X"] & & & \\
0 \arrow[r] & M_F \arrow[r, "R_F^T"] \arrow[d, "A^T"] & \mathrm{Div}_T F \arrow[r, "D_F"] \arrow[d, "B"] & \mathrm{Pic}(F) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_X \arrow[r, "R_X^T"] & \mathrm{Div}_T X \arrow[r, "R_X"] & \mathrm{Cl}(X) \arrow[r] & 0
\end{tikzcd}$$

where the two exact rows are the divisor sequences. The map $B\colon \operatorname{Div}_T F \to \operatorname{Div}_T X$ is the pull-back of torus-invariant divisors on F along the morphism $X \to F$. In terms of Cox coordinates we have the following.

$$\begin{aligned}
u_1 &\longmapsto x_2x_4\\
u_2 &\longmapsto x_3x_5\\
u_3 &\longmapsto x_7x_9\\
u_4 &\longmapsto x_8x_{10}\\
u_5 &\longmapsto x_2x_3x_6^2x_9x_{10}\\
u_6 &\longmapsto x_1x_6\\
u_7 &\longmapsto x_1^2x_4x_5x_7x_8
\end{aligned}$$

It is clear that the equation describing the image of X in F is

$$u_5u_7 = u_6^2u_1u_2u_3u_4.$$

It remains to check that the toric morphism $X \to F$ is a closed embedding. This can be checked locally by analysing the affine charts. $\square$

4. Deforming a toric variety

In Proposition 3.1 we have seen that the toric Fano 3-fold X of Proposition 2.1 is a divisor inside a smooth toric Fano 4-fold F. Now we show that a particular 4-dimensional subspace of the linear system $|\mathcal{O}_F(X)|$ gives the miniversal ($\mathbb{Q}$-Gorenstein) deformation of X.

Proposition 4.1. — *Let X be the toric Fano 3-fold in Proposition 2.1 and let F be the toric Fano 4-fold in Proposition 3.1. Let $u_1,\dots,u_7$ denote the Cox coordinates of F as in Proposition 3.1. Consider the 4-parameter flat family*

$$\mathcal{X} \longrightarrow \mathbb{A}^4 = \operatorname{Spec}\mathbb{C}[c_1,c_2,c_3,c_4]$$

given by the equation

$$u_5u_7 - u_1u_2u_3u_4u_6^2 + u_6^2(c_1u_1^2u_2^2 + c_2u_1^2u_4^2 + c_3u_2^2u_3^2 + c_4u_3^2u_4^2) = 0 \quad (4.1)$$

inside F.

Then the base change of $\mathcal{X} \to \mathbb{A}^4$ to $\operatorname{Spf}\mathbb{C}[\![c_1,c_2,c_3,c_4]\!]$ is the miniversal ($\mathbb{Q}$-Gorenstein) deformation of X. Moreover, the discriminant locus (i.e. the locus in $\mathbb{A}^4$ where the fibres of $\mathcal{X} \to \mathbb{A}^4$ are singular) and the divisor $\{c_1c_2c_3c_4 = 0\} \subset \mathbb{A}^4$ coincide in a neighbourhood of the origin in $\mathbb{A}^4$.

We put the word $\mathbb{Q}$-Gorenstein in parenthesis because the miniversal deformation of X coincides with the miniversal $\mathbb{Q}$-Gorenstein deformation of X, since every infinitesimal deformation of X is automatically $\mathbb{Q}$-Gorenstein because X is Gorenstein.

The fact that the fibre of $\mathcal{X} \to \mathbb{A}^4$ over the origin is exactly X is the content of Proposition 3.1, hence we need to prove the versality of this deformation. Before doing this we prove some preliminary lemmata.

Lemma 4.2. — *Let s_1, s_2, s_3, x, y be indeterminates. Consider the polynomial ring $R = \mathbb{C}[s_1, s_2, s_3, x, y]$ with $\mathbb{N}$-grading given by $\deg s_1 = \deg s_2 = \deg s_3 = 1$ and $\deg x = \deg y = 0$. For every $k \in \mathbb{N}$,*

- *let R_k be the homogenous summand of R of degree k, i.e. the $\mathbb{C}$-vector subspace of R with basis*
$$\left\{ s_1^{i_1} s_2^{i_2} s_3^{i_3} x^m y^n \,\middle|\, i_1, i_2, i_3, m, n \in \mathbb{N},\ i_1 + i_2 + i_3 = k \right\};$$
- *set $R_{\leqslant k} = \bigoplus_{0 \leqslant i \leqslant k} R_i$, which is the $\mathbb{C}$-vector subspace of R with basis*
$$\left\{ s_1^{i_1} s_2^{i_2} s_3^{i_3} x^m y^n \,\middle|\, i_1, i_2, i_3, m, n \in \mathbb{N},\ i_1 + i_2 + i_3 \leqslant k \right\};$$
- *set $R_{\geqslant k} = \bigoplus_{i \geqslant k} R_i$, which is the ideal $(s_1, s_2, s_3)^k$ of R.*

Then there exist two sequences $\{x_k\}_{k \in \mathbb{N}}$ and $\{y_k\}_{k \in \mathbb{N}}$ of polynomials such that:

(1) *$x_0 = x$ and $y_0 = y$,*
(2) *for every $k \in \mathbb{N}^+$, $x_k \in R_{\leqslant k} \cap (x, y)$ and $y_k \in R_{\leqslant k} \cap (x, y)$,*
(3) *for every $k \in \mathbb{N}^+$, $x_k - x_{k-1} \in R_k$ and $y_k - y_{k-1} \in R_k$,*
(4) *for every $k \in \mathbb{N}^+$, $f_k := x_k y_k - (xy + s_1 x^2 + s_2 y^2 + s_3 x^2 y^2)$ lies in the ideal $R_{\geqslant k+1} \cap (x, y)^2$.*

By (3) the sequences of the x_k's and of the y_k's give two elements of the ring $\mathbb{C}[x, y][\![s_1, s_2, s_3]\!]$. By (4) the product of these two power series is $xy + s_1 x^2 + s_2 y^2 + s_3 x^2 y^2$.

Proof of Lemma 4.2. — We proceed by induction on $k \in \mathbb{N}^+$. In the base case $k = 1$ we take $x_1 = x + s_2 y + s_3 x^2 y$ and $y_1 = y + s_1 x$. Obviously (2) and (3) hold. Moreover,
$$f_1 = x_1 y_1 - (xy + s_1 x^2 + s_2 y^2 + s_3 x^2 y^2) = s_1 s_2 xy + s_1 s_3 x^3 y$$
lies in $(s_1, s_2, s_3)^2 \cap (x, y)^2$; this gives (4).

Now we do the inductive step. Fix $k \in \mathbb{N}^+$ and assume that we have $x_0, \ldots, x_k$ and $y_0, \ldots, y_k$ which satisfy (2)–(4); we shall construct x_{k+1} and y_{k+1}.

By (4) $f_k \in (x,y)^2 \subset (x,y)$, so there exist polynomials A and B such that $f_k = Ax + By$. We can assume that there are no cancellations between Ax and By, for instance by requiring that $B \in \mathbb{C}[s_1, s_2, s_3, y]$. Since by (3) $f_k \in R_{\geqslant k+1}$, we have $A, B \in R_{\geqslant k+1}$. Let a (resp. b) the homogeneous part of A (resp. B) of degree $k+1$. So $a, b \in R_{k+1}$ and $A - a, B - b \in R_{\geqslant k+2}$. Moreover, since $f_k \in (x,y)^2$ we have $A, B \in (x,y)$, and consequently $a, b \in (x,y)$.

Set

$$x_{k+1} = x_k - b \quad \text{and} \quad y_{k+1} = y_k - a.$$

By the inductive hypothesis we have $x_k, y_k \in R_{\leqslant k} \cap (x,y)$ and by construction we have $a, b \in R_{k+1} \cap (x,y)$; therefore x_{k+1} and y_{k+1} lie in $R_{\leqslant k+1} \cap (x,y)$; this is (2) for $k+1$. Clearly we also have (3) for $k+1$.

We have

$$\begin{aligned} f_{k+1} &= x_{k+1}y_{k+1} - (xy + s_1x^2 + s_2y^2 + s_3x^2y^2) \\ &= (x_k - b)(y_k - a) - (xy + s_1x^2 + s_2y^2 + s_3x^2y^2) \\ &= f_k - by_k - ax_k + ab \\ &= (Ax + By) - ax_k - by_k + ab. \end{aligned}$$

Since $A, B, a, b, x_k, y_k \in (x,y)$ we get $f_{k+1} \in (x,y)^2$. Moreover, we can write

$$f_{k+1} = a(x - x_k) + (A - a)x + b(y - y_k) + (B - b)y + ab.$$

Since $a, b \in R_{k+1}$, $A - a, B - b \in R_{\geqslant k+2}$ and $x - x_k, y - y_k \in R_{\geqslant 1}$, we have $f_{k+1} \in R_{\geqslant k+2}$. We have obtained (4) for $k+1$. $\square$

Lemma 4.3. — *Consider the affine 3-fold $U = \operatorname{Spec} \mathbb{C}[x,y,z,w]/(xy - zw)$. Then the two formal deformations of U over* $\operatorname{Spf} \mathbb{C}[\![s_1, s_2, s_3, s_4]\!]$ *given by*

$$xy - zw + s_1x^2 + s_2y^2 + s_3x^2y^2 + s_4 = 0$$

and by

$$xy - zw + s_4 = 0,$$

respectively, are isomorphic.

Proof. — Fix $k \in \mathbb{N}$. Consider the finite $\mathbb{C}$-algebra

$$S_k = \mathbb{C}[s_1, s_2, s_3, s_4]/(s_1, s_2, s_3, s_4)^{k+1}$$

and the following two S_k-algebras

$$\begin{aligned} A_k &= S_k[x,y,z,w]/(xy - zw + s_4), \\ B_k &= S_k[x,y,z,w]/(xy - zw + s_1x^2 + s_2y^2 + s_3x^2y^2 + s_4). \end{aligned}$$

Let x_k and y_k be the polynomials constructed in Lemma 4.2. By (4) one can consider the S_k-algebra homomorphism $\phi_k \colon A_k \to B_k$ given by $x \mapsto x_k$,

$y \mapsto y_k$, $z \mapsto z$, $w \mapsto w$. By (1) and (2) we have $x - x_k \in (s_1, s_2, s_3, s_4)$, so ϕ_k is just a translation, hence it is invertible.

By (3) we have

$$x_k \equiv x_{k-1} \mod (s_1, s_2, s_3, s_4)^k \quad \text{and} \quad y_k \equiv y_{k-1} \mod (s_1, s_2, s_3, s_4)^k.$$

So ϕ_k and ϕ_{k-1} are equal modulo $(s_1, s_2, s_3, s_4)^k$. More precisely, by using the natural projection $S_k \twoheadrightarrow S_{k-1}$ we get that $\phi_k \otimes_{S_k} \mathrm{id}_{S_{k-1}}$ and ϕ_{k-1} are equal as isomorphisms from $A_{k-1} = A_k \otimes_{S_k} S_{k-1}$ to $B_{k-1} = B_k \otimes_{S_k} S_{k-1}$. Therefore the system of isomorphisms ϕ_k's gives the required isomorphism of formal deformations of U. □

Lemma 4.4. — *Consider the affine 3-fold $U = \operatorname{Spec} \mathbb{C}[x, y, z, w]/(xy - zw)$ and consider the flat deformation $\mathcal{U} \to \mathbb{A}^4 = \operatorname{Spec} \mathbb{C}[s_1, s_2, s_3, s_4]$ of U given by the equation*

$$xy - zw + s_1x^2 + s_2y^2 + s_3x^2y^2 + s_4 = 0. \tag{4.2}$$

Then the discriminant locus of $\mathcal{U} \to \mathbb{A}^4$, i.e. the locus in $\mathbb{A}^4$ over which the fibres of $\mathcal{U} \to \mathbb{A}^4$ are singular, is the vanishing locus of

$$s_4(16s_1^2s_2^2 - 32s_1s_2s_3s_4 - 8s_1s_2 + 16s_3^2s_4^2 - 8s_3s_4 + 1). \tag{4.3}$$

In particular, in a neighbourhood of 0 in $\mathbb{A}^4$ the discriminant locus coincides with the hyperplane $\{s_4 = 0\}$.

Proof. — Let $f \in \mathbb{C}[s_1, s_2, s_3, s_4, x, y, z, w]$ be the polynomial appearing in (4.2). Consider the ideal $J \subseteq \mathbb{C}[s_1, s_2, s_3, s_4, x, y, z, w]$ generated by f and by the 4 partial derivatives of f with respect to x, y, z and w. By the Jacobian criterion and [25, Section 3.2], the discriminant locus in $\mathbb{A}^4 = \operatorname{Spec} \mathbb{C}[s_1, s_2, s_3, s_4]$ is defined by the ideal $J \cap \mathbb{C}[s_1, s_2, s_3, s_4]$. By [25, Theorem 3.1.2] this last ideal can be easily computed with any computer algebra software by taking a Gröbner basis of J with respect to the lexicographic monomial order $>$ such that $x > y > z > w > s_1 > s_2 > s_3 > s_4$. The ideal $J \cap \mathbb{C}[s_1, s_2, s_3, s_4]$ is generated by the polynomial in (4.3).

The last assertion in Lemma 4.4 is obvious because the second factor of the polynomial in (4.3) does not vanish at the origin. □

Proof of Proposition 4.1. — We have already observed that Proposition 3.1 says that the fibre of $\mathcal{X} \to \mathbb{A}^4$ over the origin is X. Moreover, one can easily see that all monomials in the Cox coordinates of F appearing in (4.1) have the same degree with respect to the grading given by $\operatorname{Pic}(F)$, i.e. they are all sections of the line bundle $\mathcal{O}_F(0, 0, 2)$ on F. In other words, the fibres of $\mathcal{X} \to \mathbb{A}^4$ are elements of the linear system $|\mathcal{O}_F(X)|$.

Consider the formal deformation of X over $\operatorname{Spf} \mathbb{C}[\![c_1, c_2, c_3, c_4]\!]$ given by taking the formal completion of $\mathcal{X} \to \mathbb{A}^4$ at the central fibre.

We know that the infinitesimal deformations of X are exactly the infinitesimal deformations of its singularities, i.e. of the 4 affine open subset U_α, U_β, U_γ, U_δ which contain the 4 singular points of X. Indeed, the map in (2.3) is smooth and induces an isomorphism on tangent spaces. Therefore, in order to check that our formal deformation of X over $\operatorname{Spf}\mathbb{C}[\![c_1,c_2,c_3,c_4]\!]$ is the miniversal deformation of X, we need to check that it induces the miniversal deformation of U_α, U_β, U_γ, and U_δ when restricted to U_α, U_β, U_γ, and U_δ, respectively.

More precisely, this deformation of X over $\operatorname{Spf}\mathbb{C}[\![c_1,c_2,c_3,c_4]\!]$ must be induced by the miniversal deformation of X, which is over $\operatorname{Spf}\mathbb{C}[\![t_\alpha,t_\beta,t_\gamma,t_\delta]\!]$, via a local $\mathbb{C}$-algebra homomorphism

$$\psi\colon \mathbb{C}[\![t_\alpha,t_\beta,t_\gamma,t_\delta]\!] \longrightarrow \mathbb{C}[\![c_1,c_2,c_3,c_4]\!]. \tag{4.4}$$

We will need to prove that this ring homomorphism is an isomorphism.

Before doing that, recall all notation from Proposition 2.1 and Proposition 3.1. Denote by C_{ijkl} the cone in Σ_F generated by r_i, r_j, r_k and r_l.

Let us start from U_α. It is easy to check that the linear map $A\colon N_X \to N_F$ maps the cone σ_α to C_{1257}. Therefore, the toric morphism $X \to F$ associated to A maps U_α to the affine chart $U^F_{1257} \subset F$ associated to the cone C_{1257}. We know that the monoid $\sigma_\alpha^\vee \cap M_X$ is generated by

$$x_\alpha = (-1,1,1) \quad y_\alpha = (1,0,0) \quad z_\alpha = (0,0,1) \quad w_\alpha = (0,1,0),$$

whereas one can check that the monoid $C^\vee_{1257} \cap M_F$ is generated by

$$u_1 = (1,0,0,0) \quad u_2 = (0,1,0,0) \quad u_5 = (0,0,1,0) \quad u_7 = (0,0,0,1).$$

Here we have used the names of the Cox coordinates of F because these 4 elements correspond to the dehomogenisations of the Cox coordinates of F with respect to the cone C_{1257}. Indeed U^F_{1257} is the locus in F where $u_3 \neq 0$, $u_4 \neq 0$ and $u_6 \neq 0$. It is easy to see that the transpose $A^T: M_F \to M_X$ acts as

$$u_1 \longmapsto w_\alpha \quad u_2 \longmapsto z_\alpha \quad u_5 \longmapsto x_\alpha \quad u_7 \longmapsto y_\alpha. \tag{4.5}$$

These are exactly the assignments of the surjective $\mathbb{C}$-algebra homomorphism associated to the closed embedding

$$U_\alpha = \operatorname{Spec} \frac{\mathbb{C}[x_\alpha, y_\alpha, z_\alpha, w_\alpha]}{(x_\alpha y_\alpha - w_\alpha z_\alpha)} \longrightarrow U^F_{1257} = \operatorname{Spec}\mathbb{C}[u_1,u_2,u_5,u_7]$$

which is the restriction of the toric closed embedding $X \hookrightarrow F$. If we dehomogenise (4.1) with respect to C_{1257} we get the equation

$$u_5u_7 - u_1u_2 + c_1u_1^2u_2^2 + c_2u_1^2 + c_3u_2^2 + c_4 = 0.$$

If now we apply (4.5), we deduce that the flat family $\mathcal{X} \to \mathbb{A}^4$ induces the deformation of U_α over $\operatorname{Spf} \mathbb{C}[\![c_1, c_2, c_3, c_4]\!]$ given by the equation

$$x_\alpha y_\alpha - w_\alpha z_\alpha + c_1 w_\alpha^2 z_\alpha^2 + c_2 w_\alpha^2 + c_3 z_\alpha^2 + c_4 = 0. \tag{4.6}$$

By Lemma 4.3 this deformation of U_α over $\operatorname{Spf} \mathbb{C}[\![c_1, c_2, c_3, c_4]\!]$ is isomorphic to the one given by the equation

$$x_\alpha y_\alpha - w_\alpha z_\alpha + c_4 = 0.$$

By comparing with (2.4) we have that the homomorphism ψ in (4.4) satisfies $\psi(t_\alpha) = c_4$.

Let us now look at U_β. The analysis is very similar. The morphism associated to A maps U_β to $U_{1457}^F = \{u_2 \neq 0,\ u_3 \neq 0,\ u_6 \neq 0\}$ and the induced map on the respective $\mathbb{C}$-algebras is

$$u_1 \longmapsto y_\beta \quad u_4 \longmapsto x_\beta \quad u_5 \longmapsto z_\beta \quad u_7 \longmapsto w_\beta. \tag{4.7}$$

In the chart U_{1457}^F, equation (4.1) becomes

$$u_5 u_7 - u_1 u_4 + c_1 u_1^2 + c_2 u_1^2 u_4^2 + c_3 + c_4 u_4^2 = 0,$$

and the flat family $\mathcal{X} \to \mathbb{A}^4$ induces the deformation of U_β over $\operatorname{Spf} \mathbb{C}[\![c_1, c_2, c_3, c_4]\!]$ given by

$$z_\beta w_\beta - x_\beta y_\beta + c_1 y_\beta^2 + c_2 x_\beta^2 y_\beta^2 + c_3 + c_4 x_\beta^2 = 0. \tag{4.8}$$

By Lemma 4.3 this deformation is isomorphic to

$$z_\beta w_\beta - x_\beta y_\beta + c_3 = 0,$$

therefore we have that $\psi(t_\beta) = c_3$.

Similarly, U_γ maps to U_{3457}^F such that

$$u_3 \longmapsto x_\gamma \quad u_4 \longmapsto y_\gamma \quad u_5 \longmapsto z_\gamma \quad u_7 \longmapsto w_\gamma \tag{4.9}$$

Equation (4.1) restricted to U_{3457}^F induces the deformation of U_γ given by

$$z_\gamma w_\gamma - x_\gamma y_\gamma + c_1 + c_2 y_\gamma^2 + c_3 x_\gamma^2 + c_4 x_\gamma^2 y_\gamma^2 = 0. \tag{4.10}$$

We use Lemma 4.3 to deduce that $\psi(t_\gamma) = c_1$.

Lastly, the map $U_\delta \to U_{2357}^F$ given by

$$u_2 \longmapsto x_\delta \quad u_3 \longmapsto y_\delta \quad u_5 \longmapsto z_\delta \quad u_7 \longmapsto w_\delta \tag{4.11}$$

induces the deformation of U_δ over $\operatorname{Spf} \mathbb{C}[\![c_1, c_2, c_3, c_4]\!]$

$$z_\delta w_\delta - x_\delta y_\delta + c_1 x_\delta^2 + c_2 + c_3 x_\delta^2 y_\delta^2 + c_4 y_\delta^2 = 0. \tag{4.12}$$

Again the change of variables in Lemma 4.3 allows us to conclude that $\psi(t_\delta) = c_2$.

We have proved that the ring homomorphism ψ in (4.4) satisfies $\psi(t_\alpha) = c_4$, $\psi(t_\beta) = c_3$, $\psi(t_\gamma) = c_1$ and $\psi(t_\delta) = c_2$. Therefore ψ is an isomorphism.

This proves that the considered deformation of X is the miniversal one. This concludes the proof of the first part of the assertions in Proposition 4.1.

We need to analyse the discriminant locus of the family $\mathcal{X} \to \mathbb{A}^4$. In a neighbourhood of the origin in $\mathbb{A}^4 = \operatorname{Spec} \mathbb{C}[c_1, c_2, c_3, c_4]$, the discriminant locus coincides with the union of the discriminant locus of the deformation of the 4 affine charts of X which contain the singularities of X.

We have computed the equation of the deformation of U_α in (4.6). By Lemma 4.4 the discriminant locus of this deformation of U_α coincides with the hyperplane $\{c_4 = 0\}$ in a neighbourhood of the origin in $\mathbb{A}^4$.

By repeating the argument for U_β, U_γ and U_δ and by taking the union of these hyperplanes, we conclude. $\square$

Remark 4.5. — By analysing certain automorphisms of F which leave X invariant, it might be possible that the discussion on the deformation of U_α be sufficient in order to deduce the results about the deformations of U_β, U_γ, U_δ in the proof above. This approach might make the proof less repetitive, but not shorter.

5. Conclusion

Proof of Theorem 1.1. — Consider the flat deformation

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d] \\ 0 \arrow[r, hook] & \mathbb{A}^4 \end{tikzcd}$$

of X considered in Proposition 4.1. By Proposition 2.1 (1) X is K-polystable, in particular K-semistable. Since being K-semistable is an open property [13, 56], there exists an open neighbourhood W of the origin in $\mathbb{A}^4$ such that each fibre of $\mathcal{X}_W := \mathcal{X} \times_{\mathbb{A}^4} W \to W$ is K-semistable. Let $\mathcal{M}^{\mathrm{Kss}}_{3,28}$ be the algebraic stack which parametrises $\mathbb{Q}$-Gorenstein families of 3-dimensional K-semistable Fano varieties with anticanonical volume 28 [57]. The family $\mathcal{X}_W \to W$ is induced by a morphism $W \to \mathcal{M}^{\mathrm{Kss}}_{3,28}$.

The K-moduli space $M := M^{\mathrm{Kps}}_{3,28}$ is the good moduli space of $\mathcal{M}^{\mathrm{Kss}}_{3,28}$ in the sense of Alper, in particular there is a natural morphism $\mathcal{M}^{\mathrm{Kss}}_{3,28} \to M$. By composition we get a morphism $W \to M$ between schemes of finite type over $\mathbb{C}$. Since $\mathcal{X}_W \to W$ is essentially the miniversal deformation of X, we have that in an analytic neighbourhood of $[X]$ the morphism $W \to M$ behaves as the map $\Delta^4 \to \Delta^2$ between polydiscs described in the proof of Proposition 2.1 (9). In particular this shows that W dominates the irreducible component of M containing $[X]$. Since W is rational, the irreducible component of M containing $[X]$ is unirational.

However, we already know by Corollary 2.2 that there exists an open neighbourhood U of $[X]$ in M which is a smooth surface. Hence U must be unirational, and hence rational because $\dim U = 2$.

By Corollary 2.2 we know that the discriminant locus in U (i.e. the locus in U parametrising singular Fano 3-folds) is a smooth curve. This locus is dominated by the discriminant locus in W, which by Proposition 4.1 coincides with the reducible divisor $\{c_1c_2c_3c_4 = 0\}$ in a neighbourhood of 0 in $\mathbb{A}^4$. Since this divisor has rational components, we conclude that the discriminant locus in U is a smooth rational curve. $\square$

Appendix. Laurent inversion and global embeddings

The toric Fano 3-fold considered in Proposition 2.1 can be embedded into a toric 5-fold as a complete intersection. This is done via the Laurent inversion method, developed by Coates–Kasprzyk–Prince [24, 50, 51]. Since one of the obtained equations is a linear cone, we can then reduce to the hypersurface embedding that appears in Proposition 3.1.

Proposition A.1. — *Let X be the toric Fano 3-fold considered in Proposition 2.1. Let Y be the toric 5-fold given by the GIT quotient $\mathbb{A}^8 /\!\!/ (\mathbb{C}^*)^3$ with stability condition $\omega = (1,1,1)$ and the following weight matrix*

y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
1	0	0	1	0	0	−1	0
0	1	0	0	1	0	−1	0
0	0	1	0	0	1	1	1

Then X can be embedded into Y as the zero-section of a section of $L \oplus L^{\otimes 2}$, where L is the $(\mathbb{C}^*)^3$-linearised line bundles on Y of weights $(0,0,1)$. More specifically, the equations of X in the Cox coordinates of Y are

$$y_4y_5y_7 = y_3 \quad \textit{and} \quad y_6y_8 = y_7y_1y_2y_3.$$

Proof. — This is an immediate application of [24, Algorithm 5.1]. Consider the following 3 polytopes:

$$S_1 = \mathrm{conv}\{\rho_2, \rho_4\},$$
$$S_2 = \mathrm{conv}\{\rho_3, \rho_5\},$$
$$S_3 = \mathrm{conv}\{\rho_1, \rho_6, \rho_7, \rho_8, \rho_9, \rho_{10}\},$$

which are depicted in Figure A.1.

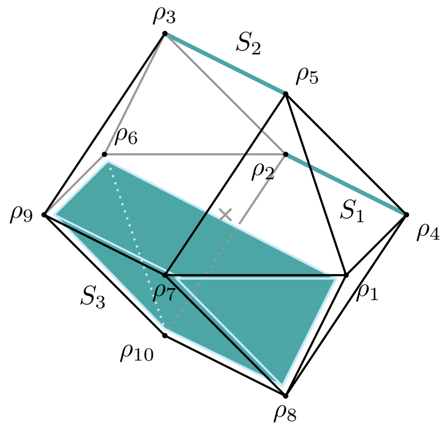

Figure A.1. Scaffolding of P735

Let Σ_Z be the normal fan of S_3. The rays of Σ_Z are the following vectors in $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$:

$$\begin{aligned} v_1 &= (0, -1, 0), \\ v_2 &= (0, 0, -1), \\ v_3 &= (-1, 1, 1), \\ v_4 &= (0, 1, 1), \\ v_5 &= (1, 0, 0). \end{aligned}$$

Let Z be the T_M-toric variety associated to the Σ_Z. It is smooth, projective of dimension 3. Since $\mathrm{conv}(S_1 \cup S_2 \cup S_3) = P$, the collection $S = \{S_1, S_2, S_3\}$ represents a *scaffolding* of P with shape Z, as introduced in [24].

We also see that Z is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$, namely $Z = \mathbb{P}_{\mathbb{P}^1}\left(\mathcal{O} \oplus \mathcal{O}(1)^{\oplus 2}\right)$. This ensures that the Laurent inversion construction embeds X as a complete intersection of codimension $\rho(Z) = 2$ (see the proof of [24, Proposition 12.2]). In particular, the relations that give the $\mathbb{P}^2$-bundle structure on $\mathbb{P}^1$ are

$$\begin{aligned} v_1 + v_2 + v_4 &= 0 \\ v_3 + v_5 &= v_4. \end{aligned} \tag{A.1}$$

Here the first relation simply describes the $\mathbb{P}^2$ fibre, while the second is a primitive relation (in the sense of Batyrev [7]) obtained by pulling back of the relation $v_3' + v_5' = 0$ in the quotient lattice induced by the projection to $\mathbb{P}^1$.

Consider the following 3 torus invariant nef divisors on Z:

$$\begin{aligned} D_1 &= E_1 - E_4, \\ D_2 &= E_2 - E_4, \\ D_3 &= E_3 + E_4 + E_5, \end{aligned} \tag{A.2}$$

where E_i are the torus-invariant divisors associated to the v_i. Their section polytopes are exactly S_1, S_2, S_3, respectively.

We consider the lattice $\widetilde{N} = \mathrm{Div}_{T_M} Z = \mathbb{Z}^5$ with basis $E_1, \dots, E_5$. This will be the ambient lattice for the fan of Y which we describe below as a GIT quotient using the expressions in (A.2). We then use the relations (A.1) to deduce the equations of X inside Y, keeping in mind the correspondence $E_i \leftrightarrow v_i$.

Following the construction in [24, Algorithm 5.1], the weight matrix of Y is of size $(r+z) \times r = 8 \times 3$, where r is the number of S_i and z is the number of rays in Σ_Z, given by:

I_1	I_2	I_3	E_1	E_2	E_3	E_4	E_5
1	0	0	1	0	0	-1	0
0	1	0	0	1	0	-1	0
0	0	1	0	0	1	1	1
y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Here a row i corresponds to a strut S_i (or, alternatively, to a divisor D_i whose polytope is S_i) and each (i,j) entry outside of the $r \times r$-identity block is the coefficient of E_j in the expression of D_i in (A.2).

The equations of X in the Cox coordinates on Y are

$$y_4 y_5 y_7 = y_3 \quad \text{and} \quad y_6 y_8 = y_7 y_1 y_2 y_3,$$

which are deduced from (A.1) as explained in [24, Proposition 12.2].

In particular, this shows that X is a complete intersection $L_1 \oplus L_2$, where L_i are $(\mathbb{C}^*)^3$-linearised line bundles with weights

$$L_1 = \mathcal{O}_Y(y_4) \otimes \mathcal{O}_Y(y_5) \otimes \mathcal{O}_Y(y_7) = \mathcal{O}_Y \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and

$$L_2 = \mathcal{O}_Y(y_6) \otimes \mathcal{O}_Y(y_8) = \mathcal{O}_Y \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

According to [24, Theorem 5.5], this is an embedding of a Fano variety; a posteriori the stability condition needed to define Y should be chosen such that

$$-K_Y - L_1 - L_2 = \mathcal{O}_Y\left(\begin{pmatrix}1\\1\\4\end{pmatrix} - \begin{pmatrix}0\\0\\1\end{pmatrix} - \begin{pmatrix}0\\0\\2\end{pmatrix}\right) = \mathcal{O}_Y\begin{pmatrix}1\\1\\1\end{pmatrix}$$

is ample, i.e. $\omega = (1,1,1)$. □

Remark A.2. — The ambient space Y thus obtained is in fact a smooth toric 5-fold. Since the first equation of the embedding in Proposition A.1 is linear in y_3, we can solve for this variable and transform the second equation into $y_6 y_8 = y_7^2 y_1 y_2 y_4 y_5$. Up to renaming the ambient Cox variables, this is exactly the embedding of X inside the 4-fold F from Proposition 3.1.

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