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# Shadow-complexity and trisection genus (*)

Hironobu Naoe (1) and Masaki Ogawa (2)

**Abstract.** — The shadow-complexity is an invariant of closed 4-manifolds defined by using 2-dimensional polyhedra called Turaev's shadows, which, roughly speaking, measures how complicated a 2-skeleton of the 4-manifold is. In this paper, we define a new version $\mathrm{sc}_r$ of shadow-complexity depending on an extra parameter $r \geqslant 0$, and we investigate the relationship between this complexity and the trisection genus $g$. More explicitly, we prove an inequality $g(W) \leqslant 2 + 2\mathrm{sc}_r(W)$ for any closed 4-manifold $W$ and any $r \geqslant 1/2$. Moreover, we determine the exact values of $\mathrm{sc}_{1/2}$ for infinitely many 4-manifolds, and also we classify all the closed 4-manifolds with $\mathrm{sc}_{1/2} \leqslant 1/2$.

**Résumé.** — La complexité ombre est un invariant de variétés de dimension 4 défini en utilisant des polyèdres de dimension 2, appelés les « ombres de Turaev », qui, de façon simplifiée, mesure la complexité d'un 2-squelette de la 4-variété. Dans cet article, nous définissons une version de la complexité ombre $\mathrm{sc}_r$ dépendant d'un paramètre supplémentaire $r \geqslant 0$, et nous investiguons les liens entre cette complexité et le genre de trisection $g$. Plus explicitement, nous prouvons l'inégalité $g(W) \leqslant 2 + 2\mathrm{sc}_r(W)$ pour toute 4-variété fermée et tout $r \geqslant 1/2$. De plus, nous déterminons les valeurs exactes de $\mathrm{sc}_{1/2}$ pour une famille infinie de 4-variétés, et nous classifions toutes les 4-variétés fermées avec $\mathrm{sc}_{1/2} \leqslant 1/2$.

## 1. Introduction

A *shadow* is a locally-flat simple polyhedron embedded in a connected closed oriented smooth 4-manifold as a 2-skeleton (see Definition 2.1), which

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was introduced by Turaev for the purpose of studying quantum invariants [20]. Afterwards, Costantino provided some applications of shadows to the topology of 3- and 4-manifolds. For example, we refer the reader to [3, 4] for the studies of Stein structures, Spin$^c$ structures and almost complex structures of connected oriented smooth 4-manifolds with boundary. In [2], he defined invariants of 3- and 4-manifolds called the *shadow-complexity* sc and the *special shadow-complexity* $\mathrm{sc}^{\mathrm{sp}}$ as the minimum numbers of certain vertices called true vertices of shadows of a fixed manifold. The shadow-complexity of 3-manifolds is closely related with the Gromov norm and stable maps of 3-manifolds [5, 8], which provided a geometric perspective on the shadow-complexity of 3-manifolds. In contrast to such studies, the shadow-complexity for 4-manifolds has been studied about the classification problem [2, 9, 13, 18, 19]. This paper aims to investigate a behavior of the shadow-complexity of 4-manifolds, and we provide a comparison between it and the *trisection genus* in particular.

A *trisection* is a decomposition of connected closed oriented smooth 4-manifold into three 4-dimensional 1-handlebodies (see Definition 2.7 for the precise definition). The intersection of the three portions forms a surface, which is called the central surface of the trisection. The *trisection genus* $g$ of a 4-manifold is defined as the minimum genus of central surfaces of trisections of the 4-manifold, and $g$ is of course an invariant of 4-manifolds. Only the 4-sphere is the closed 4-manifold with $g = 0$, and only $\pm\mathbb{CP}^2$ and $S^1 \times S^3$ are those with $g = 1$. The 4-manifolds with $g = 2$ were classified by Meier and Zupan [16]. The cases of $g \geqslant 3$ are still open, and Meier conjectured in [14] that an irreducible 4-manifold with $g = 3$ is either $\mathcal{S}_p$ or $\mathcal{S}'_p$ for some integer $p \geqslant 2$, where $\mathcal{S}_p$ and $\mathcal{S}'_p$ are 4-manifolds obtained from $S^1 \times S^3$ by surgering along a simple closed curve representing $p \in \mathbb{Z} \cong \pi_1(S^1 \times S^3)$. We also refer the reader to [21] for the decision of the trisection genera of trivial surface bundles over surfaces.

In this paper, we define a new kind of shadow-complexity called the *$r$-weighted shadow-complexity* $\mathrm{sc}_r$ for each fixed $r \in \mathbb{R}_{\geqslant 0}$, which is an invariant of 4-manifolds. It takes a value in $\{m + rn \mid m, n \in \mathbb{Z}_{\geqslant 0}\}$. The weighted shadow-complexity is defined by minimizing the sum of the number of true vertices and a "complexity" of regions of shadows, although we consider only the number of true vertices with regard to the shadow-complexity.

We establish a method to construct a trisection from a given shadow of a closed 4-manifold via a Kirby diagram. This method includes how to describe a trisection diagram, and it allows us to estimate the trisection genus of the 4-manifold from the combinatorial information of the shadow. The following is the main theorem in this paper.

Theorem 4.11. — *For any closed 4-manifold $W$ and any real number $r \geqslant 1/2$, $g(W) \leqslant 2 + 2\,\mathrm{sc}_r(W)$.*

The equality $g(W) = 2 + 2\,\mathrm{sc}_{1/2}(W)$ is attained, for instance, by $W = k_1(S^2 \times S^2)\#k_2\mathbb{CP}^2\#k_3\overline{\mathbb{CP}}^2$ for any $k_1, k_2, k_3 \in \mathbb{Z}_{\geqslant 0}$. In this sense, we can say that the inequality is the best possible (cf. Remark 4.13).

We compare the 3 series of the shadow-complexities sc, $\mathrm{sc}_r$ and $\mathrm{sc}^{\mathrm{sp}}$ with each other. More concretely, we show in Proposition 3.2 the following

$$\mathrm{sc}(W) = \mathrm{sc}_0(W) \leqslant \mathrm{sc}_r(W) \leqslant \mathrm{sc}_{r'}(W) \leqslant \mathrm{sc}_2(W) = \mathrm{sc}^{\mathrm{sp}}(W)$$

for any closed 4-manifold $W$ and $r, r' \in \mathbb{R}$ with $0 \leqslant r < r'$. It is remarkable that $\mathrm{sc}_r$ is finite-to-one invariant if $r > 0$, which will be shown in Proposition 3.4. Note that $\mathrm{sc}^{\mathrm{sp}}$ is also finite-to-one, but sc is not.

The minimum of $r$ satisfying the inequality in Theorem 4.11 is $1/2$ (cf. Remark 4.13), so we then focus on the behavior of $\mathrm{sc}_{1/2}$. Note that $\mathrm{sc}_{1/2}$ takes values in non-negative half integers. In Proposition 4.12, we determine the exact values of $\mathrm{sc}_{1/2}$ for infinitely many closed 4-manifolds by using Theorem 4.11. We also give the classification of all the 4-manifolds with 1/2-weighted shadow-complexity at most 1/2.

Theorem 5.1. — *The 1/2-weighted shadow-complexity of a closed 4-manifold $W$ is 0 if and only if $W$ is diffeomorphic to either one of $S^4$, $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, $S^2 \times S^2$, $2\mathbb{CP}^2$, $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ or $2\overline{\mathbb{CP}}^2$.*

Theorem 5.19. — *The 1/2-weighted shadow-complexity of a closed 4-manifold $W$ is $1/2$ if and only if $W$ is diffeomorphic to either one of $3\mathbb{CP}^2$, $2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$, $3\overline{\mathbb{CP}}^2$, $S^1 \times S^3$, $(S^1 \times S^3)\#\mathbb{CP}^2$, $(S^1 \times S^3)\#\overline{\mathbb{CP}}^2$, $\mathcal{S}_2$, $\mathcal{S}'_2$ or $\mathcal{S}_3$.*

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## 2. Preliminaries

### 2.1. Assumption and notations

- Any manifold is supposed to be compact, connected, oriented and smooth unless otherwise mentioned.

- For triangulable spaces $A \subset B$, let $\mathrm{Nbd}(A; B)$ denote a regular neighborhood of $A$ in $B$.
- For an $n$-manifold $W$ with $\partial W = \emptyset$ (resp. $\partial W \neq \emptyset$) and an integer $k$, we will use the notation $kW$ for the connected sum (resp. boundary connected sum) of $k$ copies of $W$ if $k > 0$, for $S^n$ (resp. $B^n$) if $k = 0$, and for the connected sum (resp. boundary connected sum) of $|k|$ copies of $W$ with the opposite orientation if $k < 0$.
- Let $\Sigma_{g,b}$ denote a compact surface of genus $g$ with $b$ boundary components. If $b = 0$, we will write it as $\Sigma_g$ simply.

## 2.2. Simple polyhedra and shadows

Let $X$ be a connected compact space. We call $X$ a *simple polyhedron* if a regular neighborhood $\mathrm{Nbd}(x; X)$ of each point $x \in X$ is homeomorphic to one of (i)–(iv) shown in Figure 2.1.

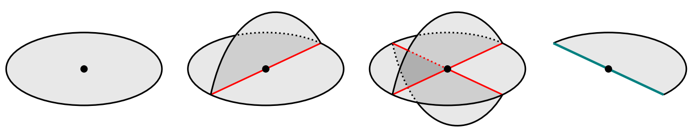



Figure 2.1. Local models of simple polyhedra.

A point of type (iii) is called a *true vertex*. The set of all points of types (ii) and (iii) is called the *singular set* of $X$, and it is denoted by $S(X)$. Note that $S(X)$ is disjoint union of circles and 4-valent graphs. A connected component of $S(X)$ with the true vertices removed is called a *triple line*. Each connected component of $X \setminus S(X)$ is called a *region*, and hence a region is homeomorphic to some surface. If $X$ has only disk regions, then $X$ is said to be *special*. The set of points of type (iv) is the *boundary* of $X$, which is denoted by $\partial X$. If $\partial X$ is empty, the simple polyhedron $X$ is said to be *closed*. If a region does not intersect $\partial X$, it is called an *internal region*, and otherwise a *boundary region*.

Before defining shadows of 4-manifold, we note that a simple polyhedron $X$ embedded in a 4-manifold $W$ is said to be *locally-flat* if a neighborhood $\mathrm{Nbd}(x; X)$ of each point $x \in X$ is contained in a smooth 3-ball in $W$.

Definition 2.1. — *A simple polyhedron $X$ embedded in a closed 4-manifold $W$ local-flatly is a* shadow *of $W$ if $W \setminus \mathrm{Int}\,\mathrm{Nbd}(X; W)$ is diffeomorphic to $k(S^1 \times B^3)$ for some $k \in \mathbb{Z}_{\geqslant 0}$.*

The notion of shadows was introduced by Turaev, who showed the following.

Theorem 2.2 (Turaev [20]). — *Any closed* 4*-manifold admits a shadow.*

The *complexity* of a simple polyhedron $X$ is the number of true vertices of $X$. Theorem 2.2 allows us to define an invariant of closed 4-manifolds like the Matveev complexity of 3-manifolds.

Definition 2.3. — *Let* $W$ *be a closed* 4*-manifold. The* shadow-complexity $\mathrm{sc}(W)$ *of* $W$ *is defined as the minimum of the complexities over all shadows of* $W$*. The* special shadow-complexity $\mathrm{sc}^{\mathrm{sp}}(W)$ *of* $W$ *is defined as the minimum of the complexities over all special shadows of* $W$*.*

This notion was introduced by Costantino in [2]. See [5, 8, 9, 13] for the studies regarding the (special) shadow-complexity.

### 2.3. Gleams and shadowed polyhedra

We then define the $\mathbb{Z}_2$*-gleam* of a simple polyhedron $X$. Let $R$ be an internal region of $X$. Then $R$ is homeomorphic to the interior of some compact surface $F$, and the homeomorphism $\mathrm{Int}\, F \to R$ will be denoted by $f$. This $f$ can extend to a local homeomorphism $\overline{f} : F \to X$. Moreover, there exists a simple polyhedron $\widetilde{F}$ obtained from $F$ by attaching an annulus or a Möbius band to each boundary component of $F$ along the core circle such that $\overline{f}$ can extend to a local homeomorphism $\widetilde{f} : \widetilde{F} \to X$. Then the number of the Möbius bands attached to $F$ modulo 2 is called the $\mathbb{Z}_2$*-gleam* of $R$ and is denoted by $\mathfrak{gl}_2(R) \in \{0, 1\}$. Note that this number is determined only by $X$ combinatorially.

Definition 2.4. — *A* gleam function*, or simply* gleam*, of a simple polyhedron* $X$ *is a function associating to each internal region* $R$ *of* $X$ *a half-integer* $\mathfrak{gl}(R)$ *satisfying* $\mathfrak{gl}(R) + \frac{1}{2}\mathfrak{gl}_2(R) \in \mathbb{Z}$*. The value* $\mathfrak{gl}(R)$ *is called the* gleam *of* $R$*. A simple polyhedron equipped with a gleam is called a* shadowed polyhedron*.*

Theorem 2.5 (Turaev [20]). — *There exists a canonical way to associate to a shadowed polyhedron* $X$ *a* 4*-manifold* $M_X$ *with boundary such that*

- $X$ *is local-flatly embedded in* $M_X$*,*
- $M_X$ *collapses onto* $X$*, and*
- $X \cap \partial M_X = \partial X$*.*

*Remark 2.6.* — The polyhedron $X$ is also called a shadow of $M_X$, and the 4-manifold $M_X$ with boundary is often called the 4-dimensional thickening of $X$.

For a shadowed polyhedron $X$, if $\partial M_X$ is diffeomorphic to $k(S^1 \times S^2)$, one can obtain a closed 4-manifold $W$ by gluing $k(S^1 \times B^3)$ to $M_X$ along their boundaries. It is easy to see that $X$ is embedded in the 4-manifold $W$ as a shadow. Due to Laudenbach and Poénaru [11], $W$ is uniquely determined up to diffeomorphism, and hence shadowed polyhedra can be treated as a description of closed 4-manifolds.

Conversely, if a shadow $X$ of a closed 4-manifold $W$ is given, there exists a canonical way to compute a gleam of $X$ such that the obtained shadowed polyhedron describes the 4-manifold $W$ in the above sense. Here we review how to compute the gleam below. Let $W$ be a closed 4-manifold and $X$ a shadow of $W$. Let $R$ be an internal region of $X$, and hence the boundary of $R$ (as a topological space) is contained in $S(X)$. Set $X_S = \mathrm{Nbd}(S(X); X)$ and $\overline{R} = R \setminus \mathrm{Int}\, X_S$. As shown in [12], there exists a 3-manifold $N_S$ with boundary satisfying

- $N_S$ is smoothly embedded in $W$,
- $N_S \cap X = X_S$, and
- $N_S$ collapses onto $X_S$.

Note that $N_S$ is homeomorphic to the disjoint union of some 3-dimensional handlebodies that are possibly non-orientable. Set $I_R = \mathrm{Nbd}(\partial\overline{R}; \partial N_S)$, which can be seen as an interval-bundle over $\partial\overline{R}$. Thus, $I_R$ is the disjoint union of some annuli and Möbius bands. Let $\overline{R}'$ be a small perturbation of $\overline{R}$ such that $\partial\overline{R}' \subset I_R$, and we can assume that $\overline{R}$ and $\overline{R}'$ intersect transversely at a finite number of points. Then the gleam we require is given by

$$\mathfrak{gl}(R) = \#\Big(\mathrm{Int}\, \overline{R} \cap \mathrm{Int}\, \overline{R}'\Big) + \frac{1}{2}\#\Big(\partial\overline{R} \cap \partial\overline{R}'\Big),$$

where the intersections are counted with signs.

## 2.4. Encoding graph

In this subsection, we review an encoding graph that is a graph describing a simple polyhedron without true vertices. Set

$$Y = \{z \in \mathbb{C} \mid \arg z \in \{0,\ 2\pi/3,\ 4\pi/3\},\ |z| \leqslant 1\} \cup \{0\},$$

and let $f_{111}$, $f_{12}$ and $f_3$ be self-homeomorphisms on $Y$ that send $z$ to, respectively, $z$, $\bar{z}$ and $e^{2\pi\sqrt{-1}/3}z$. Then, for $\sigma \in \{111, 12, 3\}$, let $Y_\sigma$ denote the mapping torus of $f_\sigma$. Note that the numbers of boundary components of $f_{111}$, $f_{12}$ and $f_3$ are 3, 2 and 1, respectively. It is easy to see that if a simple polyhedron has a circle component in the singular set, its regular neighborhood is homeomorphic to either one of $Y_{111}$, $Y_{12}$ or $Y_3$.

Let $X$ be a simple polyhedron with no true vertices. Since a connected component of $S(X)$ is homeomorphic to $S^1$, $X$ is decomposed into a finite number of $Y_{111}$, $Y_{12}$, $Y_3$, a 2-disk $D$, a pair of pants $P$ and a Möbius band $Y_2$. Such a decomposition of $X$ induces a graph consisting of vertices as shown in Figure 2.2 corresponding to the pieces in the decomposition or boundary components. An edge is associated to each circle along which $X$ is decomposed. The graph obtained in such a way is called an *encoding graph* of $X$.

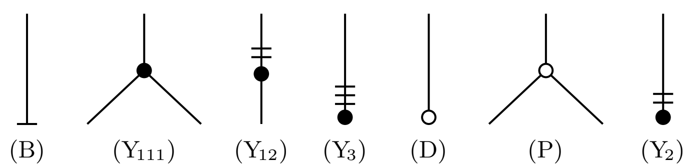



Figure 2.2. Vertices of an encoding graph.

See [9, 13] for more details.

Let $G$ be an encoding graph of a simple polyhedron $X$ with no true vertices. As mentioned in [9, 13], $X$ can not be recovered only by $G$ if $G$ has a cycle since the mapping class group of $S^1$ is $\mathbb{Z}/2\mathbb{Z}$. Actually, a pair of $G$ and a cocycle $\alpha \in H^1(G;\mathbb{Z}/2\mathbb{Z})$ can determine $X$ (here we omit the details of how they do it).

## 2.5. Trisections

Here we review the notion of trisections of closed 4-manifolds.

Definition 2.7. — *Let $W$ be a closed 4-manifold and $g, k_1, k_2, k_3$ non-negative integers with $\max\{k_1, k_2, k_3\} \leqslant g$. A $(g; k_1, k_2, k_3)$-*trisection*, or simply a* trisection*, of $W$ is a data of a decomposition of $W$ into three submanifolds $W_1, W_2$ and $W_3$ such that the following three conditions hold;*

- *for $i \in \{1,2,3\}$, $W_i$ is diffeomorphic to $k_i(S^1 \times B^3)$,*
- *for $i, j \in \{1,2,3\}$ with $i \neq j$, the intersection $H_{ij} = W_i \cap W_j$ is diffeomorphic to a genus $g$ 3-dimensional handlebody $g(S^1 \times D^2)$, and*

- *the intersection* $W_1 \cap W_2 \cap W_3$ *is diffeomorphic to* $\Sigma_g$*.*

*The surface* $W_1 \cap W_2 \cap W_3$ *is called the* central surface *of the trisection. The* genus *of a trisection is the genus of its central surface.*

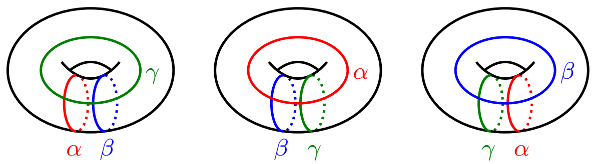



Figure 2.3. A destabilization triple.

This notion was introduced by Gay and Kirby [6], and they showed the following by using a certain generic map from 4-manifolds to the plane $\mathbb{R}^2$.

Theorem 2.8 (Gay and Kirby [6]). — *Any closed* 4*-manifold admits a trisection.*

A *trisection diagram* of the trisection $W_1 \cup W_2 \cup W_3$ is a 4-tuple $(\Sigma_g, \alpha, \beta, \gamma)$ such that $\Sigma_g$ is the central surface and that $\alpha$, $\beta$, and $\gamma$ are cut systems of $H_{31}$, $H_{12}$, and $H_{23}$, respectively. Here a *cut system* of a 3-dimensional handlebody $H$ is a collection of the boundaries of properly embedded disks in $H$ such that they cut $H$ open into a single 3-ball. We note that $\partial W_i$ is decomposed into $H_{ij} \cup H_{ik}$ for $\{i, j, k\} = \{1, 2, 3\}$, which is a genus $g$ Heegaard splitting of $\partial W_i$ since $H_{ij} \cap H_{ik} = \partial H_{ij} = \partial H_{ik}$. Therefore, $(\Sigma_g, \alpha, \beta)$, $(\Sigma_g, \beta, \gamma)$ and $(\Sigma_g, \gamma, \alpha)$ are Heegaard diagrams of $\partial W_1$, $\partial W_2$ and $\partial W_3$, respectively. We also note that a trisection diagram reconstructs the corresponding 4-manifolds and the trisection uniquely up to diffeomorphisms [6].

We here define an operation called a *stabilization* for a trisection diagram $(\Sigma_g, \alpha, \beta, \gamma)$. It is obtained by connected summing $(\Sigma_g, \alpha, \beta, \gamma)$ with either one of the diagrams shown in Figure 2.3.

By this operation, the corresponding 4-manifold does not change up to diffeomorphisms, and the genus of the corresponding trisection increases by 1. We also define an operation called a *destabilization* as the inverse of a stabilization. Note that any two trisection diagrams of the same 4-manifold are related by stabilizations, destabilizations and diffeomorphisms [6, 15].

Let $(\Sigma_g, \alpha, \beta, \gamma)$ be a trisection diagram. We note that each of $\alpha, \beta$ and $\gamma$ consists of $g$ mutually disjoint simple closed curves, so we will write $\alpha = \alpha_1 \sqcup \cdots \sqcup \alpha_g$, $\beta = \beta_1 \sqcup \cdots \sqcup \beta_g$, and $\gamma = \gamma_1 \sqcup \cdots \sqcup \gamma_g$. Suppose that there exist $h, i, j \in \{1, \ldots, g\}$ such that

- exactly two of $\alpha_h, \beta_i$ and $\gamma_j$ are parallel, and
- each of the parallel two curves intersects the other one transversely exactly once.

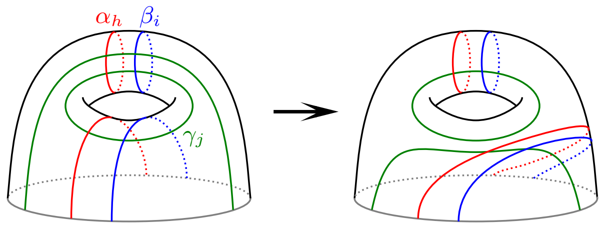



Figure 2.4. Handle sliding over a destabilization triple.

We call such a triple $(\alpha_h, \beta_i, \gamma_j)$ a *destabilization triple*. By handle sliding certain curves over $\alpha_h, \beta_i$ and $\gamma_j$ if necessary, we can assume that $\alpha_h, \beta_i$ and $\gamma_j$ do not intersect $\alpha \cup \beta \cup \gamma \setminus (\alpha_h \cup \beta_i \cup \gamma_j)$. Especially, the union of $\alpha_h, \beta_i$ and $\gamma_j$ is contained in a punctured torus after this modification, which allows the trisection diagram to be destabilized once.

See Figure 2.4 for an example. The left of this figure shows a neighborhood of a destabilization triple $(\alpha_h, \beta_i, \gamma_j)$, where $\alpha_h$ and $\beta_i$ are parallel. All the intersections with this destabilization triple can be removed by handle sliding as shown in the figure.

We close this subsection with the definition of the trisection genus of closed 4-manifolds.

Definition 2.9. — *Let $W$ be a closed 4-manifold. The* trisection genus *$g(W)$ of $W$ is defined as the minimal genus of any trisection of $W$.*

It is obvious that the trisection genus is an invariant of closed 4-manifolds that takes a value in $\mathbb{Z}_{\geqslant 0}$.

## 2.6. Handle decompositions to trisections

Meier and Zupan showed the existence of a bridge trisection for any knotted surface by constructing a trisection from a handle decomposition of the ambient 4-manifold [17]. Here we review their method to construct a trisection.

Let $W$ be a closed 4-manifold, and let us give a handle decomposition of $W$ such that each handle is attached to those with lower indices. Suppose that it has at least one 2-handle and exactly one each of 0-handle and 4-handle. Let $H^i$ denote the union of all the $i$-handles and $L \subset \partial(H^0 \cup H^1)$ the attaching link of the 2-handles. Let $\tau$ be an unknotting tunnel for $L$ in $\partial(H^0 \cup H^1)$, which means that $\partial \operatorname{Nbd}(L \cup \tau; \partial(H^0 \cup H^1))$ gives a Heegaard splitting of $\partial(H^0 \cup H^1)$. Set $\Sigma = \partial \operatorname{Nbd}(L \cup \tau; \partial(H^0 \cup H^1))$. Then $W$ is trisected by

$$W_1 = \left(H^0 \cup H^1\right) \setminus \operatorname{Int} \operatorname{Nbd}(L \cup \tau; W),$$
$$W_2 = H^2 \cup \operatorname{Nbd}(L \cup \tau; W)$$

and

$$W_3 = \left(H^3 \cup H^4\right) \setminus \operatorname{Int} \operatorname{Nbd}(L \cup \tau; W)$$

with central surface $\Sigma$. See [17, Lemma 15] for a proof.

A trisection diagram for the trisection obtained above is given by letting $\alpha$, $\beta$ and $\gamma$ be cut systems of $\partial(H^0 \cup H^1) \setminus \operatorname{Int} \operatorname{Nbd}(L \cup \tau; \partial(H^0 \cup H^1))$, $\operatorname{Nbd}(L \cup \tau; \partial(H^0 \cup H^1))$ and $\operatorname{Nbd}(L \cup \tau; \partial(H^3 \cup H^4))$, respectively. More concretely, we can describe $\beta$ and $\gamma$ as follows. Let $\tau_1, \ldots, \tau_n$ be the connected components of $\tau$, and suppose that $L \cup (\tau_1 \sqcup \cdots \sqcup \tau_{\ell-1})$ is connected, where $\ell$ is the number of components of $L$. We consider the framings of $L$ as a link $L'$ parallel to $L$, and we suppose that $L'$ lies on $\Sigma = \partial \operatorname{Nbd}(L \cup \tau; \partial(H^0 \cup H^1))$. Then, $\beta$ is given as meridians of $L$ and those of $\tau_\ell \sqcup \cdots \sqcup \tau_n$, and $\gamma$ is given as $L'$ and meridians of $\tau_\ell \sqcup \cdots \sqcup \tau_n$.

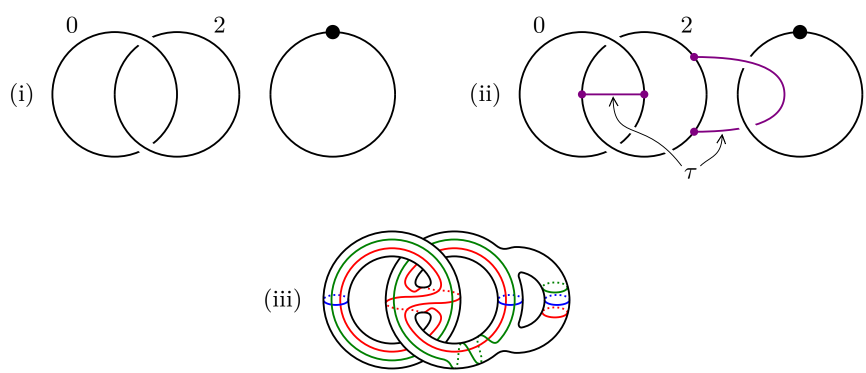



Figure 2.5. (i) A Kirby diagram of $(S^2 \times S^2)\#(S^1 \times S^3)$. (ii) An unknotting tunnel for the attaching link of the 2-handles. (iii) A trisection diagram of $(S^2 \times S^2)\#(S^1 \times S^3)$.

See Figure 2.5 for an example. The Kirby diagram depicted in (i) represents $(S^2 \times S^2)\#(S^1 \times S^3)$, where the attaching link $L$ is given as a Hopf link in $S^1 \times S^2 = \partial(S^1 \times B^3)$. We can find an unknotting tunnel for $L$ such as in (ii). Then the trisection obtained from them in the way explained in this subsection is represented by the diagram shown in (iii). Note that this diagram is of genus 3.

## 3. Cut systems and weighted complexity

In this section, we introduce a new complexity called the *weighted complexity* $c_r$, and by using it, we define the *weighted shadow-complexity* $\mathrm{sc}_r$ of closed 4-manifolds. After the definitions, we discuss some properties of $\mathrm{sc}_r$, especially, relationships with the shadow-complexity and the special shadow-complexity.

Let $X$ be a simple polyhedron with $S(X) \neq \emptyset$. We define a *cut system* for $X$ as a collection $\Gamma$ of mutually disjoint arcs embedded in $X$ such that

- each endpoint of the arcs lies in a triple line or $\partial X$,
- the interiors of the arcs are contained in $X \setminus (S(X) \cup \partial X)$,
- each component of $\partial X$ intersects exactly one arc, and
- each region with $\Gamma$ removed is simply connected.

Therefore, $\Gamma$ can be understood as a collection of cocores of 1-handles of some handle decomposition of the regions. Note that $S(X) \cup \Gamma \cup \partial X$ is connected even if $S(X)$ is not connected. It is easy to see that the number of arcs of $\Gamma$ lying in a region $R$ is exactly $1 - \chi(R)$.

Recall that the *complexity* of a simple polyhedron $X$ is defined as the number $c(X)$ of true vertices of $X$, which of course depends only on the shape of the singular set. We here introduce a new complexity to take into consideration the "non-trivialities" of regions.

Definition 3.1. — *Fix a real number $r \geqslant 0$. The* $r$-weighted complexity $c_r(X)$ *of a simple polyhedron $X$ is defined as*

$$c_r(X) = c(X) + \sum_{R:\text{region}} r(1 - \chi(R))$$

*if $X$ is not a closed surface, and set $c_r(X) = 0$ if $X$ is homeomorphic to $S^2$. The* $r$-weighted shadow-complexity $\mathrm{sc}_r(W)$ *of a 4-manifold $W$ is defined as the minimum of the $r$-weighted complexities over all shadows of $W$.*

We will show in Lemma 4.5 that any closed surface except for $S^2$ can not be a shadow of any closed 4-manifold, which is the reason why we do not define $c_r$ for closed surfaces except for $S^2$.

Note that $c_0(X) = c(X)$ and $c_r(X) \leqslant c_{r'}(X)$ if $r < r'$. We show important relationships between the weighted shadow-complexity, the shadow-complexity and the special shadow-complexity.

Proposition 3.2. — *Let $W$ be a closed 4-manifold and $r, r' \in \mathbb{R}$.*

(1) *If $0 < r < r'$, then the following hold:*

$$\mathrm{sc}(W) \leqslant \mathrm{sc}_r(W) \leqslant \mathrm{sc}_{r'}(W) \leqslant \mathrm{sc}^{\mathrm{sp}}(W).$$

(2) $\mathrm{sc}(W) = \mathrm{sc}_0(W)$*.*

(3) $\mathrm{sc}_r(W) = \mathrm{sc}^{\mathrm{sp}}(W)$ *if* $r \geqslant 2$*.*

*Proof.*

(1). — Obviously, $c(X) \leqslant c_r(X)$ for a simple polyhedron $X$, and hence the first inequality $\mathrm{sc}(W) \leqslant \mathrm{sc}_r(W)$ holds. If a simple polyhedron $X$ is special, then $c(X) = c_r(X)$. Therefore, $\mathrm{sc}_r(W) \leqslant \mathrm{sc}^{\mathrm{sp}}(W)$ holds.

(2). — It is obvious from the definition of $r$-weighted complexity.

(3). — Let $X$ be a shadow of $W$. It is enough to check that $\mathrm{sc}^{\mathrm{sp}}(W) \leqslant c_r(X)$. We first consider the case $S(X) = \emptyset$. We will show in Lemma 4.5 that a closed surface of non-zero genus can not be a shadow of any closed 4-manifold. Thus, $X$ must be homeomorphic to $S^2$ or has non-empty boundary. If $X$ is homeomorphic to $S^2$, then $W$ is diffeomorphic to $S^4$, $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$. Then $\mathrm{sc}^{\mathrm{sp}}(W) = 0 < 2 \leqslant r = c_r(X)$ holds. If $X$ has non-empty boundary, then $W$ is diffeomorphic to $k(S^1 \times S^3)$ since $X$ collapses onto a graph, where $k = 1 - \chi(X) = \frac{c_r(X)}{r}$. If $k = 0$, that is, $W$ is $S^4$, then $\mathrm{sc}^{\mathrm{sp}}(W) \leqslant c_r(X)$ also holds. Suppose $k \geqslant 1$. As shown in [19], the special shadow-complexity of $k(S^1 \times S^3)$ is equal to $k+1$. Thus, we also have $\mathrm{sc}^{\mathrm{sp}}(W) = k+1 \leqslant 2k \leqslant rk = c_r(X)$.

We next consider the case $S(X) \neq \emptyset$. Let $\Gamma$ be a cut system for $X$. Recall that $\sum_{R:\text{region}} r(1-\chi(R))$ is equal to the number of arcs of $\Gamma$. Let $e$ be one of arcs of $\Gamma$. Then $\mathrm{Nbd}(e; X)$ is shown in the leftmost part of Figure 3.1 (i) if both of the endpoints of $e$ lie in $S(X)$, and otherwise $\mathrm{Nbd}(e \cup C; X)$ is shown in the leftmost part of Figure 3.1 (ii), where $C$ is the boundary component of $X$ containing an endpoint of $e$. The move shown in Figure 3.1 (i) is called a $(0 \to 2)$-*move* (cf. [1, 20]), which creates two true vertices and decrease the number of arcs of $\Gamma$ by 1. Figure 3.1 (ii) shows the composition of three moves. The first move (ii-1) is a $(0 \to 1)$-*move* (cf. [1, 20]), and the second move (ii-2) is a $(0 \to 2)$-move. By these two moves, three true vertices and one annular boundary region are created. The move (ii-3) is a collapsing so that the annular boundary region is removed. By this collapsing, one true vertex is removed. The move (ii) that is the composition of (ii-1), (ii-2) and (ii-3) changes the simple polyhedron so that two true vertices are created and decrease the number of arcs of $\Gamma$ by 1. We apply a move (i) or a move (ii) for every arc of $\Gamma$, so that we obtain a special polyhedron $X'$ with $c(X') = c(X) + 2\sum_{R:\text{region}}(1 - \chi(R)) = c_2(X) \leqslant c_r(X)$. Therefore, we have $\mathrm{sc}^{\mathrm{sp}}(W) \leqslant c_r(X)$. □

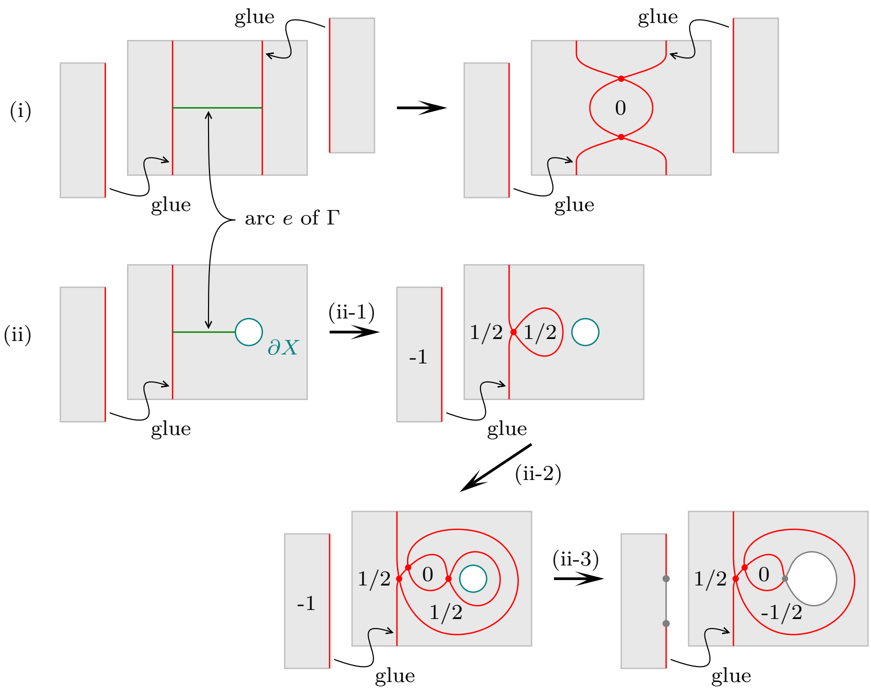



Figure 3.1. Modification of a simple polyhedron into a special polyhedron.

Let $X$ and $X'$ be shadows of closed 4-manifolds $W$ and $W'$, respectively. We choose small disks $D$ and $D'$ in regions of $X$ and $X'$, respectively. Identifying $D$ and $D'$, we obtain a new simple polyhedron $X''$. This polyhedron $X''$ is a shadow of the 4-manifold $W\#W'$. By this operation, the summation of the Euler characteristics of the regions decreases by 2, so we have the following.

Proposition 3.3. — *For any closed 4-manifolds $W$ and $W'$,*

$$\mathrm{sc}_r(W\#W') \leqslant \mathrm{sc}_r(W) + \mathrm{sc}_r(W') + 2r.$$

As shown in Proposition 4.12, the 1/2-weighted shadow-complexities of $2\mathbb{CP}^2$ and $4\mathbb{CP}^2$ are 0 and 1, respectively. These give an example satisfying the equality $\mathrm{sc}_r(W\#W') = \mathrm{sc}_r(W) + \mathrm{sc}_r(W') + 2r$ as $r = 1/2$, $W = W' = 2\mathbb{CP}^2$.

We then discuss the finiteness of the complexities. There exist infinitely many closed 4-manifolds with shadow-complexity 0. For example, the shadow-complexity of $k\mathbb{CP}^2$ is 0 for any $k \in \mathbb{Z}$ (cf. [13] and Proposition 4.12). Thus, the shadow-complexity for closed 4-manifold is not finite-to-one. On the other hand, the special shadow-complexity is finite-to-one [2, 12]. We here show that the weighted shadow-complexity is also finite-to-one.

Proposition 3.4. — *For any positive number $r$ and any non-negative number $a$, there exists a finite number of closed 4-manifolds having $r$-weighted shadow-complexity less than or equal to $a$.*

*Proof.* — Fix $r > 0$ and $a \geqslant 0$. Note that the $r$-weighted complexity $c_r$ takes a value in $\{m+rn \mid m, n \in \mathbb{Z}_{\geqslant 0}\}$. The set $\{m+rn \mid m, n \in \mathbb{Z}_{\geqslant 0}\} \cap [0, a]$ is a finite set, in which we pick arbitrary $a_0$. The number of ways to present $a_0$ in a form $m + rn$ is finite, so fix $m_0, n_0 \in \mathbb{Z}_{\geqslant 0}$ with $a_0 = m_0 + rn_0$. It is easy to check that the number of simple polyhedra with $m_0$ true vertices and $\sum_{R:\text{region}}(1 - \chi(R)) = n_0$ is finite. By Martelli's result [12, Theorem 2.4], the number of closed 4-manifolds admitting a shadow homeomorphic to a fixed simple polyhedron is finite. Therefore, the proposition holds. □

## 4. Kirby diagrams and trisections from shadows

In this section, we explain how one can draw a Kirby diagram of a 4-manifold $W$ from a given shadow of $W$. We refer the reader to [10] for the case of special shadows, and we stress that shadows we will consider can have non-empty boundary and non-disk regions. We also give the proof of Theorem 4.11 at the end of the section.

### 4.1. Shadows to Kirby diagrams

Let $X$ be a shadow of a 4-manifold $W$ with $S(X) \neq \emptyset$ and $\Gamma$ a cut system for $X$. Set $\widetilde{\Gamma} = S(X) \cup \Gamma \cup \partial X$, which will be regarded as a graph naturally. Let $T_0$ be a forest each of whose connected component is a spanning tree of a connected component of $S(X)$ as a subgraph of $\widetilde{\Gamma}$. Then let $T$ be a spanning tree of $\widetilde{\Gamma}$ obtained from $T_0$ by adding some edges of $\widetilde{\Gamma}$.

Set $X_{\widetilde{\Gamma}} = \mathrm{Nbd}(\widetilde{\Gamma}; X)$. The number of connected components of $\partial X_{\widetilde{\Gamma}} \setminus \partial X$ is the same as that of the regions of $X$, and $X$ is obtained from $X_{\widetilde{\Gamma}}$ by capping $\partial X_{\widetilde{\Gamma}} \setminus \partial X$ off by 2-disks. Especially, $X \setminus \widetilde{\Gamma}$ is the disjoint union of some open 2-disks, which gives a cell decomposition of $X$.

We consider an immersion $\varphi : X_{\tilde{\Gamma}} \to S^3$ such that:

- $\varphi|_{\tilde{\Gamma}}$ is an embedding,
- $\varphi(X_{\tilde{\Gamma}}) \subset \mathrm{Nbd}(\varphi(\widetilde{\Gamma}); S^3)$, and
- $\varphi$ is an embedding except on the neighborhood of some triple lines. As shown in Figure 4.1, the image of non-injective points of $\varphi$ form intervals, and a neighborhood of each of them is homeomorphic to the union of

  $$\{(z,t) \in \mathbb{C} \times \mathbb{R} \mid -1 \leqslant z \leqslant 0, -1 \leqslant t \leqslant 1\},$$
  $$\left\{(z,t) \in \mathbb{C} \times \mathbb{R} \,\middle|\, z = 2re^{\frac{\pi t}{3}\sqrt{-1}}, -1 \leqslant t \leqslant 1, 0 \leqslant r \leqslant 1\right\},$$

  and

  $$\left\{(z,t) \in \mathbb{C} \times \mathbb{R} \,\middle|\, z = re^{-\frac{\pi t}{3}\sqrt{-1}}, -1 \leqslant t \leqslant 1, 0 \leqslant r \leqslant 1\right\},$$

  where we identify $\mathbb{C} \times \mathbb{R} = \mathbb{R}^3$.

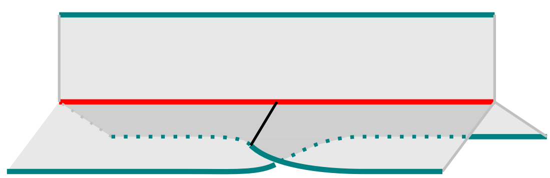


Figure 4.1. The non-injective part of $\varphi$.

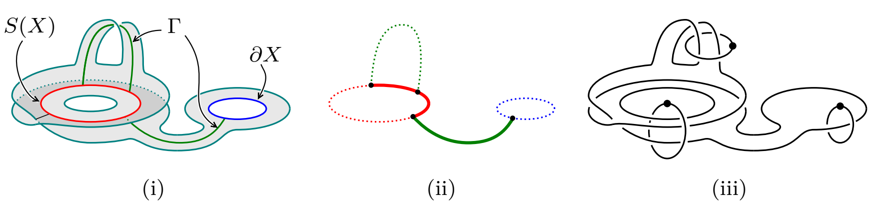
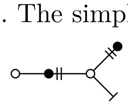



Figure 4.2. An example of how to draw a Kirby diagram. (i) The image of $X_{\tilde{\Gamma}}$ by $\varphi$. (ii) The tree graph $T$. (iii) The Kirby diagram of a 4-dimensional thickening of $X$.

We note that $\varphi|_{\partial X_{\tilde{\Gamma}}}$ is an embedding. See Figure 4.2 (i) for an example of the image of $X_{\tilde{\Gamma}}$ by $\varphi$. The simple polyhedron $X$ we use in this example is encoded by the graph .

We then encircle each arc in $\varphi(\widetilde{\Gamma} \setminus T)$ by a dotted circle so that it does not touch $\varphi(X_\Gamma)$. See Figure 4.2 for an example. If $T$ is chosen as shown in Figure 4.2 (ii), we provide dotted circles as shown in Figure 4.2 (iii).

Let $L_1$ denote the link consisting of those dotted circles, and set $L_2 = \varphi(\partial X_{\tilde{\Gamma}} \setminus \partial X)$. Note that $L_2$ is a link in $S^3 \setminus L_1$ by the assumption of $\varphi$. A Kirby diagram of $W$ that we require consists of the dotted circles $L_1$ and the link $L_2$ equipped with some framings. See Figure 4.2 (iii). Note that the framings of $L_2$ are determined by the gleam of $X$ and $\varphi$, but here we omit the details of those calculation.

*Remark 4.1.* — In the case $S(X) = \emptyset$, $X$ is a (possibly non-orientable) compact surface. Then a 4-dimensional thickening of $X$ is a disk bundle over $X$, whose Kirby diagram is easily drawn (see [7] for instance).

## 4.2. Kirby diagrams to trisections

For convenience of constructing a trisection, we start with modifying the immersion $\varphi : X_{\tilde{\Gamma}} \to S^3$.

Let $v_1, \ldots, v_n$ be the vertices of $\widetilde{\Gamma}$ as a graph. They are also vertices of the tree graph $T$, and let $e_1, \ldots, e_{n-1}$ be the edges of $T$. Let $n'$ be the number of edges of $\widetilde{\Gamma} \setminus T$, which coincides with $c_1(X) + 1$. Let $e_1^*, \ldots, e_{n'}^*$ denote the edges of $\widetilde{\Gamma} \setminus T$. We regard $X_{\tilde{\Gamma}}$ as being decomposed into

$$V_1, \ldots, V_n, \quad E_1, \ldots, E_n, \quad E_1^*, \ldots, E_{n'}^*,$$

where

$$\begin{aligned} V_i &= \mathrm{Nbd}(v_i; X), \\ E_j &= \mathrm{Nbd}(e_j; X) \setminus \mathrm{Int}(V_1 \cup \cdots \cup V_n), \\ E_k^* &= \mathrm{Nbd}(e_k^*; X) \setminus \mathrm{Int}(V_1 \cup \cdots \cup V_n) \end{aligned}$$

for $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n-1\}$ and $k \in \{1, \ldots, n'\}$. Note that $v_i$ is either a true vertex of $X$ or an endpoint of $\Gamma$. If $v_i$ is a true vertex, $V_i$ is as shown in Figure 4.3 (i). If $v_i$ is an endpoint of $\Gamma$ and is on a triple line of $X$, $V_i$ is as shown in Figure 4.3 (ii). If $v_i$ is an endpoint of $\Gamma$ and is on $\partial X$, $V_i$ is as shown in Figure 4.3 (iii). The portion $E_i$ is shown in Figure 4.3 (iv) if $e_i \subset S(X)$, and it is shown in Figure 4.3 (v) if $e_i \subset \Gamma$. An edge $e_k^*$ is contained in either $S(X)$, $\Gamma$ or $\partial X$, and hence $E_k^*$ is as shown in Figure 4.3 (iv), (v) or (vi).

We can embed each $V_i$ and $E_j$ in a 3-ball properly, and we consider the orientations of these 3-balls not to be fixed. By taking the boundary connected sums of them, we can construct the union $V_1 \cup \cdots \cup V_n \cup E_1 \cup \cdots \cup$

$E_{n-1}$ with embedded in a 3-ball properly since $T$ is a tree. The obtained 3-ball will be denoted by $B_0$, in which $\mathrm{Nbd}(T;X)$ is embedded. Let us embed $B_0$ into $S^3$. We then attach $E_1^*,\dots,E_{n'}^*$ to $V_1\cup\cdots\cup V_n\cup E_1\cup\cdots\cup E_{n-1}$ outside $B_0$ so that their neighborhood are trivial 1-handles. Note that $E_1^*,\dots,E_{n'}^*$ may have self-intersections as described in Figure 4.1. It determines the immersion $\varphi : X_{\tilde\Gamma}\to S^3$, and we define $L_1$ and $L_2$ as done in the previous subsection. The Kirby diagram $L_1\sqcup L_2$ near a dotted circle is as shown in Figure 4.4 (i), (i)′, (ii) or (iii), where each (i) and (i)′ corresponds to a subarc in $S(X)$, (ii) corresponds to an arc in $\Gamma$ and (iii) corresponds to a boundary component of $X$.

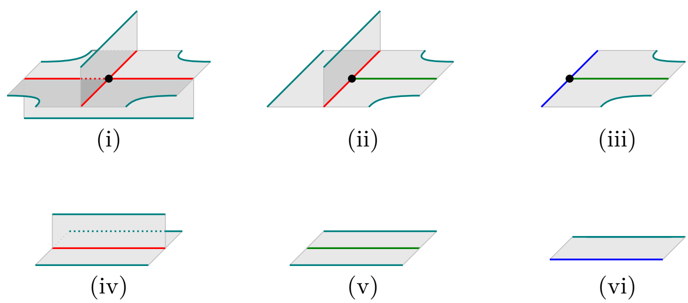



Figure 4.3. The portions $V_i$ and $E_j$.

Let $N$ be the 3-manifold obtained from $S^3$ by 0-surgery along $L_1$, and we now regard $L_2$ as a link in $N$. Note that $N$ is homeomorphic to $h(S^1\times S^2)$, where $h$ is equal to the 1-weighted complexity $c_1(X)$. We then attach some certain arcs to $L_2$ in $N\setminus B_0$ near $E_1^*,\dots,E_{n'}^*$ as shown in the left parts of Figures 4.4 (i), (i)′, (ii) and (iii). Let $\tau$ denote the collection of those arcs. Two arcs are attached for each $e_k^*$ in $S(X)$, and one arc is attached for each $e_k^*$ in $\Gamma$ and in $\partial X$. Set $\Sigma=\partial\,\mathrm{Nbd}(L_2\cup\tau;N)$.

Lemma 4.2. — *The genus of the surface $\Sigma$ is $3+2c_{1/2}(X)$.*

*Proof.* — One can see that

$$\begin{aligned}\chi(\widetilde\Gamma)&=\chi(S(X_\Gamma))+\chi(\Gamma)+\chi(\partial X)-2\chi(\partial\Gamma)\\&=-c(X_\Gamma)-\chi(\Gamma)\\&=-c(X)-\sum_{R:\text{region}}(1-\chi(R)).\end{aligned}$$

Therefore, the number $n'$ of the connected components of $\widetilde\Gamma\setminus T$ is equal to $c(X)+\sum_{R:\text{region}}(1-\chi(R))+1$. Two arcs in $\tau$ are attached for each edge $e_k^*$

in $S(X)$, and the number of such edges is $c(X)+1$. Hence, the number of arcs in $\tau$ is

$$\left(c(X)+\sum_{R:\text{region}}(1-\chi(R))+1\right)+\big(c(X)+1\big)=2+2c_{1/2}(X),$$

and the genus of $\Sigma$ is equal to this number plus 1, namely $3+2c_{1/2}(X)$. $\square$

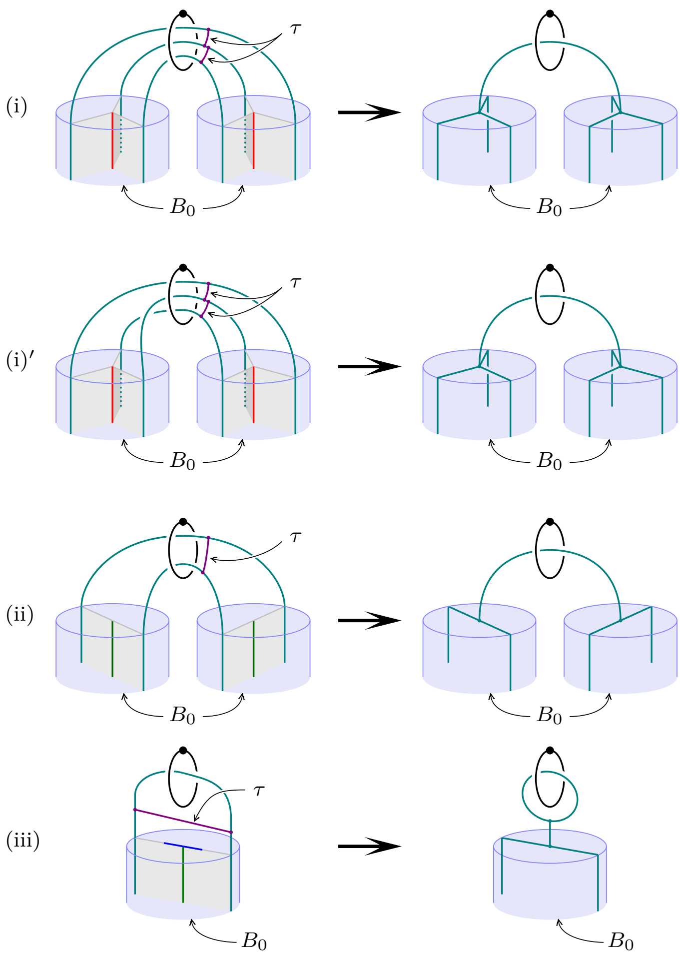



Figure 4.4. The Kirby diagram outside $B_0$.

Lemma 4.3. — *The surface $\Sigma$ is a Heegaard surface of $N$.*

*Proof.* — Outside the 3-ball $B_0$, the spacial graph $L_1 \sqcup (L_2 \cup \tau)$ can be homotoped as shown in the right parts of Figures 4.4 (i), (i)′, (ii) and (iii). By our construction, $L_1 \sqcup (L_2 \cup \tau)$ does not lie in $\mathrm{Int}\, B_0$. Hence we can assume that $\mathrm{Nbd}(L_2 \cup \tau; S^3)$ is a trivial handlebody-knot in $S^3$ after sliding $\tau$ over the 0-framed meridians $L_1$ if necessary. The dotted circles $L_1$ are meridians of $L_2 \cup \tau$. It implies that $\Sigma$ is a Heegaard surface of $N$. ☐

By Lemmata 4.2, 4.3 and the construction in Subsection 2.6, we have the following.

Proposition 4.4. — *For any closed 4-manifold $W$,*

$$g(W) \leqslant 3 + 2\,\mathrm{sc}_{1/2}(W).$$

This result will be strengthened in the next subsection.

## 4.3. Proof of Theorem 4.11

We first prove two lemmata regarding conditions for simple polyhedra to be shadows of closed 4-manifolds.

Lemma 4.5. — *A closed surface $X$ of non-zero genus is not a shadow of any closed 4-manifold.*

*Proof.* — The boundary of a 4-dimensional thickening of a closed surface $X$ is an $S^1$-bundle over $X$. Such a 3-manifold is not homeomorphic to $k(S^1 \times S^2)$ for any $k \in \mathbb{Z}$ unless the base space $X$ is the 2-sphere. ☐

Lemma 4.6. — *A closed simple polyhedron $X$ having a single region with $S(X) \neq \emptyset$ is not a shadow of any closed 4-manifold.*

*Proof.* — Let $R$ be the unique region of $X$, and set

$$\overline{R} = R \setminus \mathrm{Int}\,\mathrm{Nbd}(S(X); X).$$

Suppose that $\overline{R}$ is homeomorphic to $\Sigma_{g,b}$. Note that $b > 0$ by $S(X) \neq \emptyset$. Let $M$ be any 4-dimensional thickening of $X$ and $\pi : M \to X$ the projection. We suppose that $\partial M$ is homeomorphic to $k(S^1 \times S^2)$ for some $k \in \mathbb{Z}_{\geqslant 0}$ to lead a contradiction.

If $g = 0$ and $b = 1$, $\partial M$ is not homeomorphic to $k(S^1 \times S^2)$ for any $k \in \mathbb{Z}_{\geqslant 0}$ by [2, Collorary 3.17].

Suppose that $g \neq 0$ or that $g = 0$ and $b \geqslant 2$. Let $S_1, \dots, S_m$ be the connected components of $S(X)$, and set $N_i = \pi^{-1}(\mathrm{Nbd}(S_i; X))$ for $i \in \{1, \dots, m\}$. Set $N_0 = \partial M \setminus \mathrm{Int}(N_1 \cup \dots \cup N_m)$, which is homeomorphic to

$\overline{R} \times S^1$. Therefore, the 3-manifold $\partial M$ is decomposed into $N_0, N_1, \ldots, N_m$ along certain embedded tori. For each $i \in \{1, \ldots, m\}$, $N_i$ is homeomorphic to $\Sigma_{0,3} \times S^1$, $(\Sigma_{0,2}, (2,1))$, or $(\Sigma_{0,1}, (3,1), (3,-1))$ if $S_i$ contains no true vertices, and otherwise $N_i$ admits a complete hyperbolic structure with finite volume [5]. Hence all $N_0, N_1, \ldots, N_n$ are irreducible 3-manifolds, and the cutting tori are incompressible. It follows that $\partial M$ is also irreducible. Hence $k = 0$, that is, $\partial M$ is $S^3$. On the other hand, the decomposition $N_0 \cup N_1 \cup \cdots \cup N_m$ is the canonical one by the irreducibility, which contradicts the topology of $S^3$. □

We next prove Proposition 4.7, 4.9 and 4.10, which allows us to show Theorem 4.11.

Proposition 4.7. — *Let $X$ be a shadow of a closed 4-manifold $W$. If $X$ is the 2-sphere or is a surface with boundary, then $g(W) \leqslant 2 + 2c_{1/2}(X)$.*

*Proof.* — If $X$ is the 2-sphere, then $W$ is diffeomorphic to $S^4$, $\mathbb{CP}^2$ or $\overline{\mathbb{CP}}^2$. In either case, $g(W) \leqslant 1$, and the lemma holds.

If $X$ is a surface with boundary, $W$ is diffeomorphic to $k(S^1 \times S^3)$, where $k = 2c_{1/2}(X)$. It is easy to see that $g(k(S^1 \times S^3)) = k$, and hence the lemma holds. □

We need the following lemma for the proof of Proposition 4.9.

Lemma 4.8. — *Let $\alpha = \alpha_1 \sqcup \cdots \sqcup \alpha_g$ be a cut system of a 3-dimensional handlebody $H \cong g(S^1 \times B^2)$ and $\alpha_0, \alpha_0'$ simple closed curves in $\partial H \setminus \alpha$. Suppose there exist $i \in \{1, \ldots, g\}$ and orientations of $\alpha_0, \alpha_0'$ and $\alpha_i$ such that $[\alpha_0'] - [\alpha_0] = [\alpha_i]$ in $H_1(\partial H)$. Let $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ be the collections of curves obtained from $\alpha$ by replacing $\alpha_i$ with $\alpha_0$ and $\alpha_0'$, respectively. Then either one of $\widetilde{\alpha}$ and $\widetilde{\alpha}'$ is a cut system of $H$. Moreover, if there exists a simple closed curve $\gamma \subset \partial H$ such that $\gamma$ intersects each $\alpha_0$ and $\alpha_i$ transversely once and $\gamma \cap \alpha_j = \emptyset$ for any $j \in \{1, \ldots, g\} \setminus \{i\}$, then $\widetilde{\alpha}$ is a cut system of $H$.*

*Proof.* — We can assume that $i = 1$ without loss of generality. Let $D_1 \ldots, D_g$ be mutually disjoint disks embedded in $H$ properly such that $\partial D_j = \alpha_j$ for $j \in \{1, \ldots, g\}$. Set $V = H \setminus \bigcup_{j=2}^{g} \operatorname{Int} \operatorname{Nbd}(D_j; H)$. It is homeomorphic to a solid torus, and $\alpha_0, \alpha_0'$ and $\alpha_1$ are mutually disjoint simple closed curves in $\partial V$. Since $[\alpha_0'] - [\alpha_0] = [\alpha_1]$ in $H_1(\partial H)$, either one of $\alpha_0'$ or $\alpha_0$ is isotopic to $\alpha_1$ in $\partial V$. Assume $\alpha_0$ is isotopic to $\alpha_1$. Then, there exists a properly embedded disk $D_0$ in $H$ such that

- $\partial D_0 = \alpha_0$
- it does not intersect all $D_1 \ldots, D_g$, and
- $D_0$ is isotopic to $D_1$ in $V$.

It follows that $\alpha_0 \sqcup \alpha_2 \sqcup \cdots \sqcup \alpha_g$ is also a cut system of $H$.

Then we suppose that a simple closed curve $\gamma$ as in the statement of the lemma exists. Since $\gamma$ does not intersect $\alpha_j$ for $j \in \{2, \ldots, g\}$, it is also a simple closed curve in $\partial V$, especially a longitude of $V$. By the assumption that $\alpha_0$ intersect $\gamma$ transversely once and does not intersect $\alpha_1$, the curves $\alpha_0$ and $\alpha_1$ are parallel in $\partial V$. Thus, the lemma is proved. $\square$

PROPOSITION 4.9. — *Let $X$ be a shadow of a closed 4-manifold $W$. If $S(X) \neq \emptyset$ and $\partial X = \emptyset$, then $g(W) \leqslant 2 + 2c_{1/2}(X)$.*

*Proof.* — By Lemma 4.6, $X$ has at least two regions, and hence $X$ has a triple line $\ell_0$ such that at least one of three regions adjacent to $\ell_0$ differs from the others. Then we choose a spanning tree $T$ of $\widetilde{\Gamma}$ and an immersion $\varphi : X_{\tilde{\Gamma}} \to S^3$ as considered in Subsection 4.2, and we can assume that $\ell_0 \setminus T \neq \emptyset$ since $S(X)$ is 4-valent. Then we draw a Kirby diagram $L_1 \sqcup L_2$ of $W$ as done in Subsection 4.2. Note that, for such a Kirby diagram, we already have constructed a trisection of $W$ of genus $3 + 2c_{1/2}(X)$, so it suffices to show that the genus of this trisection can always decrease by 1.

The part of the Kirby diagram $L_1 \sqcup L_2$ corresponding to the arc $\ell_0 \setminus T$ is shown in the left of Figure 4.5 (i) (cf. Figure 4.4 (i) and (i)$'$), where $K_1, K_2$ and $K_3$ are the attaching circles of 2-handles corresponding to the regions adjacent to $\ell_0$. By the construction of a trisection in Subsection 2.6, we obtain a part of trisection diagram as shown in the right of Figure 4.5 (i), where we draw some simple closed curves $\delta_1, \ldots, \delta_{10}$. Note that $\delta_8, \delta_9$ and $\delta_{10}$ are only partially depicted in the figure. By the assumption of $\ell_0$, we can assume either one of the following;

(i) $K_1, K_2$ and $K_3$ are mutually distinct, or
(ii) $K_1$ differs from $K_2 = K_3$.

We first suppose (i). As mentioned in Subsection 2.6, the curves $\alpha = \alpha_1 \sqcup \cdots \sqcup \alpha_g$, $\beta = \beta_1 \sqcup \cdots \sqcup \beta_g$ and $\gamma = \gamma_1 \sqcup \cdots \sqcup \gamma_g$ of a trisection diagram $(\Sigma_g, \alpha, \beta, \gamma)$ of $W$ can be chosen so that

- $\alpha_1 = \delta_1$,
- $\beta_1 = \delta_3,\ \beta_2 = \delta_4$ and $\beta_3 = \delta_5$, and
- $\gamma_1 = \delta_8,\ \gamma_2 = \delta_9$ and $\gamma_3 = \delta_{10}$.

Note that $\gamma_1, \gamma_2$ and $\gamma_3$ come from $K_1, K_2$ and $K_3$, respectively. Let $\beta_1'$ be $\delta_1$, which is obtained from $\beta_1$ by handle sliding over $\beta_2$ and then over $\beta_3$. Then $(\Sigma_g, \alpha, \beta', \gamma)$ is also a trisection diagram of $W$, where $\beta' = \beta_1' \sqcup \beta_2 \sqcup \cdots \sqcup \beta_g$. Since the triple $(\alpha_1, \beta_1', \gamma_1)$ forms a destabilization triple, we obtain $g(W) \leqslant 2 + 2c_{1/2}(X)$ by a destabilization.

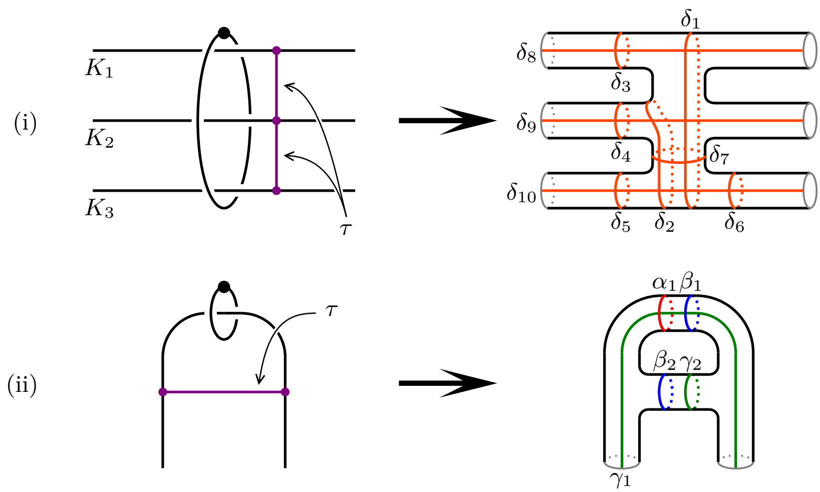



Figure 4.5. Parts of the Kirby diagram $L_1 \sqcup L_2$ and the corresponding parts of the central surface.

We next suppose (ii). As mentioned in Subsection 2.6, the curves $\alpha = \alpha_1 \sqcup \cdots \sqcup \alpha_g$, $\beta = \beta_1 \sqcup \cdots \sqcup \beta_g$ and $\gamma = \gamma_1 \sqcup \cdots \sqcup \gamma_g$ of a trisection diagram $(\Sigma_g, \alpha, \beta, \gamma)$ of $W$ can be chosen so that

- $\alpha_1 = \delta_1$,
- $\beta_1 = \delta_3$, $\beta_2 = \delta_4$ and $\beta_3 = \delta_7$, and
- $\gamma_1 = \delta_8$, $\gamma_2 = \delta_9 (= \delta_{10})$ and $\gamma_3 = \delta_7$.

Note that $\gamma_1$ and $\gamma_2$ come from $K_1$ and $K_2 (= K_3)$, respectively. By Lemma 4.8, $\beta_3$ can be replaced with another curve $\beta_3'$, where $\beta_3'$ is either $\delta_5$ or $\delta_6$. Suppose that $\beta_3' = \delta_5$, that is, three curves of $\beta$ can be chosen as $\delta_3, \delta_4$ and $\delta_5$. Then we can find a destabilization triple in the same way as in (i), and we obtain $g(W) \leqslant 2 + 2c_{1/2}(X)$ by a destabilization. Then suppose that $\beta_3' = \delta_6$, and set $\beta' = \beta_1 \sqcup \beta_2 \sqcup \beta_3' \sqcup (\beta_4 \sqcup \cdots \sqcup \beta_g)$. We note that $[\delta_1] - [\delta_2] = [\beta_1]$ in $H_1(\Sigma_g)$ for some orientations. Since $\delta_8$ is a simple closed curve intersecting $\beta'$ exactly once at a point of $\beta_1$, we can replace $\beta_1$ with $\delta_1$, which will be denoted by $\beta_1'$, by Lemma 4.8. Hence, $(\Sigma_g, \alpha, \beta'', \gamma)$ is also a trisection diagram of $W$, where $\beta'' = \beta_1' \sqcup \beta_2 \sqcup \beta_3' \sqcup (\beta_4 \sqcup \cdots \sqcup \beta_g)$. Then the triple $(\alpha_1, \beta_1'', \gamma_1)$ is a destabilization one, and we obtain $g(W) \leqslant 2+2c_{1/2}(X)$ by a destabilization. □

Proposition 4.10. — *Let $X$ be a shadow of a closed 4-manifold $W$. If $S(X) \neq \emptyset$ and $\partial X \neq \emptyset$, then $g(W) \leqslant 2 + 2c_{1/2}(X)$.*

*Proof.* — Let $T$ be a spanning tree of $\Gamma$ and $\varphi$ an immersion $X_{\tilde{\Gamma}} \to S^3$ as considered in Subsection 4.2. Then we draw a Kirby diagram $L_1 \sqcup L_2$ of $W$ as done in Subsection 4.2. For such a Kirby diagram, we have already constructed a trisection of $W$ of genus $3 + 2c_{1/2}(X)$.

Since $\partial X \neq \emptyset$, the Kirby diagram $L_1 \sqcup L_2$ contains a part as shown in the left of Figure 4.5 (ii) (cf. Figure 4.4 (iii)). By the construction of a trisection in Subsection 2.6, we obtain a part of trisection diagram as shown in the right of Figure 4.5 (ii). Moreover, a trisection diagram $(\Sigma_g, \alpha, \beta, \gamma)$ of $W$ can be drawn so that simple closed curves $\alpha_1, \beta_1, \beta_2, \gamma_1$ and $\gamma_2$ of $\alpha = \alpha_1 \sqcup \cdots \sqcup \alpha_g$, $\beta = \beta_1 \sqcup \cdots \sqcup \beta_g$ and $\gamma = \gamma_1 \sqcup \cdots \sqcup \gamma_g$ are as shown in the right of Figure 4.5 (ii). In this diagram, $(\alpha_1, \beta_1, \gamma_1)$ is a destabilization triple, and hence we get $g(W) \leqslant 2 + 2c_{1/2}(X)$. □

We are now ready to prove Theorem 4.11.

Theorem 4.11. — *For any closed 4-manifold $W$ and any real number $r \geqslant 1/2$, $g(W) \leqslant 2 + 2\,\mathrm{sc}_r(W)$.*

*Proof.* — Let $X$ be any shadow of $W$. It is enough to show the inequality $g(W) \leqslant 2 + 2c_{1/2}(X)$ since $\mathrm{sc}_{1/2}(W) \leqslant \mathrm{sc}_r(W)$ by Proposition 3.2. By Lemmata 4.5 and 4.6, at least one of the following holds;

- $X$ is the 2-sphere or a surface with boundary,
- $S(X) \neq \emptyset$ and $\partial X = \emptyset$, or
- $S(X) \neq \emptyset$ and $\partial X \neq \emptyset$.

In either case, we have $g(W) \leqslant 2 + 2\,\mathrm{sc}_{1/2}(X)$ by Propositions 4.7, 4.9 and 4.10. □

## 4.4. Examples

In this section, we will determine the exact values of $\mathrm{sc}_{1/2}$ for infinite families of certain 4-manifolds by using Theorem 4.11.

Now we define a simple polyhedron $X_k$ for $k \in \mathbb{Z}_{\geqslant 1}$. Let $X_1$ be the 2-sphere, which is encoded by a graph shown in Figure 4.6 (i). For $k \geqslant 2$, let $C_1, \ldots, C_{k-1}$ be simple closed curves in $X_1$ such that they split $X_1$ into two disks and $k-2$ annuli. Then $X_k$ is defined as a simple polyhedron obtained from $X_1$ by attaching 2-disks $D_1, \ldots, D_{k-1}$ along their boundaries to $C_1, \ldots, C_{k-1}$, respectively. The polyhedron $X_k$ is shown in Figure 4.7 and encoded in Figure 4.6 (iii). Note that $\operatorname{rank} H_2(X_k) = k$ and $c_{1/2}(X_k) = \max\{0, \frac{k-2}{2}\}$.

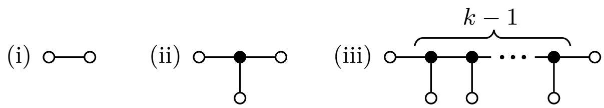



Figure 4.6. Encoding graphs of (i) $X_1$, (ii) $X_2$ and (iii) $X_k$.

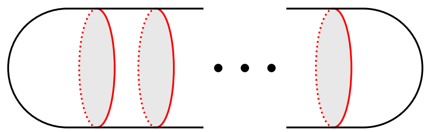


Figure 4.7. The simple polyhedron $X_k$.

Proposition 4.12. — *For any non-negative integers $k_1, k_2$ and $k_3$,*

$$\mathrm{sc}_{1/2}\Big(k_1(S^2\times S^2)\#k_2\mathbb{CP}^2\#k_3\overline{\mathbb{CP}}^2\Big) = \max\left\{0, \frac{2k_1+k_2+k_3-2}{2}\right\}.$$

*Proof.* — Set $k = 2k_1+k_2+k_3$ and $W = k_1(S^2\times S^2)\#k_2\mathbb{CP}^2\#k_3\overline{\mathbb{CP}}^2$. If $k=0$, the equality holds since $W$ $(= S^4)$ admits a shadow homeomorphic to the sphere whose 1/2-weighted complexity is 0.

Suppose $k\geqslant 1$. The simple polyhedron $X_k$ can be embedded in $W$ as a shadow (cf. Remark 4.13 (1)), and hence $\mathrm{sc}_{1/2}(W)\leqslant c_{1/2}(X_k) = \max\{0, \frac{k-2}{2}\}$. On the other hand, since $g(W) = k$, we have $\mathrm{sc}_{1/2}(W)\geqslant \frac{k-2}{2}$ by Theorem 4.11. The value of $\mathrm{sc}_{1/2}$ must not be negative. We obtain $\mathrm{sc}_{1/2}(W) = \max\{0, \frac{k-2}{2}\}$. □

*Remark 4.13.*

(1) Note that $X_1$ is a simple polyhedron forming a region itself. Equipped with a gleam $\pm1$, it corresponds to the $\pm\mathbb{CP}^2$. We also note that $X_2$ is a simple polyhedron consisting of three disk regions. If we assign gleams $+1$ to one of them and $-1$ to the others, the corresponding 4-manifold is $S^2\times S^2$. As mentioned in Section 3, a shadow of the connected sum of two 4-manifolds can be obtained from their shadows by identifying small disks chosen in regions of the shadows. Therefore, the 4-manifold $W = k_1(S^2\times S^2)\#k_2\mathbb{CP}^2\#k_3\overline{\mathbb{CP}}^2$ (for any $k_1, k_2$ and $k_3$) admits a shadow $X_k$, where $k = 2k_1+k_2+k_3$.

(2) By considering the same shadow $X_k$ of $k\mathbb{CP}^2$, for $0\leqslant r<1/2$ we also have $\mathrm{sc}_r(k\mathbb{CP}^2)\leqslant \max\{0,(k-2)r\}$. It follows that $k\mathbb{CP}^2$ violates the inequality $g\leqslant 2+2\,\mathrm{sc}_r$ for $0\leqslant r<1/2$ and $k\geqslant 3$, and the minimum of $r$ satisfying the inequality in Theorem 4.11 is $1/2$.

(3) Every pair $(g, \mathrm{sc}_{1/2}) \in \mathbb{Z}_{\geqslant 0} \times \frac{1}{2}\mathbb{Z}_{\geqslant 0}$ satisfying $g = 2 + 2\mathrm{sc}_{1/2}$ occurs among the examples in Proposition 4.12. Therefore, the inequality $g \leqslant 2 + 2\,\mathrm{sc}_{1/2}$ shown in Theorem 4.11 is the best possible result.

## 5. Closed 4-manifolds with $\mathrm{sc}_{1/2} \leqslant 1/2$

This section is mainly devoted to the proof of Theorem 5.19, which, in conjunction with Theorem 5.1, provides the classification of all closed 4-manifolds with $\mathrm{sc}_{1/2} \leqslant 1/2$. We start with exhibit simple polyhedra with $c_{1/2} \leqslant 1/2$.

### 5.1. Simple polyhedra with $c_{1/2} \leqslant 1/2$

Let $X$ be a simple polyhedron such that it is not homeomorphic to a closed surface or is homeomorphic to $S^2$.

We first consider the case $c_{1/2}(X) = 0$. Then $X$ is homeomorphic to $S^2$, or it is a special polyhedron without true vertices. The closed 4-manifolds in which $S^2$ is embedded as shadows are only $S^4$, $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$. The closed 4-manifolds with $\mathrm{sc}^{\mathrm{sp}} = 0$ are classified by Costantino in [2], and thus we have the following.

Theorem 5.1 (cf. [2, Theorem 1.1]). — *The* $1/2$*-weighted shadow-complexity of a closed* $4$*-manifold* $W$ *is* $0$ *if and only if* $W$ *is diffeomorphic to either one of* $S^4$, $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, $S^2 \times S^2$, $2\mathbb{CP}^2$, $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ *or* $2\overline{\mathbb{CP}}^2$.

We next consider the case $c_{1/2}(X) = 1/2$. Then $X$ has no true vertices, and all regions of $X$ are 2-disks except one region $R_0$. The Euler characteristic $\chi(R_0)$ of $R_0$ is 0, and hence $R_0$ is an annulus or a Möbius band. Therefore, the simple polyhedra with $c_{1/2} = 1/2$ are shown in Figure 5.1.

The simple polyhedra encoded in Figures 5.1 (a1), …, (a14), (a16), (m1), …, (m5) will be denoted by $X_{(\mathrm{a1})}, \ldots, X_{(\mathrm{a14})}, X_{(\mathrm{a16})}, X_{(\mathrm{m1})}, \ldots, X_{(\mathrm{m5})}$, respectively.

Each encoding graph shown in Figures 5.1 (a15) and (a17) has a cycle, it can not determine a simple polyhedron uniquely. Actually, each of them corresponds to exactly two simple polyhedra up to homeomorphisms. Let $X^0_{(\mathrm{a15})}$ and $X^1_{(\mathrm{a15})}$ be simple polyhedra described in Figures 5.2 (i) and (ii), respectively, which are encoded by the graph shown in Figures 5.1 (a15).

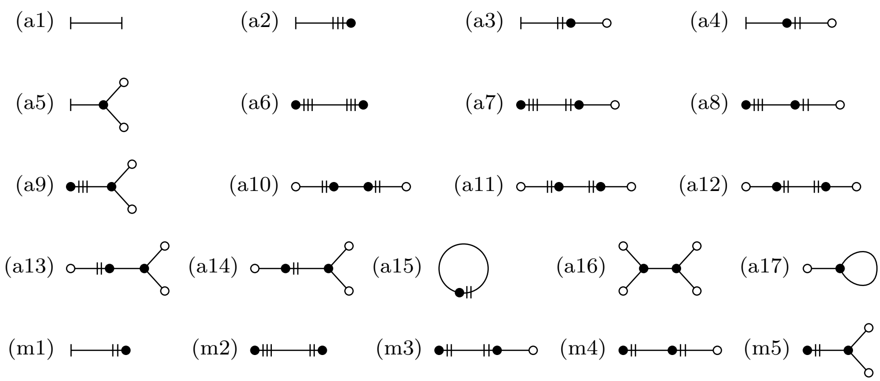



Figure 5.1. Simple polyhedra with $c_{1/2} = 1/2$.

Let $X^0_{(\mathrm{a17})}$ be a simple polyhedron obtained from a torus by gluing a 2-disk along its boundary to a meridian of the torus. We also define $X^1_{(\mathrm{a17})}$ as a simple polyhedron obtained from Klein bottle by gluing a 2-disk along its boundary to a simple closed curve representing $x$ in the fundamental group $\langle x, y \mid xyxy^{-1}\rangle$. Both $X^0_{(\mathrm{a17})}$ and $X^1_{(\mathrm{a17})}$ are simple polyhedra encoded by the graph shown in Figure 5.1 (a17).

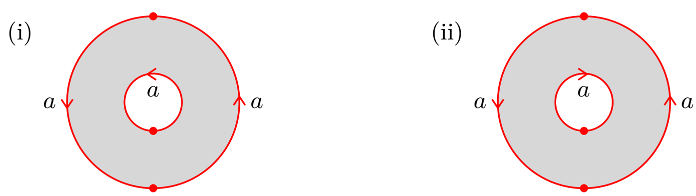



Figure 5.2. Simple polyhedra (i) $X^0_{(\mathrm{a15})}$ and (ii) $X^1_{(\mathrm{a15})}$.

## 5.2. Useful facts

Here we state some useful facts about shadows of closed 4-manifolds and the elementary ideals of finitely generated free abelian groups.

Lemma 5.2 (Costantino [2, Lemma 3.12]). — *Let $X$ be a simple polyhedron. If $H_2(X) = 0$ and $\operatorname{tor} H_1(X) \neq 0$, then $\partial M_{(X,\mathfrak{gl})}$ is not homeomorphic to $k(S^1 \times S^2)$ for any gleam $\mathfrak{gl}$ and integer $k$, especially, $X$ is not a shadow of any closed 4-manifold.*

Martelli classified all the closed 4-manifolds with sc = 0 and finite fundamental group in [13]. The following is a partial result of him.

Theorem 5.3 (Martelli [13, Theorem 1.7]). — *A closed 4-manifold $W$ has shadow-complexity 0 and $|\pi_1(W)| \leqslant 3$ if and only if $W$ is diffeomorphic to*

$$W' \# h\big(S^2 \times S^2\big) \# k\mathbb{CP}^2 \# l\overline{\mathbb{CP}}^2$$

*for some $h, k, l \in \mathbb{Z}$, where $W'$ is $S^4$, $\mathcal{S}_2$, $\mathcal{S}_2'$ or $\mathcal{S}_3$.*

Lemma 5.4. — *For any non-negative integer $k$, the $d^{\mathrm{th}}$ elementary ideal of $\pi_1(k(S^1 \times S^2))$ is isomorphic to $(0)$ if $d < k$, and $(1) = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$ if $k \leqslant d$.*

## 5.3. Non-existence

In the following Lemmata 5.5, 5.6 and 5.7, we will show that the simple polyhedra $X_{(\mathrm{a6})}$, $X_{(\mathrm{a7})}$, $X_{(\mathrm{a8})}$, $X^0_{(\mathrm{a15})}$, $X^1_{(\mathrm{a15})}$, $X_{(\mathrm{m3})}$ and $X_{(\mathrm{m4})}$ are not shadows of closed 4-manifolds.

Lemma 5.5. — *The simple polyhedra $X_{(\mathrm{a6})}, X_{(\mathrm{a7})}$, $X_{(\mathrm{a8})}$, $X^1_{(\mathrm{a15})}$, $X_{(\mathrm{m3})}$ and $X_{(\mathrm{m4})}$ are not shadows of closed 4-manifolds.*

*Proof.* — The second homology groups of simple polyhedra $X_{(\mathrm{a6})}, X_{(\mathrm{a7})}$, $X_{(\mathrm{a8})}$, $X^1_{(\mathrm{a15})}$, $X_{(\mathrm{m3})}$ and $X_{(\mathrm{m4})}$ all vanish, and their first homology groups are $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$, respectively. Hence, the lemma follows from Lemma 5.2. ☐

Lemma 5.6. — *The simple polyhedron $X^0_{(\mathrm{a15})}$ is not a shadow of closed 4-manifolds.*

*Proof.* — Suppose that there exists a closed 4-manifold $W$ admitting a shadow $X^0_{(\mathrm{a15})}$. Note that $\pi_1(W) \cong \pi_1(X^0_{(\mathrm{a15})}) \cong \langle x, y \mid xyx^{-1}y^{-2}\rangle$, which is not a cyclic group. Set $M = \mathrm{Nbd}(X^0_{(\mathrm{a15})}; W)$. Its Kirby diagram is shown in the left part of Figure 5.3 for some $m \in \mathbb{Z}$. Then we have $H_1(\partial M) = \mathbb{Z}$, and hence $\partial M$ must be $S^1 \times S^2$. Therefore, $W$ admits a handle decomposition consisting of one 0-handle, two 1-handles, one 2-handle, one 3-handle and one 4-handle. Considering the dual decomposition, we see that $\pi_1(W)$ is generated by one element, which is a contradiction. ☐

Lemma 5.7. — *The simple polyhedron $X_{(\mathrm{m2})}$ is not a shadow of closed 4-manifolds.*

*Proof.* — Suppose that there exists a closed 4-manifold $W$ admitting a shadow $X_{(\mathrm{m2})}$. Note that $\pi_1(W) \cong \pi_1(X_{(\mathrm{m2})}) \cong \langle x, y \mid x^2y^3\rangle$, which is not cyclic. Set $M = \mathrm{Nbd}(X_{(\mathrm{m2})}; W)$. Its Kirby diagram is depicted in Figure 5.4. Then the lemma can be proved in much the same way as Lemma 5.6. ☐

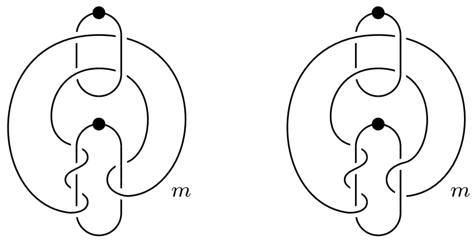



Figure 5.3. The left and right diagrams are Kirby diagrams of 4-dimensional thickenings of the simple polyhedra $X^0_{(\mathrm{a15})}$ and $X^1_{(\mathrm{a15})}$, respectively.

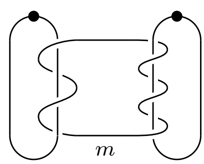



Figure 5.4. A Kirby diagram of a 4-dimensional thickening of the simple polyhedron $X_{(\mathrm{m2})}$.

## 5.4. Classification

Lemma 5.8. — *The simple polyhedra $X_{(\mathrm{a1})}$ and $X_{(\mathrm{a2})}$ are shadows only of $S^1 \times S^3$.*

*Proof.* — The simple polyhedra $X_{(\mathrm{a1})}$ and $X_{(\mathrm{a2})}$ have unique 4-dimensional thickenings, which are diffeomorphic to $S^1 \times B^3$. Hence, they are only shadows of $S^1 \times S^3$. □

Lemma 5.9. — *If a closed 4-manifold $W$ admits a shadow homeomorphic to $X_{(\mathrm{a3})}, X_{(\mathrm{a4})}$ or $X_{(\mathrm{a5})}$, then $\mathrm{sc}_{1/2}(W) = 0$.*

*Proof.* — The simple polyhedra $X_{(\mathrm{a3})}, X_{(\mathrm{a4})}$ and $X_{(\mathrm{a5})}$, respectively, collapses onto $S^2$, $\mathbb{RP}^2$ and $S^2$, whose 1/2-weighted complexities are 0. □

Lemma 5.10. — *The simple polyhedron $X_{(\mathrm{a9})}$ is a shadow only of $\mathcal{S}_3$.*

*Proof.* — We have $\pi_1(X_{(\mathrm{a9})}) \cong \mathbb{Z}/3\mathbb{Z}$, $b_2(X_{(\mathrm{a9})}) = 1$ and $c(X_{(\mathrm{a9})}) = 0$. By Theorem 5.3, if $X_{(\mathrm{a9})}$ is a shadow of a closed 4-manifold, it is nothing but $\mathcal{S}_3$. Actually, a gleam on $X_{(\mathrm{a9})}$ defined by $\mathfrak{gl}(R_1) = 1, \mathfrak{gl}(R_2) = -1$ and $\mathfrak{gl}(R_3) = 1$ provides $\mathcal{S}_3$, where $R_1$ and $R_2$ are two disk regions of $X_{(\mathrm{a9})}$ and $R_3$ is a single annular region of $X_{(\mathrm{a9})}$. □

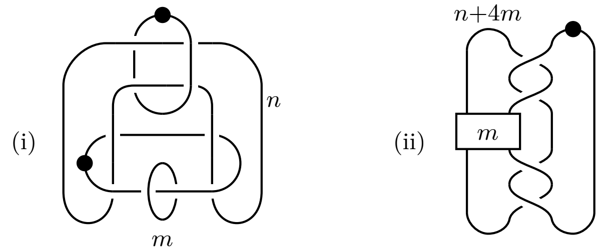



Figure 5.5. Kirby diagrams of a 4-dimensional thickening of the simple polyhedron $X^0_{(\mathrm{a17})}$.

Lemma 5.11. — *If the simple polyhedron $X_{(\mathrm{a10})}$ is a shadow of a closed 4-manifold $W$, then $W$ is $\mathcal{S}_2$ or $\mathcal{S}'_2$.*

*Proof.* — It follows from $\pi_1(X_{(\mathrm{a10})}) \cong \mathbb{Z}/2\mathbb{Z}$, $b_2(X_{(\mathrm{a10})}) = 1$, $c(X_{(\mathrm{a10})}) = 0$ and Theorem 5.3. ☐

*Remark 5.12.* — The 1/2-weighted shadow-complexities of $\mathcal{S}_2$ and $\mathcal{S}'_2$ are actually 1/2 as shown in Lemma 5.18. We have not proven in the proof of Lemma 5.11 that $\mathcal{S}_2$ or $\mathcal{S}'_2$ admits a shadow $X_{(\mathrm{a10})}$, so we do not know at this moment if the 1/2-weighted shadow-complexities of them are exactly 1/2 or not.

Lemma 5.13. — *If a closed 4-manifold $W$ admits a shadow $X$ homeomorphic to $X_{(\mathrm{a11})}$, $X_{(\mathrm{a12})}$, $X_{(\mathrm{a13})}$ or $X_{(\mathrm{a14})}$, then $\mathrm{sc}_{1/2}(W) = 0$.*

*Proof.* — In each case, we have $\pi_1(W) \cong \pi_1(X) \cong \{1\}$, $b_2(W) \leqslant b_2(X) \leqslant 2$ and $\mathrm{sc}(W) = c(X) = 0$. Therefore, $\mathrm{sc}_{1/2}(W) = 0$ by Theorem 5.3. ☐

Lemma 5.14. — *The simple polyhedron $X_{(\mathrm{a16})}$ is a shadow only of $S^2 \times S^2$ and the connected sums of at most 3 copies in $\{S^4, \mathbb{CP}^2, \overline{\mathbb{CP}}^2\}$. Especially, closed 4-manifolds with $\mathrm{sc}_{1/2} = 1/2$ admitting shadows homeomorphic to $X_{(\mathrm{a16})}$ are only $3\mathbb{CP}^2$, $2\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$ and $3\overline{\mathbb{CP}}^2$.*

*Proof.* — Note that $X_{(\mathrm{a16})}$ is homeomorphic to $X_3$ that is the simple polyhedron constructed in Subsection 4.4. By Theorem 5.3, the lemma follows. ☐

Lemma 5.15. — *The simple polyhedron $X^0_{(\mathrm{a17})}$ is a shadow only of $S^1 \times S^3$, $\mathbb{CP}^2 \# (S^1 \times S^3)$ and $\overline{\mathbb{CP}}^2 \# (S^1 \times S^3)$.*

*Proof.* — Let $M_0$ be the 4-dimensional thickening of $X^0_{(\mathrm{a17})}$ equipped with arbitrary gleam. A Kirby diagram of $M_0$ is shown in Figure 5.5 (i), where $m, n$ are some integers. The attaching circle with framing $m$ is canceled with a dotted circle, so that we get a Kirby diagram shown in Figure 5.5 (ii). By replacing the dotted circle in the figure with a 0-framed knot, we get a surgery diagram of the boundary $\partial M_0$. By Wu's result [22, Theorem 5.1], $\partial M_0$ is not homeomorphic to $k(S^1 \times S^2)$ for any $k$ unless $m = 0$. Suppose $m = 0$. The 4-manifold $M_0$ admits a Kirby diagram given by a 2-component unlink consisting of one dotted circle and one unknot with framing coefficient $n$. Therefore, $X^0_{(\mathrm{a17})}$ can be embedded in $S^1 \times S^3$, $\mathbb{CP}^2\#(S^1 \times S^3)$ and $\overline{\mathbb{CP}}^2\#(S^1 \times S^3)$ as shadows. □

Lemma 5.16. — *The simple polyhedron* $X^1_{(\mathrm{a17})}$ *is a shadow only of* $S^1 \times S^3$, $\mathbb{CP}^2\#(S^1 \times S^3)$ *and* $\overline{\mathbb{CP}}^2\#(S^1 \times S^3)$.

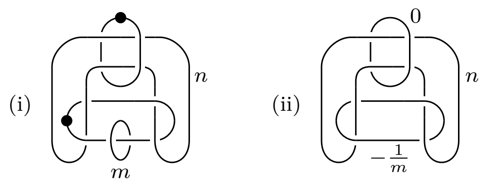



Figure 5.6. (i) A Kirby diagram of a 4-dimensional thickening of the simple polyhedron $X^1_{(\mathrm{a17})}$. (ii) A surgery diagram of the boundary of the 4-dimensional thickening of the simple polyhedron $X^1_{(\mathrm{a17})}$.

*Proof.* — Let $M_1$ be the 4-dimensional thickening of $X^1_{(\mathrm{a17})}$ equipped with arbitrary gleam, which is represented by a Kirby diagram shown in Figure 5.6 (i) for some $m, n \in \mathbb{Z}$. By replacing the dotted circles with 0-framed unknots, we obtain a surgery diagram of the 3-manifold $\partial M_1$. Performing a slum-dunk move once, we obtain the diagram of $\partial M_1$ shown in Figure 5.6 (ii). By an explicit computation from this diagram, we have

$$\pi_1(\partial M_1) \cong \left\langle x, y, z \,\middle|\, [x, z], \left[z, y^{-1}xy\right], x^n zyzy^{-1}, z^{-1}\left(xy^{-1}xy\right)^m \right\rangle.$$

Note that

$$H_1(\partial M_1) \cong \begin{cases} \mathbb{Z}\langle y\rangle & (4m+n=\pm 1) \\ \mathbb{Z}\langle y\rangle \oplus \mathbb{Z}\langle x\rangle & (4m+n=0) \\ \mathbb{Z}\langle y\rangle \oplus (\mathbb{Z}/(4m+n)\mathbb{Z}\langle x\rangle) & \text{(otherwise).} \end{cases}$$

Therefore, in order for $\partial M_1$ to be homeomorphic to $k(S^1 \times S^2)$ for some $k \in \mathbb{Z}_{\geqslant 0}$, it is necessary that $4m + n = \pm 1$ or 0.

Suppose $4m+n=\pm 1$. By explicit calculations from the presentation of $\pi_1(\partial M_1)$, we have

$$E_d(\pi_1(\partial M_1)) \cong \begin{cases} (0) & (d=0) \\ (n+m(1+t)(1+t^{-1})) & (d=1) \\ (1) & (d\geqslant 2). \end{cases}$$

By Lemma 5.4, we need $m=0$ and $n=\pm 1$. Conversely, substituting $m=0$ and $n=\pm 1$ into the diagram shown in Figure 5.6 (i), we obtain a Kirby diagram given by a 2-component unlink consisting of one dotted circle and one unknot with framing $\pm 1$ after easy Kirby calculus. It implies that $X^1_{(\mathrm{a17})}$ can be embedded in $\mathbb{CP}^2\#(S^1\times S^3)$ and $\overline{\mathbb{CP}}^2\#(S^1\times S^3)$ as shadows.

Suppose $4m+n=0$. By explicit calculations from the presentation of $\pi_1(\partial M_1)$, the Alexander matrix is given as

$$\begin{pmatrix} 1-t_2^{2m} & 0 & t_2(1-t_2) \\ t_1^{-1}t_2^{2m}\big(1-s^{2m}\big) & t_1^{-1}(1-t_2)\big(1-t_2^{2m}\big) & 1-t^2 \\ \frac{1-t_2^n}{1-t_2} & t_2^{2m+n}\big(1-t_2^{2m}\big) & t_2^n\big(1+t_1t_2^{2m}\big) \\ t_2^{-2m}\big(1+t_1^{-1}t_2\big)\frac{1-t_2^{2m}}{1-t_2} & t_1^{-1}t_2^{1-2m}(t_2-1)\frac{1-t_2^{2m}}{1-t_2} & -t_2^{-2m} \end{pmatrix}$$

where $t_1$ and $t_2$, respectively, are the images of $y$ and $x$ by the homomorphism $\mathbb{Z}\pi_1(\partial M_1)\to\mathbb{Z}[t_1^{\pm 1},t_2^{\pm 1}]$ induced by the abelianization $\pi_1(\partial M_1)\to H_1(\partial M_1)$. The upper-right $2\times 2$-minor is $t_1^{-1}t_2(1-t_2)^2(1-t_2^{2m})$, and hence $((1-t_2)^2(1-t_2^{2m}))\subset E_1(\pi_1(\partial M_1))$. By Lemma 5.4, $E_1(\pi_1(\partial M_1))$ must be $(0)$, so we need $m=0$. Since $4m+n=0$, we have $n=0$. Conversely, substituting $m=n=0$ into the diagram shown in Figure 5.6 (i), we obtain a diagram given by 2-component unlink consisting of one dotted circle and one unknot with framing 0 after easy Kirby calculus. It implies that $X^1_{(\mathrm{a17})}$ can be embedded in $S^1\times S^3$ as a shadow. □

Lemma 5.17. — *The simple polyhedron $X_{(\mathrm{m1})}$ is a shadow only of $S^1\times S^3$.*

*Proof.* — The simple polyhedron $X_{(\mathrm{m1})}$ has a unique 4-dimensional thickening, which is $S^1\times B^3$. Hence, it is a shadow only of $S^1\times S^3$. □

Lemma 5.18. — *The simple polyhedron $X_{(\mathrm{m5})}$ is shadows only of $\mathcal{S}_2$ and $\mathcal{S}_2'$.*

*Proof.* — We have $\pi_1(X_{(\mathrm{m5})})\cong\mathbb{Z}/2\mathbb{Z}$ and $b_2(X_{(\mathrm{a9})})=1$. By Theorem 5.3, if $X_{(\mathrm{m5})}$ is a shadow of a closed 4-manifold, it is nothing but $\mathcal{S}_2$ or $\mathcal{S}_2'$. Actually, a gleam on $X_{(\mathrm{a9})}$ defined by $\mathfrak{gl}(R_1)=1, \mathfrak{gl}(R_2)=-1$ and $\mathfrak{gl}(R_3)=1$ gives $\mathcal{S}_2$, where $R_1$ and $R_2$ are two disk regions of $X_{(\mathrm{a9})}$ and $R_3$ is the annular region of $X_{(\mathrm{a9})}$. If we equip $X_{(\mathrm{m5})}$ with gleams $\mathfrak{gl}(R_1)=1, \mathfrak{gl}(R_2)=-1$ and $\mathfrak{gl}(R_3)=0$, it yields $\mathcal{S}_2'$. □

Theorem 5.19. — *The* $1/2$*-weighted shadow-complexity of a closed* $4$*-manifold* $W$ *is* $1/2$ *if and only if* $W$ *is diffeomorphic to either one of* $3\mathbb{CP}^2$, $2\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$, $3\overline{\mathbb{CP}}^2$, $S^1\times S^3$, $(S^1\times S^3)\#\mathbb{CP}^2$, $(S^1\times S^3)\#\overline{\mathbb{CP}}^2$, $\mathcal{S}_2$, $\mathcal{S}_2'$ *or* $\mathcal{S}_3$.

*Remark 5.20.* — For $p \geqslant 4$, the $1/2$-weighted shadow-complexities of 4-manifolds $\mathcal{S}_p$ and $\mathcal{S}_p'$ are at least 1 by Theorems 5.1 and 5.19. On the other hand, their trisection genus are exactly 3 [16, 14]. Therefore, the strict inequality $g(W) < 2 + 2\,\mathrm{sc}_{1/2}(W)$ holds for these 4-manifolds $\mathcal{S}_p$ and $\mathcal{S}_p'$.

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