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ALEXIS ROQUEFEUIL

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Quantum K -theory of projective spaces and confluence of q -difference equations ^(*)

ALEXIS ROQUEFEUIL ⁽¹⁾

ABSTRACT. — Givental’s K -theoretical J -function can be used to reconstruct genus zero K -theoretical Gromov–Witten invariants. We view this function as a fundamental solution of a q -difference system. In the case of projective spaces, we show that we can use the confluence of q -difference systems to obtain the cohomological J -function from its K -theoretic analogue. This provides another point of view to one of the statements of Givental–Tonita’s quantum Hirzebruch–Riemann–Roch theorem. Furthermore, we compute connection numbers in the equivariant K -theoretic setting.

RÉSUMÉ. — La fonction J de Givental K -théorique peut être utilisée pour reconstruire les invariants de Gromov–Witten K -théorique de genre 0. Cette fonction peut être vue comme une solution fondamentale d’un système d’équations aux q -différences. Dans le cas des espaces projectifs, nous montrons que la confluence des équations aux q -différences peut être utilisée pour obtenir la fonction J cohomologique à partir de son analogue K -théorique. Ce procédé donne un autre point de vue à un des résultats contenu dans l’énoncé du théorème de Hirzebruch–Riemann–Roch quantique de Givental–Tonita. De plus, nous calculons les matrices de connexion dans le contexte de la K -théorie équivariante.

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⁽¹⁾ Univ Angers, CNRS, LAREMA, SFR MATHSTIC, F-49000 Angers, France —
Current address: Eindhoven, Brabant Septentrional, Pays-Bas —
alexis.roquefeuil@gmail.com

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Article proposé par Lucia Di Vizio.

1. Introduction

1.1. Some context

Gromov–Witten invariants are rational numbers that, in some situations, count the number of curves satisfying some incidence conditions inside a projective algebraic variety. Let X be a smooth projective variety, and fix $g, n \in \mathbb{Z}_{\geq 0}, d \in H_2(X; \mathbb{Z})$. Denote by $\overline{\mathcal{M}}_{g,n}(X, d)$ the moduli space of stable maps [18], and let $[\overline{\mathcal{M}}_{g,n}(X, d)]^{\text{vir}}$ be the virtual fundamental class constructed in [2, Definition 5.2]. We recall that this moduli space comes with n evaluation maps $\text{ev}_i : \overline{\mathcal{M}}_{g,n}(X, d) \rightarrow X$ and with n (orbifold) vector bundles \mathcal{L}_i called the cotangent line bundles. We also introduce the cohomological classes $\psi_i := c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}(X, d); \mathbb{Q})$.

DEFINITION ([2, 18]). — *Let $g, n \in \mathbb{Z}_{\geq 0}, d \in H_2(X; \mathbb{Z})$. Let $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$ be some integers, and let $\alpha_1, \dots, \alpha_n \in H^*(X; \mathbb{Q})$. The associated Gromov–Witten invariant is defined by the intersection product*

$$\left\langle \psi_1^{k_1} \alpha_1, \dots, \psi_n^{k_n} \alpha_n \right\rangle_{g,n,d}^{\text{coh}} = \int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{\text{vir}}} \bigcup_i \left(\psi_i^{k_i} \cup \text{ev}_i^*(\alpha_i) \right) \in \mathbb{Q},$$

where $\int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{\text{vir}}}$ denotes the cap product in cohomology with the virtual fundamental class.

More recently, in 2004, Y.-P. Lee defined new invariants by replacing the cohomological constructions in the above definition by their K -theoretic analogues. Denote by $\mathcal{O}_{g,n,d}^{\text{vir}}$ the virtual structure sheaf, constructed in [19, Subsection 2.3].

DEFINITION ([19]). — *Let $g, n \in \mathbb{Z}_{\geq 0}, d \in H_2(X; \mathbb{Z})$. Consider $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$ to be some integers, and let $\phi_1, \dots, \phi_n \in K(X)$. The associated K -theoretic Gromov–Witten invariant is given by the Euler characteristic*

$$\left\langle \mathcal{L}_1^{k_1} \phi_1, \dots, \mathcal{L}_n^{k_n} \phi_n \right\rangle_{g,n,\beta}^{\text{Kth}} = \chi \left(\overline{\mathcal{M}}_{g,n}(X, d); \mathcal{O}_{g,n,d}^{\text{vir}} \bigotimes_{i=1}^n \mathcal{L}_i^{k_i} \text{ev}_i^*(\phi_i) \right) \in \mathbb{Z}.$$

A natural question to ask upon reading these two definitions is to understand how these two invariants are related. An algebraic geometer would rightfully expect them to be related by a Riemann–Roch theorem. Due to the highly sophisticated geometry of the moduli spaces of stable maps, such formula is not easy to obtain. In 2014, A. Givental and V. Tonita [14] found a general result saying that genus zero K -theoretic Gromov–Witten invariants can be expressed with genus zero cohomological Gromov–Witten invariants (this result has been extended to all genera in [12]). However,

this formula is very technical and therefore has not seen many applications. One of its known consequences has been that a key power series expressed with K -theoretic Gromov–Witten invariants, called Givental’s K -theoretic J -function, satisfies a system q -difference equations ([14, Section 9, Theorem], see also [16, Proposition 2.12]), like it had been verified on some examples, e.g. in [13, Theorem 2].

Remark. — Another approach to obtain a comparison between cohomological and K -theoretic Gromov–Witten invariants using derived algebraic geometry has been initiated by A. A. Khan, see [17].

1.2. Goal of the article

The aim of this paper is to propose another point of view to compare K -theoretic Gromov–Witten invariants with their cohomological analogues, using the theory of q -difference equations.

We will focus on the q -difference equations satisfied by Givental’s small K -theoretic J -function of the projective space. In general, these functional equations satisfy a property called *confluence*, according to which we can take some limit $q \rightarrow 1$ of the q -difference to obtain a differential equation. A quick illustration of the confluence of q -difference equations is this identity, in which $k \in \mathbb{Z}$,

$$\lim_{q \rightarrow 1} \frac{q^{Q\partial_Q} - \text{Id}}{q - 1} \cdot Q^k = \lim_{q \rightarrow 1} \frac{q^k - 1}{q - 1} \cdot Q^k = kQ^k = Q\partial_Q \cdot Q^k.$$

Therefore, we will say that the q -difference operator $\frac{q^{Q\partial_Q} - \text{Id}}{q - 1}$ converges formally to the differential operator $Q\partial_Q$. Our goal is to obtain similar limits for the following data:

DEFINITION ([9, 10]). — Consider $X = \mathbb{P}^N$ with its usual toric action of the torus $T^{N+1} = (\mathbb{C}^*)^{N+1}$. Let $P = \mathcal{O}_{\text{eq}}(1) \in K_{T^{N+1}}(\mathbb{P}^N)$ be the anti-tautological equivariant bundle, and denote by $\lambda_0, \dots, \lambda_N$ (resp. $\Lambda_0, \dots, \Lambda_N$) the equivariant parameters in cohomology (resp. K -theory).

- (i) Let $H = c_1(\mathcal{O}_{\text{eq}}(1)) \in H_{T^{N+1}}^2(\mathbb{P}^N; \mathbb{Q})$ be the equivariant hyperplane class. Givental’s small equivariant cohomological J -function of \mathbb{P}^N is given by the expression

$$\begin{aligned} J^{\text{coh,eq}}(z, Q) &= Q^{\frac{H}{z}} \sum_{d \geq 0} \frac{Q^d}{\prod_{r=1}^d (H - \lambda_0 + rz) \cdots (H - \lambda_N + rz)} \\ &\in H_{T^{N+1}}^*(\mathbb{P}^N) \otimes \mathbb{C}[z, z^{-1}][[Q]], \end{aligned}$$

where

$$Q^{\frac{H}{z}} = \sum_{k=0}^N \frac{1}{k!} \left(\frac{H}{z} \log(Q) \right)^k.$$

(ii) Givental's small equivariant K -theoretic J -function is the function

$$J^{K\text{th,eq}}(q, Q) = P^{-\ell_q(Q)} \sum_{d \geq 0} \frac{Q^d}{(q\Lambda_0 P^{-1}, \dots, q\Lambda_N P^{-1}; q)_d} \\ \in K_{T^{N+1}}(\mathbb{P}^N) \otimes \mathbb{C}[q, q^{-1}][[Q]],$$

where

$$(q\Lambda_0 P^{-1}, \dots, q\Lambda_N P^{-1}; q)_d = \prod_{i=0}^N \prod_{r=1}^d (1 - q^r \Lambda_i P^{-1}),$$

and $P^{-\ell_q(Q)}$ is some K -theoretic function corresponding to the function $Q^{\frac{H}{z}}$, that we will introduce in Definition 3.4.

PROPOSITION ([5, 11]). — For the projective space \mathbb{P}^N ,

(i) The cohomological J -function $J^{\text{coh,eq}}$ is a solution of the differential equation

$$(3.1) : [(-\lambda_0 + zQ\partial_Q) \cdots (-\lambda_N + zQ\partial_Q) - Q] J^{\text{coh,eq}}(z, Q) = 0.$$

(ii) The K -theoretic J -function $J^{K\text{th,eq}}$ is a solution of the q -difference equation

$$(3.3) : [(1 - \Lambda_0 q^{Q\partial_Q}) \cdots (1 - \Lambda_N q^{Q\partial_Q}) - Q] J^{K\text{th}}(q, Q) = 0.$$

Applying the confluence of the q -difference equations to this data, we want to first compare the q -difference equation satisfied in K -theory with the differential equation satisfied in cohomology, then compare the two J -functions as solutions of their respective functional equations. We would like to expect that the following informal statements hold:

(i) The confluence of the q -difference equation (3.3) defines a differential equation $\lim_{q \rightarrow 1} (3.3)$ which is the same as the differential equation (3.1) satisfied by the cohomological J -function.

(ii) As a solution of the q -difference equation (3.3), Givental's K -theoretic J -function $J^{K\text{th,eq}}$ satisfies

$$\lim_{q \rightarrow 1} J^{K\text{th,eq}} = J^{\text{coh,eq}}.$$

To give a rigorous meaning to these informal identifications, we state the following theorem, which is the first goal of this article:

THEOREM 3.12. — *Consider the algebraic torus $T^{N+1} := (\mathbb{C}^*)^{N+1}$ acting on $X = \mathbb{P}^N$. Recall that Equation (3.3) (resp. (3.1)) refers to the q -difference (resp. differential) equation satisfied by the K -theoretic (resp. cohomological) J -function $J^{K^{\text{th}}, \text{eq}}$ (resp. $J^{\text{coh}, \text{eq}}$). Let $q \in \mathbb{C}$, $0 < |q| < 1$ and $z \in \mathbb{C}^*$. Assume that the relation $\Lambda_i = q^{-\frac{\lambda_i}{z}} \in \mathbb{C}$ holds for all $i \in \{0, \dots, N\}$, and that for $i \neq j$, $\lambda_i - \lambda_j \notin \mathbb{Z}$. The following statements hold:*

(i) *Consider the application $\varphi_{q,z}$ defined by*

$$\varphi_{q,z} : \begin{cases} \mathbb{C} \longrightarrow \mathbb{C} \\ Q \longmapsto \left(\frac{z}{1-q} \right)^{N+1} Q \end{cases}.$$

Then, the pullback by $\varphi_{q,z}$ of the q -difference equation (3.3) is a confluent q -difference equation. Moreover, its formal limit when $q \rightarrow 1$ is the differential equation (3.1).

(ii) *Consider the isomorphism of rings*

$$\gamma_{\text{eq}} : K_{T^{N+1}}(\mathbb{P}^N) \otimes \mathbb{C} \longrightarrow H_{T^{N+1}}^*(\mathbb{P}^N, \mathbb{C})$$

defined by, for all $i \in \{0, \dots, N\}$

$$\gamma_{\text{eq}} \left(\prod_{j \neq i} \frac{1 - \Lambda_i P^{-1}}{1 - \Lambda_i \Lambda_j^{-1}} \right) = \prod_{j \neq i} \frac{H - \lambda_i}{\lambda_j - \lambda_i}.$$

Let \mathbb{E}_q be the complex torus $\mathbb{C}^/q^{\mathbb{Z}}$ and let $\mathcal{M}(\mathbb{E}_q)$ be the space of meromorphic functions on said complex torus. Then, there exists a change of fundamental solution $P_{q,z} \in \text{GL}_{N+1}(\mathcal{M}(\mathbb{E}_q))$, whose formula is explicit, such that the fundamental solution $J^{K^{\text{th}}, \text{eq}}$ is related to the cohomological J -function $J^{\text{coh}, \text{eq}}$ by*

$$\gamma_{\text{eq}} \left(\lim_{t \rightarrow 0} P_{q^t, z} \cdot (\varphi_{q^t, z}^* J^{K^{\text{th}}, \text{eq}}(q^t, Q)) \right) = J^{\text{coh}, \text{eq}}(z, Q).$$

Once this comparison result is established, one could be interested in attempting to compute the (local) monodromy data of this q -difference equation and to compare it with the cohomological case. In the q -difference case, the monodromy data is a connection matrix, relating the solution at $Q = 0$ given by the J -function with a fundamental solution at $Q = \infty$, which we will construct in Proposition 4.12. The second main goal of this article to compute this connection matrix in the equivariant setting, as below.

THEOREM 4.13. — *Let $w = 1/Q$ and denote by g_k the fundamental solution at $Q = \infty$ constructed in Proposition 4.12. Then, the fundamental solutions at 0 and ∞ are related by the identity*

$$g_k(w) = \sum_{j=0}^N R_{k,j}^{[\lambda;q^{N+1}]}(q, w) J_{|P=\Lambda_j}^{K^{\text{th,eq}}}\left(q, \frac{1}{w}\right)$$

where the coefficients $R_{k,j}^{[\lambda;q^{N+1}]}$ are some explicit q^{N+1} -constant functions.

In order to compare with quantum cohomology, one could hope that the limit when $q \rightarrow 1$ of this connection matrix would be related to the connection matrix in the cohomology case, cf. [4, Theorem 6.7].

Remark. — After the appearance of this article as a preprint on the arXiv, similar confluence questions (solutions, connection numbers) were investigated by Y. Wen for quintic threefold in [26]; confluence of the J -function has also been proved for any smooth projective variety whose anti-canonical bundle is nef in [20].

1.3. Structure of the article

The Section 2 will be a survey on the theory of q -difference equations, which the reader might not be familiar with. The aim of the first two subsections is to introduce the definitions required to understand the statement of the main theorem, as well as the special functions that will be useful to us in quantum K -theory. Then, in a last subsection, we will explain the confluence properties of q -difference equations in the regular singular case.

In the Section 3, the reader should now have the necessary background to understand the statement of the Main Theorem. In the first subsection, we will recall the definitions of Givental’s equivariant J -functions, whose expressions are obtained by using virtual localisation theorems. Then, we will give their functional equations. In the second subsection, we state the Main Theorem and give its proof. Our proof is split in two parts: first we check the confluence of the q -difference equation, then we check the confluence of the K -theoretic J -function as a solution of the confluent q -difference equation. In the third subsection, we will explain what happens when one tries to adapt the main theorem for non equivariant J -functions.

In the Section 4, we will compute the q -monodromy of our q -difference equation in the equivariant case. In the first subsection, we construct another fundamental solution, this time at $Q = \infty$, at which the q -difference equation is irregular singular. In the second subsection, we prove a base change formula

from the J -function to this new fundamental solution, obtaining connection numbers in the equivariant case. Unfortunately, we are not able to prove a non equivalent analogue of these connection numbers, but we will be able to conjecture some formula.

2. q -difference equations

This section is structured in three subsections. In the first subsection, we give some introductory definitions regarding q -difference equations. The second subsection is dedicated to the resolution of regular singular q -difference equations. The last subsection deals with confluence of regular singular q -difference equations.

2.1. General definitions

In this subsection we recall general notions of the theory of q -difference equations from the analytical point of view.

DEFINITION 2.1. — *Let $\mathcal{M}(\mathbb{C})$ be the field of meromorphic functions on \mathbb{C} . Fix $q \in \mathbb{C}, 0 < |q| < 1$ and $n \in \mathbb{Z}_{>0}$. Let $q^{Q\partial_Q}$ be the q -difference operator acting on functions $f : \mathbb{C} \rightarrow \mathbb{C}$ by $(q^{Q\partial_Q} f)(Q) = f(qQ)$. A linear q -difference system is a functional equation*

$$q^{Q\partial_Q} X_q(Q) = A_q(Q)X_q(Q),$$

where X_q is a column vector of n functions of input Q , and the matrix $A_q \in \text{GL}_n(\mathcal{M}(\mathbb{C}))$.

From now on we will work locally at $Q = 0$. More precisely, we will look for solutions in the space of $\mathbb{C}\{Q\}[Q^{-1}]$ of Laurent series that are convergent on a punctured disk centered at $Q = 0$. The definitions and the results below would also hold for $Q = \infty$ after replacing Q with Q^{-1} .

DEFINITION 2.2. — *Let $(\dagger_q) : q^{Q\partial_Q} X_q(Q) = A_q(Q)X_q(Q)$ be a q -difference system, with $A_q \in \text{GL}_n(\mathcal{M}(\mathbb{C}))$. We define the solution space of this q -difference equation by*

$$\text{Sol}(\dagger_q) = \left\{ X_q \in (\mathbb{C}\{Q\}[Q^{-1}])^n \mid q^{Q\partial_Q} X_q(Q) = A_q(Q)X_q(Q) \right\}.$$

Example 2.3 (q-constants). — Consider the q -difference equation

$$q^{Q\partial_Q} f_q(Q) = f_q(Q).$$

Constant functions are obvious solutions to this q -difference equation. Denote by $q^{\mathbb{Z}}$ the multiplicative group $q^{\mathbb{Z}} := \{q^k \mid k \in \mathbb{Z}\}$ and choose $\tau \in \mathbb{H} \subset \mathbb{C}$ such that $q = e^{2i\pi\tau}$. The meromorphic solutions of this q -difference equation can be identified with meromorphic functions on the torus $\mathbb{C}^*/q^{\mathbb{Z}}$, where the action is given by the multiplication $q^k \cdot z = q^k z$ and the complex (torus) structure comes from the exponential ($z \mapsto \exp(2i\pi z)$), as in the diagram below.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^* \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow[\sim]{\widetilde{\exp}} & \mathbb{C}^* \\ \mathbb{Z} + \tau\mathbb{Z} & & q^{\mathbb{Z}} \end{array}$$

Solutions to this q -difference equation will be called q -constants. We denote by \mathbb{E}_q the complex torus $\mathbb{E}_q := \mathbb{C}^*/q^{\mathbb{Z}}$. The space $\mathcal{M}(\mathbb{E}_q)$ of meromorphic functions on the complex torus $\mathbb{C}^*/q^{\mathbb{Z}}$ plays a role for q -difference equations similar to the space of constant functions \mathbb{C} for differential equations.

DEFINITION 2.4. — Let $q^{Q\partial_Q} X_q(Q) = A_q(Q)X_q(Q)$ be a q -difference system of rank $n \in \mathbb{Z}_{>0}$. A fundamental solution of this system is an invertible matrix $\mathcal{X}_q \in \mathrm{GL}_n(\mathbb{C}\{Q\}[Q^{-1}])$ such that

$$q^{Q\partial_Q} \mathcal{X}_q(Q) = A_q(Q)\mathcal{X}_q(Q).$$

DEFINITION 2.5. — Let $q^{Q\partial_Q} X_q(Q) = A_q(Q)X_q(Q)$ be a q -difference system. Consider a matrix $F_q \in \mathrm{GL}_n(\mathbb{C}\{Q\}[Q^{-1}])$. The gauge transform of the matrix A_q by the gauge transformation F_q is defined to be the matrix

$$F_q \cdot [A_q] := (q^{Q\partial_Q} F_q) A_q F_q^{-1}.$$

A second q -difference system $q^{Q\partial_Q} X_q(Q) = B_q(Q)X_q(Q)$ is said to be equivalent by gauge transform to the first one if there exists a matrix $F_q \in \mathrm{GL}_n(\mathbb{C}\{Q\}[Q^{-1}])$ such that

$$B_q = F_q \cdot [A_q].$$

DEFINITION 2.6. — Let $(\ddagger_q) : q^{Q\partial_Q} X_q(Q) = A_q(Q)X_q(Q)$ be a q -difference system and let $\varphi_q : \mathbb{C} \rightarrow \mathbb{C}$ be an isomorphism. The q -pullback $(\varphi_q^* \ddagger_q)$ of (\ddagger_q) by φ_q is the q -difference system given by

$$\varphi_q^*(\ddagger_q) : q^{Q\partial_Q} X_q(Q) = A_q(\varphi_q^{-1}(Q))X_q(Q).$$

DEFINITION 2.7.

- (1) A system $q^{Q\partial_Q} X_q(Q) = A_q(Q)X_q(Q)$ is said to be regular singular at $Q = 0$ if there exists a q -gauge transform $P_q \in \mathrm{GL}_n(\mathbb{C}\{Q\}[Q^{-1}])$ after which the matrix A_q evaluated at $Q = 0$ is well-defined and invertible, i.e. $(P_q \cdot [A_q])(0) \in \mathrm{GL}_n(\mathbb{C})$.
- (2) A regular singular system is said of the first kind if $A_q(0)$ is defined and invertible.

2.2. Fundamental solution for regular singular q -difference systems

We will now mention the results regarding the fundamental solution of regular singular q -difference equations. The practical use of this part is to introduce various special functions related to the theory of q -difference equations, which will also appear in the next section dealing with quantum K -theory: Definitions 2.8 and 2.11.

DEFINITION 2.8. — The q -Pochhammer symbol is the complex function defined for $d \in \mathbb{Z}_{\geq 0}$ by

$$\begin{aligned} (Q; q)_0 &:= 1, \\ (Q; q)_d &:= \prod_{r=0}^{d-1} (1 - q^r Q), \\ (Q; q)_\infty &:= \prod_{r \geq 0} (1 - q^r Q). \end{aligned}$$

DEFINITION 2.9 ([21]). — Jacobi's theta function θ_q is the complex function defined by the convergent Laurent series

$$\theta_q(Q) := \sum_{d \in \mathbb{Z}} q^{\frac{d(d-1)}{2}} Q^d \in \mathcal{O}(\mathbb{C}^*).$$

PROPOSITION 2.10. — Jacobi's theta function θ_q is a solution of the q -difference equation

$$q^{Q\partial_Q} \theta_q(Q) = \frac{1}{Q} \theta_q(Q).$$

DEFINITION 2.11 ([24]). — The q -logarithm is the function $\ell_q \in \mathcal{M}(\mathbb{C}^*)$ defined by

$$\ell_q(Q) := \frac{-Q\theta'_q(Q)}{\theta_q(Q)}.$$

LEMMA 2.12 ([24]). — The function ℓ_q is a solution of the q -difference equation

$$q^{Q\partial_Q} \ell_q(Q) = \ell_q(Q) + 1.$$

DEFINITION 2.13 ([24]). — *A first kind regular singular q -difference system $q^{Q\partial_Q}X = A_q(Q)X$ is said to be non resonant if any couple of two different eigenvalues $\lambda_i \neq \lambda_j$ of the matrix $A_q(0)$ satisfies the condition $\lambda_i\lambda_j^{-1} \notin q^{\mathbb{Z}}$.*

Before stating a theorem of Sauloy, we need to recall some notations. Let $A \in \mathrm{GL}_n(\mathbb{C})$ a constant matrix. Recall from [24, Section 1.1.2.3]. There exists a canonical solution, denoted by $e_{q,A}$, of the equation $q^{Q\partial_Q}X_q(Q) = AX_q(Q)$ such that

- (1) $q^{Q\partial_Q}e_{q,A} = A.e_{q,A} = e_{q,A}.A$;
- (2) for the multiplicative Dunford decomposition $A = D.U$, with D semi-simple and U unipotent, we have $e_{q,A} = e_{q,D}.e_{q,U}$.
- (3) We have the following equivalence

$$\begin{aligned} [M, A] = 0 &\iff [M, D] = [M, U] = 0 \\ &\iff [M, e_{q,D}] = [M, e_{q,U}] = 0 \\ &\iff [M, e_{q,A}] = 0. \end{aligned}$$

Notice that these $e_{q,A}$ is a generalization of $e_{q,\lambda}$ (See Definition 3.8).

THEOREM 2.14 ([24, Subsection 1.1.4]). — *Let $q^{Q\partial_Q}X_q(Q) = A_q(Q)X_q(Q)$ be a first kind regular singular q -difference system which is non resonant. There exists a canonical fundamental solution of $\mathcal{X}_q \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}^*))$ of this q -difference equation of the form $Me_{q,A_q(0)}$ with $M(0) = I_n$.*

2.3. Confluence of regular singular q -difference equation

In this subsection we introduce Sauloy's confluence phenomenon. One of the main ingredient is the following asymptotic for the q -logarithm ℓ_q .

Notation. — Let $q_0 \in \mathbb{C}^*$. Choose $\tau_0 \in \mathbb{C}^*$ such that $\mathrm{Im}(\tau_0) > 0$ and $q_0 = e^{2i\pi\tau_0}$. Notice that $|q_0| = e^{-2\pi\mathrm{Im}(\tau_0)} < 1$. Denote by $q = q_0^t = e^{-2i\pi t.\tau_0}$ for $t \in \mathbb{R}$. The set $q_0^{\mathbb{R}}$ is a spiral. Note that its complementary in \mathbb{C}^* is simply connected.

PROPOSITION 2.15 ([24, Subsections 3.1.3 and 3.1.4]). — *Fix $q_0 \in \mathbb{C}^*$, $|q_0| < 1$, let $q_0^t, t \in (0, 1]$. Denote by \log the determination of the logarithm on $\mathbb{C}^* - (-1)q_0^{\mathbb{R}}$ such that $\log(1) = 0$. We have the uniform convergence, on any compact of $\mathbb{C}^* - (-1)q_0^{\mathbb{R}}$,*

$$\lim_{t \rightarrow 0} (q_0^t - 1)\ell_{q_0^t}(Q) = \log(Q).$$

Remark 2.16. — Let $\delta_q = \frac{q^{Q\partial_Q} - \text{Id}}{q-1}$. We recall that the formal limit of this q -difference operator is the differential operator $Q\partial_Q$. A motivation to consider the function $(q-1)\ell_q$ instead of the usual q -logarithm ℓ_q is that we have

$$\delta_q \begin{pmatrix} 1 \\ (q-1)\ell_q(Q) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ (q-1)\ell_q(Q) \end{pmatrix}$$

Notice that the formal limit of this q -difference system is the differential system satisfied by the logarithm, while the matrix associated to the q -difference equation of ℓ_q has no limit when $q \rightarrow 1$.

Confluence of the q -difference equation

Firstly, we work study the local case (i.e on \mathbb{C}) then the global case (i.e on \mathbb{P}^1).

Recall that we say $q = q_0^t$ for $t \in \mathbb{R}$. So when we write $q \rightarrow 1$ it means that $t \rightarrow 0$.

We follow the [6, Section 3.2] (see also [24, Section 3.3]). Let consider a regular singular q -difference equation

$$q^{Q\partial_Q} X = A_q(Q)X.$$

Assume that

- (1) $B_q := \frac{A_q - I_n}{q-1}$ has limit $\tilde{B} \in M_n(\mathbb{C}(Q))$ when $q \rightarrow 1$.
- (2) The poles of A_q goes to the poles, denoted by $Q_1(q), \dots, Q_k(q)$, of \tilde{B} when $q \rightarrow 1$.
- (3) The differential system $Q\partial_Q X = \tilde{B}X$ is fuchsian and non resonant at $Q = 0$.
- (4) There exists an invertible matrix P_q such that $B_q(0) = P_q^{-1}J_qP_q$ that converge when $q \rightarrow 1$ to $\tilde{B}(0) = \tilde{P}^{-1}\tilde{J}\tilde{P}$, where J_q and \tilde{J} are Jordan matrices with \tilde{P} invertible.

Denote by $U_0 = \mathbb{C}^* \setminus \bigcup_{i=1}^r Q_i(q)q_0^{\mathbb{R}^-} \setminus q_0^{\mathbb{R}}$.

THEOREM 2.17 ([24, Theorem 3.6]). — *Assume that $B_q \rightarrow \tilde{B}$ uniformly on any compact in U_0 . The canonical solution of Theorem 2.14, \mathcal{X}_q converge uniformly on any compact of U_0 to the canonical solution $\tilde{\mathcal{X}}$ of the differential system given by the Fuchs–Frobenius method.*

Now we go to the “global case” i.e \mathbb{P}^1 . We assume that the hypothesis (1), (2), (3) and (4) of Theorem 2.17 are also satisfied at ∞ that is for $Q^{-1} = 0$.

We put

$$U = U_0 \cap U_\infty = \mathbb{C}^* \setminus \bigcup_{i=0}^k Q_i(q)q_0^{\mathbb{R}}, \quad \text{with } Q_0 = 1.$$

We also assume that the spiral $Q_i(q)q_0^{\mathbb{R}}$ are pairwise distinct.

COROLLARY 2.18 ([6, Section 3.4]). — *We the assumption (1)–(4) and the one above, the canonical solution, $\mathcal{X}_q^{(0)}$ and $\mathcal{X}_q^{(\infty)}$ of the q -difference system converge to the solution $\tilde{\mathcal{X}}^{(0)}$ and $\mathcal{X}^{(\infty)}$ of the differential system of matrix \tilde{B} on U_0 and U_∞ .*

Remark 2.19. — Notice that not any fundamental solution of a confluent q -difference system has immediately a well defined limit when $q \rightarrow 1$. Let us give an concrete example. Consider the q -difference equation

$$q^{Q\partial_Q} f_q(Q) = f_q(Q). \tag{2.1}$$

The function $g_q(Q) = \frac{1}{q-1}$ or $h_q(Q) = 1$ are both solutions of (2.1). The q -difference equation is confluent to the differential equation

$$\partial_Q \tilde{f} = 0. \tag{2.2}$$

However the function g_q does not have a limit when $q \rightarrow 1$ whereas h_q does. The latter is the canonical of Theorem 2.14.

3. Confluence for quantum K -theory of projective spaces

3.1. Equivariant J -functions

Definitions

Let $N \in \mathbb{Z}_{>0}$ be some positive integer and consider the projective space $X = \mathbb{P}^N$ with the action of the torus $T^{N+1} := (\mathbb{C}^*)^{N+1}$ given by

$$(\lambda_0, \dots, \lambda_N) \cdot [z_0 : \dots : z_N] = [\lambda_0 z_0 : \dots : \lambda_N z_N].$$

The elementary representations, indexed by $i \in \{0, \dots, N\}$,

$$\rho_i : \begin{cases} (\mathbb{C}^*)^{N+1} \longrightarrow \mathbb{C}^* \\ (t_0, \dots, t_N) \longmapsto t_i \end{cases},$$

define $N+1$ classes in equivariant K -theory $\Lambda_0, \dots, \Lambda_N \in K_{T^{N+1}}(\text{pt})$, where $-\Lambda_i$ is the line bundle on the point with the action of the group T^{N+1}

given by ρ_i . Denote by $P = \mathcal{O}_{\text{eq}}(1) \in K_{T^{N+1}}(\mathbb{P}^N)$ the equivariant anti-tautologic bundle, $H = c_1(\mathcal{O}_{\text{eq}}(1))$ the equivariant hyperplane class and $\lambda_i = c_1(\Lambda_i) \in H_{T^{N+1}}^2(\text{pt})$. We recall that we have

$$K_{T^{N+1}}(\mathbb{P}^N) \simeq \mathbb{Z}[\Lambda_0^{\pm 1}, \dots, \Lambda_N^{\pm 1}][P^{\pm 1}] / ((1 - \Lambda_0 P^{-1}) \cdots (1 - \Lambda_N P^{-1})),$$

$$H_{T^{N+1}}^*(\mathbb{P}^N; \mathbb{Q}) \simeq \mathbb{Q}[\lambda_0, \dots, \lambda_N][H] / ((H - \lambda_0) \cdots (H - \lambda_N)).$$

A basis of the equivariant K -theory $K_{T^{N+1}}(\mathbb{P}^N)$ is given by the classes indexed by $i \in \{0, \dots, N\}$

$$\eta_i = \prod_{j \neq i} \frac{1 - \Lambda_j P^{-1}}{1 - \Lambda_j \Lambda_i^{-1}} \in K_{T^{N+1}}(\mathbb{P}^N).$$

DEFINITION 3.1 ([5, Subsection 11.2.3]). — Givental’s equivariant cohomological small J -function of the projective space \mathbb{P}^N is the function defined by

$$J^{\text{coh,eq}}(z, Q) = Q^{\frac{H}{z}} \sum_{d \geq 0} \frac{Q^d}{\prod_{r=1}^d (H - \lambda_0 + rz) \cdots (H - \lambda_N + rz)} \in H_{T^{N+1}}^*(\mathbb{P}^N) \otimes \mathbb{C}[[z, z^{-1}]].$$

Remark 3.2. — The reader familiar with Gromov–Witten theory may notice several abuses in this definition of the J -function. The proper way to define them would be from the fundamental of the quantum \mathcal{D} -module (see e.g. [5, Equation 10.28] and [16, Definition 2.4]). We also confuse the I -function and the J -function for projective spaces due to the triviality of the mirror map for complex projective spaces.

PROPOSITION 3.3. — The cohomological J -function $J^{\text{coh,eq}}$ is a solution of the differential equation

$$[(-\lambda_0 + zQ\partial_Q) \cdots (-\lambda_N + zQ\partial_Q) - Q]J^{\text{coh,eq}}(z, Q) = 0. \quad (3.1)$$

DEFINITION 3.4 ([10, p. 1]). — Givental’s small equivariant K -theoretic J -function of the projective space \mathbb{P}^N is the function defined by

$$J^{K\text{th,eq}}(q, Q) = P^{-\ell_q(Q)} \sum_{d \geq 0} \frac{Q^d}{(q\Lambda_0 P^{-1}, \dots, q\Lambda_N P^{-1}; q)_d}, \quad (3.2)$$

where

$$(q\Lambda_0 P^{-1}, \dots, q\Lambda_N P^{-1}; q)_d = \prod_{i=0}^N (q\Lambda_i P^{-1}; q)_d,$$

$$P^{-\ell_q(Q)} = \sum_{i=0}^N \Lambda_i^{-\ell_q(Q)} \eta_i.$$

PROPOSITION 3.5 ([11]). — *The K -theoretic J -function $J^{K\text{th},\text{eq}}$ is a solution of the q -difference equation, which is regular singular at $Q = 0$:*

$$[(1 - \Lambda_0 q^{Q\partial_Q}) \cdots (1 - \Lambda_N q^{Q\partial_Q}) - Q] J^{K\text{th},\text{eq}}(q, Q) = 0. \quad (3.3)$$

Remark 3.6 (On the inputs z and q). — Geometrically, the input q (resp. z) can be understood as a generator of the \mathbb{C}^* -equivariant K -theory (resp. cohomology) of the point, see [16, Section 2.6] for details. Then, these generators are related by the identity $z = -c_1(q) \in H_{\mathbb{C}^*}^*(\text{pt})$.

A remark on the choice of the function $P^{-\ell_q(Q)}$

This part will be a comparison between the K -theoretic function $P^{-\ell_q(Q)}$ we have introduced in a factor associated to the J -function in Definition 3.4 and the usual q -characters e_{q,λ_q} that appear in the analytic theory of regular singular q -difference equations. This optional part is independent of the main theorem. The reader may want to skip to Subsection 3.2.

PROPOSITION 3.7. — *The K -theoretic function defined by*

$$P^{-\ell_q(Q)} := \sum_{i=0}^N \Lambda_i^{-\ell_q(Q)} \eta_i$$

is a solution of the K -theoretically valued q -difference equation

$$q^{Q\partial_Q} f(Q) = P^{-1} f(Q).$$

Complex functions that satisfy such q -difference equations are called q -characters. Recall that Jacobi's theta function θ_q , by Proposition 2.10, is a solution of the q -difference equation $q^{Q\partial_Q} \theta_q(Q) = Q^{-1} \theta_q(Q)$. A common example of a q -character is the following function:

DEFINITION 3.8 ([24, Subsection 1.1.2]). — *Let $\lambda_q \in \mathbb{C}^*$. The corresponding q -character is the function e_{q,λ_q} defined by*

$$e_{q,\lambda_q}(Q) = \frac{\theta_q(Q)}{\theta_q(\lambda_q Q)} \in \mathcal{M}(\mathbb{C}^*).$$

PROPOSITION 3.9 ([24, Subsection 1.1.2]). — *Let $\lambda_q \in \mathbb{C}^*$. The function e_{q,λ_q} is a solution of the q -difference equation*

$$q^{Q\partial_Q} e_{q,\lambda_q}(Q) = \lambda_q e_{q,\lambda_q}(Q)$$

Remark 3.10. — For the equivariant K -theoretic J -function, instead of using the function $P^{-\ell_q(Q)}$, it would have been possible to introduce the function $e_{q,P^{-1}}$ defined by

$$e_{q,P^{-1}}(Q) = \sum_{i=0}^N e_{q,\Lambda_i^{-1}}(Q)\eta_i.$$

We chose the former to have a better basis decomposition when considering the non equivariant limit $\Lambda_i \rightarrow 1$. Indeed, the non equivariant K -theory of \mathbb{P}^N is given by $K(\mathbb{P}^N) \simeq \mathbb{Z}[P, P^{-1}]/(t(1 - P^{-1})^{N+1})$. Let us write

$$\binom{\ell_q(Q)}{k} = \frac{1}{k!} \prod_{r=0}^{k-1} (\ell_q(Q) - r).$$

The function $P^{-\ell_q(Q)}$ has the decomposition in the previous basis of the non equivariant K -theory

$$P^{-\ell_q(Q)} = (1 - (1 - P^{-1}))^{\ell_q(Q)} = \sum_{k \geq 0} (-1)^k \binom{\ell_q(Q)}{k} (1 - P^{-1})^k.$$

Let us point out that the family $(1, \ell_q(Q), \dots, \ell_q(Q)^N)$ is linearly independent over the field of q -constants $\mathcal{M}(\mathbb{E}_q)$, see [23, Lemma VI.1.1.10]. This function has to be compared with the infinite product below, whose decomposition in our basis of the non equivariant K -theory is much more technical,

$$e_{q,P^{-1}}(Q) := \theta_q(Q)\theta_q(P^{-1}Q)^{-1}.$$

Therefore, when defining the J -function, we decided to use the function $P^{-\ell_q(Q)}$ instead of the usual q -character $e_{q,P^{-1}}(Q)$.

3.2. Confluence of the J -function

We begin by making a remark on the equivariant parameters to justify the relation $\Lambda_i = q^{-\frac{\lambda_i}{z}} \in \mathbb{C}$ that will appear in our statement of the confluence of the K -theoretic J -function.

Remark 3.11. — Recall that we have $z = -c_1(q) \in H_{\mathbb{C}^*}^*(\text{pt})$ and $\lambda_i = c_1(\Lambda_i) \in H_{\mathbb{T}^{N+1}}^*(\text{pt})$. The morphism $f: T^{N+1} \rightarrow \mathbb{C}^*$ given by $f(w_0, \dots, w_N) := w_0 \cdots w_N$ induces morphisms $f_{K\text{th}}: K_{\mathbb{C}^*}(\text{pt}) \rightarrow K_{T^{N+1}}(\text{pt})$ and $f_{\text{coh}}: H_{\mathbb{C}^*}^*(\text{pt}) \rightarrow H_{T^{N+1}}^*(\text{pt})$. We have the relation in the equivariant cohomology $H_{T^{N+1}}^*(\mathbb{P}^N)$, up to degree 2 terms

$$\text{ch}(\Lambda_i) = \text{ch}(f_{K\text{th}}(q))^{-\frac{\lambda_i}{f_{\text{coh}}(z)}}.$$

Statement

THEOREM 3.12. — *Consider the algebraic torus $T^{N+1} = (\mathbb{C}^*)^{N+1}$ acting on $X = \mathbb{P}^N$. Recall that (3.3) (resp. (3.1)) denotes the q -difference (resp. differential) equation satisfied by Givental’s small equivariant K -theoretic (resp. cohomological) J -function $J^{K\text{th,eq}}$ (resp. $J^{\text{coh,eq}}$). Assume that the relation $\Lambda_i = q^{\frac{-\lambda_i}{z}} \in \mathbb{C}$ holds for all $i \in \{0, \dots, N\}$, and that for $i \neq j$, $\lambda_i - \lambda_j \notin \mathbb{Z}$. Let $q \in \mathbb{C}, 0 < |q| < 1$ and $z \in \mathbb{C}^*$. The following statements hold:*

(i) *Consider the map $\varphi_{q,z}$ defined by*

$$\varphi_{q,z} : \begin{cases} \mathbb{C} \longrightarrow \mathbb{C} \\ Q \longmapsto \left(\frac{z}{1-q} \right)^{N+1} Q. \end{cases}$$

Then, the pullback by $\varphi_{q,z}$ of the q -difference equation (3.3) is a confluent q -difference equation. Moreover, its formal limit when $q \rightarrow 1$ is the differential equation (3.1) satisfied by the cohomological J -function.

(ii) *Let \mathbb{E}_q be the complex torus $\mathbb{C}^*/q^{\mathbb{Z}}$ and $\mathcal{M}(\mathbb{E}_q)$ be the space of meromorphic functions on said complex torus. Consider the isomorphism of rings $\gamma_{\text{eq}} : K_{T^{N+1}}(\mathbb{P}^N) \otimes \mathbb{C} \rightarrow H_{T^{N+1}}^*(\mathbb{P}^N, \mathbb{C})$ defined by, for all $i \in \{0, \dots, N\}$*

$$\gamma_{\text{eq}} \left(\prod_{j \neq i} \frac{1 - \Lambda_i P^{-1}}{1 - \Lambda_i \Lambda_j^{-1}} \right) = \prod_{j \neq i} \frac{H - \lambda_i}{\lambda_j - \lambda_i}.$$

Then, there exists an explicit change of fundamental solution $P_{q,z}^{\text{eq}} \in \text{GL}_{N+1}(\mathcal{M}(\mathbb{E}_q))$, such that the fundamental solution $J^{K\text{th,eq}}$ verifies

$$\gamma_{\text{eq}} \left(\lim_{t \rightarrow 0} P_{q^t,z}^{\text{eq}} \cdot (\varphi_{q^t,z}^* J^{K\text{th,eq}}(q^t, Q)) \right) = J^{\text{coh,eq}}(z, Q).$$

The proof of this theorem consists of three computations: we begin by studying the confluence of the q -difference equation, then of the solution. Then, we compare the limit of the solution to the cohomological J -function. After these three computations, we will give a proof of this theorem.

Confluence of the q -difference equation

PROPOSITION 3.13. — *Consider the q -difference equation (3.3) satisfied by the K -theoretic J -function:*

$$[(1 - \Lambda_0 q^{Q\partial_Q}) \cdots (1 - \Lambda_N q^{Q\partial_Q}) - Q] J^{K\text{th}}(q, Q) = 0.$$

Let $\varphi_{q,z}$ be the map

$$\varphi_{q,z} : \begin{cases} \mathbb{C} \longrightarrow \mathbb{C} \\ Q \longmapsto \left(\frac{z}{1-q} \right)^{N+1} Q. \end{cases}$$

Then, the q -pullback of the q -difference equation (3.3) by the isomorphism $\varphi_{q,z}$ is confluent, and its formal limit is the differential equation satisfied by the small equivariant cohomological J -function (3.1).

Proof. — Denote by δ_q the q -difference operator $\frac{q^{Q\partial_Q} - \text{Id}}{q-1}$. We rewrite the q -difference equation (3.3) to express it with the operators δ_q instead. Using

$$q^{Q\partial_Q} = \text{Id} + (q-1)\delta_q,$$

we obtain that $\Delta(q, Q, \delta_q) J^{K^{\text{th,eq}}}(q^t, Q) = 0$, where $\Delta(q, Q, \delta_q)$ is the q -difference operator given by

$$\Delta(q, Q, \delta_q) = \left[-Q + (1-q)^{N+1} \sum_{i=0}^{N+1} \delta_q^i (-1)^i \sum_{0 \leq j_1 < \dots < j_i < N} \Lambda_{j_1} \cdots \Lambda_{j_i} \right. \\ \left. \times \prod_{k \in \{0, \dots, N\} - \{j_1, \dots, j_i\}} \frac{1 - \Lambda_k}{1 - q} \right].$$

As it is written, the formal limit when $q \rightarrow 1$ of this operator is given by $-Q$ and thus does not define a differential equation. Introduce the q -pullback

$$\varphi_{q,z} : \begin{cases} \mathbb{C} \longrightarrow \mathbb{C} \\ Q \longmapsto \left(\frac{z}{1-q} \right)^{N+1} Q. \end{cases}$$

The q -pullback by $\varphi_{q,z}$ of the above q -difference equation is given by

$$\left[-Q + z^{N+1} \sum_{i=0}^{N+1} \delta_q^i (-1)^i \sum_{0 \leq j_1 < \dots < j_i < N} \Lambda_{j_1} \cdots \Lambda_{j_i} \right. \\ \left. \times \prod_{k \in \{0, \dots, N\} - \{j_1, \dots, j_i\}} \frac{1 - \Lambda_k}{1 - q} \right] \cdot f(q, Q) = 0. \quad (3.4)$$

Since the relation $\Lambda_i = q^{-\frac{\lambda_i}{z}}$ holds for all $i \in \{0, \dots, N\}$, this q -difference equation is confluent. Using the same relation again, we can compute its formal limit when $q \rightarrow 1$. The resulting formal limit coincides with the developed expression of the differential equation (3.1) satisfied by the cohomological J -function. \square

Remark 3.14. — The q -pullback $\varphi_{q,z}$ defined in Proposition 3.13 is the only q -pullback of the form $Q \mapsto (\frac{z}{1-q})^\lambda Q$, with $\lambda \in \mathbb{Z}$, which defines a confluent q -difference system whose formal limit is non zero.

Confluence of the solution

The q -difference system associated to the q -pullbacked equation (3.4) has a fundamental solution obtained from the J -function $J^{K\text{th,eq}}(q, Q)$, which is explicitly given by

$$\mathcal{X}(q, Q) = \begin{pmatrix} J_{|P=\Lambda_0}^{K\text{th,eq}}\left(q, \left(\frac{1-q}{z}\right)^{N+1} Q\right) & \cdots & J_{|P=\Lambda_N}^{K\text{th,eq}}\left(q, \left(\frac{1-q}{z}\right)^{N+1} Q\right) \\ \vdots & \ddots & \vdots \\ \delta_q^N J_{|P=\Lambda_0}^{K\text{th,eq}}\left(q, \left(\frac{1-q}{z}\right)^{N+1} Q\right) & \cdots & \delta_q^N J_{|P=\Lambda_N}^{K\text{th,eq}}\left(q, \left(\frac{1-q}{z}\right)^{N+1} Q\right) \end{pmatrix}. \quad (3.5)$$

The condition $\Lambda_i \Lambda_j^{-1} \notin q^{\mathbb{Z}}$ for $i \neq j$ implies that this matrix is invertible.

PROPOSITION 3.15. — *There exists a change of fundamental solution, denoted by $P_{q,z}^{\text{eq}} \in \text{GL}_{N+1}(\mathcal{M}(\mathbb{E}_q))$ such that the new fundamental solution $\mathcal{X}(q, Q) P_{q,z}^{\text{eq}}$ obtained from Equation(3.5) is given by*

$$\begin{aligned} & (\mathcal{X}(q, Q) P_{q,z}^{\text{eq}})_{\text{li}} \\ &= (\delta_q)^l \Lambda_i^{-\ell_q(Q)} \sum_{d \geq 0} \frac{1}{z^{d(N+1)}} \frac{(1-q)^{d(N+1)} Q^d}{(q\Lambda_0 \Lambda_i^{-1}, \dots, q, \dots, q\Lambda_N \Lambda_i^{-1}; q)_d}. \end{aligned}$$

Moreover, this fundamental solution is confluent.

Proof. — We begin by trying to compute the limit of the fundamental solution $\mathcal{X}(q, Q)$ when q tends to 1. Let $i \in \{0, \dots, N\}$. We have

$$\begin{aligned} & J_{|P=\Lambda_i}^{K\text{th,eq}}\left(q, \left(\frac{1-q}{z}\right)^{N+1} Q\right) \\ &= \Lambda_i^{-\ell_q\left(\left(\frac{q-1}{z}\right)^{N+1} Q\right)} \sum_{d \geq 0} \frac{(1-q)^{d(N+1)}}{z^{d(N+1)} (q\Lambda_0 \Lambda_i^{-1}; q)_d \cdots (q\Lambda_N \Lambda_i^{-1}; q)_d} Q^d. \end{aligned}$$

First let us check that every term in the sum indexed by d has a well defined limit when q tends to 1: the relation $\Lambda_i = q^{-\lambda_i/z}$ gives that for any $r \in \mathbb{Z}$,

$$\lim_{q \rightarrow 1} \frac{1-q}{1 - q^r \Lambda_j \Lambda_i^{-1}} = \frac{z}{r + \lambda_i - \lambda_j}.$$

Therefore, we have

$$\lim_{q \rightarrow 1} \frac{(1-q)^{d(N+1)}}{z^{d(N+1)}(\Lambda_0 \Lambda_i^{-1}; q)_d \cdots (\Lambda_N \Lambda_i^{-1}; q)_d} Q^d = Q^d \prod_{r=1}^d \prod_{j=0}^N \frac{1}{(\lambda_i - \lambda_j + rz)}.$$

It remains to deal with the divergent coefficient $\Lambda_i^{-\ell_q((\frac{q-1}{z})^{N+1}Q)}$. Notice that the two functions given by $\ell_q(Q)$ and $\ell_q((\frac{q-1}{z})^{N+1}Q)$ are both q -logarithms, i.e. solutions of the q -difference equation $q^{\tilde{Q}\partial_Q} f_q(Q) = f_q(Q) + 1$. Therefore, there exists a change of fundamental solution

$$P_{q,z}^{\text{eq}} \in \text{GL}_{N+1}(\mathcal{M}(\mathbb{E}_q))$$

which allows us, in the formula of the fundamental solution $\mathcal{X}(q, Q)$, to change the divergent q -logarithms $\ell_q((\frac{q-1}{z})^{N+1}Q)$ into the convergent q -logarithms $\ell_q(Q)$. Then, by Proposition 2.15,

$$\lim_{t \rightarrow 0} \Lambda_i^{-\ell_{q^t}(Q)} = \lim_{t \rightarrow 0} e^{\frac{\lambda_i}{z} \log(q^t) \ell_{q^t}(Q)} = Q^{\frac{\lambda_i}{z}}.$$

Therefore, the transformed fundamental solution $\mathcal{X}(q, Q)P_{q,z}^{\text{eq}}$ is confluent, and its coefficients are given by

$$\begin{aligned} & (\mathcal{X}(q, Q)P_{q,z}^{\text{eq}})_{\text{li}} \\ &= (\delta_q)^l \Lambda_i^{-\ell_q(Q)} \sum_{d \geq 0} \frac{1}{z^{d(N+1)}} \frac{(1-q)^{d(N+1)} Q^d}{(q\Lambda_0 \Lambda_i^{-1}, \dots, q, \dots, q\Lambda_N \Lambda_i^{-1}; q)_d}. \quad \square \end{aligned}$$

Comparison between confluence of quantum K -theory and quantum cohomology

Recall that we use a basis of the equivariant K -theory given by $\eta_i = \prod_{j \neq i} \frac{1 - \Lambda_j P^{-1}}{1 - \Lambda_j \Lambda_i^{-1}} \in K_{T^{N+1}}(\mathbb{P}^N)$.

DEFINITION 3.16. — Denote by $P_{q,z}^{\text{eq}} \cdot \varphi_{q,z}^* J^{K^{\text{th,eq}}}$ the K -theoretic function obtained from the first row of the transformed fundamental solution:

$$P_{q,z}^{\text{eq}} \cdot \varphi_{q,z}^* J^{K^{\text{th,eq}}} = \sum_{i=0}^N (\mathcal{X}(q, Q)P_{q,z}^{\text{eq}})_{0i} \eta_i.$$

By Proposition 3.15, the limit when q^t tends to 1 of the function $P_{q,z}^{\text{eq}} \cdot \varphi_{q,z}^* J^{K^{\text{th,eq}}}$ is well defined. We define the K -theoretic function

$$\text{confluence}(J^{K^{\text{th,eq}}})(z, Q) := \lim_{t \rightarrow 0} P_{q^t, z}^{\text{eq}} \cdot \varphi_{q^t, z}^* J^{K^{\text{th,eq}}}(q^t, Q).$$

PROPOSITION 3.17. — *Consider the isomorphism of rings*

$$\gamma_{\text{eq}} : K_{T^{N+1}}(\mathbb{P}^N) \longrightarrow H_{T^{N+1}}^*(\mathbb{P}^N; \mathbb{Q})$$

given by $\gamma_{\text{eq}}(\eta_i) = \prod_{j \neq i} \frac{H - \lambda_i}{\lambda_j - \lambda_i}$ for all $i \in \{0, \dots, N\}$. Then,

$$\gamma_{\text{eq}}(\text{confluence}(J^{K^{\text{th,eq}}}(z, Q))) = J^{\text{coh,eq}}(z, Q).$$

Proof. — We have

$$\begin{aligned} P_{q,z}^{\text{eq}} \cdot \varphi_{q,z}^* J^{K^{\text{th,eq}}}(t, z, Q) \\ = \sum_{i=0}^N \left(\Lambda_i^{-\ell_q(Q)} \sum_{d \geq 0} \frac{1}{z^{d(N+1)}} \frac{(1-q)^{d(N+1)} Q^d}{(q\Lambda_0\Lambda_i^{-1}, \dots, q, \dots, q\Lambda_N\Lambda_i^{-1}; q)_d} \right) \eta_i. \end{aligned}$$

Thus,

$$\text{confluence}(J^{K^{\text{th,eq}}}(z, Q)) = \sum_{i=0}^N \left(Q^{\frac{\lambda_i}{z}} \sum_{d \geq 0} Q^d \prod_{r=1}^d \prod_{j=0}^N \frac{1}{(\lambda_i - \lambda_j + rz)} \right) \eta_i.$$

We conclude using $\gamma_{\text{eq}}(\eta_i) = \prod_{j \neq i} \frac{H - \lambda_i}{\lambda_j - \lambda_i}$, recalling that

$$J_{|H=\lambda_i}^{\text{coh,eq}}(z, Q) = Q^{\frac{\lambda_i}{z}} \sum_{d \geq 0} Q^d \prod_{r=1}^d \prod_{j=0}^N \frac{1}{(\lambda_i - \lambda_j + rz)}. \quad \square$$

Summary of the previous results

We have now all the ingredients to give the proof of Theorem 3.12.

Proof of Theorem 3.12.

Confluence of the equation. — Using the q -pullback $\varphi_{q,z}$ of Proposition 3.13, we obtain a confluent q -difference system. Its limit is the differential equation associated to the small equivariant cohomological J -function.

Confluence of the solution. — As done in Equation (3.5), we can encode the equivariant K -theoretic J -function as a fundamental solution of the q -pullback of the system (3.4), which we denote by $\mathcal{X}(q, Q)$ in Equation (3.5). By Proposition 3.15, there exists a q -constant transformation $P_{q,z}^{\text{eq}} \in \text{GL}_{N+1}(\mathcal{M}(\mathbb{E}_q))$ such that the fundamental solution $\mathcal{X}(q, Q)P_{q,z}^{\text{eq}}$ is confluent.

Comparison with quantum cohomology. — The first row of the fundamental solution after the q -constant transformation $P_{q,z}$, $\mathcal{X}(q, Q)P_{q,z}^{\text{eq}}$, defines

another K -theoretic function, which we denote by $P_{q,z}^{\text{eq}} \cdot \varphi_{q,z}^* J^{K^{\text{th,eq}}}$ in Definition 3.16. Since the fundamental solution was confluent, this function has a well defined limit when $q^t \rightarrow 1$. Using Proposition 3.17, we have

$$\gamma_{\text{eq}} \left(\lim_{t \rightarrow 0} P_{q^t,z}^{\text{eq}} \cdot \varphi_{q^t,z}^* J^{K^{\text{th,eq}}}(q^t, Q) \right) = J^{\text{coh,eq}}(z, Q). \quad \square$$

3.3. Confluence and non equivariant limit

Since Givental's equivariant J -functions have well defined non equivariant limit by setting $\lambda_i \rightarrow 0$ and $\Lambda_i \rightarrow 1$ for all $i \in \{0, \dots, N\}$, one may wonder if there is a statement analogue to the Theorem 3.12 for non equivariant J -function. While the answer is positive, the details are slightly more technical.

Definitions and statement of the theorem

Remark 3.18. — A basis of the non equivariant K -theory

$$K(\mathbb{P}^N) \simeq \mathbb{Z}[P, P^{-1}] / \left((1 - P^{-1})^{N+1} \right)$$

is given by the integer powers of $1 - P^{-1}$. Notice that the non equivariant limit of the equivariant basis given by $\eta_i = \prod_{j \neq i} \frac{1 - \Lambda_j P^{-1}}{1 - \Lambda_j \Lambda_i^{-1}} \in K_{T^{N+1}}(\mathbb{P}^N)$ is not a basis the non equivariant K -theory.

DEFINITION 3.19. — *Let $X = \mathbb{P}^N$ and let $P = \mathcal{O}(1) \in K(\mathbb{P}^N)$ be the anti-tautological bundle. Givental's small K -theoretic J -function is the function given by*

$$J^{K^{\text{th}}}(q, Q) = P^{-\ell_q(Q)} \sum_{d \geq 0} \frac{Q^d}{(qP^{-1}; q)_d^{N+1}} \in K(\mathbb{P}^N) \otimes \mathbb{C}(q)[[Q]],$$

where

$$P^{-\ell_q(Q)} = (1 - (1 - P^{-1}))^{\ell_q(Q)} = \sum_{k \geq 0} (-1)^k \binom{\ell_q(Q)}{k} (1 - P^{-1})^k,$$

and

$$\binom{\ell_q(Q)}{k} = \frac{1}{k!} \prod_{r=0}^{k-1} (\ell_q(Q) - r).$$

PROPOSITION 3.20 ([16, Equation 10]). — *The non equivariant J -function $J^{K^{\text{th}}}(q, Q)$ is a solution of the q -difference equation*

$$\left[(1 - q^{Q\partial_Q})^{N+1} - Q \right] \widehat{J^{K^{\text{th}}}}(q, Q) = 0. \quad (3.6)$$

This q -difference equation is regular singular at $Q = 0$.

DEFINITION 3.21 ([5, Proposition 11.2.1]). — *Givental’s small cohomological J -function is given by the expression*

$$J^{\text{coh}}(z, Q) = Q^{\frac{H}{z}} \sum_{d \geq 0} \frac{Q^d}{\prod_{r=1}^d (H + rz)^{N+1}} \in H(\mathbb{P}^N) \otimes \mathbb{C}[z, z^{-1}][[Q]].$$

PROPOSITION 3.22 ([5, Equation 10.38]). — *This function is a solution of the differential equation*

$$[(zQ\partial_Q)^{N+1} - Q]J^{\text{coh}}(z, Q) = 0. \tag{3.7}$$

THEOREM 3.23. — *Let $X = \mathbb{P}^N$. Denote by $J^{K\text{th}}$ (resp. J^{coh}) Givental’s small K -theoretic (resp. cohomological) J -function. Let $q \in \mathbb{C}, 0 < |q| < 1$ and $z \in \mathbb{C}^*$. The following statements hold:*

- (i) *Consider the application $\varphi_{q,z}$ defined by*

$$\varphi_{q,z} : \begin{cases} \mathbb{C} \longrightarrow \mathbb{C} \\ Q \longmapsto \left(\frac{z}{1-q}\right)^{N+1} Q. \end{cases}$$

The pullback by $\varphi_{q,z}$ of the q -difference equation (3.6) satisfied by the K -theoretic J -function is confluent. Moreover, its formal limit when $q \rightarrow 1$ is the differential equation (3.7) satisfied by the cohomological J -function.

- (ii) *Let \mathbb{E}_q be the complex torus $\mathbb{C}^*/q^{\mathbb{Z}}$ and $\mathcal{M}(\mathbb{E}_q)$ be the space of meromorphic functions on said complex torus. Consider the isomorphism of rings $\gamma : K(\mathbb{P}^N) \otimes \mathbb{C} \rightarrow H^*(\mathbb{P}^N, \mathbb{C})$ defined by, for all $i \in \{0, \dots, N\}$*

$$\gamma\left((1 - P^{-1})^i\right) = H^i.$$

Then, there exists an explicit change of fundamental solution $P_{q,z} \in \text{GL}_{N+1}(\mathcal{M}(\mathbb{E}_qt))$ such that the fundamental solution $J^{K\text{th}}$ verifies

$$\gamma\left(\lim_{t \rightarrow 0} P_{q^t, z} \cdot (\varphi_{q^t, z}^* J^{K\text{th}}(q^t, Q))\right) = J^{\text{coh}}(z, Q).$$

The plan of the proof is the same as in the equivariant setting (Theorem 3.12): first we study the confluence of the q -difference equation, then the confluence of Givental’s J -function as a fundamental solution, which we compare to the cohomological J -function. However, the confluence of the fundamental solution requires a different change of fundamental solution $P_{q,z} \in \text{GL}_{N+1}(\mathcal{M}(\mathbb{E}_qt))$, which is slightly more complex than in the equivariant case. For a detailed proof of this statement, we will refer to [23, Section VI.2].

Confluence of the q -difference equation

PROPOSITION 3.24. — Consider the q -difference equation (3.6) :

$$(1 - q^{Q\partial_Q})^{N+1} f(q, Q) = Qf(q, Q).$$

Let $z \in \mathbb{C}^*$ and let $\varphi_{q,z}$ be the function

$$\varphi_{q,z} : \begin{cases} \mathbb{C} \longrightarrow \mathbb{C} \\ Q \longmapsto \left(\frac{z}{1-q}\right)^{N+1} Q. \end{cases}$$

Then, the q -pullback of the q -difference equation (3.6) by $\varphi_{q,z}$ is confluent, and its limit is the differential equation (3.7) satisfied by Givental's small cohomological J -function.

The proof of this proposition can be obtained by setting $\Lambda_i \rightarrow 1, \lambda_i \rightarrow 0$ for all $i \in \{0, \dots, N\}$ in the proof of Proposition 3.13 Writing $\delta_q = \frac{q^{Q\partial_Q} - \text{Id}}{q-1}$, the pullback by $\varphi_{q,z}$ of the q -difference equation (3.6) is given by

$$\left[(z\delta_q)^{N+1} - Q \right] J^{K\text{th}}(q, \varphi_{q,z}^{-1}(Q)) = 0. \tag{3.8}$$

Confluence of the fundamental solution

Consider the decomposition

$$J^{K\text{th}}(q, Q) = \sum_{i=0}^N J_i(q, Q) (1 - P^{-1})^i \in K(\mathbb{P}^N) \otimes \mathbb{C}(q)[[Q]].$$

Givental's small K -theoretic J -functions can be encoded in the fundamental solution of the q -difference equation (3.8) given by the matrix

$$\mathcal{X}^{K\text{th}}(q, Q) = \begin{pmatrix} J_0\left(q, \left(\frac{1-q}{z}\right)^{N+1} Q\right) & J_1\left(q, \left(\frac{1-q}{z}\right)^{N+1} Q\right) & \cdots & J_N\left(q, \left(\frac{1-q}{z}\right)^{N+1} Q\right) \\ \delta_q J_0\left(q, \left(\frac{1-q}{z}\right)^{N+1} Q\right) & \delta_q J_1\left(q, \left(\frac{1-q}{z}\right)^{N+1} Q\right) & \cdots & \delta_q J_N\left(q, \left(\frac{1-q}{z}\right)^{N+1} Q\right) \\ \vdots & \vdots & \ddots & \vdots \\ \delta_q^N J_0\left(q, \left(\frac{1-q}{z}\right)^{N+1} Q\right) & \delta_q^N J_1\left(q, \left(\frac{1-q}{z}\right)^{N+1} Q\right) & \cdots & \delta_q^N J_N\left(q, \left(\frac{1-q}{z}\right)^{N+1} Q\right) \end{pmatrix} \tag{3.9}$$

PROPOSITION 3.25 ([23, Proposition VI.2.3.3]). — *There exists an explicit q -constant matrix $P_{q,z} \in \mathrm{GL}_{N+1}(\mathcal{M}(\mathbb{E}_q))$ such that the new fundamental solution $\mathcal{X}^{K\mathrm{th}}(q, Q)P_{q,z}$ obtained from Equation (3.9) is given by*

$$\begin{aligned} & (\mathcal{X}^{K\mathrm{th}}(q, Q)P_{q,z})_{ii} \\ &= (\delta_q)^i \sum_{\substack{0 \leq a, b \leq N \\ a+b=i}} \binom{q-1}{z}^a \binom{\ell_q(Q)}{a} \binom{1-q}{z}^b f_b \left(q, \binom{1-q}{z}^{N+1} Q \right), \end{aligned}$$

where the functions f_b are defined by

$$\begin{aligned} f_b(q, Q) := & \sum_{d \geq 0} \frac{Q^d}{(q; q)_d^{N+1}} \sum_{k=0}^N \sum_{\substack{0 \leq j_1, \dots, j_N \leq N \\ j_1 + \dots + j_N = k \\ j_1 + 2j_2 + \dots + Nj_N = b}} (-1)^k \frac{(N+k)!}{N!j_1! \dots j_N!} \\ & \times \prod_{l=1}^N \left(\sum_{1 \leq m_1 < \dots < m_l \leq d} \frac{q^{m_1 + \dots + m_l}}{(1-q^{m_1}) \dots (1-q^{m_l})} \right)^{j_l}. \end{aligned}$$

Moreover, this fundamental solution has a non trivial limit when q^t tends to 1.

In that case, we can not use the proof of the equivariant statement (Proposition 3.15), as the non equivariant limit of our basis in equivariant K -theory is not a basis in non equivariant K -theory. However, the technique will be similar, so we will refer to [23, Proposition VI.2.3.3] for the complete proof. Just like in Proposition 3.15, we will need to change the q -logarithms, but to obtain a well defined limit, the i -th column of the matrix (3.9) has to be multiplied by the factor $\left(\frac{q-1}{z}\right)^i$.

Comparison with quantum cohomology

To complete the proof of Theorem 3.23, it remains to compare the limit of the first row of the fundamental solution $\mathcal{X}^{K\mathrm{th}}(q, Q)P_{q,z}$ with Givental's small cohomological J -function.

DEFINITION 3.26. — *We denote by $P_{q,z} \cdot \varphi_{q,z}^* J^{K\mathrm{th}}(q, Q)$ the K -theoretic function defined by*

$$P_{q,z} \cdot \varphi_{q,z}^* J^{K\mathrm{th}}(q, Q) = \sum_{i=0}^N (\mathcal{X}^{K\mathrm{th}}(q, Q)P_{q,z})_{0i} H^i \in K(\mathbb{P}^N) \otimes \mathbb{C}(q, z)[[Q]].$$

PROPOSITION 3.27. — Consider the ring automorphism

$$\gamma : K(\mathbb{P}^N)_{\mathbb{Q}} \longrightarrow H^*(\mathbb{P}^N; \mathbb{Q})$$

defined by $\gamma(1 - P^{-1}) = H$. The following asymptotic holds

$$\gamma\left(\lim_{t \rightarrow 0} P_{q^t, z} \cdot \varphi_{q^t, z}^* J^{K\text{th}}(q^t, Q)\right) = J^{\text{coh}}(z, Q).$$

Proof. — Using the characterisation of the change of fundamental solution of Proposition 3.25, we have to compute the limits of the terms for all $i \in \{0, \dots, N\}$

$$\sum_{\substack{0 \leq a, b \leq N \\ a+b=i}} \left(\frac{q-1}{z}\right)^a \binom{\ell_q(Q)}{a} \left(\frac{1-q}{z}\right)^b f_b\left(q, \left(\frac{1-q}{z}\right)^{N+1} Q\right).$$

The decomposition of the cohomological J -function in the usual basis $(1, H, \dots, H^N)$ is given by

$$J^{\text{coh}}(z, Q) = \sum_{i=0}^N H^i \left[\sum_{\substack{0 \leq a, b \leq N \\ a+b=i}} \frac{1}{a!} \left(\frac{\log(Q)}{z}\right)^a g_b(z, Q) \right], \quad (3.10)$$

where

$$g_b(z, Q) = \sum_{d \geq 0} \frac{Q^d}{(z^d d!)^{N+1}} \frac{1}{z^b} \sum_{k=0}^N \sum_{\substack{0 \leq j_1, \dots, j_N \leq N \\ j_1 + \dots + j_N = k \\ j_1 + 2j_2 + \dots + Nj_N = b}} (-1)^k \frac{(N+k)!}{N! j_1! \dots j_N!} \\ \times \prod_{l=1}^N \left(\sum_{1 \leq m_1 < \dots < m_l \leq d} \frac{1}{m_1 \dots m_l} \right)^{j_l}.$$

We observe that

$$\lim_{t \rightarrow 0} \left(\frac{1-q^t}{z}\right)^b f_b\left(q^t, \left(\frac{1-q^t}{z}\right)^{N+1} Q\right) = g_b(z, Q).$$

Using Proposition 2.15, we also have that

$$\lim_{t \rightarrow 0} \left(\frac{q^t - 1}{z}\right)^a \binom{\ell_{q^t}(Q)}{a} = \frac{1}{a!} \left(\frac{\log(Q)}{z}\right)^a.$$

Using these two limits, we obtain that

$$\lim_{t \rightarrow 0} P_{q^t, z} \cdot \varphi_{q^t, z}^* J^{K\text{th}}(q^t, Q) = \sum_{i=0}^N (1 - P^{-1})^i \sum_{\substack{0 \leq a, b \leq N \\ a+b=i}} \frac{1}{a!} \left(\frac{\log(Q)}{z}\right)^a g_b(z, Q).$$

Applying the ring isomorphism γ and comparing with Equation (3.10), we find the desired result. \square

4. q -monodromy of the q -difference equation of projective spaces

The goal of this section is to compute some monodromy data for the q -difference equation satisfied by the small K -theoretical J -function of projective spaces. This monodromy data takes the form of a connection matrix, computing the base change from the J -function (fundamental solution at $Q = 0$) to a fundamental solution at $Q = \infty$.

Before starting, let us mention some references on q -monodromy. A treatment of the regular singular case can be found in [15], the end result being Theorem 3.4.9 p. 134. For some irregular case, we can refer to [7, 22] in general, [1] for irregular (unilateral) q -hypergeometric series, and [26] for the q -difference equation associated to Fermat's quintic threefold.

4.1. Fundamental solution at ∞

In this Subsection, we begin by constructing a fundamental solution of the q -difference equation (3.3) at $Q = \infty$. This solution is built by looking for a formal solution to the q -difference equation, then using a q -analogue of the Borel–Laplace transform to obtain an analytic solution.

4.1.1. Formal solution

We denote by $w = Q^{-1}$ our coordinate at $Q = \infty$. Notice that if f, g are two complex functions so that $g(w) = f(1/w) = f(Q)$, then $q^{Q\partial_Q} f = (q^{w\partial_w})^{-1} g$. Therefore, the q -difference equation (Δ_q) in the new local coordinate w becomes

$$\left[(-1)^{N+1} q^{N+1} \Lambda_0 \cdots \Lambda_N w (1 - \Lambda_0^{-1} q^{w\partial_w}) \cdots (1 - \Lambda_N^{-1} q^{w\partial_w}) - (q^{w\partial_w})^{N+1} \right] \times g_q(w) = 0. \quad (4.1)$$

Notation. — We recall some q -analogues of hypergeometric functions. Let $r, s \in \mathbb{Z}_{\geq 0}$ and $a_1, \dots, a_r, b_1, \dots, b_s \in \mathbb{C}$. The notation for a (unilateral) q -hypergeometric series is

$${}_r\varphi_s \left(\begin{matrix} a_1 & \cdots & a_r \\ b_1 & \cdots & b_s \end{matrix} \middle| q, z \right) = \sum_{d \geq 0} \frac{(a_1, \dots, a_r; q)_d}{(q, b_1, \dots, b_s; q)_d} \left((-1)^d q^{\frac{d(d-1)}{2}} \right)^{1+s-r} z^d.$$

Keeping the same notations, we define bilateral q -hypergeometric series by

$${}_r\psi_s \left(\begin{matrix} a_1 & \cdots & a_r \\ b_1 & \cdots & b_s \end{matrix} \middle| q, w \right) = \sum_{d \in \mathbb{Z}} \frac{(a_1, \dots, a_r; q)_d}{(b_1, \dots, b_s; q)_d} \left((-1)^d q^{\frac{d(d-1)}{2}} \right)^{s-r} w^d.$$

Let $\alpha \in \mathbb{C}^*$. We will use the following ansatz to construct our fundamental solution at $w = 0$.

LEMMA 4.1. — *Let h_q be a complex function and set the ansatz $g_q(w) := e_{q, \alpha^{-1}P}(w)h_q(w)$. The function g_q is a solution of the q -difference equation (4.1) if and only if the function h_q is a solution of the following q -difference equation:*

$$\left[\begin{array}{l} (-1)^{N+1} q^{N+1} w \Lambda_0 \cdots \Lambda_N \prod_{i=0}^N (1 - \Lambda_i^{-1} \alpha^{-1} P q^{w\partial_w}) \\ - (\alpha^{-1} P)^{N+1} (q^{w\partial_w})^{N+1} \end{array} \right] h_q(w) = 0. \quad (4.2)$$

Notice that the formula of our ansatz g_q is close to the K -theoretic J -function, as $e_{q, \alpha^{-1}P}(w)$ is a q -character and h_q will be a Laurent series.

Proof. — Assume the function g_q is a solution of the q -difference equation (4.1). The functions g_q, h_q are related by the relation

$$g_q(w) = e_{\alpha^{-1}P}(w)h_q(w).$$

Therefore,

$$q^{w\partial_w} g_q(w) = e_{\alpha^{-1}P}(w) \alpha^{-1} P q^{w\partial_w} h_q(w).$$

Thus, when applying the q -difference operator in Equation (4.1) to g_q , we obtain

$$e_{\alpha^{-1}P}(w) \cdot \left[\begin{array}{l} (-1)^{N+1} q^{N+1} w \Lambda_0 \cdots \Lambda_N \prod_{i=0}^N (1 - \Lambda_i^{-1} \alpha^{-1} P q^{w\partial_w}) \\ - (\alpha^{-1} P)^{N+1} (q^{w\partial_w})^{N+1} \end{array} \right] h_q(w),$$

which is zero by assumption that g_q is a solution of the q -difference equation (4.1). □

LEMMA 4.2. — *The q -difference equation (4.2) of the previous lemma admits as a solution the following formal Laurent series*

$$\begin{aligned} & h_q(w) \\ &= {}_{N+1}\psi_0 \left(\begin{matrix} \Lambda_0^{-1} \alpha^{-1} P & \cdots & \Lambda_N^{-1} \alpha^{-1} P \\ - & & \end{matrix} \middle| q, (\alpha P^{-1})^{N+1} \Lambda_0 \cdots \Lambda_N w \right) \end{aligned} \quad (4.3)$$

Remark 4.3. — Since $|q| < 1$, applying the ratio test to the positive part of the Laurent series (4.3) shows that its convergence ray is 0. The negative part has convergence ray ∞ .

Proof. — We look for a formal Laurent series solution to the q -difference equation (4.2). Let the Laurent series in the input w

$$h_q(w) = \sum_{d \in \mathbb{Z}} h_d(q)w^d.$$

Let's assume the Laurent series h_q is a solution of the q -difference equation (4.2). Then,

$$\begin{aligned} \sum_{d \in \mathbb{Z}} (\alpha^{-1}P)^{N+1} q^{d(N+1)} h_d w^d \\ = \sum_{d \in \mathbb{Z}} (-1)^{N+1} q^{N+1} \left[\prod_{i=0}^N \Lambda_i (1 - \alpha^{-1}P \Lambda_i^{-1} q^d) \right] h_d w^{d+1}. \end{aligned}$$

Identifying the coefficients in front of w^{d+1} , we get the following recursion relation satisfied by the family of coefficients $(h_d)_{d \in \mathbb{Z}}$.

$$(\alpha^{-1}P)^{N+1} q^{(d+1)(N+1)} h_{d+1} = (-1)^{N+1} q^{N+1} \left[\prod_{i=0}^N \Lambda_i (1 - \alpha^{-1}P \Lambda_i^{-1} q^d) \right] h_d.$$

Recall that the q -Pochhammer symbol $(a; q)_d$ is a solution of the recursion equation $(a; q)_{d+1} = (1 - aq^d)(a; q)_d$. Therefore, we get a solution h_d of the previous recursion equation given by

$$\begin{aligned} h_{d+1} = (-1)^{(N+1)(d+1)} (q^{N+1})^{-\frac{d(d+1)}{2}} (\Lambda_0^{-1} \alpha^{-1}P, \dots, \Lambda_N^{-1} \alpha^{-1}P; q)_{d+1} \\ \times \left((\alpha P^{-1})^{N+1} \Lambda_0 \cdots \Lambda_N \right)^{d+1} h_0, \end{aligned}$$

where $h_0 \in \mathbb{C}$. Setting $h_0 = 1$ produces a solution which is also the bilateral q -hypergeometric series given by Equation (4.3). \square

We can now give the formula for our fundamental solution in the proposition below.

PROPOSITION 4.4. — *Consider the q -difference equation (4.1), given by*

$$\begin{aligned} \left[(-1)^{N+1} q^{N+1} \Lambda_0 \cdots \Lambda_N w (1 - \Lambda_0^{-1} q^{w\partial_w}) \cdots (1 - \Lambda_N^{-1} q^{w\partial_w}) - (q^{w\partial_w})^{N+1} \right] \\ \times g_q(w) = 0. \end{aligned}$$

Assume that $\alpha \in \mathbb{C}^* - q^{\mathbb{Z}}$ and that for any $i \neq j \in \{0, \dots, N\}$, $\Lambda_i \Lambda_j^{-1} \notin q^{\mathbb{Z}}$. Denote by $h_q(w)$ the formal Laurent series (4.3). Then, the q -difference equation (4.1) admits a basis of formal solutions given by, for $i \in \{0, \dots, N\}$,

$$\begin{aligned} g_i(w) &= (e_{q, \alpha^{-1}P}(w)h_q(w))|_{P=\Lambda_i} \\ &= e_{q, \alpha^{-1}\Lambda_i N+1}\psi_0 \\ &\quad \times \left(\Lambda_0^{-1}\alpha^{-1}\Lambda_i \quad \cdots \quad \Lambda_N^{-1}\alpha^{-1}\Lambda_i \middle| q, (\alpha\Lambda_i^{-1})^{N+1}\Lambda_0 \cdots \Lambda_N w \right). \end{aligned}$$

Remark 4.5. — Before giving a proof of this statement, we point out that if $\alpha \in q^{\mathbb{Z}}$, then there exists a $d \in \mathbb{Z}$ such that $(\alpha^{-1}; q)_d = 0$, and therefore the expression g_i is either undefined or is not a solution of the q -difference equation.

Proof. — Let $i \in \{0, \dots, N\}$. Let us show that the function g_i is a formal solution of the q -difference equation (4.1). By setting $P = \Lambda_i$ in the statement of Lemma 4.1, we can construct one solution by solving the q -difference equation (4.2), having replaced P by Λ_i . A formal solution of this new q -difference equation can be found in Lemma 4.2 after setting $P = \Lambda_i$, which is precisely the function g_i .

Assuming the condition that for any $i \neq j \in \{0, \dots, N\}$, $\Lambda_i \Lambda_j^{-1} \notin q^{\mathbb{Z}}$, we obtain that the functions $(g_i)_{i \in \{0, \dots, N\}}$ are independent over the field of q -constants $\mathcal{M}(\mathbb{E}_q)$. □

4.1.2. Analytic solution

DEFINITION 4.6. — Let $f(w) = \sum_{d \geq 0} f_d w^d \in \mathbb{C}[[w]]$ be a formal power series. The q -Borel transform of the formal power series f is given by the expression

$$\mathcal{B}_q f(\xi) := \sum_{d \geq 0} f_d q^{\frac{d(d-1)}{2}} \xi^d \in \mathbb{C}[[\xi]].$$

DEFINITION 4.7 ([27]). — Let $[\lambda; q] \in \mathbb{C}^*/q^{\mathbb{Z}}$ be a discrete q -spiral and $f \in \mathcal{M}(\mathbb{C}^*, 0)$ be a germ of a meromorphic function with essential singularity at 0. We say the function g admits a q -Laplace transform along the q -spiral $[\lambda; q]$ if there exists a constant $\varepsilon > 0$ and an domain $\Omega \subset \mathbb{C}$ such that

(i) The domain Ω contains the domain

$$\bigcup_{m \in \mathbb{Z}} \{ \xi \in \mathbb{C}^*, |\xi - \lambda q^m| < \varepsilon |\lambda q^m| \} \subset \Omega.$$

- (ii) *The function g admits an analytic continuation \bar{g} on the domain Ω . Furthermore, we ask that there exists constants $C_1, C_2 > 0$ such that \bar{f} satisfies the bound*

$$|\bar{f}(\xi)| < C_1 \theta_{|q|}(C_2 |\xi|).$$

A function satisfying such a bound is said to have q -exponential growth at ∞ .

We will denote by $\mathcal{H}_q^{[\lambda; q]}$ the space of functions satisfying the conditions (i) and (ii).

Remark 4.8. — Notice that the definitions we gave so far are concerned with power series, while the formal fundamental solution built in Proposition 4.4 is a Laurent series. We formally extend the definition of the q -Laplace transform to Laurent series by setting

$$\mathcal{B}_q \left(\sum_{d \in \mathbb{Z}} f_d w^d \right) (\xi) = \sum_{d \in \mathbb{Z}} f_d q^{\frac{d(d-1)}{2}} \xi^d \in \mathbb{C}[[\xi^{\pm 1}]].$$

By doing so, there is a chance that the negative powers part of the Laurent series is no longer convergent, but it still is in the case of our fundamental solution.

DEFINITION 4.9 ([27, Definition 7] or [28, p. 8]). — *Let $g \in \mathcal{H}_q^{[\lambda; q]}$ be a function admitting a q -Laplace transform along the q -spiral $[\lambda; q]$. We defined the q -Laplace transform of the function f by the expression*

$$\mathcal{L}_q^{[\lambda; q]} g(w) := \sum_{m \in \mathbb{Z}} \frac{g(\lambda q^m)}{\theta_q \left(\frac{\lambda q^m}{w} \right)} \in \mathcal{M}(\mathbb{C}^*, 0).$$

We will now define a q -Borel–Laplace sum.

PROPOSITION 4.10 ([27, Lemma 6]). — *Consider a convergent power series $f \in \mathbb{C}\{w\}$ and a q -spiral $[\lambda; q] \in \mathbb{C}/q^{\mathbb{Z}}$, then*

$$\mathcal{L}_q^{[\lambda; q]} \mathcal{B}_q f(w) = f(w).$$

Note that this formula extends to formal Laurent series, as the recursion strategy used in the proof of [8, Lemma 1.7] can be used to prove that for a fixed $l \in \mathbb{Z}$ and every $a \in \mathbb{C}$, $\mathcal{L}_q^{[\lambda; q]} \mathcal{B}_q a w^{l+1} = a w^{l+1}$, then $\mathcal{L}_q^{[\lambda; q]} \mathcal{B}_q a w^l = a w^l$ also holds. Indeed, a computation gives the formulas

$$\mathcal{L}_q^{[\lambda; q]} (\xi \mathcal{B}_q(f)(\xi)) = w q^{-w \partial_w} \mathcal{L}_q^{[\lambda; q]} \mathcal{B}_q f(w) = \mathcal{L}_q^{[\lambda; q]} \mathcal{B}_q (w q^{-w \partial_w} f(w)),$$

$$\begin{aligned} & \mathcal{L}_q^{[\lambda; q]} (\xi^{-1} \mathcal{B}_q(f)(\xi)) \\ &= q^{w \partial_w} \left(w^{-1} \mathcal{L}_q^{[\lambda; q]} \mathcal{B}_q f(w) \right) = \mathcal{L}_q^{[\lambda; q]} \mathcal{B}_q (q^{w \partial_w} (w^{-1} f(w))). \end{aligned}$$

DEFINITION 4.11. — Let $f \in \mathbb{C}[[w^{\pm 1}]]$ be a formal series. We say that the function f is q -Borel–Laplace summable along the q -spiral $[\lambda; q]$ if it satisfies the condition

$$\mathcal{B}_q f \in \mathcal{H}_q^{[\lambda; q]}$$

For such a function, we define its q -Borel–Laplace resummation to be the function defined by

$$\mathcal{S}_q^{[\lambda; q]} f(w) := \left(\mathcal{L}_q^{[\lambda; q]} \mathcal{B}_q f \right)(w)$$

PROPOSITION 4.12. — Let (g_i) be the basis of formal fundamental solution of the q -difference equation (4.1) constructed in Proposition 4.4, and let $\mathcal{S}_q^{[\lambda; q]}$ denote the q -Borel–Laplace transform defined in Definition 4.11. Then, the family $(\mathcal{S}_{q^{N+1}}^{t[\lambda; q^{N+1}]} g_i)_i$ is a fundamental solution of the q -difference (4.1).

To prove such a statement, one has to check q^{N+1} -resummability of the bilateral q -series (4.2). This relies on an analytical continuation of the q^{N+1} -Borel transform, which is given in the coming Corollary 4.16. The proof that this analytical continuation has q^{N+1} -exponential growth along a domain Ω is exactly the same as in the case of unilateral q -hypergeometric series, which is given by [1, Theorem 3.1]. Indeed, one can check the analytical continuation in both cases can be written under the form, with $C_j, A_j \in \mathbb{C}$

$$\bar{g}(\xi) = \sum_j C_j \frac{\theta_q(a_j \xi)}{\theta_q(\xi)} {}_s\varphi_r \left(\begin{matrix} c_1 & \cdots & c_s \\ d_1 & \cdots & d_r \end{matrix} \middle| q, A_j \frac{1}{w} \right);$$

for which Adachi’s arguments contained in [1, Section 5] apply identically.

4.2. Connection numbers for quantum K -theory of projective spaces

We will now compute a base change formula between the J -function and the fundamental solution at $Q = \infty$ built in Proposition 4.12.

Notation. — Let $\underline{a} := (a_1, \dots, a_r) \in \mathbb{C}^r$ be a multi-index. For $d \in \mathbb{Z} \cup \{\infty\}$, $j \in \{1, \dots, r\}$ and $\gamma \in \mathbb{C}$ we will use the following notations:

$$\begin{aligned} (\underline{a}; q)_d &:= (a_1, \dots, a_r; q)_d, \\ (\gamma \underline{a}; q)_d &:= (\gamma a_1, \dots, \gamma a_r; q)_d, \\ \underline{a}^{-1} &:= (a_1^{-1}, \dots, a_r^{-1}), \\ \widehat{\underline{a}}^{(j)} &:= (a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_r) \in \mathbb{C}^{r-1}, \\ \pi(\underline{a}) &:= a_1 \cdots a_r. \end{aligned}$$

Our goal is to prove the following computation.

THEOREM 4.13. — *Write $\underline{\Lambda} := (\Lambda_0, \dots, \Lambda_N)$. Let $\alpha \in \mathbb{C}^* - q^{\mathbb{Z}}$ and let $[\lambda; q^{N+1}]$ be a q^{N+1} -spiral. Denote by $(g_k^{[\lambda; q^{N+1}]})_{k \in \{0, \dots, N\}}$ the fundamental solution of the q -difference equation for quantum K -theory at ∞ given by Proposition 4.12:*

$$g_k(w) = \left\{ e_{q, \alpha^{-1}P}(w) \left[\mathcal{S}_{q^{N+1}}^{[\lambda; q^{N+1}]} \Big|_{N+1} \psi_0 \left(\alpha^{-1}P \underline{\Lambda}^{-1} \Big|_{q, (\alpha P^{-1})^{N+1} \pi(\underline{\Lambda})w} \right) \right] (w) \right\}_{|P=\Lambda_k}.$$

Then, this fundamental solution at ∞ can be expressed with the fundamental solution at 0 given by the small J -function as in the following identity.

$$g_k^{[\lambda; q^{N+1}]}(w) = \sum_{j=0}^N R_{k,j}^{[\lambda; q^{N+1}]}(q, w) J_{|P=\Lambda_j}^{\text{Kth,eq}} \left(q, \frac{1}{w} \right),$$

where $R_{k,j}^{[\lambda; q^{N+1}]}$ is the q^{N+1} -constant function given by

$$R_{k,j}^{[\lambda; q^{N+1}]}(q, w) = \frac{\left(q, \frac{\alpha^{-1}\Lambda_k}{\underline{\Lambda}^j}; q \right)_{\infty}}{\left(q\alpha\Lambda_k^{-1}\Lambda_j, \frac{\Lambda_j}{\underline{\Lambda}^j}; q \right)_{\infty}} \frac{\theta_q \left((-1)^N \frac{\lambda \alpha^{-1} \Lambda_k}{\Lambda_j} \right)}{\theta_q \left((-1)^N \lambda \right)} \times \frac{\theta_{q^{N+1}} \left(\frac{\lambda \Lambda_j^{N+1}}{\pi(\underline{\Lambda})w} \right)}{\theta_{q^{N+1}} \left(\frac{\lambda}{(\alpha \Lambda_k^{-1})^{N+1} \pi(\underline{\Lambda})w} \right)} e_{q, \alpha^{-1}\Lambda_k}(w) \Lambda_j^{\ell_q \left(\frac{1}{w} \right)}.$$

Our strategy to prove this theorem will be the same as the one found in [1]: we start from a connection number for a regular singular bilateral q -hypergeometric series, identify some limit of these connection numbers as an identity between q -Borel transforms, then apply a q -Laplace transform to the identity.

PROPOSITION 4.14 ([25, Equation 5.2.4 p. 165]; see also [3, Theorem 2.1]). *Let $\underline{a}, \underline{b} \in \mathbb{C}^r$. Assuming the following series are finite sums, or assuming $|\frac{\pi(\underline{b})}{\pi(\underline{a})}| < |z| < 1$,*

$${}_r\psi_r \left(\frac{\underline{a}}{\underline{b}} \Big| q, w \right) = \sum_{j=1}^r C_j(q) \frac{\left(a_j w, \frac{q}{a_j w}; q \right)_{\infty}}{\left(w, \frac{q}{w}; q \right)_{\infty}} {}_r\varphi_{r-1} \left(\frac{a_j q}{\underline{b}} \Big| q, \frac{\pi(\underline{b})}{\pi(\underline{a})w} \right),$$

where

$$C_j(q) := \frac{\left(q, \widehat{a}^j, \frac{b}{a_j}; q\right)_\infty}{\left(\frac{q}{a_j}, \frac{\widehat{a}^j}{a_j}, b; q\right)_\infty} \in \mathbb{C},$$

$$\frac{a_j q}{\underline{b}} := \left(\frac{a_j q}{b_1}, \dots, \frac{a_j q}{b_r}\right) \in \mathbb{C}^r.$$

Remark 4.15. — We have

$$\frac{\left(a_j z, \frac{q}{a_j z}; q\right)_\infty}{\left(z, \frac{q}{z}; q\right)_\infty} = \frac{\theta_q(-a_j z)}{\theta_q(-z)}.$$

Over the field of q -constants $\mathcal{M}(\mathbb{E}_q)$, the corresponding function is linearly equivalent to the function given by $a_j^{-\ell_q(z)}$.

By taking the limit $\underline{b} \rightarrow \underline{0}$ in the identity of Proposition 4.14, we obtain the following corollary.

COROLLARY 4.16. — *We have the following identity of analytic functions*

$${}_r\psi_r\left(\frac{\underline{a}}{\underline{0}} \middle| q, w\right) = \sum_{j=1}^r C'_j(q) \frac{\theta_q(-a_j w)}{\theta_q(-w)} {}_0\varphi_{r-1}\left(\frac{-}{\frac{a_j q}{\widehat{a}^j}} \middle| q, \frac{q^r a_j^{r-1}}{\pi(\widehat{a}^j)w}\right),$$

where

$$C'_j(q) := \left(q, \widehat{a}^j; q\right)_\infty \left(\frac{q}{a_j}, \frac{\widehat{a}^j}{a_j}; q\right)_\infty^{-1}.$$

Remark 4.17. — The motivation for this using this corollary is the observation that, denoting \mathcal{B}_q the q -Borel transform,

$$\mathcal{B}_{q^r} \psi_0\left(\frac{\underline{a}}{-} \middle| q, w\right) = {}_r\psi_r\left(\frac{\underline{a}}{\underline{0}} \middle| q, (-1)^r w\right).$$

We also notice that the statement of this corollary does not make sense if we were to set $\underline{a} = \underline{1}$, e.g. if we were doing equivariant limit in equivariant quantum K -theory.

Proof. — We have $\lim_{b_i \rightarrow 0} C_j(q) = C'_j(q)$ by the convention $(0; q)_d = 1$. The remaining computation relies on the observation that

$$\lim_{b_i \rightarrow 0} \left(\frac{a_j q}{b_i}; q\right)_d b_i^d = \lim_{b_i \rightarrow 0} \prod_{l=1}^d (b_i - a_j q^l) = (-1)^d a_j^d q^{\frac{d(d+1)}{2}}$$

Therefore, we have

$$\begin{aligned}
 {}_r\varphi_{r-1} \left(\begin{matrix} \frac{a_j q}{b} \\ \frac{a_j q}{\widehat{a}} \end{matrix} \middle| q, \frac{\pi(b)}{\pi(\widehat{a})w} \right) &= \sum_{d \geq 0} \frac{\left(\frac{a_j q}{b}; q \right)_d \pi(b)^d}{\left(\frac{a_j q}{\widehat{a}}; q \right)_d} \left(\frac{1}{\pi(\widehat{a})^d z^d} \right) \\
 &\rightarrow \sum_{d \geq 0} \frac{(-1)^{dr} a_j^{dr} q^{r \frac{d(d-1)}{2}} q^{dr}}{\left(\frac{a_j q}{\widehat{a}}; q \right)_d} \left(\frac{1}{\pi(\widehat{a})^d z^d} \right) \\
 &= \sum_{d \geq 0} \frac{1}{\left(\frac{a_j q}{\widehat{a}}; q \right)_d} \left((-1)^d q^{\frac{d(d-1)}{2}} \right)^r \left(\frac{q^r a_j^{r-1}}{\pi(\widehat{a}^j) z} \right)^d \\
 &= {}_0\varphi_{r-1} \left(\begin{matrix} - \\ \frac{a_j q}{\widehat{a}} \end{matrix} \middle| q, \frac{q^r a_j^{r-1}}{\pi(\widehat{a}^j)w} \right). \quad \square
 \end{aligned}$$

COROLLARY 4.18. — Let $\mathcal{L}_{q^r}^{[\lambda; q^r]}$ be the q^r -Laplace transform along the q -spiral $[\lambda; q^r]$. We have the following identity of analytic functions

$$\begin{aligned}
 &\left[\mathcal{L}_{q^r}^{[\lambda; q^r]} \psi_r \left(\frac{a}{0} \middle| q, (-1)^r w \right) \right] (x) \\
 &= \sum_{j=1}^r C'_j(q) \frac{\theta_q((-1)^{r+1} a_j \lambda)}{\theta_q((-1)^{r+1} \lambda)} \frac{\theta_{q^r} \left(\frac{\lambda}{a_j^r x} \right)}{\theta_{q^r} \left(\frac{\lambda}{x} \right)} {}_r\varphi_{r-1} \left(\begin{matrix} 0 \\ \frac{a_j q}{\widehat{a}} \end{matrix} \middle| q, \frac{1}{\pi(\widehat{a})x} \right).
 \end{aligned}$$

We recall that the q^r -Laplace transform along the q -spiral $[\lambda; q^r]$ of a function g is given by

$$\left[\mathcal{L}_{q^r}^{[\lambda; q^r]} g \right] (x) := \sum_{n \in \mathbb{Z}} \frac{g(\lambda q^{rn})}{\theta_{q^r} \left(\frac{\lambda q^{rn}}{x} \right)}.$$

The main idea of the proof of the corollary is to make a change of variable for the summation on the index $n \in \mathbb{Z}$ to make the Laurent series of the function θ_{q^r} appear.

Proof. — In the expression for the q^r -Laplace transform of the right hand side of Corollary 4.18, we use the identity (deduced from the q -difference equation satisfied by the theta function)

$$\theta_q(q^r x) = \frac{1}{q^{\frac{r(r-1)}{2}} x^r} \theta_q(x).$$

We therefore obtain

$$\begin{aligned} & \left[\mathcal{L}_{q^r}^{[\lambda; q^r]} \psi_r \left(\frac{a}{0} \middle| q, (-1)^{r+1} w \right) \right] (x) \\ &= \sum_{j=1}^r \sum_{n \in \mathbb{Z}} C'_j(q) \frac{1}{a_j^{rn}} \frac{\theta_q((-1)^{r+1} a_j \lambda)}{\theta_q((-1)^{r+1} \lambda)} \left(\frac{\lambda}{x} \right)^n q^{r \frac{n(n-1)}{2}} \frac{1}{\theta_{q^r} \left(\frac{\lambda}{x} \right)} \\ & \quad \times \sum_{d \geq 0} \frac{1}{\left(\frac{a_j q}{\widehat{a}}; q \right)_d} q^{r \frac{d(d-1)}{2}} \frac{q^{rd}}{q^{drn}} \frac{a_j^{(r-1)d}}{\pi \left(\widehat{a}^j \right)^d \lambda^d}. \end{aligned}$$

Multiplying all the terms of the form $q^{(\text{exponent})}$ together, we obtain

$$q^{r \frac{n(n-1)}{2} + r \frac{r(r-1)}{2} + dr - drn} = q^{r \frac{(n-d)(n-d-1)}{2}}.$$

Setting $n' = n - d$, we have

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \frac{1}{a_j^{rn}} \left(\frac{\lambda}{x} \right)^n q^{r \frac{(n-d)(n-d-1)}{2}} &= \left(\sum_{n' \in \mathbb{Z}} \left(\frac{\lambda}{a_j^r x} \right)^{n'} q^{r \frac{(n')(n'-1)}{2}} \right) \left(\frac{\lambda}{a_j^r x} \right)^d \\ &= \theta_{q^r} \left(\frac{\lambda}{a_j^r x} \right) \left(\frac{\lambda}{a_j^r x} \right)^d. \end{aligned}$$

Therefore,

$$\begin{aligned} & \left[\mathcal{L}_{q^r}^{[\lambda; q^r]} \psi_r \left(\frac{a}{0} \middle| q, (-1)^{r+1} w \right) \right] (x) \\ &= \sum_{j=1}^r C'_j(q) \frac{\theta_q((-1)^{r+1} a_j \lambda)}{\theta_q((-1)^{r+1} \lambda)} \frac{\theta_{q^r} \left(\frac{\lambda}{a_j^r x} \right)}{\theta_{q^r} \left(\frac{\lambda}{x} \right)} \\ & \quad \times \sum_{d \geq 0} \frac{1}{\left(\frac{a_j q}{\widehat{a}}; q \right)_d} \left(\frac{\lambda}{a_j^r x} \right)^d \frac{a_j^{(r-1)d}}{\pi \left(\widehat{a}^j \right)^d \lambda^d}. \end{aligned}$$

We conclude recognizing the q -hypergeometric series

$$\sum_{d \geq 0} \frac{1}{\left(\frac{a_j q}{\widehat{a}}; q \right)_d} \left(\frac{\lambda}{a_j^r x} \right)^d \frac{a_j^{(r-1)d}}{\pi \left(\widehat{a}^j \right)^d \lambda^d} = {}_r\varphi_{r-1} \left(\frac{0}{\widehat{a}} \middle| q, \frac{1}{\pi(\widehat{a})x} \right). \quad \square$$

Applying this corollary to the case of quantum K -theory gives the following statement below.

COROLLARY 4.19. — Write $\underline{\Lambda} := (\Lambda_0, \dots, \Lambda_N) \in K_{T^{N+1}}(\mathbb{P}^N)^{N+1}$. Let $\alpha \in \mathbb{C}^* - q^{\mathbb{Z}}$ and let $[\lambda; q^{N+1}]$ be a q^{N+1} -(discrete) spiral. We have the following identity of functions

$$\begin{aligned} & e_{q, \alpha^{-1}P}(w) \left[\mathcal{S}_{q^{N+1}}^{[\lambda; q^{N+1}]} \Big|_{N+1} \psi_0 \left(\frac{\underline{\Lambda}^{-1} \alpha^{-1} P}{-} \Big| q, (\alpha P^{-1})^{N+1} \pi(\underline{\Lambda}) w \right) \right] (w) \\ &= \sum_{j=0}^N \frac{\left(q, \frac{\alpha^{-1} P}{\widehat{\underline{\Lambda}}^j}; q \right)_{\infty}}{\left(q \alpha P^{-1} \Lambda_j, \frac{\Lambda_j}{\widehat{\underline{\Lambda}}^j}; q \right)_{\infty}} \frac{\theta_q \left((-1)^N \frac{\lambda \alpha^{-1} P}{\Lambda_j} \right)}{\theta_q \left((-1)^N \lambda \right)} \frac{\theta_{q^{N+1}} \left(\frac{\lambda \Lambda_j^{N+1}}{\pi(\underline{\Lambda}) w} \right)}{\theta_{q^{N+1}} \left(\frac{\lambda}{(\alpha P^{-1})^{N+1} \pi(\underline{\Lambda}) w} \right)} \\ & \quad \times e_{q, \alpha^{-1}P}(w) \Lambda_j^{\ell_q(\frac{1}{w})} \left\{ \Lambda_j^{-\ell_q(\frac{1}{w})} \Big|_{N+1} \varphi_N \left(\frac{0}{q \Lambda_j^{-1} \widehat{\underline{\Lambda}}^j} \Big| q, \frac{1}{w} \right) \right\}. \end{aligned}$$

Notice that in the right hand side of the above identity, the function between the curly brackets is the small J -function:

$$\Lambda_j^{-\ell_q(\frac{1}{w})} \Big|_{N+1} \varphi_N \left(\frac{0}{q \Lambda_j^{-1} \widehat{\underline{\Lambda}}^j} \Big| q, \frac{1}{w} \right) = J_{|P=\Lambda_j}^{K_{\text{th,eq}}} \left(q, \frac{1}{w} \right).$$

From this observation, we obtain the identity announced in Theorem 4.13.

Remark 4.20. — If we try to obtain a non equivariant version of the formula in Theorem 4.13, the formula does not make sense as we no longer have bases of solutions on left and right hand sides. Nonetheless, let us consider the ring $K_{T^{N+1}}(\mathbb{P}^N) \otimes K_{T^{N+1}}(\mathbb{P}^N)$, denoting by $P_{(0)}$ (resp. $P_{(\infty)}$) the generator on the left (resp. right) factor. We introduce the equivariant K -theoretic number

$$\begin{aligned} \mathbf{R}^{\text{eq}}(q, w) &:= \frac{(\alpha^{-1} P_{(\infty)} \underline{\Lambda}^{-1}; q)}{(\alpha^{-1} P_{(0)} \underline{\Lambda}^{-1}; q)} \frac{(q; q)_{\infty}^2}{(q \alpha P_{(\infty)}^{-1} P_{(0)}, \alpha^{-1} P_{(\infty)} P_{(0)}^{-1}; q)} \\ &\times \frac{\theta_q \left((-1)^N \frac{\lambda \alpha^{-1} P_{(\infty)}}{P_{(0)}} \right)}{\theta_q \left((-1)^N \lambda \right)} \frac{\theta_{q^{N+1}} \left(\frac{\lambda P_{(0)}^{N+1}}{\pi(\underline{\Lambda}) w} \right)}{\theta_{q^{N+1}} \left(\frac{\lambda}{(\alpha P_{(\infty)}^{-1})^{N+1} \pi(\underline{\Lambda}) w} \right)} e_{q, \alpha^{-1}P_{(\infty)}}(w) P_{(0)}^{\ell_q(\frac{1}{w})} \\ &\in K_{T^{N+1}}(\mathbb{P}^N) \otimes K_{T^{N+1}}(\mathbb{P}^N). \end{aligned}$$

Then, one can notice that $\mathbf{R}(q, w)|_{P_{(\infty)}=\Lambda_k, P_{(0)}=\Lambda_j} = R_{k,j}$, where $R_{k,j}$ is the equivariant connection number of Theorem 4.13. The non equivariant limit

of the number $\mathbf{R}(q, w)$ is well defined and given by

$$\begin{aligned} \lim_{\Delta \rightarrow 1} \mathbf{R}(q, w) &= \frac{(\alpha^{-1}P_{(\infty)}; q)_{\infty}^{N+1}}{(\alpha^{-1}P_{(0)}; q)_{\infty}^{N+1}} \frac{(q; q)_{\infty}^2}{(q\alpha P_{(\infty)}^{-1}P_{(0)}, \alpha^{-1}P_{(\infty)}P_{(0)}^{-1}; q)_{\infty}} \\ &\quad \times \frac{\theta_q\left((-1)^N \frac{\lambda \alpha^{-1}P_{(\infty)}}{P_{(0)}}\right)}{\theta_q((-1)^N \lambda)} \frac{\theta_{q^{N+1}}\left(\frac{\lambda P_{(0)}^{N+1}}{w}\right)}{\theta_{q^{N+1}}\left(\frac{\lambda}{(\alpha P_{(\infty)}^{-1})^{N+1} w}\right)} \\ &\quad e_{q, \alpha^{-1}P_{(\infty)}}(w) P_{(0)}^{\ell_q\left(\frac{1}{w}\right)}. \end{aligned} \quad (4.4)$$

We recall that a basis of solution in the non equivariant case is obtained by taking in the formula for the J -function the coefficient in front of $(1 - P^{-1})^j$ for $j = 0, \dots, n$. Therefore, we may expect the connection numbers in the non equivariant case to be obtained by looking at the coefficients in front of $(1 - P_{(\infty)})^k \otimes (1 - P_{(0)})^j$ in the right hand side of Equation (4.4), once it is decomposed in this basis of $K(\mathbb{P}^N)^{\otimes 2}$. Unfortunately, we are currently not able to write the identity Theorem 4.13 without any choice of basis in equivariant K -theory, thus we are not able to make such a non equivariant limit.

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