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Horocyclic and geodesic orbits on geometrically infinite surfaces with variable negative curvature [(*)]

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Abstract. — We study the behaviour of the horocyclic orbit of a vector on the unit tangent bundle of a geometrically infinite surface with variable negative curvature, when the corresponding geodesic ray is almost minimizing and the injectivity radius is finite.

Résumé. — On étudie le comportement de l'orbite horocyclique d'un vecteur sur le fibré tangent unitaire d'une surface géométriquement infinie à courbure négative variable, lorsque le rayon géodésique correspondant est presque minimisant et que le rayon d'injectivité est fini.

1. Introduction

Let M be an orientable geometrically infinite surface with a complete Riemanninan metric of negative curvature, and let $\widetilde{M}$ be its universal cover. Let us suppose that Γ is the fundamental group of M. We can see it as a subgroup of the group of orientation preserving isometries of $\widetilde{M}$, that is $\Gamma < \mathrm{Isom}^+(\widetilde{M})$. We can write then

$$M = \Gamma \backslash \widetilde{M}.$$

If T^1M and $T^1\widetilde{M}$ are the unit tangent bundles of M and $\widetilde{M}$ respectively, thus we can also write:

$$T^1M = \Gamma \backslash T^1\widetilde{M}.$$

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If the curvature of $\widetilde{M}$ has an upper bound $-\kappa^2$, with $\kappa > 0$, hence the geodesic flow on $T^1\widetilde{M}$, denoted by $g_{\mathbb{R}}$, happens to be an Anosov flow (see [2, Appendix]), and this flow descends to T^1M. The strong stable manifold for the geodesic flow defines a foliation (as we shall see in Section 2), which can be parametrized into a flow, which we denote by $h_{\mathbb{R}}$, the stable horocycle flow, and it also descends to T^1M. The properties we are interested in, do not depend on the parametrization of the flow.

A work of Hedlund from the 1930s shows that if the surface M has constant negative curvature and it is compact, then the horocycle flow is minimal on the unit tangent bundle. This means that the only closed non empty invariant set for the horocycle flow is T^1M (see [8]). In the 1970s, P. Eberlein extends this result for compact manifolds of variable negative curvature. His work coversproperties of horospheres on compact and non-compact manifolds, with negative curvature and also with non-positive curvature (see [7]). Later, F. Dal'Bo generalizes it to non-compact surfaces of variable negative curvature (see [6]). She also proves that if the fundamental group is finitely generated, all the horocycles on the non-wandering set are either dense or closed, motivating the interest for studying geometrically infinite surfaces. For a surface, being geometrically infinite is equivalent to having an infinitely generated fundamental group (see [6]).

A minimal set for a flow on a metric space A, is a subset of A which is closed, invariant by the flow, and does not contain any proper subset with these properties. When the metric space is compact, any flow always admits minimal sets. But this is not true for non-compact metric spaces. An example of a flow without minimal sets on a non-compact surface was given by T. Inaba (see [9]).

In the case of the horocycle flow defined on the unit tangent bundle of a non-compact surface, the first example of such a flow without minimal sets was given by M. Kulikov in 2004 (see [11]).

Later, S. Matsumoto studied a family of geometrically infinite surfaces of constant curvature, for which he proves that the horocycle flow on the unit tangent bundle does not admit minimal sets ([12]). Also Alcalde, Dal'Bo, Martinez and Verjovsky ([1]) studied this family of surfaces which appear as leaves of foliations, and proved the same result in this context. More recently, A. Bellis studied the links between geodesic and horcycle orbits for some geometrically infinite surface of constant negative curvature ([4]). In particular, his result implies Matsumoto's result.

Our aim was to determine whether or not the results of these last works are still valid if we don't have the hypothesis of constant curvature, and to provide arguments that do no depend on specific computations which are valid only in constant negative curvature.

Let us introduce the following definitions.

Definition 1.1. — *Let p be a point on a surface M. The* injectivity radius *at p is defined as*

$$\mathrm{Inj}(p) := \frac{1}{2} \min_{Id \neq \gamma \in \Gamma} d(\widetilde{p}, \gamma(\widetilde{p})),$$

where $\widetilde{p}$ is any lift of p to the universal cover $\widetilde{M}$ of M.

Definition 1.2. — *If $v \in T^1M$, the* geodesic ray $v[0,\infty)$ *is the projection of the future geodesic orbit $\{g_t(v) : t \in [0,\infty)\}$ of v on M. Also $v(t)$ will be the projection on M of $g_t(v) \in T^1M$.*

Definition 1.3. — *A geodesic ray $v[0,\infty)$ on M is said to be* almost minimizing *if there is a positive real number c such that*

$$d(v(t), v(0)) \geqslant t - c \quad \forall\, t \geqslant 0.$$

Definition 1.4. — *Let $v[0,\infty)$ be a geodesic ray. We define its injectivity radius as*

$$\underline{\mathrm{Inj}}(v[0,\infty)) := \liminf_{t\to\infty} 2\,\mathrm{Inj}(v(t)).$$

We prove the following theorem:

Theorem 1.5. — *Let M be an orientable geometrically infinite surface with a complete Riemanninan metric of negative curvature, and such that the curvature is bounded from above by a constant $-\kappa^2$. Let $v \in T^1M$ such that $v[0,\infty)$ is an almost minimizing geodesic ray with finite injectivity radius a, and such that $h_{\mathbb{R}}(v)$ is not closed. Therefore, there is an $r_0 \in (a, 2a)$ such that $g_{r_0}(v) \in \overline{h_{\mathbb{R}(\gtrapprox)}}$.*

Corollary 1.6. — *If $r_0 > 0$ is such that $g_{r_0}(v) \in \overline{h_{\mathbb{R}(\gtrapprox)}}$, then $g_{t_n}(v) \in \overline{h_{\mathbb{R}(\gtrapprox)}}$ for all $n \in \mathbb{N}$, where $t_n := nr_0$.*

Corollary 1.7. — *If the injectivity radius of $v[0,\infty)$ is 0, then $g_{\mathbb{R}^+}(v) \subset \overline{h_{\mathbb{R}(\gtrapprox)}}$.*

These results were proved by Bellis ([4]) for surfaces of constant negative curvature. Our proof takes some ideas of Bellis's proof, but introduces a different approach. Working in a constant curvature surface has the advantage of having the hyperbolic plane as its universal cover, whose geometry is well understood. All horocycles and geodesics are Euclidean circumferences and half circumferences respectively in this context. In the case of variable curvature, the universal cover is a Hadamard space, and horocycles and geodesics could have different behaviours. Some geometric formulas that work well in the hyperbolic plane can not be used in a general Hadamard space. Since the hyperbolic plane is a symmetric space, we can also apply the tools provided by the Lie theory, which allows us to express and solve geometric problems in algebraic terms.

The following result generalizes the one proved by Matsumoto in the context of constant negative curvature.

Definition 1.8. — *Let M be a noncompact Riemannian surface with negative curvature bounded from above by a constant $-\kappa^2$, and let Γ be its fundamental group. The surface M is* tight *if Γ only has hyperbolic elements, M can be written as*

$$M := M_1 \cup M_2 \cup \dots$$

where $M_n \subset M_{n+1}$ $\forall\, n$, and each M_n is a compact, not necessarily connected submanifold of M with boundary ∂M_n and the boundary components are closed geodesics whose lengths are bounded by some uniform constant $A \in \mathbb{R}$, and the limit set of Γ is the whole boundary $\partial_\infty \widetilde{M}$ of the universal covering space $\widetilde{M}$ of M (see Definition 2.7).

Corollary 1.9. — *If M is a tight surface, there are no minimal sets for the horocycle flow on T^1M.*

An important remark is that, although some of the tools we are going to use are valid in a more general context of manifolds of pinched negative curvature, we also need some arguments which are strongly dependent on M being a 2-manifold. This is the case of Propositions 3.4 and 3.5.

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2. Preliminaries

If $u \in T^1\widetilde{M}$, we denote by $g_{\mathbb{R}}(u)$ the geodesic passing through u in $T^1\widetilde{M}$, and by $u(\mathbb{R})$ the projection of this geodesic on $\widetilde{M}$. So that $u(t)$ will denote the projection of $g_t(u)$ on $\widetilde{M}$. We denote by $h_{\mathbb{R}}(u)$ the horocycle passing through u.

We will also denote by π the projection from T^1M to M, and by $\widehat{\pi}$ the projection from $T^1\widetilde{M}$ to T^1M.

2.1. Boundary at Infinity

The boundary at infinity is the set of endpoints of all the geodesic rays. Here we give a formal definition.

Definition 2.1. — *The geodesics directed by two vectors $\widetilde{v}$ and $\widetilde{v}' \in T^1\widetilde{M}$ have the same endpoint if*

$$\sup_{t>0} d\big(\widehat{\pi}(g_t(\widetilde{v})),\ \widehat{\pi}(g_t(\widetilde{v}'))\big) < \infty.$$

When this holds, we write: $\widetilde{v} \sim_* \widetilde{v}'$.

Definition 2.2. — *The* boundary at infinity *of $\widetilde{M}$ will be the set defined by*

$$\partial_\infty \widetilde{M} := T^1\widetilde{M}/ \sim_* .$$

For any $\widetilde{v} \in T^1\widetilde{M}$, we denote by $\widetilde{v}(\infty)$ its equivalence class by the relation $\sim_*$.

Given two different points ξ and η of $\partial_\infty \widetilde{M}$, we will denote by (ξ, η) the geodesic on $\widetilde{M}$ joining them.

The action of $\mathrm{Isom}^+(\widetilde{M})$ on $\widetilde{M}$ can be naturally extended to $\widetilde{M} \cup \partial_\infty \widetilde{M}$.

An isometry of $\mathrm{Isom}^+(\widetilde{M})$ is said to be:

- *hyperbolic* if it has exactly two fixed points and both of them lie on $\partial_\infty \widetilde{M}$.
- *parabolic* if it has a unique fixed point and it lies on $\partial_\infty \widetilde{M}$.
- *elliptic* if it has at least one fixed point in $\widetilde{M}$.

Every isometry $\mathrm{Isom}^+(\widetilde{M})$ is either hyperbolic, elliptic or parabolic (see [2, Section II.3]).

2.2. The Busemann function and horocycles

The Busemann function is one of the main tools we need to describe horocycles and their properties. In this section we show how to construct this function.

Definition 2.3. — *For $x, y, z \in \widetilde{M}$ we can define*

$$\begin{aligned} b : \widetilde{M} \times \widetilde{M} \times \widetilde{M} &\longrightarrow \mathbb{R} \\ x, y, z &\longmapsto d(x, y) - d(x, z) \end{aligned}$$

As d is a continuous function, b is also continuous.

Remark 2.4. — Some properties of b that can be deduced from the definition are:

(1) $b(x, y, y) = 0$ for all $x, y \in \widetilde{M}$,
(2) $|b(x, y, z) - b(x, y, z')| \leqslant d(z, z')$, for all $x, y, z, z' \in \widetilde{M}$
(3) $b(x, y, z) = b(x, y, z') + b(x, z', z)$ for all $x, y, z, z' \in \widetilde{M}$.

See [2, Chapter II.1] for proof.

Definition 2.5. — *Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $\widetilde{M}$. Given $\xi \in \partial_\infty \widetilde{M}$, consider a fixed point $o \in M$ and the geodesic ray $u[0, \infty)$ with $u(0) = o$ and $u(\infty) = \xi$. Consider also the geodesic rays $u_n[0, \infty)$ such that $u_n(0) = o$ and $u_n(t_{j_n}) = x_n$ for some $t_{j_n} \to \infty$, and define $\xi_n := u_n(\infty)$. The sequence x_n converges to ξ if $\xi_n \to \xi$ in $\partial_\infty \widetilde{M}$.*

Given $x, y \in \widetilde{M}$, the map $b_y(x) : \widetilde{M} \to \mathbb{R}$, defined as

$$b_y(x)(z) := b(x, y, z), \ z \in \widetilde{M}$$

is a continuous function on $\widetilde{M}$. Let us consider $C(\widetilde{M})$ the space of continuous function on $\widetilde{M}$ with the topology of the uniform convergence on bounded sets. Let $\{x_n\}$ be a sequence on $\widetilde{M}$ such that $x_n \to \xi \in \partial_\infty \widetilde{M}$. Thus $b_y(x_n)$ converges on $C(\widetilde{M})$ to some function $B_\xi(y, \ )$. This is called the *Busemann function* at ξ, based at y. Explicitly, the Busemann function is defined as follows:

$$B_\xi(y, z) := \lim_{x_n \to \xi} d(x_n, y) - d(x_n, z), \tag{2.1}$$

where x_n is a sequence on $\widetilde{M}$ that goes to ξ.

This definition is independent of the choice of the sequence x_n (see [2, Chapter II.1]).

For all $\xi \in \partial_\infty \widetilde{M}$, we define $B_\xi^y : \widetilde{M} \to \mathbb{R}$, where $B_\xi^y(z) = B_\xi(y, z)$ for all $z \in \widetilde{M}$, is a continuous function, and the level set $(B_\xi^y)^{-1}(t)$ is a regular curve (meaning that it admits a C^1 arclength parametrization) for all $t \in \mathbb{R}$ (see [2, Chapter IV.3]). Let us denote by $H_y(\xi, t)$ the level set $(B_\xi^y)^{-1}(t)$, and for each $p \in H_y(\xi, t)$ consider the only vector $\widetilde{v}_\xi^p \in T_p^1\widetilde{M}$ such that $\widetilde{v}_\xi^p(\infty) = \xi$. Therefore set

$$\widehat{H}_y(\xi, t) := \left\{\widetilde{v}_\xi^p : p \in H_y(\xi, t)\right\}$$

is the horocycle through $\widetilde{v}_\xi^p$ in $T^1\widetilde{M}$, for all $p \in H_y(\xi, t)$. By definition $\pi(\widehat{H}_y(\xi, t)) = H_y(\xi, t)$, and $\widehat{H}_y(\xi, t) \subset T^1\widetilde{M}$ is the strong stable set of $\widetilde{v}_\xi^p$ for the geodesic flow, which can also be parametrized by arclength. Now, the horocycle flow $h_s(v)$, pushes a vector v along its strong stable manifold, through an arc of length s.

Given two elements u and v in $T^1\widetilde{M}$, let $z_u = u(0)$ and $z_v = v(0)$ be their respective base points in $\widetilde{M}$. Let us suppose that there are $t, s \in \mathbb{R}$ such that $g_t(u) = h_s(v)$ or, in other words, there is $t \in \mathbb{R}$ such that $g_t(h_{\mathbb{R}}(u)) = h_{\mathbb{R}}(v)$. In this case $u(\infty) = v(\infty)$. Let us suppose that $u(\infty) = v(\infty) = \xi \in \partial_\infty\widetilde{M}$. The Busemann function centered in ξ evaluated at (z_u, z_v) happens to be the real number t mentioned above. We denote it by

$$B_\xi(z_u, z_v) = t.$$

Remark 2.6. — If $u \in T^1\widetilde{M}$ is such that $u(0) = o$ and $u(\infty) = \xi$. Thus

$$B_\xi(o, z) = \lim_{t\to\infty} [d(o, u(t)) - d(z, u(t))] = \lim_{t\to\infty} t - d(z, u(t)).$$

2.3. Limit set and classification of limit points

The limit set is a special subset of the boundary at infinity. We classify limits points, and show their links with the behaviour of horocyclic orbits.

Definition 2.7. — *The* limit set $L(\Gamma)$ *of the group* Γ*, is the set of accumulation points of an orbit* Γz*, for some* $z \in \widetilde{M}$*. This is well defined because all orbits have the same accumulation points (see* [3, Chapter 1.4]*).*

One has $L(\Gamma) \subset \partial_\infty\widetilde{M}$. Otherwise, a sub-sequence of the orbit would remain on a compact region of $\widetilde{M}$, contradicting the fact that Γ acts discontinuously on $\widetilde{M}$.

In the limit set, we can distinguish two different kinds of points:

Definition 2.8. — *A limit point* $\xi \in \partial_\infty\widetilde{M}$ *is said to be* horocyclic *if given any* $z \in \widetilde{M}$*, and* $t \in \mathbb{R}$*, there is* $\gamma \in \Gamma$ *such that* $B_\xi(o, \gamma(z)) > t$*. Otherwise, the point* ξ *is a* nonhorocyclic *limit point. (See Figure 2.1.)*

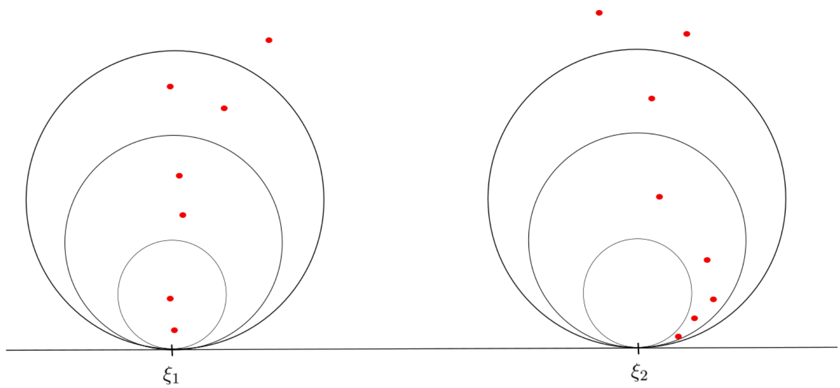

Figure 2.1. Here, ξ_1 is an horocyclic limit point. The points, representing some elements of the Γ-orbit of a point, reach all the horodisks based at ξ_1. The point ξ_2 on the other hand would be a nonhorocyclic limit point, as no element of the orbit reaches the smaller horodisk.

The sets of the form

$$\left\{z \in \widetilde{M} : B_\xi(o, \gamma(z)) > t\right\}, \quad t \in \mathbb{R}$$

are called *horodisks* based on ξ. So, in other words, a limit point ξ is horocyclic if each horodisk based on ξ intersects the orbit Γz, for all $z \in \widetilde{M}$.

Remark 2.9. — Given a point $\xi \in \partial_\infty \widetilde{M}$, if there is a sequence $\{\gamma_n\} \subset \Gamma$ such that $B_\xi(o, \gamma_n^{-1}(o)) \xrightarrow[n\to\infty]{} \infty$ for any $o \in \widetilde{M}$, then ξ is an horocyclic limit point.

This is because if $B_\xi(o, \gamma_n^{-1}(o)) \xrightarrow[n\to\infty]{} \infty$, any horodisk $\{z \in \widetilde{M} : B_\xi(o, \gamma(z)) > t\}$ will contain an element of the Γ-orbit of o.

We denote by Λ_Γ the image by $\widehat{\pi}$ of the set $\{\widetilde{v} \in T^1\widetilde{M} : \widetilde{v}(\infty) \in L(\Gamma)\}$.

Proposition 2.10. — *If $\xi \in L(\Gamma)$ is an horocyclic limit point, for all $\widetilde{v} \in T^1\widetilde{M}$ such that $\widetilde{v}(\infty) = \xi$, and $\widehat{\pi}(h_{\mathbb{R}}(\widetilde{v}))$ is dense in Λ_Γ.*

The proof of this proposition can be found in [6, Proposition B].

Given $v \in T^1M$ (or $T^1\widetilde{M}$), we denote by $v[0, \infty)$ the geodesic ray $g_t(v)_{t\in[0,\infty)}$.

2.4. Almost minimizing geodesic rays

The following proposition relates the behaviour of a geodesic ray with its endpoint.

First, we recall the following definition:

Definition 2.11. — *A geodesic ray $v[0,\infty)$ on M is said to be* almost minimizing *if there is a positive real number c such that for all $t \geqslant 0$,*

$$d(v(t), v(0)) \geqslant t - c.$$

(See Figure 2.2.)

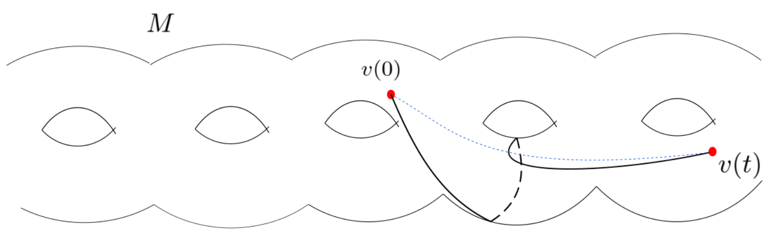

Figure 2.2. The projected geodesic ray starts at $v(0)$ and at time t passes through the point $v(t)$. The distance between this two points is less than $t-c$. The blue dotted line represents the minimizing geodesic joining $v(0)$ and $v(t)$, which obviously has length t.

Proposition 2.12. — *Let $\xi \in L(\Gamma)$ and $\widetilde{v} \in T^1\widetilde{M}$ such that $\widetilde{v}(\infty) = \xi$. Thus, the projected geodesic ray $v[0,\infty)$ over M, is almost minimizing if an only if ξ is a nonhorocyclic limit point.*

The proof of Proposition 2.12 is due to P. Eberlein.

Proof. — Take a reference point o and suppose without loss of generality that $\widetilde{v} \in T^1_o\widetilde{M}$. Let us suppose first that $\xi = \widetilde{v}(\infty)$ is a nonhoroyclic limit point. Hence, there is a horodisk H based at ξ that does not contain any point of the Γ-orbit of o. Let us take $H = \{z \in \widetilde{M} : B_\xi(o,z) > k$, with $k > 0\}$. For all $\gamma \in \Gamma$, one has $B_\xi(o,\gamma(o)) \leqslant k$. And because of Remark 2.6, this means that

$$\lim_{t\to\infty}[d(o,\widetilde{v}(t)) - d(\gamma(o),\widetilde{v}(t))] = \lim_{t\to\infty}[t - d(\gamma(o),\widetilde{v}(t))] \leqslant k.$$

This limit is not decreasing, and hence for any $t > 0$:

$$t - d(\gamma(o),\widetilde{v}(t)) \leqslant k,$$

and then

$$d(\gamma(o), \widetilde{v}(t)) \geqslant t - k.$$

As this happens for any $\gamma \in \Gamma$, down on the surface M, this implies that

$$d(v(0), v(t)) \geqslant t - k,$$

which means that $v[0, \infty)$ is an almost minimizing geodesic ray. All this implications can be reverted to prove that an almost minimizing geodesic ray, is projected from a geodesic ending on a nonhorocyclic limit point. □

2.5. Links between geodesic and horocyclic orbits

In this section, M will be an orientable geometrically infinite surface with a complete Riemanninan metric of negative curvature bounded from above by $-\kappa^2$, $\widetilde{M}$ its universal cover and Γ its fundamental group.

As we mentioned before, the geodesic flow on T^1M is an Anosov flow (see [10]), and the stable manifolds, which are contracted by this flow (see [2, Chapter IV]), have the level sets of the Busemann functions as their projections to M.

The strong stable manifold of the geodesic flow is defined as follows:

Definition 2.13. — *Consider the geodesic flow $g_t : T^1M \to T^1M$, and take $v \in T^1M$. Then the* strong stable manifold *of v will be the set*

$$W^s(v) := \left\{u \in T^1M : \ d(g_t(v), g_t(u)) \xrightarrow[t \to \infty]{} 0\right\}.$$

3. Geometric properties of horocycles

In this section, we prove some geometric properties of horocycles and the Busemann function, which are going to be useful tools in the proof of Theorem 1.5.

First, we introduce some additional notation: if $\gamma \in \Gamma$ is an hyperbolic isometry (γ^-, γ^+) is the axis of γ, where $\gamma^-, \gamma^+ \in \partial_\infty \widetilde{M}$ are its fixed points, we will denote by $\mathscr{C}_\gamma(p)$ the curve passing through p whose points are at a constant distance from (γ^-, γ^+). In general, if $c : \mathbb{R} \to \widetilde{M}$ is a geodesic, then $\mathscr{C}_c(p)$ will be the curve passing through p whose points are at a constant distance from $c(\mathbb{R})$.

For any regular connected curve $\mathscr{C}$, and p, $q \in \mathscr{C}$, we will write $[p, q]_\mathscr{C}$ to denote the arc contained in $\mathscr{C}$ joining p and q.

Finally, for $\xi \in \partial_\infty \widetilde{M}$ and $p \in \widetilde{M}$, the horocycle based at ξ passing through p will be denoted by $\mathscr{H}_\xi(p)$.

The tools for proving the following proposition can be found in [5, Chapter 9].

Proposition 3.1. — *The distance from a point $p \in \widetilde{M}$ and any closed submanifold $\widetilde{N}$ of $\widetilde{M}$ is attained by a minimizing geodesic which is orthogonal to $\widetilde{N}$.*

Proposition 3.2. — *For any hyperbolic isometry $\gamma \in \Gamma$ and any $p \in \widetilde{M}$, the curve $\mathscr{C}_\gamma(p)$ is a closed submanifold of $\widetilde{M}$.*

Proof. — Give parametrizations $a : \mathbb{R}\widetilde{M}$ and $c : \mathbb{R} \to \widetilde{M}$ of $\mathscr{C}_\gamma(p)$ and the axis of γ respectively, in such a way that $d(a(t), c(t)) = d(p, c(\mathbb{R}))$ for all $t \in \mathbb{R}$. The existence of such parametrizations of these curves, is due to the fact that the curvature is negative, Gauss–Bonnet theorem and Proposition 3.1. In fact, given a point q in $\mathscr{C}_\gamma(p)$, there is an unique t such that $d(q, c(t)) = d(q, c(\mathbb{R}))$. Otherwise, if there are two times t_1 and t_2 such that $d(q, c(t_1)) = d(q, c(t_2)) = d(q, c(\mathbb{R}))$, as the geodesic segments joining q with $c(t_1)$ and $c(t_2)$ are orthogonal to $c(\mathbb{R})$, according to Proposition 3.1, we would have a geodesic triangle with two inner angles equal to $\frac{\pi}{2}$, and because of Gauss–Bonnet theorem, this cannot happen if the curvature is negative. On the other hand, if we have two ponts q_1 and q_2 of $\mathscr{C}_\gamma(p)$ and a real number t such that $d(q_1, c(t)) = d(q_2, c(t))$, as the geodesic segments joining q_1 and q_2 with $c(t)$ are orthogonal to $c(\mathbb{R})$ at $c(t)$, and as we are in dimension 2, it must be $q_1 = q_2$.

As $c(\mathbb{R})$ is a geodesic, it is a submanifold of $\widetilde{M}$, and then there are charts $\varphi_i : I_i \to c(\mathbb{R})$. We define the map: $f : c(\mathbb{R}) \to a(\mathbb{R})$ by $f(c(t)) = a(t)$. Then, the maps $f \circ \varphi_i : I_i \to a(\mathbb{R})$ form an atlas of charts of $a(\mathbb{R})$. It follows that $a(\mathbb{R}) = \mathscr{C}_\gamma(p)$ is a submanifold of $\widetilde{M}$.

Let us see that $\mathscr{C}_\gamma(p)$ is also closed in $\widetilde{M}$. Let d be the distancce from p to the axis of γ. Suppose there is a sequence of times $\{t_n\}_{n\in\mathbb{N}}$ such that $a(t_n)$ converges to a point q which does not belong to $\mathscr{C}_\gamma(p)$. The sequence $\{t_n\}_{n\in\mathbb{N}}$ cannot be bounded, otherwise, there would be a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ converging to some T, and then $a(t_n)$ converges to $a(T) \in \mathscr{C}_\gamma(p)$. Then, the sequence $\{t_n\}_{n\in\mathbb{N}}$ must be unbounded. Therefore, there is n_0 such that for all $n > n_0$, the point $a(t_n)$ belongs to $B(q, d)$, and hence $c(t_n)$ belongs to $B(q, 2d)$. This is absurd, as $c(t_n)$ is unbounded in $\widetilde{M}$. Thus $\mathscr{C}_\gamma(p)$ contains all its accumulation points and it is closed. □

Corollary 3.3. — *Let $c : \mathbb{R} \to \widetilde{M}$ be a geodesic, and $p \in \widetilde{M}$. Give a parametrization $a(t)$ of $\mathscr{C}_c(p)$, such that $d(a(t), c(t)) = d = d(p, c(\mathbb{R}))$. Then, the geodesic joining $a(t)$ and $c(t)$ is orthogonal to both $c(\mathbb{R})$ and $a(\mathbb{R})$.*

Proof. — As the point $a(t)$ is at distance d from $c(\mathbb{R})$ and by Proposition 3.1, the geodesic joining $a(t)$ and $c(t)$ is orthogonal to $c(\mathbb{R})$. If we show that the point $c(t)$ is at a distance d from $a(\mathbb{R})$, then this geodesic will also be orthogonal to $a(\mathbb{R})$.

Suppose that $d(c(t), a(\mathbb{R})) = d' < d$, then there is t' such that $d(a(t'), c(t)) = d' < d$. But all points of $\mathscr{C}_c(p)$ are at a distance d from $c(\mathbb{R})$, hence it must be $d(c(t), a(\mathbb{R})) = d$. ☐

Proposition 3.4. — *Consider $\zeta \in \partial_\infty \widetilde{M}$, a point $p \in \widetilde{M}$, and $c : \mathbb{R} \to \widetilde{M}$ a geodesic. Then:*

(1) *$\mathscr{C}_c(p) \cap \mathscr{H}_\zeta(p)$ contains at most two points.*
(2) *Given a parametrization $a(t)$ of $\mathscr{C}_c(p)$ with $a(0) = p$, and up to changing the orientation of this curve, the Busemann function $B^p_\zeta(a(t))$ is an increasing function of t and $B^p_\zeta(a(t)) > 0$ for all $t > 0$.*

Proof. — First, let us give a parametrization $a(t)$ of the curve $\mathscr{C}_c(p)$ such that $d(a(t), c(t))$ is constant and equal to $l = d(p, c(\mathbb{R}))$ for each $t \in \mathbb{R}$, where $p = a(0)$. Let $b_t : [0, l] \to \widetilde{M}$ the geodesic joining $a(t)$ and $c(t)$ (see, Figure 3.1). Now we write $\dot{b}_t(s) = \frac{\partial b_t}{\partial s}(s)$, with $s \in [0, l]$. Also $\dot{a}$ and $\dot{c}$ will refer to $\frac{\partial a}{\partial t}$ and $\frac{\partial c}{\partial t}$ respectively. It follows that $\langle \dot{b}_t(0), \dot{a}(t)\rangle = \langle \dot{b}_t(l), \dot{c}(t)\rangle = 0$. This holds since the curves parametrized by $a(t)$ and $c(t)$ are closed submanifolds of $\widetilde{M}$, and in view of Corollary 3.3, the distance between them is assumed by a geodesic perpendicular to both $c(\mathbb{R})$ and $a(\mathbb{R})$.

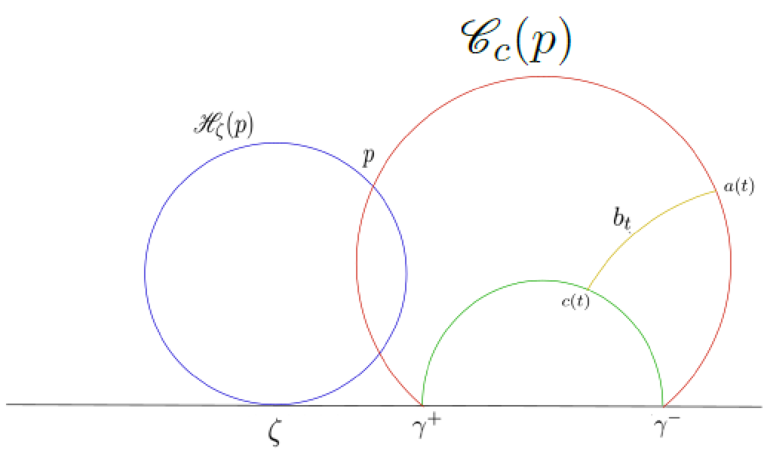

Figure 3.1.

If we look at the Busemann function $B^p_\zeta(a(t))$ along the curve $a(t)$, where $B^p_\zeta(z) := B_\zeta(p,z)$ for all $z \in \widetilde{M}$, we see that its derivative vanishes if and only if $\langle \nabla B^p_\zeta(a(t)), \dot{a}(t)\rangle = 0$, as the directional derivative of a function is zero if and only if the gradient of the function is orthogonal to the direction of the derivative. We are going to show that this derivative vanishes at most for one value of t. If we show this, then $\mathscr{C}_c(p)$ can only meet a level set of B^p_ζ at most two times, as we want to prove.

Suppose then that there is t_1 such that $\langle \nabla B^p_\zeta(a(t_1)), \dot{a}(t_1)\rangle = 0$. Then, $\nabla B^p_\zeta(a(t_1)) = K\dot{b}_{t_1}(0)$ for some $K \in \mathbb{R}$, as $\dot{b}_{t_1}(0) \perp \dot{a}(t_1)$. Then, the geodesic ray directed by $\dot{b}_{t_1}(0)$ (or $-\dot{b}_{t_1}(0)$) has the same endpoint as $\nabla B^p_\zeta(a(t_1))$, which is ζ.

Suppose now that there is an other t_2 for which $\langle \nabla B^p_\zeta(a(t_2)), \dot{a}(t_2)\rangle = 0$, then $\nabla B^p_\zeta(a(t_2)) = \widehat{K}\dot{b}_{t_2}(0)$, for some $\widehat{K} \in \mathbb{R}$, and we can assume as well that geodesic ray directed by $\dot{b}_{t_2}(0)$ has endpoint ζ. Then, the geodesic triangle with vertices ζ, $c(t_1)$ and $c(t_2)$ would have two right angles, and an angle equal to 0, contradicting Gauss–Bonnet theorem: in fact, the integral of the curvature of the surface on the interior region of the triangle, equals π minus the sum of the interior angles of the triangle (see [13, Chapter 7] for a proof). As our surfaces has negative curvature, this integral should be strictly negative, so the interior angles of the triangle cannot have sum equal to π. Then, the derivative of $B^p_\zeta(a(t))$ can only vanish for one value of t, as we wanted to see.

Now we are going to prove the second statement. As the Busemann function and $a(t)$ are continuous, $B^p_\zeta(a(t))$ is continuous as well. On one hand we have $B^p_\zeta(a(0)) = 0$, and on the other hand $\mathscr{C}_c(p)$ only meets at most in one other point the set of level 0 of B^p_ζ. Then, choosing the appropriate orientation for $a(t)$, we conclude that $B^p_\zeta(a(t))$ is an increasing function of t with $B^p_\zeta(a(0)) = 0$. This concludes the proof of Proposition 3.4. □

Proposition 3.5. — *Consider $\zeta, \eta \in \partial_\infty \widetilde{M}$ with $\zeta \neq \eta$, and $p, q \in \widetilde{M}$. Then:*

1. *The set $\mathscr{H}_\eta(q) \cap \mathscr{H}_\zeta(p)$ contains at most two points.*
2. *Let z be a point of the set $\mathscr{H}_\eta(q) \cap \mathscr{H}_\zeta(p)$, and $a : \mathbb{R} \to \widetilde{M}$ be a parametrization of $\mathscr{H}_\eta(q)$ such that $a(0) = z$. Then, up to changing the orientation of $a(t)$, we can assume that $B^p_\zeta(a(t))$ is an increasing function of t and $B^p_\zeta(a(t)) > 0$ for all $t > 0$.*

Proof. — As in Proposition 3.4, to show the first statement we are going to see that the derivative of B^p_ζ along the curve $\mathscr{H}_\eta(q)$ vanishes at most in one point, where $B^p_\zeta(z) = B_\zeta(p,z)$ for all $z \in \widetilde{M}$. And then, $\mathscr{H}_\eta(q)$ can meet a level set of B^p_ζ in at most two points.

We first give an arc length parametrization $a(t)$ to the curve $\mathscr{H}_\eta(q)$, such that $q = a(0)$. The derivative of $B^p_\zeta(z)$ on the direction of $a(t)$ vanishes on a point $a(t_0)$ if and only if $\langle \nabla B^p_\zeta(a(t_0)), \dot{a}(t_0)\rangle = 0$. Then $\nabla B^p_\zeta(a(t_0))$ is normal to the curve $\mathscr{H}_\eta(q)$, and then there is $k \in \mathbb{R}$ such that $\nabla B^p_\zeta(a(t_0)) = k\nabla B^q_\eta(a(t_0))$, since the gradient of the Busemann function B^q_η is perpendicular to its level set.

Then, the geodesic directed by the vector $\nabla B^p_\zeta(a(t_0))$ has endpoints ζ and η, since the gradient of the Busemann function B^q_η is parallel to the geodesic joining q and η (see [2, Proposition 3.2]). If there is an other point $a(t_1)$ such that $\langle \nabla B^p_\zeta(a(t_1)), \dot{a}(t_1)\rangle = 0$, then the geodesic directed by the vector $\nabla B^p_\zeta(a(t_1))$ would be the geodesic joining η and ζ. As the geodesic joining two points is unique, this geodesic would be meeting $\mathscr{H}_\eta(q)$ in two points: $a(t_0)$ and $a(t_1)$. But a geodesic ending at η only can meet a level set of B^q_η once, as B^q_η is increasing (or decreasing) along geodesics having η as one of its endpoints.(1) Then, $a(t_0) = a(t_1)$, as we wanted.

The proof of the second statement is analogous to the second statement of Proposition 3.4 (1), using that, up to changing orientation of γ, the derivative of B^p_ζ is positive along the arc of the curve $\mathscr{H}_\eta(q)$ containing z and $\gamma(z)$. ☐

4. Proof of Theorem 1.5

We are now going to prove Theorem 1.5 in several steps. First, we remind the statement of the theorem:

Let M be an orientable geometrically infinite surface with a complete Riemanninan metric of negative upper bounded curvature. Let $v \in T^1M$ such that $v[0,\infty)$ is an almost minimizing geodesic ray with finite injectivity radius a, and such that $h_{\mathbb{R}}(v)$ is not closed. Then there is a sequence of times τ_n going to ∞ such that $g_{\tau_n}(v) \in \overline{h_{\mathbb{R}(\precapprox)}}$ for all n. Moreover, the set $\mathcal{I} = \{t \in \mathbb{R} : g_t(v) \notin \overline{h_{\mathbb{R}(\precapprox)}}\}$ only contains intervals of bounded length.

Let $v \in T^1M$ be as in the hypothesis of Theorem 1.5, this is that $v[0,\infty)$ is an almost minimizing geodesic ray, with finite injectivity radius a, and such that $h_{\mathbb{R}}(v)$ is not a closed horocycle. Let $\widetilde{v}$ be a lift of v on $T^1\widetilde{M}$. We will call ξ the point $\widetilde{v}(\infty) \in \partial_\infty \widetilde{M}$.

(1) This is also a consequence of Gauss–Bonnet theorem, as a triangle could not have an angle equal to π.

Lemma 4.1. — *There is a sequence* $\{\widetilde{v}_n\}_{n\in\mathbb{N}} \subset T^1\widetilde{M}$ *such that:*

(1) $\widetilde{v}_n(0) = \gamma_n(\widetilde{v}(0))$ *for some* $\gamma_n \in \Gamma$.
(2) $\widetilde{v}_n(\infty) = \widetilde{v}(\infty) = \xi$.
(3) $v_n \xrightarrow[n\to\infty]{} v$ *in* T^1M, *where* v_n *are the projected vectors of* $\widetilde{v}_n$ *on* T^1M.
(4) $B_\xi(\widetilde{v}_n(0), \widetilde{v}(0)) \in [a, 2a]$ *for all* $n \in \mathbb{N}$.

Proof of Lemma 4.1. — From the definition of injectivity radius (1.4), as $\underline{\mathrm{Inj}}(v[0,\infty)) = a$, there is a sequence $\{t_n\}$ going to ∞ such that $\mathrm{Inj}(v(t_n)) \xrightarrow[n\to\infty]{} a$. Then, given $\epsilon > 0$, for large values of n it is $\mathrm{Inj}(v(t_n)) < a + \epsilon$. Then, if $\widetilde{v} \in T^1\widetilde{M}$ is a lift of v, as $\widetilde{v}(t_n) \in \widetilde{M}$ is a lift of $v(t_n)$, there must be an isometry $\gamma_n \in \Gamma$ such that $d(\widetilde{v}(t_n), \gamma_n(\widetilde{v}(t_n))) < a + \epsilon$. Then, the geodesic $\widetilde{\alpha}_n$ joining $\widetilde{v}(t_n)$ and $\gamma_n(\widetilde{v}(t_n))$, projects to a closed curve α_n in M.

Now, we are going to construct the sequence $\{\widetilde{v}_n\}$ as follows: consider the curve β_n^T obtained by concatenation of $v[0, t_n]$, α_n, and $v[t_n, T]$ in that order, where T is any large number. This curve β_n^T is not necessarily a geodesic. We call then $\widehat{\beta}_n^T$ the geodesic joining $v(0)$ and $v(T)$ which is homotopic to β_n^T relative to the endpoints. Now, as T goes to ∞, $\widehat{\beta}_n^T$ converges to a geodesic ray $v_n[0,\infty)$ starting at $v(0)$ which is asymptotic to $v[0,\infty)$ (see figures below). Let us see that $v_n \xrightarrow[n\to\infty]{} v$ in T^1M. In fact, if $\widetilde{v}$ is a lift of v with $\widetilde{v}(\infty) = \xi$, then it suffices to show that $\gamma_n(\xi) \xrightarrow[n\to\infty]{} \xi$. As $d(\widetilde{v}(t_n), \gamma_n(\widetilde{v}(t_n))) \xrightarrow[n\to\infty]{} a < \infty$ and $\widetilde{v}(t_n) \xrightarrow[n\to\infty]{} \xi$, then $\gamma_n(\widetilde{v}(t_n)) \xrightarrow[n\to\infty]{} \xi$. The points $\gamma_n(\widetilde{v}(t_n))$ belong to geodesic rays which endpoints are $\gamma_n(\xi)$. If $\gamma_n(\xi) \xrightarrow[n\to\infty]{} \eta \neq \xi$, then, given $\epsilon > 0$ such that it does not belong to $(\eta - \epsilon, \eta + \epsilon)$, for large values of n, the point $\gamma_n(\xi)$ belongs to $(\eta - \epsilon, \eta + \epsilon)$. Then, $\gamma_n(\widetilde{v}(t_n))$ should converge to some point in $(\eta - \epsilon, \eta + \epsilon)$, which does not occur.

Then, $\gamma_n(\xi) \xrightarrow[n\to\infty]{} \xi$.

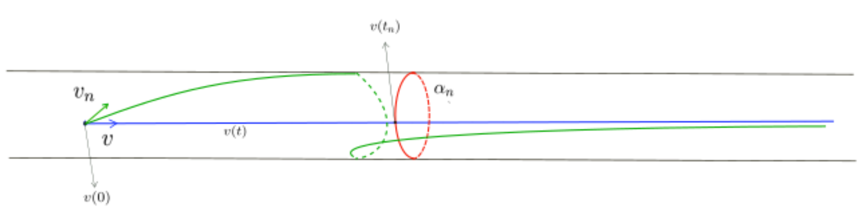

Figure 4.1. Geodesics α_n, $v[0,\infty)$ and $v_n[0,\infty)$ on the surface M.

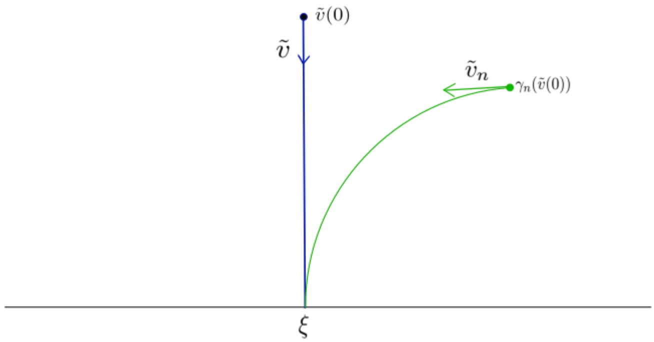

Figure 4.2. Geodesics $\widetilde{v}[0,\infty)$ and $\widetilde{v}_n[0,\infty)$ on the surface $\widetilde{M}$.

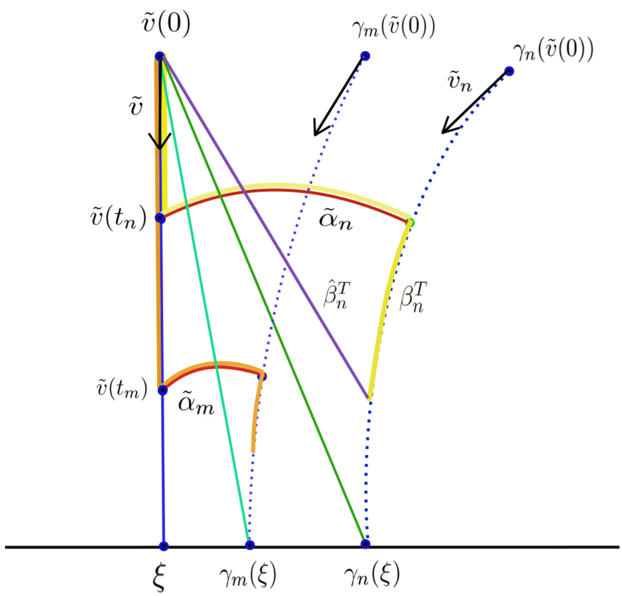

Figure 4.3. Here we see vectors $\widetilde{v}$ and $\widetilde{v}_n$. Also β_n^T (starting at $\widetilde{v}(0)$, going through $\widetilde{\alpha}_n$) and continuing through a segment contained in $\widetilde{v}_n[0,\infty)$ and β_m^T (starting at $\widetilde{v}(0)$, going through $\widetilde{\alpha}_m$) and continuing through a segment contained in $\widetilde{v}_m[0,\infty)$, with $m > n$, both ending at time T. A lift of $\widehat{\beta}_n^T$ is displayed as the geodesic segment joining $\widetilde{v}(0)$ and the endpoint of β_n^T. The geodesic ray joining $\widetilde{v}(0)$ and $\gamma_n(\xi)$ would be the lift of $v_n[0,\infty)$), and the one joining $\widetilde{v}(0)$ and $\gamma_m(\xi)$ would be the lift of $v_m[0,\infty)$.

Now consider the lift $\widetilde{v}$ of v, which has basepoint $\widetilde{v}(0)$, and let $\widetilde{v}(\infty) = \xi$. Then there is a lift $\widetilde{v}_n$ of v_n which also has endpoint ξ, because v and v_n are asymptotic. But $v_n[0,\infty)$ is the limit of $\widehat{\beta}_n^T$, and it is homotopic to β_n^T. By construction of β_n^T, a lift $\widetilde{\beta}_n^T$ of this curve that starts at $\widetilde{v}(0)$ must end at $\gamma_n(\widetilde{v}(T))$. Then $\widetilde{v}_n(0)$ must be $\gamma_n(\widetilde{v}(0))$ (see figure 4). And then $\widetilde{v}_n$ satisfies statements (1) and (2) of Lemma 4.1.

Let us see now that $B_\xi(\gamma_n(\widetilde{v}(0)), \widetilde{v}(0))$ is bounded. Given $\epsilon > 0$, and up to taking some positive power of γ_n, we can assume that the length of α_n is between $a + \epsilon$ and $2(a+\epsilon)$. Because of the construction of v_n, the length of α_n is the difference of lengths between $v[0,T]$ and β_n^T. As $\widehat{\beta}_n^T$ is a geodesic with the same initial and endpoints as β_n^T and in the same homotopy class, it is shorter than β_n^T. The difference of length between $\widehat{\beta}_n^T$ and $v[0,T]$ is, for all $T > 0$, bounded from above by the length of α_n. As T goes to ∞, this difference of length converges to $B_\xi(\gamma_n(\widetilde{v}(0)), \widetilde{v}(0))$, and the length of α_n is also an upper bound for $B_\xi(\gamma_n(\widetilde{v}(0)), \widetilde{v}(0))$. More precisely, one has:

$$\begin{aligned}
&B_\xi(\gamma_n(\widetilde{v}(0)), \widetilde{v}(0)) \\
&\qquad = \lim_{t\to\infty} d\big(\widetilde{v}(t), \gamma_n(\widetilde{v}(0))\big) - d\big(\widetilde{v}(t), \widetilde{v}(0)\big) \\
&\qquad = \lim_{n\to\infty} d\big(\widetilde{v}(t_n), \gamma_n(\widetilde{v}(0))\big) - d\big(\widetilde{v}(t_n), \widetilde{v}(0)\big) \\
&\qquad \leqslant \underbrace{d\big(\widetilde{v}(t_n), \gamma_n(\widetilde{v}(t_n))\big)}_{=\mathrm{length}(\alpha_n)} + \underbrace{d\big(\gamma_n(\widetilde{v}(t_n)), \gamma_n(\widetilde{v}(0))\big)}_{=t_n} - \underbrace{(\widetilde{v}(t), \widetilde{v}(0))}_{=t_n} \\
&\qquad = \mathrm{length}(\alpha_n).
\end{aligned}$$

Now, we are going to see that there is a lower bound for $B_\xi(\gamma_n(\widetilde{v}(0)), \widetilde{v}(0))$. There are two cases:

Case 1: γ_n is an hyperbolic isometry. — Consider the curve $\mathscr{C}_\gamma(\widetilde{v}(0))$ of points which are at distance d from the axis of γ_n, where d is the distance from $\widetilde{v}(0)$ to the axis of γ_n. If $a : \mathbb{R} \to \widetilde{M}$ is an arclength parametrization of $\mathscr{C}_\gamma(\widetilde{v}(0))$ with $a(0) = \widetilde{v}(0)$, then $\gamma_n(\widetilde{v}(0) = a(t)$ with $t > 0$. By Proposition 3.4, it follows that

$$B_\xi^{\tilde{v}(0)}(\gamma_n(\widetilde{v}(0))) = B_\xi^{\tilde{v}(0)}(\gamma_n(a(t))) > 0.$$

In case $B_\xi^{\tilde{v}(0)}(\gamma_n(\widetilde{v}(0))) > a + \epsilon$, it is $B_\xi(\gamma_n(\widetilde{v}(0)), \widetilde{v}(0)) \in [a+\epsilon, 2(a+\epsilon)]$. In case $B_\xi^{\tilde{v}(0)}(\gamma_n(\widetilde{v}(0))) < a + \epsilon$, we can take a positive power $\gamma_n^{k_n}$ of γ_n such that $B_\xi^{\tilde{v}(0)}(\gamma_n^{k_n}(\widetilde{v}(0))) \in [a+\epsilon, 2(a+\epsilon)]$. In fact, one has:

$$B_\xi^{\tilde{v}(0)}\big(\gamma_n^k(\widetilde{v}(0))\big) = B_\xi\big(\widetilde{v}(0), \gamma_n(\widetilde{v}(0))\big) + \cdots + B_\xi\big(\gamma_n^{k-1}(\widetilde{v}(0)), \gamma_n^k(\widetilde{v}(0))\big).$$

Each term of the sum on the right side, is smaller than $a+\epsilon$, and this sum goes to ∞ when k goes to ∞, as $\gamma_n^k(\widetilde{v}(0)) \xrightarrow[k\to\infty]{} \gamma_n^+ \neq \xi$, where γ_n^+ is a fixed point of γ_n. Replacing γ_n by $\gamma_n^{k_n}$ concludes the proof.

Case 2: γ_n is a parabolic isometry. — Suppose γ_n has fixed point η. As the horocyclic orbit of v is not closed, it must be $\eta \neq \xi$. Consider the curve $\mathscr{H}_\eta(\widetilde{v}(0))$, which is the projected horocycle based at η passing through $\widetilde{v}(0)$. Give a parametrization $a : \mathbb{R} \to \widetilde{M}$ of $\mathscr{H}_\eta(\widetilde{v}(0))$, such that $a(0) = \widetilde{v}(0)$. Then $\gamma_n(\widetilde{v}(0)) = a(t)$ for some $t > 0$. By Proposition 3.5, it follows that $B_\xi^{\tilde{v}(0)}(a(t)) > 0$. Following an argument similar to that of case 1, it follows that, up to taking a positive power of γ_n, we can assume that $B_\xi^{\tilde{v}(0)}(\gamma_n(\widetilde{v}(0))) \in [a+\epsilon, 2(a+\epsilon)]$. The number ϵ can be taken as small as we want, and hence $B_\xi^{\tilde{v}(0)}(\gamma_n(\widetilde{v}(0))) \in [a, 2a]$, as we wanted to see. □

Proof of Theorem 1.5. — For a sequence $\{\widetilde{v}_n\}_{n\in\mathbb{N}} \subset T^1\widetilde{M}$ satisfying points (1) to (4) of Lemma 4.1, define $r_n := B_\xi(\widetilde{v}_n(0), \widetilde{v}(0))$. As this sequence is bounded between a and $2a$, up to taking a subsequence of $\{r_n\}$, we can assume that r_n converges to some $r_0 \in (a, 2a)$. By definition of the Busemann function $g_{r_n}(\widetilde{v}_n) = h_{s_n}(v)$ for some $s_n \in \mathbb{R}$, and taking projections to T^1M, it follows that $g_{r_n}(v_n) = h_{s_n}(v)$. Then $v_n = g_{-r_n}h_{s_n}(v) \xrightarrow[n\to\infty]{} v$. Hence $d(g_{r_n}(v), h_{s_n}(v)) \xrightarrow[n\to\infty]{} 0$. As r_n converges, g_{r_n} is an equicontinuous family, and

$$h_{s_n}(v) \xrightarrow[n\to\infty]{} g_{r_0}(v).$$

We conclude that $g_{r_0}(v) \in \overline{h_{\mathbb{R}(\widetilde{v})}}$. □

Proof of Corollary 1.6. — From Theorem 1.5, there is $r_0 > 0$ such that $g_{r_0}(v) \in \overline{h_{\mathbb{R}(\widetilde{v})}}$. Now $g_{r_0}(g_{r_0}(v)) \in g_{r_0}(\overline{h_{\mathbb{R}(\widetilde{v})}}) = \overline{h_{\mathbb{R}(g_{r_0}(\widetilde{v}))}} \subset \overline{h_{\mathbb{R}(\widetilde{v})}}$, because the closure of an orbit is invariant by the horocycle flow. Applying the same argument, defining $t_n := nr_0$, we can conclude that $g_{t_n}(v) \in \overline{h_{\mathbb{R}(\widetilde{v})}}$ for all n. This completes the proof. □

5. Applications to tight surfaces

Remark 5.1. — In view of Proposition 2.10, for a tight surface M, if $\widetilde{v} \in T^1\widetilde{M}$ is such that $\widetilde{v}(\infty)$ is an horocyclic limit point, hence $\overline{\widetilde{\pi}(h_{\mathbb{R}}(\widetilde{v}))} = T^1M$.

Proposition 5.2. — *Every almost minimizing geodesic ray $v[0,\infty)$ on a tight surface has finite injectivity radius.*

Proof. — Let the geodesics δ_n be the boundaries of the submanifolds M_n of Definition 1.8. As $v[0,\infty)$ is almost minimizing, it intersects an infinite number of these geodesics. Replacing δ_n by a subsequence, we can assume that $v[0,\infty)$ intersects δ_n for all n. Let t_n be the times such that $v(t_n) \in \delta_n$. The sequence t_n goes to ∞, as the surface is the union of all M_n, each of which is compact and hence has finite diameter, and the geodesic is almost minimizing. Let $\widetilde{\delta_n}$ be a lift of δ_n on $\widetilde{M}$, and $\widetilde{v}(t_n)$ a lift of $v(t_n)$. Consider $\eta_n \in \Gamma$ the hyperbolic isometry fixing $\widetilde{\delta}_n$. By hypothesis, the lengths of δ_n are bounded by a constant A, so $d(\widetilde{v}(t_n), \eta_n(\widetilde{v}(t_n))) \leqslant A$. This means that $\mathrm{Inj}(v(t_n)) \leqslant A$ for all n, and hence $\liminf_{n\to\infty} \mathrm{Inj}(v(t_n)) \leqslant A$. It follows that $\liminf_{t\to\infty} \mathrm{Inj}(v(t)) < A$. Then

$$\underline{\mathrm{Inj}}(v[0,\infty)) \leqslant A < \infty,$$

as we wanted. □

Proposition 5.3. — *The limit set of a tight surface contains both horocyclic and nonhorocyclic limit points.*

In fact, fixed points of hyperbolic isometries are horocyclic limit ponits. Taking a Dirichlet domain $D_\Gamma(p)$ of Γ, relative to $p \in \widetilde{M}$, and because of the surface being geometrically infinite, it follows that $\overline{D_\Gamma(p)} \cap \partial_\infty \widetilde{M} \neq \emptyset$. The points $\xi \in \overline{D_\Gamma(p)} \cap \partial_\infty \widetilde{M}$ are nonhorocyclic limit points (see [12]).

Definition 5.4. — *Given a metric space Y and a flow $\{\varphi_t\}_{t\in\mathbb{R}}$, a subset $X \subset Y$ is a* minimal set *for the flow, if it is closed, invariant by φ_t and minimal with respect to the inclusion.*

Proof of Corollary 1.9. — Supose $X \subset T^1M$ is a minimal set for the horocycle flow. Consider a vector $v \in X$ and $\widetilde{v} \in T^1\widetilde{M}$ a lift of v. As X is a minimal set, $h_{\mathbb{R}}(v)$ must be dense in X, otherwise its closure would be a proper invariant subset of X, and X would not be minimal. On the other hand, X can not be T^1M, because in that case every orbit should be dense, but that can't happen since the limit set has both horocyclic and nonhorocyclic limit points, and horocycles based on nonhorocyclic limit points are not dense. So X is a proper subset of T^1M. This implies that $\widetilde{v}(\infty)$ must be a nonhorocyclic limit point, since the closure of its projected orbit is $X \neq T^1M$. The ray $v[0,\infty)$ is an almost minimizing geodesic ray, and by Theorem 1.5 and Proposition 5.2, we know that there is a t_0 such that $g_{t_0}(v) \in \overline{h_{\mathbb{R}(\approx)}}$. Then, $g_{t_0}(X) \cap X \neq \emptyset$ and $g_{t_0}(X) = X$. Therefore $g_{nt_0}(X) = X$ for all $n \in \mathbb{N}$.

Let us see that it actually implies that $\widetilde{v}(\infty)$ is horocyclic: consider an horocycle B based at $\widetilde{v}(\infty)$. As we know, we can write

$$B = \left\{ z \in \widetilde{M} :\ B_{\widetilde{v}(\infty)}(\widetilde{v}(0), z) > k \right\},$$

for some $k \in \mathbb{R}$. As the point $v(nt_0)$ belongs to $X = \overline{h_{\mathbb{R}(\approx)}}$, there is a sequence $\{\gamma_m\}_{m\in\mathbb{N}} \subset \Gamma$ such that $B_{\widetilde{v}(\infty)}(\widetilde{v}(0), \gamma_m^{-1}\widetilde{v}(0)) \xrightarrow[m\to\infty]{} nt_0$. Choosing a large

n it is $nt_0 > k$, and therefore, for a large m, it is $\gamma_m^{-1}\widetilde{v}(0) \in B$. So we can find an element of the Γ-orbit of $\widetilde{v}(0)$ on any horocycle based at $\widetilde{v}(\infty)$, and this means that $\widetilde{v}(\infty)$ is an horocyclic limit point, which is absurd as we already showed that it must be nonhorocyclic. $\square$

Bibliography

[1] F. Alcalde Cuesta, F. Dal'Bo, M. Martínez & A. Verjovsky, "Corrigendum to "Minimality of the horocycle flow on laminations by hyperbolic surfaces with non-trivial topology"", *Discrete Contin. Dyn. Syst.* **37** (2017), no. 8, p. 4585-4586.

[2] W. Ballmann, *Lectures on spaces of nonpositive curvature*, DMV Seminar, vol. 25, Birkhäuser, 1995, with an appendix by Misha Brin.

[3] T. Bedford, M. Keane & C. Series (eds.), *Ergodic theory, symbolic dynamics, and hyperbolic spaces. Lectures given at the workshop "Hyperbolic geometry and ergodic theory", held at the International Centre for Theoretical Physics in Trieste, Italy, 17-28 April, 1989*, Oxford Science Publications, Oxford University Press, 1991.

[4] A. Bellis, "On the links between horocyclic and geodesic orbits on geometrically infinite surfaces", *J. Éc. Polytech., Math.* **5** (2018), p. 443-454.

[5] M. P. do Carmo, *Riemannian geometry*, portuguese ed., Mathematics: Theory & Applications, Birkhäuser, 1992.

[6] F. Dal'Bo, "Topologie du feuilletage fortement stable", *Ann. Inst. Fourier* **50** (2000), no. 3, p. 981-993.

[7] P. Eberlein, "Geodesic flows on negatively curved manifolds. II", *Trans. Am. Math. Soc.* **178** (1973), p. 57-82.

[8] G. A. Hedlund, "Fuchsian groups and transitive horocycles", *Duke Math. J.* **2** (1936), no. 3, p. 530-542.

[9] T. Inaba, "An example of a flow on a non-compact surface without minimal set", *Ergodic Theory Dyn. Syst.* **19** (1999), no. 1, p. 31-33.

[10] A. Katok & B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and Its Applications, vol. 54, Cambridge University Press, 1995, with a supplementary chapter by Katok and Leonardo Mendoza.

[11] M. Kulikov, "The horocycle flow without minimal sets", *C. R. Math.* **338** (2004), no. 6, p. 477-480.

[12] S. Matsumoto, "Horocycle flows without minimal sets", *J. Math. Sci., Tokyo* **23** (2016), no. 3, p. 661-673.

[13] I. M. Singer & J. A. Thorpe, *Lecture notes on elementary topology and geometry*, Undergraduate Texts in Mathematics, Springer, 1976, reprint of the 1967 edition.