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Towards an asymptotic analysis of the anisotropic Ginzburg–Landau model [(*)]

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Abstract. — We develop a set of tools for the asymptotic analysis of minimizers of the anisotropic Ginzburg–Landau energy functional among the admissible competitors with Dirichlet boundary datum of negative degree $-D$. As a byproduct of our analysis, we prove that the energy of a minimizer is $K\ln(1/\varepsilon)+o(\ln(1/\varepsilon))$, where K depends only on D and the material constants that enter into the expression for the energy.

Résumé. — Nous proposons plusieurs outils pour l'analyse des minimiseurs de la fonctionnelle de Ginzburg–Landau anisotrope avec donnée de Dirichlet de degré négatif $-D$. Comme une application de notre analyse, nous montrons que l'énergie des minimiseurs est $K\ln(1/\varepsilon)+o(\ln(1/\varepsilon))$, où la constante K dépend uniquement de D et des constantes physiques qui apparaissent dans l'expression de l'énergie.

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1. Introduction

Minimizing the Ginzburg–Landau energy in a 2D domain subject to Dirichlet boundary conditions has been well understood since the seminal contribution of Bethuel, Brezis, and Hélein [2]. In particular, there is no distinction between the analysis of minimizers for boundary datum of positive and negative degree as the two cases are related by conjugation. However, somewhat surprisingly, when the Dirichlet integral is broken into the sum of the squares of the divergence and curl with arbitrary positive weights, the distinction arises. Such a decomposition of the gradient is not merely an academic exercise as it arises in modeling of nematic liquid crystals, in particular within the context of the Oseen–Frank model for uniaxial nematics, see [22].

The case of positive degree, considered by Colbert-Kelly and Phillips in [5], is reducible to the standard treatment, following the ideas of [2]. This reduction relies on existence of degree one singularities with bounded energy that are purely divergence or purely curl, and there do not exist analogous vector fields of negative degree. Other interesting cases include extreme situations of high anisotropy when the ratio of the elastic constants is vanishingly small. For example, Golovaty, Sternberg, and Venkatraman [9] show that limiting configurations may exhibit line singularities accommodating high deformation cost associated with divergence through the emergence of jumps in the tangential component. In a related work, Kowalczyk, Lamy, and Smyrnelis [12] construct entire solutions of the Euler–Lagrange equations having negative degree and possessing equivariant symmetry.

In this paper, we develop some tools that we believe should be helpful in completing the asymptotic analysis of the minimization problem for the anisotropic Ginzburg–Landau functional, subject to boundary datum of negative degree. We begin by providing the precise statement of the problem and establishing some necessary notation.

1.1. The problem

We let $\Omega \subset \mathbb{R}^2 \sim \mathbb{C}$ be a smooth bounded domain that we assume to be simply connected. For technical reasons, we occasionally also assume that Ω is strictly star-shaped. For $K_1, K_3 > 0$, $\omega \subset \Omega$ and $u : \omega \to \mathbb{C}$, we consider the energies

$$E_0(u) = E_0(u, \omega) := \frac{K_1}{2} \int_\omega (\operatorname{div} u)^2 + \frac{K_3}{2} \int_\omega (\operatorname{curl} u)^2$$

and

$$E_\varepsilon(u) = E_\varepsilon(u,\omega) := E_0(u) + \frac{1}{4\varepsilon^2}\int_\omega \left(1-|u|^2\right)^2,$$

where $\varepsilon > 0$. With no loss of generality, we assume that $K_1 + K_3 = 2$. Noting that

$$\frac{K_1}{2}(\operatorname{div} u)^2 + \frac{K_3}{2}(\operatorname{curl} u)^2 = \frac{K_3}{2}(\operatorname{div} \imath u)^2 + \frac{K_1}{2}(\operatorname{curl} \imath u)^2,$$

we may also assume in the analysis below that $K_1 \geqslant K_3$, so that we can write

$$K_1 = 1+\delta,\ \ K_3 = 1-\delta,\ \text{ with } 0 \leqslant \delta < 1. \tag{1.1}$$

For $u : \omega \to \mathbb{C}$, we denote by

$$G_0(u) = G_0(u,\omega) := \frac{1}{2}\int_\omega |\nabla u|^2,$$

and

$$G_\varepsilon(u) = G_\varepsilon(u,\omega) := G_0(u) + \frac{1}{4\varepsilon^2}\int_\omega \left(1-|u|^2\right)^2,$$

the standard Dirichlet and Ginzburg–Landau energies, respectively. When $\delta = 0$, the functional E_0 reduces to G_0 while the functional E_ε reduces to G_ε (after integration by parts and modulo a fixed boundary term; see the proof of Lemma 2.1). In what follows, we denote by lower case letters the energy densities, e.g.,

$$e_0(u) := \frac{K_1}{2}(\operatorname{div} u)^2 + \frac{K_3}{2}(\operatorname{curl} u)^2,$$

and

$$g_\varepsilon(u) := \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left(1-|u|^2\right)^2,$$

for E_0 and G_ε, respectively.

Given $g : \partial\Omega \to \mathbb{S}^1$ a smooth map of degree $-D < 0$, we let u_ε denote a minimizer of E_ε in the class $H^1_g(\Omega;\mathbb{C}) := \{u \in H^1(\Omega;\mathbb{C});\, \operatorname{tr} u = g\}$. We are interested in the asymptotic properties of u_ε as $\varepsilon \to 0$, and our main purpose is to extend to E_ε some of the analysis achieved for G_ε in [2].

1.2. The main results

Although our results are far from being as complete as those in [2], we feel that they may have some interest and give impetus for subsequent research. Most of the techniques that we use have roots in [2] and subsequent works. In particular, several proofs are in the spirit of Struwe [20, 21] or Sandier and Serfaty [19, Chapter 5] (see also Han and Shafrir [10], Jerrard [11], and Sandier [18]). Part of the analysis consists of establishing a priori estimates. Such estimates are also obtained for critical points of E_ε, either assuming an energy bound or assuming that Ω is strictly star-shaped.

A significant part of our analysis is valid for every K_1 and K_3. For example, we prove that minimizers u_ε of E_ε satisfy, for small ε, the bounds $|u_\varepsilon| \leqslant C_1$, $|\nabla u_\varepsilon| \leqslant C_2/\varepsilon$ (Lemma 4.7). In the case of the standard Ginzburg–Landau equation, this follows from a maximum principle that does not seem to be available in our case. This is derived *via* various Pohozaev identities (see, e.g., Lemma 4.1) and elliptic estimates (see, e.g., Lemma 2.4).

We establish an η-ellipticity result (Lemma 3.1) similar to the one for the standard Ginzburg–Landau equation, asserting, roughly speaking, that if the energy of a minimizer u_ε is small when compared to $\ln(1/\varepsilon)$, then u_ε has no vortices. We also prove that critical points of E_ε satisfying a logarithmic energy bound (and, in particular, minimizers) display a controlled bad discs structure (Lemma 5.1). These bad discs are far away from the boundary (Corollary 7.2). We also prove the existence of bounded entire local minimizers of negative degree (Corollary 8.1).

Sharper results are established under the assumption that K_1 and K_3 are “close”, i.e., for sufficiently small $|\delta|$. For example, we prove that, when $|\delta|$ is small, the local minimizers in Corollary 8.1 have degree -1 (Corollary 8.5). Moreover, when $|\delta|$ and ε are small, we prove that the bad discs structure associated with a minimizer of E_ε with respect to a boundary datum of degree $-D < 0$ consists of exactly D bad discs, each of degree -1 (Theorem 9.1).

When $|\delta|$ and ε are small and $0 < \alpha < 1$, in Section 10 we prove that the bad discs are at distance $\geqslant \varepsilon^\alpha$ from each other and from the boundary (Theorem 10.1). We complement these results in Section 12, where we are also able to show that the energy density concentrates on bad discs as $\varepsilon \to 0$ (Theorem 12.4).

Another series of results concerns the energy of minimizers of E_ε with boundary datum of degree $-D$. For arbitrary δ, we introduce the concept of giant bad discs, that allows us to obtain the asymptotic expansion of this energy up to an $o(\ln(1/\varepsilon))$ term (Theorem 13.2). When $|\delta|$ is sufficiently small, we prove that the leading term in the expansion of the energy is $DC_\delta \ln(1/\varepsilon)$. It is well-known that, for the standard Ginzburg–Landau functional investigated in [2] and which corresponds to $\delta = 0$, we have $C_0 = \pi$, and the above term is the leading term for both positive D (as in our work) and negative D. When $\delta \neq 0$ and D is negative, it was proved in [5] that the leading order is $D(1 - |\delta|)\pi \ln(1/\varepsilon)$. We prove that, when $\delta \neq 0$ is small, the cost of negative degrees is different from the one of positive degrees. More specifically, we prove that, when $|\delta|$ is small, we have $C_\delta > (1-|\delta|)\pi$ (Lemma 11.1 and Theorem 13.1).

2. Preliminaries

We will repeatedly use the following observations.

Lemma 2.1. — *Let ω be a bounded Lipschitz domain and $u = v + \imath w \in H^1(\omega;\mathbb{C})$. Then*

$$\begin{aligned} E_0(u,\omega) &= \frac{K_1}{2}\int_\omega \big([v_x]^2+[w_y]^2\big) + \frac{K_3}{2}\int_\omega \big([v_y]^2+[w_x]^2\big) \\ &\quad + (K_1-K_3)\int_\omega v_x w_y + K_3\int_\omega (v_x w_y - v_y w_x) \\ &= \frac{K_1}{2}\int_\omega \big([v_x]^2+[w_y]^2\big) + \frac{K_3}{2}\int_\omega \big([v_y]^2+[w_x]^2\big) \\ &\quad + (K_1-K_3)\int_\omega v_x w_y + \frac{K_3}{2}\int_{\partial\omega} u\wedge\frac{\partial u}{\partial\tau}, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} E_0(u,\omega) &= \frac{K_1}{2}\int_\omega \big([v_x]^2+[w_y]^2\big) + \frac{K_3}{2}\int_\omega \big([v_y]^2+[w_x]^2\big) \\ &\quad + (K_1-K_3)\int_\omega v_y w_x + K_1\int_\omega (v_x w_y - v_y w_x) \\ &= \frac{K_1}{2}\int_\omega \big([v_x]^2+[w_y]^2\big) + \frac{K_3}{2}\int_\omega \big([v_y]^2+[w_x]^2\big) \\ &\quad + (K_1-K_3)\int_\omega v_y w_x + \frac{K_1}{2}\int_{\partial\omega} u\wedge\frac{\partial u}{\partial\tau}. \end{aligned} \tag{2.2}$$

Moreover,

$$\begin{aligned} (1-\delta)G_0(u,\omega) - \frac{1-\delta}{2}\left|\int_{\partial\omega} u\wedge\frac{\partial u}{\partial\tau}\right| &\leqslant E_0(u,\omega) \\ &\leqslant (1+\delta)G_0(u,\omega) + \frac{1+\delta}{2}\left|\int_{\partial\omega} u\wedge\frac{\partial u}{\partial\tau}\right|. \end{aligned} \tag{2.3}$$

Proof. — Identities (2.1) and (2.2) are straightforward consequences of

$$\int_\omega (v_x w_y - v_y w_x) = \frac{1}{2}\int_{\partial\omega} u\wedge\frac{\partial u}{\partial\tau}. \tag{2.4}$$

The first and the second inequality in (2.3) follow from the second identity in (2.1) and (2.2), respectively, once we observe that

$$(K_1-K_3)\int_\omega v_x w_y \geqslant -\delta\int_\omega \big([v_x]^2+[w_y]^2\big)$$

and

$$(K_1 - K_3)\int_\omega v_y w_x \leqslant \delta \int_\omega \big([v_y]^2 + [w_x]^2\big). \qquad \square$$

Next, recalling that $\deg g = -D < 0$, we prove the following lemma.

Lemma 2.2. — *For small ε, we have the δ-independent bound*

$$\min\big\{E_\varepsilon(u);\, u \in H^1_g(\Omega;\mathbb{C})\big\} \leqslant \pi D \ln\frac{1}{\varepsilon} + C(g). \tag{2.5}$$

Proof. — Using the standard construction of competitors for the Ginzburg–Landau energy, it suffices to prove the result when Ω is the unit disc, $D = 1$, and $g(z) = \bar z$. Consider, for $0 < \varepsilon < 1$, the competitor

$$u(z) = \begin{cases} \bar z/|z|, & \text{if } |z| \geqslant \varepsilon \\ \bar z/\varepsilon, & \text{if } |z| \leqslant \varepsilon. \end{cases}$$

Then

$$E_\varepsilon(u) = \pi \ln\frac{1}{\varepsilon} + \frac{\pi}{2}\int_0^1 r\big(1-r^2\big)^2\,\mathrm{d}r. \qquad \square$$

The following is straightforward.

Lemma 2.3. — *A critical point $u_\varepsilon = v_\varepsilon + \imath w_\varepsilon$ of E_ε in $H^1_g(\Omega;\mathbb{C})$ satisfies*

$$\begin{cases} \mathcal{L}_1(v_\varepsilon, w_\varepsilon) := -K_1(v_{\varepsilon,x} + w_{\varepsilon,y})_x - K_3(v_{\varepsilon,y} - w_{\varepsilon,x})_y \\ \qquad\qquad\quad = \varepsilon^{-2} v_\varepsilon\big(1 - |u_\varepsilon|^2\big), \\ \mathcal{L}_2(v_\varepsilon, w_\varepsilon) := -K_1(v_{\varepsilon,x} + w_{\varepsilon,y})_y + K_3(v_{\varepsilon,y} - w_{\varepsilon,x})_x \\ \qquad\qquad\quad = \varepsilon^{-2} w_\varepsilon\big(1 - |u_\varepsilon|^2\big). \end{cases} \tag{2.6}$$

Here and in what follows, we use a subscript notation for partial or directional derivatives: $w_x = \frac{\partial w}{\partial x}$, $u_\tau := \frac{\partial u}{\partial \tau}$, etc.

We next note that the second order constant coefficients linear system $\mathcal{L} := (\mathcal{L}_1, \mathcal{L}_2)$ is elliptic, in the sense that it satisfies the strong Legendre–Hadamard ellipticity condition (see, e.g., [7, Chapter I, (1.9)]). To justify this observation, we note that $\mathcal{L}$ arises from the energy functional $E_0(u)$. Writing (only in this paragraph)

$$u = \big(u^1, u^2\big), \quad p^i = \big(p^i_1, p^i_2\big) = \nabla^t u^i, \quad i = 1, 2,$$

the energy density $e_0(u)$ may be identified with the following function of (p^1, p^2):

$$e_0\big(p^1, p^2\big) = \frac{K_1}{2}\big(p^1_1 + p^2_2\big)^2 + \frac{K_3}{2}\big(p^2_1 - p^1_2\big)^2,$$

and thus, for every $\xi = (\xi_1, \xi_2)$ and $\lambda = (\lambda^1, \lambda^2)$, we have

$$\begin{aligned}\sum_{1\leqslant i,j,\alpha,\beta\leqslant 2} \frac{\partial^2 e_0}{\partial p^i_\alpha \partial p^j_\beta}\xi_\alpha\xi_\beta\lambda^i\lambda^j &= K_1\big(\xi_1\lambda^1 + \xi_2\lambda^2\big)^2 + K_3\big(\xi_1\lambda^2 - \xi_2\lambda^1\big)^2 \\ &\geqslant K_3\Big[\big(\xi_1\lambda^1 + \xi_2\lambda^2\big)^2 + \big(\xi_1\lambda^2 - \xi_2\lambda^1\big)^2\Big] \\ &= K_3|\xi|^2|\lambda|^2,\end{aligned} \tag{2.7}$$

which shows that, indeed, $\mathcal{L}$ is elliptic.

An alternative route to ellipticity consists of identifying u with the 1-form $\zeta = v\,\mathrm{d}x + w\,\mathrm{d}y$, noting that

$$E_0(u) = \frac{K_1}{2}\int_\Omega |\mathrm{d}^*\zeta|^2 + \frac{K_3}{2}\int_\Omega |\mathrm{d}\zeta|^2,$$

and then using the ellipticity of the Hodge system $\begin{cases} \mathrm{d}\zeta = f, \\ \mathrm{d}^*\zeta = f^*. \end{cases}$

This observation allows us to apply to $\mathcal{L}$ the regularity theory for elliptic systems as in [6] or [1]. However, since we will rely on estimates in variable domains and with variable operators, we present here the statements instrumental for our purposes, with elements of proofs.

We first quantify the uniform ellipticity of the operator $\mathcal{L}$, by introducing the assumption

$$0 \leqslant \delta \leqslant \delta_1 < 1, \tag{2.8}$$

where δ_1 is a fixed constant.

We fix a smooth bounded domain Ω and a boundary datum $g \in C^\infty(\partial\Omega; \mathbb{C})$. A ball $B = B_r(x)$ is *admissible* if either $B \subset \Omega$, or the center of B is on $\partial\Omega$. We set

$$B_* := B_{r/2}(x). \tag{2.9}$$

Consider a solution u of

$$\begin{cases} \mathcal{L}u = f & \text{in } B\cap\Omega \\ \quad u = g & \text{on } B\cap\partial\Omega \end{cases} \tag{2.10}$$

(the last condition being empty if $B \subset \Omega$). Note that, for small r, if the ball B is centered at some $x \in \partial\Omega$, then $B\cap\Omega$ is a Lipchitz open set and $B\cap\partial\Omega$ is a Lipschitz portion of $\partial(B\cap\Omega)$. Therefore, the second condition in (2.10) makes sense provided, say, $u \in H^1(B\cap\Omega)$. In what follows, we always make the implicit assumption that r is sufficiently small so that these considerations apply. Note that this smallness assumption does not depend on ε or δ.

LEMMA 2.4. — *Assume* (2.8)*. Let* $0 < \alpha < 1$ *and set* $q = q(\alpha) := \frac{2}{2-\alpha} \in (1, 2)$*. Let* $p > 2$*. Let* $B = B_r(x)$ *be an admissible ball and consider a solution* $u \in H^1(B \cap \Omega)$ *of* (2.10)*.*

(1) Interior estimates: *If* $B \subset \Omega$*, then (for some absolute constants* C_{α,δ_1} *and* C_{p,δ_1}*)*

$$r^\alpha \frac{|u(y)-u(z)|}{|y-z|^\alpha} \leqslant C_{\alpha,\delta_1}\Big(\|\nabla u\|_{L^2(B)} + r^\alpha \|f\|_{L^q(B)}\Big), \qquad \forall\, y, z \in B_*, \quad (2.11)$$

$$r^\alpha \frac{|u(y)-u(z)|}{|y-z|^\alpha} \leqslant C_{\alpha,\delta_1}\Big(\|\nabla u\|_{L^2(B)} + r \|f\|_{L^2(B)}\Big), \qquad \forall\, y, z \in B_*, \quad (2.12)$$

$$r|\nabla u(y)| \leqslant C_{p,\delta_1}\Big(\|\nabla u\|_{L^2(B)} + r^{2-2/p} \|f\|_{L^p(B)}\Big), \ \forall\, y \in B_*. \quad (2.13)$$

(2) Boundary estimates: *There exists some finite* $r_0 > 0$ *such that, if* $r \leqslant r_0$ *and* $x \in \partial\Omega$*, then (for some constants* $C_{\alpha,\delta_1,\Omega}$ *and* $C_{p,\delta_1,\Omega}$*)*

$$r^\alpha \frac{|u(y)-u(z)|}{|y-z|^\alpha} \leqslant C_{\alpha,\delta_1,\Omega}\Big(\|\nabla u\|_{L^2(B\cap\Omega)} + r^\alpha \|f\|_{L^q(B\cap\Omega)} + r|g|_{\mathrm{Lip}(B\cap\partial\Omega)}\Big), \quad \forall\, y, z \in B_*, \quad (2.14)$$

$$r^\alpha \frac{|u(y)-u(z)|}{|y-z|^\alpha} \leqslant C_{\alpha,\delta_1,\Omega}\Big(\|\nabla u\|_{L^2(B\cap\Omega)} + r \|f\|_{L^2(B\cap\Omega)} + r|g|_{\mathrm{Lip}(B\cap\partial\Omega)}\Big), \quad \forall\, y, z \in B_*, \quad (2.15)$$

$$r|\nabla u(y)| \leqslant C_{p,\delta_1,\Omega}\Big(\|\nabla u\|_{L^2(B\cap\Omega)} + r^{2-2/p} \|f\|_{L^p(B\cap\Omega)} + r|g|_{\mathrm{Lip}(B\cap\partial\Omega)} + r^2|\partial g/\partial\tau|_{\mathrm{Lip}(B\cap\partial\Omega)}\Big), \ \forall\, y \in B_*. \quad (2.16)$$

Note the scaling (in the radius r) of the estimates, which comes from the fact that we work in two dimensions and that $\mathcal{L}$ is a homogeneous second order system.

Idea of proof of Lemma 2.4. — After scaling, (1) is a special case of the interior estimates for elliptic systems [6], [7, Chapter 3, Theorem 2.2], combined with the embedding $H^2_{\mathrm{loc}} \hookrightarrow C^\alpha_{\mathrm{loc}}$, $0 < \alpha < 1$. Note that here the scaling argument relies on the homogeneity of $\mathcal{L}$. Again after scaling and (for small r) flattening of the boundary, item (2) follows from the model case $B \cap \Omega = \{(x, y) \in B_1(0);\ y > 0\}$. Some care is needed since the flattening depends on x and r, and one has to make sure that one can choose constants independent of x, r, and δ_1 in the method of freezing of the coefficients.

This is indeed possible for sufficiently small r (see, e.g., the detailed proofs in [4, Cap. III] or [8, proof of Theorem 9.13]). $\square$

Iterating the proof of Lemma 2.4 for our specific system (2.6) and taking $r = \varepsilon$, we obtain the following result, that we state here without proof.

LEMMA 2.5. — *Assume* (2.8). *Fix* $g \in C^\infty(\partial\Omega; \mathbb{S}^1)$. *Let* $u = u_\varepsilon$, $0 < \varepsilon \leqslant 1$, *be critical points of* E_ε *in* $H^1_g(\Omega)$ *satisfying the a priori bound*

$$|u(x)| \leqslant M < \infty, \quad \forall\, \varepsilon, \quad \forall\, x \in \Omega. \tag{2.17}$$

Then there exist finite constants C_k *depending on* M, δ_1, Ω, *and* g *such that*

$$\left|D^k u(x)\right| \leqslant C_k \varepsilon^{-k}, \quad \forall\, x \in \overline{\Omega}, \quad \forall\, k \in \mathbb{N}. \tag{2.18}$$

Moreover, with finite constants $\widetilde{C}_k$ *depending on* M *and* δ_1 *(but not on* Ω *or* g*), we have*

$$\left|D^k u(x)\right| \leqslant \widetilde{C}_k \varepsilon^{-k}, \quad \forall\, x \in \Omega \ s.t.\ \operatorname{dist}(x, \partial\Omega) \geqslant \varepsilon, \quad \forall\, k \in \mathbb{N}. \tag{2.19}$$

Next, we note an important consequence of the ellipticity of $\mathcal{L}$. The system (2.6) is of the form

$$\mathcal{L}u = F(u), \text{ with } F(u) = \varepsilon^{-2} u (1 - |u|)^2. \tag{2.20}$$

Noting that F is analytic, we have the following result, essentially established by Morrey [14] (see also Petrowsky [15]).

LEMMA 2.6. — *Let* $U \subset \mathbb{R}^2$ *be an open set. If* $u \in H^1_{\mathrm{loc}}(U)$ *is a weak solution of* (2.20)*, then* u *is analytic.*

Proof. — Let us note that, by standard regularity theory [6], the 2D-embedding $H^1_{\mathrm{loc}} \hookrightarrow L^p_{\mathrm{loc}}$, $\forall\, p < \infty$, and the fact that our F has polynomial growth, we have $u \in C^\infty$. We next note that the Legendre–Hadamard ellipticity condition checked in (2.7) implies the ellipticity in the sense of Douglis and Nirenberg [6, Section 1]. This is a general fact, but we illustrate it in our special case. For a second order 2D-variational system with energy density $e_0(p^1, p^2)$, the ellipticity in the sense of [6] requires that the following determinant

$$D(\xi) := \det \left(\sum_{1 \leqslant \alpha, \beta \leqslant 2} \frac{\partial^2 e_0}{\partial p^i_\alpha \partial p^j_\beta} \xi_\alpha \xi_\beta \right)_{1 \leqslant i,j \leqslant 2} \tag{2.21}$$

does not vanish when $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$.

Considering the left-hand side of (2.7) as a quadratic form in λ with ξ-depending coefficients, the determinant in (2.21) is nothing but the determinant of this quadratic form. Thus, by (2.7), $D(\xi) > 0$, $\forall\, \xi \neq 0$, as claimed. (Of course, one could check (2.21) directly by noting that $D(\xi) = K_1 K_3 |\xi|^2$.)

Finally, the main result in Morrey [14] asserts that smooth solutions of analytic elliptic systems are analytic, implying thus the conclusion of the lemma. $\square$

3. η-ellipticity

Throughout this section, we assume (2.8). Let Ω and the boundary datum $g \in C^\infty(\partial\Omega; \mathbb{S}^1)$ be fixed. Let u_ε be a minimizer of E_ε in $H^1_g(\Omega; \mathbb{C})$. We will establish conditional a priori estimates on u_ε, with constants depending on δ_1, but not on δ satisfying (2.8). These constants will possibly depend on Ω or g and the estimates will be valid for $\varepsilon \leqslant \varepsilon_0$, with ε_0 possibly depending on Ω and g.

The main result of this section is the following.

Lemma 3.1. — *Let $0 < \alpha < 1$ and $\lambda > 0$ be fixed. Then there exist absolute constants $\eta > 0$ and $M < \infty$ (depending only on δ_1, α, λ) and a constant $\varepsilon_0 > 0$ depending on g and Ω such that:*

$$\begin{aligned} &\left[0 < \varepsilon \leqslant \varepsilon_0,\ B_{\varepsilon^\alpha}(x) \textit{ admissible},\ E_\varepsilon(u_\varepsilon, B_{\varepsilon^\alpha}(x) \cap \Omega) \leqslant \eta |\ln \varepsilon|\right] \\ &\qquad\qquad \Longrightarrow \left[\big||u_\varepsilon(x)| - 1\big| \leqslant \lambda,\ |\nabla u_\varepsilon(x)| \leqslant M/\varepsilon\right]. \quad (3.1) \end{aligned}$$

Moreover, we may choose M independent of $0 < \lambda < 1$.

We next state some intermediate results (to be proved later) that will be needed in the proof of Lemma 3.1. The first result is well-known in the Ginzburg–Landau literature.

Lemma 3.2.

(1) *Let $\mu > 0$ be fixed. Then there exists an absolute finite positive constant ν (depending only on μ) such that:*

$$\begin{aligned} &\begin{bmatrix} B_r(x) \subset \Omega,\ 0 < \varepsilon \leqslant r,\ f : C_r(x) \\ \quad \longrightarrow \mathbb{C}, r\displaystyle\int_{C_r(x)} |f_\tau|^2 + \frac{r}{\varepsilon^2}\int_{C_r(x)} \left(1 - |f|^2\right)^2 \leqslant \nu \end{bmatrix} \\ &\qquad\qquad \Longrightarrow \left[\left|\int_{C_r(x)} f \wedge f_\tau\right| \leqslant \mu \ \&\ \begin{matrix} \exists\, h \in H^1_f(B_r(x)) \\ \quad s.t.\ G_\varepsilon(h, B_r(x)) \leqslant \mu \end{matrix}\right]. \quad (3.2) \end{aligned}$$

(2) *Let $\mu > 0$ be fixed. Then there exists a finite positive absolute constant ν (depending only on μ) and a constant r_0 depending on μ, Ω, and g, such that:*

$$\begin{bmatrix} x \in \partial\Omega, \quad 0 < \varepsilon \leqslant r \leqslant r_0, f \in H^1(\partial(B_r(x) \cap \Omega); \mathbb{C}), \\ f = g \text{ on } B_r(x) \cap \partial\Omega,\ r \displaystyle\int_{C_r(x)\cap\Omega} |f_\tau|^2 + \frac{r}{\varepsilon^2} \int_{C_r(x)\cap\Omega} \left(1 - |f|^2\right)^2 \leqslant \nu \end{bmatrix}$$
$$\Longrightarrow \left[\left| \int_{C_r(x)\cap\Omega} f \wedge f_\tau \right| \leqslant \mu \ \& \ \begin{array}{r} \exists\, h \in H^1_f(B_r(x) \cap \Omega) \\ \text{s.t. } G_\varepsilon(h, B_r(x) \cap \Omega) \leqslant \mu \end{array} \right]. \quad (3.3)$$

The proof of Lemma 3.2 also leads to Lemmata 3.3 and 3.4, that we note, without proof, for further use.

Lemma 3.3. — *Let $B = B_r(x)$. Fix some $s > 0$. Then there exists a finite constant $t > 0$ (depending only on s) and a finite constant $r_1 > 0$ depending on s, Ω, and g such that*

$$\begin{bmatrix} r \leqslant r_1,\ B \text{ admissible},\ v \in H^1(\partial(B \cap \Omega); \mathbb{C}), \\ v = g \text{ on } B \cap \partial\Omega,\ r \displaystyle\int_{\partial B\cap\Omega} |v_\tau|^2 + \frac{1}{r} \int_{\partial B\cap\Omega} (1 - |v|^2)^2 \leqslant t \end{bmatrix}$$
$$\Longrightarrow \left| \int_{\partial B\cap\Omega} v \wedge v_\tau \right| \leqslant s. \quad (3.4)$$

Lemma 3.4. — *Let $B = B_r(x)$. Fix some $t > 0$. Then there exists a finite constant $s > 0$ (depending only on t) and a finite constant $r_1 > 0$ depending on t, Ω, and g such that*

$$\begin{bmatrix} r \leqslant r_1,\ B \text{ admissible},\ v \in H^1(\partial(B \cap \Omega); \mathbb{C}), \\ v = g \text{ on } B \cap \partial\Omega,\ r \displaystyle\int_{\partial(B\cap\Omega)} |v_\tau|^2 + \frac{1}{r} \int_{\partial(B\cap\Omega)} \left(1 - |v|^2\right)^2 \leqslant t \end{bmatrix}$$
$$\Longrightarrow \left| \int_{\partial(B\cap\Omega)} v \wedge v_\tau \right| \leqslant s. \quad (3.5)$$

Note that, in Lemma 3.3 we prove existence of t, given s, while the opposite is shown in Lemma 3.4, where t is given and existence of s follows.

The final auxiliary result used in the proof of Lemma 3.1 relies on Lemma 2.4.

Lemma 3.5. — *Let $B = B_r(x)$. Let $u = u_\varepsilon$ be a minimizer of E_ε in $H^1_g(\Omega; \mathbb{C})$. Let $0 < s \leqslant 1$. Then there exists some finite constant $t > 0$ (depending only on s) and a finite constant $r_1 > 0$ depending on s, Ω, and*

g such that

$$\left[\begin{array}{l} 0 < 4\varepsilon \leqslant r \leqslant r_1,\ B \text{ admissible},\ E_\varepsilon(u, B\cap\Omega) \leqslant s, \\ \qquad\qquad r\displaystyle\int_{\partial B\cap\Omega} |u_\tau|^2 + \frac{1}{r}\int_{\partial B\cap\Omega}\left(1-|u|^2\right)^2 \leqslant s \end{array}\right] \\ \Longrightarrow |1-|u(z)|\,| \leqslant t,\ \forall\, z\in B_*. \tag{3.6}$$

Moreover, we may choose $t = t(s)$ such that $\lim_{s\to 0} t(s) = 0$.

(Recall that $B_* := B_{r/2}(x)$.)

We now return to the proof of Lemma 3.1. In what follows, C_j is a generic constant independent of ε or the center of the ball.

Proof of Lemma 3.1. — We use Lemmata 2.1 and 3.2–3.5. Fix some constant α_1 such that $0 < \alpha < \alpha_1 < 1$. We distinguish the cases $B_{\varepsilon^{\alpha_1}}(x) \subset \Omega$, respectively $B_{\varepsilon^{\alpha_1}}(x) \not\subset \Omega$. In what follows, ε is sufficiently small and not fixed, while η and $s > 0$ are constants that we will select at the end of the proof.

Case 1: $B_{\varepsilon^{\alpha_1}}(x) \subset \Omega$. — Clearly, we have $E_\varepsilon(u_\varepsilon, \Omega) \leqslant C_1|\ln\varepsilon|$ and thus, by Lemma 2.1 applied with $\omega = \Omega$, we have

$$G_\varepsilon(u_\varepsilon, \Omega) \leqslant C_2|\ln\varepsilon|. \tag{3.7}$$

Fix $\alpha_1 < \beta < \gamma < 1$. By (3.7) and the mean value theorem, there exists some $\varepsilon^\beta < r_1 < \varepsilon^{\alpha_1}$ such that

$$r_1\int_{C_{r_1}(x)} |u_{\varepsilon,\tau}|^2 + \frac{r_1}{\varepsilon^2}\int_{C_{r_1}(x)}\left(1-|u_\varepsilon|^2\right)^2 \leqslant C_3. \tag{3.8}$$

By Lemmata 2.1 and 3.4, this implies, for sufficiently small ε,

$$\begin{aligned} G_\varepsilon(u_\varepsilon, B_{r_1}(x)) &\leqslant \frac{1}{1-\delta_1}E_\varepsilon(u_\varepsilon, B_{r_1}(x)) + C_4 \\ &\leqslant \frac{1}{1-\delta_1}E_\varepsilon(u_\varepsilon, B_{\varepsilon^\alpha}(x)) + C_4 \\ &\leqslant C_5\eta|\ln\varepsilon|. \end{aligned} \tag{3.9}$$

Note the important fact that, while C_1, C_2, C_3, C_4 depend on g, C_5 and the constant C_6 below are universal constants, depending only on $\delta_1, \alpha, \alpha_1, \beta, \gamma$. By (3.9) and the mean value theorem, there exists some $\varepsilon^\gamma < r_2 < \varepsilon^\beta$ such that

$$r_2\int_{C_{r_2}(x)} |u_{\varepsilon,\tau}|^2 + \frac{r_2}{\varepsilon^2}\int_{C_{r_2}(x)}\left(1-|u_\varepsilon|^2\right)^2 \leqslant C_6\eta. \tag{3.10}$$

By (3.10), and Lemmata 3.3 and 3.2, for sufficiently small η (depending on s) we have (with h the competitor given by Lemma 3.2)

$$E_\varepsilon(u_\varepsilon, B_{r_2}(x)) \leqslant E_\varepsilon(h, B_{r_2}(x)) \leqslant C_7 s. \tag{3.11}$$

The first conclusion in (3.1) follows from (3.10), (3.11) (with sufficiently small s), and Lemma 3.5.

The second part of (3.1) follows from the first part of (3.1) and estimate (2.13) in Lemma 2.4 (1) (applied with $r = \varepsilon$).

Case 2: $B_{\varepsilon^{\alpha_1}}(x) \not\subset \Omega$. — The idea is similar, but this time we rely on estimate (2.16) in Lemma 3.2 (2). Let $\alpha < \alpha_2 < \alpha_1$. Let y be the nearest point projection of x on $\partial\Omega$. Clearly, for small ε, the admissible ball $B_{\varepsilon^{\alpha_2}}(y)$ is contained in $B_{\varepsilon^{\alpha}}(x)$ and contains $B_{\varepsilon}(x)$. We proceed as in the proof of (3.11) and find that $E_\varepsilon(u_\varepsilon, B_{\varepsilon^{\alpha_2}}(y) \cap \Omega) \leqslant C_7 s$, which is the analogue of (3.11) adapted to Case 2. We conclude as above. □

We now proceed to the proofs of the auxiliary results.

Proof of Lemma 3.2 (2). — Set $\mathbb{D} := B_1(0)$. By scaling, we have to prove the following, for a sufficiently small ν, and with $t := \varepsilon/r \leqslant 1$:

$$\begin{aligned}&\left[0 < t \leqslant 1,\ f : \mathbb{S}^1 \longrightarrow \mathbb{C}, \int_{\mathbb{S}^1} |f_\tau|^2 + \frac{1}{t^2}\int_{\mathbb{S}^1} \left(1 - |f|^2\right)^2 \leqslant \nu\right] \\ &\qquad \Longrightarrow \left[\left|\int_{\mathbb{S}^1} f \wedge f_\tau\right| \leqslant \mu \ \&\ \ \exists\, h \in H^1_f(\mathbb{D}; \mathbb{C}) \text{ s.t. } G_t(h, \mathbb{D}) \leqslant \mu\right]. \end{aligned} \tag{3.12}$$

We first note that

$$\begin{aligned}|f|^2 = \left(|f|^2 - 1\right) + 1 &\leqslant \frac{1}{2}\left(1 - |f|^2\right)^2 + \frac{1}{2} + 1 \\ &\leqslant \frac{1}{2t^2}\left(1 - |f|^2\right)^2 + \frac{3}{2}.\end{aligned} \tag{3.13}$$

Combining (3.13) with Cauchy–Schwarz, we find that

$$\left|\int_{\mathbb{S}^1} f \wedge f_\tau\right|^2 \leqslant \left(\frac{1}{2t^2}\int_{\mathbb{S}^1} \left(1 - |f|^2\right)^2 + 3\pi\right) \int_{\mathbb{S}^1} |f_\tau|^2 \leqslant (\nu/2 + 3\pi)\nu,$$

whence the first part of (3.12) if $(\nu/2 + 3\pi)\nu \leqslant \mu^2$.

Concerning the second part of (3.12), we first note that, for small ν independent of $t \leqslant 1$, under the assumption of (3.12) we have

$$1/2 \leqslant |f| \leqslant 3/2. \tag{3.14}$$

A cheap way to establish this fact consists of noting that, if a sequence satisfies

$$\int_{\mathbb{S}^1} |f_{j,\tau}|^2 + \int_{\mathbb{S}^1} \left(1 - |f_j|^2\right)^2 \longrightarrow 0 \quad \text{as } j \longrightarrow \infty,$$

then $|f_j| \to 1$ uniformly as $j \to \infty$. Alternatively, one may use the inequality

$$\left|f(e^{\imath\theta}) - f(e^{\imath\varphi})\right|^2 \leqslant |\theta - \varphi| \int_{\mathbb{S}^1} |f_\tau|^2, \quad \forall\, \theta - \pi \leqslant \varphi \leqslant \theta + \pi,$$

and check that (3.14) holds, e.g., when $\nu \leqslant 7\sqrt{2}/64$.

Consider ν such that (3.14) holds for every f satisfying the assumption of (3.12). Writing, locally, $f = \rho e^{\imath\psi}$, we have

$$\frac{1}{4}\int_{\mathbb{S}^1} |\psi_\tau|^2 \leqslant \int_{\mathbb{S}^1} |f_\tau|^2 \leqslant \nu,$$

and thus

$$\int_{\mathbb{S}^1} |\psi_\tau| < 2\pi,$$

provided $\nu < \pi/2$. Therefore, for small ν, $f/|f|$ has zero degree and ψ is globally defined.

We now define our competitor

$$h\big(re^{\imath\theta}\big) := F\big(re^{\imath\theta}\big)e^{\imath L\left(re^{\imath\theta}\right)}, \quad 0 \leqslant r \leqslant 1, \quad \theta \in \mathbb{R},$$

where

$$\begin{aligned} F\big(re^{\imath\theta}\big) &:= (1-r) + r\rho\big(e^{\imath\theta}\big) = (1-r) + r|f|\big(e^{\imath\theta}\big), \\ L\big(re^{\imath\theta}\big) &:= (1-r)a + r\psi\big(e^{\imath\theta}\big), \quad \text{with } a := \fint \psi. \end{aligned}$$

Clearly, thanks to (3.14), we have

$$\Big(1 - \big|F\big(re^{\imath\theta}\big)\big|^2\Big)^2 \leqslant \Big(1 - \big|f\big(e^{\imath\theta}\big)\big|^2\Big)^2, \tag{3.15}$$

$$\begin{aligned} \big|\nabla F\big(re^{\imath\theta}\big)\big|^2 &= \big(1 - \big|f\big(e^{\imath\theta}\big)\big|\big)^2 + \left|\frac{\mathrm{d}\big|f\big(e^{\imath\theta}\big)\big|}{\mathrm{d}\theta}\right|^2 \\ &\leqslant \Big(1 - \big|f\big(e^{\imath\theta}\big)\big|^2\Big)^2 + \left|\frac{\mathrm{d}f\big(e^{\imath\theta}\big)}{\mathrm{d}\theta}\right|^2, \end{aligned} \tag{3.16}$$

$$\big|\nabla L\big(re^{\imath\theta}\big)\big|^2 = \big(a - \psi\big(e^{\imath\theta}\big)\big)^2 + \left|\frac{\mathrm{d}\psi\big(e^{\imath\theta}\big)}{\mathrm{d}\theta}\right|^2,$$

and thus, using the definition of a and Poincaré's inequality,

$$\int_{\mathbb{D}} |\nabla L|^2 \leqslant 2\int_{\mathbb{S}^1}\left|\frac{\mathrm{d}\psi}{\mathrm{d}\theta}\right|^2. \tag{3.17}$$

For small ν (depending on μ), the second part of (3.12) follows from the estimates (3.15)–(3.17). □

Proof of Lemma 3.2(2). — The first part of (3.3) is proved exactly as the first part of (3.2).

We will reduce the second part of (3.3) to the situation considered in item (1). Let r_0 be sufficiently small (depending on Ω) and C_0 be a sufficiently large universal constant such that, for $x \in \partial\Omega$ and $0 < r \leqslant r_0$, there exists a bi-Lipschitz homeomorphism $\Phi = \Phi_{x,r} : \overline{B}_r(x) \cap \overline{\Omega} \to \overline{B}_r(0)$ such that $\|D\Phi\|_\infty \leqslant C_0$, $\|D\Phi^{-1}\|_\infty \leqslant C_0$, and $\Phi(\overline{B}_r(x) \cap \partial\Omega) = \{x + \imath y \in C_r(0);\ y \leqslant 0\}$. After composing with Φ^{-1} and using scale invariance, the second part of (3.3) amounts to proving (3.18) below. Set $\mathbb{S}^1_+ := \{x + \imath y \in \mathbb{D};\ y \geqslant 0\}$, and define similarly $\mathbb{S}^1_-$. Then, for a sufficiently small ν_1 (depending only on μ) and a sufficiently small r_1 (depending on μ and on a fixed given constant M), we have

$$\begin{bmatrix} 0 < t \leqslant 1, f \in H^1(\mathbb{S}^1; \mathbb{C}), |f| = 1 \quad \text{and} \quad |f_\tau| \leqslant M r_1 \text{ on } \mathbb{S}^1_-, \\ \displaystyle\int_{\mathbb{S}^1_+} |f_\tau|^2 + \frac{1}{t^2} \int_{\mathbb{S}^1_+} \left(1 - |f|^2\right)^2 \leqslant \nu_1 \end{bmatrix} \\ \Longrightarrow \exists\, h \in H^1_f(\mathbb{D}; \mathbb{C}) \text{ s.t. } G_t(h, \mathbb{D}) \leqslant \mu. \quad (3.18)$$

(In our case, the constant M itself depends only on C_0 and on the Lipschitz constant of g.)

In order to prove the existence of ν_1 and r_1 (and thus to complete the proof of the lemma), we note that, if ν is as in item (1), then (3.18) holds provided $\nu_1 + \pi(Mr_1)^2 < \nu$. It then suffices to let $\nu_1 < \nu/2$ and $r_1 < \sqrt{2\pi\nu}/M$. ☐

Proof of Lemma 3.5. — We consider only the case where $B \subset \Omega$. As explained in the proof of Lemma 3.2 (2), the other case is similar.

By estimate (2.3) in Lemma 2.1 and Lemma 3.4, there exists some finite constant $C_1 > 0$ independent of $s \leqslant 1$ such that, if the assumptions of (3.6) hold for such s, then

$$G_\varepsilon(B) = \frac{1}{2} \int_B |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_B \left(1 - |u|^2\right)^2 \leqslant C_1 s \leqslant C_1. \quad (3.19)$$

We next note that, for some appropriate constant C_2, we have

$$\left|w\left(1 - |w|^2\right)\right|^{4/3} \leqslant C_2 \left(\left(1 - |w|^2\right)^2 + 1\right), \quad \forall\, w \in \mathbb{C}. \quad (3.20)$$

Let B' be a ball of size 2ε contained in B. Applying (3.20) with $z = u(x)$, integrating over B', and using (3.19), we find that

$$\left\|\varepsilon^{-2} u\left(1 - |u|^2\right)\right\|^{4/3}_{L^{4/3}(B')} \leqslant C_2 \varepsilon^{-8/3} \int_{B'} \left(\left(1 - |u|^2\right)^2 + 1\right) \leqslant C_3 \varepsilon^{-2/3}. \quad (3.21)$$

Combining estimate (2.11) (applied, in B', with $\alpha=1/2$ and thus $q=4/3$), (3.19), and (3.21), we find that

$$\varepsilon^{1/2}\frac{|u(y)-u(z)|}{|y-z|^{1/2}} \leqslant C_4, \quad \forall\, y,z\in(B')_*, \tag{3.22}$$

and thus, in particular,

$$|u(y)-u(z)| \leqslant C_5, \quad \forall\, y,z\in(B')_*, \tag{3.23}$$

where C_5 is independent of $s\leqslant 1$.

Combining now (3.23) with (3.19), we find that

$$|u(y)| \leqslant C_6, \quad \forall\, y\in(B')_*, \tag{3.24}$$

again with C_6 independent of $s\leqslant 1$.

We next note that, for small w, (3.20) can be improved as follows:

$$|w|\leqslant C_6 \Longrightarrow \big|w\big(1-|w|^2\big)\big|^2 \leqslant (C_6)^2\big(1-|w|^2\big)^2. \tag{3.25}$$

Arguing as above and using the first inequality in (3.19), (3.25) (instead of (3.20)), (3.24), and (2.12) (instead of (2.11)), we find that

$$|u(y)-u(z)| \leqslant C_7\sqrt{s}, \quad \forall\, y,z\in(B')_*, \tag{3.26}$$

with C_7 independent of $0<s\leqslant 1$.

Finally, (3.26) and (3.19) imply (3.6), with $t(s)\to 0$ as $s\to 0$. □

4. Pohozaev type identities and a priori estimates

In this section, we derive the Pohozaev identity corresponding to the operator $\mathcal{L}$ in (2.6). As for the Dirichlet integral, the identity is obtained by multiplying (2.6) with $(x-x^0)u_x+(y-y^0)u_y$. For simplicity, we perform our calculations when $x^0 = y^0 = 0$, but in subsequent results we may take other values of x^0 and y^0. Remarkably, the Pohozaev identity implies a priori estimates merely under the δ-independent assumption that Ω is star-shaped. The idea of using the Pohozaev identity is natural in this context. For the standard Ginzburg–Landau equation, it was successfully used in [2] and subsequently [20] and [19]. For our specific system and in a disc, it appears in Kowalczyk, Lamy, and Smyrnelis [12, Section 5].

Lemma 4.1 (General Pohozaev identity). — *Let ω be a Lipschitz bounded domain. Let $X = (x, y)$ denote the "generic" point in $\mathbb{R}^2$. Let ν, respectively τ, denote the unit outward normal, respectively the unit directly oriented tangent vector to $\partial\omega$.*

Set

$$V := xv_x + yv_y, \quad W := xw_x + yw_y, \quad Z = (V, W) \sim xu_x + yu_y.$$

Let $u \in C^3(\overline{\omega}; \mathbb{C})$ be a critical point of E_ε. Then

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_\omega \big(1-|u|^2\big)^2 = \frac{1}{2\varepsilon^2}&\int_{\partial\omega}\big(1-|u|^2\big)^2(X\cdot\nu)\\ &- 2K_1\int_{\partial\omega}(\operatorname{div} u)(Z\cdot\nu) - 2K_3\int_{\partial\omega}(\operatorname{curl} u)(Z\cdot\tau)\\ &+ K_1\int_{\partial\omega}(\operatorname{div} u)^2(X\cdot\nu) + K_3\int_{\partial\omega}(\operatorname{curl} u)^2(X\cdot\nu). \quad (4.1)\end{aligned}$$

Proof. — We mimic the proof of the Pohozaev identity. We rewrite (2.6) as

$$\begin{cases} -K_1(\operatorname{div} u)_x + K_3(\operatorname{curl} u)_y = \varepsilon^{-2} v\big(1-|v|^2\big) \\ -K_1(\operatorname{div} u)_y - K_3(\operatorname{curl} u)_x = \varepsilon^{-2} w\big(1-|u|^2\big). \end{cases} \quad (4.2)$$

We let B_j denote a boundary term that we will make explicit at the end of the proof.

Multiplying the first equation in (4.2) by V, and the second one by W, integrating once by parts and summing up the results, we find that

$$\begin{aligned}-B_1 + K_1\int_\omega(\operatorname{div} u)(\operatorname{div} Z) + K_3\int_\omega(\operatorname{curl} u)(\operatorname{curl} Z)&\\ = -B_2 + \frac{1}{2\varepsilon^2}\int_\omega\big(1-|u|^2\big)^2.& \quad (4.3)\end{aligned}$$

We next note the 2D-identities

$$(\operatorname{div} u)(\operatorname{div} Z) = \frac{1}{2}\operatorname{div}\big[(\operatorname{div} u)^2 X\big], \quad (4.4)$$

$$(\operatorname{curl} u)(\operatorname{curl} Z) = \frac{1}{2}\operatorname{div}\big[(\operatorname{curl} u)^2 X\big]. \quad (4.5)$$

Inserting (4.4)–(4.5) into (4.3) and integrating, we find that

$$-B_1 + K_1 B_3 + K_3 B_4 = -B_2 + \frac{1}{2\varepsilon^2}\int_\omega \left(1 - |u|^2\right)^2.$$

We obtain the conclusion of the lemma by noting that

$$\begin{aligned}
B_1 &= K_1 \int_{\partial\omega} (\operatorname{div} u)(Z\cdot\nu) + K_3 \int_{\partial\omega} (\operatorname{curl} u)(Z\cdot\tau),\\
B_2 &= \frac{1}{4\varepsilon^2}\int_{\partial\omega} \left(1 - |u|^2\right)^2 (X\cdot\nu),\\
B_3 &= \frac{1}{2}\int_{\partial\omega} (\operatorname{div} u)^2 (X\cdot\nu),\\
B_4 &= \frac{1}{2}\int_{\partial\omega} (\operatorname{curl} u)^2 (X\cdot\nu).
\end{aligned}$$

□

We next rewrite the identity (4.1) in normal and tangential coordinates on $\partial\omega$.

We note the following identities, with $(\mathbf{i},\mathbf{j})$ the canonical basis of $\mathbb{R}^2$:

$$\begin{aligned}
\operatorname{div} u &= (\nabla v)\cdot\mathbf{i} + (\nabla w)\cdot\mathbf{j} = (v_\tau\tau + v_\nu\nu)\cdot\mathbf{i} + (w_\tau\tau + w_\nu\nu)\cdot\mathbf{j}\\
&= u_\tau\cdot\tau + u_\nu\cdot\nu.
\end{aligned}\tag{4.6}$$

We write $\nu = (\nu_x, \nu_y)$ and $\tau = (\tau_x, \tau_y)$. Using (4.6) and the identities

$$\nu_x = \tau_y,\ \ \nu_y = -\tau_x,\ \ \operatorname{curl} u = \operatorname{div}(w, -v),$$

we find that

$$\operatorname{curl} u = u_\nu\cdot\tau - u_\tau\cdot\nu. \tag{4.7}$$

Similarly, we have

$$Z = (X\cdot\tau)u_\tau + (X\cdot\nu)u_\nu, \tag{4.8}$$

$$Z\cdot\nu = (X\cdot\tau)(u_\tau\cdot\nu) + (X\cdot\nu)(u_\nu\cdot\nu), \tag{4.9}$$

$$Z\cdot\tau = (X\cdot\tau)(u_\tau\cdot\tau) + (X\cdot\nu)(u_\nu\cdot\tau). \tag{4.10}$$

Inserting (4.6)–(4.10) into (4.1) and rearranging the terms, we obtain the following consequence of (4.1).

Lemma 4.2. — *With the notation in Lemma 4.1, we have, for any* $X^0 \in \mathbb{R}^2$,

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_\omega \big(1-|u|^2\big)^2 = \frac{1}{2\varepsilon^2}&\int_{\partial\omega}\big(1-|u|^2\big)^2\big((X-X^0)\cdot\nu\big)\\ &+\int_{\partial\omega} Q_1\big(X-X^0, u_\tau\cdot\tau, u_\tau\cdot\nu\big)\\ &-\int_{\partial\omega} Q_2\big(X-X^0, u_\nu\cdot\tau, u_\nu\cdot\nu\big)\\ &+\int_{\partial\omega} Q_3\big(X-X^0, u_\tau\cdot\tau, u_\tau\cdot\nu, u_\nu\cdot\tau, u_\nu\cdot\nu\big),\end{aligned}\tag{4.11}$$

where the Q_j*'s are quadratic forms with coefficients depending on* $X-X^0$, *explicitly given by*

$$\begin{aligned}&Q_1\big(X-X^0,\xi_1,\xi_2\big)\\ &\qquad = K_1\big((X-X^0)\cdot\nu\big)(\xi_1)^2 + K_3\big((X-X^0)\cdot\nu\big)(\xi_2)^2\\ &\qquad\qquad - 2(K_1-K_3)\big((X-X^0)\cdot\tau\big)\xi_1\xi_2,\end{aligned}\tag{4.12}$$

$$\begin{aligned}&Q_2\big(X-X^0,\eta_1,\eta_2\big)\\ &\qquad = K_3\big((X-X^0)\cdot\nu\big)(\eta_1)^2 + K_1\big((X-X^0)\cdot\nu\big)(\eta_2)^2,\end{aligned}\tag{4.13}$$

$$\begin{aligned}&Q_3\big(X-X^0,\xi_1,\xi_2,\eta_1,\eta_2\big)\\ &\qquad = -2K_3\big((X-X^0)\cdot\tau\big)\xi_1\eta_1 - 2K_1\big((X-X^0)\cdot\tau\big)\xi_2\eta_2.\end{aligned}\tag{4.14}$$

Specializing to the case where ω is a disc, respectively a half-disc, we obtain the following consequences of our calculations.

Lemma 4.3. — *Assume* (2.8). *Let* $u \in C^3(\overline{B})$ *be a critical point of* E_ε *in a disc* B *of radius* r. *Then*

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_B\big(1-|u|^2\big)^2 + (1-\delta_1)r\int_{\partial B}|u_\nu|^2&\\ \leqslant \frac{r}{2\varepsilon^2}\int_{\partial B}\big(1-|u|^2\big)^2 &+ (1+\delta_1)r\int_{\partial B}|u_\tau|^2.\end{aligned}$$

Lemma 4.4. — *Assume* (2.8). *Then there exist some finite positive constants* $C_j = C_j(\delta_1)$, $j=1,2,3$, *such that, if* $u\in C^3(\overline{H})$ *is a critical point of* E_ε *in a half-disc* H *of radius* r, *then*

$$\frac{1}{\varepsilon^2}\int_H\big(1-|u|^2\big)^2 + C_1 r\int_{\partial H}|u_\nu|^2 \leqslant \frac{C_2 r}{\varepsilon^2}\int_{\partial H}\big(1-|u|^2\big)^2 + C_3 r\int_{\partial H}|u_\tau|^2.$$

Proof of Lemma 4.3. — The conclusion follows from (4.11)–(4.14) (with X^0 the center of B), combined with the observation that, in the case of a disc of radius r, we have $Q_3 = 0$ and

$$Q_1\big(X - X^0, \xi_1, \xi_2\big) = K_1 r(\xi_1)^2 + K_3 r(\xi_2)^2 \leqslant (1+\delta_1) r|\xi|^2,$$
$$Q_2\big(X - X^0, \eta_1, \eta_2\big) = K_1 r(\eta_2)^2 + K_3 r(\eta_1)^2 \geqslant (1-\delta_1) r|\eta|^2. \qquad \square$$

Proof of Lemma 4.4. — In what follows, C_j denotes a generic positive constant depending possibly on δ_1.

With no loss of generality, we may assume that $r = 1$ and

$$H = \big\{X = (x,y) \in \mathbb{R}^2; \quad |X| < 1,\, y > 0\big\}.$$

Let $0 < a < 1$ be any fixed number, and set $X^0 = (0, a)$. It is easy to see that

$$\big(X - X^0\big)\cdot\nu \geqslant C_3 > 0, \quad \forall\, X \in \partial H, \tag{4.15}$$
$$\big|\big(X - X^0\big)\cdot\nu\big| \leqslant C_4, \quad \forall\, X \in \partial H, \tag{4.16}$$
$$\big|\big(X - X^0\big)\cdot\tau\big| \leqslant C_5, \quad \forall\, X \in \partial H. \tag{4.17}$$

Combining (4.11)–(4.14) and (4.15)–(4.17), we find that

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_H \big(1-|u|^2\big)^2 + C_3(1-\delta_1)\int_{\partial H}|u_\nu|^2 \\
&\quad\leqslant \frac{1}{\varepsilon^2}\int_H \big(1-|u|^2\big)^2 + \int_{\partial H} Q_2\big(X - X^0, u_\nu\cdot\tau, u_\nu\cdot\nu\big) \\
&\quad\leqslant \frac{C_4}{2\varepsilon^2}\int_{\partial H}\big(1-|u|^2\big)^2 + 2(C_4+C_5)\int_{\partial H}|u_\tau|^2 \\
&\qquad + 4C_4\int_{\partial H}\big(|u_\tau\cdot\tau||u_\nu\cdot\tau| + |u_\tau\cdot\nu||u_\nu\cdot\nu|\big) \\
&\quad\leqslant \frac{C_4}{2\varepsilon^2}\int_{\partial H}\big(1-|u|^2\big)^2 + 2(C_4+C_5)\int_{\partial H}|u_\tau|^2 + 4C_4\int_{\partial H}|u_\tau||u_\nu| \\
&\quad\leqslant \frac{C_4}{2\varepsilon^2}\int_{\partial H}\big(1-|u|^2\big)^2 + 2(C_4+C_5)\int_{\partial H}|u_\tau|^2 \\
&\qquad + \frac{1}{2}C_3(1-\delta_1)\int_{\partial H}|u_\nu|^2 + C_6\int_{\partial H}|u_\tau|^2,
\end{aligned}$$

whence the conclusion of the lemma. $\square$

By a straightforward modification of the proof of Lemma 4.4, we obtain the following result in a fixed bounded domain Ω, that we state without proof.

Lemma 4.5. — *Assume* (2.8)*. Then there exist some finite positive constants* $C_j = C_j(\delta_1)$*,* $j = 1,2,3$*, and* $r_0 = r_0(\delta_1,\Omega)$ *such that, if* $u \in C^3(\overline{B_r(x_0)\cap\Omega})$ *is a critical point of* E_ε *in* $B_r(x_0)\cap\Omega$*, with* $r \leqslant r_0$ *and* $x_0 \in \partial\Omega$*, then*

$$\frac{1}{\varepsilon^2}\int_{B_r(x_0)\cap\Omega}\left(1-|u|^2\right)^2 + C_1 r\int_{\partial(B_r(x_0)\cap\Omega)}|u_\nu|^2 \\ \leqslant \frac{C_2 r}{\varepsilon^2}\int_{\partial(B_r(x_0)\cap\Omega)}\left(1-|u|^2\right)^2 + C_3 r\int_{\partial(B_r(x_0)\cap\Omega)}|u_\tau|^2. \quad (4.18)$$

When Ω is strictly star-shaped, the local estimate (4.18) upgrades to a global estimate.

Lemma 4.6. — *Assume* (2.8)*. Assume that* Ω *is strictly star-shaped, i.e., that there exist some* $X^0 \in \Omega$ *and* $C_3 > 0$ *such that*

$$\left(X - X^0\right)\cdot\nu \geqslant C_3,\ \forall\, X \in \partial\Omega. \quad (4.19)$$

Let $u = u_\varepsilon$ *be a critical point of* E_ε *in* $H^1_g(\Omega)$*. Then there exists a finite positive constant* $C = C(\delta_1, C_3)$ *such that*

$$\frac{1}{\varepsilon^2}\int_\Omega\left(1-|u|^2\right)^2 + \int_{\partial\Omega}|u_\nu|^2 \leqslant C\int_{\partial\Omega}|g_\tau|^2. \quad (4.20)$$

Proof. — It suffices to copy the proof of Lemma 4.4. There, the existence of C_3 follows from the geometry of H. In our case, the existence of C_3 is an assumption. □

Lemmata 4.3 and 4.5 yield the following a priori estimates for critical points of E_ε satisfying a natural bound on the energy. In particular, thanks to the energy estimate (2.5), these bounds apply to minimizers of E_ε in $H^1_g(\Omega;\mathbb{C})$. Note, however, that the estimates below do not imply (3.1), since the constants we obtain below depend on the energy bound (which in turn depends on the boundary datum g).

Lemma 4.7. — *Assume* (2.8)*. Let* $u = u_\varepsilon$ *be critical points of* E_ε *in* $H^1_g(\Omega)$ *satisfying the bound*

$$E_\varepsilon(u) \leqslant K|\ln\varepsilon|. \quad (4.21)$$

Then, with finite constants $C_j = C_j(K,\delta_1)$*,* $j = 1,2$*, we have, for small* ε*,*

$$|u| \leqslant C_1, \quad (4.22)$$

$$|\nabla u| \leqslant C_2/\varepsilon. \quad (4.23)$$

In particular, if u *minimizes* E_ε *in* $H^1_g(\Omega)$*, then* C_1*,* C_2 *may be chosen to depend only on* $\deg g$ *and* δ_1*.*

Combining Lemma 4.7 with Lemma 2.5, we obtain the following

Corollary 4.8. — *Assume* (2.8). *Fix* $g \in C^\infty(\partial\Omega;\mathbb{S}^1)$. *Let* $u = u_\varepsilon$, $0 < \varepsilon \leqslant 1$, *be critical points of* E_ε *in* $H^1_g(\Omega)$ *satisfying the energy bound* (4.21). *Then* (2.18) *holds.*

Proof of Lemma 4.7. — The proof is similar to the one of Lemma 3.5, with the variation that, for clarity, we perform a blow-up.

In what follows, C_j, $j \geqslant 3$, denotes a finite positive constant depending only on K and possibly δ_1, and D_j a finite positive universal constant. Let $x \in \overline{\Omega}$. With the notation in the proof of Lemma 3.1, we have either $B_{\varepsilon^{\alpha_1}}(x) \subset \Omega$, or $B_{\varepsilon^{\alpha_1}}(x) \not\subset \Omega$. We consider only the first case, the other one being similar. Pick some $4\varepsilon \leqslant \varepsilon^\beta < r < \varepsilon^{\alpha_1}$ such that

$$r\int_{C_r(x)} |\nabla u|^2 + \frac{r}{\varepsilon^2}\int_{C_r(x)} \left(1-|u|^2\right)^2 \leqslant C_3. \tag{4.24}$$

By (4.24) and Lemma 4.3, we have

$$\frac{1}{\varepsilon^2}\int_{B_r(x)} \left(1-|u|^2\right)^2 \leqslant C_4. \tag{4.25}$$

Assume, for simplicity, that $x = 0$. Set $\widetilde{u}(y) := u(\varepsilon y)$, $y \in B := B_4(0)$. Then

$$\mathcal{L}\widetilde{u} = \widetilde{f} := \widetilde{u}\left(1-|\widetilde{u}|^2\right) \quad \text{in } B,$$

so that, by standard elliptic estimates,

$$\|\widetilde{u}\|_{C^{1/2}(B_*)} \leqslant D_1\|\widetilde{f}\|_{L^{4/3}(B)} + D_2\|\widetilde{u}\|_{L^4(B)}, \tag{4.26}$$

where B_* is as defined in (2.9). By (4.25), (4.26), (3.20), and the inequality

$$|w|^4 \leqslant D_3\left(1-|w|^2\right)^2 + 1, \quad \forall\, w \in \mathbb{C},$$

we find (going back to u) that

$$|u(z) - u(t)| \leqslant C_5, \quad \forall\, z, \quad t \in B_{2\varepsilon}(x). \tag{4.27}$$

Combining (4.27) with (4.25), we find that (4.22) holds in $B_{2\varepsilon}(x)$. Then, using (4.22) in $B_{2\varepsilon}(x)$ and the estimate

$$\|\nabla\widetilde{u}\|_{L^\infty(B_{**})} \leqslant D_4\|\widetilde{f}\|_{L^2(B_*)} + D_4\|\widetilde{u}\|_{L^\infty(B_*)},$$

(with $B_* = B_2(0)$, $B_{**} = B_1(0)$) and going back to u, we find that (4.23) holds in $B_\varepsilon(x)$. $\square$

Lemma 4.9. — *Assume* (2.8). *Assume that* Ω *is strictly star-shaped. Consider critical points* $u = u_\varepsilon$ *of* E_ε *in* $H^1_g(\Omega)$ *satisfying the energy bound* (4.21). *Then, for some finite constant* $C_1 = C_1(K, \delta_1, \Omega, g)$ *and small* ε*, we have*

$$\int_\Omega |1 - |u|^2| \times |\nabla u|^2 \leqslant C_1. \tag{4.28}$$

Proof. — By the Gagliardo–Nirenberg inequality, (4.22), standard elliptic estimates, and Lemma 4.6, we have the following (global in Ω) estimates, with constants depending only on K, δ_1, Ω, g:

$$\begin{aligned} \|\nabla u\|_4^2 \leqslant & C_2 \|u\|_\infty \|u\|_{H^2} \leqslant C_3 \|u\|_{H^2} \leqslant C_4(\|g\|_{H^{3/2}} + \|\mathscr{L} u\|_2) \\ \leqslant & C_5\big(1 + \varepsilon^{-2} \big\|1 - |u|^2\big\|_2\big) \leqslant \frac{C_6}{\varepsilon}. \end{aligned} \tag{4.29}$$

We obtain (4.28) from (4.20), (4.29), and Cauchy–Schwarz. □

5. Bad discs structure

In this section, we provide some easy consequences of the a priori estimates established in the previous sections. We first define the notion of bad disc. A bad disc $B = B_{C\varepsilon}(x_\varepsilon)$ is a disc of radius $C\varepsilon$, centered at x_ε, such that $|u_\varepsilon(x_\varepsilon)| \leqslant 1/2$ and $|u_\varepsilon(x)| \geqslant 1/2$ on $\partial(B \cap \Omega)$. Note that in our situation we have $u_\varepsilon \in H^1_g(\Omega)$, and thus the latter condition is equivalent to $|u_\varepsilon(x)| \geqslant 1/2$ on $\partial B \cap \Omega$. Here, the constant C could possibly depend on a sequence $\varepsilon_\ell \to 0$, but its size is controlled by the a priori estimates available on u.

Lemma 5.1. — *Assume that* (2.8) *holds. We have:*

(1) *Suppose that* Ω *is strictly star-shaped. Consider critical points* $u = u_\varepsilon$ *of* E_ε *in* $H^1_g(\Omega)$ *satisfying the energy bound* (4.21), *where* $\varepsilon = \varepsilon_\ell \to 0$. *Set* $A_\varepsilon := \{x \in \Omega;\, |u(x)| \leqslant 1/2\}$. *Then there exist finite constants* $N = N(K, \delta_1, \Omega, \|g_\tau\|_2)$ *and* $L = L(K, \delta_1)$ *such that, possibly along a subsequence* (ε_{ℓ_m}),

$$A_\varepsilon \text{ can be covered with at most } N \text{ bad discs } B_{C\varepsilon}\big(x^j_\varepsilon\big), \tag{5.1}$$

for some constant C *possibly depending on* (ε_ℓ) *such that*

$$3 \leqslant C \leqslant L, \tag{5.2}$$

and

$$\text{for } j \neq k, \quad \big|x^j_\varepsilon - x^k_\varepsilon\big| \geqslant 4C\varepsilon. \tag{5.3}$$

Moreover, there exists some finite number $C' \geqslant C$, possibly depending on (ε_ℓ), such that, possibly along a further subsequence $(\varepsilon_{\ell_{m_n}})$,

$$A_\varepsilon \text{ can be covered with at most } N \text{ ``enlarged'' bad discs } B_{C'\varepsilon}\big(x_\varepsilon^j\big) \tag{5.4}$$

such that

$$\text{for } j \neq k, \big|x_\varepsilon^j - x_\varepsilon^k\big| \gg \varepsilon \text{ as } \varepsilon \longrightarrow 0. \tag{5.5}$$

(2) *The same conclusions hold if Ω is arbitrary and u is a minimizer of E_ε in $H_g^1(\Omega)$, where this time $N = N(\deg g, \delta_1)$ and $L = L(\deg g, \delta_1)$.*

(3) *For each ε, consider: either (i) critical points of E_ε in a strictly star-shaped domain Ω, satisfying the energy bound* (4.21) *; or (ii) minimizers of E_ε. Then there exists a $C = C(\varepsilon)$ satisfying* (5.1)–(5.2).

Note that the price to pay in order to have (5.5) is the lack of control on the constant C'.

Proof.

(1). — By (4.23) in Lemma 4.7, there exists some $\lambda > 0$ such that, if $\varepsilon > 0$ is small, we have

$$\left[x \in \Omega, \int_{B_\varepsilon(x)\cap\Omega} \left(1 - |u|^2\right)^2 \leqslant \lambda\right] \Longrightarrow [|u| \geqslant 1/2 \text{ in } B_\varepsilon(x)]. \tag{5.6}$$

Combining this fact with the a priori estimate (4.20), we find that any disjoint family of discs $B = B_{\varepsilon/3}(x)$ such that $|u(x)| \leqslant 1/2$ has at most N elements, where $N = N(K, \delta_1, \Omega, \|g_\tau\|_2)$. Therefore, the set A_ε can be covered with at most N discs $B_i^\varepsilon = B_\varepsilon(x_i^\varepsilon)$, satisfying $|u(x_i^\varepsilon)| \leqslant 1/2$.

We next enlarge these discs in order to obtain bad discs. For this purpose, let us note the following. Fix an integer M and consider, for each ε, at most M intervals $I_1^\varepsilon, \ldots, I_k^\varepsilon$, each of length $\leqslant 2\varepsilon$. Then, up to a sequence $\varepsilon_\ell \to 0$, there exists some $3 \leqslant C \leqslant 3(M+1)$ such that $C\varepsilon \notin \bigcup_k I_k^\varepsilon$. (A similar conclusion can be drawn if we start from a sequence $\varepsilon_\ell \to 0$, possibly after passing to a subsequence.) Indeed, the union of the I_k^ε's cannot contain all the points $n\varepsilon$, with $n = 3, 6, \ldots, 3(M+1)$, and thus, up to a subsequence, we may take $C = 3n$, for one of these n's. Applying this observation to the sets $\{|x - x_i^\varepsilon|; x \in B_j^\varepsilon\}$, we find that there exists some $3 \leqslant C \leqslant 3(N+1)(N+2)/2$ such that the discs $B_{C\varepsilon}(x_i^\varepsilon)$ cover A_ε and, in addition, $C_{C\varepsilon}(x_i^\varepsilon)$ does not intersect any of the $B_\varepsilon(x_j^\varepsilon)$. Therefore, we have $C_{C\varepsilon}(x_i^\varepsilon) \cap A_\varepsilon = \emptyset$. We find that each $B_{C\varepsilon}(x_i^\varepsilon)$ is a bad disc.

Finally, the existence of enlarged bad discs satisfying the additional properties (5.3) or (5.5) is then obtained as in [2, Chapter IV]. (This may require passing to a further subsequence.)

(2). — The proof is essentially the same, thanks to the upper bound (2.5). The only difference arises from the argument leading to the existence of $N = N(\delta_1, \deg g)$, since we are not in position to rely on the assumption (4.19). Let $0 < \beta < \alpha < 1$ be fixed. Let $0 < \lambda < 1/2$ and let η be as in Lemma 3.1 (corresponding to this λ). If ε is sufficiently small and $x \in A_\varepsilon$, then $E_\varepsilon(u_\varepsilon, B_{\varepsilon^\alpha/5}(x)) \geqslant \eta|\ln\varepsilon|$. Therefore, there exists some $N_1 = N_1(\deg g, \delta_1)$ such that A_ε can be covered with at most N_1 admissible balls $B_{\varepsilon^\alpha}(x_i^\varepsilon)$. By a mean value argument, there exist a constant $C_0 = C_0(\deg g, \alpha, \beta)$ and radii $r = r_i^\varepsilon$ such that $\varepsilon^\alpha \leqslant r_i^\varepsilon \leqslant \varepsilon^\beta$ and

$$r\int_{\partial\left(B_r(x_i^\varepsilon)\cap\Omega\right)} |u_\tau|^2 + \frac{r}{\varepsilon^2}\int_{\partial\left(B_r(x_i^\varepsilon)\cap\Omega\right)} \left(1-|u|^2\right)^2 \leqslant C_0. \tag{5.7}$$

Combining (5.7) with (4.18), we find that, for some constant $D = D(\deg g, \delta_1)$, we have, for small ε,

$$\frac{1}{\varepsilon^2}\int_{B_{\varepsilon^\alpha}\left(x_i^\varepsilon\right)\cap\Omega} \left(1-|u|^2\right)^2 \leqslant D, \quad \forall\, i. \tag{5.8}$$

Using (5.8) and arguing as in the proof of (1), we find that $A_\varepsilon \cap B_{\varepsilon^\alpha}(x_i^\varepsilon)$ can be covered with at most $N_2 = N_2(\deg g, \delta_1)$ discs $B_\varepsilon(x)$. This yields a covering of A_ε with at most N_1N_2 discs, each of radius ε.

(3). — By the argument used in the proof of (1), we may actually choose some $C(\varepsilon) \in [3, 3(N+1)(N+2)/2]$. □

6. The energy is bounded on "intermediate" balls away from the bad discs

In this section, we derive an easy consequence of the results in Section 4 combined with the bad discs structure. We consider the setting of Section 3 *and we assume* (2.8). Let Ω and the boundary datum $g \in C^\infty(\partial\Omega; \mathbb{S}^1)$ be fixed. *Let $u = u_\varepsilon$ be a minimizer of E_ε in $H^1_g(\Omega;\mathbb{C})$.* By Lemma 5.1 (2), we may find an integer N and a finite constant C, depending only on $\deg g$, such that, for small ε (depending on Ω and g), (5.1) holds.

An inspection of the proofs in Section 3 shows that the smallness of the constant η plays a role mainly in the existence of a suitable radius r such that, on $\partial(B_r(x)\cap\Omega)$, (i) $|u_\varepsilon|$ is far away from 0 and (ii) $u_\varepsilon/|u_\varepsilon|$ has degree 0. This is especially useful in Lemmata 3.2 and 3.3. *If we know that the* *assumptions (i) and (ii) are valid for all "useful" radii r (i.e., for the radii obtained* via *a mean value argument, as, e.g., in* (3.8)*), then Lemmata 3.2 and 3.3 hold without the smallness assumption on η.* These considerations lead to the following result.

Lemma 6.1. — *Let $0 < \alpha < 1 < \beta < 1/\alpha$ and $\varepsilon < 1/2$. Let S_ε denote the union of the bad discs in* (5.1) *and suppose that $B = B_R(x)$ is a ball such that*

$$\varepsilon^{1/\beta} \leqslant R \leqslant \varepsilon^{\alpha} \quad \textit{and} \quad \operatorname{dist}(x, S_\varepsilon) \geqslant R. \tag{6.1}$$

Then, for sufficiently small ε and some finite constant $C_1 = C_1(\deg g, \delta_1, \alpha, \beta)$, we have

$$E_\varepsilon(u, B_{R^\beta}(x) \cap \Omega) \leqslant C_1 \quad \textit{and} \quad G_\varepsilon(u, B_{R^\beta}(x) \cap \Omega) \leqslant C_1. \tag{6.2}$$

Sketch of proof. — In what follows, $C_j = C_j(\deg g, \delta_1, \alpha, \beta)$ is a finite constant, and ε is sufficiently small. By Lemma 2.2, the assumption $\varepsilon^{1/\beta} \leqslant R \leqslant \varepsilon^{\alpha}$, and a mean value argument, there exists some $R^\beta < r < R/2$ such that

$$r \int_{\partial B_r(x) \cap \Omega} |u_\tau|^2 + \frac{r}{\varepsilon^2} \int_{\partial B_r(x) \cap \Omega} \left(1 - |u|^2\right)^2 \leqslant C_2. \tag{6.3}$$

Let C be the constant in (5.1). Since $\operatorname{dist}(\overline{B}_r(x), S_\varepsilon)) \geqslant R/2 \geqslant C\varepsilon$, we find that $|u| \geqslant 1/2$ in $\overline{B}_r(x) \cap \overline{\Omega}$, and thus, in particular, $u/|u|$ has degree zero on $\partial(B_r(x) \cap \Omega)$. By the above, we are now in position to repeat the proof of (3.3) in Lemma 3.2 and obtain the estimates

$$\left| \int_{C_r(x) \cap \Omega} u \wedge u_\tau \right| \leqslant C_3 \quad \text{and} \quad G_\varepsilon(u, B_r(x) \cap \Omega) \leqslant C_4. \tag{6.4}$$

Combining (6.4) and (2.3), we find that

$$E_\varepsilon(u, B_r(x) \cap \Omega) \leqslant C_5. \tag{6.5}$$

Finally, (6.2) follows from the second inequality in (6.4) and (6.5). $\square$

In Section 13, we will encounter an avatar of the above considerations; see Lemma 13.4 there.

7. Blow-up near the boundary

In this section, we prove that the bad discs described in the previous section are far away from the boundary at the ε scale. This fact is an obvious consequence of the following result.

Lemma 7.1. — *Let $u = u_\varepsilon$ be critical points of E_ε in $H^1_g(\Omega)$ satisfying the energy bound (4.21). Let C be a fixed constant. Consider, for each ε, a point $y_\varepsilon \in \Omega$ such that* $\operatorname{dist}(y_\varepsilon, \partial\Omega) \leqslant C\varepsilon$ *and let, for small ε, z_ε be the nearest point projection of y_ε on $\partial\Omega$. Then*

$$u_\varepsilon(y_\varepsilon) - g(z_\varepsilon) \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0. \tag{7.1}$$

Corollary 7.2. — *Under the assumptions of Lemma 5.1, the centers x^j_ε of the bad discs satisfy* $\operatorname{dist}(x^j_\varepsilon, \partial\Omega) \gg \varepsilon$ *as $\varepsilon \to 0$.*

Proof of Lemma 7.1. — It suffices to obtain (7.1) up to a subsequence. This is obtained *via* a blow-up analysis. Consider the rescaled maps

$$v_\varepsilon(z) := u_\varepsilon(\varepsilon z + z_\varepsilon), \quad \forall\, z \in U_\varepsilon := \varepsilon^{-1}(-z_\varepsilon + \Omega),$$

extended with the same formula to $\overline{U}_\varepsilon$.

Note that $0 \in \partial U_\varepsilon$. Up to a subsequence, we have $g(z_\varepsilon) \to \widetilde{g}$ for some constant $\widetilde{g} \in \mathbb{S}^1$, and the unit outer normal to U_ε at the origin converges to some $\xi \in \mathbb{S}^1$. We work with such a subsequence. Consider the half-plane $H := \{X \in \mathbb{R}^2;\ X \cdot \xi < 0\}$.

We next note that, by Corollary 4.8, v_ε has bounded derivatives, at any order. Moreover, the tangential derivative of v_ε on ∂U_ε is (uniformly) of the order of ε. By the above, possibly up to a further subsequence, we have $v_\varepsilon \to v$ pointwise in H and uniformly on bounded sets, where the map v is smooth in $\overline{H}$, satisfies $v = \widetilde{g}$ on ∂H, and is a solution of

$$\mathcal{L}v = v\big(1 - |v|^2\big) \quad \text{in } H. \tag{7.2}$$

(To be specific, uniform convergence on bounded sets means that

$$\max\big\{|v_\varepsilon(x) - v(x)|; x \in K \cap \overline{U}_\varepsilon\big\} \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0, \qquad \forall\, K \subset H \text{ compact.}) \tag{7.3}$$

We note that the conclusion of the lemma amounts to $v = \widetilde{g}$. In order to obtain this conclusion, we first establish an additional property of v. By Lemma 4.5, the energy bound (4.21) and a mean value argument, there exist a finite constant C (possibly depending on δ_1, K, Ω, and g) and some $r = r_\varepsilon \in (\varepsilon^{1/2}, \varepsilon^{1/3})$ such that

$$r \int_{B_r(z_\varepsilon)\cap\partial\Omega} |u_{\varepsilon,\nu}|^2 \leqslant C. \tag{7.4}$$

Estimate (7.4) is equivalent to

$$\int_{B_{r/\varepsilon}(0)\cap\partial U_\varepsilon} |v_{\varepsilon,\nu}|^2 \leqslant C\frac{\varepsilon}{r}. \tag{7.5}$$

Using (7.5), Corollary 4.8, and the fact that $\varepsilon/r \to 0$ as $\varepsilon \to 0$, we find that v satisfies

$$\begin{cases} v \in C^\infty(\overline{H}) \\ \mathcal{L}v = v(1-|v|^2) & \text{in } H \\ v = \widetilde{g} \in \mathbb{S}^1 & \text{on } \partial H \\ v_\nu = 0 & \text{on } \partial H. \end{cases} \tag{7.6}$$

We now complete the proof of the lemma.

Claim. The only solution of (7.6) *is* $v \equiv \widetilde{g}$. — In order to prove the claim, we extend v to $\mathbb{R}^2$ with the value $\widetilde{g}$ on $H_- := \mathbb{R}^2 \setminus \overline{H}$, and still denote the extension by v. We note that $v \in H^1_{\text{loc}}(\mathbb{R}^2)$ and that $\mathcal{L}v = v(1-|v|^2)$ in H and also in H_-. The key observation is that we actually have

$$\mathcal{L}v = v\big(1-|v|^2\big) \quad \text{in the weak sense in } \mathbb{R}^2. \tag{7.7}$$

Indeed, since $v \equiv \widetilde{g}$ =constant on ∂H and $v_\nu = 0$ on ∂H, we find that $\nabla v = 0$ on ∂H. Therefore, if $\varphi \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ and we write $\widetilde{g} = (\widetilde{g}^1, \widetilde{g}^2)$, $\varphi = (\varphi^1, \varphi^2)$, then (using the fact that $\mathcal{L}$ is formally self-adjoint)

$$\int_H v \cdot {}^t\mathcal{L}\varphi = \int_H v \cdot \mathcal{L}\varphi = T + \int_H \mathcal{L}v \cdot \varphi = T + \int_H \varphi \cdot v\big(1-|v|^2\big),$$

$$\begin{aligned} \int_{H_-} v \cdot {}^t\mathcal{L}\varphi = \int_{H_-} v \cdot \mathcal{L}\varphi = -T + \int_{H_-} \mathcal{L}v \cdot \varphi &= -T \\ &= -T + \int_{H_-} \varphi \cdot v\big(1-|v|^2\big), \end{aligned}$$

$$\int_{\mathbb{R}^2} v \cdot {}^t\mathcal{L}\varphi = \int_{\mathbb{R}^2} v \cdot \mathcal{L}\varphi = \int_{\mathbb{R}^2} \varphi \cdot v\big(1-|v|^2\big),$$

where T is the boundary term

$$\begin{aligned} T := \int_{\partial H} \Big[-K_1 \widetilde{g}^1 \nu_x \big(\varphi^1_x + \varphi^2_y\big) - K_3 \widetilde{g}^1 \nu_y \big(\varphi^1_y - \varphi^2_x\big) \\ - K_1 \widetilde{g}^2 \nu_y \big(\varphi^1_x + \varphi^2_y\big) + K_3 \widetilde{g}^2 \nu_x \big(\varphi^1_y - \varphi^2_x\big) \Big], \end{aligned}$$

whence (7.7).

We complete the proof of the claim, and thus of the lemma, by noting that the definition of v on H_- combined with Lemma 2.6 implies that $v \equiv \widetilde{g}$. $\square$

8. Blow-up of enlarged bad discs

The results in this section are valid under the assumption (2.8)*. We consider maps $u = u_\varepsilon \in H^1_g(\Omega)$ satisfying the assumptions of Lemma 5.1, i.e.: either (i) critical points of E_ε in $H^1_g(\Omega)$ in a strictly star-shaped domain Ω, satisfying the a priori bound* (4.21)*; or (ii) minimizers of E_ε in $H^1_g(\Omega)$.* In particular, the conclusions of Lemma 5.1 hold.

Consider an enlarged bad disc as in Lemma 5.1. Consider the rescaled maps v_ε as in the proof of Lemma 7.1, with z_ε replaced with the center of the bad disc. By Lemma 2.5, Corollary 7.2 and the a priori estimates (4.22), (4.23), and (2.19) (all valid, for small ε, as a consequence of the assumptions considered above), the following hold, with constants independent of small ε, δ satisfying (2.8), the boundary datum g, and with $R_\varepsilon \to \infty$ as $\varepsilon \to 0$:

$$v_\varepsilon \text{ is defined in } B_{R_\varepsilon}(0), \tag{8.1}$$

$$|v_\varepsilon(0)| \leqslant 1/2, \tag{8.2}$$

$$\left|D^k v_\varepsilon(x)\right| \leqslant \widetilde{C}_k \quad \text{in } B_{R_\varepsilon}(0),\ \forall\, k, \tag{8.3}$$

$$\mathcal{L} v_\varepsilon = v_\varepsilon\big(1 - |v_\varepsilon|^2\big) \quad \text{in } B_{R_\varepsilon}(0). \tag{8.4}$$

Moreover, there exists finite constants D_1, D_2 (possibly depending on g if u_ε is merely a critical point of E_ε) such that

$$\int_{B_{R_\varepsilon}(0)} \big(1 - |v_\varepsilon|^2\big)^2 \leqslant D_1, \tag{8.5}$$

$$|v_\varepsilon(x)| \geqslant 1/2 \quad \text{if } D_2 \leqslant |x| < R_\varepsilon. \tag{8.6}$$

In addition, if u_ε is a minimizer of E_ε,

$$\begin{gathered} v_\varepsilon \text{ is a minimizer of } E_1 \text{ in } B_{R_\varepsilon}(0) \text{ with} \\ \text{respect to its own boundary condition.} \end{gathered} \tag{8.7}$$

It follows that, possibly up to a subsequence, (v_ε) converges in $C^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ to a smooth map $v : \mathbb{R}^2 \to \mathbb{C}$ such that

$$|v(0)| \leqslant 1/2, \tag{8.8}$$

$$|D^k v| \leqslant \widetilde{C}_k,\ \forall\, k, \tag{8.9}$$

$$\mathcal{L} v = v\big(1 - |v|^2\big), \tag{8.10}$$

$$\int_{\mathbb{R}^2} \big(1 - |v|^2\big)^2 < \infty, \tag{8.11}$$

and, if u_ε is a minimizer of E_ε,

$$v \text{ is an entire local minimizer of } E_1 \text{ (in the sense of De Giorgi).} \tag{8.12}$$

For further use, let us note that any map v satisfying (8.9) and (8.11) satisfies $\lim_{|x|\to\infty}|v(x)|=1$, and thus v has a "degree at ∞", in the sense that, for large R (depending on v), the integer $\deg v := \deg(v/|v|, C_R(0))$ is well-defined and independent of R.

In what follows, we derive some easy consequences of the analysis developed up to now.

Corollary 8.1. — *For every δ, there exists a bounded entire local minimizer of E_1 satisfying* (8.11) *and of negative degree.*

Proof. — Consider any domain Ω and any boundary datum of negative degree. The enlarged bad discs in Lemma 5.1 satisfy

$$\sum_j \deg\big(u_\varepsilon/|u_\varepsilon|, C_{C'\varepsilon}(x_j^\varepsilon)\big) = \deg(g, \partial\Omega)$$

(since u_ε does not vanish in $\overline{\Omega}\setminus\bigcup_j B_{C'\varepsilon}(x_\varepsilon^j)$). Therefore, at least one of them has negative degree on $C_{C'\varepsilon}(x_j^\varepsilon)$. Blowing-up this bad disc and possibly after passing to a subsequence, we obtain a v as in the above statement. ☐

Remark 8.2. — With more work, it is possible to remove the assumption (8.11) and establish the following analogue of the main result in Sandier [17]. Let v be a bounded entire local minimizer of E_1. Then $\int_{\mathbb{R}^2}(1-|v|^2)^2<\infty$. However, it is unclear how to remove the boundedness assumption on v.

We next note a first "small δ" result.

Lemma 8.3. — *We fix a boundary datum $g\in C^\infty(\partial\Omega;\mathbb{S}^1)$. Let u_ε be a minimizer of E_ε in $H^1_g(\Omega;\mathbb{C})$. Let $0<\lambda<2\pi$. There exist finite constants δ_0, C_1, C_2 depending only on λ, such that, if $\delta<\delta_0$ and $\varepsilon<\varepsilon_0(\delta,\lambda)$, and v_ε is as above, then:*

$$\int_{B_{C_1}(0)}\big(1-|v_\varepsilon|^2\big)^2 \geqslant \lambda, \tag{8.13}$$

$$|v_\varepsilon(x)|\geqslant 1/2 \quad \textit{if} \quad C_2\leqslant |x|\leqslant R_\varepsilon, \tag{8.14}$$

$$\deg(v_\varepsilon/|v_\varepsilon|, C_{C_2}(0)) = \pm 1. \tag{8.15}$$

Corollary 8.4. — *If $\delta<\delta_0$ and u_ε minimizes E_ε in $H^1_g(\Omega;\mathbb{C})$, then we may replace, in the definition of the enlarged bad discs, the condition $|u_\varepsilon(x_\varepsilon^j)|\leqslant 1/2$ with $u_\varepsilon(x_\varepsilon^j)=0$.*

Proof. — This follows by noting that, by (8.14) and (8.15), v has to vanish in $B_{C_2}(0)$. ☐

Combining Lemma 8.3 with the proof of Corollary 8.1, we obtain the following

Corollary 8.5. — *If $\delta < \delta_0$, then there exists an entire local minimizer satisfying* (8.11) *and of degree* -1.

Proof of Lemma 8.3. — For a fixed δ, consider any v^δ arising as a $C^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ limit of v_ε (possibly up to a subsequence $\varepsilon_k \to 0$). The conclusion of the lemma follows provided any such v^δ has, for small δ and with respect to appropriate constants C_1 and C_2, the following properties:

$$\int_{B_{C_1}(0)} \left(1 - |v^\delta|^2\right)^2 > \lambda, \tag{8.16}$$

$$\left|v^\delta(x)\right| \geqslant 2/3 \quad \text{if} \quad |x| \geqslant C_2, \tag{8.17}$$

$$\deg\left(v^\delta/|v^\delta|, C_{C_2}(0)\right) = \pm 1. \tag{8.18}$$

In turn, (8.16)–(8.18) hold if any v arising as a $C^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ limit of v^δ (possibly up to a subsequence) satisfies

$$\int_{B_{C_1}(0)} \left(1 - |v|^2\right)^2 > \lambda, \tag{8.19}$$

$$|v(x)| \geqslant 3/4 \quad \text{if} \quad |x| \geqslant C_2, \tag{8.20}$$

$$\deg(v/|v|, C_{C_2}(0)) = \pm 1. \tag{8.21}$$

In order to prove (8.19)–(8.21), we note that (8.8) and (8.12) applied to v^δ yield, by letting $\delta \to 0$, that

$$|v(0)| \leqslant 1/2 \tag{8.22}$$

and v is an entire local minimizer of G_1. (Here, we use the fact that, when $\delta = 0$, the minimization of E_1 is equivalent to the minimization of G_1; see the proof of Lemma 2.1.) Such minimizers are either constants of modulus 1 (which cannot happen in our case, by (8.22)) or, up to a rotation and translation, of the form $v(re^{\imath\theta}) = f(r)e^{\pm\imath\theta}$ [13]. Here, $f \geqslant 0$ is (strictly) increasing and uniquely determined by the equation $-\Delta v = v(1 - |v|^2)$ and the condition $f(r) \to 1$ as $r \to \infty$. Moreover, for such v we have, by a straightforward application of Pohozaev's identity,

$$\int_{\mathbb{R}^2} \left(1 - |v|^2\right)^2 = 2\pi \tag{8.23}$$

(see also Brezis, Merle, and Rivière [3] for a more general result). If, for $0 < t < 1$, we let r_t be the unique solution of $f(r_t) = t$, then, by (8.23) and the monotonicity of f, (8.19) holds for large C_1, while (using (8.22)) (8.20) and (8.21) hold with $C_2 := r_{3/4} + r_{1/2}$. □

9. Small δ analysis. Bad discs structure for minimizers

Throughout this section, we consider minimizers $u = u_\varepsilon$ of E_ε in $H^1_g(\Omega;\mathbb{C})$, with boundary datum of degree $-D < 0$. The main result of this section is the following.

Theorem 9.1. — *There exists some $0 < \delta_2 < 1$, possibly depending on D, but not on Ω or g, such that, if $0 \leqslant \delta \leqslant \delta_2$ and ε is small, then u has exactly D enlarged bad discs, all of degree -1.*

The proof of Theorem 9.1, which is somewhat similar to [19, Proof of Theorem 5.4], is slightly easier in the case where Ω is strictly star-shaped. We start with this case, and later we present the minor modifications to be made in order to treat the general case. A first key ingredient is the following straightforward variant of Lemma 4.3 combined with Lemma 4.6.

Lemma 9.2. — *Let $0 \leqslant \delta \leqslant \delta_2 < 1$. If u is a critical point of E_ε in $H^1_g(\Omega)$, then there exists some finite constant C depending only on δ_2 such that, for every $x \in \Omega$ and $r > 0$,*

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{B_r(x)\cap\Omega}\left(1-|u|^2\right)^2 + (1-\delta_2)r\int_{C_r(x)\cap\Omega}|u_\nu|^2 \\ \leqslant \frac{r}{2\varepsilon^2}\int_{C_r(x)\cap\Omega}\left(1-|u|^2\right)^2 + (1+\delta_2)r\int_{C_r(x)\cap\Omega}|u_\tau|^2 \\ + Cr\int_{B_r(x)\cap\partial\Omega}|\nabla u|^2. \quad (9.1)\end{aligned}$$

If, moreover, Ω is strictly star-shaped, then there exists some finite constant $\widetilde{C}$, depending on δ_2, Ω and g, such that, for every $x \in \Omega$ and $r > 0$,

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{B_r(x)\cap\Omega}\left(1-|u|^2\right)^2 + (1-\delta_2)r\int_{C_r(x)\cap\Omega}|u_\nu|^2 \\ \leqslant \frac{r}{2\varepsilon^2}\int_{C_r(x)\cap\Omega}\left(1-|u|^2\right)^2 + (1+\delta_2)r\int_{C_r(x)\cap\Omega}|u_\tau|^2 + \widetilde{C}r, \quad (9.2)\end{aligned}$$

and therefore

$$\begin{aligned}\int_{C_r(x)\cap\Omega}\left[\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left(1-|u|^2\right)^2\right] \\ \geqslant \frac{1}{r}\left[\frac{1}{2(1+\delta_2)\varepsilon^2}\int_{B_r(x)\cap\Omega}\left(1-|u|^2\right)^2 - \frac{\widetilde{C}r}{2(1+\delta_2)}\right]. \quad (9.3)\end{aligned}$$

A second ingredient is reminiscent of the expanding balls technique in [19, Chapter 4]. Although we could adapt to our context the more general arguments in [19], we rely on a very simple result, sufficient for our purposes. Since the proof does not "see" the space dimension, we state the result in $\mathbb{R}^n$ (and use it in $\mathbb{R}^2$).

Lemma 9.3. — *Let $n \geqslant 2$, $R_0 > 0$, and $X \subset \mathbb{R}^n$. Set $U := \{x \in \mathbb{R}^n; \operatorname{dist}(x, X) \leqslant R_0\}$. Consider an integer N, a radius $0 < R \leqslant 3^{1-N} R_0$, and N (not necessarily disjoint) closed balls of radius R, $B_j = \overline{B}_R(x_j)$, $1 \leqslant j \leqslant N$, such that $x_j \in X$, $\forall\, j$. For each $x \in X$ and $0 < r \leqslant R_0$, set*

$$J(x, r) := \{j;\ B_j \subset B_r(x)\}.$$

Let $\lambda_1, \dots, \lambda_N \geqslant 0$. Suppose that a non-negative Borel function h defined on $U \setminus \bigcup_j B_j$ has the following property:

$$[S_r(x) \cap B_j = \emptyset, \quad \forall\, j] \\ \implies \int_{S_r(x)} h \geqslant \frac{1}{r} \sum_{j \in J(x,r)} \lambda_j, \forall\, x \in X, \quad \forall\, 0 < r \leqslant R_0. \tag{9.4}$$

Then

$$\int_{U \setminus \bigcup_j B_j} h \geqslant \left(\ln \frac{R_0}{3^{N-1} R}\right) \sum_j \lambda_j. \tag{9.5}$$

Proof of Lemma 9.3. — With no loss of generality, we may assume that $R_0 = 1$. The proof is by induction on N. When $N = 1$, we note that

$$\int_{U \setminus B_R(x_1)} h \geqslant \int_{B_1(x_1) \setminus B_R(x_1)} h = \int_R^1 \int_{S_r(x_1)} h(y)\, \mathrm{d}\sigma(y) \mathrm{d}r \geqslant \left(\ln \frac{1}{R}\right) \lambda_1,$$

whence the conclusion.

Assume next that (9.5) holds for $(N-1)$ balls and consider N balls as in the statement of the lemma. Set $\mathbf{m} := \frac{1}{2} \min_{i \neq j} |x_i - x_j|$.

Case 1: We have $\mathbf{m} \leqslant R$. — Equivalently, we have $B_i \cap B_j \neq \emptyset$ for some $i \neq j$. With no loss of generality, we may assume that $B_{N-1} \cap B_N \neq \emptyset$. Consider the balls $\widetilde{B}_j := \overline{B}_{3R}(x_j)$, $1 \leqslant j \leqslant N-1$. Then $\bigcup_{1 \leqslant j \leqslant N} B_j \subset \bigcup_{1 \leqslant j \leqslant N-1} \widetilde{B}_j$ and $3R \leqslant 3^{1-(N-1)}$. Associate with these balls the numbers $\widetilde{\lambda}_1 := \lambda_1, \dots, \widetilde{\lambda}_{N-2} := \lambda_{N-2}, \widetilde{\lambda}_{N-1} := \lambda_{N-1} + \lambda_N$. Then clearly the radius $3R$, the $(N-1)$ balls $\widetilde{B}_j$, and the $(N-1)$ numbers $\widetilde{\lambda}_j$ satisfy the adapted version of (9.4). By the induction assumption, we find that

$$\int_{U \setminus \bigcup_j B_j} h \geqslant \int_{U \setminus \bigcup_j \tilde{B}_j} h \geqslant \left(\ln \frac{1}{3^{N-2}(3R)}\right) \sum_j \widetilde{\lambda}_j = \left(\ln \frac{1}{3^{N-1} R}\right) \sum_j \lambda_j,$$

whence the desired conclusion in Case 1.

Case 2: We have $R < \mathbf{m} \leqslant 3^{1-N}$. — Consider the balls $\overline{B}_{\mathbf{m}}(x_j)$, $1 \leqslant j \leqslant N$. Then (by definition of $\mathbf{m}$) two of these balls have a common point. By the conclusion of Case 1, we have

$$\int_{U\setminus\bigcup_j B_{\mathbf{m}}(x_j)} h \geqslant \left(\ln \frac{1}{3^{N-1}\mathbf{m}}\right) \sum_j \lambda_j. \tag{9.6}$$

On the other hand, we have (using (9.4))

$$\int_{\bigcup_j (B_{\mathbf{m}}(x_j)\setminus B_R(x_j))} = \sum_j \int_R^{\mathbf{m}} \int_{S_r(x_j)} h(y)\,\mathrm{d}\sigma(y)\mathrm{d}r \geqslant \left(\ln \frac{\mathbf{m}}{R}\right) \sum_j \lambda_j. \tag{9.7}$$

We complete Case 2 by combining (9.6) and (9.7).

Case 3: We have $\mathbf{m} > 3^{1-N}$. — Then the balls $\overline{B}_{3^{1-N}}(x_j)$ are mutually disjoint and therefore (using (9.4))

$$\begin{aligned}\int_{U\setminus\bigcup_j B_j} h \geqslant & \sum_j \int_{B_{3^{1-N}}(x_j)\setminus B_R(x_j)} h = \sum_j \int_R^{3^{1-N}} \int_{S_r(x_j)} h(y)\,\mathrm{d}\sigma(y)\mathrm{d}r \\ \geqslant & \left(\ln \frac{1}{3^{N-1}R}\right) \sum_j \lambda_j. \end{aligned}$$

□

Proof of Theorem 9.1 when Ω is strictly star-shaped. — Let $\lambda < 2\pi$ be a number to be fixed later (sufficiently close to 2π). Let $\delta < \delta_0 = \delta_0(\lambda)$, with δ_0 as at the end of Section 8. By Lemma 8.3, if we prove that there are *at most* D enlarged bad discs, then there are *exactly* D enlarged bad discs, and their respective degrees are all -1.

Let C_1 be as in Lemma 8.3. (Note that C_1 depends only on λ.) Let x_ε^j, $1 \leqslant j \leqslant N(\varepsilon) \leqslant N$, be the centers of the enlarged bad discs (as in Lemma 5.1 (2)). For a sufficiently small ε, the enlarged bad discs are contained in Ω (Corollary 7.2) and, by Lemma 8.3 and the convergence results derived at the beginning of Section 8, we have (after rescaling back the functions v_ε)

$$\liminf_{\varepsilon\to 0} \inf_j \frac{1}{\varepsilon^2} \int_{B_{C_1\varepsilon}(x_\varepsilon^j)} \left(1 - |u_\varepsilon|^2\right)^2 \geqslant \lambda. \tag{9.8}$$

Consider some smooth $\mathbb{S}^1$-valued extension G of g to $\mathbb{R}^2 \setminus \overline{\Omega}$. (Recall that, for simplicity, we have assumed Ω simply connected, and therefore such an extension does exist.) We still denote $u = u_\varepsilon$ the extension of u with the value G outside $\overline{\Omega}$.

Consider a small number $a > 0$, to be fixed later. Set

$$X := \overline{\Omega}, \quad U := \{x \in \mathbb{R}^2; \quad \operatorname{dist}(x, \Omega) \leqslant 1\}, \tag{9.9}$$

$$R := C_1\varepsilon, \quad B_j := \overline{B}_R(x_\varepsilon^j), \quad 1 \leqslant j \leqslant N(\varepsilon), \tag{9.10}$$

$$h := \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left(1 - |u|^2\right)^2, \tag{9.11}$$

$$\lambda_1 = \cdots = \lambda_{N(\varepsilon)} := \frac{\lambda}{2(1+\delta_2)} - a. \tag{9.12}$$

Let a be sufficiently small such that $\lambda_1 > 0$ and

$$R_0 := \frac{2(1+\delta_2)a}{\widetilde{C}} \leqslant 1. \tag{9.13}$$

By (9.8) and (9.3), when $x \in \Omega$, $0 < r \leqslant R_0$, and $B_r(x)$ contains at least one enlarged bad disc B_{j_0}, we have

$$\int_{C_r(x)} h \geqslant \frac{1}{r}\left[\sum_{j \in J(x,r)} \frac{\lambda}{2(1+\delta_2)} - \frac{\widetilde{C}R_0}{2(1+\delta_2)}\right] \geqslant \frac{1}{r}\sum_{j \in J(x,r)} \lambda_j. \tag{9.14}$$

Using (9.14) and Lemma 9.3 (with R_0 given by (9.13)), we find that, for sufficiently small ε, we have

$$\int_U h \geqslant \left[\frac{\lambda}{2(1+\delta_2)} - a\right] N(\varepsilon) \ln \frac{R_0}{C_1\varepsilon}, \tag{9.15}$$

and therefore (using the definition of h and the fact that G is smooth and $\mathbb{S}^1$-valued), the Ginzburg–Landau energy of u satisfies

$$G_\varepsilon(u, \Omega) \geqslant \left[\frac{\lambda}{2(1+\delta_2)} - a\right] N(\varepsilon) \ln \frac{1}{\varepsilon} - C(a, \varepsilon). \tag{9.16}$$

Here, $C(a, \varepsilon)$ is bounded as $\varepsilon \to 0$.

On the other hand, Lemma 2.2 combined with Lemma 2.1 yields, with a constant C_3 depending on δ_2, Ω, and g, the following bound for the standard Ginzburg–Landau energy:

$$G_\varepsilon(u, \Omega) \leqslant \frac{\pi}{1-\delta_2} D \ln \frac{1}{\varepsilon} + C_3, \quad \forall\, 0 < \varepsilon \leqslant 1. \tag{9.17}$$

We finally choose λ, a, and δ_2 such that

$$(D+1)\left[\frac{\lambda}{2(1+\delta_2)} - a\right] > D\frac{\pi}{1-\delta_2}. \tag{9.18}$$

(This is possible, provided λ is sufficiently close to 2π and δ_2 and a are sufficiently small.)

By (9.16), (9.17), and (9.18), for small ε we have $N(\varepsilon) < D+1$, i.e., $N(\varepsilon) \leqslant D$. □

Proof of Theorem 9.1 in the general case. — Let r_0 be as in Lemma 4.5. Let $x_0 \in \partial\Omega$. Using the upper bound (9.17), (4.18), and a mean value argument, we find that (with C_1 as in (4.18), $C_4 = C_4(D, \delta_2)$, and $\varepsilon \leqslant \varepsilon_0 = \varepsilon_0(D)$) there exists some $r_0/2 < r < r_0$ such that

$$C_1 r \int_{B_r(x_0)\cap\partial\Omega} |u_\nu|^2 \leqslant C_4 r \ln\frac{1}{\varepsilon}.$$

Covering $\partial\Omega$ with a finite number (independent of $\varepsilon \leqslant \varepsilon_0$) of balls $B_r(x_0)$ as above, we find that

$$\int_{\partial\Omega} |u_\nu|^2 \leqslant C_5 \ln\frac{1}{\varepsilon}. \tag{9.19}$$

Combining (9.19) with (9.1), we obtain, for small ε, the following versions of (9.2) and (9.3):

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{B_r(x)\cap\Omega}\left(1-|u|^2\right)^2 + (1-\delta_2)\int_{C_r(x)\cap\Omega}|u_\nu|^2 \\ \leqslant \frac{r}{2\varepsilon^2}\int_{C_r(x)\cap\Omega}\left(1-|u|^2\right)^2 + (1+\delta_2)r\int_{C_r(x)\cap\Omega}|u_\tau|^2 + \widetilde{C} r\ln\frac{1}{\varepsilon}\end{aligned} \tag{9.20}$$

and

$$\begin{aligned}\int_{C_r(x)\cap\Omega}\left[\frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left(1-|u|^2\right)^2\right] \\ \geqslant \frac{1}{r}\frac{1}{2(1+\delta_2)\varepsilon^2}\int_{B_r(x)\cap\Omega}\left(1-|u|^2\right)^2 - \frac{\widetilde{C} r\ln(1/\varepsilon)}{(1+\delta_2)r}.\end{aligned} \tag{9.21}$$

Set

$$R_0 := \frac{(1+\delta_2)a}{\widetilde{C}\ln(1/\varepsilon)}. \tag{9.22}$$

For small ε, we have $R_0 \leqslant 1$. Repeating the argument in the star-shaped case and using (9.21), we see that, with R_0 as in (9.22), (9.14) still holds. Therefore, we are in position to derive the analogue of (9.16), which in our case (using (9.22)) is

$$G_\varepsilon(u,\Omega) \geqslant \left[\frac{\lambda}{2(1+\delta_2)} - a\right] N(\varepsilon)\ln\frac{1}{\varepsilon} - C(a,\varepsilon) - \widetilde{C}(a,\varepsilon)\ln\ln\frac{1}{\varepsilon}, \tag{9.23}$$

with $C(a,\varepsilon)$ and $\widetilde{C}(a,\varepsilon)$ bounded as $\varepsilon \to 0$. Finally, we choose λ, a, and δ_2 as in (9.18) and complete the proof *via* (9.17), (9.18), and (9.23). □

10. Small δ analysis. Insight on the locations of the bad discs

Throughout this section, we consider minimizers $u = u_\varepsilon$ of E_ε in $H^1_g(\Omega;\mathbb{C})$, with boundary datum of degree $-D < 0$. *We let δ_2 be as in Theorem 9.1.* The main result of this section is the following theorem that will subsequently be sharpened in several directions in Sections 12 and 13.

Theorem 10.1. — *Let $0 < \alpha < 1$. There exists some $0 < \delta_3 \leqslant \delta_2$, possibly depending on D and α, but not on Ω or g, such that, if $0 \leqslant \delta \leqslant \delta_3$ and ε is small, then the centers x^j_ε, $j = 1, \dots, D$, of the enlarged bad discs satisfy*

$$\mathbf{m} := \frac{1}{2}\min_{j\neq k}\left|x^j_\varepsilon - x^k_\varepsilon\right| \geqslant \varepsilon^\alpha, \tag{10.1}$$

$$\operatorname{dist}\left(x^j_\varepsilon, \partial\Omega\right) \geqslant \varepsilon^\alpha,\ \forall\, j. \tag{10.2}$$

The proof of (10.1) relies on the following generalization of Lemma 9.3.

Lemma 10.2. — *We use the same notation as in Lemma 9.3. Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a superadditive function. Let $N \geqslant 2$ and $\lambda_1, \dots, \lambda_N \geqslant 0$. Consider the numbers*

$$b = b(\lambda_1, \dots, \lambda_N) := \min_{i\neq j}[\Phi(\lambda_i + \lambda_j) - \Phi(\lambda_i) - \Phi(\lambda_j)] \geqslant 0,$$

$$\mathbf{m} := \frac{1}{2}\min_{i\neq j}|x_i - x_j|, \quad \rho := \max(R, \mathbf{m}).$$

Suppose that a non-negative Borel function h on $U \setminus \bigcup_j B_j$ has the following property:

$$[S_r(x) \cap B_j = \emptyset, \quad \forall\, j]$$
$$\Longrightarrow \int_{S_r(x)} h \geqslant \frac{1}{r}\Phi\left(\sum_{j\in J(x,r)}\lambda_j\right), \quad \forall\, x \in X, \quad \forall\, 0 < r \leqslant R_0. \tag{10.3}$$

Then

$$\int_{U\setminus\bigcup_j B_j} h \geqslant \left(\ln\frac{R_0}{3^{N-1}R}\right)\sum_j \Phi(\lambda_j) + b\ln\frac{R_0}{3^{N-1}\rho}. \tag{10.4}$$

Note that one recovers Lemma 9.3 by taking $\Phi = \mathrm{Id}$.

Proof of Lemma 10.2. — We may assume that $R_0 = 1$. The proof is by induction on N. We mainly rely on Lemma 9.3, using the fact that, thanks to the superadditivity of Φ, the assumption (9.4) is satisfied with λ_j replaced with $\Phi(\lambda_j)$. The case $N = 1$ is a special case of Lemma 9.3 if we take by convention $b = 0$ when $N = 1$. Assuming that the result holds for $(N-1)$ balls, we argue as in the proof of Lemma 9.3.

Case 1: We have $\mathbf{m} \leqslant R$ *(and thus* $\rho = R$*).* — Consider the enlarged balls $\widetilde{B}_j$ as in Case 1 in the proof of Lemma 9.3. Using the conclusion of Lemma 9.3 (with λ_j replaced with $\Phi(\lambda_j)$), we find that

$$\begin{aligned}
&\int_{U\setminus\bigcup_j B_j} h \\
&\geqslant \left(\ln\frac{1}{3^{N-1}R}\right)\left(\sum_{j\leqslant N-2}\Phi(\lambda_j)+\Phi(\lambda_{N-1}+\lambda_N)\right)\\
&= \left(\ln\frac{1}{3^{N-1}R}\right)\left(\sum_j\Phi(\lambda_j)+\left[\Phi(\lambda_{N-1}+\lambda_N)-\Phi(\lambda_{N-1})-\Phi(\lambda_N)\right]\right)\\
&\geqslant \left(\ln\frac{1}{3^{N-1}R}\right)\sum_j\Phi(\lambda_j)+b\ln\frac{1}{3^{N-1}R}\\
&= \left(\ln\frac{1}{3^{N-1}R}\right)\sum_j\Phi(\lambda_j)+b\ln\frac{1}{3^{N-1}\rho}.
\end{aligned}$$

Case 2: We have $R < \mathbf{m} \leqslant 3^{1-N}$. — Arguing as in Case 2 in the proof of Lemma 9.3 and using the conclusion of Case 1 above, we find that

$$\begin{aligned}
\int_{U\setminus\bigcup_j B_j} h &\geqslant \left(\ln\frac{1}{3^{N-1}\mathbf{m}}\right)\sum_j\Phi(\lambda_j)+b\ln\frac{1}{3^{N-1}\mathbf{m}}+\left(\ln\frac{\mathbf{m}}{R}\right)\sum_j\Phi(\lambda_j)\\
&=\left(\ln\frac{1}{3^{N-1}R}\right)\sum_j\Phi(\lambda_j)+b\ln\frac{1}{3^{N-1}\rho}.
\end{aligned}$$

Case 3: We have $\mathbf{m} > 3^{1-N}$. — The conclusion follows from Lemma 9.3 noting that in this case we have $b\ln\frac{1}{3^{N-1}\rho} \leqslant 0$. □

We may now proceed with the proof of (10.1). As for Theorem 9.1, the argument is slightly easier when Ω is strictly star-shaped, and we start with this case.

Proof of (10.1) *when* Ω *is strictly star-shaped.* — Thanks to the assumption $\delta \leqslant \delta_2$, for small ε the function u has exactly D enlarged bad discs, each of degree -1 (Theorem 9.1). We extend u to $\mathbb{R}^2$ as in the proof of Theorem 9.1. Let X, U, R, and B_j be as in (9.9)–(9.10) (with $N(\varepsilon) = D$). Let

$$w := \frac{u}{|u|}, \quad h := \frac{1}{2}|\nabla w|^2 \quad \text{in } U \setminus \bigcup_j B_j, \tag{10.5}$$

$$\lambda_1 = \cdots = \lambda_D := 1, \tag{10.6}$$

$$\Phi(t) := \pi t^2, \quad \forall\, t \in \mathbb{R}. \tag{10.7}$$

By Theorem 9.1, we have

$$\begin{aligned}\deg(w, C_r(x)) = -\#J(x,r) \ \text{ if } x \in X,\ 0 < r \leqslant 1,& \\ \text{and} \quad C_r(x) \cap B_j = \emptyset, \quad \forall\, j.&\end{aligned} \tag{10.8}$$

For x and r as above, we have, by the Cauchy–Schwarz inequality and (10.8),

$$2\pi r \int_{C_r(x)} |w_\tau|^2 \geqslant \left(\int_{C_r(x)} |w_\tau|\right)^2 \geqslant (2\pi \#J(x,r))^2, \tag{10.9}$$

and thus (10.3) holds with λ_j and Φ as above.

We use the notation in Lemma 10.2. By Lemma 10.2, using the fact that w is smooth and fixed outside Ω and that, by the construction of the enlarged bad discs, we have $\rho = \mathbf{m}$ for small ε, we find that

$$\frac{1}{2}\int_{\Omega\setminus\bigcup_j B_j} |\nabla w|^2 \geqslant \pi D \ln\frac{1}{\varepsilon} + 2\pi \ln\frac{1}{\mathbf{m}} - C, \tag{10.10}$$

where C is a finite constant independent of small ε.

On the other hand, one easily checks the following:

$$|u| \geqslant \frac{1}{2} \Longrightarrow \frac{1}{2}|\nabla u|^2 \geqslant \frac{1}{2}|\nabla w|^2 - 2|1-|u|^2||\nabla u|^2. \tag{10.11}$$

Combining (10.10) and (10.11) with the upper bound (4.28) and the fact that $|u| \geqslant 1/2$ outside the bad discs, we find that

$$G_\varepsilon(u,\Omega) \geqslant \frac{1}{2}\int_{\Omega\setminus\bigcup_j B_j} |\nabla u|^2 \geqslant \pi D \ln\frac{1}{\varepsilon} + 2\pi \ln\frac{1}{\mathbf{m}} - \widetilde{C}, \tag{10.12}$$

where $\widetilde{C}$ is a finite constant independent of small ε. On the other hand, if $\delta \leqslant \delta_3$, with $\delta_3 \leqslant \delta_2$ to be fixed later, (9.17) (with δ_3 instead of δ_2) holds, and thus, using (10.12), we find that

$$\ln\frac{1}{\mathbf{m}} \leqslant \frac{D}{2}\left(\frac{1}{1-\delta_3} - 1\right)\ln\frac{1}{\varepsilon} + \frac{\widetilde{C} + C_3}{2\pi}. \tag{10.13}$$

From (10.13), we find that, for small ε, (10.1) holds provided

$$\frac{D}{2}\left(\frac{1}{1-\delta_3}-1\right)<\alpha. \qquad \square$$

When Ω is a general domain, we do not have (4.28) at our disposal anymore. We sketch below the adapted argument.

Sketch of proof of (10.1) *in the general case.* — The inequality (10.10) still holds in the general case. However, the strategy for obtaining an analogue of (10.12) from (10.10) is different. Consider some number $0<a<1$ to be fixed later. Define the (enlarged) bad discs as in Section 5, but with $1/2$ replaced with a. One can see that Lemma 5.1 still holds, possibly with some N depending on a. Also, the analysis in Section 8 holds, for $\delta \leqslant \delta_0$, with δ_0 possibly depending on a. So does Theorem 9.1. At this stage, using Theorem 9.1 and Corollary 8.4 we conclude that, for small δ and two different a's, the corresponding enlarged bad discs coincide, up to a multiplicative constant factor of their radii. Therefore, the estimate (10.1) does not depend on the specific value of a we choose.

We next explain how to choose a. We have the following substitute of (10.11):

$$|u|\geqslant a \Longrightarrow |\nabla u|^2 \geqslant a^2|\nabla w|^2. \tag{10.14}$$

From (10.10) (with the enlarged bad discs corresponding to a) and (10.14), we find that

$$G_\varepsilon(u,\Omega)\geqslant \pi a^2 D\ln\frac{1}{\varepsilon}+2\pi a^2\ln\frac{1}{\mathbf{m}}-C(a). \tag{10.15}$$

In order to obtain (10.1) from (10.15) and (9.17), it then suffices to choose a and δ_3 such that

$$\frac{D}{2a^2}\left(\frac{1}{1-\delta_3}-a^2\right)<\alpha. \qquad \square$$

Remark 10.3. — For a different approach to the case of general (i.e., not necessarily strictly star-shaped) domains Ω, see also Lemma 12.2 and its applications in Section 12.

The basic ingredient of the proof of (10.2) is the following simple result.

LEMMA 10.4. — *Let $0<\lambda<2\pi$. Let $g\in C^1(\partial\Omega;\mathbb{S}^1)$. Then there exists some $r_0=r_0(\lambda,\Omega,g)$ such that the following holds. Let $0<r\leqslant r_0$ and $x_0\in\partial\Omega$. Consider a Lipschitz map $w:\partial(B_r(x_0)\cap\Omega)\to\mathbb{S}^1$ such that $w=g$ on $B_r(x_0)\cap\partial\Omega$ and* $\deg w=-1$. *Then*

$$\frac{r}{2}\int_{C_r(x_0)\cap\Omega}|w_\tau|^2\geqslant\lambda. \tag{10.16}$$

Similarly if $\deg w = d \in \mathbb{Z}$ *and* $0 < \lambda < 2d^2\pi$.

Proof. — For small r_0 and with finite constants C_0, $\widetilde{C}_0$, all depending only on Ω, and for r, x_0 as above, the following hold:

$$C_r(x_0) \cap \Omega \quad \text{consists of a single arc of endpoints } a = a(r, x_0),\ b = b(r, x_0), \tag{10.17}$$

$$\mathscr{H}^1(C_r(x_0)) \leqslant \pi r + C_0 r^2, \tag{10.18}$$

$$\operatorname{dist}(a, b) \leqslant \widetilde{C}_0 r \tag{10.19}$$

(in the last line, dist is the geodesic distance on $\partial\Omega$).

Using: (i) (10.17)–(10.19); (ii) the degree condition on w; (iii) the fact that g is Lipschitz; (iv) the Cauchy–Schwarz inequality, we find successively, possibly after considering a smaller r_0:

$$\int_{\partial(B_r(x_0)\cap\Omega)} |w_\tau| \geqslant 2\pi,$$

$$\int_{C_r(x_0)\cap\Omega} |w_\tau| \geqslant 2\pi - \int_{B_r(x_0)\cap\partial\Omega} |w_\tau| \geqslant 2\pi - C(g)\widetilde{C}_0 r,$$

$$\left(\pi r + C_0 r^2\right) \int_{C_r(x_0)\cap\Omega} |w_\tau|^2 \geqslant \left(2\pi - C(g)\widetilde{C}_0 r\right)^2,$$

and the last line implies (10.16) (since $\lambda < 2\pi$), provided r_0 is sufficiently small. □

In the proof of (10.2), we will use Lemma 10.4 in conjunction with the following lower bound, which is a simple consequence of the Cauchy–Schwarz inequality:

$$\begin{bmatrix} 0 < r_1 < r_2,\ x \in \mathbb{R}^2,\ w : B_{r_2}(x) \setminus B_{r_1}(x) \longrightarrow \mathbb{S}^1, \\ \hfill w \text{ Lipschitz},\ \deg(w, C_{r_1}(x)) = d \in \mathbb{Z} \end{bmatrix} \\ \Longrightarrow \frac{1}{2} \int_{B_{r_2}(x)\setminus B_{r_1}(x)} |\nabla w|^2 \geqslant \pi d^2 \ln \frac{r_2}{r_1}. \tag{10.20}$$

Proof of (10.2). — Let $0 < \alpha_1 < \alpha$ be a constant to be fixed later. We take δ_3 such that, when $0 \leqslant \delta \leqslant \delta_3$, (10.1) holds for α_1 instead of α. Let ε be sufficiently small. Let $B_j = B_{C_1\varepsilon}(x_\varepsilon^j)$, $1 \leqslant j \leqslant D$, be the enlarged bad discs. Assume, with no loss of generality, that the enlarged bad disc closest to $\partial\Omega$ is B_1. If $\operatorname{dist}(x_\varepsilon^1, \partial\Omega) \geqslant \varepsilon^\alpha$, then we are done. Otherwise, by choosing appropriate values of δ_3, α_1, and of λ in Lemma 10.4, we will obtain a contradiction for small ε. For this purpose, we first use (10.20) and obtain the following inequalities

$$\frac{1}{2}\int_{B_{\varepsilon^{\alpha_1}}(x_\varepsilon^j)\setminus B_j}|\nabla w|^2 \geqslant \pi \ln\frac{\varepsilon^{\alpha_1}}{C_1\varepsilon}, \quad \forall\, j\geqslant 2, \tag{10.21}$$

$$\frac{1}{2}\int_{B_{\varepsilon^{\alpha}}(x_\varepsilon^1)\setminus B_1}|\nabla w|^2 \geqslant \pi \ln\frac{\varepsilon^{\alpha}}{C_1\varepsilon}. \tag{10.22}$$

We next use Lemma 10.4 and obtain, for small ε and with x_0 the nearest point projection of x_ε^1 on $\partial\Omega$, the bound

$$\frac{1}{2}\int_{B_{\varepsilon^{\alpha_1}/2}(x_0)\setminus B_{2\varepsilon^{\alpha}}(x_0)}|\nabla w|^2 \geqslant \lambda \ln\frac{\varepsilon^{\alpha_1}/2}{2\varepsilon^{\alpha}}. \tag{10.23}$$

We next note that, for small ε, the integration domains in (10.21)–(10.23) are mutually disjoint. Combining this observation with (10.21)–(10.23) and using the fact that w is smooth and fixed outside Ω, we obtain the lower bound

$$\frac{1}{2}\int_{\Omega\setminus\bigcup_j B_j}|\nabla w|^2 \geqslant \big[\pi(1-\alpha_1)D+(\lambda-\pi)(\alpha-\alpha_1)\big]\ln\frac{1}{\varepsilon}-C, \tag{10.24}$$

where C is a finite numerical constant (depending on C_1 and on the extension of u outside Ω).

On the other hand, we know from the proof of (10.1) in a general domain that, possibly after modifying the construction of the enlarged bad discs as explained there, we have, for a given $0<a<1$ and sufficiently small ε,

$$G_\varepsilon(u,\Omega)\geqslant a^2\frac{1}{2}\int_{\Omega\setminus\bigcup_j B_j}|\nabla w|^2. \tag{10.25}$$

The estimates (10.24) and (10.25) contradict, for small ε, the upper bound (9.17) (with δ_3 instead of δ_2), provided we have

$$a^2\big[\pi(1-\alpha_1)D+(\lambda-\pi)(\alpha-\alpha_1)\big] > \frac{\pi}{1-\delta_3}D. \tag{10.26}$$

We finally note that (10.26) holds for any constant $\lambda>\pi$, provided we let α_1 and δ_3 sufficiently small and a sufficiently close to 1. □

Remark 10.5. — For the standard Ginzburg–Landau energy and in a strictly star-shaped domain Ω, the method of proof of (10.1)–(10.2) allows us to recover a "repelling effect" initially established in [2]: for small ε, the mutual distances between the bad discs and the distances from the bad discs to $\partial\Omega$ is above some positive constant.

Indeed, combining (10.10) with the upper bound (2.5) and the inequalities (10.11) and (4.28) (for the latter one, we rely on the fact that Ω is strictly star-shaped), we see that

$$\liminf_{\varepsilon\to 0} \min_{j\neq k} \left|x_\varepsilon^j - x_\varepsilon^k\right| \geqslant C > 0 \tag{10.27}$$

for some constant C depending on Ω and g.

It remains to prove that the bad discs cannot get close to the boundary. Argue by contradiction and assume, e.g., that, possibly up to a subsequence, $\mathbf{m} := \operatorname{dist}(x_\varepsilon^1, \partial\Omega) \to 0$. With C as above, we may repeat the proof of (10.21)–(10.23), with ε^{α_1} replaced with $C/4$, and find (*via* Lemma 10.4 and (10.20)) that

$$\frac{1}{2}\int_{\bigcup_j \left(B_{C/4}(x_\varepsilon^j)\setminus B_{C_1\varepsilon}(x_\varepsilon^j)\right)} |\nabla w|^2 \geqslant \pi D \ln\frac{1}{\varepsilon} + (\lambda-\pi)\ln\frac{1}{\mathbf{m}} - C. \tag{10.28}$$

Once $\lambda > \pi$ is fixed, the inequality (10.28) contradicts, for small m, the upper bound (2.5).

11. Toy minimization problems on an annulus

There exists a natural construction of competitors for the minimization problem $\min_{H^1_g(\Omega;\mathbb{S}^1)} E_\varepsilon$ when $\deg g = -D < 0$. More specifically, set, for $0 < R_1 < R_2 < \infty$,

$$A_{R_1,R_2} := \overline{B}_{R_2}(0) \setminus B_{R_1}(0).$$

Consider the class

$$\mathscr{H}_{R_1,R_2,C} := \left\{ \begin{array}{l} v \in H^1\big(A_{R_1,R_2};\mathbb{S}^1\big);\ |v_\theta| \leqslant C \text{ on } C_{R_1}(0) \text{ and} \\ C_{R_2}(0),\ \deg(v, C_{R_1}(0)) = \deg(v, C_{R_2}(0)) = -1 \end{array} \right\} \tag{11.1}$$

and the minimization problem

$$I_{R_1,R_2,C} := \min\left\{E_0(v);\ v \in \mathscr{H}_{R_1,R_2,C}\right\}. \tag{11.2}$$

The class $\mathscr{H}_{R_1,R_2,C}$ is non-empty if $C \geqslant 1$, since for such C it contains the map

$$u_0 : A_{R_1,R_2} \longrightarrow \mathbb{S}^1,\ u_0(z) := \frac{\bar{z}}{|z|},\ \forall\, z \in A_{R_1,R_2}.$$

From now on, we assume that $C \geqslant 1$.

It is straightforward that there exists a minimizer $u_{R_1,R_2,C}$ in (11.2). In the special case where $\delta = 0$, (11.2) is equivalent to the minimization of the standard Dirichlet integral $G_0(v)$, and the above u_0 is a minimizer.

We conjecture that even when $\delta > 0$, a minimizer likewise could be a 0-homogeneous map (thus a function depending only on θ and independent of $r = |x|$, R_1, R_2, and C), but we are not aware of a proof of this fact.

Starting from a minimizer $w := u_{\varepsilon,R_2,C}$ in (11.2), where $0 < \varepsilon < R_2$ and R_2 is sufficiently small (depending on Ω) and fixed, we construct a competitor u in $H^1_g(\Omega;\mathbb{S}^1)$ as follows. Consider D disjoint closed balls $\overline{B}_{R_2}(x_j)$, $j = 1,\dots,D$, contained in Ω. Let v be the restriction of w to $C_\varepsilon(0)$. We first define u in each $\overline{B}_{R_2}(x_j)$ by setting

$$u(x) = u\big(re^{\imath\theta} + x_j\big) := \begin{cases} w(x - x_j), & \text{if } \varepsilon \leqslant r \leqslant R_2 \\ (r/\varepsilon)v\big(\varepsilon e^{\imath\theta}\big), & \text{if } r \leqslant \varepsilon. \end{cases} \tag{11.3}$$

We next extend u to Ω by considering, in $\Omega \setminus \bigcup_j B_{R_2}(x_j)$, an $\mathbb{S}^1$-valued map, still denoted u, agreeing with the above map on each $C_{R_2}(x_j)$ and taking the value g on $\partial\Omega$. It is straightforwards that this is possible such that, in addition,

$$|\nabla u| \leqslant C_1(C,\Omega,g) \quad \text{in } \overline{\Omega} \setminus \bigcup_j B_{R_2}(x_j). \tag{11.4}$$

Fixing the value of C and using u as a competitor, we obtain the upper bound

$$m_\varepsilon := \min\big\{E_\varepsilon(u);\, u \in H^1_g(\Omega;\mathbb{C})\big\} \leqslant DI_{\varepsilon,R_2,C} + C_2(\Omega,g). \tag{11.5}$$

A remarkable result of Bethuel, Brezis, and Hélein [2] asserts that, when $\delta = 0$ and Ω is strictly star-shaped, this construction provides the correct asymptotics of m_ε up to a bounded error, that is,

$$m_\varepsilon = DI_{\varepsilon,R_2,C} + O(1) = \pi D \ln\frac{1}{\varepsilon} + O(1) \text{ as } \varepsilon \longrightarrow 0. \tag{11.6}$$

Among other ingredients of the proof of (11.6) in [2], there is the exact formula for $I_{\varepsilon,R_2,C}$. Although we are not aware of such a formula when $\delta \neq 0$, we will establish, in the next two sections, analogues of (11.6) valid for small δ.

In the current section, we investigate some basic properties of $I_{R_1,R_2,C}$, that we collect in the following simple result.

Lemma 11.1. — *Let $C, C' \geqslant 1$. Let $0 < R_1 \leqslant R_2 \leqslant R_3 \leqslant R_4 < \infty$. Then the following properties hold, with the constants C_j depending only on the variables indicated below.*

$$(1-\delta)\pi \ln\frac{R_2}{R_1} - (1-\delta)\pi \leqslant I_{R_1,R_2,C} \leqslant \pi \ln\frac{R_2}{R_1}. \tag{11.7}$$

$$I_{tR_1,tR_2,C} = I_{R_1,R_2,C},\ \forall\, t > 0. \tag{11.8}$$

If $R_3/R_2 \geqslant 2$, then

$$I_{R_1,R_4,C} \leqslant I_{R_1,R_2,C} + I_{R_3,R_4,C} + C_1(C,\delta)\ln\frac{R_3}{R_2} + C_2(C,\delta). \tag{11.9}$$

If $t \geqslant 2$, then $I_{R_1,tR_2,C} \leqslant I_{R_1,R_2,C} + C_1(C,\delta)\ln t + C_2(C,\delta)$. (11.10)

If $t \geqslant 2$, then $I_{R_1,tR_2,C'} \leqslant I_{R_1,R_2,C} + C_3(C,\delta)\ln t + C_4(C,\delta)$. (11.11)

There exists a constant $(1-\delta)\pi \leqslant C_\delta \leqslant \pi$

$$\textit{such that } \lim_{t\to\infty}\frac{I_{R_1,tR_1,C}}{\ln t} = C_\delta,\ \forall\, R_1 > 0,\ \forall\, C \geqslant 1. \tag{11.12}$$

$$\textit{If } \delta > 0, \textit{ then } C_\delta < \pi. \tag{11.13}$$

$$\textit{If } \delta > 0, \textit{ then } C_\delta > (1-\delta)\pi. \tag{11.14}$$

Proof of (11.7). — For the left-hand side, we use (2.4) combined with the fact that u_0 is a minimizer of G_0 in the class $\mathscr{H}_{R_1,R_2,C}$. For the right-hand side, we consider, as in the proof of Lemma 2.2, the competitor u_0. □

Proof of (11.8). — This identity follows from the fact that E_0 is invariant under homotheties, and so is the condition $|v_\theta| \leqslant C$. □

Proof of (11.9). — Let u_2, respectively u_3, be a minimizer in $\mathscr{H}_{R_1,R_2,C}$, respectively in $\mathscr{H}_{R_3,R_4,C}$. Let v_2, respectively v_3, be the trace of u_2 on $C_{R_2}(0)$, respectively of u_3 on $C_{R_3}(0)$.

Since we have $|(v_j)_\theta| \leqslant C$, $j = 2,3$, and $\deg v_2 = \deg v_3 = -1$, we may write $v_j(R_j e^{\imath\theta}) = \exp(\imath(-\theta + \psi_j(\theta)))$, with

$$|(\psi_j)_\theta - 1| \leqslant C \quad \text{and} \quad |\psi_j| \leqslant (1+C)\pi. \tag{11.15}$$

We next "interpolate" between v_2 and v_3 by setting

$$u\big((1-\sigma)R_2 + \sigma R_3)e^{\imath\theta}\big) := \exp(\imath(-\theta + (1-\sigma)\psi_2(\theta) + \sigma\psi_3(\theta))),$$
$$\forall\, 0 \leqslant \sigma \leqslant 1.$$

Clearly, u agrees with v_2 on $C_{R_2}(0)$ and with v_3 on $C_{R_3}(0)$. On the other hand, one has (from (11.15))

$$|u_r| \leqslant \frac{2(1+C)\pi}{R_3 - R_2} \quad \text{and} \quad |u_\theta| \leqslant C. \tag{11.16}$$

Considering the competitor

$$\begin{cases} u_2, & \text{in } A_{R_1,R_2} \\ u, & \text{in } A_{R_2,R_3} \\ u_3, & \text{in } A_{R_3,R_4}, \end{cases}$$

which belongs to the class $\mathscr{H}_{R_1,R_4,C}$, and using, in A_{R_2,R_3}, (2.3) and(11.16), we obtain that (11.9) holds with $C_1(C,\delta) := (1+\delta)\pi C^2$ and $C_2(C,\delta) := 6(1+C)^2\pi^3 + 2\pi(1+\delta)$. □

Proof of (11.10). — This is a special case of (11.9), with $R_3 := tR_2$ and $R_4 = R_3$. □

Proof of (11.11). — We essentially repeat the proof of (11.9). Fix $1 < \tau < 2$. Given a minimizer u_1 in $\mathscr{H}_{\tau R_1,\tau R_2,C}$ and letting v_1(respectively v_2) be the restriction of u_1 to $C_{R_1}(0)$ (respectively of u_1 to $C_{R_2}(0)$), we interpolate, in $\overline{B}_{\tau t R_1}(0) \setminus B_{R_1}(0)$ (respectively in $\overline{B}_{tR_2}(0) \setminus B_{R_2}(0)$), between $\theta \mapsto e^{-\imath\theta}$ and v_1 (respectively v_2 and $\theta \mapsto e^{-\imath\theta}$) in order to construct a competitor in $\mathscr{H}_{R_1,tR_2,C'}$. □

Proof of (11.12). — By (11.8), it suffices to investigate the case where $R_1 = 1$. Set

$$C_\delta := \liminf_{t\to\infty} \frac{I_{1,t,1}}{\ln t}. \tag{11.17}$$

We will prove that (11.12) holds for this C_δ. To start with, we note that, by (11.7), we have $(1-\delta)\pi \leqslant C_\delta \leqslant \pi$.

We next prove that

$$\limsup_{t\to\infty} \frac{I_{1,t,1}}{\ln t} \leqslant \liminf_{t\to\infty} \frac{I_{1,t,1}}{\ln t}. \tag{11.18}$$

Let $\varepsilon > 0$ and let $M > 1$ to be fixed later in function of ε. Let $t_0 \geqslant M$ be such that

$$\frac{I_{1,t_0,1}}{\ln t_0} < C_\delta + \frac{\varepsilon}{2}. \tag{11.19}$$

For $t \geqslant 2t_0$, let

$$k := \left\lfloor \frac{\ln t}{\ln(2t_0)} \right\rfloor \geqslant 1, \tag{11.20}$$

so that

$$(2t_0)^k \leqslant t < (2t_0)^{k+1}. \tag{11.21}$$

By applying repeatedly (11.8)–(11.10), we obtain, by a straightforward induction on k, that

$$I_{1,t,1} \leqslant kI_{1,t_0,1} + (k-1)[C_1(1,\delta)\ln 2 + C_2(1,\delta)] + C_1(1,\delta)\ln(4t_0) + C_2(1,\delta). \tag{11.22}$$

To illustrate the proof of (11.22), we detail, for example, the case where $k=2$. Then, using successively (11.9), (11.21), (11.9), (11.8), we find that

$$\begin{aligned}
&I_{1,t,1}\\
&\leqslant I_{1,2t_0^2,1}+C_1(1,\delta)\ln\frac{t}{2t_0^2}+C_2(1,\delta)\\
&\leqslant I_{1,2t_0^2,1}+C_1(1,\delta)\ln(4t_0)+C_2(1,\delta)\\
&\leqslant I_{1,t_0,1}+I_{2t_0,2t_0^2,1}+C_1(1,\delta)\ln 2+C_2(1,\delta)+C_1(1,\delta)\ln(4t_0)+C_2(1,\delta)\\
&=I_{1,t_0,1}+I_{1,t_0,1}+C_1(1,\delta)\ln 2+C_2(1,\delta)+C_1(1,\delta)\ln(4t_0)+C_2(1,\delta),
\end{aligned}$$

and the last line coincides with the right-hand side of (11.22) with $k=2$.

Dividing (11.22) by $k\ln(2t_0)$, letting $k\to\infty$, and taking (11.20) into account, we find that

$$\limsup_{t\to\infty}\frac{I_{1,t,1}}{\ln t}\leqslant\frac{I_{1,t_0,1}}{\ln t_0}\frac{\ln t_0}{\ln(2t_0)}+\frac{C_1(1,\delta)\ln 2+C_2(1,\delta)}{\ln(2t_0)}. \tag{11.23}$$

We next note that, when $M>1$ is sufficiently large (depending on ε), $t_0\geqslant M$, and (11.19) holds, the right-hand side of (11.23) is $<C_\delta+\varepsilon$. Therefore, (11.18) holds.

To complete the proof of (11.12), we note the straightforward inequality $I_{1,t,C}\leqslant I_{1,t,1}$. Combining this with (11.11), we find that, when $t\geqslant 2$, we have

$$I_{1,t/2t,1}-C_3(C,\delta)\ln 2-C_4(C,\delta)\leqslant I_{1,t,C}\leqslant I_{1,t,1}. \tag{11.24}$$

We obtain (11.18) for an arbitrary constant $C\geqslant 1$ *via* (11.17), (11.18), and (11.24). □

Proof of (11.13). — We consider a C^1 map $f:\mathbb{S}^1\to\mathbb{S}^1$, of degree -1, and the competitor $u(re^{\imath\theta}):=f(e^{\imath\theta})$, $\forall\ r>0$. Clearly, for some C depending on f, we have $u\in\mathscr{H}_{1,t,C}$. On the other hand, if we write $f(e^{\imath\theta})=\exp(\imath(-\theta+\psi(\theta)))$, with ψ of class C^1 and 2π-periodic, we have

$$\begin{aligned}
(\operatorname{div}u)\big(re^{\imath\theta}\big)&=\frac{1}{r}\big(\psi'(\theta)-1\big)\cos(\psi(\theta)-2\theta),\\
(\operatorname{curl}u)\big(re^{\imath\theta}\big)&=\frac{1}{r}\big(\psi'(\theta)-1\big)\sin(\psi(\theta)-2\theta),
\end{aligned}$$

and thus

$$\frac{E_0(u;A_{1,t})}{\ln t}=\mathscr{I}(\psi), \tag{11.25}$$

where

$$\mathscr{I}(\psi) := \int_0^{2\pi} \frac{K_1\cos^2(\psi(\theta)-2\theta)+K_3\sin^2(\psi(\theta)-2\theta)}{2}\left(\psi'(\theta)-1\right)^2 \mathrm{d}\theta. \quad (11.26)$$

From (11.25)–(11.26) and (11.12), we find that

$$\begin{aligned} C_\delta &\leqslant \inf\left\{\mathscr{I}(\psi);\quad \psi\in C^1([0,2\pi];\mathbb{R}),\quad \psi(0)=\psi(2\pi)\right\}\\ &= \min\left\{\mathscr{I}(\psi);\quad \psi\in H^1([0,2\pi];\mathbb{R}),\quad \psi(0)=\psi(2\pi)\right\}. \end{aligned} \quad (11.27)$$

By the direct method in the calculus of variations, the min in the second line of (11.27) is achieved, and any minimizer satisfies

$$\begin{aligned} &F_\psi(\psi(\theta),\theta)(\psi'(\theta)-1)^2 - 2\left[F(\psi(\theta),\theta)(\psi'(\theta)-1)\right]_\theta = 0,\\ &\qquad \text{where}\quad F(\psi,\theta) := \frac{K_1}{2}\cos^2(\psi-2\theta)+\frac{K_3}{2}\sin^2(\psi-2\theta). \end{aligned} \quad (11.28)$$

Let us note that

$$F_\psi(\psi,\theta) = -2\delta\sin(\psi-2\theta)\cos(\psi-2\theta). \quad (11.29)$$

Using (11.29), we find that, when $\psi=0$, the left-hand side of the first line of (11.28) equals $-6\delta\sin(2\theta)\cos(2\theta)$, and thus, when $\delta\neq 0$, $\psi=0$ is not a minimizer of $\mathscr{I}$. Combining this with the fact that, when $\psi=0$, we have $E_0(u)=\pi\ln t$ (see the proof of (11.7)) and with (11.27), we obtain that $C_\delta<\pi$ when $\delta>0$. □

Proof of (11.14). — Let $R_1 := 1$, $R_2 := t > 1$. Any competitor v in (11.2) is of the form

$$v = \exp\left(-\imath(\theta+\psi)\right),\ \text{with } \psi\in H^1(A_{1,t}).$$

For v as above, we find, with $\Omega := A_{1,t}$, using the fact that $\operatorname{Jac} v = 0$ in Ω (since v is $\mathbb{S}^1$-valued):

$$\begin{aligned}
\int_\Omega [(\operatorname{div} v)^2 + (\operatorname{curl} v)^2] &= \int_\Omega |\nabla v|^2 + 2\int_\Omega \operatorname{Jac} v = \int_\Omega |\nabla v|^2,\\
\int_\Omega |\nabla v|^2 = \int_\Omega |\nabla(\theta+\psi)|^2 &= \int_\Omega |\nabla \theta|^2 + \int_\Omega |\nabla\psi|^2 - 2\int_\Omega \frac{1}{r}\psi_\tau\\
&= \int_\Omega |\nabla \theta|^2 + \int_\Omega |\nabla\psi|^2 = 2\pi \ln t + \int_\Omega |\nabla\psi|^2,\\
E_0(v) &= \frac{1-\delta}{2}\int_\Omega [(\operatorname{div} v)^2 + (\operatorname{curl} v)^2] + \delta \int_\Omega (\operatorname{div} v)^2\\
&= (1-\delta)\pi \ln t + \frac{1-\delta}{2}\int_\Omega |\nabla\psi|^2\\
&\quad + \delta\int_\Omega \left[\frac{1}{r}\cos(\psi - 2\theta) + \begin{pmatrix}\sin(\theta-\psi)\\ \cos(\theta-\psi)\end{pmatrix}\cdot\nabla\psi\right]^2.
\end{aligned}$$

Assume, by contradiction, that $C_\delta = (1-\delta)\pi$. Then there exist $t_j \to \infty$ and $\psi_j \in H^1(A_{1,t_j})$ such that, with $\Omega_j := A_{1,t_j}$, we have

$$\begin{aligned}
\int_{\Omega_j} |\nabla\psi_j|^2 + \int_{\Omega_j}\left[\frac{1}{r}\cos(\psi_j - 2\theta) + \begin{pmatrix}\sin(\theta-\psi_j)\\ \cos(\theta-\psi_j)\end{pmatrix}\cdot\nabla\psi_j\right]^2&\\
= o(\ln t_j) \text{ as } j \longrightarrow \infty.&
\end{aligned}\tag{11.30}$$

By (11.30) we obtain (using Cauchy–Schwarz on the third line) that

$$\int_{\Omega_j}\left[\frac{1}{r}\psi_{j,\theta}\right]^2 \leqslant \int_{\Omega_j} |\nabla\psi_j|^2 = o(\ln t_j) \ \text{ as } j \longrightarrow \infty, \tag{11.31}$$

$$\int_{\Omega_j}\left[\begin{pmatrix}\sin(\theta-\psi_j)\\ \cos(\theta-\psi_j)\end{pmatrix}\cdot\nabla\psi_j\right]^2 = o(\ln t_j) \ \text{ as } j \longrightarrow \infty,$$

$$\int_{\Omega_j}\frac{1}{r}\cos(\psi_j - 2\theta)\begin{pmatrix}\sin(\theta-\psi_j)\\ \cos(\theta-\psi_j)\end{pmatrix}\cdot\nabla\psi_j = o(\ln t_j) \ \text{ as } j \longrightarrow \infty,$$

$$\int_{\Omega_j}\left[\frac{1}{r}\cos(\psi_j - 2\theta)\right]^2 = o(\ln t_j) \ \text{ as } j \longrightarrow \infty. \tag{11.32}$$

Combining (11.31) and (11.32) with a mean value argument, we find that there exist radii $1 < r_j < t_j$ such that

$$\frac{1}{r_j}\int_{C_{r_j}(0)} \left\{(\psi_{j,\theta})^2 + [\cos(\psi_j - 2\theta)]^2\right\} \longrightarrow 0 \ \text{ as } j \longrightarrow \infty.$$

Therefore, if we set

$$g_j\big(e^{\imath\theta}\big) := \psi_j\big(r_j e^{\imath\theta}\big),\ \forall\, j,\ \forall\, \theta,$$

we have

$$\int_{\mathbb{S}^1} \big\{(g_{j,\theta})^2 + [\cos(g_j - 2\theta)]^2\big\} \longrightarrow 0 \ \text{ as } j \longrightarrow \infty. \tag{11.33}$$

After subtracting a suitable multiple of 2π from g, we may assume that $0 \leqslant g_j(e^{\imath\theta_j}) \leqslant 2\pi$ for some θ_j, and then (11.33) implies that, possibly up to a subsequence, there exists some constant C such that $g_j \to C$ uniformly. We obtain from (11.33) that

$$\int_{\mathbb{S}^1} [\cos(C - 2\theta)]^2 = 0,$$

which is impossible. This contradiction completes the proof of (11.14). □

While the above considerations will suffice to yield the correct asymptotics of the minimal energy m_ε when $D = 1$, for higher degrees we rely on the study of a "cousin" of $\mathscr{H}_{R_1,R_2,C}$. More specifically, when $C \geqslant 1$, we consider the class

$$\widetilde{\mathscr{H}}_{R_1,R_2,C} := \left\{\begin{array}{c} v \in H^1\big(A_{R_1,R_2};\mathbb{S}^1\big);\ \displaystyle\int_{C_{R_j}(0)} |v_\theta|^2 \leqslant 2\pi R_j C^2, \\ j = 1,2, \deg(v, C_{R_1}(0)) = \deg(v, C_{R_2}(0)) = -1 \end{array}\right\} \tag{11.34}$$

and the minimization problem

$$\widetilde{I}_{R_1,R_2,C} := \min\Big\{E_0(v);\ v \in \widetilde{\mathscr{H}}_{R_1,R_2,C}\Big\}. \tag{11.35}$$

The following result is a straightforward version of (part of) Lemma 11.1, and its proof is omitted.

Lemma 11.2. — *Let $C \geqslant 1$. Let $0 < R_1 \leqslant R_2 \leqslant R_3 \leqslant R_4 < \infty$. Then the following properties hold, with constants C_j depending only on the variables indicated below.*

$$\mathscr{H}_{R_1,R_2,C} \subset \widetilde{\mathscr{H}}_{R_1,R_2,C}, \quad \textit{and therefore} \quad \widetilde{I}_{R_1,R_2,C} \leqslant I_{R_1,R_2,C}. \tag{11.36}$$

$$\widetilde{I}_{tR_1,tR_2,C} = \widetilde{I}_{R_1,R_2,C},\ \forall\, t > 0. \tag{11.37}$$

If $R_3/R_2 \geqslant 2$, then $I_{R_1,R_4,C}$

$$\leqslant \widetilde{I}_{R_1,R_2,C} + \widetilde{I}_{R_3,R_4,C} + C_1(C,\delta)\ln\frac{R_3}{R_2} + C_2(C,\delta). \tag{11.38}$$

With C_δ as in (11.12), $$\lim_{t\to\infty} \frac{\widetilde{I}_{R_1,tR_1,C}}{\ln t} = C_\delta,\ \forall\, R_1 > 0,\ \forall\, C \geqslant 1. \tag{11.39}$$

12. Small δ analysis. More on the location of bad discs. Asymptotic expansion of the energy

We derive here a number of consequences of the results established in Sections 9–11; in particular, we improve the conclusion of (10.1) in Theorem 10.1.

In what follows, we consider some integer $D \geqslant 1$. Given a domain Ω and a boundary condition $g : \partial\Omega \to \mathbb{S}^1$ of degree $-D$, we let

$$m_\varepsilon = m_{\varepsilon,\Omega,g} := \min\big\{E_\varepsilon(u);\ u \in H^1_g(\Omega;\mathbb{C})\big\}. \tag{12.1}$$

Let $\delta_2 = \delta_2(D)$ be as is defined in Theorem 9.1, and let $0 \leqslant \delta \leqslant \delta_2$. By Theorem 9.1, for small ε, a map $u = u_\varepsilon$ achieving m_ε has exactly D enlarged bad discs of centers $x^1_\varepsilon, \ldots, x^D_\varepsilon$.

We start with an easy result.

Theorem 12.1. — *Let $D = 1$. Let $\delta \leqslant \delta_2(1)$. Then, for any $C \geqslant 1$, we have*

$$m_\varepsilon = I_{\varepsilon,1,C} + O(1) \quad \text{as } \varepsilon \longrightarrow 0. \tag{12.2}$$

In particular, we have

$$m_\varepsilon = C_\delta \ln\frac{1}{\varepsilon} + o\left(\ln\frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \longrightarrow 0. \tag{12.3}$$

In the above, $O(1)$ stands for a quantity such that $|O(1)| \leqslant C(\delta,\Omega,g) < \infty$ as $\varepsilon \to 0$.

We continue with a significant improvement of (10.1).

Theorem 12.2. — *Let $D \geqslant 2$. Let $0 < \alpha_0 < 1$. Assume that $\delta_3 = \delta_3(D) < \min(\delta_2(D), 2/(D+2))$ is such that: if $0 \leqslant \delta \leqslant \delta_3$, there exists some $0 < \alpha_0 = \alpha_0(\delta,\Omega,g) < 1$ with the property that, when ε is small (smallness depending on δ), the centers x^j_ε, $j = 1,\ldots,D$, of the enlarged bad discs satisfy*

$$\mathbf{m} := \frac{1}{2}\min_{j\neq k}\big|x^j_\varepsilon - x^k_\varepsilon\big| \geqslant \varepsilon^{\alpha_0}. \tag{12.4}$$

Then, for every $0 < \alpha < 1$ and for $\delta \leqslant \delta_3$ as above, we have, for small ε (smallness depending on α and δ),

$$\mathbf{m} \geqslant \varepsilon^{\alpha}. \tag{12.5}$$

In particular, there exists some $\delta_3 > 0$ such that (12.5) holds for each $0 < \alpha < 1$ provided ε is sufficiently small (smallness depending on α and δ).

The heart of the matter consists of establishing the first part of Theorem 12.2; the second part of Theorem 12.2 follows from the first part combined with (10.1).

Note that, while in Theorem 10.1 δ_3 depends on both D and α, the conclusion of the second part of Theorem 12.2 is that δ_3 can be chosen to depend only on D.

An easy consequence of Theorem 12.2 is the following.

Theorem 12.3. — *Let $D \geqslant 2$. Let $\delta \leqslant \delta_3(D)$, with $\delta_3(D)$ as in Theorem 12.2. Then*

$$m_\varepsilon = DC_\delta \ln\frac{1}{\varepsilon} + o\left(\ln\frac{1}{\varepsilon}\right) \text{ as } \varepsilon \longrightarrow 0. \tag{12.6}$$

Our next result complements Theorems 12.1 and 12.3.

Theorem 12.4. — *Let $D \geqslant 1$. If $D = 1$, let $\delta \leqslant \delta_2(1)$. If $D \geqslant 2$, let $\delta \leqslant \delta_3(D)$. Let u_ε achieve m_ε. If, up to a subsequence, $x_\varepsilon^j \to a_j \in \overline{\Omega}$, $j = 1, \dots, D$, then*

$$\frac{e_\varepsilon(u_\varepsilon)}{\ln(1/\varepsilon)} \rightharpoonup C_\delta \sum_j \delta_{a_j} \quad *\text{-weakly in } \mathscr{M}(\overline{\Omega}). \tag{12.7}$$

(Recall that $e_\varepsilon(u) = \frac{K_1}{2}(\operatorname{div} u)^2 + \frac{K_3}{2}(\operatorname{curl} u)^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2$ is the energy density.)

A basic tool used in the proofs of the above results is the following substitute of (10.14).

Lemma 12.5. — *Let $1/2 \leqslant a < 1$ and $C = C(a) < \infty$ be such that, for the corresponding enlarged bad discs, we have $|u| \geqslant a$ in $\omega := \Omega \setminus \bigcup_j B_{C\varepsilon}(x_\varepsilon^j)$. Let $w := u/|u|$ in ω. Then*

$$E_0(u,\omega) \geqslant E_0(w,\omega) - C_1(\delta,\Omega,g)\frac{1}{\varepsilon^2}\int_\Omega \left(1-|u|^2\right)^2 - C_2(\delta,a,\Omega,g). \tag{12.8}$$

Proof of Lemma 12.5. — If $z = (z_1, z_2) \in \mathbb{R}^2 \sim \mathbb{C}$, we set $z^\perp := (-z_2, z_1)$.

Let $\rho := |u|$ so that $u = \rho w$ in ω and

$$(\operatorname{div} u)^2 = (\rho \operatorname{div} w + \nabla\rho\cdot w)^2 \geqslant \rho^2(\operatorname{div} w)^2 + \nabla\big(\rho^2-1\big)\cdot((\operatorname{div} w)w) \tag{12.9}$$

$$\begin{aligned}(\operatorname{curl} u)^2 &= \big(\rho \operatorname{curl} w - \nabla\rho\cdot w^\perp\big)^2\\ &\geqslant \rho^2(\operatorname{curl} w)^2 - \nabla(\rho^2-1)\cdot\big((\operatorname{curl} w)w^\perp\big).\end{aligned} \tag{12.10}$$

We integrate (12.9) over ω, using an integration by parts for the last term. We proceed similarly for (12.10). Combining the two results, we find that

$$\begin{aligned} E_0(u,\omega) \geqslant E_0(w,\omega) &- \frac{K_1}{2}\int_\omega \big(1-\rho^2\big)(\operatorname{div} w)^2 \\ &- \frac{K_3}{2}\int_\Omega \big(1-\rho^2\big)(\operatorname{curl} w)^2 + \frac{K_1}{2}\int_{\partial\omega} \big(\rho^2-1\big)(\operatorname{div} w) w\cdot\nu \\ &- \frac{K_3}{2}\int_{\partial\omega} \big(\rho^2-1\big)(\operatorname{curl} w) w^\perp\cdot\nu - \frac{K_1}{2}\int_\omega \big(\rho^2-1\big)(\operatorname{div} w)^2 \\ &- \frac{K_3}{2}\int_\omega \big(\rho^2-1\big)(\operatorname{curl} w)^2 - \frac{K_1}{2}\int_\omega \big(\rho^2-1\big) w\cdot\nabla(\operatorname{div} w) \\ &+ \frac{K_3}{2}\int_\omega \big(\rho^2-1\big) w^\perp\cdot\nabla(\operatorname{curl} w). \end{aligned} \tag{12.11}$$

Using: (i) Corollary 4.8 ; (ii) the fact that $|u| \geqslant 1/2$ in $\overline{\omega}$; (iii) the fact that $\rho = 1$ on $\partial\Omega$, we find that the sum of the third and fourth term on the right-hand side in (12.11) is $\geqslant -C_3$, where $C_3 = C_3(\delta, a, \deg g)$. Using, for the other integrals in (12.11), the fact that $|u| \geqslant 1/2$ in ω, we find that

$$\begin{aligned} E_0(u,\omega) &\geqslant E_0(w,\omega) - C_3 - C_4\int_\omega |1-|u|^2| \times \big(|\nabla u|^2 + |D^2 u|\big) \\ &\geqslant E_0(w,\omega) - C_3 - C_4\int_\Omega |1-|u|^2| \times \big(|\nabla u|^2 + |D^2 u|\big), \end{aligned} \tag{12.12}$$

where C_4 is a universal constant.

It remains to dominate the last integral in (12.12). Using: (i) Cauchy–Schwarz; (ii) formula (4.29) (except for the final inequality in (4.29), which requires that Ω is strictly star-shaped), we find that

$$\int_\Omega |1-|u|^2| \times \big(|\nabla u|^2 + |D^2 u|\big) \leqslant C_5\big(1 + \varepsilon^{-2}\big\|1-|u|^2\big\|_2\big), \tag{12.13}$$

where $C_5 = C_5(\delta, \Omega, g)$. We obtain (12.8) from (12.12) and (12.13). □

Proof of Theorem 12.1.

Proof when Ω is strictly star-shaped and $C = 1$. — Let ε be sufficiently small and let $B_{C_1\varepsilon}(x_\varepsilon^1)$ be the enlarged bad disc corresponding to $u = u_\varepsilon$. By choosing if needed a larger (but fixed) C_1, we may assume that, $\Omega \subset B_{C_1/4}(0)$ and thus $\Omega \subset B_{C_1/2}(x_\varepsilon^1)$. Extend u to $\mathbb{R}^2$ as explained in the proof of Theorem 9.1 (after formula (9.8)). Assume, to simplify the forthcoming formulas, that $x_\varepsilon^1 = 0$. Using: (i) estimate (4.20); (ii) the competitor $w := u/|u|$ in the minimization problem (11.2) in $A_{C_1\varepsilon, C_1/2}$ (where

C defining the class $\mathscr{H}_{C_1\varepsilon,C_2/2,C}$ is a sufficiently large fixed constant depending on g *via* (4.23) and the extension G); (iii) the upper bound (4.20); (iv) Lemma 12.5; (v) (11.8); (vi) (11.11), we find that

$$m_\varepsilon \geqslant I_{\varepsilon,1/2,C} + O(1) \geqslant I_{\varepsilon,1,1} + O(1). \tag{12.14}$$

On the other hand, by using (12.14) with (11.5), (11.8), and (11.10), we find that

$$m_\varepsilon \leqslant I_{\varepsilon,1,1} + O(1). \tag{12.15}$$

We complete the proof via (12.14) and (12.15).

Boundedness of the potential term in a general domain. — This follows from a principle devised by Del Pino and Felmer [16]. Assume for simplicity that $0 \in \Omega$. Let $v_\varepsilon(x) := u_\varepsilon(2x)$, $x \in \mathbb{R}^2$ (where u_ε has been extended to $\mathbb{R}^2$ as above). Let $B_R(0)$ be a large fixed ball containing $\overline{\Omega}$. By the first part of the proof, we have

$$E_\varepsilon(u_\varepsilon, B_{2R}(0)) \leqslant I_{\varepsilon,1,1} + O(1), \tag{12.16}$$

$$E_\varepsilon(v_\varepsilon, B_R(0)) \geqslant I_{\varepsilon,1,1} + O(1). \tag{12.17}$$

Subtracting the inequalities (12.16) and (12.17) and noting that

$$\begin{aligned} E_\varepsilon(u_\varepsilon, B_{2R}(0)) - E_\varepsilon(v_\varepsilon, B_R(0)) &= \frac{3}{16\varepsilon^2}\int_{B_R(0)} \left(1-|u_\varepsilon|^2\right)^2 \\ &= \frac{3}{16\varepsilon^2}\int_\Omega \left(1-|u_\varepsilon|^2\right)^2, \end{aligned}$$

we find that

$$\frac{1}{\varepsilon^2}\int_\Omega \left(1-|u_\varepsilon|^2\right)^2 \leqslant C_6(\delta,\Omega,g). \tag{12.18}$$

Proof in a general domain when $C = 1$. — We proceed as in the case of a strictly star-shaped domain, using (12.18) instead of (4.20).

Proof in a general domain for arbitrary C. — The inequality $m_\varepsilon \leqslant I_{\varepsilon,1,C} + O(1)$ is straightforward. On the other hand, we have (arguing as in the first step and using (11.11))

$$m_\varepsilon \geqslant I_{\varepsilon,1/2,C} + O(1) \geqslant I_{\varepsilon,1,C} + O(1). \qquad \square$$

Proof of Theorem 12.2 when Ω is strictly-starshaped. — We argue by contradiction. Then there exists some $\alpha > 0$ such that, passing to a subsequence $\varepsilon_\ell \to 0$ and relabeling the enlarged bad discs if necessary, we have

$$\lim_{\varepsilon\to 0} \frac{\ln\left|x_\varepsilon^1 - x_\varepsilon^2\right|}{\ln\varepsilon} = \alpha, \tag{12.19}$$

$$\liminf_{\varepsilon\to 0} \frac{\ln\left|x_\varepsilon^i - x_\varepsilon^j\right|}{\ln\varepsilon} \leqslant \alpha, \quad \forall\, i \neq j. \tag{12.20}$$

Note that, by assumption, we have

$$0 < \alpha \leqslant \alpha_0 < 1. \tag{12.21}$$

Possibly after passing to further subsequences, there exist a partition consisting of non-empty sets,

$$\{1, \dots, D\} = \mathscr{G}_1 \sqcup \dots \sqcup \mathscr{G}_\ell$$

(with, possibly, $\ell = 1$), with each $\mathscr{G}_k$ non-empty, and a number $0 < \beta < \alpha$ such that

$$1, 2 \in \mathscr{G}_1, \tag{12.22}$$

$$[i, j \in \mathscr{G}_k,\, i \neq j] \Longrightarrow \lim_{\varepsilon \to 0} \frac{\ln\left|x_\varepsilon^i - x_\varepsilon^j\right|}{\ln \varepsilon} = \alpha, \tag{12.23}$$

$$[i \in \mathscr{G}_k,\, j \in \mathscr{G}_n,\, k \neq n] \Longrightarrow \limsup_{\varepsilon \to 0} \frac{\ln\left|x_\varepsilon^i - x_\varepsilon^j\right|}{\ln \varepsilon} \leqslant \beta. \tag{12.24}$$

Consider now constants $\beta < \alpha_1 < \alpha_2 < \alpha < \alpha_3 < \alpha_4 < 1$ to be fixed later (in order to obtain a contradiction). We extend u to $\mathbb{R}^2$ as explained earlier in this section. By (2.3), the upper bound in Lemma 2.2, and a mean value argument, there exists a finite constant C' depending on D and on all the above constants such that: for small ε, there exist radii $\varepsilon^{\alpha_4} < R_1 < \varepsilon^{\alpha_3} < \varepsilon^{\alpha_2} < R_2 < \varepsilon^{\alpha_1}$ satisfying

$$R_j \int_{C_{R_j}(x_\varepsilon^i)} |\nabla u|^2 \leqslant C', \quad j = 1, 2, \quad \forall\, 1 \leqslant i \leqslant D. \tag{12.25}$$

Note that (by definition of α and choice of R_1),

$$\text{for small } \varepsilon, B_{R_1}\left(x_\varepsilon^i\right) \cap B_{R_1}\left(x_\varepsilon^j\right) = \emptyset \text{ if } i \neq j. \tag{12.26}$$

For simplicity, assume, only in this paragraph, that $x_\varepsilon^i = 0$. From (12.25) and the fact that $|u| \geqslant 1/2$ on $C_{R_j}(0)$, $j = 1, 2$, we find that, in the annulus $A_{C_1\varepsilon, R_1}$, $w := u/|u|$ is a competitor in the class $\widetilde{\mathscr{H}}_{C_1\varepsilon, R_1, C}$ (where C_1 is the constant in the definition of the enlarged bad discs and the constant C depends on C' and on the constant C_2 in (4.23)).

By the above, we find that

$$E_0\left(w, B_{R_1}\left(x_\varepsilon^i\right) \setminus B_{C_1\varepsilon}\left(x_\varepsilon^i\right)\right) \geqslant \widetilde{I}_{C_1\varepsilon, R_1, C} - C_3,\ \forall\, i, \tag{12.27}$$

with C_3 a finite constant depending only on the extension of u. (Same for the constants C_4,C_5, C_6 below.)

Set $D_k := \#\mathscr{G}_k$, so that

$$D_1 \geqslant 2, \quad \sum_k D_k = D, \quad \sum_k (D_k)^2 \geqslant D + 2. \tag{12.28}$$

Choose, for each $1 \leqslant k \leqslant \ell$, an index $i_k \in \mathscr{G}_k$. Note that

$$\text{for small } \varepsilon, \quad B_{R_2}\big(x_\varepsilon^{i_k}\big) \cap B_{R_2}\big(x_\varepsilon^{i_j}\big) = \emptyset \text{ if } k \neq j. \tag{12.29}$$

We are therefore in position to apply Lemma 9.3 with:

$$\begin{gathered} X := \overline{\Omega}, \quad U := \big\{x \in \mathbb{R}^2;\, \operatorname{dist}(x, \Omega) \leqslant 1\big\}, \\ R := R_2, \quad B_k := \overline{B}_{R_2}\big(x^{i_k}\varepsilon\big), \quad 1 \leqslant k \leqslant \ell, \\ h := \frac{1}{2}|\nabla w|^2, \\ \lambda_k := (D_k)^2, \quad 1 \leqslant k \leqslant \ell. \end{gathered}$$

(The fact that the assumption (9.4) is satisfied follows from (12.29) and (10.9).) Using Lemma 9.3, we find that

$$G_0\left(w, \Omega \setminus \bigcup_k B_{R_2}\big(x_\varepsilon^{i_k}\big)\right) \geqslant \sum_k (D_k)^2 \ln \frac{1}{R_2} - C_4. \tag{12.30}$$

Combining (12.30) with: (i) Lemma 2.1 applied in $\Omega \setminus \bigcup_k B_{R_2}(x_\varepsilon^{i_k})$; (ii) the upper bounds (12.25) and (4.22); (iii) the fact that $|u| \geqslant 1/2$ on the complement of the enlarged bad discs, we find that

$$E_0\left(w, \Omega \setminus \bigcup_k B_{R_2}\big(x_\varepsilon^{i_k}\big)\right) \geqslant \sum_k (D_k)^2 (1 - \delta_3) \ln \frac{1}{R_2} - C_5. \tag{12.31}$$

Collecting (12.26), (12.27), and (12.31), we find that

$$E_0(w, \Omega) \geqslant D\widetilde{I}_{C_1\varepsilon, R_1, C} + \sum_k (D_k)^2 (1 - \delta_3) \ln \frac{1}{R_2} - C_6. \tag{12.32}$$

Using: (i) (12.32); (ii) Lemma 11.2; (iii) Lemma 12.5; (iv) the fact that Ω is strictly star-shaped (and thus (4.20) holds); (v) the inequalities satisfied by R_1 and R_2; (vi) the last inequality in (12.28), we find that

$$E_0(u, \Omega) \geqslant DC_\delta \ln \frac{1}{\varepsilon} + [(D+2)(1-\delta_3)\alpha_1 - DC_\delta \alpha_4] \ln \frac{1}{\varepsilon} + o\left(\ln \frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \longrightarrow 0. \tag{12.33}$$

Since we have $C_\delta \leqslant \pi$ (see Lemma 11.1), and, by assumption, $\delta_3 < 2/(D+2)$, we find that

$$(D+2)(1-\delta_3)\alpha_1 - DC_\delta \alpha_4 > 0 \quad \text{provided } \alpha_1 \text{ and } \alpha_4 \text{ are sufficiently close to } \alpha. \tag{12.34}$$

We obtain the desired contradiction *via* (12.33), (12.34), and the upper bound (11.6). □

Proof of Theorem 12.3 when Ω is strictly star-shaped. — Let $0 < \alpha_1 < \alpha < 1$. In what follows, constants are finite and independent of small ε. Consider, for small ε, a radius $\varepsilon^{\alpha} < R < \varepsilon^{\alpha_1}$ and a finite constant C' satisfying

$$R \int_{C_R(x_\varepsilon^i)} |\nabla u|^2 \leqslant C', \ \forall\, 1 \leqslant i \leqslant D. \tag{12.35}$$

Arguing as for (12.27), we have

$$E_0\big(w, B_R\big(x_\varepsilon^i\big) \setminus B_{C_1\varepsilon}\big(x_\varepsilon^i\big)\big) \geqslant \widetilde{I}_{C_1\varepsilon,R,C} - C_3. \tag{12.36}$$

From (12.36), (11.39), and the fact that $R > \varepsilon^{\alpha}$, we find that

$$E_0(w, \Omega) \geqslant (1-\alpha) D C_\delta \ln \frac{1}{\varepsilon} + o\bigg(\ln \frac{1}{\varepsilon}\bigg). \tag{12.37}$$

The parameter α being arbitrary in $(0,1)$, we obtain from (12.37) that

$$E_0(w, \Omega) \geqslant D C_\delta \ln \frac{1}{\varepsilon} + o\bigg(\ln \frac{1}{\varepsilon}\bigg). \tag{12.38}$$

Since Ω is strictly star-shaped, (12.38), the upper bound (4.20), and Lemma 12.8 imply that

$$E_0(u, \Omega) \geqslant D C_\delta \ln \frac{1}{\varepsilon} + o\bigg(\ln \frac{1}{\varepsilon}\bigg). \tag{12.39}$$

On the other hand, (11.5) and (11.39) yield

$$E_\varepsilon(u, \Omega) \leqslant D C_\delta \ln \frac{1}{\varepsilon} + o\bigg(\ln \frac{1}{\varepsilon}\bigg). \tag{12.40}$$

The conclusion then follows from (12.39) and (12.40). □

Proof of Theorem 12.2 in a general domain. — Arguing as in the proof of (12.18) and using the upper bound (12.40) (valid in any domain) and the lower bound (12.39) (valid in a ball, by the preceding proof), we find that

$$\frac{1}{\varepsilon^2} \int_\Omega (1 - |u_\varepsilon|^2)^2 = o\bigg(\ln \frac{1}{\varepsilon}\bigg). \tag{12.41}$$

We next repeat the proof of Theorem 12.2 in a strictly star-shaped domain. The only difference arises in the justification of (12.33): when Ω is strictly star-shaped, we use the upper bound (4.20), while, for a general Ω, we rely on (12.41). □

Proof of Theorem 12.3 in a general domain. — As for the preceding proof, we repeat the proof in the strictly star-shaped case, except when it comes to justify (12.39), for which we rely on (12.41) instead of (4.20). □

The proof of Theorem 12.4 relies on the following straightforward variant of Lemma 12.5, whose proof is left to the reader.

Lemma 12.6. — *Let $1/2 \leqslant a < 1$ and $C = C(a) < \infty$ be such that, for the corresponding enlarged bad discs, we have $|u| \geqslant a$ in $\omega := \Omega \setminus \bigcup_j B_{C\varepsilon}(x_\varepsilon^j)$. Let $C\varepsilon < R < \mathbf{m}$. Then*

$$\begin{aligned} E_0\big(u, B_R\big(x_\varepsilon^i\big) \setminus B_{C\varepsilon}\big(x_\varepsilon^i\big)\big) \geqslant E_0\big(w, B_R\big(x_\varepsilon^i\big) \setminus B_{C\varepsilon}\big(x_\varepsilon^i\big)\big) \\ - C_1(\delta, \Omega, g) \frac{1}{\varepsilon^2} \int_\Omega \big(1 - |u|^2\big)^2 \\ - C_2(\delta, a, \Omega, g) - C_3(\delta, \Omega, g) \int_{C_R(x_\varepsilon^i)} |\nabla u|. \end{aligned} \tag{12.42}$$

Proof of Theorem 12.4. — Let $0 < \alpha_1 < \alpha < 1$ and let R be as in the proof of Theorem 12.3. By (11.39), (12.35), (12.36), and (12.42), we find that

$$E_\varepsilon\big(u, B_{\varepsilon^\alpha}\big(x_\varepsilon^i\big) \setminus B_{C\varepsilon}\big(x_\varepsilon^i\big)\big) \geqslant (1 - \alpha) C_\delta \ln \frac{1}{\varepsilon} + o\left(\ln \frac{1}{\varepsilon}\right). \tag{12.43}$$

Therefore, for any fixed $r > 0$ and for any $J \subset \{1, \ldots, D\}$, we have

$$E_\varepsilon\left(u, \bigcup_{j \in J} B_r\big(x_\varepsilon^i\big) \setminus B_{C\varepsilon}\big(x_\varepsilon^i\big)\right) \geqslant \# J C_\delta \ln \frac{1}{\varepsilon} + o\left(\ln \frac{1}{\varepsilon}\right). \tag{12.44}$$

Combining (12.44) with the upper bound (12.40), we find that, for any fixed $r > 0$,

$$E_\varepsilon\left(u, \Omega \setminus \bigcup_j B_r(a_j)\right) = o\left(\ln \frac{1}{\varepsilon}\right). \tag{12.45}$$

From (12.44) and (12.45), we obtain that there exist numbers $b_j \geqslant C_\delta$, $\forall\, j$, such that, possibly up to a subsequence,

$$\frac{e_\varepsilon(u_\varepsilon)}{\ln(1/\varepsilon)} \rightharpoonup \sum_j b_j \delta_{a_j} \quad *\text{-weakly in } \mathscr{M}(\overline{\Omega}). \tag{12.46}$$

The fact that $b_j \leqslant C_\delta$, and thus (12.7) holds for the full sequence, is a consequence of (12.46) and (12.40). □

13. Arbitrary δ analysis. Asymptotic expansion of the energy

Throughout this section, we consider minimizers $u = u_\varepsilon$ of E_ε in $H^1_g(\Omega;\mathbb{C})$, with boundary datum of degree $-D < 0$. A first main goal is to generalize the formula (12.6) to any δ (without any smallness assumption on δ). We will also obtain variants of Theorems 12.2 and 12.4, under weaker assumption on δ. However, the results below are not necessarily, strictly speaking, generalizations of the results in the previous section: while they hold either for any δ or under *explicit* smallness conditions on δ, the picture we get is "blurred", in the sense that it involves, instead of enlarged bad discs (as up to now), giant bad discs (that we define below), whose radii can be much bigger than ε. (Recall that, in the previous section, we assume that $\delta \leqslant \delta_2 = \delta_2(D)$, and the existence of δ_2 is established *via* a proof by contradiction. Therefore, the smallness conditions on δ in the previous section are *not explicit.*)

In order to state the main results of this section, we introduce new notation and several definitions.

Fix δ. Consider the enlarged bad discs constructed in Lemma 5.1. Possibly after passing to a subsequence, we may assume that the number M of bad discs is independent of ε, and that all the limits

$$\lim_{\varepsilon\to 0} \frac{\ln\left|x^i_\varepsilon - x^j_\varepsilon\right|}{\ln \varepsilon} := L_{ij}, \quad i \neq j,$$

exist. Note that we have $0 \leqslant L_{ij} \leqslant 1$, $\forall\ i \neq j$. There exists a partition consisting of non-empty sets,

$$\{1, \ldots, D\} = \mathscr{G}^0_1 \sqcup \ldots \sqcup \mathscr{G}^0_{\ell_0}$$

(with, possibly $\ell_0 = 1$) such that

$$\left[i, j \in \mathscr{G}^0_k,\ i \neq j\right] \iff L_{ij} = 1, \tag{13.1}$$

$$\left[i \in \mathscr{G}^0_k,\ j \in \mathscr{G}^0_n,\ k \neq n\right] \iff L_{ij} < 1. \tag{13.2}$$

If $\mathscr{G}^0_k$ consists of a single index i, the corresponding giant bad disc is simply the enlarged bad disc $B_{C\varepsilon}(x^i_\varepsilon)$. Otherwise, we choose (arbitrarily) $i \in \mathscr{G}^0_k$. We set

$$R_k = R_k(\varepsilon) := 2\max\left\{\left|x^i_\varepsilon - x^j_\varepsilon\right|; \quad j \in \mathscr{G}^0_k, \quad j \neq i\right\}$$

and define the giant bad disc associated with $\mathscr{G}^0_k$ as $B_{R_k}(x^i_\varepsilon)$. Note that, while there is some ambiguity in this definition (since it depends on the choice of i), the giant bad discs have two common features: (i) for small ε, if $j \in \mathscr{G}^0_k$, then $B_{C\varepsilon}(x^j_\varepsilon) \subset B_{R_k}(x^i_\varepsilon)$; (ii) for small ε, two giant bad discs corresponding to two different $\mathscr{G}^0_k$ are disjoint.

We extend u to $\mathbb{R}^2 \setminus \Omega$ as explained in the previous sections and define the degree D_k^0 of the giant bad disc associated with $\mathscr{G}_k^0$ through the formula

$$D_k^0 := \deg\big(u/|u|, C_{R_k}(x_\varepsilon^i)\big).$$

It is straightforward the definition does not depend on the choice of the extension of u.

To give a flavor of the results in this section and how they do compare with the results in the previous sections, we start with a special case of more general assertions below.

Theorem 13.1. — *Assume that $\delta \leqslant 2/(D+2)$. Let $0 < \alpha < 1$. Then, for small ε (smallness depending on α), we have*

$$D_k^0 = -1, \ \forall\, k, \quad \textit{and thus } \ell_0 = D, \tag{13.3}$$

$$m_\varepsilon = DC_\delta \ln\frac{1}{\varepsilon} + o\left(\ln\frac{1}{\varepsilon}\right) \ \textit{as } \varepsilon \longrightarrow 0, \tag{13.4}$$

If x_ε^i, x_ε^j are the centers of two different giant bad discs,

$$\textit{then } |x_\varepsilon^i - x_\varepsilon^j| \geqslant \varepsilon^\alpha, \tag{13.5}$$

If, up to a subsequence, the centers of the giant bad discs satisfy

$$\begin{gathered} x_\varepsilon^j \longrightarrow a_j \in \overline{\Omega}, \quad j = 1, \dots, D, \\ \textit{then } \frac{e_\varepsilon(u_\varepsilon)}{\ln(1/\varepsilon)} \rightharpoonup C_\delta \sum_j \delta_{a_j} \ \ast\textit{-weakly in } \mathscr{M}(\overline{\Omega}). \end{gathered} \tag{13.6}$$

This is to be compared respectively with Theorems 9.1, 12.3, 12.2, and 12.4.

We next introduce a quantity that will play the role of DC_δ in the general case (i.e., without any size assumption on δ). To start with, let $d \in \mathbb{Z}$ be an integer. Associate with d the classes $\mathscr{H}^d_{R_1,R_2,C}$ and $\widetilde{\mathscr{H}}^d_{R_1,R_2,C}$, by replacing, in the definition of $\mathscr{H}_{R_1,R_2,C}$ and $\widetilde{\mathscr{H}}_{R_1,R_2,C}$ (see (11.1) and (11.34)), the condition $\deg(v, C_{R_1}(0)) = \deg(v, C_{R_2}(0)) = -1$, with the condition $\deg(v, C_{R_1}(0)) = \deg(v, C_{R_2}(0)) = d$. (In order to have non-empty classes, one has to suppose that $C \geqslant |d|$.) Consider the corresponding minima $I^d_{R_1,R_2,C}$ and $\widetilde{I}^d_{R_1,R_2,C}$. The analysis in Section 11 (which corresponds to the special case $d = -1$) can be readily extended to the case of an arbitrary degree condition and yields full analogues of the results in Section 11.

We quote e.g., without proof, straightforward generalizations of some of the results there.

$$d^2(1-\delta)\pi \ln \frac{R_2}{R_1} - |d|(1-\delta)\pi \leqslant \widetilde{I}_{R_1,R_2,C} \leqslant d^2\pi \ln \frac{R_2}{R_1}, \tag{13.7}$$

There exists some $d^2(1-\delta)\pi \leqslant C_\delta^d \leqslant d^2\pi$ such that

$$\lim_{t\to\infty} \frac{\widetilde{I}_{R_1,tR_1,C}}{\ln t} = C_\delta^d, \quad \forall\, R_1 > 0, \quad \forall\, C \geqslant |d|. \tag{13.8}$$

Consider now the quantity

$$K(\delta,-D) := \inf \left\{ \sum_{j=1}^{M} C_\delta^{d_j};\ M \geqslant 1, \ \ d_j \in \mathbb{Z}, \ \ \sum_j d_j = -D \right\}. \tag{13.9}$$

We will see later that it suffices to consider, in (13.9), only M and degrees d_j satisfying a priori bounds depending only on δ and D, and thus, in (13.9), inf is actually a min. We will also see that, under the assumption $\delta \leqslant 2/(D+2)$, we have $K(\delta,-D) = DC_\delta$.

A main result in this section is the following.

Theorem 13.2. — *We have*

$$m_\varepsilon = K(\delta,-D) \ln \frac{1}{\varepsilon} + o\left(\ln \frac{1}{\varepsilon}\right) \quad \textit{as } \varepsilon \longrightarrow 0. \tag{13.10}$$

We now proceed to the proofs and establish, on the way, some auxiliary results of independent interest. Since the techniques and arguments used in this section are essentially variants of the ones presented in Sections 9, 10, and 12, the proofs will be rather sketchy.

We start with a straightforward result.

Lemma 13.3. — *The infimum in* (13.9) *is achieved, and every minimal configuration* $(d_j)_{1\leqslant j\leqslant M}$ *such that* $d_j \neq 0$, $\forall\ j$, *satisfies* $M \leqslant C_1(\delta,D)$, $|d_j| \leqslant C_2(\delta,D)$.

Proof. — This follows from the lower bound $d^2(1-\delta)\pi \leqslant C_\delta^d$ (see (13.8)). □

Lemma 13.4. — *For small* ε, *a giant bad disc has a non-zero degree.*

Sketch of proof. — Proof by contradiction. Suppose that, possibly after a subsequence and relabeling the giant bad discs, we have $D_1^0 = 0$ and $x_\varepsilon^1 \in \mathscr{G}_1^0$. Let $0 < \alpha < 1$ be such that, for sufficiently small ε,

$$\left[i \in \mathscr{G}_k^0,\ j \in \mathscr{G}_n^0,\ k \neq n\right] \Longrightarrow \left|x_\varepsilon^i - x_\varepsilon^j\right| \geqslant \varepsilon^\alpha. \tag{13.11}$$

Let $\alpha < \beta < 1$. Using (13.11) and the assumption $D_1^0 = 0$, we are in position to repeat the arguments leading to (6.2) in the proof of Lemma 6.1, and find that

$$E_\varepsilon\big(u, B_{\varepsilon^\beta}(x_\varepsilon^1)\big) \leqslant C_1 \quad \text{and} \quad G_\varepsilon\big(B_{\varepsilon^\beta}(x_\varepsilon^1)\big) \leqslant C_1, \tag{13.12}$$

for some finite constant $C_1 = C_1(\alpha, \beta, \deg g)$. For small ε, estimate (13.12), the fact that $|u(x_\varepsilon^1)| \leqslant 1/2$, and the η-ellipticity Lemma 3.1 yield a contradiction. □

Proof of Theorem 13.2. — An argument similar to the one leading to (11.5) yields the upper bound

$$m_\varepsilon \leqslant K(\delta, -D) \ln \frac{1}{\varepsilon} + C(\Omega, g). \tag{13.13}$$

The heart of the proof consists of establishing the lower bound

$$m_\varepsilon \geqslant K(\delta, -D) \ln \frac{1}{\varepsilon} + o\bigg(\ln \frac{1}{\varepsilon}\bigg) \quad \text{as } \varepsilon \longrightarrow 0. \tag{13.14}$$

Construction of nested groups of bad discs. — Define $L_{ii} := 1$. Possibly after passing to a further subsequence in ε, we may assume that there exist numbers $0 \leqslant \alpha_p < \cdots < \alpha_1 < \alpha_0 = 1$ (with possibly $p = 0$) such that

$$\{L_{ij}; 1 \leqslant i, j \leqslant M\} = \{\alpha_0, \ldots, \alpha_p\}. \tag{13.15}$$

For $0 \leqslant q \leqslant p$, we define the equivalence relation

$$i \sim_q j \iff L_{ij} \geqslant \alpha_q.$$

This equivalence relation defines a partition

$$\{1, \ldots, D\} = \mathscr{G}_1^q \sqcup \ldots \sqcup \mathscr{G}_{\ell_q}^q;$$

for $q = 0$, we recover the partition defined at the beginning at this section, and the corresponding equivalence classes define the giant bad discs. Note that these equivalence classes are nested, in the sense that, if $i \sim_q j$ (and thus i and j are in the same equivalence class at the q level), then $i \sim_r j$, $\forall\, r > q$ (and thus i and j are in the same equivalence class at any higher level).

Proof of (13.14) *when* Ω *is strictly star-shaped.* — If $\alpha_p > 0$, define $\alpha_{p+1} := 0$; otherwise we do not define α_{p+1}. We extend u to $\mathbb{R}^2$ as in the previous sections. For $0 \leqslant q \leqslant p-1$ (if $\alpha_p = 0$), respectively $0 \leqslant q \leqslant p$ (if $\alpha_p > 0$), let

$$\alpha_{q+1} < \beta_q' < \beta_q'' < \gamma_q' < \gamma_q'' < \alpha_q$$

be (arbitrary, but fixed at this stage) constants. Consider a radius R such that

$$\varepsilon^{\gamma_q''} < R < \varepsilon^{\gamma_q'} \quad \text{or} \quad \varepsilon^{\beta_q''} < R < \varepsilon^{\beta_q'}. \tag{13.16}$$

Then, for small ε (smallness depending only on the above constants, not on R),

$$\text{If } i \not\sim_q j, \quad \text{then } B_R\big(x_\varepsilon^i\big) \cap B_R\big(x_\varepsilon^j\big) = \emptyset. \tag{13.17}$$

Consider now, for each q and k, some $i = i(k,q)$ such that $i \in \mathscr{G}_k^q$. We define the "degree of the class $\mathscr{G}_k^q$" as

$$D_k^q := \deg\big(u/|u|, C_R\big(x_\varepsilon^i\big)\big);$$

the definition does not depend on the choice of i or of R satisfying (13.16), and is independent of the extension of u to $\mathbb{R}^2 \setminus \Omega$. When $q = 0$, we recover the definition of the degree of a giant bad disc. Moreover, by (13.17) we have

$$\sum_k D_k^q = -D, \quad \forall\, q. \tag{13.18}$$

We next choose, using a mean value argument, radii

$$\varepsilon^{\gamma_q''} < \rho^q < \varepsilon^{\gamma_q'} < \varepsilon^{\beta_q''} < R^q < \varepsilon^{\beta_q'} \tag{13.19}$$

such that

$$\begin{aligned} &\rho^q \int_{C_{\rho^q}\left(x_\varepsilon^{i(q,k)}\right)} |\nabla u|^2 \leqslant C(D) \quad \text{and} \\ &R^q \int_{C_{R^q}\left(x_\varepsilon^{i(q,k)}\right)} |\nabla u|^2 \leqslant C(D), \quad \forall\, q,\ \forall\, k. \end{aligned} \tag{13.20}$$

Let $w := u/|u|$, well-defined outside the enlarged bad discs. By (13.17), if $\rho^q \leqslant R \leqslant R^q$ and ε is small, then $\deg(w, C_R(x_\varepsilon^{i(q,k)})) = D_k^q$. Combining this fact with (13.19), (13.20), and (13.8), we obtain

$$\begin{aligned} E_0\Big(w, B_{R^q}\Big(x_\varepsilon^{i(q,k)}\Big)\setminus B_{\rho^q}\Big(x_\varepsilon^{i(q,k)}\Big)\Big) \\ \geqslant \big(\gamma_q' - \beta_q''\big) C_\delta^{D_k^q} \ln\frac{1}{\varepsilon} + o\bigg(\ln\frac{1}{\varepsilon}\bigg) \quad \text{as } \varepsilon \longrightarrow 0. \end{aligned} \tag{13.21}$$

Summing (13.21) over q and k, and using (13.17), (13.18), and (13.9), and the fact that u is fixed and smooth in $\mathbb{R}^2 \setminus \Omega$, we obtain

$$\begin{aligned} E_0(w,\Omega) &\geqslant \sum_q \big(\gamma_q' - \beta_q''\big) \sum_k C_\delta^{D_k^q} \ln\frac{1}{\varepsilon} + o\bigg(\ln\frac{1}{\varepsilon}\bigg) \\ &\geqslant K(\delta,-D) \sum_q \big(\gamma_q' - \beta_q''\big) \ln\frac{1}{\varepsilon} + o\bigg(\ln\frac{1}{\varepsilon}\bigg) \quad \text{as } \varepsilon \longrightarrow 0. \end{aligned} \tag{13.22}$$

Letting, in (13.22), $\gamma_q' \to \alpha_q$ and $\beta_q'' \to \alpha_{q+1}$, and using the fact that $\sum_q(\alpha_q - \alpha_{q+1}) = 1$, we find that

$$E_0(w,\Omega) \geqslant K(\delta,-D)\ln\frac{1}{\varepsilon} + o\left(\ln\frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \longrightarrow 0. \tag{13.23}$$

We obtain (13.14) from (13.23), (4.20), and Lemma 12.5.

Proof of (13.14) *in a general domain.* — As in the proof of Theorem 12.2, we rely on the previous step to derive first (12.41), then (13.14) in a general domain. □

An inspection of the proof of (13.22) leads to the following

Corollary 13.5. — *With the above notation, we have*

$$\sum_k C_\delta^{D_k^q} = K(\delta,-D), \quad \forall\, q. \tag{13.24}$$

Sketch of proof of Theorem 13.1. — By (13.24), we have

$$\sum_k C_\delta^{D_k^0} = K(\delta,-D). \tag{13.25}$$

On the other hand, if $\delta < 2/(D+2)$, then, by the first part of (13.8), when $d \neq 0$ we have

$$C_\delta^d \geqslant d^2(1-\delta)\pi > \frac{D}{D+2}d^2\pi. \tag{13.26}$$

Using (13.26), we find that

$$\begin{aligned}\left[D_k \neq 0,\ \forall\, k, \quad \sum_k D_k = -D\right] &\Longrightarrow \text{ either } D_k = -1,\ \forall\, k,\\ &\text{or} \quad \sum_k (D_k)^2 \geqslant D+2 \quad \text{and} \quad \sum_k C_\delta^{D_k} > \pi D.\end{aligned} \tag{13.27}$$

Since, on the other hand, we have (using (11.12))

$$K(\delta,-D) \leqslant DC_\delta \leqslant \pi D, \tag{13.28}$$

we find, from (13.25)–(13.28) and Lemma 13.4, that each giant bad disc is of degree -1 (i.e., (13.3) holds) and that, moreover,

$$K(\delta,-D) = DC_\delta. \tag{13.29}$$

Combining (13.29) with (13.10), we obtain (13.4) when $\delta < 2/(D+2)$.

When $\delta = 2/(D+2)$, we argue similarly (using (11.13) instead of (11.12)), and find that (13.3) and (13.4) still hold in this case.

On the other hand, by construction, the giant bad discs satisfy the assumption (12.4). (With the notation in the proof of Theorem 13.2, the role of α_0 in (12.4) can be played by any constant β with $\alpha_1 < \beta < 1$.) We are in position to repeat the proof of Theorem 12.2 and obtain, for the centers of giant bad discs and ε small, the validity of (13.5), which is the analogue of (12.5). Combining this with (13.29), we are in position to repeat the proof of Theorem 12.4, and obtain the validity of (13.6). □

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