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W. TIMOTHY GOWERS, BEN J. GREEN, FREDDIE MANNERS AND TERENCE TAO

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Marton's conjecture in abelian groups with bounded torsion ^(*)

W. TIMOTHY GOWERS ⁽¹⁾, BEN J. GREEN ⁽²⁾,
FREDDIE MANNERS ⁽³⁾ AND TERENCE TAO ⁽⁴⁾

ABSTRACT. — We prove a Freiman–Ruzsa-type theorem with polynomial bounds in arbitrary abelian groups with bounded torsion, thereby proving (in full generality) a conjecture of Marton. Specifically, let G be an abelian group of torsion m (meaning $mg = 0$ for all $g \in G$) and suppose that A is a non-empty subset of G with $|A + A| \leq K|A|$. Then A can be covered by at most $(2K)^{O(m^3)}$ translates of a subgroup $H \leq G$ of cardinality at most $|A|$. The argument is a variant of that used in the case $G = \mathbf{F}_2^n$ in a recent paper of the authors.

RÉSUMÉ. — Nous démontrons un théorème de type Freiman–Ruzsa avec des bornes polynomiales pour des groupes abéliens arbitraires à torsion bornée, prouvant ainsi (en toute généralité) une conjecture de Marton. Plus précisément, soit G un groupe abélien de torsion m (c'est-à-dire $mg = 0$ pour tout g dans G) et soit A un sous-ensemble non vide de G tel que $|A + A| \leq K|A|$. Alors A peut être couvert par au plus $(2K)^{O(m^3)}$ translatés d'un sous-groupe $H \leq G$ de taille au plus $|A|$. L'argument est une variante de celui utilisé dans le cas $G = \mathbf{F}_2^n$ dans un article récent des auteurs.

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⁽¹⁾ Collège de France 11, place Marcelin-Berthelot 75231 Paris 05 France and Centre for Mathematical Sciences Wilberforce Road Cambridge CB3 0WB UK — wtg10@dpmms.cam.ac.uk

⁽²⁾ Mathematical Institute Andrew Wiles Building Radcliffe Observatory Quarter Woodstock Rd Oxford OX2 6QW UK — ben.green@maths.ox.ac.uk

⁽³⁾ Department of Mathematics University of California San Diego 9500 Gilman Drive # 0112 La Jolla, CA 92093-0112 USA — fmanners@ucsd.edu

⁽⁴⁾ Math Sciences Building 520 Portola Plaza Box 951555 Los Angeles, CA 90095 USA — tao@math.ucla.edu

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1. Introduction

This paper is a companion to the recent work [3] of the authors. Here, we adapt the techniques of that paper to prove a conjecture of Marton, also known as the polynomial Freiman–Ruzsa conjecture, in all groups with bounded torsion. We refer to our previous paper [3] for further discussion of this conjecture.

Throughout the paper, $m \geq 2$ will be an integer. We say that an abelian group $G = (G, +)$ has *torsion* m if $mg = 0$ for all $g \in G$. Note that we do not require m to be minimal with this property.

Here is our main result.

THEOREM 1.1. — *Let G be an abelian group with torsion m for some $m \geq 2$. Suppose that $A \subseteq G$ is a finite non-empty set with $|A + A| \leq K|A|$. Then A is covered by at most⁽¹⁾ $(2K)^{O(m^3)}$ cosets of some subgroup $H \leq G$ of size at most $|A|$. Moreover, H is contained in $\ell A - \ell A$ for some $\ell \ll (2 + m \log K)^{O(m^3 \log m)}$.*

The $m = 2$ case of this theorem (without the containment $H \subseteq \ell A - \ell A$, but with the bound $(2K)^{O(m^3)}$ replaced by the more explicit quantity $2K^{12}$) was⁽²⁾ established in our previous paper [3]. The requirement that G be finite can be easily removed if A is finite, since the torsion hypothesis then implies that the group generated by A is also finite with torsion m . The dependence of the exponents in this theorem on m can probably be improved with additional effort.

In [13, Theorem 11.1] it was shown under the hypotheses of Theorem 1.1 that $2A - 2A$ contains a subgroup H' of cardinality $\geq \exp(-C_m \log^4(2K))|A|$, for some C_m depending on m , where here and throughout the paper we write $\log^k(x)$ for $(\log x)^k$. By standard arguments this also implies that $|H'| \leq K^{O(1)}|A|$ and that A can be covered by $\exp(C_m \log^4(2K))$ cosets of H' (possibly after increasing C_m slightly). A somewhat stronger bound in the direction of Theorem 1.1 was obtained by Konyagin (see [14]), who proved a corresponding result with $\exp(C_{m,\varepsilon} \log^{3+\varepsilon}(2K))$ translates of H' for any $\varepsilon > 0$; however, in Konyagin’s result, unlike in the work of Sanders, one does not have $H' \subseteq 2A - 2A$.

It is a well-known conjecture, the polynomial Bogolyubov conjecture, that one can find a subgroup $H \subseteq 2A - 2A$ with size as large as $K^{-O_m(1)}|A|$.

⁽¹⁾ See Section 1.1 for our conventions on asymptotic notation and sumsets.

⁽²⁾ This bound was subsequently improved by Jun-Jie Liao, first to $2K^{11}$ and then to $2K^9$; see [8].

We have been unable to achieve this even in the $m = 2$ case. However, our method naturally gives the weaker statement given in Theorem 1.1, i.e., that we can find a subgroup H of this size in $\ell A - \ell A$ for ℓ polylogarithmic in K (rather than $\ell = 2$ as in the conjecture).

The special case of Theorem 1.1 that is likely to be of most interest is when $G = \mathbf{F}_p^n$ for some finite field \mathbf{F}_p , a situation which has been widely studied in the literature. Here of course we can take $m = p$. This special case of the result may be formulated in various equivalent ways, analogously to [3, Corollary 1.3]. For applications, the most useful one is likely to be a polynomially effective inverse theorem for the $U^3(\mathbf{F}_p^n)$ -norm

$$\|f\|_{U^3(\mathbf{F}_p^n)} := \left(\mathbf{E}_{x, h_1, h_2, h_3 \in \mathbf{F}_p^n} \Delta_{h_1} \Delta_{h_2} \Delta_{h_3} f(x) \right)^{1/8},$$

where $\Delta_h f(x) := f(x) \overline{f(x+h)}$ and we use the averaging notation $\mathbf{E}_{x \in A} := \frac{1}{|A|} \sum_{x \in A}$.

COROLLARY 1.2. — *Let p be odd. Let $f: \mathbf{F}_p^n \rightarrow \mathbf{C}$ be a 1-bounded function, and suppose that $\|f\|_{U^3(\mathbf{F}_p^n)} \geq \eta$ for some η , $0 < \eta \leq \frac{1}{2}$. Then there is a quadratic polynomial $\phi: \mathbf{F}_p^n \rightarrow \mathbf{F}_p$ such that*

$$|\mathbf{E}_{x \in \mathbf{F}_p^n} f(x) e_p(-\phi(x))| \gg \eta^{O(p^3)},$$

where $e_p(x) := e^{2\pi i x/p}$.

For a proof of this corollary (which is well-known to follow from a result such as Theorem 1.1, but this deduction is not explicitly in the literature), see Appendix C. The $p = 2$ version of this corollary was established in [3].

As in [3], entropy notions will play a key role in the proof of Theorem 1.1. For any set G , define a G -valued random variable to be a discrete random variable taking finitely many values in G . The *entropy* $\mathbf{H}[X]$ of a G -valued random variable is defined by the formula

$$\mathbf{H}[X] := \sum_{x \in G} p_X(x) \log \frac{1}{p_X(x)}$$

where $p_X(x) := \mathbf{P}(X = x)$ is the distribution function of X , with \log denoting the natural logarithm, and with the usual convention $0 \log \frac{1}{0} = 0$. The *entropic Ruzsa distance* $d[X; Y]$ between two G -valued random variables X, Y , first introduced in [11] and then studied further in [3, 4, 15], is defined by the formula

$$d[X; Y] := \mathbf{H}[\tilde{X} + \tilde{Y}] - \frac{1}{2} \mathbf{H}[\tilde{X}] - \frac{1}{2} \mathbf{H}[\tilde{Y}], \quad (1.1)$$

where \tilde{X}, \tilde{Y} are independent copies of X, Y respectively.

Some basic inequalities involving entropic Ruzsa distance (valid for arbitrary abelian groups G) are collected in Appendix A. For now, we give three representative examples of these inequalities that will be used repeatedly in our arguments. The first is the entropic Ruzsa triangle inequality

$$d[X; Y] \leq d[X; Z] + d[Z; Y]$$

for any G -valued random variables X, Y, Z ; see Proposition A.1(i). The second is a Kaimonovich–Vershik–Madiman type inequality

$$d\left[X; \sum_{i=1}^n Y_i\right] \leq 2 \sum_{i=1}^n d[X; Y_i]$$

for any G -valued random variables X, Y_1, \dots, Y_n with Y_1, \dots, Y_n independent; see Proposition A.1(viii). The third is the inequality

$$\mathbf{H}[X - aY] - \mathbf{H}[X] \ll (1 + \log |a|)d[X; Y] \quad (1.2)$$

for any non-zero integer a and independent G -valued random variables X, Y ; see Lemma A.3(ii). We note that a cruder inequality with $O(|a|)$ in place of $O(\log |a|)$ on the right-hand side is rather easier to prove (see Lemma A.3(i)), and leads to the same exponents in our main results. However, (1.2) may be of some independent interest.

Another Ruzsa distance inequality worth mentioning is the contraction property

$$d[\pi(X); \pi(Y)] \leq d[X; Y],$$

valid whenever X, Y are G -valued random variables and $\pi: G \rightarrow G'$ is a homomorphism. While we do not directly use this inequality in this paper, a more precise version of this inequality, which we call the “fibring inequality”, played a crucial role in our previous paper [3], and a variant of that inequality, which we call the “multidistance chain rule”, will play a similarly crucial role in our current arguments; see Lemma 4.1 below.

In [4], it was shown (over \mathbf{F}_2) that the polynomial Freiman–Ruzsa conjecture can be formulated using the language of entropy, and this is the form of the conjecture we will work with here.

THEOREM 1.3. — *Suppose that G is an abelian group of torsion m . Suppose that X, Y are G -valued random variables. Then there exists a subgroup $H \leq G$ such that*

$$d[X; U_H], d[Y; U_H] \ll m^3 d[X; Y].$$

Moreover, if X, Y take values in some symmetric set $S \subseteq G$ containing the origin, then H can be taken to be contained in ℓS for some $\ell \ll (2 + m d[X; Y])^{O(m^3 \log m)}$.

The equivalence of this result and Theorem 1.1 can be obtained by making simple modifications to the arguments of [4, Section 8], so that they work over arbitrary torsion groups rather than \mathbf{F}_2 . The important direction for us, namely that Theorem 1.3 implies Theorem 1.1, is also the easier one. We give the argument in full in Appendix B.

A variant of Theorem 1.3 in the case $m = 2$ was established in [3]. There, the idea was to work with minimizers X, Y of a certain functional

$$\tau[X; Y] := d[X; Y] + \frac{1}{9}d[X^0; X] + \frac{1}{9}d[Y^0; Y]$$

where X^0, Y^0 were the original random variables of interest. If one took $X, Y, \tilde{X}, \tilde{Y}$ to be four independent copies of X, Y, X, Y respectively, it was shown in [3] that entropy inequalities (and in particular the aforementioned “fibring identity”) could be used to establish that any two of the random variables $X + Y, \tilde{X} + Y, X + \tilde{X}$ were nearly independent of each other relative to $X + Y + \tilde{X} + \tilde{Y}$. In an “endgame” step, the above fact was then shown to be in conflict with the characteristic 2 identity

$$(X + Y) + (\tilde{X} + Y) + (X + \tilde{X}) = 0,$$

except when X, Y were already translates of a uniform distribution on a subspace.

In this paper we adopt a slight variant of the approach. We replace the τ functional by a “multidistance” functional

$$D[(X_i)_{i=1}^m] = \mathbf{H}\left[\sum_{i=1}^m X_i\right] - \frac{1}{m} \sum_{i=1}^m \mathbf{H}[X_i]$$

(where we take X_1, \dots, X_m to be independent here for simplicity of notation). Instead of working with minimizers, we work with approximate minimizers, for which the multidistance cannot be significantly improved without making significant changes to the X_i . Considering a tuple of m^2 independent random variables $X_{i,j}$ with $i, j \in \mathbf{Z}/m\mathbf{Z}$, with each $X_{i,j}$ an independent copy of X_i , we will use a kind of iterated variant of the fibring identity, which we call the *multidistance chain rule*, to show that the variables

$$\sum_{i,j} iX_{i,j}, \quad \sum_{i,j} jX_{i,j}, \quad \sum_{i,j} (-i - j)X_{i,j}$$

are pairwise approximately independent relative to $\sum_{i,j} X_{i,j}$. The “endgame” is then based on finding a conflict with the obvious identity

$$\sum_{i,j} iX_{i,j} + \sum_{i,j} jX_{i,j} + \sum_{i,j} (-i - j)X_{i,j} = 0.$$

This turns out to be enough to conclude using similar tools to those wielded in [3], that is to say entropic sumset estimates (which we informally term

“entropic Ruzsa calculus”) and in particular the entropic Balog–Szemerédi–Gowers theorem (established in [15, Lemma 3.3], although we use a minor variant detailed in [3]).

A further outline of the structure of the argument may be found at the end of Section 2, after we have introduced the key notion of multidistance.

1.1. Notation and terminology

The notations $X = O(Y)$, $X \ll Y$, or $Y \gg X$ all mean that $|X| \leq CY$ for an absolute constant C . Different instances of the notation may imply different constants C . We use the usual sumset notation $A \pm B := \{a \pm b : a \in A, b \in B\}$ and $-A := \{-a : a \in A\}$, and let $\ell A = A + \cdots + A$ denote the sum of ℓ copies of A . A set S is symmetric if $S = -S$.

We define the *mutual information* $\mathbf{I}[X : Y]$ by the formula

$$\mathbf{I}[X : Y] := \mathbf{H}[X] + \mathbf{H}[Y] - \mathbf{H}[X, Y].$$

The *conditional entropy* $\mathbf{H}[X|Y]$ of X relative to Y is given by the formula

$$\mathbf{H}[X|Y] := \sum_y p_Y(y) \mathbf{H}[X | Y = y]$$

where y ranges over the support of p_Y , and $(X | Y = y)$ denotes the random variable X conditioned to $Y = y$; we observe the chain rule

$$\mathbf{H}[X, Y] = \mathbf{H}[X|Y] + \mathbf{H}[Y] \tag{1.3}$$

and conditional chain rule

$$\mathbf{H}[X, Y|Z] = \mathbf{H}[X|Y, Z] + \mathbf{H}[Y | Z] \tag{1.4}$$

for arbitrary random variables X, Y, Z defined on the same sample space; see for instance [3, Appendix A]. Here and in the sequel we are adopting the convention of omitting parentheses when the meaning is clear from context, for instance abbreviating $\mathbf{H}[(X, Y)|(Z, W)]$ as $\mathbf{H}[X, Y|Z, W]$, $\mathbf{D}[X_I|(Y_I, Z_I)]$ as $\mathbf{D}[X_I|Y_I, Z_I]$, and so forth.

The *conditional mutual information* $\mathbf{I}[X : Y|Z]$ of two random variables X, Y relative to a third Z (that are all defined on the same sample space) is similarly defined by

$$\mathbf{I}[X : Y|Z] := \sum_z p_Z(z) \mathbf{I}[(X | Z = z) : (Y | Z = z)]$$

or equivalently

$$\begin{aligned} \mathbf{I}[X : Y|Z] &= \mathbf{H}[X|Z] + \mathbf{H}[Y|Z] - \mathbf{H}[X, Y|Z] \\ &= \mathbf{H}[X, Z] + \mathbf{H}[Y, Z] - \mathbf{H}[X, Y, Z] - \mathbf{H}[Z]. \end{aligned} \tag{1.5}$$

We recall the submodularity inequality

$$\mathbf{I}[X : Y | Z] \geq 0; \quad (1.6)$$

see e.g., [3, (A.8)].

In a similar vein, if X and (Y, W) are random variables, with X, Y taking values in an abelian group G , we define the *conditional Ruzsa distance* $d[X; Y|W]$ by the formula

$$d[X; Y|W] = \sum_w p_W(w) d[X; (Y | W=w)]. \quad (1.7)$$

Here, we do not require that X and (Y, W) are defined on the same sample space. The definition (1.7) is a special case of a somewhat more general conditional Ruzsa distance $d[X|Z; Y|W]$ in which one conditions X and Y separately; see [3]. However, we will only need the partially conditional Ruzsa distance (1.7) in this paper.

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2. Induction on multidistance

To describe our argument, we first introduce the notion of *multidistance*. The need for this notion is a key innovation in this paper that was not required in [3].

It is convenient to introduce the following notational convention: if I is a finite indexing set, then X_I denotes a tuple $(X_i)_{i \in I}$ of random variables (and similarly, for example, Y_I denotes $(Y_i)_{i \in I}$ and X'_I denotes $(X'_i)_{i \in I}$). Usually all the variables in such a tuple will be G -valued for some abelian group G .

DEFINITION 2.1. — *Let G be an abelian group and let X_I be a non-empty finite tuple of G -valued random variables. Then we define*

$$D[X_I] := \mathbf{H}[\sum_{i \in I} \tilde{X}_i] - \frac{1}{|I|} \sum_{i \in I} \mathbf{H}[\tilde{X}_i],$$

where the \tilde{X}_i are independent copies of the X_i .

From Proposition A.1(ii) we see that $\mathbf{H}[\sum_{i \in I} \tilde{X}_i] \geq \mathbf{H}[X_i]$ for all $i \in I$, and so on averaging we conclude that the multidistance is always non-negative. It is also clearly invariant with respect to permutations of the X_i . We remark that, in the case $I = \{1, 2\}$, $D[X_{\{1, 2\}}]$ is equal to $d[X_1; -X_2]$ (and hence, if G is a vector space of characteristic 2, is the same as $d[X_1; X_2]$). This observation explains why we use the term multidistance; however, one should not take this terminology too seriously. The basic properties of multidistance are further developed in Section 3 below.

We will deduce Theorem 1.3 from the following statement involving multidistance.

PROPOSITION 2.2. — *Let G be an abelian group of torsion m , and let $I = \{1, 2, \dots, m\}$. If X_I is a tuple of G -valued random variables then there exists a subspace $H \leq G$ such that*

$$\sum_{i \in I} d[X_i; U_H] \ll m^3 D[X_I].$$

Moreover, if all the X_i take values in some symmetric set $S \subseteq G$ containing the origin, then H can be taken to be contained in ℓS for some $\ell \ll (2 + D[X_I])^{O(m^3 \log m)}$.

The deduction of Theorem 1.3 from this is fairly routine and is given in Section 3.

We will prove Proposition 2.2 by a kind of induction on multidistance. The key technical result which drives this is the following proposition (cf. [3, Proposition 2.1], as well as [3, Remarks 2.3, 2.4]).

PROPOSITION 2.3. — *Let G be an abelian group with torsion m . Set $\eta := c/m^3$ for a sufficiently small absolute constant $c > 0$, and set $I := \{1, 2, \dots, m\}$. If X_I is a tuple of G -valued random variables with $D[X_I] > 0$, then there exists a tuple X'_I of G -valued random variables such that we have the multidistance decrement*

$$D[X'_I] \leq (1 - \eta) D[X_I] - \eta \sum_{i \in I} d[X_i; X'_i]. \quad (2.1)$$

Moreover, if all the X_i take values in some symmetric set $S \subseteq G$ containing the origin, then the X'_i can be chosen to take values in $m^3 S$.

The proof of this proposition occupies the bulk of the paper.

We will also need the following result concerning very small values of the multidistance, which forms the base case of the induction.

PROPOSITION 2.4. — *Let G be a finite abelian group. Suppose that I is an indexing set of size $m \geq 2$. Suppose that X_I is a tuple of G -valued random*

variables with $D[X_I] < c_0$ for a sufficiently small absolute constant $c_0 > 0$. Then there is some subgroup $H \leq G$ such that $\sum_{i \in I} d[X_i; U_H] \ll m D[X_I]$. Moreover, if all the X_i take values in some symmetric set $S \subseteq G$ containing the origin, then we can take $H \subseteq 6S$.

This is a relatively straightforward consequence of [4, Theorem 1.3]; for the details, see Section 3.

To conclude this outline, we show how our main result follows from Proposition 2.3 and Proposition 2.4.

Proof of Proposition 2.2 assuming Propositions 2.3 and 2.4. — We apply Proposition 2.3 iteratively, obtaining for each $t \geq 0$ a tuple of random variables $X_I^{(t)}$ supported in $m^{3t}S$ with $X_I^{(0)} = X_I$ (that is, $X_i^{(0)} = X_i$ for $i \in I$), and

$$D[X_I^{(t+1)}] \leq (1 - \eta) D[X_I^{(t)}] - \eta \sum_{i \in I} d[X_i^{(t)}; X_i^{(t+1)}], \quad (2.2)$$

which in particular implies that $D[X_I^{(t+1)}] \leq (1 - \eta) D[X_I^{(t)}]$. Set $k := D[X_I]$. By an easy induction it therefore follows that

$$D[X_I^{(t)}] \leq (1 - \eta)^t k. \quad (2.3)$$

We apply this iteration until t reaches the value

$$s := \lfloor Cm^3 \log(2 + k) \rfloor, \quad (2.4)$$

where C is a sufficiently large absolute constant. By (2.3), we will have $D[X_I^{(s)}] < c_0$ for C large enough, where c_0 is the constant in Proposition 2.4. From that proposition, we see that there is some subgroup $H \leq G$ such that $\sum_{i \in I} d[X_i^{(s)}; U_H] \ll m D[X_I^{(s)}]$. Moreover, we can take $H \subseteq \ell S$ for some $\ell \leq 6m^{3s} \ll (2 + k)^{O(Cm^3 \log m)}$.

From several applications of the triangle inequality (that is, Proposition A.1(i)) and (2.2) we now obtain

$$\begin{aligned} \sum_{i \in I} d[X_i; U_H] &\leq \sum_{i \in I} d[X_i^{(s)}; U_H] + \sum_{t=0}^{s-1} \sum_{i \in I} d[X_i^{(t)}; X_i^{(t+1)}] \\ &\ll m D[X_I^{(s)}] + \sum_{t=0}^{s-1} \frac{1}{\eta} \left((1 - \eta) D[X_I^{(t)}] - D[X_I^{(t+1)}] \right) \\ &\ll (m + \eta^{-1}) D[X_I^{(s)}] + \sum_{t=1}^{s-1} D[X_I^{(t)}]. \end{aligned}$$

By (2.3) we deduce

$$\sum_{i \in I} d[X_i; U_H] \ll ((m + \eta^{-1})(1 - \eta)^s + \eta^{-1})k \ll m^3 k.$$

This concludes the proof. \square

Remark 2.5. — The argument here is a little different from (but on some level equivalent to) that in [3], where a compactness argument was used. We hope that the reader will find it instructive to see the different arguments. The more hands-on iterative argument here allows one to retain some control of H in terms of the support of the X_i , though at the cost of a slight degradation of the absolute constants (but not of the basic form of the dependence on m).

The tasks for the remainder of the paper, then, are to prove Proposition 2.3 and Proposition 2.4, and to deduce Theorem 1.1 from Proposition 2.2. The second and third of these tasks are relatively straightforward and will be dealt with in the next section, leaving only the proof of Proposition 2.3 for the remaining sections of the paper.

3. Relating Ruzsa distance and multidistance

We develop some basic properties of multidistance (Definition 2.1), starting by relating this notion to the more standard notion of entropic Ruzsa distance.

LEMMA 3.1. — *Let G be an abelian group, let I be an indexing set of size $m \geq 2$, and let X_I be a tuple of G -valued random variables. Then*

- (i) $\sum_{\substack{j, k \in I \\ j \neq k}} d[X_j; -X_k] \leq m(m-1) D[X_I];$
- (ii) $\sum_{j \in I} d[X_j; X_j] \leq 2m D[X_I];$
- (iii) *if $(X_i)_{i \in I}$ all have the same distribution, $D[X_I] \leq m d[X_j; X_j]$ for any $j \in I$.*

Proof. — Without loss of generality we may take the X_i to be jointly independent. From Proposition A.1(ii), we see that for any distinct $j, k \in I$, we have

$$\mathbf{H}[X_j + X_k] \leq \mathbf{H}\left[\sum_{i \in I} X_i\right],$$

and hence by (1.1) we have

$$d[X_j; -X_k] \leq \mathbf{H}\left[\sum_{i \in I} X_i\right] - \frac{1}{2}\mathbf{H}[X_j] - \frac{1}{2}\mathbf{H}[X_k].$$

Summing this over all pairs (j, k) , $j \neq k$, gives (i).

From the triangle inequality (Proposition A.1(i)) we have

$$d[X_j; X_j] \leq 2d[X_j; -X_k],$$

and applying this to every summand in (i) gives (ii) (after dividing by $m-1$).

For (iii), we apply Proposition A.1(viii) with X a further independent copy of $-X_j$, and Y_1, \dots, Y_n a relabeling of $(X_i)_{i \in I}$, to get

$$\mathbf{H}[-X + \sum_{i \in I} X_i] - \mathbf{H}[X] \leq m d[X; X].$$

Since

$$\mathbf{H}[-X + \sum_{i \in I} X_i] \geq \mathbf{H}[\sum_{i \in I} X_i]$$

by Proposition A.1(ii), this implies (iii). \square

We are now in a position to give the reduction of Theorem 1.3 to Proposition 2.2.

Proof of Theorem 1.3 assuming Proposition 2.2. — First, we claim that it suffices to establish Theorem 1.3 in the case $X = Y$ (at the expense of worsening the implicit constants by a little more than a factor of 2). Indeed, for general X, Y , by the triangle inequality we have $d[X; X] \leq 2d[X; Y]$, and so (assuming the case $X = Y$ of Theorem 1.3) there is some H with

$$d[X; U_H] \ll m^3 d[X; Y].$$

We then also have

$$d[Y; U_H] \leq d[X; U_H] + d[X; Y] \ll m^3 d[X; Y].$$

This proves the claim.

Suppose then that $X = Y$. Take $I = \{1, 2, \dots, m\}$, and let $X_i = X$ for all $i \in I$. From Lemma 3.1(iii) we have

$$D[X_I] \leq m d[X; X],$$

and hence by Proposition 2.2 we can find a subspace H of G such that $H \subseteq \ell S$ for some

$$\ell \ll (2 + D[X_I])^{O(m^3 \log m)} \leq (2 + m d[X; X])^{O(m^3 \log m)}$$

and such that

$$\sum_{i \in I} d[X; U_H] \ll m^3 D[X_I] \leq m^4 d[X; X].$$

Since the left-hand side is $m d[X; U_H]$, the theorem follows. \square

We are also in a position to establish Proposition 2.4.

Proof of Proposition 2.4. — Write $\varepsilon := D[X_I]$. Then by Lemma 3.1 (i) and averaging, we can find $i \in I$ such that

$$\sum_{k \neq i} d[X_i; -X_k] \leq (m-1)\varepsilon, \quad (3.1)$$

and by further averaging we can find $j \neq i$ such that

$$d[X_i; -X_j] \leq \varepsilon.$$

For ε small enough, we may apply [4, Theorem 1.3] to conclude that there exists a finite subgroup H of G with $d[X_i; U_H] \leq 12\varepsilon$. An inspection of the proof of that theorem (in [4, Section 5]) also reveals that H is of the form $H = S' - S'$, where all the elements y of S' have Kullback–Leibler divergences⁽³⁾ $D_{\text{KL}}(y - X_j || X_i - X_j)$ finite; this implies that $S' \subseteq 3S$ and hence that $H \subseteq 6S$. Since $d[-Y; U_H] = d[Y; -U_H] = d[Y; U_H]$ for any random variable Y , (3.1) and the triangle inequality (Proposition A.1(i)) then give us

$$\sum_{k \in I} d[X_k; U_H] \leq (13m-1)\varepsilon,$$

as required. \square

4. The multidistance chain rule

In this section we establish a key inequality for the behaviour of multidistance under homomorphisms, together with some consequences of it. The key lemma, Lemma 4.1, is a “chain rule” for multidistance, analogous to the chain rule (1.3) for Shannon entropy, as well as the “fibring lemma” for entropic Ruzsa distance in [3, Proposition 4.1] and [4, Proposition 1.4].

We have already remarked on our convention of writing $X_I = (X_i)_{i \in I}$ for a tuple of random variables indexed by some finite set I . If these random variables are G -valued, it is convenient to introduce two further notational conventions. First, if $\pi: G \rightarrow H$ is a homomorphism, we write $\pi(X_I) := (\pi(X_i))_{i \in I}$. Second, if $Y_I = (Y_i)_{i \in I}$ is another tuple of G -valued random variables, we write $X_I + Y_I := (X_i + Y_i)_{i \in I}$.

We will also need to introduce a notion of *conditional multidistance*. If X_I and Y_I are tuples of random variables, with the X_i being G -valued, then we define

$$D[X_I | Y_I] := \mathbf{H} \left[\sum_{i \in I} \tilde{X}_i \mid (\tilde{Y}_j)_{j \in I} \right] - \frac{1}{|I|} \sum_{i \in I} \mathbf{H}[\tilde{X}_i | \tilde{Y}_i] \quad (4.1)$$

(3) We refer the reader to [4] for a definition of Kullback–Leibler divergence.

where $(\tilde{X}_i, \tilde{Y}_i)$, $i \in I$ are independent copies of (X_i, Y_i) , $i \in I$ (but note here that we do *not* assume X_i are independent of Y_i , or \tilde{X}_i independent of \tilde{Y}_i). Equivalently, one has

$$D[X_I | Y_I] = \sum_{(y_i)_{i \in I}} \left(\prod_{i \in I} p_{Y_i}(y_i) \right) D[(X_i | Y_i = y_i)_{i \in I}] \quad (4.2)$$

where each y_i ranges over the support of p_{Y_i} for $i \in I$.

Here is the first key result of this section, the chain rule for multidistance.

LEMMA 4.1. — *Let $\pi: G \rightarrow H$ be a homomorphism of abelian groups and let X_I be a tuple of jointly independent G -valued random variables. Then $D[X_I]$ is equal to*

$$D[X_I | \pi(X_I)] + D[\pi(X_I)] + \mathbf{I}\left[\sum_{i \in I} X_i : \pi(X_I) \mid \pi(\sum_{i \in I} X_i)\right]. \quad (4.3)$$

Proof. — For notational brevity during this proof, write $S := \sum_{i \in I} X_i$.

Expanding out the definition (1.5) of $\mathbf{I}[S : \pi(X_I) | \pi(S)]$ and using the fact that $\pi(S)$ is determined both by S and by $\pi(X_I)$, we obtain

$$\mathbf{I}[S : \pi(X_I) | \pi(S)] = \mathbf{H}[S] + \mathbf{H}[\pi(X_I)] - \mathbf{H}[S, \pi(X_I)] - \mathbf{H}[\pi(S)],$$

and by the chain rule (1.3) the right-hand side is equal to

$$\mathbf{H}[S] - \mathbf{H}[S | \pi(X_I)] - \mathbf{H}[\pi(S)].$$

Therefore,

$$\mathbf{H}[S] = \mathbf{H}[S | \pi(X_I)] + \mathbf{H}[\pi(S)] + \mathbf{I}[S : \pi(X_I) | \pi(S)]. \quad (4.4)$$

From a further application of the chain rule (1.3) we have

$$\mathbf{H}[X_i] = \mathbf{H}[X_i | \pi(X_i)] + \mathbf{H}[\pi(X_i)] \quad (4.5)$$

for all $i \in I$. Averaging (4.5) in i and subtracting this from (4.4), we obtain Lemma 4.1 as a consequence of the definition of multidistance (Definition 2.1). \square

We will need to iterate the multidistance chain rule, so it is convenient to observe a conditional version of this rule, as follows.

LEMMA 4.2. — *Let $\pi: G \rightarrow H$ be a homomorphism of abelian groups. Let I be a finite index set and let X_I be a tuple of G -valued random variables. Let Y_I be another tuple of random variables (not necessarily G -valued). Suppose that the pairs (X_i, Y_i) are jointly independent of one another (but X_i need not be independent of Y_i). Then*

$$\begin{aligned} D[X_I | Y_I] &= D[X_I | \pi(X_I), Y_I] + D[\pi(X_I) | Y_I] \\ &\quad + \mathbf{I}\left[\sum_{i \in I} X_i : \pi(X_I) \mid \pi(\sum_{i \in I} X_i), Y_I\right]. \end{aligned} \quad (4.6)$$

Indeed, for each y_i in the support of p_{Y_i} , we may apply Lemma 4.1 with X_i replaced by the conditioned random variable $(X_i|Y_i = y_i)$, and the claim (4.6) follows by averaging (4.3) in the y_i using the weights p_{Y_i} .

We can iterate the above lemma as follows.

LEMMA 4.3. — *Let m be a positive integer. Suppose one has a sequence*

$$G_m \longrightarrow G_{m-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 = \{0\} \quad (4.7)$$

of homomorphisms between abelian groups G_0, \dots, G_m , and for each $d = 0, \dots, m$, let $\pi_d: G_m \rightarrow G_d$ be the homomorphism from G_m to G_d arising from this sequence by composition (so for instance π_m is the identity homomorphism and π_0 is the zero homomorphism). Let I be a finite index set and let $X_I = (X_i)_{i \in I}$ be a jointly independent tuple of G_m -valued random variables. Then

$$\begin{aligned} D[X_I] &= \sum_{d=1}^m D[\pi_d(X_I) \mid \pi_{d-1}(X_I)] \\ &\quad + \sum_{d=1}^{m-1} \mathbf{I}\left[\sum_i X_i : \pi_d(X_I) \mid \pi_d(\sum_i X_i), \pi_{d-1}(X_I)\right]. \end{aligned} \quad (4.8)$$

In particular, since all the $\mathbf{I}[-]$ terms are nonnegative (see (1.6)), we have

$$D[X_I] \geq \sum_{d=1}^m D[\pi_d(X_I) \mid \pi_{d-1}(X_I)] + \mathbf{I}\left[\sum_i X_i : \pi_1(X_I) \mid \pi_1(\sum_i X_i)\right]. \quad (4.9)$$

Proof. — Indeed, from Lemma 4.2 (taking $Y_I = \pi_{d-1}(X_I)$ and $\pi = \pi_d$ there, and noting that $\pi_d(X_I)$ determines Y_I) we have

$$\begin{aligned} D[X_I \mid \pi_{d-1}(X_I)] &= D[X_I \mid \pi_d(X_I)] + D[\pi_d(X_I) \mid \pi_{d-1}(X_I)] \\ &\quad + \mathbf{I}\left[\sum_{i \in I} X_i : \pi_d(X_I) \mid \pi_d(\sum_{i \in I} X_i), \pi_{d-1}(X_I)\right] \end{aligned}$$

for $d = 1, \dots, m-1$. The claim follows by telescoping series, noting that $D[X_I \mid \pi_0(X_I)] = D[X_I]$ and that $\pi_m(X_I) = X_I$. \square

In our application we will need the following special case of the above lemma.

COROLLARY 4.4. — *Let G be an abelian group and let $m \geq 2$. Suppose that $X_{i,j}$, $1 \leq i, j \leq m$, are independent G -valued random variables. Then*

$$\begin{aligned} & \mathbf{I}\left[\left(\sum_{i=1}^m X_{i,j}\right)_{j=1}^m : \left(\sum_{j=1}^m X_{i,j}\right)_{i=1}^m \mid \sum_{i=1}^m \sum_{j=1}^m X_{i,j}\right] \\ & \leq \sum_{j=1}^{m-1} \left(D\left[\left(X_{i,j}\right)_{i=1}^m\right] - D\left[\left(X_{i,j}\right)_{i=1}^m \mid \left(X_{i,j} + \cdots + X_{i,m}\right)_{i=1}^m\right] \right) \\ & \quad + D\left[\left(X_{i,m}\right)_{i=1}^m\right] - D\left[\left(\sum_{j=1}^m X_{i,j}\right)_{i=1}^m\right], \end{aligned}$$

where all the multidistances here involve the indexing set $\{1, \dots, m\}$.

Proof. — In Lemma 4.3 we take $G_d := G^d$ with the maps $\pi_d: G^m \rightarrow G^d$ for $d = 1, \dots, m$ defined by

$$\pi_d(x_1, \dots, x_m) := (x_1, \dots, x_{d-1}, x_d + \cdots + x_m)$$

with $\pi_0 = 0$. Since $\pi_{d-1}(x)$ can be obtained from $\pi_d(x)$ by applying a homomorphism, we obtain a sequence of the form (4.7).

Now we apply Lemma 4.3 with $I = \{1, \dots, m\}$ and $X_i := (X_{i,j})_{j=1}^m$. Using joint independence, we find that

$$D[X_I] = \sum_{j=1}^m D\left[\left(X_{i,j}\right)_{i \in I}\right].$$

On the other hand, for $1 \leq j \leq m-1$, we see that once $\pi_j(X_i)$ is fixed, $\pi_{j+1}(X_i)$ is determined by $X_{i,j}$ and vice versa, so

$$D\left[\pi_{j+1}(X_I) \mid \pi_j(X_I)\right] = D\left[\left(X_{i,j}\right)_{i \in I} \mid \pi_j(X_I)\right].$$

Since the $X_{i,j}$ are jointly independent, we may further simplify:

$$D\left[\left(X_{i,j}\right)_{i \in I} \mid \pi_j(X_I)\right] = D\left[\left(X_{i,j}\right)_{i \in I} \mid \left(X_{i,j} + \cdots + X_{i,m}\right)_{i \in I}\right].$$

Putting all this into the conclusion of Lemma 4.3, we obtain

$$\begin{aligned} \sum_{j=1}^m D\left[\left(X_{i,j}\right)_{i \in I}\right] & \geq \sum_{j=1}^{m-1} D\left[\left(X_{i,j}\right)_{i \in I} \mid \left(X_{i,j} + \cdots + X_{i,m}\right)_{i \in I}\right] \\ & \quad + D\left[\left(\sum_{j=1}^m X_{i,j}\right)_{i \in I}\right] \\ & \quad + \mathbf{I}\left[\left(\sum_{i=1}^m X_{i,j}\right)_{j=1}^m : \left(\sum_{j=1}^m X_{i,j}\right)_{i=1}^m \mid \sum_{i=1}^m \sum_{j=1}^m X_{i,j}\right] \end{aligned}$$

and the claim follows by rearranging. \square

5. The main argument

We now begin a preliminary discussion of the main task for the rest of the paper, the proof of Proposition 2.3. As in [3], it is convenient to work in the contrapositive, which allows us to take advantage of the notation of conditional entropy.

Suppose, for the rest of the paper, that we have some G -valued random variables X_i , $i \in I = \{1, \dots, m\}$, taking values in a set $S \subseteq G$, with

$$k := D[X_I], \quad (5.1)$$

and suppose that they cannot be decremented in the manner stated in Proposition 2.3. That is to say,

$$D[X'_I] \geq (1 - \eta)k - \eta \sum_{i \in I} d[X_i; X'_i] \quad (5.2)$$

for every tuple X'_I of G -valued random variables taking values in $m^3 S$. The aim is then to show that $k = 0$; this is equivalent to Proposition 2.3.

We now observe that (5.2) implies a conditioned variant of itself, namely

$$D[X'_I|Y_I] \geq (1 - \eta)k - \eta \sum_{i \in I} d[X_i; X'_i|Y_i] \quad (5.3)$$

for any tuple X'_I of G -valued random variables taking values in $m^3 S$ and for any tuple Y_I of random variables. Here, the conditioned multidistance $D[X'_I|Y_I]$ is defined as in (4.1), (4.2), and the conditional Ruzsa distance $d[X; Y|Z]$ is defined in (1.7).

To obtain (5.3) from (5.2), simply replace X'_i by $(X'_i|Y_i = y_i)$ and then sum weighted by $\prod_{i \in I} p_{Y_i}(y_i)$.

The inequality (5.3) may be rearranged in the following convenient way:

$$k - D[X'_I|Y_I] \leq \eta \left(k + \sum_{i \in I} d[X_i; X'_i|Y_i] \right).$$

It also turns out to be convenient to note that

$$k - D[X'_I|Y_I] \leq \eta \left(k + \sum_{i \in I} d[X_{\sigma(i)}; X'_i|Y_i] \right) \quad (5.4)$$

for any permutation $\sigma : I \rightarrow I$, since the multidistance is permutation invariant.

5.1. Bounding the mutual information

The first main step of the argument is to observe that (5.4) combines with Corollary 4.4 to give the following inequality.

PROPOSITION 5.1. — *Let G be an abelian group. Let $m \geq 2$, and suppose that $X_{i,j}$, $1 \leq i, j \leq m$, are jointly independent G -valued random variables, such that for each $j = 1, \dots, m$, the random variables $(X_{i,j})_{i=1}^m$ coincide in distribution with some permutation of the random variables $X_I = (X_i)_{i=1}^m$. Write*

$$\mathcal{I} := \mathbf{I}\left[\left(\sum_{i=1}^m X_{i,j}\right)_{j=1}^m : \left(\sum_{j=1}^m X_{i,j}\right)_{i=1}^m \mid \sum_{i=1}^m \sum_{j=1}^m X_{i,j}\right].$$

Then (assuming (5.1) and (5.2) hold) we have

$$\mathcal{I} \leq 2\eta m \left(k + \sum_{i=1}^m d[X_i; X_i] \right) \leq 2m(2m+1)\eta k. \quad (5.5)$$

For each $j \in \{1, \dots, m\}$ we call the tuple $(X_{i,j})_{i=1}^m$ a *column* and for each $i \in \{1, \dots, m\}$ we call the tuple $(X_{i,j})_{j=1}^m$ a *row*. Hence, by hypothesis, each column is a permutation of $X_I = (X_i)_{i=1}^m$.

Proof. — Corollary 4.4 states that

$$\mathcal{I} \leq \sum_{j=1}^{m-1} A_j + B, \quad (5.6)$$

where

$$A_j := D[(X_{i,j})_{i=1}^m] - D\left[(X_{i,j})_{i=1}^m \mid (X_{i,j} + \dots + X_{i,m})_{i=1}^m\right]$$

and

$$B := D[(X_{i,m})_{i=1}^m] - D\left[\left(\sum_{j=1}^m X_{i,j}\right)_{i=1}^m\right].$$

We first consider the A_j , for fixed $j \in \{1, \dots, m-1\}$. By permutation symmetry of the multidistance, and our hypothesis on columns, we have

$$D[(X_{i,j})_{i=1}^m] = D[(X_i)_{i=1}^m] = D[X_I] = k.$$

Let $\sigma = \sigma_j: I \rightarrow I$ be a permutation such that $X_{i,j} = X_{\sigma(i)}$, and write $X'_i := X_{i,j}$ and $Y_i := X_{i,j} + \dots + X_{i,m}$. Note that all these random variables take values in mS . By (5.4), we conclude that

$$A_j \leq \eta \left(k + \sum_{i=1}^m d[X_{i,j}; X_{i,j} \mid X_{i,j} + \dots + X_{i,m}] \right). \quad (5.7)$$

We similarly consider B . By permutation symmetry on the m -th column,

$$D[(X_{i,m})_{i=1}^m] = D[X_I] = k.$$

For $i \in I$, denote the sum of row i by

$$V_i := \sum_{j=1}^m X_{i,j};$$

if we apply (5.4) again, now with $X_{\sigma(i)} = X_{i,m}$, $X'_i := V_i$, and with the variable Y_i being trivial (that is, using the unconditioned statement (5.2)), we obtain

$$B \leq \eta \left(k + \sum_{i=1}^m d[X_{i,m}; V_i] \right). \quad (5.8)$$

It remains to bound the distances appearing in (5.7) and (5.8) further using Ruzsa calculus. For $1 \leq j \leq m-1$ and $1 \leq i \leq m$, by Proposition A.1(vii) we have

$$\begin{aligned} & d[X_{i,j}; X_{i,j} \mid X_{i,j} + \cdots + X_{i,m}] \\ & \leq d[X_{i,j}; X_{i,j}] + \frac{1}{2} (\mathbf{H}[X_{i,j} + \cdots + X_{i,m}] - \mathbf{H}[X_{i,j+1} + \cdots + X_{i,m}]). \end{aligned}$$

For each i , summing over $j = 1, \dots, m-1$ gives

$$\begin{aligned} & \sum_{j=1}^{m-1} d[X_{i,j}; X_{i,j} \mid X_{i,j} + \cdots + X_{i,m}] \\ & \leq \sum_{j=1}^{m-1} d[X_{i,j}; X_{i,j}] + \frac{1}{2} (\mathbf{H}[V_i] - \mathbf{H}[X_{i,m}]). \end{aligned} \quad (5.9)$$

On the other hand, by Lemma A.2(i) (since $X_{i,m}$ appears in the sum V_i) we have

$$d[X_{i,m}; V_i] \leq d[X_{i,m}; X_{i,m}] + \frac{1}{2} (\mathbf{H}[V_i] - \mathbf{H}[X_{i,m}]). \quad (5.10)$$

Combining (5.6), (5.7) and (5.8) with (5.9) and (5.10) (the latter two summed over i), we get

$$\begin{aligned} \frac{1}{\eta} \mathcal{I} & \leq mk + \sum_{i,j=1}^m d[X_{i,j}; X_{i,j}] + \sum_{i=1}^m (\mathbf{H}[V_i] - \mathbf{H}[X_{i,m}]) \\ & = mk + m \sum_{i=1}^m d[X_i; X_i] + \sum_{i=1}^m \mathbf{H}[V_i] - \sum_{i=1}^m \mathbf{H}[X_i]. \end{aligned} \quad (5.11)$$

By Lemma A.2(ii) (with f taking each j to the index j' such that $X_{i,j}$ is a copy of $X_{j'}$) we obtain the bound

$$\mathbf{H}[V_i] \leq \mathbf{H} \left[\sum_{j=1}^m X_j \right] + \sum_{j=1}^m d[X_{i,j}; X_{i,j}].$$

Finally, summing over i and using $D[X_I] = k$ gives

$$\begin{aligned} \sum_{i=1}^m \mathbf{H}[V_i] - \sum_{i=1}^m \mathbf{H}[X_i] &\leq \sum_{i,j=1}^m d[X_{i,j}; X_{i,j}] + mk \\ &= m \sum_{i=1}^m d[X_i; X_i] + mk, \end{aligned}$$

where in the second step we used the permutation hypothesis. Combining this with (5.11) gives the first inequality in (5.5). The second follows immediately by Lemma 3.1(ii). \square

Remark 5.2. — The choices of the X'_i, Y_i used in this argument are analogous to the choices we called “sums” and “fibres” in the informal discussion in [3, Section 3].

5.2. The endgame

We now define a tuple of independent random variables $(Y_{i,j})_{i,j \in \mathbf{Z}/m\mathbf{Z}}$ as follows: by a slight abuse of notation, we identify $\mathbf{Z}/m\mathbf{Z}$ and $\{1, \dots, m\}$ in the obvious way, and let $Y_{i,j}$ be an independent copy of X_i .

We will be interested in the following random variables derived from $(Y_{i,j})_{i,j \in \mathbf{Z}/m\mathbf{Z}}$:

$$W := \sum_{i,j \in \mathbf{Z}/m\mathbf{Z}} Y_{i,j}$$

and

$$Z_1 := \sum_{i,j \in \mathbf{Z}/m\mathbf{Z}} iY_{i,j}, \quad Z_2 := \sum_{i,j \in \mathbf{Z}/m\mathbf{Z}} jY_{i,j}, \quad Z_3 := \sum_{i,j \in \mathbf{Z}/m\mathbf{Z}} (-i-j)Y_{i,j}.$$

The addition $(-i-j)$ takes place over $\mathbf{Z}/m\mathbf{Z}$. Note that, because we are assuming G is m -torsion, it is well-defined to multiply elements of G by elements of $\mathbf{Z}/m\mathbf{Z}$. Moreover, we note that the Z_i are all supported on m^3S .

Because they will arise frequently, we will also define for $i, j, r \in \mathbf{Z}/m\mathbf{Z}$ the variables

$$P_i := \sum_{j \in \mathbf{Z}/m\mathbf{Z}} Y_{i,j}, \quad Q_j := \sum_{i \in \mathbf{Z}/m\mathbf{Z}} Y_{i,j}, \quad R_r := \sum_{\substack{i,j \in \mathbf{Z}/m\mathbf{Z} \\ i+j=-r}} Y_{i,j}. \quad (5.12)$$

Note the identities

$$Z_1 = \sum_{i \in \mathbf{Z}/m\mathbf{Z}} iP_i, \quad Z_2 = \sum_{j \in \mathbf{Z}/m\mathbf{Z}} jQ_j, \quad Z_3 = \sum_{r \in \mathbf{Z}/m\mathbf{Z}} rR_r. \quad (5.13)$$

There are several key facts to note concerning this situation. One is the easily verified statement that

$$Z_1 + Z_2 + Z_3 = 0 \quad (5.14)$$

holds identically. Another is the following statement, which roughly says that Z_1, Z_2, Z_3 are “almost” pairwise independent conditional on W .

PROPOSITION 5.3. — *Assuming still that (5.1) and (5.2) hold, we have*

$$\mathbf{I}[Z_1 : Z_2 | W], \mathbf{I}[Z_2 : Z_3 | W], \mathbf{I}[Z_1 : Z_3 | W] \leq t$$

where

$$t := 2m(2m+1)\eta k. \quad (5.15)$$

Proof. — We analyze these variables by applying Proposition 5.1 in several different ways. In the first application, take $X_{i,j} = Y_{i,j}$. Note that each column $(X_{i,j})_{i=1}^m$ is indeed a permutation of X_1, \dots, X_m ; in fact, the trivial permutation. Note also that for each $i \in \mathbf{Z}/m\mathbf{Z}$, the row sum is

$$\sum_{j=1}^m X_{i,j} = \sum_{j \in \mathbf{Z}/m\mathbf{Z}} Y_{i,j} = P_i$$

and for each $j \in \mathbf{Z}/m\mathbf{Z}$, the column sum is

$$\sum_{i=1}^m X_{i,j} = \sum_{i \in \mathbf{Z}/m\mathbf{Z}} Y_{i,j} = Q_j.$$

Finally note that $\sum_{i,j=1}^m X_{i,j} = W$. The conclusion of Proposition 5.1 then states that

$$\mathbf{I}\left[(P_i)_{i \in \mathbf{Z}/m\mathbf{Z}} : (Q_j)_{j \in \mathbf{Z}/m\mathbf{Z}} \mid W\right] \leq t,$$

with t as in (5.15). Since Z_1 is a function of $(P_i)_{i \in \mathbf{Z}/m\mathbf{Z}}$ by (5.13), and similarly Z_2 is a function of $(Q_j)_{j \in \mathbf{Z}/m\mathbf{Z}}$, it follows immediately from the data processing inequality, Lemma A.4, that

$$\mathbf{I}[Z_1 : Z_2 | W] \leq t.$$

In the second application of Proposition 5.1, we instead consider $X'_{i,j} = Y_{i-j,j}$. Again, for each fixed j , the tuple $(X'_{i,j})_{i=1}^m$ is a permutation of $(X_i)_{i=1}^m$. This time the row sums for $i \in \{1, \dots, m\}$ are

$$\sum_{j=1}^m X'_{i,j} = \sum_{j \in \mathbf{Z}/m\mathbf{Z}} Y_{i-j,j} = R_{-i}.$$

Similarly, the column sums for $j \in \{1, \dots, m\}$ are

$$\sum_{i=1}^m X'_{i,j} = \sum_{i \in \mathbf{Z}/m\mathbf{Z}} Y_{i-j,j} = Q_j.$$

As before, $\sum_{i,j=1}^m X'_{i,j} = W$. Hence, using (5.13) and the data processing inequality again, the conclusion of Proposition 5.1 tells us

$$\mathbf{I}[Z_3 : Z_2 | W] \leq \mathbf{I}\left[(R_i)_{i \in \mathbf{Z}/m\mathbf{Z}} : (Q_j)_{j \in \mathbf{Z}/m\mathbf{Z}} \mid W\right] \leq t.$$

In the third application⁽⁴⁾ of Proposition 5.1, take $X''_{i,j} = Y_{i,j-i}$. The column and row sums are respectively

$$\sum_{j=1}^m X''_{i,j} = \sum_{j \in \mathbf{Z}/m\mathbf{Z}} Y_{i,j-i} = P_i$$

and

$$\sum_{i=1}^m X''_{i,j} = \sum_{i \in \mathbf{Z}/m\mathbf{Z}} Y_{i,j-i} = R_{-j}.$$

Hence, Proposition 5.1 and data processing gives

$$\mathbf{I}[Z_1 : Z_3 | W] \leq \mathbf{I}\left[(P_i)_{i \in \mathbf{Z}/m\mathbf{Z}} : (R_j)_{j \in \mathbf{Z}/m\mathbf{Z}} \mid W\right] \leq t,$$

which completes the proof. \square

At this point, we are in a very similar situation to [3, Section 7]: conditioning on a typical value $W = w$, the random variables Z_1, Z_2, Z_3 are “almost pairwise independent”, in the sense that the mutual information of any two of them is small. Using the entropic Balog–Szemerédi–Gowers lemma, we can find related variables with very small doubling, and then use these as candidates in (5.3) to obtain a contradiction (unless $k = 0$).

To put this into practice, we first collect some Ruzsa calculus type estimates about the variables W and Z_2 .

LEMMA 5.4. — *For W and Z_2 as above, the following hold:*

- (i) $\mathbf{H}[W] \leq (2m-1)k + \frac{1}{m} \sum_{i=1}^m \mathbf{H}[X_i]$;
- (ii) $\mathbf{H}[Z_2] \leq (28(m-1) \log_2 m)k + \frac{1}{m} \sum_{i=1}^m \mathbf{H}[X_i]$;
- (iii) $\mathbf{I}[W : Z_2] \leq 2(m-1)k$;
- (iv) $\sum_{i=1}^m \mathbf{d}[X_i; Z_2 | W] \leq 15(m^2 \log_2 m)k$.

Proof. — Without loss of generality, we may take X_1, \dots, X_m to be independent. Write $S = \sum_{i=1}^m X_i$. Note that for each $j \in \mathbf{Z}/m\mathbf{Z}$, the sum Q_j from (5.12) above has the same distribution as S . By Proposition A.1(viii)

(4) In fact, by permuting the variables $(Y_{i,j})_{i,j \in \mathbf{Z}/m\mathbf{Z}}$, one can see that the random variables (W, Z_1, Z_2) and (W, Z_1, Z_3) have the same distribution, so this is in some sense identical to – and can be deduced from – the first application.

we have

$$\begin{aligned}\mathbf{H}[W] &= \mathbf{H}\left[\sum_{j \in \mathbf{Z}/m\mathbf{Z}} Q_j\right] \leq \mathbf{H}[S] + \sum_{j=2}^m (\mathbf{H}[Q_1 + Q_j] - \mathbf{H}[S]) \\ &= \mathbf{H}[S] + (m-1)\mathbf{d}[S; -S].\end{aligned}$$

By Lemma A.2(iii), we have

$$\mathbf{d}[S; -S] \leq 2k \quad (5.16)$$

and hence

$$\mathbf{H}[W] \leq 2k(m-1) + \mathbf{H}[S]. \quad (5.17)$$

By the definition (Definition 2.1) of multidistance, and recalling that $k := \mathbf{D}[X_I]$, we have

$$\mathbf{H}[S] = k + \frac{1}{m} \sum_{i=1}^m \mathbf{H}[X_i]. \quad (5.18)$$

This and (5.17) imply (i).

Turning to (ii), we observe

$$\mathbf{H}[Z_2] = \mathbf{H}\left[\sum_{j \in \mathbf{Z}/m\mathbf{Z}} jQ_j\right].$$

Applying Proposition A.1(viii) again gives

$$\mathbf{H}[Z_2] \leq \sum_{i=2}^{m-1} \mathbf{H}[Q_1 + iQ_i] - (m-2)\mathbf{H}[S].$$

Using Lemma A.3(ii) and (5.16) we get

$$\begin{aligned}\mathbf{H}[Z_2] &\leq \mathbf{H}[S] + (10\lfloor \log_2 m \rfloor + 4)(m-2)\mathbf{d}[S; -S] \\ &\leq \mathbf{H}[S] + (20\lfloor \log_2 m \rfloor + 8)(m-2)k.\end{aligned}$$

Applying (5.18) (and crude estimates to tidy the terms involving m) proves (ii). We remark that using the weaker but easier Lemma A.3(i) in place of Lemma A.3(ii) here would lead to similar bounds in our main results.

Turning to (iii), we of course have $\mathbf{I}[W : Z_2] = \mathbf{H}[W] - \mathbf{H}[W|Z_2]$, and since $Z_2 = \sum_{j=1}^{m-1} jQ_j$ and $W = \sum_{j=1}^m Q_j$,

$$\mathbf{H}[W|Z_2] \geq \mathbf{H}[W | Q_1, \dots, Q_{m-1}] = \mathbf{H}[Q_m] = \mathbf{H}[S].$$

Hence, by (i) and (5.18),

$$\mathbf{I}[W : Z_2] \leq \mathbf{H}[W] - \mathbf{H}[S] \leq 2(m-1)k,$$

which is (iii).

Finally we turn to (iv). For each $i \in \{1, \dots, m\}$, using Lemma A.2(i) (noting the sum Z_2 contains X_i as a summand) we have

$$d[X_i; Z_2] \leq d[X_i; X_i] + \frac{1}{2}(\mathbf{H}[Z_2] - \mathbf{H}[X_i]) \quad (5.19)$$

and using Proposition A.1(vi) we have

$$d[X_i; Z_2|W] \leq d[X_i; Z_2] + \frac{1}{2}\mathbf{I}[W : Z_2].$$

Combining with (5.19) and (iii) gives

$$d[X_i; Z_2|W] \leq d[X_i; X_i] + \frac{1}{2}(\mathbf{H}[Z_2] - \mathbf{H}[X_i]) + (m-1)k.$$

Summing over i and applying (ii) gives

$$\sum_{i=1}^m d[X_i; Z_2|W] \leq \sum_{i=1}^m d[X_i; X_i] + \frac{m}{2}(28(m-1)\log_2 m)k + m(m-1)k.$$

Finally, applying Lemma 3.1(ii) (and crude bounds for the terms involving m) gives (iv). \square

We next prove the following analogue of [3, Lemma 7.2].

LEMMA 5.5. — *Let G be an abelian group, let (T_1, T_2, T_3) be a G^3 -valued random variable such that $T_1 + T_2 + T_3 = 0$ holds identically, and write*

$$\delta := \mathbf{I}[T_1 : T_2] + \mathbf{I}[T_1 : T_3] + \mathbf{I}[T_2 : T_3].$$

Let Y_1, \dots, Y_n be some further G -valued random variables and let $\alpha > 0$ be a constant. Then there exists a random variable U , with the support of U contained in that of T_2 , such that

$$d[U; U] + \alpha \sum_{i=1}^n d[Y_i; U] \leq \left(2 + \frac{\alpha n}{2}\right)\delta + \alpha \sum_{i=1}^n d[Y_i; T_2]. \quad (5.20)$$

Proof. — Apply entropic Balog–Szemerédi–Gowers (Proposition A.1(iii)) with $X = T_1$ and $Y = T_2$. Since $T_1 + T_2 = -T_3$, we find that

$$\begin{aligned} \sum_z p_{T_3}(z) d[T_1 | T_3 = z; T_2 | T_3 = z] \\ \leq 3\mathbf{I}[T_1 : T_2] + 2\mathbf{H}[T_3] - \mathbf{H}[T_1] - \mathbf{H}[T_2] \\ = \mathbf{I}[T_1 : T_2] + \mathbf{I}[T_1 : T_3] + \mathbf{I}[T_2 : T_3] = \delta, \end{aligned} \quad (5.21)$$

where the last line follows by observing

$$\mathbf{H}[T_1, T_2] = \mathbf{H}[T_1, T_3] = \mathbf{H}[T_2, T_3] = \mathbf{H}[T_1, T_2, T_3]$$

since any two of T_1, T_2, T_3 determine the third, and by unpacking the definition of mutual information.

By (5.21) and the triangle inequality,

$$\sum_z p_{T_3}(z) d[T_2 \mid T_3 = z; T_2 \mid T_3 = z] \leq 2\delta$$

and by Proposition A.1(vi), for each Y_i ,

$$\begin{aligned} \sum_z p_{T_3}(z) d[Y_i; T_2 \mid T_3 = z] &= d[Y_i; T_2 \mid T_3] \\ &\leq d[Y_i; T_2] + \frac{1}{2} \mathbf{I}[T_2 : T_3] \leq d[Y_i; T_2] + \frac{\delta}{2}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_z p_{T_3}(z) \left(d[T_2 \mid T_3 = z; T_2 \mid T_3 = z] + \alpha \sum_{i=1}^n d[Y_i; T_2 \mid T_3 = z] \right) \\ \leq \left(2 + \frac{\alpha n}{2} \right) \delta + \alpha \sum_{i=1}^n d[Y_i; T_2], \end{aligned}$$

and the result follows by setting $U = (T_2 \mid T_3 = z)$ for some z such that the quantity in parentheses on the left-hand side is at most the weighted average value. \square

Finally, we can put this all together. For each value $W = w$, apply Lemma 5.5 to

$$T_1 = (Z_1 \mid W = w), \quad T_2 = (Z_2 \mid W = w), \quad T_3 = (Z_3 \mid W = w)$$

with $Y_i = X_i$ and $\alpha = \eta/m$, with η the constant in the statement of Proposition 2.3. Write

$$\delta_w := \mathbf{I}[T_1 : T_2] + \mathbf{I}[T_1 : T_3] + \mathbf{I}[T_2 : T_3]$$

for this choice, and note that

$$\begin{aligned} \delta_* &:= \sum_w p_W(w) \delta_w = \mathbf{I}[Z_1 : Z_2 \mid W] + \mathbf{I}[Z_1 : Z_3 \mid W] + \mathbf{I}[Z_2 : Z_3 \mid W] \\ &\leq 6m(2m+1)\eta k \ll \eta m^2 k \end{aligned} \tag{5.22}$$

by Proposition 5.3. Write U_w for the random variable guaranteed to exist by Lemma 5.5, so that (5.20) gives

$$d[U_w; U_w] \leq \left(2 + \frac{\alpha m}{2} \right) \delta_w + \alpha \sum_{i=1}^m (d[X_i; T_2] - d[X_i; U_w]). \tag{5.23}$$

Note that (by Lemma 5.5) the support of U_w is contained in that of Z_2 . Recalling that $Z_2 = \sum_{i,j \in \mathbf{Z}/m\mathbf{Z}} j Y_{i,j}$, with the $Y_{i,j}$ all being copies of X_i and hence supported on S , we see that the support of U_w is contained in $m^3 S$.

Let $(U_w)_I$ denote the tuple consisting of the same variable U_w repeated m times. By Lemma 3.1(iii),

$$D[(U_w)_I] \leq m d[U_w; U_w]. \quad (5.24)$$

On the other hand, applying our hypothesis (5.2) gives

$$D[(U_w)_I] \geq (1 - \eta)k - \eta \sum_{i=1}^m d[X_i; U_w]. \quad (5.25)$$

Combining (5.23), (5.24) and (5.25) and averaging over w (with weight $p_w(w)$), and recalling the value $\alpha = \eta/m$, gives

$$m \left(2 + \frac{\eta}{2}\right) \delta_* + \eta \sum_{i=1}^m d[X_i; Z_2|W] \geq (1 - \eta)k$$

since the terms $d[X_i; U_w]$ cancel out by our choice of α . Substituting in Lemma 5.4(iv) and (5.22), and using the fact that $2 + \frac{\eta}{2} < 3$, we have

$$m^3 \eta k + \eta(m^2 \log_2 m)k \gg k.$$

Recall that, in the statement of Proposition 2.3, η was taken to be c/m^3 . If the constant c is sufficiently small, this gives a contradiction unless $k = 0$. This is what we needed to prove, and the proof of Proposition 2.3 is complete.

Appendix A. Entropic Ruzsa calculus

In this appendix we collect a number of useful inequalities involving entropic Ruzsa distance and related quantities.

PROPOSITION A.1. — *Let G be an abelian group, and let X, Y, Z be G -valued random variables. Then we have the following statements.*

- (i) $d[X; Y] = d[Y; X] \geq 0$ and $d[X; Z] \leq d[X; Y] + d[Y; Z]$.
- (ii) $\max(\mathbf{H}[X], \mathbf{H}[Y]) - \mathbf{I}[X : Y] \leq \mathbf{H}[X \pm Y]$ for either choice of sign \pm .
In particular, if X, Y are independent, then $\mathbf{H}[X], \mathbf{H}[Y]$ are both at most $\mathbf{H}[X \pm Y]$.
- (iii) (Entropic Balog-Szemerédi-Gowers)

$$\begin{aligned} \sum_z p_{X+Y}(z) d[X|X+Y = z; Y|X+Y = z] \\ \leq 3\mathbf{I}[X : Y] + 2\mathbf{H}[X+Y] - \mathbf{H}[X] - \mathbf{H}[Y]. \end{aligned}$$

- (iv) If (X, Z) are independent of Y (but X and Z are not necessarily independent of each other), then

$$\mathbf{H}[X - Z] \leq \mathbf{H}[X - Y] + \mathbf{H}[Y - Z] - \mathbf{H}[Y].$$

- (v) $d[X; -Y] \leq 3d[X; Y]$.

(vi) $d[X; Y|Z] \leq d[X; Y] + \frac{1}{2}\mathbf{I}[Y : Z]$.
(vii) *If Y, Z are independent, then*

$$d[X; Y|Y + Z] \leq d[X; Y] + \frac{1}{2}(\mathbf{H}[Y + Z] - \mathbf{H}[Z]).$$

(viii) *If X, Y_1, \dots, Y_n are jointly independent random variables, then*

$$\mathbf{H}\left[X + \sum_{i=1}^n Y_i\right] - \mathbf{H}[X] \leq \sum_{i=1}^n (\mathbf{H}[X + Y_i] - \mathbf{H}[X]) \quad (\text{A.1})$$

and

$$d\left[X; \sum_{i=1}^n Y_i\right] \leq 2 \sum_{i=1}^n d[X; Y_i]. \quad (\text{A.2})$$

Proof. — For (i)–(iii), see [3, Appendix A]. For (iv), see [4, Lemma 1.1 (i)] which, after unpacking the definitions there, is the same statement. For (v), see [15, Theorem 1.10]. For (vi), see [3, Lemma 5.1]. To prove (vii) (which was also established in [3, Lemma 5.2]), we apply part (vi) and calculate

$$\begin{aligned} \mathbf{I}[Y : Y + Z] &= \mathbf{H}[Y] + \mathbf{H}[Y + Z] - \mathbf{H}[Y, Y + Z] \\ &= \mathbf{H}[Y] + \mathbf{H}[Y + Z] - \mathbf{H}[Y, Z] \\ &= \mathbf{H}[Y + Z] - \mathbf{H}[Z]. \end{aligned}$$

Now we establish (viii), which is essentially due to Madiman [10], and related to an earlier inequality of Kaimanovich and Vershik [7]. The inequality (A.1) follows easily by induction from the $n = 2$ case, which is [3, Lemma A.1]. We turn now to (A.2). From the definition of distance, and replacing all Y_i by $-Y_i$, we may rewrite (A.1) as

$$\begin{aligned} d[X; \sum_{i=1}^n Y_i] + \frac{1}{2}(\mathbf{H}\left[\sum_{i=1}^n Y_i\right] - \mathbf{H}[X]) \\ \leq \sum_{i=1}^n \left(d[X; Y_i] + \frac{1}{2}(\mathbf{H}[X] - \mathbf{H}[Y_i]) \right). \end{aligned}$$

However, by the nonnegativity of multidistance we have $\mathbf{H}\left[\sum_{i=1}^n Y_i\right] \geq \frac{1}{n} \sum_{i=1}^n \mathbf{H}[Y_i]$, so after rearranging we obtain

$$d[X; \sum_{i=1}^n Y_i] \leq \sum_{i=1}^n d[X; Y_i] + \frac{n-1}{2n} \sum_{i=1}^n (\mathbf{H}[Y_i] - \mathbf{H}[X]).$$

The claimed bound (A.2) now follows using the inequality $\mathbf{H}[Y_i] - \mathbf{H}[X] \leq 2d[X; Y_i]$ (see [3, (A.12)]); in fact we can replace the constant 2 by the slightly sharper constant $(2n-1)/n$. \square

In a similar spirit, we give some further inequalities relating to multidistance.

LEMMA A.2. — *Let G be an abelian group, suppose I is a finite index set, $|I| \geq 2$, and that $(X_i)_{i \in I}$ are jointly independent G -valued random variables.*

(i) *For any further random variable Y , and any $i_0 \in I$, we have*

$$d\left[Y; \sum_{i \in I} X_i\right] \leq d[Y; X_{i_0}] + \frac{1}{2} \left(\mathbf{H}\left[\sum_{i \in I} X_i\right] - \mathbf{H}[X_{i_0}] \right).$$

(ii) *For any finite index set J , any jointly independent random variables $(Y_j)_{j \in J}$ independent of the $(X_i)_{i \in I}$, and any function $f: J \rightarrow I$, we have*

$$\mathbf{H}\left[\sum_{j \in J} Y_j\right] \leq \mathbf{H}\left[\sum_{i \in I} X_i\right] + \sum_{j \in J} (\mathbf{H}[Y_j - X_{f(j)}] - \mathbf{H}[X_{f(j)}]).$$

(iii) *If we write $W = \sum_{i \in I} X_i$ then $d[W; -W] \leq 2D[X_I]$.*

Proof. — We may assume without loss of generality that $Y, (Y_j)_{j \in J}$ and X_I are all independent of each other. Applying (A.1) with $n = 2$, $X = X_{i_0}$, $Y_1 = -Y$ and $Y_2 = \sum_{i \neq i_0} X_i$ gives

$$\mathbf{H}\left[-Y + \sum_{i \in I} X_i\right] \leq \mathbf{H}[X_{i_0} - Y] + \mathbf{H}\left[\sum_{i \in I} X_i\right] - \mathbf{H}[X_{i_0}]. \quad (\text{A.3})$$

Using the definition of Ruzsa distance, this gives (i).

For (ii), write $W := \sum_{i \in I} X_i$. Then

$$\begin{aligned} \mathbf{H}\left[\sum_{j \in J} Y_j\right] &\leq \mathbf{H}\left[-W + \sum_{j \in J} Y_j\right] \\ &\leq \mathbf{H}[W] + \sum_{j \in J} (\mathbf{H}[Y_j - W] - \mathbf{H}[W]) \\ &\leq \mathbf{H}[W] + \sum_{j \in J} (\mathbf{H}[Y_j - X_{f(j)}] - \mathbf{H}[X_{f(j)}]) \end{aligned}$$

where in the penultimate step we used (A.1) and in the last step (A.3).

For (iii), take $(X'_i)_{i \in I}$ to be further independent copies of $(X_i)_{i \in I}$ and write $W' = \sum_{i \in I} X'_i$. Fix any distinct $a, b \in I$.

Applying (A.1) with $n = 2$, $X = X_a$, $Y_1 = \sum_{i \neq a} X_i$ and $Y_2 = W'$, we obtain

$$\mathbf{H}[W + W'] \leq \mathbf{H}[W] + \mathbf{H}[X_a + W'] - \mathbf{H}[X_a]. \quad (\text{A.4})$$

Applying (A.1) with $n = 2$, $X = X'_b$, $Y_1 = X_a$ and $Y_2 = \sum_{i \neq b} X'_i$ gives

$$\mathbf{H}[X_a + W'] \leq \mathbf{H}[X_a + X_b] + \mathbf{H}[W'] - \mathbf{H}[X'_b].$$

Combining this with (A.4) and then applying Proposition A.1 (ii) gives

$$\begin{aligned} \mathbf{H}[W + W'] &\leq 2\mathbf{H}[W] + \mathbf{H}[X_a + X_b] - \mathbf{H}[X_a] - \mathbf{H}[X_b] \\ &\leq 3\mathbf{H}[W] - \mathbf{H}[X_a] - \mathbf{H}[X_b]. \end{aligned}$$

Averaging this over all choices of (a, b) gives $\mathbf{H}[W] + 2D[X_I]$, and rearranging gives (iii). \square

Next, we obtain an entropic analogue of some estimates of Bukh [1] on sums of dilates.

LEMMA A.3. — *Let G be an abelian group, let X, Y be independent G -valued random variables, and let $a \in \mathbf{Z}$. Then we have the following two inequalities.*

- (i) $\mathbf{H}[X - aY] - \mathbf{H}[X] \leq 4|a|d[X; Y].$
- (ii) $\mathbf{H}[X - aY] - \mathbf{H}[X] \leq (4 + 10\lfloor \log_2 |a| \rfloor)d[X; Y].$

Proof. — Let a be any integer, and let X' be another independent copy of X . We will show two key inequalities:

$$\mathbf{H}[X - (a \pm 1)Y] \leq \mathbf{H}[X - aY] + (\mathbf{H}[X - Y - X'] - \mathbf{H}[X]) \quad (\text{A.5})$$

and

$$\mathbf{H}[X - 2aY] \leq \mathbf{H}[X - aY] + (\mathbf{H}[X - 2X'] - \mathbf{H}[X]). \quad (\text{A.6})$$

We first show why these suffice. By (A.1) with $n = 2$ we have

$$\mathbf{H}[Y - X + X'] \leq \mathbf{H}[Y - X] + \mathbf{H}[Y + X'] - \mathbf{H}[Y],$$

which rearranges to give

$$\mathbf{H}[X - Y - X'] - \mathbf{H}[X] \leq d[X; Y] + d[X; -Y].$$

By Proposition A.1(v), this is at most $4d[X; Y]$, and so (A.5) gives

$$\mathbf{H}[X - (a \pm 1)Y] \leq \mathbf{H}[X - aY] + 4d[X; Y]. \quad (\text{A.7})$$

By a straightforward induction on a , this proves (i).

Now let X'' be a further independent copy of X . By the case $a = 1$ of (A.5) applied to X, X' we obtain

$$\mathbf{H}[X - 2X'] \leq \mathbf{H}[X - X'] + \mathbf{H}[X - X' - X''] - \mathbf{H}[X],$$

and another application of (A.1) with $n = 2$ gives

$$\mathbf{H}[X - X' - X''] \leq \mathbf{H}[X - X'] + \mathbf{H}[X - X''] - \mathbf{H}[X] = \mathbf{H}[X] + 2d[X; X].$$

Hence,

$$\mathbf{H}[X - 2X'] \leq \mathbf{H}[X] + 3d[X; X] \leq \mathbf{H}[X] + 6d[X; Y]$$

where in the second bound we use the triangle inequality. Hence, (A.6) applied to X and Y implies

$$\mathbf{H}[X - 2aY] \leq \mathbf{H}[X - aY] + 6d[X; Y]. \quad (\text{A.8})$$

Inequalities (A.7) and (A.8) imply that we may recursively bound

$$\mathbf{H}[X - aY] - \mathbf{H}[X] \leq f(a)d[X; Y],$$

where $f: \mathbf{Z} \rightarrow \mathbf{Z}_{\geq 0}$ is the smallest function obeying $f(0) = 0$, $f(a \pm 1) \leq f(a) + 4$ and $f(2a) \leq f(a) + 6$. Certainly, we may insert a new binary digit at the start of a at the cost of increasing $f(a)$ by at most 10, which gives the bound $f(a) \leq 4 + 10\lfloor \log_2 |a| \rfloor$, as required for (ii).

It remains to establish (A.5) and (A.6). For the former, we write $X - (a+1)Y$ as $(X - Y) - aY$ and apply Proposition A.1(iv) with the variables (X, Y, Z) there replaced by $(X - Y, X', aY)$ to obtain

$$\mathbf{H}[X - Y - aY] \leq \mathbf{H}[X - Y - X'] + \mathbf{H}[X' - aY] - \mathbf{H}[X'].$$

On relabeling this is exactly (A.5) in the case $a+1$. For $a-1$, we simply replace all terms $X - Y$ with $X + Y$, noting that $\mathbf{H}[X - X' + Y] = \mathbf{H}[X - X' - Y]$.

For (A.6) we apply the triangle inequality Proposition A.1(i) (or if you prefer, Proposition A.1(iv) again) to get

$$\mathbf{H}[X - 2aY] \leq \mathbf{H}[X - 2X'] + \mathbf{H}[2X' - 2aY] - \mathbf{H}[2X'].$$

It thus suffices to show that

$$\mathbf{H}[2X' - 2aY] - \mathbf{H}[2X'] \leq \mathbf{H}[X' - aY] - \mathbf{H}[X']. \quad (\text{A.9})$$

To see this, observe that by submodularity we have

$$\begin{aligned} \mathbf{H}[X' - aY \mid 2X' - 2aY] &\geq \mathbf{H}[X' - aY \mid 2X' - 2aY, aY] \\ &= \mathbf{H}[X' \mid 2X' - 2aY, aY] \\ &= \mathbf{H}[X' \mid 2X', aY] = \mathbf{H}[X' \mid 2X'], \end{aligned}$$

and this rearranges to give the claimed inequality (A.9). This completes the proof. \square

Finally, for reference we recall the *data processing inequality*, a standard result on mutual information.

LEMMA A.4. — *Let X, Y, Z be random variables. For any functions f, g on the ranges of X, Y respectively, we have $\mathbf{I}[f(X) : g(Y) \mid Z] \leq \mathbf{I}[X : Y \mid Z]$.*

Proof. — It suffices to prove the unconditional version $\mathbf{I}[f(X) : g(Y)] \leq \mathbf{I}[X : Y]$ of this inequality, as the conditional version then follows by conditioning to the events $Z = z$, multiplying by $p_Z(z)$, and summing over z . By symmetry and iteration it will suffice to prove the one-sided data processing inequality $\mathbf{I}[f(X) : Y] \leq \mathbf{I}[X : Y]$, or equivalently that $\mathbf{H}[Y \mid X] \leq \mathbf{H}[Y \mid f(X)]$. But as X determines $f(X)$, we have $\mathbf{H}[Y \mid X] = \mathbf{H}[Y \mid f(X), X]$, and the claim follows from submodularity (1.6). \square

Appendix B. From entropic PFR to combinatorial PFR

In this appendix we repeat the arguments from [4] (which were specialized to the \mathbf{F}_2 case) to derive Theorem 1.1 from Theorem 1.3.

Let m, A, K be as in Theorem 1.1. By translation we may assume without loss of generality that A contains 0. Let U_A be the uniform distribution on A . The doubling condition $|A + A| \leq K|A|$ and Jensen's inequality give

$$d[U_A; -U_A] \leq \log K.$$

By Theorem 1.3 (with S equal to $A \cup -A$, which is clearly symmetric and contains 0), we may thus find a subspace H of G , with $H \subseteq \ell S \subseteq \ell A - \ell A$ for some $\ell \ll (2 + m \log K)^{O(m^3 \log m)}$, such that

$$d[U_A; U_H] \ll m^3 \log K.$$

Now, since $\frac{1}{2}|\mathbf{H}[X] - \mathbf{H}[Y]| \leq d[X; Y]$ we conclude that

$$\log |H| = \log |A| + O(m^3 \log K)$$

and also

$$\mathbf{H}[U_A - U_H] = \log |H| + O(m^3 \log K).$$

Applying [4, (A.2)], we conclude the existence of a point $x_0 \in \mathbf{F}_p^n$ such that

$$p_{U_A - U_H}(x_0) \geq e^{-\mathbf{H}[U_A - U_H]} \geq K^{-O(m^3)} / |H|,$$

or in other words

$$|A \cap (H + x_0)| \geq K^{-O(m^3)} |H|.$$

Applying the Ruzsa covering lemma [16, Lemma 2.14], we may thus cover A by at most $K^{O(m^3)}$ translates of

$$(A \cap (H + x_0)) - A \cap (H + x_0) \subseteq H.$$

By subdividing H into cosets H' of a subgroup of cardinality between $|A|/m$ and $|A|$ if necessary, the claim then follows.

Appendix C. Inverse theorem for U^3

In this appendix we sketch a derivation of Corollary 1.2 from Theorem 1.1. In the even characteristic case $p = 2$ this implication was worked out in [6, 9, 12]. In odd characteristic, the implication is almost worked out in [5], except that the argument there contained a somewhat gratuitous invocation of the Bogolyubov theorem, which currently does not have polynomial bounds and so does not recover the full strength Corollary 1.2. However, it is not difficult to modify the argument in [5] to avoid the use of Bogolyubov's theorem, and we do so here. We will assume familiarity with the notation in that paper.

Let the notation be as in Corollary 1.2. Applying [5, Proposition 5.4] (and identifying \mathbf{F}_p^n with its Pontryagin dual $\widehat{\mathbf{F}}_p^n$ in the standard fashion), one can find a subset H' of \mathbf{F}_p^n , a function $\xi : H' \rightarrow \mathbf{F}_p^n$ whose graph $\Gamma' := \{(h, \xi_h) : h \in H'\} \subseteq \mathbf{F}_p^n \times \mathbf{F}_p^n$ obeys the estimates

$$|\Gamma'| \gg \eta^{O(1)} p^n$$

and

$$|2\Gamma'| \ll \eta^{-O(1)} p^n,$$

and such that

$$|\mathbf{E}_{x \in \mathbf{F}_p^n} f(x + h) \overline{f(x)} e_p(-\xi_h \cdot x)| \gg \eta^{O(1)}$$

for all $(h, \xi_h) \in \Gamma'$, where $\cdot : \mathbf{F}_p^n \times \mathbf{F}_p^n \rightarrow \mathbf{F}_p$ is the usual inner product.

Applying Theorem 1.1, we may cover Γ' by $O(\eta^{-O(p^3)})$ translates of a subgroup H of $\mathbf{F}_p^n \times \mathbf{F}_p^n$ of cardinality at most $|H'| \leq p^n$. If we introduce the groups

$$H_1 := \{y \in \mathbf{F}_p^n : (0, y) \in H\}$$

and

$$H_0 := \{x \in \mathbf{F}_p^n : (x, y) \in H \text{ for some } y \in \mathbf{F}_p^n\}$$

then $|H| = |H_0||H_1|$, and H' can be covered by $O(\eta^{-O(p^3)})$ translates of H_0 , hence

$$|H_0| \gg \eta^{O(p^3)} p^n$$

and hence

$$|H_1| = |H|/|H_0| \ll \eta^{-O(p^3)}.$$

By considering a complementing subspace of $0 \times H_1$ in H , we may write

$$H = \{(x, y) : x \in H_0; y - M_0 x \in H_1\}$$

for some linear transformation $M_0 : H_0 \rightarrow \mathbf{F}_p^n$, which we may then extend (somewhat arbitrarily) to a homomorphism from \mathbf{F}_p^n to \mathbf{F}_p^n . As we are in odd characteristic, we can write $M_0 = 2M$ for some other linear map $M : \mathbf{F}_p^n \rightarrow \mathbf{F}_p^n$. One can then cover H by $O(|H_1|) = O(\eta^{-O(p^3)})$ translates of the graph $\{(x, 2Mx) : x \in \mathbf{F}_p^n\}$, and hence Γ' can also be covered by $O(\eta^{-O(p^3)})$ translates of this graph. We conclude that

$$\mathbf{E}_{h \in \mathbf{F}_p^n} 1_H(h) 1_{\xi_h = 2Mh + \xi_0} \gg \eta^{O(p^3)}$$

for some $\xi_0 \in \mathbf{F}_p^n$. This is a variant of the conclusion of [5, Proposition 6.1], but without the restriction to the subspace V (and with slightly worse exponents). Repeating the “symmetry argument” in [5, Section 6, Step 2], one can then find a subspace W of \mathbf{F}_p^n with

$$|W|/p^n \gg \eta^{O(p^3)} \tag{C.1}$$

such that $Mw \cdot w' = Mw' \cdot w$ for all $w, w' \in W$. Repeating the arguments in [5, Section 6, Step 3], one then concludes that

$$\mathbf{E}_{y \in \mathbf{F}_p^n} \|f\|_{u^3(y+W)} \gg \eta^{O(p^3)}.$$

By⁽⁵⁾ [5, Theorem 2.3(ii)] we have

$$\|f\|_{u^3(\mathbf{F}_p^n)} \geq p^{-n}|W| \|f\|_{u^3(y+W)}$$

for all $y \in \mathbf{F}_p^n$, hence by (C.1)

$$\|f\|_{u^3(\mathbf{F}_p^n)} \gg \eta^{O(p^3)}.$$

By the definition of the u^3 norm, this gives the claim.

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⁽⁵⁾ This result relies on [5, Corollary 3.2]. As pointed out very recently in [2, Remark 4.3], this corollary is false in the generality stated, but holds when W is a complemented subgroup. Fortunately, in \mathbf{F}_p^n , all subgroups are complemented, so this is not an issue for the current argument.

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