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Good height functions on quasi-projective varieties: equidistribution and applications in dynamics [(*)]

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Abstract. — In the present article, we define a notion of *good height functions* on quasi-projective varieties V defined over number fields and prove an equidistribution theorem of small points for such height functions. Those good height functions are defined as limits of height functions associated with semi-positive adelic metrization on big and nef $\mathbb{Q}$-line bundles on projective models of V satisfying mild assumptions.

Building on a recent work of the author and Vigny as well as on a classical estimate of Call and Silverman, and inspiring from recent works of Kühne and Yuan and Zhang, we deduce the equidistribution of generic sequence of preperiodic parameters for families of polarized endomorphisms with marked points.

Résumé. — Dans cet article, nous définissons une notion de *bonne fonction hauteur* sur une variété quasi-projective V définie sur un corps de nombres et nous prouvons un théorème d'équidistribution des petits points pour de telles fonctions hauteurs. Ces bonnes fonctions hauteurs sont définies comme des limites de fonctions hauteurs associées à des suites de $\mathbb{Q}$-fibrés en droites munis de métrisations adéliques semi-positives sur des modèles projectifs de V satisfaisant des hypothèses assez générales.

En nous appuyant sur un récent travail de l'auteur et Vigny, ainsi que sur des estimées classiques de Call et Silverman, et en nous inspirant de travaux récents de Kühne et de Yuan et Zhang, nous en déduisons un résultat d'équidistribution pour les suites génériques de paramètres prépériodiques pour des familles d'endomorphismes polarisés munis de points marqués.

1. Introduction

Let X be a projective variety defined over number field $\mathbb{K}$ and L be an ample line bundle on X. When L is endowed with an adelic semi-positive

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continuous metric $\{\|\cdot\|_v\}_{v\in M_{\mathbb{K}}}$ with induced height function $h_{\overline{L}}: X(\overline{\mathbb{Q}}) \to \mathbb{R}$, a fundamental result is the existence of a systematic equidistribution of small and generic sequences: if $\{x_n\}_n$ is a sequence of points of $X(\overline{\mathbb{Q}})$ such that $h_{\overline{L}}(x_n) \to h_{\overline{L}}(X)$ and if for any subvariety $Z \subset X$ defined over $\mathbb{K}$ there is $n_0 \geqslant 1$ such that the Galois orbit $\mathsf{O}(x_n)$ of x_n is disjoint from Z for all $n \geqslant n_0$, Yuan [43] proved that for any place $v \in M_{\mathbb{K}}$, we have

$$\frac{1}{\deg(x_n)} \sum_{x\in \mathsf{O}(x_n)} \delta_x \to \frac{c_1(\overline{L})_v^{\dim X}}{\mathrm{vol}(L)},$$

in the weak sense of probability measures on the Berkovich analytic space X_v^{an}.

This result, as well as previous existing results concerning the equidistribution of small points, has shown many important implications in arithmetic geometry and dynamics. Historically, a first striking example is the proof by Ullmo [42] and Zhang [47] of the Bogomolov conjecture. An emblematic example in dynamics is the following: let $f_t(z) = z^d + t$ for $(z,t) \in \mathbb{C}^2$ and pick any two complex numbers $a, b \in \mathbb{C}$. Baker and DeMarco [1] prove that the set of parameters $t \in \mathbb{C}$ such that a and b are both preperiodic points of f_t is infinite if and only if $a^d = b^d$. Building on this work, they propose in [2] a dynamical analogue of the André–Oort conjecture. Let us mention that, relying also on Yuan's Theorem, Favre and the author [25] recently proved this so-called Dynamical André–Oort conjecture for curves of polynomials.

When trying to prove this conjecture for general families of rational maps $f_t : \mathbb{P}^1 \to \mathbb{P}^1$, this strategy fails for several reasons. Given such a family parametrized by a quasi-projective curve together with a marked point $a : S \to \mathbb{P}^1$ (viewed as a moving dynamical point), we still have a candidate height function. However, we don't even know whether this function is a Weil height associated with an $\mathbb{R}$-divisor. Worse, in some cases when we can build a metrized line bundle inducing this height function, the continuity of the metric fails [20] or the metric is not anymore adelic [21].

In the present article, we introduce a notion of good height function on a quasi-projective variety defined over a number field and prove an equidistribution of small points for such heights, allowing us for example to prove a general equidistribution statement in families of polarized endomorphisms of projective varieties with marked points, which applies in particular in the above mentionned cases where Yuan's result does not apply.

Good height functions and equidistribution

Let V be a smooth quasi-projective variety defined over a number field $\mathbb{K}$ and place $v \in M_{\mathbb{K}}$ and let $h : V(\overline{\mathbb{Q}}) \to \mathbb{R}$ be a function. A sequence $(F_i)_i$ of Galois-invariant finite subsets of $V(\overline{\mathbb{Q}})$ is *h-small* if

$$h(F_i) := \frac{1}{\#F_i} \sum_{x \in F_i} h(x) \longrightarrow 0, \quad \text{as } i \longrightarrow \infty.$$

Definition 1.1. — *We say h is a* good height *at v if for any $n \geqslant 0$, there is a projective model X_n of V together with a birational morphism $\psi_n : X_n \to X_0$ which is an isomorphism above V and a big and nef $\mathbb{Q}$-line bundle L_n on X_n endowed with an adelic semi-positive continuous metrization $\overline{L}_n$, such that the following holds :*

1. *For any generic h-small sequence $(F_i)_i$ of Galois-invariant finite subsets of $V(\overline{\mathbb{Q}})$, the sequence $\varepsilon_n(\{F_i\}_i) := \limsup_i h_{\overline{L}_n}(\psi_n^{-1}(F_i)) - h_{\overline{L}_n}(X_n)$ satisfies $\varepsilon_n(\{F_i\}) \to 0$ as $n \to \infty$,*
2. *the sequence of volumes $\mathrm{vol}(L_n)$ converges to $\mathrm{vol}(h) > 0$ as $n \to \infty$ and if $c_1(\overline{L}_n)_v$ is the curvature form of $\overline{L}_n$ on $X_{n,v}^{\mathrm{an}}$, then the sequence of finite measures $\left(\mathrm{vol}(L_n)^{-1}(\psi_n)_* c_1(\overline{L}_n)_v^k\right)_n$ converges weakly on V_v^{an} to a probability measure μ_v,*
3. *If $k := \dim V > 1$, for any ample line bundle M_0 on X_0 and any adelic semi-positive continuous metrization $\overline{M}_0$ on M_0, there is a constant $C \geqslant 0$ such that*
$$\left(\psi_n^*(\overline{M}_0)\right)^j \cdot \left(\overline{L}_n\right)^{k+1-j} \leqslant C,$$
for any $2 \leqslant j \leqslant k+1$ and any $n \geqslant 0$.

We say that $\mathrm{vol}(h)$ *is the* volume *of h and that μ_v is the measure* induced by *h at the place v. We finally say h is a* good height *if it is v-good for all $v \in M_{\mathbb{K}}$. In this case, we say $\{\mu_v\}_{v \in M_{\mathbb{K}}}$ is the* global measure *induced by h.*

We prove here the following general equidistribution result.

Theorem 1.2 (Equidistribution of small points). — *Let V be a smooth quasiprojective variety defined over a number field $\mathbb{K}$, let $v \in M_{\mathbb{K}}$ and let h be a v-good height on V with induced measure μ_v. For any h-small sequence $(F_m)_m$ of Galois-invariant finite subsets of $V(\overline{\mathbb{Q}})$ such that for any hypersurface $H \subset V$ defined over $\mathbb{K}$, we have*

$$\#(F_n \cap H) = o(\#F_n), \quad \text{as } n \longrightarrow +\infty,$$

the probability measure $\mu_{F_m,v}$ on V_v^{an} which is equidistributed on F_m converges to μ_v in the weak sense of measures, i.e. for any continuous function *with compact support* $\varphi \in \mathscr{C}^0_c(V_v^{\mathrm{an}})$*, we have*

$$\lim_{m\to\infty} \frac{1}{\#F_m} \sum_{y\in F_m} \varphi(y) = \int_{V_v^{\mathrm{an}}} \varphi\, \mu_v.$$

This result is inspired by Kühne's work [34] where he establishes an equidistribution statement in families of abelian varieties and from [44], where Yuan and Zhang develop a general theory of adelic line bundles on quasi-projective varieties. Among other results, Yuan and Zhang prove an equidistribution theorem for small point in this context. They also deduce Theorem 1.3 and Theorem 1.4 from this general result. The aim of this article is to provide a more naive and independent approach which seems particularly adapted to a dynamical setting. I also have to mention that, even though he focuses on the case of families of abelian varieties in his paper, Kühne has a strategy to generalize his relative equidistribution theorem to a dynamical setting.

It is worth mentioning there are many arithmetic equidistribution statement for small points in the past decades see, e.g., [3, 4, 7, 8, 13, 15, 18, 26, 36, 37, 40, 41, 43]. It si also worth mentioning that, prior to Kühne's recent work only the equidistribution results of arithmetic nature from [37] and [4] do not rely on the continuity of the underlying metrics and compactness of the variety. Also, in both [37] and [4], the results are stated on $\mathbb{P}^1$ and each metric is continuous outside a polar set and is bounded. The generalization of their approach remains unexplored in a more general context. It should also be noted that Mavraki and Ye [36] were the first to get rid of the assumption that the metrization is *adelic*. Theorem 1.2 gives a general criterion to have such an equidistribution statement. What is important here is that a good height function is not necessarily induced by a metrization on a projective model of the variety, or given by an adelic datum.

The strategy of the proof of Theorem 1.2 follows more or less that of Yuan's result. Let us now quickly sketch the proof. Fix an integer n and let φ be a test function at place v with compact support in V_v^{an} and endow the trivial bundle of X_n with the metric induced by $\varphi_n := \varphi \circ \psi_n$ at place v and with the trivial metric at all places $w \neq v$. As it is classical, we first use the adelic Minkowski's second Theorem to compare the $\liminf_i h_{\overline{L}_n(\varphi)}(F_i)$ with the arithmetic volume $\widehat{\mathrm{vol}}_\chi(\overline{L}_n(\varphi))$ of the metrized $\mathbb{Q}$-line bundle $\overline{L}_n(\varphi)$. Then, we rely on the arithmetic version of Siu's bigness criterion proved by Yuan [43] to get a lower bound on the expansion of $\widehat{\mathrm{vol}}_\chi(\overline{L}_n(t\varphi))$ with respect to $t > 0$ of the form $t\int_{X_{n,v}^{\mathrm{an}}} \varphi_n \cdot \mathrm{vol}(L_n)^{-1} c_1(\overline{L}_n)^k_v + C_n(\varphi, t)$.

Our main input here is to get an explicit control of the term $C_n(\varphi,t)$ in terms of $\mathrm{vol}(L_n)$, $\sup_{V_v^{\mathrm{an}}} |\varphi|$ only, for all $t \in (0,1]$. This allows us to find

$C \geqslant 1$ depending only on φ such that for all $t \in (0,1]$, we have

$$\limsup_{i\to\infty}\left|\int_{X_v^{\mathrm{an}}} \varphi\cdot\mu_{F_i,v} - \int_{X_{n,v}^{\mathrm{an}}} \varphi_n \frac{c_1(\overline{L}_n)_v^k}{\mathrm{vol}(L_n)}\right| \leqslant \frac{\varepsilon_n(\{F_i\}_i)}{t} + Ct.$$

When $\dim V > 1$, this is ensured by assumption (3). The hypothesis on h-small sequences and the assumption that the sequence of measures $\mathrm{vol}(L_n)^{-1}(\psi_n)_* c_1(\overline{L}_n)_v^k$ converges to μ_v allow us to conclude.

Applications in families of dynamical systems

Our motivation for proving Theorem 1.2 comes from the study of families of dynamical systems. More precisely, we want to apply Theorem 1.2 to families of polarized endomorphisms. Let S be a smooth quasi-projective variety of dimension $p \geqslant 1$ and let $\pi : \mathcal{X} \to S$ be a family of smooth projective varieties. We say $(\mathcal{X}, f, S)$ is a family of *polarized endomorphisms* if $f : \mathcal{X} \to \mathcal{X}$ is a morphism with $\pi\circ f = \pi$ and if there is a relatively ample line bundle $\mathcal{L}$ on $\mathcal{X}$ and an integer $d \geqslant 2$ such that $f^*\mathcal{L} \simeq \mathcal{L}^{\otimes d}$. Given a family $(\mathcal{X}, f, \mathcal{L})$ and a collection $\mathfrak{a} := (a_1, \ldots, a_q)$ of sections $a_j : S \to \mathcal{X}$ of π are all defined over $\overline{\mathbb{Q}}$, we can define a height function on the variety S by letting

$$h_{f,\mathfrak{a}}(t) := \sum_{j=1}^{q} \widehat{h}_{f_t}(a_j(t))), \quad t \in S(\overline{\mathbb{Q}}),$$

where f_t is the restriction of f to the fiber X_t of $\pi : \mathcal{X} \to S$ and $\widehat{h}_{f_t}$ is the canonical height of the endomorphism $f_t : X_t \to X_t$, as defined by Call–Silverman [14]. If $v \in M_{\mathbb{K}}$ is archimedean, we can also define a *bifurcation current* T_{f,a_i} on S_v^{an} for each dynamical pair $(\mathcal{X}, f, \mathcal{L}, a_i)$ by letting

$$T_{f,a_i} := \pi_*\Big(\widehat{T}_f \wedge [a_i(S_v^{\mathrm{an}})]\Big).$$

This is a closed positive $(1,1)$-current with continuous potential, see Section 4.1 for more details.

As an application of Theorem 1.2, we prove the following.

Theorem 1.3. — *Let $(\mathcal{X}, f, \mathcal{L})$ be a family of polarized endomorphisms parametrized by a smooth quasiprojective variety S and let $\mathfrak{a} := (a_1, \ldots, a_q)$ be a collection of sections of $\pi : \mathcal{X} \to S$, all defined over a number field $\mathbb{K}$. Assume the following properties hold:*

(1) *there exists $v \in M_{\mathbb{K}}$ archimedean such that the measure*

$$\mu_{f,\mathfrak{a}} := (T_{f,a_1} + \cdots + T_{f,a_q})^{\dim S}$$

is non-zero with mass $\mathrm{vol}_f(\mathfrak{a}) := \mu_{f,a}(S_v^{\mathrm{an}}) > 0$,

(2) *there is a generic $h_{f,\mathfrak{a}}$-small sequence $\{F_i\}_i$ of finite Galois-invariant subsets of $S(\overline{\mathbb{Q}})$.*

Pick any $v \in M_{\mathbb{K}}$ and let $(F_m)_m$ be a h-small sequence of Galois-invariant finite subsets of $S(\overline{\mathbb{Q}})$ such that for any hypersurface $H \subset S$ defined over $\mathbb{K}$, we have

$$\#(F_n \cap H) = o(\#F_n), \quad \text{as } n \longrightarrow +\infty.$$

Then the sequence $(\mu_{F_m,v})_m$ of probability measure on S_v^{an} converges to the probability measure $\frac{1}{\mathrm{vol}_f(\mathfrak{a})}\mu_{f,a,v}$ in the weak sense of measures on S_v^{an}.

Let us make a few comments on the proof. By Theorem 1.2, it is sufficient to prove $h_{f,\mathsf{a}}$ is a good height function with associated global measure $\{\mu_{f,\mathfrak{a},v}\}_{v\in M_{\mathbb{K}}}$. The construction of $(X_n, \overline{L}_n, \psi_n)$ is quite easy and just consists in revisiting the convergence

$$\widehat{h}_{f_t}(a_j(t)) = \lim_{n\to\infty} d^{-n} h_{\mathcal{X},\mathcal{L}}(f_t^n(a_j(t))).$$

The convergence of measures $c_1(\overline{L}_n)_v^{\dim S}$ towards $\mu_{f,a,v}$ and the upper bound on the local intersection numbers are also not difficult to establish. The two key facts are the convergence of volumes, which relies on the key estimates of [31], and the fact that for any small generic sequence $\{F_i\}_i$, the sequence $\varepsilon_n(\{F_i\}_i)$ converges to 0. To establish this last point, we use a comparison of $\widehat{h}_{f_t}$ with $h_{\mathcal{X},\mathcal{L}}|_{X_t}$ established by Call and Silverman [14], as well as the convergence of the volumes and Siu's classical bigness criterion. This proof directly inspires from the strategy of [34], where the above mentioned arguments replace his use of a deep and recent result on families of abelian varieties due to Gao and Habegger [27].

To emphasize the strength of Theorem 1.3, we finish here with a general equiditribution towards the *bifurcation measure* μ_{bif} of the moduli space $\mathcal{M}_d$ as introduced by [5]. Recall that the moduli space $\mathcal{M}_d$ of degree d rational maps is the space of PGL(2) conjugacy classes of degree d rational maps, and that it is an irreducible affine variety of dimension $2d-2$ defined over $\mathbb{Q}$, see e.g. [39]. The bifurcation measure μ_{bif} can be build as the measures $\mu_{f,\mathfrak{a}}$ above, where the marked points $a_1, \ldots, a_{2d-2}$ are those parametrizing the critical set $\mathrm{Crit}(f)$. This measure detects the most drastic changes in the global dynamical behavior under small perturbations of the parameter. This is the equivalent in the present context of the equilibrium measure of the Mandelbrot set. We refer to Section 5.3 for a more complete description. Recall also that a parameter $\{f\}$ is *post-critically finite* (or PCF) if its post-critical set $\bigcup_{n\geqslant 1} f^{\circ n}(\mathrm{Crit}(f))$ is finite.

Theorem 1.4. — *Fix a sequence $(F_n)_n$ of finite subsets of the moduli space $\mathcal{M}_d(\overline{\mathbb{Q}})$ of degree d rational maps of $\mathbb{P}^1$ such that F_n is Galois-invariant* *for all n and such that, for any hypersurface H of $\mathcal{M}_d$ that is defined over $\mathbb{Q}$, then*

$$\#(F_n \cap H) = o(\#F_n), \quad \text{as } n \longrightarrow \infty.$$

Assume that for any $\{f\} \in \bigcup_n F_n$, the map f is PCF. Then the measure $\frac{1}{\#F_n}\sum_{\{f\}\in F_n}\delta_{\{f\}}$ converges weakly to the normalized bifurcation measure μ_{bif}.

A variant of Theorem 1.2 on projective varieties

The strength of the proof we give is that it allows to get rid of the existence of a “height function” to get an equidistribution result in the spirit of Theorem 1.2, at least when working on a projective variety. To highlight this observation, we are now going to give a variant of Theorem 1.2 on *projective* varieties, where the open set over which the model X_n is isomorphic to X actually *depends* on n.

Let us now be more precise. We let X be a projective variety of dimension k defined over $\overline{\mathbb{Q}}$ and we fix a place $v \in M_{\mathbb{K}}$. For any $n \geqslant 0$, we let be a birational morphism $\psi_n : X_n \to X$ and we let L_n be a big and nef $\mathbb{Q}$-line bundle endowed with a semi-positive adelic continuous metrization $\overline{L}_n$. We assume that

(1) the sequence $\mathrm{vol}(L_n)$ converges to constant $\mathrm{V} > 0$ and the sequence of probability measures $(\mathrm{vol}(L_n)^{-1}(\psi_n)_* c_1(\overline{L}_n)_v^k)_n$ converges weakly to a probability measure μ_v on X_v^{an},
(2) If $k := \dim X > 1$, for any ample line bundle M_0 on X and any adelic semi-positive continuous metrization $\overline{M}_0$ on M_0, there is a constant $C \geqslant 0$ such that

$$\left(\psi_n^*(\overline{M}_0)\right)^j \cdot \left(\overline{L}_n\right)^{k+1-j} \leqslant C,$$

for any $2 \leqslant j \leqslant k+1$ and any $n \geqslant 0$.

Definition 1.5. — *The data $(X, \mu_v, X_n, \overline{L}_n)$ is a* quasi-height *on X at place v.*

A sequence $(F_i)_i$ of Galois-invariant finite subsets of $X(\overline{\mathbb{Q}})$ is *quasi-small* if $\psi_n^{-1}\{F_i\}$ is a finite subset of $X_n(\overline{\mathbb{Q}})$ for any $n \geqslant 0$ and any i and if the sequence

$$\varepsilon_n(\{F_i\}_i) := \limsup_i h_{\overline{L}_n}(\psi_n^{-1}(F_i)) - h_{\overline{L}_n}(X_n)$$

satisfies $\varepsilon_n(\{F_i\}) \to 0$ as $n \to \infty$.

As in the case of good height functions on a quasi-projective variety, quasi-small points equidistribute the measure μ_v. The precise statement is the following.

Theorem 1.6 (Equidistribution of quasi-small points). — *Let X be a projective variety defined over a number field $\mathbb{K}$, and let $(X,\mu_v,X_n,\overline{L}_n)$ is a quasi-height on X at a given place $v\in M_{\mathbb{K}}$. For any quasi-small sequence $(F_m)_m$ of Galois-invariant finite subsets of $X(\overline{\mathbb{Q}})$ such that for any hypersurface $H\subset V$ defined over $\mathbb{K}$, we have*

$$\#(F_n\cap H)=o(\#F_n),\quad \textit{as } n\longrightarrow+\infty,$$

the probability measure $\mu_{F_m,v}$ on X_v^{an} which is equidistributed on F_m converges to μ_v in the weak sense of measures, i.e. for any continuous function with compact support $\varphi\in\mathscr{C}^0(X_v^{\mathrm{an}})$, we have

$$\lim_{m\to\infty}\frac{1}{\#F_m}\sum_{y\in F_m}\varphi(y)=\int_{X_v^{\mathrm{an}}}\varphi\,\mu_v.$$

The proof of this result follows closely that of Theorem 1.2 and we will explain how to adapt the arguments, when needed.

There are two essential differences between this definition of quasi-height and the definition of good height at a place v. The first one is that we do not require here that a height function is actually defined on a Zariski open subset $U\subseteq X$, but only on the Zariski dense subset of $\overline{\mathbb{Q}}$-points of X that we want to equidistribute. This is why we chose to call the data $(X,\mu_v,X_n,\overline{L}_n)$ a quasi-height. The second important difference is that, since there is no Zariski open set U above which $\psi_n: X_n\to X$ is an isomorphism, we have to require a *stronger* convergence properties of the sequence of measures $(\mathrm{vol}(L_n)^{-1}(\psi_n)_*c_1(\overline{L}_n)_v^k)_n$: we have to require that this sequence converges to μ_v on the compact space X_v^{an}. This was not required in the case of good heights.

To motivate this variant, let me mention one case where Theorem 1.2 does not apply but Theorem 1.6 applies: with Vigny, we show in [30] that, under mild assumptions, for a birational map $f:\mathbb{P}^k\dashrightarrow\mathbb{P}^k$ defined over a number field, generic sequences of periodic points converge to the maximal entropy measure μ_f of f weakly on $\mathbb{P}^k(\mathbb{C})$. We also exhibit examples of such maps such that the set $\bigcup_n(I_{f^n}\cup I_{f^{-n}})$ is Zariski dense in $\mathbb{P}^k$, where I_{f_n} (resp. $I_{f^{-n}}$) is the indeterminacy locus of f^n (resp. of f^{-n}). In the construction of the quasi-height, the map $\psi_n: X_n\to\mathbb{P}^k$ can be an isomorphism only above $\bigcup_{k\leqslant n}(I_{f^k}\cup I_{f^{-k}})$ (see [30] for more details).

Organization of the paper

Section 2 is devoted to quantitative height inequalities which are use in the proof of Theorem 1.2. Section 3 is dedicated to the proof of Theorem 1.2, and we prove in Section 3.3 that quasi-adelic measures induce good height functions. Theorem 1.3 is proved in Section 4. Finally, in Section 5 we discuss the assumptions of Theorem 1.3, proving they are sharp and we prove Theorem 1.4.

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2. Quantitative height inequalities

In the whole section, we let X be a smooth projective variety of dimension k defined over a number field $\mathbb{K}$.

2.1. Arithmetic intersection and heights

For the material of this section, we refer to [16] and [45]. Let $L_0, \dots, L_k$ be $\mathbb{Q}$-line bundle on X. Assume L_i is equipped with an adelic continuous metric $\{\|\cdot\|_{v,i}\}_{v\in M_{\mathbb{K}}}$ and we denote $\overline{L}_i := (L_i, \{\|\cdot\|_v\}_{v\in M_{\mathbb{K}}})$. Assume $\overline{L}_i$ is semi-positive for $1 \leqslant i \leqslant k$ and $\overline{L}_0$ is integrable, i.e. can be written as a difference of semi-positive adelic line bundles.

Fix a place $v \in M_{\mathbb{K}}$. Denote by X_v^{an} the Berkovich analytification of X at the place v. We also let $c_1(\overline{L}_i)_v$ be the curvature form of the metric $\|\cdot\|_{v,i}$ on X_v^{an}. We will use in the sequel that the arithmetic intersection number $\left(\overline{L}_0 \cdots \overline{L}_k\right)$ is symmetric and multilinear with respect to the L_i and that

$$\left(\overline{L}_0\right) \cdots \left(\overline{L}_k\right) = \left(\overline{L}_1|_{\mathrm{div}(s)}\right) \cdots \left(\overline{L}_k|_{\mathrm{div}(s)}\right) + \sum_{v\in M_{\mathbb{K}}} \int_{X_v^{\mathrm{an}}} \log\|s\|_v^{-1} \bigwedge_{j=1}^{k} c_1(\overline{L}_i)_v,$$

for any global section $s \in H^0(X, L_0)$ (whenever such a section exists). In particular, if L_0 is the trivial bundle and $\|\cdot\|_{v,0}$ is the trivial metric at all places but v_0, this gives

$$\left(\overline{L}_0\right)\cdots\left(\overline{L}_k\right) = \int_{X_{v_0}^{\mathrm{an}}} \log\|1\|_{v_0,0}^{-1} \bigwedge_{j=1}^{k} c_1(\overline{L}_i)_{v_0}.$$

When $\overline{L}$ is a big and nef $\mathbb{Q}$-line bundle endowed with a semi-positive continuous adelic metric, following Zhang [45], we define $h_{\overline{L}}(X)$ as

$$h_{\overline{L}}(X) := \frac{\left(\overline{L}\right)^{k+1}}{(k+1)[\mathbb{K}:\mathbb{Q}]\operatorname{vol}(L)},$$

where $\operatorname{vol}(L) = (L)^k$ is the volume of the line bundle L (also denoted by $\deg_X(L)$ sometimes). We also define the height of a closed point $x \in X(\overline{\mathbb{Q}})$ as

$$h_{\overline{L}}(x) = \frac{\left(\overline{L}|x\right)}{[\mathbb{K}:\mathbb{Q}]} = \frac{1}{[\mathbb{K}:\mathbb{Q}]\#\mathsf{O}(x)} \sum_{v\in M_{\mathbb{K}}} \sum_{\sigma:\mathbb{K}(x)\hookrightarrow\mathbb{C}_v} \log\|s(\sigma(x))\|_v^{-1},$$

where $\mathsf{O}(x)$ is the Galois orbit of x, for any section $s \in H^0(X, L)$ which does not vanish at x. Finally, for any Galois-invariant finite set $F \subset X(\overline{\mathbb{Q}})$, we define $h_{\overline{L}}(F)$ as

$$h_{\overline{L}}(F) := \frac{1}{\#F}\sum_{y\in F} h_{\overline{L}}(y).$$

A fundamental estimate is the following, called Zhang's inequalities [45]:

Lemma 2.1 (Zhang). — *If L is ample and*

$$e(\overline{L}) = \sup_{H} \inf_{x\in(X\setminus H)(\overline{\mathbb{Q}})} h_{\overline{L}}(x),$$

where the supremum is taken over all hypersurfaces H of X defined over $\mathbb{K}$, then

$$\frac{1}{k+1}\left(e(\overline{L}) + k \inf_{y\in X(\overline{\mathbb{Q}})} h_{\overline{L}}(y)\right) \leqslant h_{\overline{L}}(X) \leqslant e(\overline{L}).$$

In particular, if $h_{\overline{L}}(x) \geqslant 0$ for all $x \in X(\overline{\mathbb{Q}})$, then $h_{\overline{L}}(X) \geqslant 0$.

In particular, we can deduce the following.

Corollary 2.2. — *Assume L is big and nef $\mathbb{Q}$-line bundle on X which is endowed with an adelic semi-positive continuous metrization $\{\|\cdot\|_v\}_{v\in M_{\mathbb{K}}}$. Assume $h_{\overline{L}}(x) \geqslant 0$ for any point $x \in X(\overline{\mathbb{Q}})$, then $h_{\overline{L}}(X) \geqslant 0$.*

Proof. — Let $\overline{A}$ be an ample hermitian line bundle on X with $h_{\overline{A}} \geqslant 0$ on X and $\epsilon > 0$ rational. Then $\overline{L}(\epsilon) := \overline{L} + \epsilon\overline{A}$ is an ample adelically metrized

$\mathbb{Q}$-line bundle and $h_{\overline{L}(\epsilon)} \geqslant 0$ on $X(\overline{\mathbb{Q}})$. By Lemma 2.1, we have $h_{\overline{L}(\epsilon)}(X) \geqslant 0$. First, note that

$$\mathrm{vol}(L(\epsilon)) = (L + \epsilon A)^k = \sum_{j=0}^{k} \binom{k}{j} (L)^j (\epsilon A)^{k-j} = \mathrm{vol}(L) + O(\epsilon).$$

Also, we can compute similarly

$$\left(\overline{L}(\epsilon)\right)^{k+1} = \sum_{j=0}^{k+1} \binom{k+1}{j} \epsilon^j \left(\overline{L}\right)^{k+1-j} \left(\overline{A}\right)^j = \left(\overline{L}\right)^{k+1} + O(\epsilon).$$

We thus deduce that

$$0 \leqslant h_{\overline{L}(\epsilon)}(X) = h_{\overline{L}}(X) + O(\epsilon)$$

and the conclusion follows making $\epsilon \to 0$. □

2.2. Test functions

Let L be a big and nef $\mathbb{Q}$-line bundle on X. We equip L with an adelic continuous semi-positive metric $\{\|\cdot\|_v\}_{v \in M_{\mathbb{K}}}$ and we denote $\overline{L} := (L, \{\|\cdot\|_v\}_{v \in M_{\mathbb{K}}})$. We also fix a Zariski open set $V \subset X$. Let $v \in M_{\mathbb{K}}$ be any place of the field $\mathbb{K}$.

Definition 2.3. — *A function $\varphi : X_v^{\mathrm{an}} \to \mathbb{R}$ is called a* test function *on X_v^{an} if it is*

- *a $\mathscr{C}^\infty$-smooth function on X_v^{an} if v is archimedean,*
- *a $\mathbb{Q}$-model function on X_v^{an} if v is non-archimedean.*

We also say φ is a test function on V_v^{an} *is it is a test function on X_v^{an} and if its support is compactly contained in V_v^{an}.*

If v is archimedean, as the space $\mathscr{C}_c^\infty(V_v^{\mathrm{an}})$ of smooth functions on V_v^{an} with compact support is dense in the space $\mathscr{C}_c^0(V_v^{\mathrm{an}})$ of continuous functions on V_v^{an} with compact support, it is sufficient to test the convergence against smooth functions. If v is non-archimedean, we use a result of Gubler [32, Theorem 7.12]: $\mathbb{Q}$-model functions with compact support in V_v^{an} are dense in the space $\mathscr{C}_c^0(V_v^{\mathrm{an}})$ of continuous functions on V_v^{an} with compact support, it is sufficient to test the convergence against compactly supported model functions. In particular, we have the following:

Lemma 2.4 (Gubler). — *Test functions are dense in the space $\mathscr{C}_c^0(V_v^{\mathrm{an}})$ (resp. in $\mathscr{C}^0(X_v^{\mathrm{an}})$) of continuous functions on V_v^{an} with compact support (resp. of continuous functions on X_v^{an}).*

2.3. Arithmetic volume and the adelic Minkowski Theorem

We want here to use the arithmetic Minkowski's second Theorem to relate the limit of $h_{\overline{M}}(F_i)$ along a generic sequence of finite subsets $\{F_i\}_i$ of $X(\overline{\mathbb{Q}})$ and the arithmetic volume of $\overline{M}$, for any continuous adelic metrized $\mathbb{Q}$-line bundle. For any $v \in M_{\mathbb{K}}$, and any section $s \in H^0(X, M)$, set $\|s\|_{v,\sup} := \sup_{x\in X(\overline{\mathbb{Q}}_v)} \|s(x)\|_v$. Assume $H^0(X, M) \neq 0$. If $\mathbb{A}_{\mathbb{K}}$ is the ring of adels of $\mathbb{K}$ and if μ is a Haar measure on the locally compact abelian group $H^0(X, M)\otimes \mathbb{A}_{\mathbb{K}}$, let

$$\chi_{\sup}(\overline{M}) = -\log \frac{\mu(H^0(X, M) \otimes \mathbb{A}_{\mathbb{K}}/H^0(X, M))}{\mu(\prod_v B_v)},$$

where B_v is the (closed) unit ball of $H^0(X, M) \otimes \mathbb{K}_v$ for the norm induced by $\|\cdot\|_{v,\sup}$, see e.g. [18] or [46]. The *arithmetic volume* $\widehat{\mathrm{vol}}_\chi(\overline{M})$ of $\overline{M}$ is the defined as

$$\widehat{\mathrm{vol}}_\chi(\overline{M}) := \limsup_{N\to\infty} \frac{\chi_{\sup}(N\overline{M})}{N^{k+1}/(k + 1)!}.$$

Choose a place $v_0 \in M_{\mathbb{K}}$ and let $\varphi : X^{\mathrm{an}}_{v_0} \to \mathbb{R}$ be a test functions, i.e. $\varphi \in \mathscr{C}^\infty(X^{\mathrm{an}}_{v_0})$ if v_0 is archimedean and φ is a model function otherwise. We define a twisted metrized line bundle $\overline{M}(\varphi)$ by changing the metric at the place v_0 as follows: let $\|\cdot\|_{v_0,\varphi} := \|\cdot\|_{v_0} e^{-\varphi}$.

Recall that a sequence $(F_i)_i$ of Galois-invariant subsets of $X(\overline{\mathbb{Q}})$ is *generic* if for any hypersurface Z of X defined over $\mathbb{K}$, there is $i_0 \geqslant 0$ such that $F_i \cap Z(\overline{\mathbb{Q}}) = \varnothing$ for all $i \geqslant i_0$. Using the arithmetic Minkowski's second Theorem, we can prove the next lemma using a classical argument (see, e.g., [7]).

Lemma 2.5. — *For any big and nef line bundle $\overline{M}$ endowed with an adelic semi-positive continuous metrization, any $v_0 \in M_{\mathbb{K}}$, any test function $\varphi : X^{\mathrm{an}}_{v_0} \to \mathbb{R}$ and any generic sequence $(F_i)_i$ of Galois-invariant subsets of $X(\overline{\mathbb{Q}})$, we have*

$$\liminf_{i\to\infty} h_{\overline{M}(\varphi)}(F_i) \geqslant \frac{\widehat{\mathrm{vol}}_\chi(\overline{M}(\varphi))}{[\mathbb{K} : \mathbb{Q}](k + 1)\,\mathrm{vol}(M)}.$$

Proof. — Taking c large enough, we have

$$\widehat{\mathrm{vol}}_\chi(\overline{M}(\varphi + c)) = \widehat{\mathrm{vol}}_\chi(\overline{M}(\varphi)) + c(k + 1)\,\mathrm{vol}(M) > 0.$$

By Minkowski's theorem [9, Theorem C.2.11], as soon as $\widehat{\mathrm{vol}}_\chi(\overline{M}(\varphi+c)) > 0$, there exists a non-zero small section $s \in H^0(X, NL)$ such that

$$\log \|s\|_{v_0} \leqslant -\frac{\widehat{\mathrm{vol}}_\chi(\overline{M}(\varphi))}{[\mathbb{K} : \mathbb{Q}](k + 1)\,\mathrm{vol}(M)} N + o(N)$$

and $\log\|s\|_v \leqslant 0$ for any other place $v \in M_{\mathbb{K}}$, where $\|\cdot\|_v$ is the metrization of $N\overline{M}(\varphi+c)$ at the place v induced by that of $\overline{M}$. As $(F_i)_i$ is generic, there is i_0 such that $F_i \cap \mathrm{supp}(\mathrm{div}(s)) = \varnothing$ for $i \geqslant i_0$. We thus can compute the height $h_{N\overline{M}(\varphi+c)}$ of F_i using this small section s to find

$$h_{\overline{M}(\varphi+c)}(F_i) = \frac{1}{N} h_{N\overline{M}(\varphi+c)}(F_i) \geqslant \frac{\widehat{\mathrm{vol}}_\chi(\overline{M}(\varphi+c))}{[\mathbb{K}:\mathbb{Q}](k+1)\,\mathrm{vol}(M)} + o(1).$$

Making $N \to \infty$, this gives

$$h_{\overline{M}(\varphi+c)}(F_i) \geqslant \frac{\widehat{\mathrm{vol}}_\chi(\overline{M}(\varphi+c))}{[\mathbb{K}:\mathbb{Q}](k+1)}. \tag{2.1}$$

To conclude, we use $\widehat{\mathrm{vol}}_\chi(\overline{M}(c+\varphi)) = \widehat{\mathrm{vol}}_\chi(\overline{M}(\varphi)) + c(k+1)\,\mathrm{vol}(M)$ and we remark that by definition, for any $c > 0$ and any closed point $x \in X(\overline{\mathbb{Q}})$,

$$h_{\overline{M}(c+\varphi)}(x) = h_{\overline{M}(\varphi)}(x) + \frac{c}{[\mathbb{K}:\mathbb{Q}]},$$

so that the inequality (2.1) is the expected result. □

2.4. A lower bound on the arithmetic volume

Let now L be a big and nef $\mathbb{Q}$-line bundle on X. We equip L with an adelic continuous semi-positive metric $\{\|\cdot\|_v\}_{v\in M_{\mathbb{K}}}$ and we denote $\overline{L} := (L, \{\|\cdot\|_v\}_{v\in M_{\mathbb{K}}})$. We also fix a Zariski open set $V \subset X$.

Fix a place $v \in M_{\mathbb{K}}$ and pick any test function $\varphi : X_v^{\mathrm{an}} \to \mathbb{R}$ with compact support in V_v^{an}. Let $\overline{\mathcal{O}}_v(\varphi)$ the trivial line bundle on X equipped with the trivial metric at all places but v and equipped with the metric induced by φ at the place v. For any $t \in \mathbb{R}$, we let

$$\overline{L}(t\varphi) := \overline{L} + t\overline{\mathcal{O}}_v(\varphi).$$

The key fact of this section is the next proposition which relies on Yuan's arithmetic bigness criterion à la Siu [43, Theorem 2.2] :

Proposition 2.6. — *Let M be a big and nef line bundle on X. For any non-constant test function $\varphi : X_v^{\mathrm{an}} \to \mathbb{R}$, and any decomposition $\overline{\mathcal{O}}(\varphi) = \overline{M}_+ - \overline{M}_-$ as a difference of big and nef adelic metrized line bundle with* *underlying line bundle M and any $t>0$, we have*

$$\frac{\widehat{\mathrm{vol}}_\chi(\overline{L}(t\varphi))}{[\mathbb{K}:\mathbb{Q}](k+1)\,\mathrm{vol}(L)} \geqslant h_{\overline{L}}(X) + \frac{t}{[\mathbb{K}:\mathbb{Q}]}\int_{X_v^{\mathrm{an}}} \varphi\,\frac{c_1(\overline{L})_v^k}{\mathrm{vol}(L)} - t\frac{\sup_{X_v^{\mathrm{an}}}|\varphi|}{[\mathbb{K}:\mathbb{Q}]}\left(\frac{\mathrm{vol}(L+tM)}{\mathrm{vol}(L)}-1\right) - \sum_{j=2}^{k+1}\frac{(k+2-j)}{[\mathbb{K}:\mathbb{Q}]\,\mathrm{vol}(L)}\binom{k+1}{j}\big(\overline{L}\big)^{k-j+1}\cdot\big(t\overline{M}_+\big)^j.$$

Moreover, when φ is compactly supported in V_v^{an}, $\sup_{X_v^{\mathrm{an}}}|\varphi|$ can be replaced by $\sup_{V_v^{\mathrm{an}}}|\varphi|$ and the integral can be computed on V_v^{an}.

Proof. — Write $\varphi=\psi_+-\psi_-$, where $\psi_\pm$ is a smooth psh metric on M_v^{an}. On then can write $\overline{\mathcal{O}}_v(t\varphi)=t\overline{M}_+-t\overline{M}_-$, where $\overline{M}_\pm$ are the induced metrizations on M. Extend both $t\overline{M}_\pm$ as adelic metrized line bundles which coincide at all places $w\neq v$ and pick any ample arithmetic line bundle $\overline{L}_0$ on X. As M is nef, for a given $q\geqslant 1$, $q\overline{M}_\pm+\overline{L}_0$ is an ample hermitian line bundle. The line bundle $q\overline{L}(t\varphi)$ is then the difference of two ample hermitian line bundles:

$$q\overline{L}(t\varphi)=\big(q\overline{L}+tq\overline{M}_+\big)-tq\overline{M}_-=\big(q\overline{L}+t\big(q\overline{M}_++\overline{L}_0\big)\big)-t\big(q\overline{M}_-+\overline{L}_0\big).$$

Apply Yuan's arithmetic bigness criterion [43, Theorem 2.2] to multiples of $N\overline{L}(t\varphi)$, where q divides N and making $N\to\infty$ gives

$$\widehat{\mathrm{vol}}_\chi(\overline{L}(t\varphi))\geqslant\big(\overline{L}+t\overline{M}_+\big)^{k+1}-(k+1)\big(\overline{L}+t\overline{M}_+\big)^k\cdot\big(t\overline{M}_-\big) \tag{2.2}$$

Remark that $\big(\overline{L}+t\overline{M}_+\big)^{k+1}-(k+1)\big(\overline{L}+t\overline{M}_+\big)^k\cdot\big(t\overline{M}_-\big)$ expands as

$$\big(\overline{L}\big)^{k+1}+\sum_{j=1}^{k+1}\binom{k+1}{j}\big(\overline{\mathcal{O}}_v(\varphi)\big)\cdot\big(\overline{L}\big)^{k-j+1}\cdot\big(t\overline{M}_+\big)^{j-1} - \sum_{j=2}^{k+1}(k-j)\binom{k+1}{j}\big(\overline{L}\big)^{k-j+1}\cdot\big(t\overline{M}_+\big)^j,$$

where we used that $t\overline{M}_+-t\overline{M}_-=\overline{\mathcal{O}}_v(t\varphi)$. As the underlying line bundle of $\overline{\mathcal{O}}_v(\varphi)$ is trivial, we can compute this intersection number with the use of the constant section 1. The term $j=1$ is exactly the integral

$$\big(\overline{\mathcal{O}}_v(t\varphi)\big)\cdot\big(\overline{L}\big)^k=t\int_{X_v^{\mathrm{an}}}\varphi\,c_1(\overline{L})_v^k$$

and for $j \geqslant 2$, we can compute

$$\begin{aligned}\left(\overline{\mathcal{O}}_v(t\varphi)\right) \cdot \left(\overline{L}\right)^{k-j+1} \cdot \left(t\overline{M}_+\right)^{j-1} & \\ &= t\int_{X_v^{\mathrm{an}}} \varphi\, c_1(\overline{L})_v^{k-j+1} \wedge (tc_1(M,\psi_+)_v)^{j-1} \\ &\geqslant -t \sup_{X_v^{\mathrm{an}}} |\varphi| \int_{X_v^{\mathrm{an}}} c_1(\overline{L})_v^{k-j+1} \wedge (tc_1(M,\psi_+)_v)^{j-1} \\ &\geqslant -t \sup_{X_v^{\mathrm{an}}} |\varphi| (L^{k+1-j} \cdot (tM)^{j-1}).\end{aligned}$$

Summing from $j = 2$ to $k+1$, we find

$$\begin{aligned}\sum_{j=2}^{k+1} \binom{k+1}{j} \left(\overline{\mathcal{O}}_v(\varphi)\right) \cdot \left(\overline{L}\right)^{k-j+1} \cdot \left(t\overline{M}_+\right)^{j} & \\ &\geqslant -t \sup_{X_v^{\mathrm{an}}} |\varphi| \sum_{j=2}^{k+1} \binom{k+1}{j} (L^{k+1-j} \cdot (tM)^{j-1}) \\ &\geqslant -t \sup_{X_v^{\mathrm{an}}} |\varphi| \sum_{j=1}^{k} \frac{k+1}{j+1} \binom{k}{j} (L^{k-j} \cdot (tM)^{j}) \\ &\geqslant -t(k+1) \sup_{X_v^{\mathrm{an}}} |\varphi| (\mathrm{vol}(L+tM) - \mathrm{vol}(L)).\end{aligned}$$

The above summarizes as

$$\begin{aligned}\widehat{\mathrm{vol}}_{\chi}(\overline{L}(t\varphi)) \geqslant \left(\overline{L}\right)^{k+1} + (k+1)t \int_{X_v^{\mathrm{an}}} \varphi\, c_1(\overline{L})_v^{k} & \\ &- t(k+1) \sup_{X_v^{\mathrm{an}}} |\varphi| (\mathrm{vol}(L+tM) - \mathrm{vol}(L)) \\ &- \sum_{j=2}^{k+1} (k-j) \binom{k+1}{j} \left(\overline{L}\right)^{k-j+1} \cdot \left(t\overline{M}_+\right)^{j}.\end{aligned}$$

The conclusion follows dividing by $(k+1)[\mathbb{K}:\mathbb{Q}]\,\mathrm{vol}(L)$.

When φ is compactly supported in V_v^{an}, it is obvious that the sup and the integral can be taken over V_v^{an}. □

3. Good height functions and equidistribution

In this section, we prove Theorem 1.2 using the estimates established in Section 2. Let $\mathbb{K}$ be z number field, let V be a smooth quasi-projective variety of dimension k defined over $\mathbb{K}$. Let h be a v-good height function on V with induced measure μ_v on V_v^{an} and $\mathrm{vol}(h) > 0$ be its *volume*. Let $(X_n, \overline{L}_n, \psi_n)$ be given by the definition of good height function.

3.1. A preliminary lemma

Let $\iota_0 : X_0 \hookrightarrow \mathbb{P}^N$ be an embedding defined over $\mathbb{K}$ and let M_0 be an ample line bundle on X_0. For $n \geqslant 0$, let

$$\overline{M}_n := \psi_n^*(\overline{M}_0).$$

Since ψ_n is a birational morphism and since M_0 is ample, M_n is big and nef.

An important ingredient is the following:

Lemma 3.1. — *The sequence* $\mathrm{vol}(L_n + M_n)$ *converges to a limit* $\ell > 0$. *In particular, there exists a constant* $C \geqslant 1$ *such that* $0 \leqslant \mathrm{vol}(L_n + M_n) \leqslant C$, *for all* $n \geqslant 0$.

Proof. — By definition of the line bundle M_n, M_n is big and nef, so that $\mathrm{vol}(M_n) = (M_n)^k = (M_0)^k = \mathrm{vol}(M_0)$, hence it is independent of n. By the assumption (3) of the definition of good height function, we also have

$$(M_0)^k = \frac{(M_0)^k}{\mathrm{vol}(h)} \mathrm{vol}(L_n) + o(\mathrm{vol}(L_n)) = \frac{\mathrm{vol}(M_0)}{\mathrm{vol}(h)} \mathrm{vol}(L_n) + o(\mathrm{vol}(L_n)).$$

In particular, there exists a constant $C_1 \geqslant 1$ independent of n such that $\mathrm{vol}(M_n) \leqslant C_1 \mathrm{vol}(L_n)$ and thus

$$\mathrm{vol}(L_n + M_n) = (L_n + M_n)^k \leqslant (1 + C_1)^k (L_n)^k = (1 + C_1)^k \mathrm{vol}(L_n).$$

As $\mathrm{vol}(L_n)$ is bounded, the conclusion follows. □

3.2. Proof of Theorem 1.2 and of Theorem 1.6

We give here the proof of Theorem 1.2 and we explain where to adapt the arguments to prove Theorem 1.6. Let $v \in M_{\mathbb{K}}$ be such that h is v-good and let $\varphi : V_v^{\mathrm{an}} \to \mathbb{R}$ be a test function on V_v^{an}. For $t > 0$, we define an adelic metrized line bundle by

$$\overline{L}_n(t\varphi_n) := \overline{L}_n + \overline{\mathcal{O}}(t\varphi_n).$$

Let us first prove the result for a sequence $(F_i)_i$ of generic and h-small Galois-invariant finite subsets $F_i \subset V(\overline{\mathbb{Q}})$ and let $F_{i,n} := \psi_n^{-1}(F_i)$. (Here we assume $\{F_i\}_i$ is quasi-small in the quasi-height case).

We apply Proposition 2.6 and Lemma 2.5 to find

$$\begin{aligned}\liminf_{i\to\infty} h_{\overline{L}_n(t\varphi_n)}(F_{i,n}) \geqslant h_{\overline{L}_n}(X_n) - \frac{t}{[\mathbb{K}:\mathbb{Q}]}\int_{V_v^{\mathrm{an}}} \varphi_n \frac{c_1(\overline{L}_n)_v^k}{\mathrm{vol}(L_n)} \\ - t^2 \frac{\sup_{V_v^{\mathrm{an}}}|\varphi|}{[\mathbb{K}:\mathbb{Q}]}\left(\frac{\mathrm{vol}(L_n+M_n)}{\mathrm{vol}(L_n)} - 1\right) \\ - \sum_{j=2}^{k+1} \frac{(k-j)}{[\mathbb{K}:\mathbb{Q}]\,\mathrm{vol}(L)}\binom{k+1}{j}\left(\overline{L}_n\right)^{k-j+1}\cdot\left(t\overline{M}_{n,+}\right)^j\end{aligned}$$

for any decomposition $\overline{\mathcal{O}}_v(\varphi_n) = \overline{M}_{n,+} - \overline{M}_{n,-}$, where $\overline{M}_{n,\pm}$ is a semi-positive continuous metrized big and nef line bundle on $(X_n)_v^{\mathrm{an}}$ with underlying line bundle M_n, as defined in Section 3.1. If we write $\varphi = \psi_+ - \psi_-$ where $\psi_\pm$ are metrizations on $M_{0,v}^{\mathrm{an}}$, then one can obviously write $\varphi_n = \psi_+ \circ \psi_n - \psi_- \circ \psi_n$. We thus have $\overline{M}_{n,+} = \psi_n^*(\overline{M}_{0,+})$.

As $\mathrm{vol}(L_n) \to \mathrm{vol}(h) > 0$, by Lemma 3.1, there exists $C_1 \geqslant 1$ independent of n such that

$$\frac{\mathrm{vol}(L_n+M_n)}{\mathrm{vol}(L_n)} \leqslant C_1,$$

for any $n \geqslant 0$. Using hypothesis 3, we deduce there is $C_2 \geqslant 1$ independent of n such that for any $0 < t < 1$ we find

$$\liminf_{i\to\infty} h_{\overline{L}_n(t\varphi_n)}(F_{i,n}) \geqslant h_{\overline{L}_n}(X_n) - \frac{t}{[\mathbb{K}:\mathbb{Q}]}\int_{(X_n)_v^{\mathrm{an}}} \varphi_n \frac{c_1(\overline{L}_n)_v^k}{\mathrm{vol}(L_n)} - C_2 t^2.$$

By definition of $\overline{L}_n(t\varphi_n)$, we can compute

$$h_{\overline{L}_n(t\varphi_n)}(F_{i,n}) = h_{\overline{L}_n}(F_{i,n}) + \frac{t}{[\mathbb{K}:\mathbb{Q}]\#F_i}\sum_{y\in F_i}\varphi(y).$$

Using that the sequence $(F_i)_i$ is h-small and generic, assumption 1. gives

$$\liminf_{i\to\infty} h_{\overline{L}_n(t)}(F_{i,n}) \leqslant \varepsilon_n(\{F_i\}_i) + h_{\overline{L}_n}(X_n) + \liminf_{i\to+\infty}\frac{t}{[\mathbb{K}:\mathbb{Q}]F_i}\sum_{y\in F_i}\varphi(y),$$

for $0 < t < 1$. Combined with the above and divided by $t/[\mathbb{K}:\mathbb{Q}]$, this gives

$$\liminf_{i\to+\infty}\left(\frac{1}{\#F_i}\sum_{y\in F_i}\varphi(y) - \int_{(X_n)_v^{\mathrm{an}}}\varphi_n\frac{c_1(\overline{L}_n)_v^k}{\mathrm{vol}(L_n)}\right) \geqslant -\frac{\varepsilon_n(\{F_i\}_i)}{t} - C_2 t$$

for all $0 < t < 1$. Replacing φ by $-\varphi$ gives the converse inequality, whence

$$\limsup_{i\to+\infty}\left|\frac{1}{\#F_i}\sum_{y\in F_i}\varphi(y) - \int_{(X_n)_v^{\mathrm{an}}}\varphi_n\frac{c_1(\overline{L}_n)_v^k}{\mathrm{vol}(L_n)}\right| \leqslant \frac{\varepsilon_n(\{F_i\}_i)}{t} + C_1 t,$$

for all $0 < t < 1$. One can also remark that since ψ_n is an isomorphism over V, we have

$$\int_{(X_n)_v^{\mathrm{an}}} \varphi_n \frac{c_1(\overline{L}_n)_v^k}{\mathrm{vol}(L_n)} = \int_{X_v^{\mathrm{an}}} \varphi \frac{(\psi_n)_* c_1(\overline{L}_n)_v^k}{\mathrm{vol}(L_n)}.$$

In the quasi-height, we use here that the measure $c_1(\overline{L}_n)_v^k$ doesn't give mass to Zariski closed subsets of X_n, since it is induced by a smooth metrization on L_n.

Fix now $\varepsilon > 0$ small. We now let $t := \varepsilon/2C_2 > 0$. By assumption, we can choose $n_0 \geqslant 1$ such that we have $\frac{\varepsilon_n(\{x_i\}_i)}{t} \leqslant \varepsilon/2$ for any $n \geqslant n_0$. Therefore, for $n \geqslant n_0$,

$$\limsup_{i\to+\infty} \left| \frac{1}{\#F_i} \sum_{y\in F_i} \varphi(y) - \int_{V_v^{\mathrm{an}}} \varphi \frac{(\psi_n)_* c_1(\overline{L}_n)_v^k}{\mathrm{vol}(L_n)} \right| \leqslant \varepsilon. \tag{3.1}$$

By assumption (3) of the definition of good height, we can choose n_0 such that for any $n \geqslant n_0$,

$$\left| \int_{X_v^{\mathrm{an}}} \varphi \frac{(\psi_n)_* c_1(\overline{L}_n)_v^k}{\mathrm{vol}(L_n)} - \int_{X_v^{\mathrm{an}}} \varphi \, \mu_v \right| \leqslant (1 + \|\varphi\|_{L^\infty})\varepsilon.$$

Combined with (3.1), this completes the proof for generic and h-small sequences (we use that φ is compactly supported in V_v^{an} in the case of good heights).

We now show how to deduce the full statement of Theorem 1.2, proceeding as in [24, Section 5.5]. Let us enumerate all irreducible hypersurfaces $(H_\ell)_\ell$ of V that are defined over $\mathbb{K}$. We use the next lemma, see e.g. [24, Lemma 5.12].

Lemma 3.2. — *Take a sequence $(F_n)_n$ of Galois-invariant finite subsets of $V(\overline{\mathbb{Q}})$ with*

$$\lim_{n\to\infty} \frac{\#(F_n \cap H_\ell)}{\#F_n} = 0,$$

for any ℓ. Then, for any $\epsilon > 0$, there exists a sequence of sets $F'_{n,\epsilon} \subset F_n$ such that:

1. *$\#F'_{n,\epsilon} \geqslant (1-\epsilon)\#F_n$ for all n,*
2. *$F'_{n,\epsilon}$ is Galois-invariant,*
3. *for any ℓ there exists $N(\ell) \geqslant 1$, such that $F'_{n,\epsilon} \cap H_\ell = \varnothing$ for all $n \geqslant N(\ell)$.*

Fix $\epsilon > 0$. The last condition of Lemma 3.2 implies $F'_{n,\epsilon}$ to be generic. Now pick any continuous function $\varphi \in \mathscr{C}_c^0(V_v^{\mathrm{an}})$. The above implies that there is $n_0 \geqslant 1$ such that

$$\left|\int_{V_v^{\mathrm{an}}} \varphi\, \mu_{F_n,v} - \int_{V_v^{\mathrm{an}}} \varphi\, \mu_v\right| \leqslant \left|\int_{V_v^{\mathrm{an}}} \varphi\, \mu_{F_n,v} - \int_{V_v^{\mathrm{an}}} \varphi\, \mu_{F'_{n,\epsilon},v}\right| + \epsilon.$$

We infer from Lemma 3.2 that

$$\begin{aligned}\left|\int_{V_v^{\mathrm{an}}} \varphi\, \mu_{F_n,v} - \int_{V_v^{\mathrm{an}}} \varphi\, \mu_v\right| &\leqslant \frac{1}{\#F_n}\int_{V_v^{\mathrm{an}}} |\varphi| \sum_{x\in F_n\setminus F'_{n,\epsilon}} \delta_x \\ &\quad + \left(\frac{1}{\#F'_{n,\epsilon}} - \frac{1}{\#F_n}\right)\int_{V_v^{\mathrm{an}}} |\varphi| \sum_{x\in F'_{n,\epsilon}} \delta_x \\ &\leqslant 2\,\epsilon \sup_{V_v^{\mathrm{an}}} |\varphi|\ .\end{aligned}$$

This shows that for $n \geqslant n_0$, we have

$$\left|\int_{V_v^{\mathrm{an}}} \varphi\, \mu_{F_n,v} - \int_{V_v^{\mathrm{an}}} \varphi\, \mu_v\right| \leqslant \left(1 + 2\sup_{V_v^{\mathrm{an}}} |\varphi|\right)\epsilon,$$

which concludes the proof.

3.3. Quasi-adelic measures on the projective line are good

Let $\mathbb{K}$ be a number field. We prove here that quasi-adelic measures on $\mathbb{P}^1$ as defined by Mavraki and Ye [36], and their induced height functions, satisfy the assumptions of Theorem 1.2.

Let us defined quasi-adelic measures and their height functions following Mavraki and Ye. Pick $v \in M_{\mathbb{K}}$. Write $\log^+|\cdot|_v := \log\max\{|\cdot|_v, 1\}$ on $\mathbb{A}^1(\mathbb{C}_v)$. This function extends to $\mathbb{A}_v^{1,\mathrm{an}}$ and $dd^c \log^+|\cdot|_v = \delta_\infty - \lambda_v$, as a function from $\mathbb{P}_v^{1,\mathrm{an}}$ to $\mathbb{R}_+ \cup \{+\infty\}$, where λ_v is the Lebesgue measure on $\{|z|_v = 1\}$ if v is archimedean, and $\lambda_v = \delta_{G,v}$ is the dirac mass at the Gauß point of $\mathbb{P}_v^{1,\mathrm{an}}$ otherwise.

A probability measure on $\mathbb{P}_v^{1,\mathrm{an}}$ has continuous potential if $\mu_v - \lambda_v = dd^c g_v$ for some $g_v \in \mathscr{C}^0(\mathbb{P}_v^{1,\mathrm{an}}, \mathbb{R})$. In this case, there is a unique function $g_{\mu_v} : \mathbb{P}_v^{1,\mathrm{an}} \to \mathbb{R} \cup \{+\infty\}$ such that $g_v = \log^+|\cdot|_v - g_{\mu_v}$ with the following normalization: if we let

$$G_{\mu_v}(x,y) := \begin{cases} g_{\mu_v}(x/y) + \log|y|_v & \text{for } (x,y) \in \mathbb{C}_v \times (\mathbb{C}_v \setminus \{0\}), \\ \log|x|_v - g_v(\infty) & \text{for } (x,y) \in (\mathbb{C}_v \setminus \{0\}) \times \{0\}, \\ -\infty & \text{for } (x,y) = (0,0), \end{cases}$$

then the set $M_{\mu_v} := \{(x,y) \in \mathbb{C}_v^2 \ : \ G_{\mu_v}(x,y) \leqslant 0\}$ has homogeneous logarithmic capacity 1. Note that for $\alpha \in \mathbb{C}_v \setminus \{0\}$ and $(x,y) \in \mathbb{C}_v^2 \setminus \{(0,0)\}$, we have

$$G_{\mu_v}(\alpha x, \alpha y) = G_{\mu_v}(x,y) + \log|\alpha|_v.$$

We then define the inner (resp. outer) radius of μ_v as

$$\begin{cases} r_{\text{in}}(\mu_v) := \sup\{r > 0 \ : \ \overline{D}_v(0,r) \times \overline{D}_v(0,r) \subset M_{\mu_v}\}, \\ r_{\text{out}}(\mu_v) := \inf\{r > 0 \ : \ M_{\mu_v} \subset \overline{D}_v(0,r) \times \overline{D}_v(0,r)\} \end{cases}$$

Definition 3.3. — *We say that $\mu = \{\mu_v\}_{v \in M_{\mathbb{K}}}$ is a* quasi-adelic *measure on $\mathbb{P}^1$ if*

- *for any $v \in M_{\mathbb{K}}$, the measure μ_v has continuous potential,*
- *both series $\sum_{v \in M_{\mathbb{K}}} |\log r_{\text{in}}(\mu_v)|$ and $\sum_{v \in M_{\mathbb{K}}} |\log r_{\text{out}}(\mu_v)|$ converge.*

The height function *h_μ induced by a quasi-adelic measure $\mu = \{\mu_v\}_{v \in M_{\mathbb{K}}}$ is defined as*

$$h_\mu(z) := \frac{1}{[\mathbb{K} : \mathbb{Q}]\#\mathsf{O}(x)} \sum_{v \in M_{\mathbb{K}}} \sum_{\sigma : \mathbb{K}(x) \hookrightarrow \mathbb{C}_v} G_{\mu_v}(\sigma(x), \sigma(y))),$$

where $(x,y) \in \mathbb{A}^2(\overline{\mathbb{Q}}) \setminus \{(0,0)\}$ is any point with $z = [x : y]$.

As the functions G_{μ_v} are homogeneous, the product formula implies the height function h_μ is well-defined and independent of the choice of (x,y).

We prove here the following.

Proposition 3.4. — *Let $\mathbb{K}$ be a number field and let $\mu := \{\mu_v\}_{v \in M_{\mathbb{K}}}$ be a quasi-adelic measure on $\mathbb{P}^1$ with induced height function h_μ. Then h_μ is a good height function on $\mathbb{P}^1$ with induced global measure $\{\mu_v\}_{v \in M_{\mathbb{K}}}$.*

Proof. — Enumerate the places of $\mathbb{K}$ as $M_{\mathbb{K}} := \{v_n, \ n \geqslant 0\}$ and define $X_n = \mathbb{P}^1$ and $\overline{L}_n$ as $\mathcal{O}_{\mathbb{P}^1}(1)$ endowed with the adelic continous semi-positive metrization $\{|\cdot|_{v,n}\}_{v \in M_{\mathbb{K}}}$ defined by the following conditions:

- for any $j \geqslant n+1$, the metric $|\cdot|_{v_j,n}$ is the usual naive metric on $\mathcal{O}_{\mathbb{P}^1}(1)$ at place v_j,
- for any $j \leqslant n$, the metric is that induced by $g_{\mu_{v_j}}$,

so that for any $z \in \mathbb{P}^1(\overline{\mathbb{Q}})$ and any $(x,y) \in \mathbb{A}^2(\overline{\mathbb{Q}}) \setminus \{(0,0)\}$ with $z = [x : y]$, we have

$$h_{\overline{L}_n}(z) = \frac{1}{[\mathbb{K}:\mathbb{Q}]\#\mathsf{O}(z)} \sum_{\sigma:\mathbb{K}(x)\hookrightarrow\mathbb{C}_v} \Bigg(\sum_{j=0}^{n} G_{\mu_{v_j}}(\sigma(x),\sigma(y)) + \sum_{\ell=n+1}^{\infty} \log \|\sigma(x),\sigma(y))\|_{v_\ell} \Bigg).$$

For any $n \geqslant 0$, we have $\mathrm{vol}(L_n) = 1$ and for all $v \in M_{\mathbb{K}}$, we have $c_1(\overline{L}_n)_v = \mu_v$ if n is large enough. All there is left to prove is that there is a h-small sequence and that the condition of pointwise approximation is satisfied on $\mathbb{P}^1$.

The definitions of $r_{\mathrm{in}}(\mu_v)$ and $r_{\mathrm{out}}(\mu_v)$ and the product formula give

$$\sum_{j=n+1}^{\infty} \frac{\log(r_{\mathrm{in}}(\mu_v))}{[\mathbb{K}:\mathbb{Q}]} \leqslant h_{\overline{L}_n} - h_\mu \leqslant \sum_{j=n+1}^{\infty} \frac{\log(r_{\mathrm{out}}(\mu_v))}{[\mathbb{K}:\mathbb{Q}]},$$

on $\mathbb{P}^1(\overline{\mathbb{Q}})$. In particular, if we let

$$\varepsilon(n) := \sum_{j=n+1}^{\infty} \frac{1}{[\mathbb{K}:\mathbb{Q}]} \max\{|\log r_{\mathrm{in}}(\mu_v)|, |\log r_{\mathrm{out}}(\mu_v)|\},$$

then $\varepsilon(n) \to 0$ as $n \to \infty$, by assumption, and

$$|h_{\overline{L}_n} - h_\mu| \leqslant \varepsilon(n), \quad \text{on } \mathbb{P}^1(\overline{\mathbb{Q}}). \tag{3.2}$$

We are thus left with justifying the existence of a small h_μ-sequence to conclude the proof. Let $\overline{L}_n$ be a model of $\overline{L}_n$ over $\mathbb{P}^1_{\mathscr{O}_{\mathbb{K}}}$. By the arithmetic Hilbert–Samuel theorem as stated in [38, Theorem A], we have

$$0 = -\sum_{j=0}^{n} \log \mathrm{cap}_{v_j}(M_{\mu_{v_j}}) = \widehat{\mathrm{vol}}(\overline{L}_n) = \left(\overline{L}_n\right)^2,$$

since we assumed $\log \mathrm{cap}_{v_j}(M_{\mu_{v_j}}) = 0$ for all j. In particular, $h_{\overline{L}_n}(\mathbb{P}^1) = 0$ and the assumption 1 of the definition of good height function is satsified.

Finally, for any $n \geqslant 0$, there is a generic sequence $(F_{i,n})_i$ of Galois invariant finite subsets of $\mathbb{P}^1(\overline{\mathbb{Q}})$ such that $h_{\overline{L}_n}(F_{i,n}) \to 0$, as $i \to \infty$. In particular, By inequality (3.2),

$$\limsup_{i\to\infty} |h_\mu(F_{i,n})| \leqslant \varepsilon(n),$$

and a diagonal extraction argument implies there exists $(\widetilde{F}_i)_i$ which is generic and such that $\limsup_{i\to\infty} |h_\mu(\widetilde{F}_i)| \to 0$, and the proof is complete. □

4. Equidistribution in families of dynamical systems

We give here an application of Theorem 1.2 in families of polarized endomorphisms with marked points. We begin with a general result and then we focus on special cases.

4.1. Families of polarized endomorphisms

Let S be a smooth quasi-projective variety of dimension $p \geqslant 1$ and let $\pi : \mathcal{X} \to S$ be a family of smooth projective varieties of dimension $k \geqslant p$. We say $(\mathcal{X}, f, \mathcal{L})$ is a family of *polarized endomorphisms* if $f : \mathcal{X} \to \mathcal{X}$ is a morphism with $\pi \circ f = \pi$ and if there is a relatively ample line bundle $\mathcal{L}$ on $\mathcal{X}$ and an integer $d \geqslant 2$ such that $f^*\mathcal{L} \simeq \mathcal{L}^{\otimes d}$. When $(\mathcal{X}, f, \mathcal{L})$ is defined over $\overline{\mathbb{Q}}$, following Call and Silverman [14], for any $t \in S(\overline{\mathbb{Q}})$ we can define a *canonical height* function for the restriction $f_t : X_t \to X_t$ of f to the fiber $X_t := \pi^{-1}\{t\}$ of $\pi : \mathcal{X} \to S$ as

$$\widehat{h}_{f_t}(x) = \lim_{n\to\infty} \frac{1}{d^n} h_{X_t,L_t}(f_t^{\circ n}(x)), \quad x \in X_t(\overline{\mathbb{Q}}).$$

If $\mathbb{L}$ is a finite extension of $\mathbb{K}$ such that $t \in S(\mathbb{L})$, the canonical height $\widehat{h}_{f_t}$ is induced by an adelic semi-positive continuous metrization $\{\|\cdot\|_{f_t,v}\}_{v\in M_{\mathbb{L}}}$ on the ample line bundle $\mathcal{L}_t$ of X_t. We then have

(1) $\widehat{h}_{f_t}(x) \geqslant 0$ for all $x \in X_t(\overline{\mathbb{Q}})$,
(2) $\widehat{h}_{f_t} - h_{X_t,L_t} = O(1)$ on $X_t(\overline{\mathbb{Q}})$, where $O(1)$ depends on t, and

the function $\widehat{h}_{f_t}$ is characterized by those two properties. Moreover, by the Northcott property, for any $x \in X_t(\overline{\mathbb{Q}})$ we have

(3) $\widehat{h}_{f_t}(x) = 0$ if and only if x is preperiodic under iteration of f_t.

When $\pi : \mathcal{X} \to S$, $\mathcal{L}$ and f are all defined over $\mathbb{C}$, we can associate to $(\mathcal{X}, f, \mathcal{L})$ a closed positive $(1,1)$-current on $\mathcal{X}(\mathbb{C})$ with continuous potentials which we can define as

$$\widehat{T}_f := \lim_{n\to\infty} \frac{1}{d^n} (f^{\circ n})^*(\iota^*\omega_{\mathbb{P}^N}), \tag{4.1}$$

where $\iota : \mathcal{X} \hookrightarrow \mathbb{P}^N$ induces (a large power of) $\mathcal{L} \otimes \pi^*(\mathcal{M})$, where $\mathcal{M}$ is ample on a projective model $\overline{S}$ of S. Moreover, the convergence towards $\widehat{T}_f$ is uniform local for potentials, see, e.g. [31, Proposition 2.7]. This current $\widehat{T}_f$ restricts on fibers $X_t(\mathbb{C})$ of π as the *Green current* of the endomorphism f_t of $X_t(\mathbb{C})$ which is polarized by L_t. Moreover, the closed positive (k,k)-current

$\widehat{T}_f^k$ on $\mathcal{X}(\mathbb{C})$ restricts to the fiber $X_t(\mathbb{C})$ as $\mathrm{vol}(L_t)\cdot\mu_{f_t}$, where μ_{f_t} is the unique maximal entropy probability measure of f_t.

To a section $a : S \to \mathcal{X}$ of π which is regular on the quasi-projective variety S, as introduced in [31], we associate a *bifurcation current* defined as

$$T_{f,a} := (\pi)_*\Big(\widehat{T}_f \wedge [a(S)]\Big).$$

Compare with [23] for the case when $\mathcal{X}$ has relative dimension 1.

4.2. Step 1 of the proof of Theorem 1.3: reduction and definition of $(B_n, \bar{L}_n, \psi_n)$

According to Theorem 1.2, all there is to do is to prove that $h_{f,\mathfrak{a}}$ is a good height and that, for any archimedean place $v \in M_{\mathbb{K}}$, the induced measure on S_v^{an} is $\mathrm{vol}_f(\mathfrak{a})^{-1}\mu_{f,\mathfrak{a},v}$.

Recall that we are given $q \geqslant 1$ sections $a_1, \dots a_q : S \to \mathcal{X}$. We first justify that we can reduce to the case when $q = 1$. Indeed, if $\pi_{[q]} : \mathcal{X}^{[q]} \to S$ is the q-fibered product of $\pi : \mathcal{X} \to S$, let $f^{[q]} : \mathcal{X}^{[q]} \to \mathcal{X}^{[q]}$ be defined by

$$f^{[q]}(x) = (f_t(x_1), \dots, f_t(x_q)), \quad x = (x_1, \dots, x_q) \in \pi_{[q]}^{-1}\{t\}.$$

Then $(\mathcal{X}^{[q]}, f^{[q]}, \mathcal{L}^{[q]})$ is a family of polarized endomorphisms parametrized by S. Moreover, an easy computation gives, for all $x = (x_1, \dots, x_q) \in \mathcal{X}^{[q]}$ with $\pi_{[q]}(x) = t$,

$$\widehat{h}_{f_t^{[q]}}(x_1, \dots, x_q) = \sum_{j=1}^{q} \widehat{h}_{f_t}(x_j) \geqslant 0 \quad \text{and} \quad \widehat{h}_{f_t^{[q]}}\Big(f_t^{[q]}(x)\Big) = d \cdot \widehat{h}_{f_t^{[q]}}(x). \quad (4.2)$$

Let $\mathfrak{a} : S \to \mathcal{X}^{[q]}$ be the section of $\pi_{[q]}$ induced by $a_1, \dots, a_q$. As before, we define the bifurcation current of $\mathfrak{a}$ as

$$T_{f,\mathfrak{a}} := (\pi_{[q]})_*\Big(\widehat{T}_{f^{[q]}} \wedge [\mathfrak{a}(S)]\Big).$$

An easy computation gives

$$T_{f,\mathfrak{a}} = T_{f,a_1} + \dots + T_{f,a_q}. \quad (4.3)$$

Combining equations (4.3) and (4.2), we deduce that $\mu_{f,\mathfrak{a},v} = T_{f,\mathfrak{a}}^p$ and that $h_{f,\mathfrak{a}} = \sum_{j=1}^q h_{f,a_j}$. In particular, up to replacing f by $f^{[q]}$ and $a_1, \dots, a_q$ by $\mathfrak{a}$, we can assume $q = 1$.

Let $\iota : \mathcal{X} \hookrightarrow \mathbb{P}^{M_1} \times S \hookrightarrow \mathbb{P}^{M_1} \times \mathbb{P}^{M_2} \hookrightarrow \mathbb{P}^N$, where the last embedding is the Segre embedding. Let also $\overline{S}$ be the closure of S in $\mathbb{P}^{M_2}$ induced by this embedding. Up to taking a large multiple of $\mathcal{L}$, we can assume the first embedding ι_1 is induced by $\mathcal{L}$ and

$$h_\iota = h_{\mathcal{X},\mathcal{L}} + h_S \quad \text{on } \mathcal{X}(\overline{\mathbb{Q}}),$$

where h_S is the ample height on $\overline{S}$ induced by the second embedding. For any $n \geqslant 1$, define a section $\mathfrak{a}_n : S \to \mathcal{X}$ by

$$\mathfrak{a}_n(t) := f_t^{\circ n}(a(t)), \quad t \in S.$$

As $\mathfrak{a}_n$ is a section of $\pi : \mathcal{X} \to S$, it is injective. In particular, $\iota \circ \mathfrak{a}_n : S \to \mathbb{P}^N$ is finite.

We can assume $\mathfrak{a}$ extends as a morphism from $B_0 := \overline{S}$ to $\overline{\mathcal{X}}$, which is then generically finite. Set then $\psi_0 := \mathrm{id}$. We can define B_n and ψ_n inductively as follows: if $\iota \circ \mathfrak{a}_{n+1}$ extends as a morphism $B_n \to \mathbb{P}^N$, let $B_{n+1} := B_n$ and $\psi_{n+1} = \psi_n$. Otherwise, let $p_{n+1} : B_{n+1} \to B_n$ be a finite sequence of blowups such that $\iota \circ \mathfrak{a}_{n+1} \circ \psi_n \circ p_{n+1}$ extends as a morphism $B_{n+1} \to \mathbb{P}^N$. Let then $\psi_{n+1} := \psi_n \circ p_{n+1}$. The morphism $\iota \circ \mathfrak{a}_{n+1} \circ \psi_{n+1} : B_{n+1} \to \mathbb{P}^N$ is then generically finite by construction. We equip $\mathcal{O}_{\mathbb{P}^N}(1)$ with its standard metrization and denote it by $\overline{\mathcal{O}}(1)$. Define

$$\overline{L}_n := \frac{1}{d^n}(\iota \circ \mathfrak{a}_n \circ \psi_n)^* \overline{\mathcal{O}}(1).$$

As $\overline{\mathcal{O}}(1)$ is ample and $\iota \circ \mathfrak{a}_n \circ \psi_n$ is generically finite, $\overline{L}_n$ is big and nef. Moreover, by construction, the variety B_n is defined over $\mathbb{K}$.

4.3. Step 2 of the proof of Theorem 1.3: convergence of the measures

Fix a place $v \in M_{\mathbb{K}}$. We now prove that the sequence $(c_1(\overline{L}_n)_v^p)_n$ of positive measures on S_v^{an} converges weakly to a positive measure μ_v.

Assume first v is archimedean. We follow classical arguments we summarize here: we can write $d^{-1} f^*(\iota^*\omega_{\mathbb{P}^N}) - \iota^*\omega_{\mathbb{P}^N} = dd^c u$, where $u \in \mathscr{C}^\infty(\mathcal{X}_v^{\mathrm{an}})$. An easy induction gives, for any integer $n \geqslant 1$,

$$\widehat{T}_f - \frac{1}{d^n}(f^{\circ n})^*(\iota^*\omega_{\mathbb{P}^N}) = \frac{1}{d^n} dd^c g \circ f^{\circ n} \quad \text{on } \mathcal{X}_v^{\mathrm{an}},$$

where $g = \sum_{j \geqslant 0} d^{-j} \cdot u \circ f^{\circ j} \in \mathscr{C}^0(\mathcal{X}_v^{\mathrm{an}})$. Pulling back by $\mathfrak{a} : S \to \mathcal{X}$, we find $T_{f,\mathfrak{a}} - (\psi_n)_* c_1(\overline{L}_n)_v = d^{-n} \cdot dd^c g \circ f^{\circ n}$ on S_v^{an}. In particular, $(\psi_n)_* c_1(\overline{L}_n)_v$ converges to $T_{f,\mathfrak{a}}$ with a local uniform convergence of potentials. Thus the sequence of measures $((\psi_n)_* c_1(\overline{L}_n)_v^p)_n$ converges weakly on S_v^{an} to $\mu_{f,\mathfrak{a},v}$.

When v is non-archimedean, we rely on the work [17] of Chambert-Loir and Ducros, and we employ freely their vocabulary. We follow the strategy used in the archimedean case, as presented in [10]: let $\|\cdot\|_v$ be the naive (Fubini–Study) metric on $\mathcal{O}_{\mathbb{P}^N}(1)$ at the place v so that the metric on L_n is $d^{-n}(\iota\circ\mathfrak{a}_n\circ\psi_n)^*\|\cdot\|_v$. On the space of metrics on $\mathcal{L}$, the map $(d^{-1}f^*-\mathrm{id})$ is $\frac{C_K}{d}$-contracting over any compact subset K of $\mathcal{X}_v^{\mathrm{an}}$. In particular, we have

$$\left|\frac{1}{d^n}(f^{\circ n})^*(\iota^*\|\cdot\|_v)-\frac{1}{d^{n+1}}\left(f^{\circ(n+1)}\right)^*(\iota^*\|\cdot\|_v)\right|\leqslant\frac{C_K}{d^{n+1}},$$

over any compact subset $K\Subset\mathcal{X}_v^{\mathrm{an}}$. In particular, the sequence

$$\left(\frac{1}{d^n}(f^{\circ n})^*(\iota^*\|\cdot\|_v)\right)_n$$

converges uniformly over compact subsets of $\mathcal{X}_v^{\mathrm{an}}$ to a continuous and semi-positive metric $\|\cdot\|_{f,v}$ on $\mathcal{L}$ which satisfies

$$\left|\frac{1}{d^n}(f^{\circ n})^*(\iota^*\|\cdot\|_v)-\|\cdot\|_{f,v}\right|\leqslant\frac{C'_K}{d^n},$$

over any compact subset $K\Subset\mathcal{X}_v^{\mathrm{an}}$. To conclude, we remark that on S_v^{an}, one can write

$$\begin{aligned}(\psi_n)_*c_1(\overline{L}_n)_v&=\mathfrak{a}_0^*\left(\frac{1}{d^n}(f^{\circ n})^*\iota^*c_1(\overline{\mathcal{O}}_{\mathbb{P}^N}(1))_v\right)\\&=\mathfrak{a}_0^*(c_1(\mathcal{L},\|\cdot\|_{f,v}))+dd^cu_n,\end{aligned}$$

and the above implies that (u_n) converges uniformly on any compact subset $K\Subset S_v^{\mathrm{an}}$ to the constant function $0\in\mathscr{C}^0(S_v^{\mathrm{an}})$ (which is obviously locally approachable). By [17, Corollaire (5.6.5)], this implies

$$(\psi_n)_*c_1(\overline{L}_n)^p\longrightarrow\mu_v:=(\mathfrak{a}_0^*(c_1(\mathcal{L},\|\cdot\|_{f,v})))^p$$

in the weak sense of measures on S_v^{an}, as granted.

4.4. Step 3 of the proof of Theorem 1.3: convergence of volumes

By the above, $\overline{L}_n$ is an ample $\mathbb{Q}$-line bundle equipped with a semi-positive adelic continuous metrization. Moreover, we have $h_{\overline{L}_n}(t)\geqslant 0$ for all $t\in B_n(\overline{\mathbb{Q}})$, so that Corollary 2.2 implies $h_{\overline{L}_n}(B_n)\geqslant 0$. Next, we prove that $\mathrm{vol}(L_n)\to\mathrm{vol}_f(\mathfrak{a})$ as $n\to\infty$ to conclude. To do so, we rely on [31].

Lemma 4.1. — *There is $C_1>0$ depending only on $(\mathcal{X},f,\mathcal{L})$, S, $\mathfrak{a}$ and ι such that*

$$|\mathrm{vol}(L_n)-\mathrm{vol}_f(\mathfrak{a})|\leqslant\frac{C_1}{d^n}.$$

Moreover, for any ample class H on X_0, there is a constant $C_2>0$ depending only on $(\mathcal{X},f,\mathcal{L})$, S, $\mathfrak{a}$, ι and H such that for any $n\geqslant 1$,

$$\left(\psi_n^*H\cdot(L_n)^{p-1}\right)\leqslant C_2$$

Proof. — Let us prove the lemma by computing masses of currents on $\mathcal{X}_v^{\mathrm{an}}$ with respect to the Kähler form $\iota^*(\omega_{\mathbb{P}^N})$. One can interpret $\mathrm{vol}(L_n)$ as

$$\begin{aligned}\mathrm{vol}(L_n) &= \int_{S_v^{\mathrm{an}}} \left(\frac{1}{d^n}(\iota\circ\mathfrak{a}_n)^*\omega_{\mathbb{P}^N}\right)^p \\ &= \frac{1}{d^{np}}\int_{(\mathcal{X})_v^{\mathrm{an}}} [\mathfrak{a}(S_v^{\mathrm{an}})]\wedge\big((f^{\circ n})^*\iota^*(\omega_{\mathbb{P}^N})\big)^p.\end{aligned}$$

First, point 1 of Proposition 3.3 of [31] implies $\|\widehat{T}_f\|$ is finite and

$$\left| d^n\|\widehat{T}_f\| - \|(f^{\circ n})^*(\iota^*\omega_{\mathbb{P}^N})\|\right| \leqslant C_0$$

for some constant C_0 depending only on $(\mathcal{X},f,\mathcal{L})$. Then, the second point of Proposition 3.3 of [31] rereads as

$$\begin{aligned}&|\mathrm{vol}(L_n)-\mathrm{vol}_f(\mathfrak{a})| \\ &\qquad \leqslant C_0'\sum_{j=0}^{p-1} d^{n(j-p)}\left\|\widehat{T}_f^{\wedge j}\wedge[\mathfrak{a}(S_v^{\mathrm{an}})]\wedge((f^{\circ n})^*\iota^*(\omega_{\mathbb{P}^N}))^{\wedge(p-j-1)}\right\|\end{aligned}$$

for some constant $C_0'>0$ depending only on $(\mathcal{X},f,\mathcal{L})$. This gives

$$|\mathrm{vol}(L_n)-\mathrm{vol}_f(\mathfrak{a})| \leqslant C_0'\sum_{j=0}^{p-1} d^{n(j-p)}\|\widehat{T}_f\|^j\cdot\Big(d^n\|\widehat{T}_f\|+C_0\Big)^{p-j-1} \leqslant \frac{C_1}{d^n},$$

by Bezout Theorem.

Let now ω_H be a smooth $(1,1)$-form on S_v^{an} cohomologous to H. Then

$$\big(H_n\cdot(L_n)^{p-1}\big) = \frac{1}{d^{n(p-1)}}\int_{\mathcal{X}_v^{\mathrm{an}}} [\mathfrak{a}(S_v^{\mathrm{an}})]\wedge\big((f^{\circ n})^*\iota^*(\omega_{\mathbb{P}^N})\big)^{p-1}\wedge(\pi^*\omega_H)$$

and as above, for any $n\geqslant 1$, we have

$$\begin{aligned}&\big(H_n\cdot(L_n)^{p-1}\big) \\ &\quad\leqslant C_0'\sum_{j=0}^{p-2} d^{n(j-p+1)}\left\|\widehat{T}_f^j\wedge[\mathfrak{a}(S_v^{\mathrm{an}})]\wedge((f^{\circ n})^*\iota^*(\omega_{\mathbb{P}^N}))^{p-j-2}\wedge(\pi^*\omega_H)\right\| \\ &\qquad\qquad + \left\|\widehat{T}_f^{p-1}\wedge[\mathfrak{a}(S_v^{\mathrm{an}})]\wedge(\pi^*\omega_H)\right\|,\end{aligned}$$

so that we deduce as in the proof of the first point of the Lemma that

$$\big(H_n\cdot(L_n)^{p-1}\big)\leqslant C',$$

for some constant $C'>0$ depending only on $(\mathcal{X},f,\mathcal{L})$, $\mathfrak{a}$, ι and H. □

4.5. Step 4 of the proof of Theorem 1.3: an upper bound on intersection numbers

Let $\overline{M}_0$ be any ample adelic semi-positive line bundle on B_0. We prove here the next lemma.

LEMMA 4.2. — *There is a constant $C \geqslant 0$ such that for any $n \geqslant 0$ and any $2 \leqslant j \leqslant p+1$,*

$$\left(\psi_n^*(\overline{M}_0)\right)^j \cdot \left(\overline{L}_n\right)^{p+1-j} \leqslant C.$$

Proof. — Up to changing the initial projective model B_0 of S, we can assume $B_0 \setminus S$ is the support of an effective divisor D of B_0 and that $\overline{E} := f^*(\iota^*\overline{\mathcal{O}}(1)) - d\iota^*\overline{\mathcal{O}}(1) = \pi^*(\overline{E}_0)$, for some adelic metrize line bundle $\overline{E}_0$ on B_0 which can be represented by a divisor supported on $\mathrm{supp}(D)$, where $\pi : \overline{\mathcal{X}} \to B_0$ is the extension of $\pi : \mathcal{X} \to S$. We may also assume $\mathfrak{a}_0$ extends as a morphism from B_0 to $\overline{\mathcal{X}}$. Let $\overline{\mathcal{X}}_{n,1} := \overline{\mathcal{X}} \times_{B_0} B_n$ and there is a normal projective variety $\overline{\mathcal{X}}_n$, a birational morphism $\overline{\mathcal{X}}_n \to \overline{\mathcal{X}}_{n,1}$ and a projective morphism $\pi_n : \overline{\mathcal{X}}_n \to B_n$ which is flat over $\psi_n^{-1}(S)$ such that $f^{\circ j}$ extends as a morphism $F_{j,n} : \overline{\mathcal{X}}_n \to \overline{\mathcal{X}}$ for all $j \leqslant n$. Let also $\mathfrak{a}_{0,(n)}$ be the section of π_n induced by $\mathfrak{a}_0$. Up to blowing up B_n, we can assume $\mathfrak{a}_{0,(n)}$ extends as a morphism from B_n to $\overline{\mathcal{X}}_n$. Let $\Psi_n : \overline{\mathcal{X}}_n \to \overline{\mathcal{X}}$ be the birational morphism induced by this construction. Note that Ψ_n is an isomorphism from $\pi_n^{-1}(\psi_n^{-1}(S))$ to $\pi^{-1}(S)$ and that $\pi \circ \Psi_n = \psi_n \circ \pi_n$ and $\Psi_n \circ \mathfrak{a}_{0,(n)} = \mathfrak{a}_0 \circ \psi_n$.

Let $\overline{N}_0 := \mathfrak{a}_0^*\overline{E}$. By construction of $\overline{L}_n$ and $\overline{L}_{n+1}$ we can write

$$\begin{aligned}
\overline{L}_{n+1} - p_{n+1}^*\overline{L}_n &= \frac{1}{d^{n+1}}\mathfrak{a}_{0,(n+1)}^*(F_{n,n+1})^*\left(f^*\iota^*\overline{\mathcal{O}}(1)\right) \\
&\qquad - \frac{1}{d^n}\cdot\mathfrak{a}_{0,(n+1)}^*(F_{n,n+1})^*\left(\iota^*\overline{\mathcal{O}}(1)\right) \\
&= \frac{1}{d^{n+1}}\mathfrak{a}_{0,(n+1)}^*(\Psi_n)^*\left(f^*\iota^*\overline{\mathcal{O}}(1) - d\cdot\iota^*\overline{\mathcal{O}}(1)\right) \\
&= \frac{1}{d^{n+1}}\mathfrak{a}_{0,(n+1)}^*(\Psi_n)^*(\overline{E}) \\
&= \frac{1}{d^{n+1}}\psi_{n+1}^*\left(\overline{N}_0\right).
\end{aligned}$$

If we let $\alpha(n) := \frac{1-d^{-n}}{d-1}$, using that $p_{n+1} \circ \psi_n = \psi_{n+1}$, an easy induction gives

$$\overline{L}_n = \psi_n^*\left(\overline{L}_0 + \alpha(n)\cdot\overline{N}_0\right). \tag{4.4}$$

In particular, if $\overline{\mathscr{N}}_0$ is a metrized line bundle on $\mathscr{B}_0$ which restricts to $\overline{N}_0$ on the special fiber of $\mathscr{B}_0 \to \mathrm{Spec}(\mathscr{O}_{\mathbb{K}})$ andf if we let $I_n^j := \left(\psi_n^*(\overline{M}_0)\right)^j \cdot$

$\left(\overline{L}_n\right)^{p+1-j}$, using first the equation (4.4) and then the projection formula, we find

$$\begin{aligned} I_n^j &= \left(\psi_n^*(\overline{M}_0)\right)^j \cdot \left(\psi_n^*\left(\overline{L}_0+\alpha(n)\cdot\overline{\mathscr{N}}_0\right)\right)^{p+1-j} \\ &= \left(\overline{M}_0\right)^j \cdot \left(\overline{L}_0+\alpha(n)\cdot\overline{\mathscr{N}_0}\right)^{p+1-j} \\ &= \sum_{\ell=0}^{p+1-j}\binom{p+1-j}{\ell}(\alpha(n))^{p+1-j-\ell}\left(\overline{M}_0\right)^j\cdot\left(\overline{L}_0\right)^\ell\cdot\left(\overline{\mathscr{N}_0}\right)^{p+1-j-\ell}. \end{aligned}$$

Take $C_1 \geqslant 0$ such that $\left(\overline{M}_0\right)^j\cdot\left(\overline{L}_0\right)^\ell\cdot\left(\overline{\mathscr{N}_0}\right)^{p+1-j-\ell} \leqslant C_1$ for any $2 \leqslant j \leqslant p+1$ and any $0 \leqslant \ell \leqslant p+1-j$. As $\alpha(n) \leqslant 1$ for any $n \geqslant 0$, the above gives

$$I_n^j \leqslant \left(\overline{M}_0\right)^j \cdot \left(\overline{L}_0\right)^{p+1-j} + C_2,$$

where C_2 depends only on p and C_1. The conclusion of the lemma follows. □

4.6. Step 5 of the proof of Theorem 1.3: end of the proof

All there is thus left to prove is that, for any given generic and h-small sequence $\{x_i\}_i$, the induced sequence $\varepsilon_n(\{x_i\}_i) := \limsup h_{\overline{L}_n}(x_i)$ converges to 0 as $n \to \infty$. We rely on [14] and again on [31] and we use Siu's classical bigness criterion as, e.g., in [29, Section 7].

The key point, in this step is the following in the spirit of [27, Theorem 1.4]:

Lemma 4.3. — *If $\mathrm{vol}_f(\mathfrak{a}) > 0$, there is a non-empty Zariski open subset $U \subset \mathfrak{a}(S)$ and a constant $c > 0$ depending only on $(\mathcal{X}, f, \mathcal{L})$, S, $\mathfrak{a}$ and ι such that*

$$h_S(\pi(x)) \leqslant c\Big(1 + \widehat{h}_{f_{\pi(x)}}(x)\Big), \quad x \in U(\overline{\mathbb{Q}}).$$

In particular, there is $c' > 0$ depending only on $(\mathcal{X}, f, \mathcal{L})$, S, $\mathfrak{a}$ and ι such that

$$\Big|\widehat{h}_{f_{\pi(x)}}(x) - h_{\mathcal{X},\mathcal{L}}(x)\Big| \leqslant c'\Big(1 + \widehat{h}_{f_{\pi(x)}}(x)\Big), \quad x \in U(\overline{\mathbb{Q}}).$$

Befor proving this lemma, we can remark that the open set U may be in fact $\mathfrak{a}(S)$, but that the strategy of the proof does not allow to prove it. It would be interesting to clarify the situation here, but it is not needed in the present proof.

Proof. — Define a line bundle H_n on B_n by letting

$$H_n := \psi_n^*(H),$$

where H is the ample line bundle on X_0 inducing h_S. As ψ_n is a birational morphism, H_n is big and nef. By construction, L_n is also big and nef. Fix $n_0 \geqslant 1$ large enough to that Lemma 4.1 gives

$$\mathrm{vol}(L_n) \geqslant \frac{1}{2}\mathrm{vol}_f(\mathfrak{a}), \quad \text{for all } n \geqslant n_0.$$

Fix now an integer $M \geqslant 1$ such that $M\,\mathrm{vol}_f(\mathfrak{a}) > 2pC_2$. By multilinearity of the intersection number, we find

$$\mathrm{vol}(ML_n) = M^p\,\mathrm{vol}(L_n) \geqslant \frac{M^p}{2}\mathrm{vol}_f(\mathfrak{a}) > pM^{p-1}C_2 \geqslant p\big(H_n \cdot (ML_n)^{p-1}\big),$$

so that by the classical bigness criterion of Siu [35, Theorem 2.2.15], $ML_n - H_n$ is big. In particular, there exists an integer $\ell \geqslant 1$ such that $\ell(ML_n - H_n)$ is effective. According to [33, Theorem B.3.2 (e)], this implies there exists a Zariski open subset $U_n \subset B_n$ such that

$$h_{\ell(ML_n - H_n)} \geqslant O(1) \quad \text{on } U_n(\overline{\mathbb{Q}}).$$

Also, by functorial properties of Weil heights, see e.g. [33, Theorem B.3.2 (b-c)],

$$\begin{aligned} h_{\ell(ML_n-H_n)} &= h_{\ell ML_n} - h_{\ell H_n} + O(1) = \ell\Big(Mh_{\overline{L}_n} - h_{H_n}\Big) + O(1) \\ &= \ell M h_{\overline{L}_n} - \ell h_S \circ \psi_n + O(1), \end{aligned}$$

so that we have proved that

$$Mh_{\overline{L}_n} \geqslant h_S \circ \psi_n + O(1), \quad \text{on } U_n(\overline{\mathbb{Q}}).$$

Up to removing a Zariski closed subset of U_n, we can assume $U_n \subset \psi_n^{-1}(S)$. Since ψ_n is an isomorphism from U_n to $S_n := \psi_n(U_n) \subset S$, this gives

$$h_S(t) \leqslant Mh_{\overline{L}_n}(\psi_n^{-1}(t)) + N,\ t \in S_n(\overline{\mathbb{Q}}),$$

for some constant $M \geqslant 1$ independent of n and some constant N which depends a priori on n. In particular, when $t \in S_n(\overline{\mathbb{Q}})$ and $h_S(t) \to \infty$, we find

$$\liminf_{\substack{h_S(t)\to\infty \\ t\in S_n(\overline{\mathbb{Q}})}} \frac{h_{\overline{L}_n}(\psi_n^{-1}(t))}{h_S(t)} \geqslant \frac{1}{M} > 0. \tag{4.5}$$

According to [14, Theorem 3.1], there is a constant $C_3 \geqslant 1$ depending only on $(\mathcal{X}, f, \mathcal{L})$, S, q and ι such that for all $t \in S(\overline{\mathbb{Q}})$ and all $x \in \mathcal{X}(\overline{\mathbb{Q}})$ with $\pi(x) = t$,

$$\left|\widehat{h}_{f_t}(x) - h_{\iota^*\overline{\mathcal{O}}(1)}(x)\right| \leqslant C_3(h_S(t) + 1). \tag{4.6}$$

Evaluating the above for $x = \mathfrak{a}_n(t)$, we find

$$\left|\widehat{h}_{f_t}(\mathfrak{a}_n(t)) - h_{\iota^*\overline{\mathcal{O}}(1)}(\mathfrak{a}_n(t))\right| \leqslant C_3(h_S(t) + 1),$$

which, by invariance of $\widehat{h}_{f_t}$ and by definition of $\overline{L}_n$, gives

$$\left|h_{f,\mathfrak{a}}(t) - h_{\overline{L}_n}(\psi_n^{-1}(t))\right| \leqslant \frac{C_3}{d^n}(h_S(t)+1), \quad t \in S(\overline{\mathbb{Q}}). \tag{4.7}$$

We now fix $n_1 \geqslant n_0$ such that $2MC_3 \leqslant d^n$ for all $n \geqslant n_1$. Pick an integer $n \geqslant n_1$. Using (4.5) and (4.7), we deduce that

$$\liminf_{\substack{h_S(t)\to\infty \\ t\in S_n(\overline{\mathbb{Q}})}} \frac{h_{f,\mathfrak{a}}(t)}{h_S(t)} \geqslant \liminf_{\substack{h_S(t)\to\infty \\ t\in S_n(\overline{\mathbb{Q}})}} \frac{h_{\overline{L}_n}(\psi_n^{-1}(t))}{h_S(t)} - \frac{C_3}{d^n} \geqslant \frac{1}{M} - \frac{C_3}{d^n} \geqslant \frac{1}{2M} > 0.$$

This concludes the first assertion the proof of Lemma 4.3. The second follows using again (4.6). □

To conclude the proof of Theorem 1.3, we now rewrite (4.7) as

$$\left|h_{f,\mathfrak{a}}(t) - h_{\overline{L}_n}(\psi_n^{-1}(t))\right| \leqslant \frac{C_4}{d^n}(1+h_{f,\mathfrak{a}}(t)), \quad t \in S_n(\overline{\mathbb{Q}}),$$

where $C_4 > 0$ is a constant independent of n and t. In particular, if $(F_i)_i$ is a generic and $h_{f,\mathfrak{a}}$-small sequence of finite Galois-invariant subsets of $S(\overline{\mathbb{Q}})$, then $F_i \subset S_n(\overline{\mathbb{Q}})$ for i large enough and, using that $h_{\overline{L}_n}(B_n) \geqslant 0$, we deduce that

$$\limsup_{i\to\infty} h_{\overline{L}_n}(\psi_n^{-1}(F_i)) - h_{\overline{L}_n}(B_n) \leqslant \limsup_{i\to\infty} h_{\overline{L}_n}(\psi_n^{-1}(F_i)) \leqslant \frac{C_4}{d^n},$$

as required. We have proved that $h_{f,\mathfrak{a}}$ is a good height on S with associated global measure $\{\mu_{f,a,v}\}_{v\in\mathbb{K}}$ and Theorem 1.3 follows from Theorem 1.2.

5. Applications and sharpness of the assumptions

We want finally to explore some specific cases. First, we consider the case of one parameter families of rational maps of $\mathbb{P}^1$. In a second time, we focus on the critical height on the moduli space of rational maps.

5.1. One-dimensional families of rational maps

In the case when $f : \mathbb{P}^1 \times S \to \mathbb{P}^1 \times S$ is a family of rational maps of $\mathbb{P}^1$ parametrized by a smooth quasi-projective curve, Theorem 1.3 reduces to the following:

Theorem 5.1. — *Let $f : \mathbb{P}^1 \times S \to \mathbb{P}^1 \times S$ be a family of degree $d \geqslant 1$ rational maps parametrized by a smooth quasi-projective curve S and let $a : S \to \mathbb{P}^1$ be a rational function, all defined over a number field $\mathbb{K}$. Assume*

$\widehat{h}_{f_\eta}(a_\eta) > 0$, *where* η *is the generic point of* S. *Then the set* $\mathrm{Preper}(f,a) := \{t \in S(\overline{\mathbb{Q}}) \, : \, \widehat{h}_{f_t}(a(t)) = 0\}$ *is infinite.*

Moreover, for any $v \in M_{\mathbb{K}}$ *and for any non-repeating sequence* $F_n \subset S(\overline{\mathbb{Q}})$ *of Galois-invariant finite sets such that*

$$\frac{1}{\# F_n} \sum_{t \in F_n} \widehat{h}_{f_t}(a(t)) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,$$

the sequence $\mu_{F_n,v}$ *of probability measures on* S_v^{an} *equidistributed on* F_n *converges weakly to* $\frac{1}{\widehat{h}_{f_\eta}(a_\eta)}\mu_{f,a,v}$ *as* $n \to \infty$.

Proof. — Choose any $v \in M_{\mathbb{K}}$ archimedean. In this case, the author and Vigny [31, Theorem B] proved that

$$\widehat{h}_{f_\eta}(a_\eta) = \int_{S_v^{\mathrm{an}}} \mu_{f,a,v}.$$

In particular, when S, f and a are defined over a number field $\mathbb{K}$, the assumption that $\mu_{f,a,v} > 0$ for some archimedean $v \in M_{\mathbb{K}}$ is satisfied whenever $\widehat{h}_{f_\eta}(a_\eta) > 0$. To be able to apply Theorem 1.3, it is thus sufficient to be able to produce a sequence $t_n \in S(\overline{\mathbb{Q}})$ such that $\widehat{h}_{f_{t_n}}(a(t_n)) = 0$ for all n and such that, if $\mathsf{O}(t_n)$ is the Galois orbit of t_n, then $\mathsf{O}(t_n) \cap \mathsf{O}(t_m) = \varnothing$ for all $n \neq m$. This is now classical and it can be done using Montel's Theorem.

We reproduce here the argument for completeness: let $v \in M_{\mathbb{K}}$ be an archimedean place. Let $U \subset S_v^{\mathrm{an}}$ be any euclidean open set with $\mu_{f,a,v}(U) > 0$. By e.g. [19, Theorem 9.1] or [23, Proposition-Definition 3.1], this is equivalent to the fact that the sequence of rational functions $(a_n)_n$ defined by $a_n(t) := f_t^{\circ n}(a(t))$ is not a normal family on U. Up to reducing U, by the Implicit Function Theorem, we can assume there exists $N \geqslant 3$ and holomorphic function $z : U \to \mathbb{P}^{1,\mathrm{an}}_v$ such that N is minimal such that$f_t^{\circ n} z(t) = z(t)$ for all $t \in U$, and $z(t)$ is repelling for f_t, i.e. $|(f_t)^{\circ n}(z(t))| > 1$. By Montel's Theorem, one can define inductively $t_{j+1} \in U \setminus \{t_\ell, \ell \leqslant j\}$ such that for all $j \geqslant 1$, there is $n(j) \geqslant 1$ and $a_{n(j)}(t_j) \in \{z(t_j), f_{t_j}(z(t_j)), f_{t_j}^{\circ 2}(z(t_j))\}$. We thus have defined an infinite sequence (t_n) of parameters for which $a(t_n)$ is preperiodic. In particular, we have $\widehat{h}_{f_{t_n}}(a(t_n)) = 0$ for all $n \geqslant 1$ and the proof is complete. □

5.2. Sharpness of the assumptions

We now explain why the assumptions are sharp. When $(\mathcal{X}, f, \mathcal{L})$ is a family of polarized endomorphisms of degree d parametrized by a smooth complex quasi-projective curve S, we still have

$$\widehat{h}_{f_\eta}(a_\eta) = \int_{S(\mathbb{C})} \mu_{f,a} = \int_{\mathcal{X}(\mathbb{C})} \widehat{T}_f \wedge [a(S(\mathbb{C}))]$$

for any regular section $a : S \to \mathcal{X}$ by [31, Theorem B]. However when the relative dimension of $\pi : \mathcal{X} \to S$ is $k > 1$, the most probable situation is that there are at most finitely many $t \in S(\overline{\mathbb{Q}})$ such that $\widehat{h}_{f_t}(a(t)) = 0$.

Lemma 5.2. — *There is a family of polarized endomorphisms $(\mathcal{X}, f, \mathcal{L})$ parametrized by $\mathbb{A}^1$ of relative dimension 2 and a section $a : \mathbb{A}^1 \to \mathcal{X}$, all defined over $\mathbb{Q}$, such that*

$$\widehat{h}_{f_\eta}(a_\eta) = 1 \quad \textit{and} \quad \mathrm{Preper}(f,a) := \{t \in \mathbb{A}^1(\overline{\mathbb{Q}}) \ : \ \widehat{h}_{f_t}(a(t)) = 0\} = \varnothing.$$

Proof. — let $f : (\mathbb{P}^1)^2 \times \mathbb{A}^1 \to (\mathbb{P}^1)^2 \times \mathbb{A}^1$ be defined by

$$f_t(z,w) = (z^2 + t, w^2 + t), \quad (z,w,t) \in \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1.$$

Define now a section $a : \mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1$ of the canonical projection $\pi : \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ by letting $a(t) := (t, 0, 4)$ for all $t \in \mathbb{A}^1$. Write $|\cdot| := |\cdot|_\infty$. For any $t \in \mathbb{A}^1$, let $p_t(z) := z^2 + t$ we define the

$$G_t(z) := \lim_{n\to\infty} \frac{1}{2^n} \log^+ |p_t^{\circ n}(z)|, \quad (t,z) \in \mathbb{C} \times \mathbb{C}.$$

Then $\mu_{f,a} = dd^c_t(G_t(0) + G_t(4))$. Since for $u \in \{0;4\}$ we have

$$G_t(u) = \frac{1}{2} \log^+ |t| + O(1), \quad \text{as } |t| \longrightarrow \infty,$$

the measure $\mu_{f,a}$ is a probability measure on $\mathbb{A}^{1,\mathrm{an}}$, whence $\widehat{h}_{f_\eta}(a_\eta) = 1$. Also, by an elementary computation, we have

$$h_{f,a}(t) = \widehat{h}_{p_t}(0) + \widehat{h}_{p_t}(4), \quad t \in \mathbb{A}^1(\overline{\mathbb{Q}}).$$

In particular, if $h_{f,a}(t) = 0$, then $G_t(0) = G_t(4) = 0$. The condition $G_t(0) = 0$ implies $|t| \leqslant 2$ and by [11, Lemma 7] the condition $G_t(4) = 0$ implies that

- either $|t| \leqslant 2$ and $|t \cdot p_t^{\circ n}(4)| \leqslant 2$ for all $n \geqslant 0$,
- or $|t| > 2$ and $|t \cdot p_t^{\circ n}(4)| < 1$ for all $n \geqslant 0$.

The second condition is empty since, for $n = 0$, this implies $2 < 1/4$. For $n = 0$, the first condition implies $|t| \leqslant 1/2$ and for $n = 1$, it gives $|t| \leqslant 5/32$. In particular, the polynomial p_t has an attracting fixed point and the only case when $\widehat{h}_{f_t}(0) = 0$ is the case $t = 0$. Finally, for $t = 0$, $G_0(4) = \log|4| > 0$ ending the proof. ☐

We thus can ask whether the condition of existence of a Zariski dense set of small points is reasonable in families with relative dimension > 1. Following the proof of Theorem 0.1 of [22] exactly as adapted in [28], we can prove the next proposition.

Proposition 5.3. — *Let $(\mathcal{X}, f, \mathcal{L})$ be a family of polarized endomorphisms of degree d parametrized by a smooth complex quasi-projective variety S and let $q \geqslant 1$ be an integer. Assume* $\dim S = qk$, *where k is the relative dimension of $\mathcal{X}$. Assume there are q sections $a_1, \ldots, a_q : S \to \mathcal{X}$ with $\mu_{f,\mathfrak{a}} > 0$. Then set*

$$\mathrm{Preper}(f, a_1, \ldots, a_q) := \left\{ t \in S(\overline{\mathbb{Q}}) \ : \ \widehat{h}_{f_t}(a_1(t)) = \cdots = \widehat{h}_{f_t}(a_q(t)) = 0 \right\}$$

is Zariski dense in $S(\overline{\mathbb{Q}})$. In particular, if S, $(\mathcal{X}, f, \mathcal{L})$ and $a_1, \ldots, a_q$ are defined over a number field, assumption 2 *of Theorem 1.3 is satisfied.*

We omit the proof since it copies verbatim that of [28, Theorem 2.2].

Finally, fix $k \geqslant 1$ and $d \geqslant 2$ and let $(\mathcal{X}, f, \mathcal{L})$ be a family of polarized endomorphisms of degree d, parametrized by a smooth complex quasi-projective variety S with $\dim S > 1$, where $\mathcal{X}$ has relative dimension k and let $a_1, \ldots, a_q : S \to \mathcal{X}$ be $q \geqslant 1$ sections. Given a Kähler form ω on S which is cohomologous to $c_1(M)$ with M ample on S, [31, Theorem B] reads as

$$\int_{S(\mathbb{C})} \left(T_{f,a_1} + \cdots + T_{f,a_q}\right) \wedge \omega_S^{\dim S - 1} = \sum_{j=1}^{q} \widehat{h}_{f_\eta}(a_{j,\eta}).$$

The hypothesis that $\mu_{f,a,v} > 0$ for some archimedean $v \in M_{\mathbb{K}}$ is thus stronger than only assuming $\sum_{j=1}^{q} \widehat{h}_{f_\eta}(a_{j,\eta}) > 0$, which – by the above formula – is equivalent to assuming that $T_{f,a_1} + \cdots + T_{f,a_q} > 0$. We can prove the following.

Lemma 5.4. — *There is a family of $(\mathcal{X}, f, \mathcal{L})$ of polarized endomorphisms parametrized by $\mathbb{A}^2$ and a section $a : \mathbb{A}^2 \to \mathcal{X}$, all defined over $\mathbb{Q}$ where $\mathcal{X}$ has relative dimension* 1 *and such that:*

(1) *if $\omega_{\mathbb{P}^2}$ is the Fubini–Study form of $\mathbb{P}^2(\mathbb{C})$, the current $T_{f,a}$ satisfies*

$$\int_{\mathbb{C}^2} T_{f,a} \wedge \omega_{\mathbb{P}^2} = 1.$$

(2) *the bifurcation measure $\mu_{f,a}$ vanishes identically,*

(3) *the set* $\mathrm{Preper}(f, a)$ *is Zariski dense in* $\mathbb{A}^2(\overline{\mathbb{Q}})$.

The idea behind the proof is that, in relative dimension k, the current $\widehat{T}_f^{k+1}$ vanishes identically. In particular, the current $T_{f,a}^{k+1}$ also vanishes.

Proof. — Define a family of degree 4 polynomials $f : \mathbb{P}^1 \times \mathbb{A}^2 \to \mathbb{P}^1 \times \mathbb{A}^2$ by letting

$$f_{s,t}(z) = \frac{1}{4}z^4 - \frac{2}{3}sz^3 + \frac{s^2}{2}z^2 + s^4, \quad z \in \mathbb{A}^1 \text{ and } (s,t) \in \mathbb{A}^2,$$

and define $a : \mathbb{A}^2 \to \mathbb{A}^1 \times \mathbb{A}^2$ by $a(s,t) = (s,(s,t))$. Let $|\cdot| := |\cdot|_\infty$. As in the proof of Lemma 5.2, define $G : \mathbb{C} \times \mathbb{C}^2 \to \mathbb{R}_+$ by letting

$$G_{s,t}(z) := \lim_{n\to\infty} \frac{1}{4^n} \log^+ |f_{s,t}^{\circ n}(z)|, \ (s,t) \in \mathbb{C}^2, \ z \in \mathbb{C}.$$

We then have $T_{f,a} = dd^c G_{s,t}(s)$ and $\mu_{f,a} = \left(dd^c G_{s,t}(s)\right)^2$.

We now follows arguments from [23, Sections 5–6]. Let us first justify that $\mu_{f,a} = 0$. This follows from Bedford–Taylor theory. Let $g(s,t) := G_{s,t}(s)$ so that the current $T_{f,a}$ is $dd^c g$. As $g \geqslant 0$, we have $g = \lim_{n\to\infty} \max\{g, \frac{1}{n}\}$ and since $\mathrm{supp}(dd^c g) \subset \{g = 0\}$ and $\mathrm{supp}(dd^c \max\{g, 1/n\}) \subset \{g = 1/n\}$, we have

$$\mu_{f,a} = \lim_{n\to\infty} dd^c g \wedge dd^c \max\left\{g, \frac{1}{n}\right\} = 0.$$

Since $g(s,t) \leqslant \log^+ \max\{|s|,|t|\} + O(1)$ as $\|(s,t)\| \to \infty$, we have

$$\int_{\mathbb{C}^2} T_{f,a} \wedge \omega_{\mathbb{P}^2} \leqslant 1.$$

In particular, by Siu's extension Theorem, the trivial extension of $T_{f,a}$ to $\mathbb{P}^2(\mathbb{C})$ is a closed positive $(1,1)$-current S which decomposes at $\widetilde{T} + \alpha[L_\infty]$, where $[L_\infty]$ is the integration current on the lien at infinity and $\widetilde{T} \wedge \omega_{\mathbb{P}^2}$ gives no mass to L_∞. But [6, Theorem 4.2] implies that $\alpha = 0$, whence

$$\int_{\mathbb{C}^2} T_{f,a} \wedge \omega_{\mathbb{P}^2} = 1.$$

To prove the last assertion, for $n > m \geqslant 0$, we let

$$\mathrm{Preper}(n,m) := \{(s,t) \in \mathbb{C}^2 \ : \ f_{s,t}^{\circ n}(s) = f_{s,t}^{\circ m}(s)\}.$$

For $n > m \geqslant 0$, the set $\mathrm{Preper}(n,m)$ is a (possibly reducible) plane curve of degree 4^n which is defined over $\mathbb{Q}$. In particular, the set $\mathrm{Preper}(n,m)(\overline{\mathbb{Q}})$ is infinite. Also [23, Theorem 1] implies that for any sequence $\{m(n)\}_n$ with $0 \leqslant m(n) < n$,

$$T_{f,a} = \lim_{n\to\infty} \frac{1}{4^n} [\mathrm{Preper}(n, m(n))]$$

in the weak sense of currents. As $T_{f,a} = dd^c g$ where g is continuous, the set $\mathrm{Preper}(f,a)$ is Zariski dense. □

5.3. In the moduli space of degree d rational maps

We finally focus on the case of the moduli space $\mathcal{M}_d$ of degree d rational maps and we give a very quick proof of Theorem 1.4. The variety $\mathcal{M}_d$ is the space of PGL(2) conjugacy classes of rational maps of degree d. By [39], the variety $\mathcal{M}_d$ is irreducible, affine has dimension $2d-2$, and is defined over $\mathbb{Q}$.

The good setting to apply Theorem 1.3 is actually the *critically marked moduli space* $\mathcal{M}_d^{\mathrm{cm}}$. As in [12], define first

$$\mathrm{Rat}_d^{\mathrm{cm}} := \left\{ (f, c_1, \dots, c_{2d-2}) \in \mathrm{Rat}_d \times (\mathbb{P}^1)^{2d-2} \; : \; \mathrm{Crit}(f) = \sum_j [c_j] \right\},$$

where $\mathrm{Crit}(f)$ stands for the *critical divisor* of f. The space $\mathrm{Rat}_d^{\mathrm{cm}}$ is an quasi-projective variety of dimension $2d+1$ which is a finite branched cover of Rat_d. We then define the *critically marked moduli space* $\mathcal{M}_d^{\mathrm{cm}}$ as

$$\mathcal{M}_d^{\mathrm{cm}} := \mathrm{Rat}_d^{\mathrm{cm}} / \mathrm{PGL}(2),$$

where $\mathrm{PGL}(2)$ acts by $\phi \cdot (f, c) = (\phi \circ f \circ \phi^{-1}, \phi(c_1), \dots, \phi(c_{2d-2}))$, and the quotient is geometric in the sense of Invariant Geometric Theory as in [39]. Again, it is an irreducible affine variety defined over $\mathbb{Q}$. Moreover, we can directly apply Theorem 1.3 to the good height function $h_{\mathrm{Crit}} : \mathcal{M}_d^{\mathrm{cm}} \to \mathbb{R}_+$ defined by

$$h_{\mathrm{Crit}}(f, c_1, \dots, c_{2d-2}) = \sum_{j=1}^{2d-2} \widehat{h}_f(c_j),$$

for all $(f, c_1, \dots, c_{2d-2}) \in \mathcal{M}_d^{\mathrm{cm}}(\overline{\mathbb{Q}})$. Indeed, we have a natural map $f : (\mathbb{P}^1)^{2d-2} \times \mathcal{M}_d^{\mathrm{cm}} \to (\mathbb{P}^1)^{2d-2} \times \mathcal{M}_d^{\mathrm{cm}}$ together with a section $\mathfrak{c}$ defined as

$$\mathfrak{c} : \{(f, c_1 \dots, c_{2d-2})\} \longmapsto ((c_1, \dots, c_{2d-2}), \{(f, c_1 \dots, c_{2d-2})\}).$$

The current $T_{f^{[2d-2]}, \mathfrak{c}}$ is then the bifurcation current T_{bif} of the family.

We now justify quickly why we are in the domain of application of Theorem 1.3. For any irreducible subvariety $V \subset \mathcal{M}_d$, the measure $\mu_{\mathrm{bif},V} := T_{\mathrm{bif}}^{\dim V}$ is non zero if and only if V does not coincide with the curve $\mathcal{L}_d$ of flexible Lattès maps, by [29, Lemma 6.8]. Here $\mathcal{L}_d$ is, when $d = N^2$, the family of maps induced by the multiplication by N on a non-isotrivial elliptic surface $\mathcal{E} \to S$. In particular, $T_{\mathrm{bif}}^{2d-2} > 0$ on $\mathcal{M}_d^{\mathrm{cm}}$.

As $\dim \mathcal{M}_d^{\mathrm{cm}} = 2d - 2$, Proposition 5.3 implies the set of $h_{f,\mathrm{crit}}$-small points form a Zariski dense subset of $\mathcal{M}_d^{\mathrm{cm}}(\overline{\mathbb{Q}})$. In particular, we are in position to apply Theorem 1.3 in the family $\mathcal{M}_d^{\mathrm{cm}}$. To conclude the proof of Theorem 1.4, we just need to recall that

(1) the canonical projection $p : \mathcal{M}_d^{\mathrm{cm}} \to \mathcal{M}_d$ is a finite branched cover,
(2) a conjugacy class $\{f\}$ is post-critically finite (PCF) iff and only if $\{f, c_1, \dots, c_{2d-2}\}$ is $h_{f,\mathrm{crit}}$-small,
(3) the bifurcation measures μ_{bif} and $\mu_{\mathrm{bif,cm}}$ respectively of $\mathcal{M}_d(\mathbb{C})$ and of $\mathcal{M}_d^{\mathrm{cm}}(\mathbb{C})$ are related by $\mu_{\mathrm{bif,cm}} = p^*(\mu_{\mathrm{bif}})$.

In particular, we deduce that for any sequence $F_n \subset \mathcal{M}_d(\overline{\mathbb{Q}})$ of Galois-invariant finite sets of post-critically finite parameters such that

$$\frac{\#(F_n \cap H)}{\#F_n} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,$$

for any hypersurface H of $\mathcal{M}_d$ which is defined over $\mathbb{Q}$, for any place $v \in M_{\mathbb{Q}}$ the sequence of probability measures $\frac{1}{\#F_n}\sum_{\{f\}\in F_n}\delta_{\{f\}}$ on $\mathcal{M}_{d,v}^{\mathrm{an}}$ converges weakly to $\mathrm{vol}(\mu_{\mathrm{bif}})^{-1}\mu_{\mathrm{bif},v}$ in the weak sense of probability measures on $\mathcal{M}_{d,v}^{\mathrm{an}}$.

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