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Motivic Gauss and Jacobi sums [(*)]

Noriyuki Otsubo [(1)] and Takao Yamazaki [(2)]

Abstract. — We study the Gauss and Jacobi sums from a viewpoint of motives. We exhibit isomorphisms between Chow motives arising from the Artin–Schreier curve and the Fermat varieties over a finite field, that can be regarded as (and yield a new proof of) classically known relations among Gauss and Jacobi sums such as Davenport–Hasse's multiplication formula. As a key step, we define motivic analogues of the Gauss and Jacobi sums as algebraic correspondences, and show that they represent the Frobenius endomorphisms of such motives. This generalizes Coleman's result for curves. These results are applied to investigate the group of invertible Chow motives with coefficients in a cyclotomic field.

Résumé. — Nous étudions les sommes de Gauss et de Jacobi du point de vue des motifs. Nous démontrons des isomorphismes entre les motifs de Chow associés à la courbe d'Artin–Schreier et les variétés de Fermat sur un corps fini, qui peuvent être considérés comme (et fournissent une nouvelle preuve de) relations classiquement connues entre les sommes de Gauss et de Jacobi telles que la formule de multiplication de Davenport–Hasse. Comme étape clé, nous définissons des analogues motiviques des sommes de Gauss et de Jacobi comme des correspondances algébriques, et montrons qu'ils représentent les endomorphismes de Frobenius de tels motifs. Cela généralise le résultat de Coleman pour les courbes. Ces résultats sont appliqués à l'étude du groupe des motifs de Chow inversibles à coefficients dans un corps cyclotomique.

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Keywords: Gauss sums, Jacobi sums, Davenport–Hasse relation, Chow motives, Weil numbers.

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1. Introduction

Let κ be a finite field of cardinality q and of characteristic p, and d a positive divisor of $q-1$. Take a non-trivial additive character $\psi\colon \kappa \to \mathbb{C}^*$, and multiplicative characters $\chi, \chi_1, \ldots, \chi_n \colon \mu_d \to \mathbb{C}^*$, where $\mu_d := \{m \in \kappa^* \mid m^d = 1\}$. We consider the Gauss sum $g(\psi, \chi) \in \mathbb{Q}(\zeta_{pd})$ and the Jacobi sum $j(\chi_1, \ldots, \chi_n) \in \mathbb{Q}(\zeta_d)$ (see (2.1), (2.4) for the definitions), where $\zeta_k := e^{2\pi i/k} \in \mathbb{C}$. In this introduction, we discuss the following relations among $g(\psi, \chi)$ and $j(\chi_1, \ldots, \chi_n)$:

- Assume that none of $\chi_1, \ldots, \chi_n, \prod_{i=1}^n \chi_i$ is trivial. Then we have (cf. (2.5))

$$\prod_{i=1}^{n} g(\psi, \chi_i) = g(\psi, \chi_1 \cdots \chi_n) j(\chi_1, \ldots, \chi_n). \tag{1.1}$$

- Assume $n \mid d$ and let $\alpha\colon \mu_d \to \mathbb{C}^*$ be a character such that $\alpha^n \neq 1$. Then we have the Davenport–Hasse multiplication formula (cf. (7.5))

$$\alpha^n(n) j(\underbrace{\alpha, \ldots, \alpha}_{n \text{ times}}) = \prod_{\chi^n=1, \chi\neq 1} j(\alpha, \chi), \tag{1.2}$$

  where χ ranges over all non-trivial characters of μ_d such that $\chi^n = 1$.

(See (2.2), (2.3), (2.6), (6.3), (7.1), (7.2), (7.4), (7.5) for other relations considered in the body of the text.) The aim of the present note is to upgrade these relations to *motives*. This is achieved in two steps. The first step is to construct isomorphisms between suitable motives, and the second is to relate the Frobenius endomorphisms with the motivic Gauss and Jacobi sums.

To state our results, we introduce more notations. Let $\mathbf{Chow}(\kappa, \Lambda)$ be the category of Chow motives over κ with coefficients in a field Λ of characteristic zero. Let A_d be the (smooth projective) Artin–Schreier curve defined by $x^q - x = y^d$. We construct an object $h(A_d)^{(\psi,\chi)}$ of $\mathbf{Chow}(\kappa, \mathbb{Q}(\zeta_{pd}))$ as a direct factor of the motive $h(A_d)$ of A_d cut out by the action of $\kappa \times \mu_d$. Similarly, for each $c \in \kappa^*$ we construct an object $h(F_d^{(n)}\langle c\rangle)^{(\chi_1,\ldots,\chi_n)}$ of $\mathbf{Chow}(\kappa, \mathbb{Q}(\zeta_d))$ as a direct factor of the motive $h(F_d^{(n)}\langle c\rangle)$ of the Fermat variety $F_d^{(n)}\langle c\rangle \subset \mathbb{P}^n$ defined by $u_1^d + \cdots + u_n^d = cu_0^d$ cut out by the action of μ_d^n. (See Section 4 for details.) We drop $\langle c\rangle$ from the notation when $c = 1$. Our first main result is the following.

Theorem 1.1. —

(i) *Let ψ be a character of κ and $\chi_1, \dots, \chi_n$ characters of μ_d. If none of $\psi, \chi_1, \dots, \chi_n$, $\prod_{i=1}^n \chi_n$ is trivial, then there exists an isomorphism between invertible objects of* $\mathbf{Chow}(\kappa, \mathbb{Q}(\zeta_{pd}))$

$$\bigotimes_{i=1}^n h(A_d)^{(\psi,\chi_i)} \simeq h(A_d)^{\left(\psi, \prod_{i=1}^n \chi_i\right)} \otimes h(F_d^{(n)})^{(\chi_1,\dots,\chi_n)}.$$

(ii) *Suppose that n divides d and let α be a character of μ_d such that $\alpha^n \neq 1$. Then there exists an isomorphism between invertible objects of* $\mathbf{Chow}(\kappa, \mathbb{Q}(\zeta_d))$

$$h(F_d^{(n)}\langle n\rangle)^{(\alpha,\dots,\alpha)} \simeq \bigotimes_{\chi^n=1, \chi\neq 1} h(F_d^{(2)})^{(\alpha,\chi)}.$$

In fact, the isomorphism in (ii) holds over arbitrary base field κ as long as κ contains a primitive dth root of unity (see Section 7.2). When $\kappa = \mathbb{C}$, it has an implication on the gamma function (see Remark 7.5).

To state the second main result, we introduce an element of the group ring $\mathbb{Z}[\mu_d^n]$

$$j_d^{(n)}\langle c\rangle = (-1)^{n-1} \sum_{m_1,\dots,m_n\in\kappa^*, \sum m_i = c} \left(m_1^{\frac{q-1}{d}}, \dots, m_n^{\frac{q-1}{d}}\right),$$

which we call the *twisted Jacobi sum element*. It follows from the definition that $j_d^{(n)}\langle c\rangle$ acts on $h(F_d^{(n)}\langle c\rangle)^{(\chi_1,\dots,\chi_n)}$ as the multiplication by

$$\chi_1 \cdots \chi_n(c^{\frac{q-1}{d}}) j(\chi_1, \dots, \chi_n).$$

The following theorem generalizes Coleman's result [5, Theorem A] for $n = 2$ and $c = 1$.

Theorem 1.2. — *Let $\chi_1, \dots, \chi_n$ be characters of μ_d such that none of $\chi_1, \dots, \chi_n, \prod_{i=1}^n \chi_i$ is trivial. Then the endomorphism of $h(F_d^{(n)}\langle c\rangle)^{(\chi_1,\dots,\chi_n)}$ induced by $j_d^{(n)}\langle c\rangle$ agrees with the Frobenius endomorphism.*

Coleman also proved that the *Gauss sum element*

$$g_d = -\sum_{m\in\kappa^*} \left(m, m^{\frac{q-1}{d}}\right) \in \mathbb{Z}[\kappa \times \mu_d] \tag{1.3}$$

induces the Frobenius endomorphism on $h(A_d)^{(\psi,\chi)}$ if ψ and χ are non-trivial. Since the Frobenius endomorphism commutes with any morphisms (see (3.4) below), (1.1) and (1.2) can be deduced from Coleman's result and Theorems 1.1, 1.2. See Remark 7.4 for details.

We conclude this introduction by a discussion on the relations among Weil numbers and motives. Recall that $\alpha \in \mathbb{C}$ is called a *q-Weil number* of weight $w \in \mathbb{Z}$ if there exists $m \in \mathbb{Z}$ such that $q^m\alpha$ is an algebraic integer and $|\sigma(\alpha)| = q^{w/2}$ for all $\sigma\colon \mathbb{Q}(\alpha) \to \mathbb{C}$. Let $W_q(\Lambda)$ be the subgroup of Λ^* consisting of all q-Weil numbers (of arbitrary weight) belonging to a given subfield Λ of $\mathbb{C}$. It is conjectured by Beilinson [2, 1.0] that the rational equivalence and numerical equivalence should agree over a finite field κ (with coefficients in Λ). If we assume this as well as the Tate conjecture, it follows from [12, Proposition 2.21] that all simple objects of $\mathbf{Chow}(\kappa, \overline{\mathbb{Q}})$ should be invertible, and their isomorphism classes would form a group (with respect to the tensor product) isomorphic to $W_q(\overline{\mathbb{Q}})$. In particular, the group $\mathbf{Pic}(\mathbf{Chow}(\kappa, \Lambda))$ of all isomorphism classes of invertible objects of $\mathbf{Chow}(\kappa, \Lambda)$ should be isomorphic to a subgroup of $W_q(\Lambda)$ by Lemma 3.3 below. Therefore, the multiplicative relations among q-Weil numbers (such as (1.1) and (1.2)) *should* come from relations among motives, as demonstrated by Theorem 1.1.

Using our motivic relations, we shall deduce the following two results on $\mathbf{Pic}(\mathbf{Chow}(\kappa, \mathbb{Q}(\zeta_d)))$ where κ is the residue field of $\mathbb{Q}(\zeta_d)$ at a prime $v \nmid d$, both conditional to the conjectures of Beilinson and Tate (see Corollary 8.4 and the discussion after Proposition 8.5):

- $\mathbf{Pic}(\mathbf{Chow}(\kappa, \mathbb{Q}(\zeta_d))$ should be generated by the Fermat motives $h(F_d^{(2)})^{(\chi_1,\chi_2)}$, up to powers and Artin motives.
- All the relations among $h(F_d^{(2)})^{(\chi_1,\chi_2)}$ in $\mathbf{Pic}(\mathbf{Chow}(\kappa, \mathbb{Q}(\zeta_d)))$ should be implied by Theorem 1.1 (ii) and the reflection relation (4.8), up to powers and Artin motives.

The key input here is a result of Iwasawa–Sinnott [19] on Stickelberger's ideal.

The paper is organized as follows. After a brief recollection on the Gauss and Jacobi sums, we define their motivic variants in Section 2. We prepare a few basic facts on the Chow and Voevodsky motives in Section 3. We then extensively study the motives of the Artin–Schreier curves and the Fermat varieties in Section 4, where a crucial ingredient is the inductive structure of Fermat varieties due to Katsura–Shioda [18]. We complete the proof of Theorem 1.1 (i), (ii) and Theorem 1.2 in Sections 5, 7 and 6, respectively. The last Section 8 is devoted to a discussion on Weil numbers and $\mathbf{Pic}(\mathbf{Chow}(\kappa, \mathbb{Q}(\zeta_d)))$.

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2. Gauss and Jacobi sums

Let κ be a finite field of characteristic p with q elements, and let d be a positive divisor of $q-1$. We write $\widehat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$ when G is a finite group, and $\overline{\chi} := \chi^{-1}$ for $\chi \in \widehat{G}$. For $\psi \in \widehat{\kappa} \setminus \{1\}$ and $\chi \in \widehat{\mu}_d$, the *Gauss sum* is defined by

$$g(\psi, \chi) = -\sum_{m \in \kappa^*} \psi(m)\chi\Big(m^{\frac{q-1}{d}}\Big) \in \mathbb{Q}(\zeta_{pd}). \tag{2.1}$$

Note that $g(\psi, 1) = 1$. We have for any $\chi \neq 1$

$$g(\psi, \chi)g(\overline{\psi}, \overline{\chi}) = q. \tag{2.2}$$

In particular, $g(\psi, \chi)$ is a q-Weil number of weight one if $\chi \neq 1$. We have also

$$g(\overline{\psi}, \chi) = \chi\Big((-1)^{\frac{q-1}{d}}\Big)g(\psi, \chi). \tag{2.3}$$

If $d' \mid d$ and if $\chi' \in \widehat{\mu}_{d'}$ is such that $\chi(m) = \chi'(m^{d/d'})$ for all $m \in \mu_d$, then we have $g(\psi, \chi) = g(\psi, \chi')$.

For $\chi_1, \dots, \chi_n \in \widehat{\mu}_d$, the *Jacobi sum* is defined by

$$\begin{aligned} &j(\chi_1, \dots, \chi_n) \\ &\quad = (-1)^{n-1} \sum_{m_i \in \kappa^*, \sum_{i=1}^n m_i = 1} \chi_1\left(m_1^{\frac{q-1}{d}}\right) \cdots \chi_n\left(m_n^{\frac{q-1}{d}}\right) \in \mathbb{Q}(\zeta_d). \end{aligned} \tag{2.4}$$

If none of $\chi_1, \dots, \chi_n, \prod_{i=1}^n \chi_i$ is trivial, we have (cf. [16, Proposition 2.2])

$$j(\chi_1, \dots, \chi_n) = \frac{g(\psi, \chi_1) \cdots g(\psi, \chi_n)}{g(\psi, \chi_1 \cdots \chi_n)}, \tag{2.5}$$

$$j(\chi_1, \dots, \chi_n) j(\overline{\chi_1}, \dots, \overline{\chi_n}) = q^{n-1}. \tag{2.6}$$

(In particular, the right member of (2.5) is independent of ψ). If $d' \mid d$ and if $\chi'_i \in \widehat{\mu}_{d'}$ is such that $\chi_i(m) = \chi'_i(m^{d/d'})$ for all $m \in \mu_d$ and $i = 1, \dots, n$, then we have $j(\chi_1, \dots, \chi_n) = j(\chi'_1, \dots, \chi'_n)$.

Remark 2.1. — The relations (2.2) and (2.5) are finite field analogues of the functional equations for the gamma and beta functions:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad B(s_1, \dots, s_n) = \frac{\Gamma(s_1)\cdots\Gamma(s_n)}{\Gamma(s_1 + \cdots + s_n)}.$$

Define the *Gauss sum element* in the group ring $\mathbb{Z}[\kappa \times \mu_d]$ by

$$g_d = -\sum_{m \in \kappa^*} \left(m, m^{\frac{q-1}{d}}\right).$$

Define for $c \in \kappa$ the *twisted Jacobi sum element* in the group ring $\mathbb{Z}[\mu_d^n]$ by

$$j_d^{(n)}\langle c\rangle = (-1)^{n-1} \sum_{m_1, \dots, m_n \in \kappa^*, \sum_{i=1}^n m_i = c} \left(m_1^{\frac{q-1}{d}}, \dots, m_n^{\frac{q-1}{d}}\right). \tag{2.7}$$

We write $j_d^{(n)} = j_d^{(n)}\langle 1\rangle$. Note that if $c \neq 0$

$$j_d^{(n)}\langle c\rangle = \left(c^{\frac{q-1}{d}}, \dots, c^{\frac{q-1}{d}}\right) \cdot j_d^{(n)}.$$

If $d' \mid d$, then the map $\mathbb{Z}[\kappa \times \mu_d] \to \mathbb{Z}[\kappa \times \mu_{d'}]; (a, m) \mapsto (a, m^{d/d'})$ sends g_d to $g_{d'}$, and the map $\mathbb{Z}[\mu_d^n] \to \mathbb{Z}[\mu_{d'}^n]; (m_i) \mapsto (m_i^{d/d'})$ sends $j_d^{(n)}\langle c\rangle$ to $j_{d'}^{(n)}\langle c\rangle$.

If G is a finite group and $\chi \in \widehat{G}$, we write

$$e^\chi = e_G^\chi = \frac{1}{|G|} \sum_{g \in G} \bar{\chi}(g) g \ \in \mathbb{C}[G] \tag{2.8}$$

for the corresponding projector. We have $g \cdot e^\chi = \chi(g) e^\chi$ and $e^\chi e^{\chi'} = \delta(\chi, \chi') e^\chi$ in $\mathbb{C}[G]$ for any $g \in G$ and $\chi, \chi' \in \widehat{G}$, where δ is the Kronecker delta. If G is abelian, we also have $\sum_{\chi \in \widehat{G}} e^\chi = 1$. The following lemma is an immediate consequence of the definitions and will be used frequently without further notice.

Lemma 2.2. —

(i) *For any* $\psi \in \widehat{\kappa} \setminus \{1\}$ *and* $\chi \in \widehat{\mu}_d$, *we have*

$$g_d \cdot e^{(\psi, \chi)} = g(\psi, \chi) e^{(\psi, \chi)} \quad \text{in } \mathbb{C}[\kappa \times \mu_d].$$

(ii) *For any* $\chi_1, \dots, \chi_n \in \widehat{\mu}_d$, *we have*

$$j_d^{(n)}\langle c\rangle \cdot e^{(\chi_1, \dots, \chi_n)} = \chi_1 \cdots \chi_n(c^{\frac{q-1}{d}}) j(\chi_1, \dots, \chi_n) e^{(\chi_1, \dots, \chi_n)} \quad \text{in } \mathbb{C}[\mu_d^n].$$

3. Preliminaries on motives

In this section κ is an arbitrary field, which we assume to be perfect from Section 3.2 onward. Let $\mathbf{SmProj}(\kappa)$ be the category of smooth projective varieties over κ. We also fix a field Λ of characteristic zero. We write $A_\Lambda := A \otimes_{\mathbb{Z}} \Lambda$ when A is an abelian group.

3.1. Chow motives

Let $\mathbf{Chow}(\kappa, \Lambda)$ be the *homological* category of Chow motives over κ with coefficients in Λ (cf. e.g. [11, Chapter 20]; this is opposite of the one in [17]). This is a Λ-linear rigid tensor pseudo-abelian category. Recall that an object of $\mathbf{Chow}(\kappa, \Lambda)$ can be written as a triple (X, π, r) where $X \in \mathbf{SmProj}(\kappa)$ is equi-dimensional, $\pi \in \mathrm{CH}_{\dim X}(X \times X)_\Lambda$ is such that $\pi^2 = \pi$ (with respect to the composition of algebraic correspondences), and $r \in \mathbb{Z}$. For two such triples (X, π, r) and (Y, ρ, s), we have

$$\mathrm{Hom}_{\mathbf{Chow}(\kappa,\Lambda)}((X, \pi, r), (Y, \rho, s)) = \rho \circ \mathrm{CH}_{\dim X + r - s}(X \times Y)_\Lambda \circ \pi.$$

For $r = 0$ we abbreviate $(X, \pi, 0) = (X, \pi)$.

The tensor product on $\mathbf{Chow}(\kappa, \Lambda)$ is given by

$$(X, \pi, r) \otimes (Y, \rho, s) = (X \times Y, \pi \times \rho, r + s).$$

We put $\Lambda(r) := (\mathrm{Spec}\, \kappa, \mathrm{id}_{\mathrm{Spec}\, \kappa}, r)$ and $\Lambda := \Lambda(0)$. For any $M \in \mathbf{Chow}(\kappa, \Lambda)$, we set $M(r) := M \otimes \Lambda(r)$ and write $M^\vee$ for the (strong) dual of M. We have $(X, \pi, r)^\vee(\dim X) = (X, {}^t\pi, -r)$, where ${}^t\pi$ denotes the transpose of π.

Suppose $X \in \mathbf{SmProj}(\kappa)$ is connected of dimension m. Given a κ-rational point $x_0 \in X(\kappa)$, we define objects in $\mathbf{Chow}(\kappa, \Lambda)$ by

$$h_0(X) := (X, [X \times x_0]) \simeq \Lambda, \quad h_{2m}(X) := (X, [x_0 \times X]) \simeq \Lambda(m). \tag{3.1}$$

If $m = 1$, we further put

$$h_1(X) := (X, \mathrm{id}_X - [X \times x_0] - [x_0 \times X]) \in \mathbf{Chow}(\kappa, \Lambda). \tag{3.2}$$

We do not indicate x_0 to ease the notation, although these objects depend on the class of x_0 in $\mathrm{CH}_0(X)_\Lambda$.

There is a *covariant* functor

$$h \colon \mathbf{SmProj}(\kappa) \longrightarrow \mathbf{Chow}(\kappa, \Lambda), \quad h(X) = (X, \mathrm{id}_X).$$

For a morphism $f \colon X \to Y$ in $\mathbf{SmProj}(\kappa)$, we have

$$f_* := h(f) = [\Gamma_f] \colon h(X) \longrightarrow h(Y),$$

where $\Gamma_f \subset X \times Y$ denotes the graph of f. If X, Y are equi-dimensional, we also have

$$f^* := [{}^t\Gamma_f] \colon h(Y) \longrightarrow h(X)(\dim Y - \dim X).$$

LEMMA 3.1. — *Let $f \colon X \to Y$ be a generically finite morphism of degree d, where $X, Y \in \mathbf{SmProj}(\kappa)$ are both irreducible and of the same dimension m. We define a group homomorphism*

$$f_\# \colon \operatorname{End}(h(X)) \longrightarrow \operatorname{End}(h(Y)), \qquad f_\#(\alpha) = \frac{1}{d}(f_* \circ \alpha \circ f^*), \tag{3.3}$$

where End *denotes the endomorphism ring in* $\mathbf{Chow}(\kappa, \Lambda)$.

(i) *We have a commutative diagram*

$$\begin{array}{ccc} \operatorname{End}(h(X)) & \xrightarrow{f_\#} & \operatorname{End}(h(Y)) \\ \| & & \| \\ \operatorname{CH}_m(X \times X)_\Lambda & \xrightarrow{(1/d)(f\times f)_*} & \operatorname{CH}_m(Y \times Y)_\Lambda. \end{array}$$

(ii) *Let σ_X (resp. σ_Y) be an automorphism of X (resp. Y) such that $f \circ \sigma_X = \sigma_Y \circ f$. Then, $f_\#((\sigma_X)_*) = (\sigma_Y)_*$.*

Proof. — (i) is a consequence of Lieberman's lemma (cf. e.g. [13, Lemma 2.1.2]). To see (ii), we consider the commutative diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{f\times f} & Y \times Y \\ {\scriptstyle \mathrm{id}_X \times \sigma_X}\uparrow & & \uparrow{\scriptstyle \mathrm{id}_Y \times \sigma_Y} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

We then compute

$$\begin{aligned}(f \times f)_*([\Gamma_{\sigma_X}]) &= ((f \times f) \circ (\mathrm{id}_X \times \sigma_X))_*([X]) \\ &= ((\mathrm{id}_Y \times \sigma_Y) \circ f)_*([X]) = (\mathrm{id}_Y \times \sigma_Y)_*(d[Y]) = d[\Gamma_{\sigma_Y}].\end{aligned}$$

We now apply (i) to conclude (ii). □

Example 3.2. — We will use this lemma in the following situations.

(i) If f is an isomorphism, then $f^* = (f_*)^{-1}$ and $f_\#$ is a ring isomorphism.
(ii) If finite groups G and G' act on X and Y respectively, and the actions are compatible under f and a homomorphism $g \colon G \to G'$,

then the diagram

$$\begin{array}{ccc} \Lambda[G] & \xrightarrow{\quad g \quad} & \Lambda[G'] \\ \downarrow & & \downarrow \\ \mathrm{End}(h(X)) & \xrightarrow{f_\#} & \mathrm{End}(h(Y)) \end{array}$$

commutes. Hence the restriction of $f_\#$ to the image of $\Lambda[G]$ is a ring homomorphism.

(iii) Let κ be a finite field with q elements and Fr_X be the qth power Frobenius endomorphism of X. Then $f_\#(\mathrm{Fr}_X) = \mathrm{Fr}_Y$ holds, because we have

$$f \circ \mathrm{Fr}_X = \mathrm{Fr}_Y \circ f \tag{3.4}$$

for any morphism $f\colon X \to Y$ in $\mathbf{Chow}(\kappa, \Lambda)$ by [9, p. 80] (see also [20, Proposition 2]).

For later use, we state an elementary lemma.

Lemma 3.3. — *Let Λ' be a field extension of Λ. Then the scalar extension functor $\mathbf{Chow}(\kappa, \Lambda) \to \mathbf{Chow}(\kappa, \Lambda')$ is conservative.*

Proof. — Let $f\colon M \to N$ be a morphism in $\mathbf{Chow}(\kappa, \Lambda)$ and assume that its scalar extension $f_{\Lambda'}\colon M_{\Lambda'} \to N_{\Lambda'}$ is an isomorphism in $\mathbf{Chow}(\kappa, \Lambda')$. We must show that f is an isomorphism in $\mathbf{Chow}(\kappa, \Lambda)$. By Yoneda's lemma, it suffices to show that

$$f_*\colon \mathrm{Hom}_{\mathbf{Chow}(\kappa,\Lambda)}(L, M) \longrightarrow \mathrm{Hom}_{\mathbf{Chow}(\kappa,\Lambda)}(L, N)$$

is bijective for any $L \in \mathbf{Chow}(\kappa, \Lambda)$. This follows from the bijectivity of

$$f_{\Lambda' *}\colon \mathrm{Hom}_{\mathbf{Chow}(\kappa,\Lambda')}(L_{\Lambda'}, M_{\Lambda'}) \longrightarrow \mathrm{Hom}_{\mathbf{Chow}(\kappa,\Lambda')}(L_{\Lambda'}, N_{\Lambda'})$$

since Λ' is faithfully flat over Λ. □

3.2. Voevodsky motives

From now on we assume that κ is perfect. Let $\mathbf{DM}_{\mathrm{gm}}(\kappa, \Lambda)$ be Voevodsky's category of geometric mixed motives over κ with coefficients in Λ (cf., e.g. [11]). This is a Λ-linear rigid tensor pseudo-abelian triangulated category equipped with a covariant functor $M\colon \mathbf{Sm}(\kappa) \to \mathbf{DM}_{\mathrm{gm}}(\kappa, \Lambda)$, where $\mathbf{Sm}(\kappa)$ is the category of smooth separated schemes of finite type over κ. There is a fully faithful tensor functor

$$\overline{M}\colon \mathbf{Chow}(\kappa, \Lambda) \longrightarrow \mathbf{DM}_{\mathrm{gm}}(\kappa, \Lambda) \tag{3.5}$$

such that $M \circ i = \overline{M} \circ h$, where $i\colon \mathbf{SmProj}(\kappa) \to \mathbf{Sm}(\kappa)$ is the inclusion functor. This fact is first proved by Voevodsky [22, Corollary 4.2.6] (see also [11, Proposition 20.1]) under the assumption that κ admits resolution of singularities. See [3, 6.7.3] for a proof over an arbitrary perfect base field. For each $i \in \mathbb{Z}$ and an object N of $\mathbf{DM}_{\mathrm{gm}}(\kappa, \Lambda)$, we put $\Lambda(i) := \overline{M}(\Lambda(i))[-2i]$ and $N(i) := N \otimes \Lambda(i)$, where $[\,\cdot\,]$ denotes the shift functor.

For later use, we record the blow-up formula:

Proposition 3.4. — *Let V be a smooth variety over κ and $Z \subset V$ a smooth closed subvariety of pure codimension c. Let $f\colon U \to V$ be the blow-up along Z. Then we have an isomorphism in $\mathbf{DM}_{\mathrm{gm}}(\kappa, \Lambda)$*

$$M(U) \simeq M(V) \oplus \left(\bigoplus_{i=1}^{c-1} M(Z)(i)[2i] \right).$$

If further V is projective over κ, then f also induces an isomorphism $h(U) \simeq h(V) \oplus \left(\bigoplus_{i=1}^{c-1} h(Z)(i)\right)$ in $\mathbf{Chow}(\kappa, \Lambda)$.

Proof. — The first statement is the blow-up formula [11, Corollary 15.13], and the second follows from (3.5). (The latter is also seen from [17, Theorems 2.5, 2.8].) □

If a finite group G acts (from left) on a motive M (that is, an object of either $\mathbf{Chow}(\kappa, \Lambda)$ or $\mathbf{DM}_{\mathrm{gm}}(\kappa, \Lambda)$), we write M^{χ} for the image of the projector e^{χ} from (2.8) for $\chi \in \widehat{G}$.

Proposition 3.5. — *Let G be a finite group and $\chi \in \widehat{G}$. Suppose that Λ is large enough to contain the values of χ. Let U, V be smooth varieties with G-action, and let $f\colon U \to V$ be a G-equivariant morphism. If one of the following conditions is satisfied, then f induces an isomorphism $M(U)^{\chi} \simeq M(V)^{\chi}$ in $\mathbf{DM}_{\mathrm{gm}}(k, \Lambda)$.*

(i) *$f\colon U \to V$ is an open immersion such that each irreducible component T of $V \setminus f(U)$ is smooth and G-stable with $M(T)^{\chi} = 0$.*
(ii) *$f\colon U \to V$ is a finite generically Galois morphism with* $\mathrm{Gal}(U/V) \subset G$ *such that $\chi|_{\mathrm{Gal}(U/V)} = 1$.*

If further U, V are projective over κ, then f also induces an isomorphism $h(U)^{\chi} \simeq h(V)^{\chi}$ in $\mathbf{Chow}(\kappa, \Lambda)$.

Proof. — The first statement for the case (i) and (ii) follows respectively from the localization sequence [11, Theorem 15.15] and Lemma 3.6 below. The second follows from (3.5). □

In the following lemma, we denote by $\mathbf{Cor}(X, Y)$ the group of finite correspondences for $X, Y \in \mathbf{Sm}(\kappa)$ (cf. [11, Definition 1.1]).

LEMMA 3.6. — *If $f\colon U \to V$ is a finite generically Galois morphism in $\mathbf{Sm}(\kappa)$, then we have equalities*

$$f_* \circ f^* = (\deg f) \cdot \mathrm{id}_V \ \text{ in } \mathbf{Cor}(V,V), \quad f^* \circ f_* = \sum_{g \in \mathrm{Gal}(U/V)} g_* \ \text{ in } \mathbf{Cor}(U,U).$$

Proof. — For an open dense immersion $j\colon V' \to V$, we have injections

$$\mathbf{Cor}(V',V') \xhookrightarrow[j\circ -]{} \mathbf{Cor}(V',V) \xhookleftarrow[-\circ j]{} \mathbf{Cor}(V,V).$$

Putting $U' := f^{-1}(V')$, we have

$$(f_* \circ f^*) \circ j = j \circ (f|_{U'*} \circ f|_{U'}^*), \quad \mathrm{id}_V \circ j = j \circ \mathrm{id}_{V'} \quad \text{in } \mathbf{Cor}(V',V),$$

which reduces the first statement to the case V is the spectrum of a field. A similar argument reduces the second statement to the same case. Then both statements are found in [11, Exercise 1.11]. □

3.3. Invertible objects

Let $\mathscr{C}$ be a Λ-linear rigid tensor pseudo-abelian category such that the endomorphism ring of the unit object is canonically isomorphic to Λ. (We shall apply the following discussion to $\mathscr{C} = \mathbf{Chow}(\kappa,\Lambda), \mathbf{DM}_{\mathrm{gm}}(\kappa,\Lambda)$.) Recall that an object L of $\mathscr{C}$ is called *invertible* if the evaluation map $L^\vee \otimes L \to \Lambda$ is an isomorphism, where $L^\vee$ denotes the (strong) dual of L. It then follows that $\mathrm{End}(L) \simeq \Lambda$ and hence L is indecomposable (that is, $\mathrm{End}(L)$ has no projectors other than $0,1$). We will use the following result of Krull–Schmidt type.

PROPOSITION 3.7. — *Let $L_1, \dots, L_n$ be invertible objects of $\mathscr{C}$, and put $M := L_1 \oplus \cdots \oplus L_n$. Let N_1, N_2 be objects of $\mathscr{C}$ such that there are isomorphisms $M \simeq N_1 \oplus N_2$ and $N_1 \simeq L_1 \oplus \cdots \oplus L_r$ for some $1 \leqslant r \leqslant n$. Then there is an isomorphism $N_2 \simeq L_{r+1} \oplus \cdots \oplus L_n$.*

Proof. — This follows from [1, Chapter 1, Theorem 3.6]. □

Artin motives provide basic examples of invertible motives, as seen in the following lemma. We shall see more examples in Propositions 4.5 and 4.11 below.

LEMMA 3.8. — *Let K/κ be a finite Galois extension and put $X = \mathrm{Spec}\, K$, $G = \mathrm{Gal}(X/\mathrm{Spec}\,\kappa)$. For any χ_1, $\chi_2 \in \widehat{G}$, there is an isomorphism*

$$h(X)^{\chi_1\chi_2} \simeq h(X)^{\chi_1} \otimes h(X)^{\chi_2}$$

in $\mathbf{Chow}(\kappa,\Lambda)$, where Λ is large enough to contain the values of χ_i's. In particular, $h(X)^\chi$ is invertible for any $\chi \in \widehat{G}$.

Proof. — Let $\Delta\colon X \to X \times X$ be the diagonal and consider the morphisms

$$\Delta_*\colon h(X) \longrightarrow h(X \times X), \quad \Delta^*\colon h(X \times X) \longrightarrow h(X).$$

Put $d := |G|$. We shall show that

$$\begin{aligned} d \cdot e^{\chi_1\chi_2} \circ \Delta^* \circ (e^{\chi_1} \times e^{\chi_2}) &: h(X)^{\chi_1} \otimes h(X)^{\chi_2} \longrightarrow h(X)^{\chi_1\chi_2}, \\ (e^{\chi_1} \times e^{\chi_2}) \circ \Delta_* \circ e^{\chi_1\chi_2} &: h(X)^{\chi_1\chi_2} \longrightarrow h(X)^{\chi_1} \otimes h(X)^{\chi_2} \end{aligned} \tag{3.6}$$

are isomorphisms mutually inverse to each other. Since

$$\Delta^* \circ (g_1, g_2)_* \circ \Delta_* \circ g_* = ((g_1^{-1}, g_2^{-1}) \circ \Delta)^* \circ (\Delta \circ g)_* = \begin{cases} (g_1 g)_* & (g_1 = g_2), \\ 0 & (g_1 \neq g_2), \end{cases}$$

for any $g, g_1, g_2 \in G$, we have

$$\begin{aligned} \Delta^* \circ (e^{\chi_1} \times e^{\chi_2}) \circ \Delta_* \circ e^{\chi_1\chi_2} &= \frac{1}{d^3} \sum_{g, g_1 = g_2 \in G} \overline{\chi_1\chi_2}(g)\overline{\chi_1}(g_1)\overline{\chi_2}(g_2)(g_1 g)_* \\ &= \frac{1}{d^2} \sum_{g' \in G} \overline{\chi_1\chi_2}(g') g'_* = \frac{1}{d} e^{\chi_1\chi_2}. \end{aligned}$$

Hence $e^{\chi_1\chi_2} \circ \Delta^* \circ (e^{\chi_1} \times e^{\chi_2}) \circ \Delta_* \circ e^{\chi_1\chi_2} = d^{-1} e^{\chi_1\chi_2}$.

On the other hand, for any $g_1, g_2, g, h_1, h_2 \in G$, the 0-cycle on X^4

$$\begin{aligned} \Gamma(g_1, g_2, g, h_1, h_2) &:= (h_1, h_2)_* \circ \Delta_* \circ g_* \circ \Delta^* \circ (g_1, g_2)_* \\ &= ((h_1, h_2) \circ \Delta \circ g)_* \circ ((g_1^{-1}, g_2^{-1}) \circ \Delta)^* \end{aligned}$$

is the image of $X \to X^4$; $x \mapsto (g_1^{-1}x, g_2^{-1}x, h_1 g x, h_2 g x)$, and we have

$$\begin{aligned} &(e^{\chi_1} \times e^{\chi_2}) \circ \Delta_* \circ e^{\chi_1\chi_2} \circ \Delta^* \circ (e^{\chi_1} \times e^{\chi_2}) \\ &\qquad = \frac{1}{d^5} \sum_{g_1, g_2, g, h_1, h_2 \in G} \overline{\chi_1}(h_1 g g_1)\overline{\chi_2}(h_2 g g_2)\Gamma(g_1, g_2, g, h_1, h_2). \end{aligned}$$

Note that $\Gamma(g_1, g_2, g, h_1, h_2)$ is contained in the graph $(h_1 g g_1, h_2 g g_2)_*$. When g_1, g_2, g, h_1, h_2 range over G with fixed values $g'_1 = h_1 g g_1$ and $g'_2 = h_2 g g_2$, the cycles $\Gamma(g_1, g_2, g, h_1, h_2)$ sum up to $d^2(g'_1, g'_2)_*$. Hence we obtain

$$(e^{\chi_1} \times e^{\chi_2}) \circ \Delta_* \circ e^{\chi_1\chi_2} \circ \Delta^* \circ (e^{\chi_1} \times e^{\chi_2}) = d^{-1}(e^{\chi_1} \times e^{\chi_2}).$$

This proves that (3.6) are isomorphisms.

The last statement follows by letting $\chi_1 = \chi$ and $\chi_2 = \bar{\chi}$, since $h(X)^1 \simeq \Lambda$ by Proposition 3.5 (ii). □

Let d be a positive integer such that $\mu_d \subset \kappa$ and assume that $\mathbb{Q}(\zeta_d) \subset \Lambda$. For $c \in \kappa^*$ and $\chi \in \widehat{\mu}_d$, let

$$\mathrm{Km}(c) = \mathrm{Km}_d(c)\colon\ \mathrm{Gal}(\kappa(c^{1/d})/\kappa) \longrightarrow \mu_d; \quad g \longmapsto g(c^{1/d})/c^{1/d}$$

be the Kummer character associated with c, and define

$$\Lambda\langle c\rangle^{\chi} = h(\operatorname{Spec}\kappa(c^{1/d}))^{\chi\circ\mathrm{Km}(c)}, \tag{3.7}$$

an invertible object in $\mathbf{Chow}(\kappa, \Lambda)$.

Lemma 3.9. —

(i) *For $c_1, c_2 \in \kappa^*$ and $\chi \in \widehat{\mu}_d$, we have $\Lambda\langle c_1c_2\rangle^{\chi} \simeq \Lambda\langle c_1\rangle^{\chi} \otimes \Lambda\langle c_2\rangle^{\chi}$.*
(ii) *For $c \in \kappa^*$ and $\chi_1, \chi_2 \in \widehat{\mu}_d$, we have $\Lambda\langle c\rangle^{\chi_1\chi_2} \simeq \Lambda\langle c\rangle^{\chi_1} \otimes \Lambda\langle c\rangle^{\chi_2}$.*
(iii) *If χ factors through $\chi' \in \widehat{\mu}_{d'}$ for some $d' \mid d$, i.e. $\chi(m) = \chi'(m^{d/d'})$ for any $m \in \mu_d$, then we have $\Lambda\langle c\rangle^{\chi} \simeq \Lambda\langle c\rangle^{\chi'}$.*

Proof. — These follow from Proposition 3.5 (ii) and Lemma 3.8. □

4. Artin–Schreier and Fermat motives

4.1. Artin–Schreier motives

In this subsection let κ be a finite field with q elements of characteristic p, and let d be a positive divisor of $q-1$. Let A_d° be the affine *Artin–Schreier curve* over κ defined by

$$x^q - x = y^d. \tag{4.1}$$

It admits an action of $\kappa \times \mu_d$ given by $(a, m).(x, y) = (x + a, my)$. There is no fixed point of $(a, m) \in \kappa \times \mu_d$ if $a \neq 0$, while μ_d fixes the points in $A_d^\circ(\kappa) = \{(x, 0) \mid x \in \kappa\}$. The projectivization $X^q - XZ^{q-1} = Y^dZ^{q-d}$ is non-singular at the unique point at infinity $[0 : 1 : 0]$ if and only if $d = q-1$. Let A_d be the projective smooth curve obtained by normalizing the singularity. It has a unique point at infinity, written as ∞, which we take as the distinguished point. The action of $\kappa \times \mu_d$ extends to A_d and fixes ∞.

If $d' \mid d$, we have a finite surjective morphism

$$A_d \longrightarrow A_{d'}, \qquad (x, y) \longmapsto (x, y^{d/d'}) \tag{4.2}$$

of degree d/d', compatible with the group actions via the homomorphism $\kappa\times\mu_d \to \kappa\times\mu_{d'}$, $(a, m) \mapsto (a, m^{d/d'})$. It is generically Galois with the Galois group $\operatorname{Ker}(\mu_d \twoheadrightarrow \mu_{d'}) = \mu_{d/d'}$. The genus of A_d is $(q-1)(d-1)/2$, which can be seen by the Riemann–Hurwitz formula for the covering $A_d \to A_1 \simeq \mathbb{P}^1$.

Let Λ be a field containing $\mathbb{Q}(\zeta_{pd})$. We have the decomposition in $\mathbf{Chow}(\kappa, \Lambda)$

$$h(A_d) = \bigoplus_{(\psi,\chi)\in\widehat{\kappa}\times\widehat{\mu}_d} h(A_d)^{(\psi,\chi)}, \quad h(A_d)^{(\psi,\chi)} := (A_d, e^{(\psi,\chi)}).$$

Here, $e^{(\psi,\chi)}$ means the algebraic correspondence induced by the group-ring element defined in (2.8). Define a projector $e_{\mathrm{prim}} \in \mathbb{Q}[\kappa \times \mu_d]$ (with coefficients in $\mathbb{Q}$) by

$$e_{\mathrm{prim}} = \sum_{(\psi,\chi)\in\widehat{\kappa}\times\widehat{\mu}_d, \psi\neq 1, \chi\neq 1} e^{(\psi,\chi)} = (1-e^1_\kappa)(1-e^1_{\mu_d}). \tag{4.3}$$

Note that, for any projectors $e, f \in \mathbb{Q}[G]$ where G is an abelian group, $1-e$ and ef are also projectors. Put

$$h(A_d)_{\mathrm{prim}} = (A_d, e_{\mathrm{prim}}) = \bigoplus_{\psi\neq 1,\chi\neq 1} h(A_d)^{(\psi,\chi)}.$$

Proposition 4.1. — *Suppose that $d' \mid d$ and that $\chi \in \widehat{\mu}_d$ factors through $\chi' \in \widehat{\mu}_{d'}$ (i.e. $\chi(m) = \chi'(m^{d/d'})$). Then we have $h(A_d)^{(\psi,\chi)} \simeq h(A_{d'})^{(\psi,\chi')}$.*

Proof. — This follows from Proposition 3.5 (ii) applied to (4.2). □

Proposition 4.2. — *Let $\psi \in \widehat{\kappa}$ and $\chi \in \widehat{\mu}_d$. We use the notations from* (3.1) *and* (3.2).

(i) *If $\psi = 1$ and $\chi = 1$, then $h(A_d)^{(1,1)} = h_0(A_d) \oplus h_2(A_d) = \Lambda \oplus \Lambda(1)$.*
(ii) *If $\psi \neq 1$ or $\chi \neq 1$, then $h(A_d)^{(\psi,\chi)} = h_1(A_d)^{(\psi,\chi)}$.*
(iii) *If only one of ψ, χ is trivial, then $h(A_d)^{(\psi,\chi)} = 0$.*

Proof. — Proposition 3.5 (ii) applied to $A_d \to A_d/(\kappa \times \mu_d) = \mathbb{P}^1$ yields $h(A_d)^{(1,1)} = h(\mathbb{P}^1) = \Lambda \oplus \Lambda(1)$, showing (i). This also implies $h_i(A_d) = h_i(A_d)^{(1,1)}$ $(i = 0, 2)$, from which we obtain (ii). To see (iii), it suffices to apply Proposition 3.5 (ii) to $A_d \to A_d/\mu_d = \mathbb{P}^1$ (resp. $A_d \to A_d/\kappa = \mathbb{P}^1$) when $\chi = 1$ (resp. $\psi = 1$). □

Lemma 4.3. — *Let*

$$(\,\cdot\,,\cdot\,)\colon \mathrm{CH}_1(A_{q-1} \times A_{q-1}) \times \mathrm{CH}_1(A_{q-1} \times A_{q-1}) \longrightarrow \mathbb{Z}$$

be the intersection number pairing. For $(a, m) \in \kappa \times \kappa^*$, let $\Gamma_{(a,m)} \in \mathrm{CH}_1(A_{q-1} \times A_{q-1})$ be the class of its graph and $\Delta = \Gamma_{(0,1)}$. Then,

$$(\Delta, \Gamma_{(a,m)}) = \begin{cases} 3q - q^2 & ((a, m) = (0, 1)), \\ q + 1 & (a = 0, m \neq 1), \\ q & (a \neq 0, m = 1), \\ 1 & (a \neq 0, m \neq 1). \end{cases}$$

Proof. — First, (Δ, Δ) equals the Euler–Poincaré characteristic of A_d, i.e. $2-(q-1)(q-2)$. Secondly if $m \neq 1$, then Δ and $\Gamma_{(0,m)}$ meet transversally at the $q+1$ points $\{(P, P) \mid P \in A_d(\kappa)\}$, where $A_d(\kappa) = \{(x, 0) \mid x \in \kappa\} \cup \{\infty\}$. Thirdly if $a \neq 0$, then $\Delta \cap \Gamma_{(a,1)} = \{(\infty, \infty)\}$. Since $v_\infty(x) = 1 - q$,

$v_\infty(y) = -q$, the completed local ring of A_d at ∞ is $\widehat{\mathscr{O}}_{A_d,\infty} = \kappa[\![t]\!]$ where $t = x/y$, and (a, m) maps t to $m^{-1}(t + ay^{-1})$. We have

$$\kappa[\![t,s]\!]/(t-s) \otimes_{\kappa[\![t,s]\!]} \kappa[\![t,s]\!]/(mt - s - ay^{-1}) = \kappa[\![t]\!]/((m-1)t - ay^{-1}).$$

Its length is q if $m = 1$, and is 1 otherwise since $(m-1)t - ay^{-1} \in t\kappa[\![t]\!]^\times$. The proof is complete. □

Lemma 4.4. — *Let S be a connected smooth projective surface over κ, and take*

$$e, e' \in \mathrm{CH}_1(S)_\Lambda = \mathrm{Hom}_{\mathbf{Chow}(\kappa,\Lambda)}(\Lambda(1), h(S)) = \mathrm{Hom}_{\mathbf{Chow}(\kappa,\Lambda)}(h(S), \Lambda(1)).$$

Suppose that the intersection number $n := (e, e')$ is not zero, and define

$$\alpha := \frac{1}{n}(e \times e') \in \mathrm{CH}_2(S \times S)_\Lambda = \mathrm{Hom}_{\mathbf{Chow}(\kappa,\Lambda)}(h(S), h(S)).$$

(i) *We have $\alpha^2 = \alpha$ in $\mathrm{Hom}_{\mathbf{Chow}(\kappa,\Lambda)}(h(S), h(S))$.*
(ii) *Set $M := (S, \alpha) \in \mathbf{Chow}(\kappa, \Lambda)$ and let $\overline{\alpha}\colon M \to h(S)$ and $\underline{\alpha}\colon h(S) \to M$ be the inclusion and projection (so that $\alpha = \overline{\alpha} \circ \underline{\alpha}$). Then*

$$\underline{\alpha} \circ e\colon \Lambda(1) \longrightarrow M \quad \textit{and} \quad \frac{1}{n} e' \circ \overline{\alpha}\colon M \longrightarrow \Lambda(1)$$

are isomorphisms mutually inverse to each other.

Proof. — This is shown by a straightforward computation using the definition of the composition of correspondences. □

The following are motivic analogues of (2.2) and (2.3), respectively (see also Remark 6.6 below).

Proposition 4.5. — *Suppose that $\psi \in \widehat{\kappa}$, $\psi \neq 1$ and $\chi \in \widehat{\mu}_d$, $\chi \neq 1$.*

(i) *There is an isomorphism*

$$h(A_d)^{(\psi,\chi)} \otimes h(A_d)^{(\overline{\psi},\overline{\chi})} \simeq \Lambda(1).$$

In particular, $h(A_d)^{(\psi,\chi)}$ is invertible.
(ii) *There is an isomorphism*

$$h(A_d)^{(\overline{\psi},\chi)} \simeq h(A_d)^{(\psi,\chi)} \otimes \Lambda\langle -1\rangle^\chi,$$

where $\Lambda\langle -1\rangle^\chi$ is from (3.7).

Proof.

(i). — By Proposition 4.1, we can assume that $d = q-1$. Regard $e^{(\psi,\chi)}$ and $e^{(\bar\psi,\bar\chi)}$ as elements of $\mathrm{CH}_1(A_{q-1}\times A_{q-1})_\Lambda$. Then the intersection number is computed using Lemma 4.3 as

$$\begin{aligned}
&(e^{(\psi,\chi)}, e^{(\bar\psi,\bar\chi)})\\
&= \frac{1}{q^2(q-1)^2}\sum_{a,a'\in\kappa,m,m'\in\kappa^*}\psi(a'-a)\chi(m'/m)(\Gamma_{(a,m)},\Gamma_{(a',m')})\\
&= \frac{1}{q^2(q-1)^2}\sum_{a,a'\in\kappa,m,m'\in\kappa^*}\psi(a'-a)\chi(m'/m)(\Delta,\Gamma_{(a'-a,m'/m)})\\
&= \frac{1}{q(q-1)}\sum_{a\in\kappa,m\in\kappa^*}\psi(a)\chi(m)(\Delta,\Gamma_{(a,m)})\\
&= \frac{1}{q(q-1)}\left((3q-q^2)+(q+1)\sum_{m\neq1}\chi(m)+q\sum_{a\neq0}\psi(a)+\sum_{a\neq0,m\neq1}\psi(a)\chi(m)\right)\\
&= \frac{1}{q(q-1)}\big((3q-q^2)-(q+1)-q+(-1)^2\big) = -1.
\end{aligned}$$

Now Lemma 4.4 completes the proof of (i).

(ii). — If $p = 2$, then the statement is trivial since $\bar\psi = \psi$ and $\Lambda\langle -1\rangle^\chi = \Lambda$. Suppose that p is odd. We may further suppose that $d = q-1$ by Lemma 3.9 (iii) and Proposition 4.1. Put $K = \kappa(\mu_{2(q-1)})$, $A_{q-1,K} = A_{q-1}\times_{\mathrm{Spec}\,\kappa}\mathrm{Spec}\,K$, and fix a primitive $2(q-1)$th root of unity $\zeta\in K$. Let f be a K-automorphism of $A_{q-1,K}$ defined by $f(x,y) = (-x,\zeta y)$. Then, we have

$$f\circ((a,m)\times \mathrm{id}_K) = ((-a,m)\times\mathrm{id}_K)\circ f, \quad f\circ((a,m)\times\sigma) = ((-a,-m)\times\sigma)\circ f$$

as κ-automorphisms of $A_{q-1,K}$, where σ is the generator of $\mathrm{Gal}(K/\kappa)$. It follows that

$$f_\#\left(e^{(\psi,\chi)}\otimes\frac{1+\sigma}{2}\right) = \left(e^{(\bar\psi,\chi)}\otimes\frac{1+\chi(-1)\sigma}{2}\right),$$

where $f_\#$ is from (3.3). We conclude

$$\begin{aligned}
h(A_{q-1})^{(\psi,\chi)} &\simeq \left(e^{(\psi,\chi)}\otimes\frac{1+\sigma}{2}\right)h(A_{q-1,K})\\
&\overset{f}{\simeq}\left(e^{(\bar\psi,\chi)}\otimes\frac{1+\chi(-1)\sigma}{2}\right)h(A_{q-1,K}) \simeq h(A_{q-1})^{(\bar\psi,\chi)}\otimes\Lambda\langle -1\rangle^\chi,
\end{aligned}$$

as desired. □

4.2. Fermat motives

In this subsection we let κ be an arbitrary perfect field and assume that d is a positive integer such that $\mu_d := \{m \in \kappa^* \mid m^d = 1\}$ has d elements. For each positive integer n and $c \in \kappa^*$, let $F_d^{(n)}\langle c\rangle \subset \mathbb{P}^n$ be the (twisted) projective Fermat hypersurface of degree d and dimension $n-1$ defined by

$$u_1^d + \cdots + u_n^d = cu_0^d.$$

When $c = 1$, we just write $F_d^{(n)}$ instead of $F_d^{(n)}\langle 1\rangle$. Let μ_d^n act on $F_d^{(n)}\langle c\rangle$ by

$$(m_1, \ldots, m_n)[u_0 : u_1 : \ldots : u_n] = [u_0 : m_1u_1 : \ldots : m_nu_n].$$

If $d' \mid d$, we have a finite surjective morphism

$$F_d^{(n)}\langle c\rangle \longrightarrow F_{d'}^{(n)}\langle c\rangle; \quad [u_0 : \ldots : u_n] \longmapsto [u_0^{d/d'} : \ldots : u_n^{d/d'}], \tag{4.4}$$

compatible with the group actions via the homomorphism $\mu_d^n \to \mu_{d'}^n$, $(m_i) \mapsto (m_i^{d/d'})$. It is generically Galois (étale over $u_0 \cdots u_n \neq 0$) with the Galois group $\operatorname{Ker}(\mu_d^n \to \mu_{d'}^n)$.

Let Λ be a field containing $\mathbb{Q}(\zeta_d)$. We have the decomposition in **Chow**(κ, Λ)

$$h(F_d^{(n)}\langle c\rangle) = \bigoplus_{\boldsymbol{\chi} \in \widehat{\mu_d^n}} h(F_d^{(n)}\langle c\rangle)^{\boldsymbol{\chi}}, \quad h(F_d^{(n)}\langle c\rangle)^{\boldsymbol{\chi}} := (F_d^{(n)}\langle c\rangle, e^{\boldsymbol{\chi}}).$$

Proposition 4.6. — *Suppose that $d' \mid d$ and $\boldsymbol{\chi} \in \widehat{\mu}_d^n$ factors through $\boldsymbol{\chi}' \in \widehat{\mu}_{d'}^n$ (i.e. $\boldsymbol{\chi}(m) = \boldsymbol{\chi}'(m^{d/d'})$). Then we have $h(F_d^{(n)}\langle c\rangle)^{\boldsymbol{\chi}} \simeq h(F_{d'}^{(n)}\langle c\rangle)^{\boldsymbol{\chi}'}$.*

Proof. — This follows from Proposition 3.5 (ii) applied to (4.4). □

Put

$$\mathfrak{X}_d^{(n)} = \left\{ \boldsymbol{\chi} = (\chi_1, \ldots, \chi_n) \in \widehat{\mu}_d^n \;\middle|\; \chi_1, \ldots, \chi_n, \prod_{i=1}^n \chi_i \neq 1 \right\},$$

and define a projector $e_{\mathrm{prim}} \in \mathbb{Q}[\mu_d^n]$ (with coefficients in $\mathbb{Q}$) by

$$e_{\mathrm{prim}} = \sum_{\boldsymbol{\chi} \in \mathfrak{X}_d^{(n)}} e^{\boldsymbol{\chi}} = \prod_{i=0}^n (1 - e^1_{\iota_i(\mu_d)}), \tag{4.5}$$

where $\iota_i \colon \mu_d \to \mu_d^n$ is the embedding of the ith factor if $i \neq 0$ and ι_0 is the diagonal embedding. Define

$$h(F_d^{(n)}\langle c\rangle)_{\mathrm{prim}} = (F_d^{(n)}\langle c\rangle, e_{\mathrm{prim}}) = \bigoplus_{\boldsymbol{\chi} \in \mathfrak{X}_d^{(n)}} h(F_d^{(n)}\langle c\rangle)^{\boldsymbol{\chi}}.$$

Proposition 4.7. — *For any $\boldsymbol{\chi} = (\chi_1, \dots, \chi_n) \in \widehat{\mu}_d^n$, we have an isomorphism*

$$h(F_d^{(n)}\langle c\rangle)^{\boldsymbol{\chi}} \simeq h(F_d^{(n)})^{\boldsymbol{\chi}} \otimes \Lambda\langle c\rangle^{\prod_{i=1}^n \chi_i}.$$

In particular, $h(F_d^{(1)}\langle c\rangle)^{\chi} \simeq \Lambda\langle c\rangle^{\chi}$ is invertible for any $\chi \in \widehat{\mu}_d$.

Proof. — Let α be a dth root of c, put $K = \kappa(\alpha)$, and consider the K-isomorphism

$$f\colon F_d^{(n)}\langle c\rangle \times \operatorname{Spec} K \longrightarrow F_d^{(n)} \times \operatorname{Spec} K$$

defined by $f([u_0 : u_1 : \cdots : u_n]) = [u_0 : \alpha^{-1}u_1 : \cdots : \alpha^{-1}u_n]$. Then f sends the graph of $(\xi, g) \in \mu_d^n \times \operatorname{Gal}(K/\kappa)$ to the graph of $(\iota_0(\operatorname{Km}(c)(g)^{-1})\xi, g)$. Hence the projector $e^{\boldsymbol{\chi}} \times e^1$ is mapped to

$$\begin{aligned}\frac{1}{d^n d}\sum_{\xi,g}\overline{\boldsymbol{\chi}}(\xi)(\iota_0(\operatorname{Km}(c)(g)^{-1})\xi, g) &= \frac{1}{d^n d}\sum_{\xi,g}\overline{\boldsymbol{\chi}}\big(\iota_0(\operatorname{Km}(c)(g))\xi\big)(\xi, g)\\ &= \frac{1}{d^n d}\sum_{\xi,g}\overline{\boldsymbol{\chi}}(\xi)\left(\prod_{i=1}^n \overline{\chi_i}(\operatorname{Km}(c)(g))\right)(\xi, g) = e^{\boldsymbol{\chi}} \times e^{\prod_{i=1}^n \chi_i\circ\operatorname{Km}(c)}.\end{aligned}$$

Hence the first assertion follows. Since $h(F_d^{(1)}) \simeq \Lambda[\mu_d]$, the Artin motive of the regular representation of μ_d, we have $h(F_d^{(1)})^{\chi} \simeq \Lambda$, and the second assertion follows. □

The following proposition will be generalized in Proposition 4.11 below.

Proposition 4.8. — *If $\boldsymbol{\chi} \in \mathfrak{X}_d^{(2)}$, there is an isomorphism*

$$h(F_d^{(2)}\langle c\rangle)^{\boldsymbol{\chi}} \otimes h(F_d^{(2)}\langle c\rangle)^{\overline{\boldsymbol{\chi}}} \simeq \Lambda(1).$$

In particular, $h(F_d^{(2)}\langle c\rangle)^{\boldsymbol{\chi}}$ is invertible.

Proof. — This can be proved similarly as Proposition 4.5 (i). By Lemma 3.9 and Proposition 4.7, we can assume that $c = 1$. For $(m_1, m_2) \in \mu_d^2$, the intersection numbers on $F_d^{(2)} \times F_d^{(2)}$ are computed as:

$$(\Delta, \Gamma_{(m_1,m_2)}) = \begin{cases} 2-(d-1)(d-2) & (m_1 = m_2 = 1),\\ d & (\text{if only one of } m_1,\, m_2,\, m_1m_2^{-1} \text{ is } 1),\\ 0 & (\text{otherwise}).\end{cases}$$

Then it follows as before that $(e^{(\chi_1,\chi_2)}, e^{(\overline{\chi_1},\overline{\chi_2})}) = -1$, and the proposition follows. □

To study $h(F_d^{(n)}\langle c\rangle)$ for general n, we use the inductive structure of Katsura–Shioda [18]. By Proposition 4.7, it suffices to consider the case

$c = 1$. Let $n \geqslant 2$ and $f_0\colon F_d^{(n)} \times F_d^{(2)} \dashrightarrow F_d^{(n+1)}$ be the rational map defined by

$$\begin{aligned}&([u_0 : \cdots : u_n], [v_0 : v_1 : v_2]) \\ &\qquad \longmapsto [w_0 : \cdots : w_{n+1}] = [u_0 v_0 : u_1 v_0 : \cdots : u_{n-1} v_0 : u_n v_1 : u_n v_2].\end{aligned}$$

Then f_0 is compatible with the actions of $\mu_d^n \times \mu_d^2$ and μ_d^{n+1} via the map

$$\begin{aligned}&\nu\colon \mu_d^n \times \mu_d^2 \longrightarrow \mu_d^{n+1}; \\ &\qquad ((\xi_1, \dots, \xi_n), (\eta_1, \eta_2)) \longmapsto (\xi_1, \dots, \xi_{n-1}, \xi_n\eta_1, \xi_n\eta_2). \qquad (4.6)\end{aligned}$$

We have the following commutative diagram:

$$\begin{array}{ccc} F_d^{(n,2)} & \xrightarrow{\ f\ } & F_d^{(n,2)}/H \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ F_d^{(n)} \times F_d^{(2)} & \overset{f_0}{\dashrightarrow} & F_d^{(n+1)}. \end{array}$$

Here,

- The morphism α is the blow-up along $Z = \{u_n = v_0 = 0\}$. We have

$$Z \simeq F_d^{(n-1)} \times F_d^{(1)}\langle -1 \rangle; \quad ([u_i]_{i=0}^n, [v_i]_{i=0}^2) \longmapsto ([u_i]_{i=0}^{n-1}, [v_2 : v_1]). \qquad (4.7)$$

  Since Z is smooth, $F_d^{(n,2)}$ is also smooth.
- The action of $\mu_d^n \times \mu_d^2$ on $F_d^{(n)} \times F_d^{(2)}$ respects Z and extends to $F_d^{(n,2)}$.
- The morphism β is the blow-up along $Z_1 \sqcup Z_2$, where Z_i are disjoint smooth closed subschemes of $F_d^{(n+1)}$ defined by

$$\begin{aligned} Z_1 &= \{w_0 = \cdots = w_{n-1} = 0\} \simeq F_d^{(1)}\langle -1 \rangle; && [w_i]_{i=0}^{n+1} \longmapsto [w_{n+1} : w_n], \\ Z_2 &= \{w_n = w_{n+1} = 0\} \simeq F_d^{(n-1)}; && [w_i]_{i=0}^{n+1} \longmapsto [w_i]_{i=0}^{n-1}. \end{aligned}$$

- The action of μ_d^{n+1} on $F_d^{(n+1)}$ respects Z_i's and extends to $F_d^{(n,2)}/H$.
- The morphism f is finite and generically Galois with the Galois group

$$H := \operatorname{Ker} \nu = \{((1, \dots, 1, \xi), (\xi^{-1}, \xi^{-1})) \mid \xi \in \mu_d\} \simeq \mu_d.$$

  Also, f is compatible with the group actions via ν.

Proposition 4.9. —

(i) *Let* $\boldsymbol{\chi} = (\chi_1, \ldots, \chi_{n+1}) \in \widehat{\mu}_d^{n+1}$. *Then*

$$h(Z_1)^{\boldsymbol{\chi}} \simeq \begin{cases} \Lambda\langle -1\rangle^{\chi_n} & \text{if } \chi_1 = \cdots = \chi_{n-1} = \chi_n\chi_{n+1} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$h(Z_2)^{\boldsymbol{\chi}} \simeq \begin{cases} h(F_d^{(n-1)})^{(\chi_1, \ldots, \chi_{n-1})} & \text{if } \chi_n = \chi_{n+1} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *Let* $\boldsymbol{\chi}' = ((\chi_1, \ldots, \chi_n), (\chi'_1, \chi'_2)) \in \widehat{\mu}_d^n \times \widehat{\mu}_d^2$. *Then*

$$h(F_d^{(n,2)}/H)^{\boldsymbol{\chi}'} \simeq \begin{cases} h(F_d^{(n,2)})^{\boldsymbol{\chi}'} & \text{if } \chi_n = \chi'_1\chi'_2, \\ 0 & \text{otherwise.} \end{cases}$$

$$h(Z)^{\boldsymbol{\chi}'} \simeq \begin{cases} h(F_d^{(n-1)})^{(\chi_1, \ldots, \chi_{n-1})} \otimes \Lambda\langle -1\rangle^{\chi'_1} & \text{if } \chi_n = \chi'_1\chi'_2 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

(i). — The stabilizer of Z_1 (resp. Z_2) in μ_d^{n+1} is $\mu_d^{n-1} \times \{(\zeta, \zeta) \mid \zeta \in \mu_d\}$ (resp. $\{1\}^{n-1} \times \mu_d^2$) and $\boldsymbol{\chi}$ is trivial on this group if and only if $\chi_1 = \cdots = \chi_{n-1} = \chi_n\chi_{n+1} = 1$ (resp. $\chi_n = \chi_{n+1} = 1$), and then we have respectively $h(Z_1)^{\boldsymbol{\chi}} \simeq \Lambda\langle -1\rangle^{\chi_n}$ by Proposition 4.7, and $h(Z_2)^{\boldsymbol{\chi}} \simeq h(F_d^{(n-1)})^{(\chi_1, \ldots, \chi_{n-1})}$. Otherwise, $h(Z_i)^{\boldsymbol{\chi}} = 0$.

(ii). — The first formula follows from Proposition 3.5 (ii) since $(\mu_d^n \times \mu_d^2)/H \simeq \mu_d^{n+1}$ and the pull-back of the characters of μ_d^{n+1} are of the form

$$((\chi_1, \ldots, \chi_{n-1}, \chi_n\chi_{n+1}), (\chi_n, \chi_{n+1})).$$

On the other hand, the stabilizer of Z in $\mu_d^n \times \mu_d^2$ is $\{1\}^{n-1} \times \mu_d \times \{(\eta, \eta) \mid \eta \in \mu_d\}$, and $\boldsymbol{\chi}'$ is trivial on this subgroup if and only if $\chi_n = 1$ and $\chi'_1\chi'_2 = 1$. If the condition is satisfied, we have $h(Z)^{\boldsymbol{\chi}'} \simeq h(F_d^{(n-1)})^{(\chi_1, \ldots, \chi_{n-1})} \otimes \Lambda\langle -1\rangle^{\chi'_1}$ by Proposition 4.7, and $h(Z)^{\boldsymbol{\chi}'} = 0$ otherwise. Hence the second formula follows. □

Proposition 4.10. — *Let* $\boldsymbol{\chi} = (\chi_1, \ldots, \chi_{n+1}) \in \widehat{\mu}_d^{n+1}$ *and put*

$$\boldsymbol{\chi}^{(n)} = (\chi_1, \ldots, \chi_{n-1}, \chi_n\chi_{n+1}),\ \boldsymbol{\chi}^{(2)} = (\chi_n, \chi_{n+1}),\ \boldsymbol{\chi}^{(n,2)} = (\boldsymbol{\chi}^{(n)}, \boldsymbol{\chi}^{(2)}).$$

Then we have an isomorphism

$$h\Big(F_d^{(n+1)}\Big)^{\boldsymbol{\chi}} \oplus \left(\bigoplus_{i=1}^{n-1} h(Z_1)^{\boldsymbol{\chi}}(i)\right) \oplus h(Z_2)^{\boldsymbol{\chi}}(1) \simeq \Big(h(F_d^{(n)})^{\boldsymbol{\chi}^{(n)}} \otimes h(F_d^{(2)})^{\boldsymbol{\chi}^{(2)}}\Big) \oplus h(Z)^{\boldsymbol{\chi}^{(n,2)}}(1).$$

Proof. — Compute $h(F_d^{(n,2)})^{\boldsymbol{\chi}^{(n,2)}}$ in two ways using Propositions 3.4 and 4.9. □

Proposition 4.11. — *Let $n \geqslant 2$, $\boldsymbol{\chi} = (\chi_1, \ldots, \chi_n) \in \widehat{\mu}_d^n$ and write $\mathbf{1} = (1, \ldots, 1)$.*

(i) $h(F_d^{(n)}\langle c\rangle)^{\mathbf{1}} \simeq \bigoplus_{i=0}^{n-1} \Lambda(i)$.

(ii) *If $\chi_{n-1}\chi_n \neq 1$, then*

$$h(F_d^{(n)}\langle c\rangle)^{\boldsymbol{\chi}} \simeq h(F_d^{(n-1)}\langle c\rangle)^{(\chi_1,\ldots,\chi_{n-2},\chi_{n-1}\chi_n)} \otimes h(F_d^{(2)})^{(\chi_{n-1},\chi_n)}.$$

(iii) *If $\chi_{n-1}\chi_n = 1$ and $\boldsymbol{\chi} \in \mathfrak{X}_d^{(n)}$, then*

$$h(F_d^{(n)}\langle c\rangle)^{\boldsymbol{\chi}} \simeq h(F_d^{(n-2)}\langle c\rangle)^{(\chi_1,\ldots,\chi_{n-2})} \otimes \Lambda\langle -1\rangle^{\chi_{n-1}}(1).$$

(iv) *If $\boldsymbol{\chi} \notin \mathfrak{X}_d^{(n)} \cup \{\mathbf{1}\}$, then $h(F_d^{(n)}\langle c\rangle)^{\boldsymbol{\chi}} = 0$.*

In particular, if $\boldsymbol{\chi} \in \mathfrak{X}_d^{(n)}$, there is an isomorphism

$$h(F_d^{(n)}\langle c\rangle)^{\boldsymbol{\chi}} \otimes h(F_d^{(n)}\langle c\rangle)^{\overline{\boldsymbol{\chi}}} \simeq \Lambda(n-1), \tag{4.8}$$

and hence $h(F_d^{(n)}\langle c\rangle)^{\boldsymbol{\chi}}$ is invertible.

Proof. — As before, we can assume that $c = 1$. The case $n = 2$ is proved in [15, Proposition 2.9]. Let $n \geqslant 2$ and we prove the statements for $\boldsymbol{\chi} = (\chi_1, \ldots, \chi_{n+1}) \in \widehat{\mu}_d^{n+1}$ by induction on n. Put $\boldsymbol{\chi}^{(n-1)} = (\chi_1, \ldots, \chi_{n-1})$. We also use the results and notations of Propositions 4.9 and 4.10. It should be possible to trace the isomorphisms arising from these propositions, but we avoid it by resorting to the Krull–Schmidt principle, i.e. Proposition 3.7.

(1). — If $\boldsymbol{\chi} = \mathbf{1}$, then by the induction hypothesis, $h(Z_1)^{\boldsymbol{\chi}} \simeq \Lambda$, $h(Z_2)^{\boldsymbol{\chi}} \simeq \bigoplus_{i=0}^{n-2} \Lambda(i) \simeq h(Z)^{\boldsymbol{\chi}^{(n,2)}}$, and $h(F_d^{(n)})^{\boldsymbol{\chi}^{(n)}} \otimes h(F_d^{(2)})^{\boldsymbol{\chi}^{(2)}} \simeq \big(\bigoplus_{i=0}^{n-1} \Lambda(i)\big) \otimes \big(\bigoplus_{i=0}^{1} \Lambda(i)\big)$. Hence (i) follows.

From now on, we suppose $\boldsymbol{\chi} \neq \mathbf{1}$.

(2). — If $\chi_{n-1}\chi_n \neq 1$, then $h(Z_1)^{\boldsymbol{\chi}} = h(Z_2)^{\boldsymbol{\chi}} = h(Z)^{\boldsymbol{\chi}^{(n,2)}} = 0$, and (ii) follows.

(2-1). — If moreover $\boldsymbol{\chi} \notin \mathfrak{X}_d^{(n+1)}$, then we have either $\boldsymbol{\chi}^{(n)} \notin \mathfrak{X}_d^{(n)} \cup \{\mathbf{1}\}$ or $\boldsymbol{\chi}^{(2)} \notin \mathfrak{X}_d^{(2)} \cup \{\mathbf{1}\}$. Hence $h(F_d^{(n+1)})^{\boldsymbol{\chi}} = 0$ by (ii) and the induction hypothesis.

(3). — Suppose $\chi_n\chi_{n+1} = 1$, so that $\boldsymbol{\chi}^{(n)} \notin \mathfrak{X}_d^{(n)}$.

(3-1). — If $\chi_n = \chi_{n+1} = 1$, then $\boldsymbol{\chi} \notin \mathfrak{X}_d^{(n+1)}$ and $\boldsymbol{\chi}^{(n)} \neq \mathbf{1}$. We have $h(Z_1)^{\boldsymbol{\chi}} = 0$, $h(Z_2)^{\boldsymbol{\chi}} \simeq (F_d^{(n-1)})^{\boldsymbol{\chi}^{(n-1)}} \simeq h(Z)^{\boldsymbol{\chi}^{(n,2)}}$, and $h(F_d^{(n)})^{\boldsymbol{\chi}^{(n)}} = 0$ by the induction hypothesis, hence $h(F_d^{(n+1)})^{\boldsymbol{\chi}} = 0$.

(3-2). — If $\chi_n \neq 1$ (so $\chi_{n+1} \neq 1$), then $h(Z_2)^{\boldsymbol{\chi}} = 0$, and $h(F_d^{(2)})^{\boldsymbol{\chi}^{(2)}} = 0$ by the induction hypothesis.

(3-2-1). — If $\boldsymbol{\chi}^{(n-1)} = \mathbf{1}$ (so $\boldsymbol{\chi} \notin \mathfrak{X}_d^{(n+1)}$), then $h(Z_1)^{\boldsymbol{\chi}} \simeq \Lambda\langle -1\rangle^{\chi_n}$ and $h(Z)^{\boldsymbol{\chi}^{(n,2)}} \simeq \bigoplus_{i=0}^{n-2} \Lambda\langle -1\rangle^{\chi_n}(i)$ by the induction hypothesis, which implies $h(F_d^{(n+1)})^{\boldsymbol{\chi}} = 0$.

(3-2-2). — If $\boldsymbol{\chi}^{(n-1)} \neq \mathbf{1}$, then $h(Z_1)^{\boldsymbol{\chi}} = 0$. Hence (iii) follows. Moreover if $\boldsymbol{\chi} \notin \mathfrak{X}_d^{(n+1)}$ (so $n \geqslant 3$), then $\boldsymbol{\chi}^{(n-1)} \notin \mathfrak{X}_d^{(n-1)}$ and

$$h(F_d^{(n+1)})^{\boldsymbol{\chi}} \simeq h(Z)^{\boldsymbol{\chi}^{(n,2)}}(1) \simeq h(F_d^{(n-1)})^{\boldsymbol{\chi}^{(n-1)}} \otimes \Lambda\langle -1\rangle^{\chi_n}(1) = 0$$

by the induction hypothesis. This finishes the proof of (iv). □

Remark 4.12. — The relations (ii), (iii) are motivic analogues of the functional equations

$$B(s_1, \dots, s_{n-1}, s_n) = B(s_1, \dots, s_{n-2}, s_{n-1} + s_n) B(s_{n-1}, s_n),$$
$$B(s_1, \dots, s_{n-1}, 1 - s_{n-1}) = \frac{B(s_1, \dots, s_{n-2})}{s_1 + \cdots + s_{n-2}} \cdot \frac{\pi}{\sin \pi s_{n-1}},$$

which follows by Remark 2.1.

Corollary 4.13. — *We have*

$$h(F_d^{(n)}\langle c\rangle) \simeq h(F_d^{(n)}\langle c\rangle)_{\mathrm{prim}} \oplus \bigoplus_{i=0}^{n-1} \mathbb{Q}(i)$$

in $\mathbf{Chow}(\kappa, \mathbb{Q})$, *where* $(\,\cdot\,)_{\mathrm{prim}}$ *denotes the direct factor defined by* (4.5).

Proof. — This follows from Proposition 4.11 and Lemma 3.3. □

Remark 4.14. — Let L be the class of the hyperplane section defined by the embedding $\iota\colon F_d^{(n)}\langle c\rangle \hookrightarrow \mathbb{P}^n$, and define the objects of $\mathbf{Chow}(\kappa, \mathbb{Q})$ for $i = 0, \dots, 2n-2$ by

$$h_i(F_d^{(n)}\langle c\rangle) := \begin{cases} (F_d^{(n)}\langle c\rangle, \pi_i) & \text{if } i \neq n-1, \\ (F_d^{(n)}\langle c\rangle, e_{\mathrm{prim}}) & \text{if } i = n-1 \text{ and } n \text{ is even}, \\ (F_d^{(n)}\langle c\rangle, \pi_{n-1} + e_{\mathrm{prim}}) & \text{if } i = n-1 \text{ and } n \text{ is odd}, \end{cases}$$

where

$$\pi_i := \frac{1}{d}[L^i \times L^{n-i-1}] \in \mathrm{CH}_{n-1}(F_d^{(n)}\langle c\rangle \times F_d^{(n)}\langle c\rangle)_{\mathbb{Q}}$$

is a projector of $h(F_d^{(n)}\langle c\rangle)$. We have $(h(F_d^{(n)}\langle c\rangle), \pi_i) \subset h(F_d^{(n)}\langle c\rangle)^{\mathbf{1}}$ since L is fixed by the μ_d^n-action. It follows from Proposition 4.11 that $h(F_d^{(n)}\langle c\rangle) = \bigoplus_{i=0}^{2n-2} h_i(F_d^{(n)}\langle c\rangle)$ is a Chow–Künneth decomposition of $F_d^{(n)}\langle c\rangle$ ([13, Definition 6.1.1]). Note also that, if a Weil cohomology theory H^* satisfies the hard Lefschetz, e_{prim} acts on $H^*(F_d^{(n)}\langle c\rangle)$ as the projection to the primitive part $\mathrm{Ker}(H^{n-1}(F_d^{(n)}\langle c\rangle) \xrightarrow{-\cup L} H^{n+1}(F_d^{(n)}\langle c\rangle)(1))$, as is seen from a formula in [8, Proposition 1.4.7 (i)].

Remark 4.15. — With no difficulty, we can generalize the results in this subsection to the general diagonal hypersurface $c_1u_1^d + \cdots + c_nu_n^d = u_0^d$ ($c_i \in \kappa^*$). We restricted ourselves, however, to the situation as above, which will be needed in Section 7.2.

5. Proof of Theorem 1.1 (i)

In this section we assume κ is a finite field of characteristic p and of order q, and d is a positive divisor of $q-1$. We will complete the proof of Theorem 1.1 (i).

5.1. Reduction to the case $n=2$

We proceed by induction on n. The case $n=1$ follows immediately from Proposition 4.7. The case $n=2$ will be proved in the next subsection. Let $n \geqslant 3$. First assume that $\chi_{n-1}\chi_n \neq 1$, Then we have by the case $n=2$ and the induction hypothesis

$$\begin{aligned}\bigotimes_{i=1}^{n} h(A_d)^{\chi_i} &\simeq \left(\bigotimes_{i=1}^{n-2} h(A_d)^{\chi_i}\right) \otimes h(A_d)^{\chi_{n-1}\chi_n} \otimes h(F_d^{(2)})^{(\chi_{n-1},\chi_n)} \\ &\simeq h(A_d)^{\prod_{i=1}^{n}\chi_i} \otimes h(F_d^{(n-1)})^{(\chi_1,\dots,\chi_{n-2},\chi_{n-1}\chi_n)} \otimes h(F_d^{(2)})^{(\chi_{n-1},\chi_n)}.\end{aligned}$$

By Proposition 4.11 (ii), the formula follows. Secondly, assume that $\chi_{n-1}\chi_n = 1$. Then we have by the induction hypothesis and Proposition 4.5

$$\bigotimes_{i=1}^{n} h(A_d)^{\chi_i} \simeq \left(h(A_d)^{\prod_{i=1}^{n-2}\chi_i} \otimes h(F_d^{(n-2)})^{(\chi_1,\dots,\chi_{n-2})}\right) \otimes \Lambda\langle -1\rangle^{\chi_{n-1}}(1).$$

By Proposition 4.11 (iii), the theorem follows.

5.2. Proof of the case $n=2$

By Propositions 4.1 and 4.6, we can suppose $d=q-1$, so that $\mu_d = \kappa^*$. Here we need $\mathbf{DM}_{\mathrm{gm}}(\kappa,\Lambda)$ from Section 3.2 to treat open varieties. We just write $A = A_{q-1}$, $F = F_{q-1}^{(2)}$. Let $A^\circ \subset A$ (resp. $F^\circ \subset F$) be the affine open subscheme defined in (4.1) (resp. by $u_0 \neq 0$). Write $A^\circ F = A^\circ \times F$, $A^\circ F^\circ = A^\circ \times F^\circ$. Define a closed subscheme $\Gamma \subset (A^\circ)^2 \times A^\circ F$ by

$$x_1 + x_2 = x, \quad u_0 y_1 = u_1 y, \quad u_0 y_2 = u_2 y.$$

Here, the coordinates of the ith factor of $(A^\circ)^2$ are given by (x_i, y_i) subject to the relation $x_i^q - x_i = y_i^{q-1}$. Those of the first and second factors of $A^\circ F$ are (x, y) and $[u_0 : u_1 : u_2]$, which are subject to the relations $x^q - x = y^{q-1}$ and $u_1^{q-1} + u_2^{q-1} = u_0^{q-1}$, respectively. Let $\mathrm{pr}_1 \colon \Gamma \to (A^\circ)^2$ and $\mathrm{pr}_2 \colon \Gamma \to A^\circ F$ be the projections. Put

$$\Gamma_1 = \mathrm{pr}_1^{-1}((A^\circ)^2 \setminus Z), \quad \Gamma_2 = \mathrm{pr}_2^{-1}(A^\circ F^\circ),$$

where

$$Z := \bigsqcup_{a \in \kappa} \big\{ (x_i, y_i)_i \in (A^\circ)^2 \,\big|\, x_1 + x_2 = a \big\} \subset (A^\circ)^2.$$

Note that $\Gamma_1 \subset \Gamma$ is defined by $y \neq 0$. Since $u_0 = 0$ implies $u_1, u_2 \neq 0$, hence $y = 0$, we have $\Gamma_1 \subset \Gamma_2$.

Put $G_1 = (\kappa \times \kappa^*)^2$, $G_2 = (\kappa \times \kappa^*) \times (\kappa^*)^2$ and $G = (\kappa \times \kappa^*)^2 \times \kappa^*$. Let $\pi_1 \colon G \to G_1$ be the first projection and define $\pi_2 \colon G \to G_2$ by

$$\pi_2((a_i, m_i)_i, m) = \big((a_1 + a_2, m), (m_i m^{-1})_i\big).$$

Then G acts on $(A^\circ)^2 \times A^\circ F$ via $\pi_1 \times \pi_2 \colon G \to G_1 \times G_2$ and it respects Γ, Γ_1 and Γ_2. Besides, Z is stable under the action of G_1 on A^2.

Lemma 5.1. —

(i) *The singular locus of* Γ *is given by* $y_1 = y_2 = y = u_0 = 0$ *(geometrically* $q^2(q-1)$ *points). In particular,* Γ_1 *and* Γ_2 *are non-singular.*
(ii) Γ_1 *is finite Galois over* $(A^\circ)^2 \setminus Z$ *with the Galois group* $\kappa^* \subset G$ *(embedded as the last component).*
(iii) Γ_2 *is finite Galois over* $A^\circ F^\circ$ *with the Galois group* $\kappa \subset G$ *(embedded as the image of* $\kappa \xrightarrow{(\mathrm{id}, -\mathrm{id})} \kappa^2 \subset G$*).*

Proof.

(i). — This follows from a straightforward computation of the Jacobian matrix.

(ii). — We have $(A^\circ)^2 \setminus Z = \mathrm{Spec}\, R$, where

$$R = \kappa\big[x_i, y_i, ((x_1 + x_2)^q - (x_1 + x_2))^{-1}\big] / \big(x_i^q - x_i - y_i^{q-1}\big)$$

and $\Gamma_1 = \mathrm{Spec}\, R[y]/((x_1 + x_2)^q - (x_1 + x_2) - y^{q-1})$. Note that on Γ_1, $x = x_1 + x_2$ and $u_i/u_0 = y_i/y$. Therefore, $\Gamma_1 \to (A^\circ)^2 \setminus Z$ is the base change by $\mathrm{Spec}\, R \to \mathrm{Spec}\, \kappa[s, s^{-1}]$ ($s = (x_1 + x_2)^q - (x_1 + x_2)$) of $\mathrm{Spec}\, \kappa[s, s^{-1}, y]/(s - y^{q-1}) \to \mathrm{Spec}\, \kappa[s, s^{-1}]$, which has the desired property, and the assertion follows.

(iii). — We have $A^\circ F^\circ = \operatorname{Spec} R$, $R = \kappa[x, y, t_1, t_2]/(x^q - x - y^{q-1}, t_1^{q-1} + t_2^{q-1} - 1)$ $(t_i = u_i/u_0)$, and $\Gamma_2 = \operatorname{Spec} R[x_1]/(x_1^q - x_1 - (t_1 y)^{q-1})$. Note that on Γ_2, we have $y_i = t_i y$, $x_2 = x - x_1$ and $x_2^q - x_2 = (x - x_1)^q - (x - x_1) = y^{q-1} - (t_1 y)^{q-1} = y_2^{q-1}$. Therefore, $\Gamma_2 \to A^\circ F^\circ$ is the base change by $\operatorname{Spec} R \to \operatorname{Spec} \kappa[s]$ $(s = (t_1 y)^{q-1})$ of $\operatorname{Spec} \kappa[s, x_1]/(x_1^q - x_1 - s) \to \operatorname{Spec} \kappa[s]$, which has the desired property, and the assertion follows. $\square$

Proposition 5.2. — *Let $\psi \in \widehat{\kappa} \setminus \{1\}$, $\chi_1, \chi_2 \in \widehat{\kappa^*}$ and put $\boldsymbol{\chi} = ((\psi, \chi_i)_i, 1) \in \widehat{G} = (\widehat{\kappa} \times \widehat{\kappa^*})^2 \times \widehat{\kappa^*}$.*

(i) *If one of χ_1, χ_2 is non-trivial, then $M(\Gamma_1)^{\boldsymbol{\chi}} \simeq M(\Gamma_2)^{\boldsymbol{\chi}}$.*
(ii) *If none of $\chi_1, \chi_2, \chi_1\chi_2$ is trivial, then $M(\Gamma_1)^{\boldsymbol{\chi}} \simeq M(A)^{(\psi,\chi_1)} \otimes M(A)^{(\psi,\chi_2)}$.*
(iii) *If $\chi_1\chi_2 \neq 1$, then $M(\Gamma_2)^{\boldsymbol{\chi}} \simeq M(A)^{(\psi,\chi_1\chi_2)} \otimes M(F)^{(\chi_1,\chi_2)}$.*

Proof.

(i). — The complement $\Gamma_2 \setminus \Gamma_1$ is given by $y = y_1 = y_2 = 0$, on which $(\kappa^*)^3 \subset G$ acts trivially and we have $M(\Gamma_2 \setminus \Gamma_1)^{\boldsymbol{\chi}} = 0$. The result follows by Proposition 3.5 (i).

(ii). — Define $A^* \subset A^\circ$ by $y \neq 0$. Then $Z^* := Z \cap (A^*)^2$ is smooth over κ. First, we have by Proposition 3.5 (ii) and Lemma 5.1 (ii)

$$M(\Gamma_1)^{\boldsymbol{\chi}} \simeq M((A^\circ)^2 \setminus Z)^{(\psi,\chi_i)_i} = M((A^*)^2 \setminus Z^*)^{(\psi,\chi_i)_i}.$$

We will prove $M(Z^*)^{(\psi,\chi_i)_i} = 0$. Then it follows by Proposition 3.5 (i) that

$$M((A^*)^2 \setminus Z^*)^{(\psi,\chi_i)_i} \simeq M((A^*)^2)^{(\psi,\chi_i)_i} = M(A^*)^{(\psi,\chi_1)} \otimes M(A^*)^{(\psi,\chi_2)}.$$

Since $A \setminus A^*$ is fixed by the κ^*-action and $\chi_i \neq 1$, we have $M(A^*)^{(\psi,\chi_i)} \simeq M(A)^{(\psi,\chi_i)}$ by Proposition 3.5 (i), and the assertion follows.

Now we prove $M(Z^*)^{(\psi,\chi_i)_i} = 0$. Note that $(x_1 + x_2)^q = x_1 + x_2$ is equivalent to $y_1^{q-1} + y_2^{q-1} = 0$. Let K be a quadratic extension of κ and write X_K for $X \times \operatorname{Spec} K$. It suffices to prove $M(Z_K^*)^{(\psi,\chi_i)_i} = 0$ by Proposition 3.5 (ii). Choose $\zeta \in K$ such that $\zeta^{q-1} = -1$. For $(a, m) \in \kappa \times \kappa^*$, let $f_{a,m}$ be the K-automorphism of A_K^* defined by $f_{a,m}(x, y) = (a - x, m\zeta y)$, and $Z_{a,m}^* \subset (A^*)_K^2$ be its graph, regarded as a κ-scheme. Then, $Z_K^* = \coprod_{(a,m)\in\kappa\times\kappa^*} Z_{a,m}^*$ and

$$M(Z_K^*)^{(\psi,\chi_i)_i} \simeq \left(\bigoplus_{(a,m)} M(Z_{a,m}^*) \right)^{(\psi,\chi_i)_i} \quad \text{in } \mathbf{DM}_{\mathrm{gm}}(\kappa, \Lambda). \tag{5.1}$$

By the isomorphism

$$A_K^* \simeq Z_{a,m}^*; \quad (x, y) \longmapsto ((x, y), (f_{a,m}(x, y))),$$

we have $\bigoplus_{(a,m)} M(A_K^*) \simeq M(Z_K^*)$. The action of $\kappa \times \kappa^*$ on A_K^* and that of G_1 on Z_K^* are compatible under the isomorphism as above and the homomorphism

$$\delta\colon \kappa \times \kappa^* \longrightarrow G_1; \quad (a', m') \longmapsto ((a', m'), (-a', m')).$$

In particular, the action of $\mathrm{Im}(\delta) \subset G_1$ preserves the components $Z_{a,m}^*$ of Z_K^*. Since $((\psi, \chi_i)_i) \circ \delta = (1, \chi_1\chi_2)$ holds in $\widehat{\kappa} \times \widehat{\kappa^*}$, the right hand side of (5.1) is isomorphic to a subobject of $\bigoplus_{(a,m)} M(A_K^*)^{(1,\chi_1\chi_2)}$. Since $\chi_1\chi_2 \neq 1$, we have $M(A_K^*)^{(1,\chi_1\chi_2)} \simeq M(A_K)^{(1,\chi_1\chi_2)}$ as above, and this is trivial by Proposition 4.2 (iii) and (3.5). It follows that $M(Z_K^*)^{(\psi,\chi_i)_i} = 0$, as desired.

(iii). — We have

$$M(\Gamma_2)^{\boldsymbol{\chi}} \simeq M(A^\circ F^\circ)^{((\psi,\chi_1\chi_2),(\chi_1,\chi_2))} = M(A^\circ)^{(\psi,\chi_1\chi_2)} \otimes M(F^\circ)^{(\chi_1,\chi_2)}$$

by Proposition 3.5 (ii) and Lemma 5.1 (iii). Since κ^* (resp. the diagonal $\kappa^* \subset (\kappa^*)^2$) acts trivially on $A \setminus A^\circ$ (resp. $F \setminus F^\circ$) and $\chi_1\chi_2 \neq 1$, we have $M(A^\circ)^{(\psi,\chi_1\chi_2)} \simeq M(A)^{(\psi,\chi_1\chi_2)}$ (resp. $M(F^\circ)^{(\chi_1,\chi_2)} \simeq M(F)^{(\chi_1,\chi_2)}$) by Proposition 3.5 (i), and the result follows. □

Proof of Theorem 1.1 (i). — We are already reduced to the case $n = 2$ in Section 5.1. By the proposition, we have

$$\bigotimes_{i=1}^{2} M(A)^{(\psi,\chi_i)} \simeq M(A)^{(\psi,\chi_1\chi_2)} \otimes M(F)^{(\chi_1,\chi_2)}$$

for $(\chi_1, \chi_2) \in \mathfrak{X}_{q-1}^{(2)}$. The theorem follows from the full faithfulness of (3.5). □

6. Frobenius endomorphisms

We continue to assume κ is a finite field of characteristic p and of order q, and d is a positive divisor of $q-1$. The following extends slightly Coleman's result [5, Theorem A]. He only considers the Artin–Schreier curves of the form $x^p - x = y^d$ over κ, so that only Gauss sums with additive characters factoring through the trace $\mathrm{Tr}_{\kappa/\mathbb{F}_p}$ are involved.

Proposition 6.1. —

(i) *If $\infty \in A_d$ denotes the unique point at infinity, then we have in* $\mathrm{CH}_1(A_d \times A_d)$

$$[\mathrm{Fr}_{A_d}] = [g_d] + q[A_d \times \infty] + (2q-1)[\infty \times A_d].$$

(ii) *If we put* $Z_d^0 = \{u_0 = 0\} \subset F_d^{(2)}$*, then we have in* $\mathrm{CH}_1(F_d^{(2)} \times F_d^{(2)})$

$$[\mathrm{Fr}_{F_d^{(2)}}] = [j_d^{(2)}] + \frac{q-1}{d}\Big([F_d^{(2)} \times Z_d^0] + 2[Z_d^0 \times F_d^{(2)}]\Big).$$

Proof.

(i). — Define $Z = \{y = 0\} \subset A_d$ so that $\mathrm{div}(y) = Z - q\cdot\infty$. In particular, we have $[Z] = q[\infty]$ in $\mathrm{CH}_0(A_d)$. Let $((x,y),(x',y'))$ be the coordinates of $A_d \times A_d$, and define a function on $A_d \times A_d$ as

$$f = (x'-x)^{\frac{q-1}{d}} - \frac{y'}{y}.$$

By abuse of notation, we write the graph of a morphism by the same letter. We claim that

$$\mathrm{div}(f) = \mathrm{Fr}_{A_d} + \sum_{m\in\kappa^*}(m, m^{\frac{q-1}{d}}) - q(A_d \times \infty) - (q-1)(\infty \times A_d) - (Z \times A_d), \tag{6.1}$$

from which the statement follows.

Write D for the left hand side minus the right hand side of (6.1). Using $\mathrm{div}(x) = d((0,0) - \infty)$ and $\mathrm{div}(y) = Z - q\cdot\infty$, it is straightforward to see that D is effective. To show $D = 0$, we consider the map

$$\Pi\colon\ \mathrm{Div}(A_d \times A_d) \xrightarrow{\pi_*} \mathrm{Div}(\mathbb{P}^1 \times \mathbb{P}^1) \twoheadrightarrow \mathrm{CH}_1(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} \times \mathbb{Z},$$

where the first map is the push-forward along the self-product $\pi\colon A_d \times A_d \to A_1 \times A_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ of (4.2). Observe that an effective divisor E on $A_d \times A_d$ is trivial if $\Pi(E) = (0,0)$, since we have a strict inequality $\Pi(C) > (0,0)$ for any integral curve C on $A_d \times A_d$. On the other hand, we compute

$$\Pi(\mathrm{Fr}_{A_d}) = (d, qd), \qquad \Pi(g) = (d,d)\ (g \in \kappa \times \kappa^*),$$
$$\Pi(A_d \times \infty) = (d, 0), \quad \Pi(P \times A_d) = (0, d)\ (P \in Z \cup \{\infty\}).$$

It follows that $\Pi(D) = (0,0)$. This completes the proof of (i).

(ii). — Put $Z_d^i = \{u_i = 0\} \subset F_d^{(2)}$ so that $[Z_d^1] = [Z_d^2] = [Z_d^0]$ in $\mathrm{CH}_1(F_d^{(2)})$ since $\mathrm{div}(u_i) = Z_d^i - Z_d^0$.

First, let $d = q-1$ and let $([u_i]_{i=0}^2, [u_i']_{i=0}^2)$ be the coordinates of $F_{q-1}^{(2)} \times F_{q-1}^{(2)}$. Define a function on $F_{q-1}^{(2)} \times F_{q-1}^{(2)}$ as

$$f = 1 - \frac{u_0u_1'}{u_0'u_1} - \frac{u_0u_2'}{u_0'u_2}.$$

By a similar argument as (i), one verifies the equality of divisors

$$\mathrm{div}(f) = \mathrm{Fr}_{F_{q-1}^{(2)}} - j_{q-1}^{(2)} - F_{q-1}^{(2)} \times Z_{q-1}^0 - Z_{q-1}^1 \times F_{q-1}^{(2)} - Z_{q-1}^2 \times F_{q-1}^{(2)}, \tag{6.2}$$

from which the statement for $d = q-1$ follows.

For general d, we look at the image of the both sides of (6.2) under the push-forward along $F_{q-1}^{(2)} \times F_{q-1}^{(2)} \to F_d^{(2)} \times F_d^{(2)}$. For the left hand side this is the divisor of the norm of f in $\kappa(F_d^{(2)} \times F_d^{(2)})$ (hence vanishes in $\mathrm{CH}_1(F_d^{(2)} \times F_d^{(2)})$), while for the right this is

$$N^2\Big(\mathrm{Fr}_{F_d^{(2)}} - j_d^{(2)}\Big) - N^3\Big(F_d^{(2)} \times Z_d^0 + Z_d^1 \times F_d^{(2)} + Z_d^2 \times F_d^{(2)}\Big),$$

where $N := (q-1)/d$. We are done. $\square$

Corollary 6.2. — *We have equalities of endomorphisms of $h(A_d)$ (resp. $h(F_d^{(2)})$)*

$$g_d \cdot e_{\mathrm{prim}} = \mathrm{Fr}_{A_d} \cdot e_{\mathrm{prim}} \quad (\textit{resp.} \quad j_d^{(2)} \cdot e_{\mathrm{prim}} = \mathrm{Fr}_{F_d^{(2)}} \cdot e_{\mathrm{prim}})$$

in $\mathbf{Chow}(\kappa, \mathbb{Q})$.

Proof. — Let us first prove (i). We have $[A_d \times \infty] \cdot e^{(\psi,\chi)} = [\infty \times A_d] \cdot e^{(\psi,\chi)} = 0$ if none of $\psi \in \widehat{\kappa}, \chi \in \widehat{\kappa^*}$ is trivial. Hence the statement follows from the proposition by Lemma 3.3. The proof of (ii) is similar. $\square$

To prove Theorem 1.2, we prepare computations in group rings. Let

$$\nu\colon \mathbb{Q}[\mu_d^n \times \mu_d^2] \longrightarrow \mathbb{Q}[\mu_d^{n+1}], \qquad \rho\colon \mathbb{Q}[\mu_d^n \times \mu_d^2] \longrightarrow \mathbb{Q}[\mu_d^{n-1} \times \mu_d]$$

be the ring homomorphisms induced respectively by (4.6) and

$$\rho((\xi_1, \dots, \xi_n), (\eta_1, \eta_2)) = ((\xi_1, \dots, \xi_{n-1}), \eta_1/\eta_2).$$

Lemma 6.3. — *For $\boldsymbol{\chi} = (\chi_1, \dots, \chi_{n+1}) \in \mathfrak{X}_d^{(n+1)}$, let $\boldsymbol{\chi}^{(n)}$, $\boldsymbol{\chi}^{(2)}$ be as in Proposition 4.10, and put $\boldsymbol{\chi}^{(n-1)} = (\chi_1, \dots, \chi_{n-1})$.*

(i) *If $\chi_n\chi_{n+1} \neq 1$, then $\nu(j_d^{(n)} e^{\boldsymbol{\chi}^{(n)}} \times j_d^{(2)} e^{\boldsymbol{\chi}^{(2)}}) = j_d^{(n+1)} e^{\boldsymbol{\chi}}$.*

(ii) *There exists an element $j_d^{(n,2)} \in \mathbb{Q}[\mu_d^n \times \mu_d^2]$ independent of $\boldsymbol{\chi}$, such that $\nu(j_d^{(n,2)}) = j_d^{(n+1)}$ and*

$$\rho\Big(j_d^{(n,2)}\Big)\Big(e^{\boldsymbol{\chi}^{(n-1)}} \times e^{\chi_n}\Big) = \chi_n\Big((-1)^{\frac{q-1}{d}}\Big) q\Big(\Big(j_d^{(n-1)} e^{\boldsymbol{\chi}^{(n-1)}}\Big) \times e^{\chi_n}\Big)$$

unless $\boldsymbol{\chi}^{(n-1)} = \mathbf{1}$.

Proof.

(i). — Put, for $c \in \kappa$, $S_c^{(n)} = \{(m_i)_i \in (\kappa^*)^n \mid \sum_{i=1}^n m_i = c\}$, so that

$$j_d^{(n)}\langle c \rangle = (-1)^{n-1} \sum_{(m_i)_i \in S_c^{(n)}} \Big(m_i^{\frac{q-1}{d}}\Big)_i$$

where $j_d^{(n-1)}\langle c\rangle$ is from (2.7). Then, ν induces a bijection $S_1^{(n)} \times S_1^{(2)} \to S_1^{(n+1)} \setminus (S_1^{(n-1)} \times S_0^{(2)})$. By assumption, we have

$$\sum_{(m_n, m_{n+1}) \in S_0^{(2)}} \chi_n\Big(m_n^{\frac{q-1}{d}}\Big)\chi_{n+1}\Big(m_{n+1}^{\frac{q-1}{d}}\Big)$$
$$= \chi_n(-1)^{\frac{q-1}{d}} \sum_{m\in\kappa^*} \chi_n\chi_{n+1}\Big(m^{\frac{q-1}{d}}\Big) = 0.$$

Hence $\sum_{(m_i)_i \in S_1^{(n-1)} \times S_0^{(2)}} \boldsymbol{\chi}((m_i^{\frac{q-1}{d}})_i) = 0$. One verifies easily $\nu(e^{\boldsymbol{\chi}^{(n)}} \times e^{\boldsymbol{\chi}^{(2)}}) = e^{\boldsymbol{\chi}}$. Now it follows

$$\nu\Big(j_d^{(n)} e^{\boldsymbol{\chi}^{(n)}} \times j_d^{(2)} e^{\boldsymbol{\chi}^{(2)}}\Big) = \nu\Big(j_d^{(n)} \times j_d^{(2)}\Big)e^{\boldsymbol{\chi}}$$
$$= \left(j_d^{(n+1)} - \sum_{(m_i)_i \in S_1^{(n-1)} \times S_0^{(2)}} \Big(m_i^{\frac{q-1}{d}}\Big)_i\right)e^{\boldsymbol{\chi}} = j_d^{(n+1)} e^{\boldsymbol{\chi}}.$$

(ii). — If we put

$$j_d^{(n,2)} = (-1)^n \sum_{(m_i)_i \in S_1^{(n+1)}} \Big(\Big(m_1^{\frac{q-1}{d}}, \dots, m_{n-1}^{\frac{q-1}{d}}, 1\Big), \Big(m_n^{\frac{q-1}{d}}, m_{n+1}^{\frac{q-1}{d}}\Big)\Big),$$

then $\nu(j_d^{(n,2)}) = j_d^{(n+1)}$. On the other hand,

$$\rho(j_d^{(n,2)}) = \sum_{s,t\in\kappa, s+t=1} j_d^{(n-1)}\langle s\rangle \times h(t),$$

$$h(t) := \sum_{(k,l)\in S_t^{(2)}} [(k/l)^{\frac{q-1}{d}}] = \begin{cases} (q-1)[(-1)^{\frac{q-1}{d}}] & \text{if } t=0,\\ \sum_{m\in\kappa^*}[m^{\frac{q-1}{d}}] - [(-1)^{\frac{q-1}{d}}] & \text{if } t\neq 0.\end{cases}$$

Here we write $[\zeta] \in \mathbb{Q}[\mu_d]$ for the image of $\zeta \in \mu_d$. Since $\chi_n \neq 1$,

$$\sum_{m\in\kappa^*} \Big[m^{\frac{q-1}{d}}\Big] e^{\chi_n} = \frac{q-1}{d} \sum_{\zeta\in\mu_d} [\zeta] e^{\chi_n} = \frac{q-1}{d} \sum_{z\in\mu_d} \chi_n(\zeta) e^{\chi_n} = 0.$$

Since $\boldsymbol{\chi}^{(n-1)} \neq \mathbf{1}$, we have

$$\sum_{s\in\kappa} j_d^{(n-1)}\langle s\rangle e^{\boldsymbol{\chi}^{(n-1)}} = (-1)^n \sum_{(m_i)_i\in(\kappa^*)^{n-1}} \Big(m_i^{\frac{q-1}{d}}\Big)_i \cdot e^{\boldsymbol{\chi}^{(n-1)}} = 0.$$

Therefore, only the term with $(s,t) = (1,0)$ remains and

$$\rho(j_d^{(n,2)})(e^{\boldsymbol{\chi}^{(n-1)}} \times e^{\chi_n}) = j_d^{(n-1)} e^{\boldsymbol{\chi}^{(n-1)}} \times q\Big[(-1)^{\frac{q-1}{d}}\Big] e^{\chi_n}$$
$$= j_d^{(n-1)} e^{\boldsymbol{\chi}^{(n-1)}} \times q\chi_n\Big((-1)^{\frac{q-1}{d}}\Big) e^{\chi_n}. \qquad \square$$

Proof of Theorem 1.2. — First, we prove the case $c = 1$ by induction on n. The case $n = 2$ follows from Corollary 6.2. It suffices to prove $j_d^{(n+1)} \cdot e^{\boldsymbol{\chi}} = \mathrm{Fr}_{F_d^{(n+1)}} \cdot e^{\boldsymbol{\chi}}$ for each $\boldsymbol{\chi} = (\chi_1, \dots, \chi_{n+1}) \in \mathfrak{X}_d^{(n+1)}$ $(n \geqslant 2)$. First, assume that $\chi_n \chi_{n+1} \neq 1$. Then by Proposition 4.11 (ii), we have an isomorphism

$$h(F_d^{(n)})^{\boldsymbol{\chi}^{(n)}} \otimes h(F_d^{(2)})^{\boldsymbol{\chi}^{(2)}} \xrightarrow{\simeq} h(F_d^{(n+1)})^{\boldsymbol{\chi}}.$$

Since any morphism in $\mathbf{Chow}(\kappa, \Lambda)$ commutes with the Frobenius endomorphisms, the assertion follows by the induction hypothesis and Lemma 6.3 (i). Secondly, assume that $\chi_n \chi_{n+1} = 1$ (so $\boldsymbol{\chi}^{(n-1)} \neq \mathbf{1}$). Then by Proposition 4.11 (iii), we have an isomorphism

$$h\big(F_d^{(n-1)}\big)^{\boldsymbol{\chi}^{(n-1)}} \otimes \Lambda\langle -1\rangle^{\chi_n}(1) \xrightarrow{\simeq} h\big(F_d^{(n+1)}\big)^{\boldsymbol{\chi}}.$$

Here we used the identification $Z \simeq F_d^{(n-1)} \times F_d^{(1)}\langle -1\rangle$ given by (4.7), under which the actions of $\mu_d^n \times \mu_d^2$ and $\mu_d^{n-1} \times \mu_d$ are compatible via the homomorphism ρ. Since the Frobenius acts on $\Lambda\langle -1\rangle^{\chi_n}(1)$ as the multiplication by $\chi_n \circ \mathrm{Km}(-1)q$ by definition, the assertion follows by the induction hypothesis and Lemma 6.3 (ii).

The general case is reduced to the previous case using Proposition 4.7. Note that the Frobenius acts on $\Lambda\langle c\rangle^{\prod_{i=1}^n \chi_i}$ as the multiplication by

$$\prod_{i=1}^n \chi_i(\mathrm{Km}(c)(\mathrm{Fr})) = \prod_{i=1}^n \chi_i\Big(c^{\frac{q-1}{d}}\Big) = \boldsymbol{\chi}\Big(c^{\frac{q-1}{d}}, \dots, c^{\frac{q-1}{d}}\Big),$$

and that $j_d^{(n)}\langle c\rangle = \big(c^{\frac{q-1}{d}}, \dots, c^{\frac{q-1}{d}}\big) j_d^{(n)}$. Hence the proof of Theorem 1.2 is complete. □

The following corollary follows immediately by Lemma 3.3.

Corollary 6.4. — *For any $c \in \kappa^*$, we have*

$$j_d^{(n)}\langle c\rangle \cdot e_{\mathrm{prim}} = \mathrm{Fr}_{F_d^{(n)}\langle c\rangle} \cdot e_{\mathrm{prim}}$$

in $\mathrm{End}_{\mathbf{Chow}(\kappa,\mathbb{Q})}(h(F_d^{(n)}\langle c\rangle))$.

Corollary 6.5. — *Under the isomorphism of Theorem 1.1 (i), we have for $(\chi_1, \dots, \chi_n) \in \mathfrak{X}_d^{(n)}$*

$$\bigotimes_{i=1}^n \Big(g_d \cdot e^{(\psi,\chi_i)}\Big) = \Big(g_d \cdot e^{(\psi, \prod_{i=1}^n \chi_i)}\Big) \otimes \Big(j_d^{(n)} \cdot e^{(\chi_1,\dots,\chi_n)}\Big)$$

in $\mathrm{End}(h(A_d)^{\otimes n}) = \mathrm{End}(h(A_d) \otimes h(F_d^{(n)}))$.

Proof. — Since the isomorphism of Theorem 1.1 (i) is compatible with the Frobenius endomorphisms, the statement follows by Theorem 1.2 and Corollary 6.2. □

Remark 6.6. — Proposition 6.1 (i) shows that the Frobenius acts on $h(A_d)_{\mathrm{prim}}$ as g_d, and hence on $h(A_d)^{(\psi,\chi)}$ by the multiplication by $g(\psi,\chi)$ if $\psi \in \widehat{\kappa}$ and $\chi \in \widehat{\mu}_d$ are non-trivial. From this fact, together with Proposition 4.5, we can deduce (2.2) and (2.3). Similarly, we can deduce (2.5), (2.6), and for $(\chi_1,\dots,\chi_n) \in \mathfrak{X}_d^{(n)}$

$$j(\chi_1,\dots,\chi_n) = \begin{cases} j(\chi_1,\dots,\chi_{n-2},\chi_{n-1}\chi_n)j(\chi_{n-1},\chi_n) & \text{if } \chi_{n-1}\chi_n \neq 1, \\ j(\chi_1,\dots,\chi_{n-2})\chi_{n-1}((-1)^{\frac{q-1}{d}})q & \text{if } \chi_{n-1}\chi_n = 1 \end{cases} \tag{6.3}$$

from Corollary 6.5, (4.8), and Proposition 4.11 (ii), (iii), respectively.

The invertibility (Proposition 4.11) implies that, for any Weil cohomology theory H whose coefficient field contains μ_d, the $\boldsymbol{\chi}$-eigenspace $H(F_d^{(n)})^{\boldsymbol{\chi}}$ for $\boldsymbol{\chi} \in \mathfrak{X}_d^{(n)}$ is one-dimensional and the Frobenius acts on this space as the multiplication by $j(\boldsymbol{\chi})$ by Theorem 1.2. Classically, this fact is proved by point counting and the Grothendieck–Lefschetz trace formula. We have given a purely motivic proof that avoids the use of Weil cohomology. A similar remark applies to $H(A_d)^{(\psi,\chi)}$.

Remark 6.7. — If $\psi, \psi' \in \widehat{\kappa}$ are nontrivial, there exists a unique $c \in \kappa^*$ such that $\psi'(x) = \psi(cx)$ for any $x \in \kappa$. This follows from the non-degeneracy of the paring $\kappa^2 \to \mathbb{F}_p$; $(c,x) \mapsto \mathrm{Tr}_{\kappa/\mathbb{F}_p}(cx)$. One shows easily

$$g(\psi',\chi) = \overline{\chi}\Big(c^{\frac{q-1}{d}}\Big)g(\psi,\chi). \tag{6.4}$$

This identity is interpreted motivically as follows. Let $A_d\langle c\rangle$ be defined by $x^q - x = cy^d$. Let $\kappa \times \kappa^*$ act on this as before and $h(A_d\langle c\rangle)^{(\psi,\chi)}$ be the associated motive (see Section 4.1). Then the isomorphism $A_d \to A_d\langle c\rangle$; $(x,y) \mapsto (cx,y)$ induces an isomorphism

$$h(A_d)^{(\psi',\chi)} \simeq h(A_d\langle c\rangle)^{(\psi,\chi)}$$

of motives. The Gauss sum element g_d acting on $h(A_d)$ corresponds to the element

$$-\sum_{m\in\kappa^*}\Big(cm, m^{\frac{q-1}{d}}\Big) = -\sum_{m\in\kappa^*}\Big(m, (c^{-1}m)^{\frac{q-1}{d}}\Big)$$

acting on $h(A_d\langle c\rangle)$, hence (6.4) follows by the argument in Remark 6.6.

7. Davenport–Hasse relations

7.1. Base-change formula

Let κ be a finite field of characteristic p and of order q and K be a degree r extension of κ. To avoid the confusion, we write the Gauss and Jacobi sums over κ as $g_\kappa(\psi,\chi)$ and $j_\kappa(\chi_1,\dots,\chi_n)$, and similarly over K. For $\psi\in\widehat{\kappa}$, let

$$\psi_K=\psi\circ\mathrm{Tr}_{K/\kappa}\in\widehat{K}$$

be the lifted character. For $\psi\in\widehat{\kappa}$, $\psi\neq 1$ and $\chi,\chi_1,\dots,\chi_n\in\widehat{\mu}_d$, $(\chi_1,\dots,\chi_n)\neq(1,\dots,1)$, we have the classical Davenport–Hasse base-change formulas [6]

$$g_K(\psi_K,\chi)=g_\kappa(\psi,\chi)^r, \tag{7.1}$$

$$j_K(\chi_1,\dots,\chi_n)=j_\kappa(\chi_1,\dots,\chi_n)^r. \tag{7.2}$$

The latter follows from the former by (2.5). See [23] for an elementary proof and [7, Sommes trig., 4.12] for an l-adic sheaf-theoretic proof.

We give their motivic proofs. Let $\lambda\colon\,\mathrm{Spec}\,K\to\mathrm{Spec}\,\kappa$ be the structure morphism. Then λ induces a functor

$$\lambda^*\colon\mathbf{Chow}(\kappa,\Lambda)\longrightarrow\mathbf{Chow}(K,\Lambda)$$

such that $\lambda^*(h(X))=h(X_K)$, where $X_K:=X\times_{\mathrm{Spec}\,\kappa}\mathrm{Spec}\,K$.

Define $g_{d,K/\kappa}\in\mathrm{End}(h(A_{d,K}))$ by the element

$$-\sum_{m\in K^*}\left(\mathrm{Tr}_{K/\kappa}(m),m^{\frac{q^r-1}{d}}\right)\ \in\mathbb{Q}[\kappa\times\mu_d],$$

and for $c\in K^*$, $j^{(n)}_{d,K}\langle c\rangle\in\mathrm{End}(h(F^{(n)}_{d,K}\langle c\rangle)$ by the element (as before)

$$(-1)^{n-1}\sum_{m_1,\dots,m_n\in K^*,\sum_{i=1}^n m_i=c}\left(m_1^{\frac{q^r-1}{d}},\dots,m_n^{\frac{q^r-1}{d}}\right)\in\mathbb{Q}[\mu_d^n].$$

Let $e_{\mathrm{prim}}\in\mathrm{End}(h(A_{d,K}))$ (resp. $e_{\mathrm{prim}}\in\mathrm{End}(h(F^{(n)}_{d,K}\langle c\rangle))$ be the projector onto the primitive part with respect to the $\kappa\times\mu_d$-action (resp. μ_d^n-action).

Theorem 7.1. — *We have equalities of endomorphisms of $h(A_{d,K})$ (resp. $h(F^{(n)}_{d,K}\langle c\rangle)$)*

$$\lambda^*(g_d)^r\cdot e_{\mathrm{prim}}=g_{d,K/\kappa}\cdot e_{\mathrm{prim}}\quad\Big(\textit{resp.}\ \ \lambda^*\big(j^{(n)}_d\langle c\rangle\big)^r\cdot e_{\mathrm{prim}}=j^{(n)}_{d,K}\langle c\rangle\cdot e_{\mathrm{prim}}\Big)$$

in $\mathbf{Chow}(K,\mathbb{Q})$.

Proof. — Let A' be the Artin–Schreier curve over K of degree q^r with the affine equation $u^{q^r}-u=v^d$. Let $f\colon A'\to A_{d,K}$ be the morphism defined by $x=\sum_{i=0}^{r-1}u^{q^i}$, $y=v$. It is finite of degree q^{r-1}. Let

$$g'=-\sum_{m\in K^*}\left(m,m^{\frac{q^r-1}{d}}\right),\quad e'_{\mathrm{prim}}\ \in\mathrm{End}_{\mathbf{Chow}(K,\mathbb{Q})}(h(A'))$$

be the Gauss sum element and the projector to the primitive part (with respect to the $K\times\mu_d$-action), respectively. Since f is compatible with the group actions via the homomorphism $\mathrm{Tr}_{K/\kappa}\times\mathrm{id}\colon K\times\mu_d\to\kappa\times\mu_d$, we have $f_{\#}(g') = g_{d,K/\kappa}$. Since $\mathrm{Tr}_{K/\kappa}$ is a surjective homomorphism, the map $\mathbb{Q}[K] \to \mathbb{Q}[\kappa]$ maps e_K^1 to e_κ^1, and we have $f_{\#}(e'_{\mathrm{prim}}) = e_{\mathrm{prim}}$. We have also $f_{\#}(\mathrm{Fr}_{A'}) = \mathrm{Fr}_{A_{d,K}}$. Since $g' \cdot e'_{\mathrm{prim}} = \mathrm{Fr}_{A'} \cdot e'_{\mathrm{prim}}$ by Corollary 6.2, we obtain $g_{d,K/\kappa} \cdot e_{\mathrm{prim}} = \mathrm{Fr}_{A_{d,K}} \cdot e_{\mathrm{prim}}$. On the other hand, we have

$$\lambda^*(g_d)^r \cdot e_{\mathrm{prim}} = \lambda^*(g_d^r \cdot e_{\mathrm{prim}}) = \lambda^*(\mathrm{Fr}_{A_d}^r \cdot e_{\mathrm{prim}}) = \mathrm{Fr}_{A_{d,K}} \cdot e_{\mathrm{prim}}$$

by Corollary 6.2. Note that the action of $\mathbb{Q}[\kappa \times \mu_d]$, in particular of e_{prim}, is compatible with λ^* and commute with the Frobenius. Hence the first assertion is proved. The second assertion follows similarly (and more easily) by Corollary 6.4. ☐

We can recover the classical formulas (7.1), (7.2) from the theorem by multiplying the both sides by $e^{(\psi,\chi)}$ (resp. by $e^{(\chi_1,\dots,\chi_n)}$), together with the fact that $h(A_{q-1,K})^{(\psi_K,\chi)} = \lambda^*(h(A_{q-1})^{(\psi,\chi)})$ (resp. $h(F_{q-1,K}^{(n)})^{(\chi_1,\dots,\chi_n)} = \lambda^*(h(F_d^{(n)})^{(\chi_1,\dots,\chi_n)})$) is invertible.

7.2. Multiplication formula

Recall the multiplication formula for the gamma function

$$\frac{\Gamma(ns)}{\Gamma(n)} = n^{n(s-1)} \prod_{i=0}^{n-1} \frac{\Gamma(s+\frac{i}{n})}{\Gamma(1+\frac{i}{n})}. \tag{7.3}$$

Its finite analogue is due to Davenport–Hasse [6]. Let κ be as in the preceding subsection and suppose $n \mid d \mid q-1$. Then we have

$$g(\psi, \alpha^n) = \alpha^n(n) \prod_{\chi^n=1} \frac{g(\psi, \alpha\chi)}{g(\psi,\chi)} \tag{7.4}$$

for any $\alpha \in \widehat{\mu}_d$. The statement is evident if $\alpha^n = 1$, so we assume $\alpha^n \neq 1$. Then it is equivalent by (2.5) to

$$\alpha^n(n) j(\underbrace{\alpha, \dots, \alpha}_{n \text{ times}}) = \prod_{\chi^n=1, \chi\neq 1} j(\alpha,\chi). \tag{7.5}$$

Its first elementary proof which does not use Stickelberger's theorem on the prime decomposition of Gauss sums (see (8.5) below) or cohomology was given recently in [16, Appendix A]. It applies a point counting argument based on the geometric construction of Terasoma [21] in his cohomological proof.

We prove the following motivic analogue, which includes Theorem 1.1 (ii).

Theorem 7.2. — *Let κ be a field and assume that d is a positive integer such that $\mu_d := \{m \in \kappa^* \mid m^d = 1\}$ has d elements and $\mathbb{Q}(\zeta_d) \subset \Lambda$. For any $n \mid d$ and $\alpha \in \widehat{\mu}_d$ such that $\alpha^n \neq 1$, there is an isomorphism in* $\mathbf{Chow}(\kappa, \Lambda)$

$$h(F_d^{(n)}\langle n\rangle)^{(\alpha,\dots,\alpha)} \simeq \bigotimes_{\chi\in\widehat{\mu}_d, \chi^n=1, \chi\neq 1} h(F_d^{(2)})^{(\alpha,\chi)}. \tag{7.6}$$

Proof. — We may assume κ is a finite field or $\kappa = \mathbb{Q}(\zeta_d)$. (In particular, κ is perfect.) We use the following notations after [16, Appendix A]:

- $S \subset \mathbb{A}^n$ is a hyperplane defined by $s_1 + \cdots + s_n = n$, $s_1 \cdots s_n \neq 0$.
- $T \to S$ is a covering defined by $t^d = s_1 \cdots s_n$. It is Galois with the natural identification $\mathrm{Gal}(T/S) = \mu_d$.
- $X \subset \mathbb{A}^n$ is a hypersurface defined by $t_1^d + \cdots + t_n^d = n$, $t_1 \cdots t_n \neq 0$. There is a morphism

  $$X \longrightarrow T; \quad s_i = t_i^d,\ t = t_1 \cdots t_n.$$

  Then X is Galois over S with the natural identification $\mathrm{Gal}(X/S) = \mu_d^n$, under which $\mathrm{Gal}(X/S) \to \mathrm{Gal}(T/S)$ sends $(\xi_1, \dots, \xi_n)$ to $\prod_{i=1}^n \xi_i$.
- C is an affine curve defined by $x^d + y^n = 1$, $x \neq 0$. There is a morphism

  $$C^{n-1} \longrightarrow T; \quad s_i = \prod_{j=1}^{n-1}(1 - \zeta^i y_j),\ t = x_1 \cdots x_{n-1},$$

  where (x_j, y_j) is the coordinate of the jth component of C^{n-1}, and $\zeta \in \kappa$ is a fixed primitive nth root of unity. Then C^{n-1} is generically Galois over S with the natural identification $\mathrm{Gal}(C^{n-1}/S) = \mu_d^{n-1} \rtimes S_{n-1}$, under which $\mathrm{Gal}(C^{n-1}/S) \to \mathrm{Gal}(T/S)$ sends $((\xi_1, \dots, \xi_{n-1}), \sigma)$ to $\prod_{i=1}^{n-1} \xi_i$.

First, X is an open subscheme of $F_d^{(n)}\langle n\rangle$ and we have isomorphisms

$$M(T)^\alpha \simeq M(X)^{(\alpha,\dots,\alpha)} \simeq M(F_d^{(n)}\langle n\rangle)^{(\alpha,\dots,\alpha)}$$

by Proposition 3.5. For the latter, note that the diagonal in μ_d^n acts trivially on $F_d^{(n)}\langle n\rangle \setminus X$ but $(\alpha, \dots, \alpha)$ is non-trivial on the diagonal since $\alpha^n \neq 1$. In particular, $M(T)^\alpha$ is invertible by Proposition 4.11 (iv).

On the other hand, if $\beta \in \widehat{\mu_d^{n-1} \rtimes S_{n-1}}$ denotes the pull-back of α, then

$$M(T)^\alpha \simeq M(C^{n-1})^\beta$$

by Proposition 3.5 (ii). The restriction of β to the first (resp. the second) component is $(\alpha, \dots, \alpha)$ (resp. 1), and the corresponding projectors $e^{(\alpha,\dots\alpha)}$

and e^1 satisfy

$$e^{\beta} = e^{(\alpha,\ldots,\alpha)} e^1 = e^1 e^{(\alpha,\ldots,\alpha)} \quad \text{in } \Lambda[\mu_d^{n-1} \rtimes S_{n-1}].$$

Hence e^1 restricts to an idempotent endomorphism of $M := M(C^{n-1})^{(\alpha,\ldots,\alpha)}$. Since C admits an action of $\mu_d \times \mu_n$, we can further decompose M with respect to the μ_n^{n-1}-action. Let $\nu_1, \ldots, \nu_{n-1} \in \widehat{\mu}_n \setminus \{1\}$ be distinct characters (such a choice is unique up to permutations), and $e^{\boldsymbol{\nu}}$ be the corresponding projector. One easily verifies that $\sigma e^{\boldsymbol{\nu}} = e^{\sigma\boldsymbol{\nu}}\sigma$ for any $\sigma \in S_{n-1}$, where S_{n-1} acts on $\widehat{\mu}_n^{n-1}$ as permutations. Hence

$$e^{\boldsymbol{\nu}} e^1 e^{\boldsymbol{\nu}} = \frac{1}{(n-1)!} \sum_{\sigma} e^{\boldsymbol{\nu}} e^{\sigma\boldsymbol{\nu}} \sigma = \frac{1}{(n-1)!} e^{\boldsymbol{\nu}} \quad \text{in } \operatorname{End}(M).$$

Note that $e^{\boldsymbol{\nu}} e^{\sigma\boldsymbol{\nu}} = 0$ unless $\sigma = 1$ by the assumption on $\boldsymbol{\nu}$. If we put $L = M^{\boldsymbol{\nu}}$ and $N = M^1 \simeq M(T)^{\alpha}$, then the composite $L \to N \to L$ of the natural morphisms (via M) is the multiplication by $1/(n-1)!$, hence is an isomorphism. Since N is invertible, it follows that $L \simeq N$ once we show that L is also invertible. We have by definition $L = \bigotimes_{i=1}^{n-1} M(C)^{(\alpha,\nu_i)}$. If $\widetilde{C} \subset F_d^{(2)}$ denotes the open curve defined by $u_0 u_1 \neq 0$, C is the quotient of $\widetilde{C}$ by $1 \times \mu_{d/n} \subset \mu_d^2$, and we have isomorphisms by Propositions 3.5

$$M(C)^{(\alpha,\nu_i)} \simeq M(\widetilde{C})^{(\alpha,\chi_i)} \simeq M(F_d^{(2)})^{(\alpha,\chi_i)},$$

where χ_i is the pull-back of ν_i to μ_d. For the latter isomorphism, note that the diagonal in μ_d^2 (resp. $\mu_d \times 1$) acts trivially on $\{u_0 = 0\}$ (resp. $\{u_1 = 0\}$) and $\alpha\chi_i$ (resp. α) is non-trivial on the subgroup. Since $M(F_d^{(2)})^{(\alpha,\chi_i)}$ is invertible by Proposition 4.11, L is also invertible and we have $\bigotimes_{i=1}^{n-1} M(F_d^{(2)})^{(\alpha,\chi_i)} \simeq N \simeq M(F_d^{(n)}\langle n\rangle)^{(\alpha,\ldots,\alpha)}$, which implies the desired isomorphism of Chow motives. □

Corollary 7.3. — *Suppose further that κ is a finite field. Under the isomorphism* (7.6)*, the endomorphism $j_d^{(n)}\langle n\rangle$ on the left hand side corresponds to $\bigotimes_{\chi^n=1,\chi\neq 1} j_d^{(2)}$ on the right hand side.*

Proof. — This follows by comparing the Frobenius endomorphisms using Theorem 1.2. □

Remark 7.4. — One can deduce (1.1) and (1.2) from Coleman's theorem (1.3) and Theorems 1.1, 1.2 as follows. Since both sides of Theorem 1.1 (i) are invertible, their endomorphism rings are canonically isomorphic to Λ. The Frobenius endomorphisms on both sides, regarded as elements of Λ, yield the same element because the Frobenius endomorphism commutes with any morphism (see (3.4)). We conclude (1.1) by observing that the Frobenius endomorphisms agree with the left and right hand sides of (1.1) by (1.3) and Theorem 1.2, respectively.

Similarly, one recovers (1.2) (which is (7.5)) from Corollary 7.3 and the invertibility of the motives, noting

$$j_d^{(n)}\langle n\rangle\cdot e^{(\alpha,\dots,\alpha)}=\alpha^n(n)j(\alpha,\dots,\alpha)\cdot e^{(\alpha,\dots,\alpha)},\quad j_d^{(2)}\cdot e^{(\alpha,\chi)}=j(\alpha,\chi)\cdot e^{(\alpha,\chi)}.$$

Remark 7.5. — If $\kappa=\mathbb{C}$, the complex period of an invertible motive in $\mathbf{Chow}(\mathbb{C},\mathbb{Q}(\zeta_d))$ is an element of $\mathbb{C}^*/\mathbb{Q}(\zeta_d)^*$, defined by the de Rham–Betti comparison isomorphism. Since the periods of Fermat motives are special values of the beta function, the isomorphism (7.6) implies (7.3) for any $s\in d^{-1}\mathbb{Z}$, up to $\mathbb{Q}(\zeta_d)^*$.

8. Weil numbers

In this section, we start with the cyclotomic field $F=\mathbb{Q}(\zeta_d)$ for an integer $d\geqslant 3$. We have an isomorphism

$$(\mathbb{Z}/d\mathbb{Z})^*\simeq G:=\mathrm{Gal}(F/\mathbb{Q});\quad h\longmapsto\sigma_h,$$

where σ_h is defined by $\sigma_h(\zeta_d)=\zeta_d^h$. Note that σ_{-1} agrees with the complex conjugation (for any embedding $F\hookrightarrow\mathbb{C}$). Let $\mu(F)$ denote the group of all the roots of unity in F. Note that $|\mu(F)|=d$ or $2d$ according to the parity of d. Let v be a prime of F over a rational prime $p\nmid d$, κ be the residue field at v and put $q=p^f=|\kappa|$. Let $D=\langle\sigma_p\rangle\subset G$ be the decomposition subgroup of v. Then G/D is bijective to the set of primes of F over p by $\sigma\mapsto\sigma v$. We have $|D|=f$ and $|G/D|=\varphi(d)/f$, where φ denotes Euler's totient function.

Let $\chi_d\colon\kappa^*\to F^*$ be the dth power residue character modulo v, i.e. $\chi_d(x \bmod v)\equiv x^{\frac{q-1}{d}}\pmod{v}$ for any $x\in F^*$ such that $v(x)=0$. For any $\mathbf{a}=(a_1,\dots,a_n)\in(\mathbb{Z}/d\mathbb{Z})^n$ $(n\geqslant 2)$, put

$$j_d(\mathbf{a})=j(\chi_d^{a_1},\dots,\chi_d^{a_n})\ \in F^*.$$

Define

$$A_d^{(n)}=\big\{\mathbf{a}=(a_1,\dots,a_n)\in(\mathbb{Z}/d\mathbb{Z})^n\ \big|\ a_0,a_1,\dots,a_n\neq 0\big\}$$

where $a_0:=-\sum_{i=1}^n a_i$. If $\mathbf{a}\in A_d^{(n)}$, then $j_d(\mathbf{a})$ is a q-Weil number of weight $n-1$ by (2.6). Let

$$W=W_q(F)\subset F^*$$

be the subgroup of q-Weil numbers and $J\subset W$ be the subgroup generated by $\big\{j_d(\mathbf{a})\ \big|\ \mathbf{a}\in A_d^{(n)},n\geqslant 2\big\}$. Since $\sigma_h j_d(\mathbf{a})=j_d(h\mathbf{a})$, J is G-stable as well as W. In fact, J is generated by $\big\{j_d(\mathbf{a})\ \big|\ \mathbf{a}\in A_d^{(2)}\big\}\cup\big\{(-1)^{\frac{q-1}{d}}\big\}$ by (6.3).

Define the group homomorphism

$$\Phi\colon W\longrightarrow\mathbb{Z}[G/D]$$

by $\Phi(\alpha)v = (\alpha)$ (the principal divisor of α). (Note that we have $v'(\alpha) = 0$ for any finite place $v' \nmid p$ and $\alpha \in W$.) Then for α of weight w, we have

$$\Phi(\alpha) + \sigma_{-1}\Phi(\alpha) = \Phi(\alpha \cdot \sigma_{-1}(\alpha)) = \Phi(q^w) = wfT, \tag{8.1}$$

where $T := \sum_{\tau \in G/D} \tau$ is the trace element. Kronecker's theorem [10] shows

$$\operatorname{Ker} \Phi = \mu(F). \tag{8.2}$$

In particular, W is a finitely generated abelian group whose torsion part is precisely $\mu(F)$ (see (8.7) below for its rank). Since W and J are G-stable and Φ is G-equivariant, $\Phi(W)$ and $\Phi(J)$ are ideals of $\mathbb{Z}[G/D]$.

The case when f is even is easy.

Proposition 8.1. — *If f is even, then $\Phi(W) = \Phi(J) = \mathbb{Z} \cdot \frac{f}{2}T$, and W is generated by $\mu(F)$ and $\sqrt{q} = p^{f/2}$.*

Proof. — Since $\sigma_{-1} = (\sigma_p)^{f/2} \in D$, we have $\Phi(\alpha) = \frac{wf}{2}T$ for any $\alpha \in W$ of weight w by (8.1). Since we assumed $d \geqslant 3$, there exists a Jacobi sum of weight 1 (e.g. $j_d(1,1)$), and the proposition follows. □

For any $a \in \mathbb{Z}/d\mathbb{Z}$, define the Stickelberger element by

$$\theta_d(a) = \sum_{h \in (\mathbb{Z}/d\mathbb{Z})^*} \left\{-\frac{ha}{d}\right\} \sigma_h^{-1} \in \mathbb{Q}[G],$$

where $\{x\}$ denotes the fractional part, i.e. $x = \{x\} + \lfloor x \rfloor$. Note that

$$\sigma_h \theta_d(a) = \theta_d(ha).$$

Define the trace element as $\widetilde{T} = \sum_{\sigma \in G} \sigma \in \mathbb{Z}[G]$. Then

$$\theta_d(a) + \theta_d(-a) = \widetilde{T} \quad \text{if } a \neq 0. \tag{8.3}$$

If $\pi \colon \mathbb{Z}[G] \to \mathbb{Z}[G/D]$ denotes the natural surjection, then $\pi(\widetilde{T}) = fT$. For $\mathbf{a} = (a_1, \dots, a_n) \in A_d^{(n)}$, put

$$\begin{aligned}\theta_d(\mathbf{a}) &= \theta_d(a_1) + \cdots + \theta_d(a_n) - \theta_d(a_1 + \cdots + a_n)\\ &= \theta_d(a_0) + \theta_d(a_1) + \cdots + \theta_d(a_n) - \widetilde{T}.\end{aligned}$$

Then $\theta_d(\mathbf{a}) \in \mathbb{Z}[G]$, and we have by (8.3)

$$\theta_d(\mathbf{a}) + \theta_d(-\mathbf{a}) = (n-1)\widetilde{T}. \tag{8.4}$$

Stickelberger's theorem states that (see, e.g. [4, Theorem 11.2.3])

$$(j_d(\mathbf{a})) = \theta_d(\mathbf{a})v \quad (\mathbf{a} \in A_d^{(n)}). \tag{8.5}$$

In other words, $\Phi(j_d(\mathbf{a})) = \pi(\theta_d(\mathbf{a}))$. Therefore, $\Phi(J)$ is generated as a $\mathbb{Z}$-module by $\{\theta_d(\mathbf{a}) \mid \mathbf{a} \in A_d^{(2)}\}$.

Let $S \subset \mathbb{Z}[G]$ be the $\mathbb{Z}$-module generated by $\{\theta_d(\mathbf{a}) \mid \mathbf{a} \in A_d^{(2)}\}$. Since $\sigma_h\theta_d(\mathbf{a}) = \theta_d(h\mathbf{a})$, S is an ideal of $\mathbb{Z}[G]$ and is called the *Stickelberger ideal.* By (8.5), we have $\pi(S) = \Phi(J)$. For any G-module M, write $M^{\pm} = \{m \in M \mid \sigma_{-1}m = \pm m\}$. By Iwasawa and Sinnott [19], S^- is of finite index in $\mathbb{Z}[G]^-$. More precisely, let r be the number of prime factors of d, put $s = \max\{0, r-2\}$, and let h_d^- be the minus part of the class number of $F = \mathbb{Q}(\zeta_d)$. Then

$$m_d := (\mathbb{Z}[G]^- : S^-) = 2^r h_d^-. \tag{8.6}$$

As a corollary, we obtain the following.

Proposition 8.2. — *For any $\alpha \in W$, there exists $\zeta \in \mu(F)$ such that $\zeta\alpha^{2m_d} \in J$. Moreover, J is of finite index in W and we have*

$$\operatorname{rank} J = \operatorname{rank} W = \begin{cases} 1 + \frac{1}{2f}\varphi(d) & \textit{if } f \textit{ is odd,} \\ 1 & \textit{if } f \textit{ is even.} \end{cases} \tag{8.7}$$

Proof. — It is a consequence of Proposition 8.1 if f is even. Suppose that f is odd. Then we have $\sigma_{-1} \notin D$ and hence the right vertical map in the commutative diagram

$$\begin{array}{ccc} S^- & \hookrightarrow & \mathbb{Z}[G]^- \\ \downarrow & & \downarrow\!\!\downarrow \\ \Phi(J)^- & \hookrightarrow & \mathbb{Z}[G/D]^- \end{array}$$

is surjective. By (8.6), we have $(\mathbb{Z}[G/D]^- : \Phi(J)^-) \mid m_d$. On the other hand, we have $\Phi(J)^+ = \Phi(W)^+ = \mathbb{Z} \cdot fT$. For any $\alpha \in W$ of weight w, we have by (8.1)

$$\begin{aligned} 2\Phi(\alpha) &= (\Phi(\alpha) + \sigma_{-1}\Phi(\alpha)) + (\Phi(\alpha) - \sigma_{-1}\Phi(\alpha)) \\ &= wfT + (\Phi(\alpha) - \sigma_{-1}\Phi(\alpha)). \end{aligned}$$

Since $fT = \Phi(q) \in \Phi(J)^+$ and $m_d(\Phi(\alpha) - \sigma_{-1}\Phi(\alpha)) \in \Phi(J)^-$, the assertion follows by (8.2) and $\operatorname{rank} \mathbb{Z}[G/D]^- = \varphi(d)/(2f)$. □

We have an obvious inequality $(W : J) \leqslant |\mu(F)| \cdot 2^{\operatorname{rank} W} \cdot m_d$. It might be an interesting problem to find a better upper bound for $(W : J)$. To understand its motivic meaning, we consider the group homomorphism

$$\mathfrak{Fr} \colon \mathbf{Pic}(\mathbf{Chow}(\kappa, F)) \longrightarrow W$$

which associates to an invertible motive M its Frobenius eigenvalue in $\Lambda \simeq \operatorname{End}(M)$. As a result of Theorem 1.2 and Proposition 8.2, we obtain the following.

Corollary 8.3. — *For any $\alpha \in W$, there exist Fermat motives $M_i = h(F_d^{(2)})^{\boldsymbol{\chi}_i}$ $(i = 1, \dots, s,\ \boldsymbol{\chi}_i \in \mathfrak{X}_d^{(2)})$, an Artin motive $h(\operatorname{Spec} K)^{\chi}$ for a character $\chi\colon \operatorname{Gal}(K/\kappa) \to \mu(F)$ of a finite extension K/κ, and an integer r, such that*

$$\alpha^{2m_d} = \mathfrak{Fr}\left(\bigotimes_{i=1}^{s} M_i \otimes h(\operatorname{Spec} K)^{\chi}(r)\right).$$

If we assume the conjectures of Beilinson and Tate as in the introduction, then $\mathfrak{Fr}$ should be injective. It follows that any multiplicative relation among Weil numbers in the image of $\mathfrak{Fr}$ should lift to a relation of invertible motives. In this paper, we have exhibited such lifts for basic relations in J unconditionally.

Corollary 8.4. — *Assume that $\mathfrak{Fr}$ is injective. Then $\mathbf{Pic}(\mathbf{Chow}(\kappa, F))_{\mathbb{Q}}$ is isomorphic to $W_{\mathbb{Q}}$ and is generated as a $\mathbb{Q}$-vector space by the classes of Fermat motives $h(F_d^{(2)})^{\boldsymbol{\chi}}$ $(\boldsymbol{\chi} \in \mathfrak{X}_d^{(2)})$.*

Consider the normalization $(a \in \mathbb{Z}/d\mathbb{Z},\ a \neq 0)$

$$\widetilde{\theta}_d(a) = \theta_d(a) - \frac{1}{2}\widetilde{T} \ \in \mathbb{Q}[G].$$

First, we have by (8.3)

$$\widetilde{\theta}_d(-a) = -\widetilde{\theta}_d(a), \tag{8.8}$$

i.e. $\widetilde{\theta}_d(a) \in \mathbb{Q}[G]^-$. Since $\Phi(J)^-_{\mathbb{Q}} = \mathbb{Q}[G]^-$ by (8.6), $\mathbb{Q}[G]^-$ is generated by $\{\widetilde{\theta}_d(a) \mid a \in \mathbb{Z}/d\mathbb{Z},\ a \neq 0\}$. Secondly, for any positive divisor n of d and $a \in \mathbb{Z}/d\mathbb{Z}$ such that $na \neq 0$, we have

$$\widetilde{\theta}_d(na) = \sum_{i=0}^{n-1} \widetilde{\theta}_d\left(a + \frac{d}{n}i\right). \tag{8.9}$$

(This is easy to prove and is also a consequence of (7.5) and (8.5).)

Proposition 8.5. — *The set*

$$\left\{\widetilde{\theta}_d(a) \,\middle|\, a \in (\mathbb{Z}/d\mathbb{Z})^*, \left\{\frac{a}{d}\right\} < \frac{1}{2}\right\}$$

is a $\mathbb{Q}$-basis of $\mathbb{Q}[G]^-$.

Proof. — We already proved that $\mathbb{Q}[G]^-$ is generated by $\widetilde{\theta}_d(a)$ if a ranges over $\mathbb{Z}/d\mathbb{Z} \setminus \{0\}$. One shows by induction using (8.8) and (8.9) that the set in the proposition already generates $\mathbb{Q}[G]^-$. It is a basis since $\dim \mathbb{Q}[G]^- = \varphi(d)/2$. □

The last proposition shows that the reflection formula (8.8) and the multiplication formula (8.9) generate all the $\mathbb{Q}$-linear relations among $\theta_d(\mathbf{a})$ ($\mathbf{a} \in A_d^{(n)}, n \geqslant 2$). Hence the corresponding relations (2.6) and (7.5) together with (6.3) generate all the relations among $j_d(\mathbf{a})$ up to torsion. Assuming the conjectures of Beilinson and Tate, it follows that (4.8) and Theorem 1.1 (ii) (under Proposition 4.11 (ii), (iii)) should generate all the relations among the Fermat motives up to torsion (i.e. up to powers and Artin motives).

On the other hand, finding all the *integral* relations appears to be subtler. Indeed, it was first observed by Yamamoto [24] that the relations among the Gauss sums are *not* exhausted by the reflection formula (2.2) and the multiplication formula (7.4), disproving Hasse's conjecture. Its counterpart for Jacobi sums can be found in [14]. We leave it as a future problem to study the corresponding relations among motives.

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