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Amenable actions of real and p-adic algebraic groups (*)

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Abstract. — Let K be a locally compact field of characteristic 0. Let G be a linear algebraic group defined over K, acting algebraically on an algebraic variety V. We prove that the action of $G(K)$ (the group of K-rational points of G) on $V(K)$ is topologically amenable, if and only if all point stabilizers in $G(K)$ are solvable-by-compact. This follows by combining a result by Borel–Serre [5]with the following fact: let G be a second countable locally compact group acting continuously on a second countable locally compact space Y. If the action $G \curvearrowright Y$ is smooth (i.e. the Borel structure on $G\backslash Y$ is countably separated), then topological amenability of $G \curvearrowright Y$ is equivalent to amenability of all point stabilizers in G.

Résumé. — Soit K un corps localement compact de caractéristique 0. Soit G un groupe algébrique linéaire défini sur K, agissant algébriquement sur une variété algébrique V. Nous montrons que l'action de $G(K)$ (le groupe des points K-rationnels de G) sur $V(K)$ est topologiquement moyennable, si et seulement si tous les stabilisateurs de points dans $G(K)$ sont résolubles-par-compacts. Pour ce faire, on combine un résultat de Borel–Serre [5] avec le fait suivant : soit G un groupe localement compact dénombrable à l'infini, agissant continument sur un espace localement compact Y, dénombrable à l'infini. Si l'action $G \curvearrowright Y$ est lisse (c-à-d. la structure borélienne sur $G\backslash Y$ est dénombrablement séparée), la moyennabilité topologique de $G \curvearrowright Y$ est équivalente à la moyennabilité des stabilisateurs de points.

1. Introduction

Amenability of group actions is a far-reaching generalization of classical amenability for locally compact groups. It was proposed by R. J. Zimmer ([12, Definition 4.3.1]) in the measurable setting, and in the topological setting by J. Renault (measurewise amenability, see [10, Definition 2.3.6])

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and C. Anantharaman-Delaroche (topological amenability, see [2, Definition 2.1]). We recall the relevant definitions.

Definition 1.1[(1)]. — *Let G be a locally compact group acting measurably on a measure space (X,μ), where μ is a quasi-invariant measure. We say that the action $G \curvearrowright X$ is* amenable in the sense of Zimmer *if there exists a G-equivariant conditional expectation $L^\infty(X\times G,\mu\times\lambda_G)\to L^\infty(X,\mu)$, where λ_G denotes Haar measure on G.*

Definition 1.2. — *Let G be a locally compact group acting continuously on a locally compact space Y.*

(1) *The action $G \curvearrowright Y$ is* measurewise amenable *if the action of G on (Y,μ) is amenable in the sense of Zimmer for every quasi-invariant measure μ on Y.*
(2) *We denote by* $\mathrm{Prob}(G)$ *the set of probability measures on G, equipped with the weak *-topology. We say that the action $G \curvearrowright Y$ is* topologically amenable *if there exists a net $(m^i)_{i\in I}$ of continuous functions $m^i : Y \to \mathrm{Prob}(G) : y \mapsto m^i_y$ such that* $\lim_{i\to\infty}\|g_* m^i_y - m^i_g y\| = 0$ *uniformly on compact subsets of $Y\times G$ (where the norm corresponds to the total variation distance).*

It follows from any of the three definitions above, that a locally compact group is amenable if and only if it acts amenably on a one-point space.

It was proved by Anantharaman-Delaroche and Renault that topological amenability implies measurewise amenability ([3, Proposition 3.3.5]), and that they are equivalent for actions of countable discrete groups ([3, Corollary 3.3.8]). The equivalence for general group actions was a long-standing open question that was settled only recently through combined work by Buss–Echterhoff–Willett [6] and Bearden–Crann [4]: *for a locally compact second countable group G acting countinuously on a locally compact second countable space Y, the action $G \curvearrowright Y$ is topologically amenable if and only if it is measurewise amenable* (see [6, Corollary 3.29]).

The goal of this note is to prove:

Theorem 1.3. — *Let K be a locally compact field of characteristic 0. Let G be a linear algebraic group defined over K, acting algebraically on an algebraic variety V. The action of $G(K)$ (the group of K-rational points of G) on $V(K)$ is topologically amenable, if and only if all points stabilizers in $G(K)$ are solvable-by-compact.*

Recall that, for a locally compact second countable group G acting continuously on a locally compact second countable space Y, the action of G on Y

(1) This is not Zimmer's original definition, but it is equivalent to it by [1, Theorem A].

is *smooth* if the quotient Borel structure on $G\backslash Y$ is countably separated. In Section 2 we prove that, for smooth actions, topological amenability is equivalent to amenability of stabilizers (see Proposition 2.4[2]). As explained in Section 3, Theorem 1.3 follows then easily by combining this result with a deep result by Borel–Serre [5] asserting that the action $G(K) \curvearrowright V(K)$ is always smooth.

We end this Introduction with the example that motivated the present study.

Example 1.4. — For $n \geqslant 0$, let ρ_n denote the $(n+1)$-dimensional irreducible representation of $\mathrm{SL}_2(K)$ on the space $P_n(K)$ of homogeneous polynomials of degree n with coefficients in K in the 2 variables X, Y. So, for $P(X,Y) \in P_n(K)$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(K)$, we define

$$\left(\rho_n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)P\right)(X,Y) = P(aX + cY, bX + dY).$$

Going to the projective space of $P_n(K)$, we get an action of $\mathrm{SL}_2(K)$ on the projective space $\mathbb{P}^n(K)$. In [9, Proposition 3.5], the authors gave a direct but *ad hoc* proof of the fact that, for $K = \mathbb{R}, \mathbb{C}$, the action $\mathrm{SL}_2(K) \curvearrowright \mathbb{P}^n(K)$ is topologically amenable. Now, for any local field K of characteristic 0, stabilizers of points in $\mathbb{P}^n(K)$ are proper algebraic subgroups of dimension at most 2, so they are solvable. By Theorem 1.3, the action $\mathrm{SL}_2(K) \curvearrowright \mathbb{P}^n(K)$ is topologically amenable.

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2. Amenable actions vs. amenable equivalence relations

The next lemma is probably well-known.

Lemma 2.1. — *Let the locally compact group G act continuously on a locally compact space Y. If the action of G on Y is topologically amenable, then for every $y \in Y$ the stabilizer G_y is amenable.*

[2] After reading the first version of this paper, Jean Renault was able to prove that, for a smooth, second countable, locally compact groupoid endowed with a Borel Haar system, measurewise amenability is equivalent to amenability of the isotropy subgroups.

Proof. — As the action $G \curvearrowright Y$ is topologically amenable, by [2, Proposition 2.5] there exists a net $(h_i)_{i\in I}$ of continuous, compactly supported, positive type functions(3) on $Y \times G$ that converges to 1 uniformly on compact subsets on $Y \times G$. For $t \in G_y$, set $g_i(t) = h_i(y,t)$. Then $(g_i)_{i\in I}$ is a net of continuous, compactly supported, positive type functions on G_y that converges to 1 on compact subsets of G_y: the existence of such a net characterizes amenability of G_y. □

We must now discuss amenable equivalence relations as introduced in [8]. So let X be a standard Borel space and $\mathcal{R} \subset X \times X$ be a Borel equivalence relation. A transverse measure Λ for $\mathcal{R}$ is defined as in [7, Definition 1 in Chapter II]; our aim is to define the amenability of the pair $(\mathcal{R}, \text{class of}\,\Lambda)$. Denote by $p_1 : X \times X \to X : (x,y) \mapsto x$ the first projection map. We choose an auxiliary *transverse function* ν for $\mathcal{R}$, i.e. a map $x \mapsto \nu^x$ from X to the space of positive measures on $\mathcal{R}$, such that:

- For every $x \in X$, the measure ν^x is non-zero and supported on $[x]_{\mathcal{R}} =: p_1^{-1}(x) \cap \mathcal{R}$. Moreover ν is invariant in the sense that $\gamma\nu^x = \nu^y$ for every $\gamma = (y,x) \in \mathcal{R}$.
- For every Borel set A in $\mathcal{R}$, the map $X \to [0,+\infty] : x \mapsto \nu^x(A)$ is measurable.
- ν is proper in the sense that $\mathcal{R}$ is a countable union of Borel sets $(A_n)_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, the function $x \mapsto \nu^x(A_n)$ is bounded.

Theorem 3 in [7, Chapter II] then provides a bijection between transverse measures on $\mathcal{R}$ and ordinary measures μ on X such that the measure $m =: \int_X \nu^x \, d\mu(x)$ on $\mathcal{R}$ is invariant under the flip on $\mathcal{R}$.

Definition 2.2 (See [8, p. 446]). — *The pair $(\mathcal{R}, \text{class of}\,\Lambda)$ is amenable if, for almost every $x \in X$ there exists a state p_x on $L^\infty([x]_{\mathcal{R}}, \nu^x)$ such that:*

(1) *The family $(p_x)_{x\in X}$ is invariant in the sense that $\gamma p_x = p_y$ for every $\gamma = (y,x) \in \mathcal{R}$.*
(2) *With $m = \int_X \nu^x \, \mathrm{d}\mu(x)$ as above, for every $f \in L^\infty(\mathcal{R}, m)$ the function $x \mapsto p_x(f)$ is μ-measurable.*

It is proved in [8, Lemma 15] that this definition is in fact independent of the choice of the transverse function ν.

With $X, \mathcal{R}$ as above, we say that the equivalence relation $\mathcal{R}$ is *smooth* if the quotient Borel structure on $\mathcal{R}\backslash X$ is countably separated (see [12,

(3) Recall that a function h on $Y \times G$ is positive type if, for every $n \geqslant 1, y \in Y$, $t_1, \dots, t_n \in G, \lambda_1, \dots, \lambda_n \in \mathbb{C}$, we have $\sum_{i,j} \overline{\lambda_i}\lambda_j h(t_i^{-1}y, t_i^{-1}t_j) \geqslant 0$.

Definition 2.1.9]). Extending the first part of Definition 1.2, we say that $\mathcal{R}$ is *measurewise amenable* if $(\mathcal{R}, \text{class of}\,\Lambda)$ is amenable for every transverse measure Λ.

LEMMA 2.3. — *Any smooth Borel equivalence relation is measurewise amenable.*

Proof. — Let $\mathcal{R}$ be a smooth equivalence relation on the standard Borel space X.

- We denote by $\pi : X \to Y =: \mathcal{R}\backslash X$ the quotient map. By the discussion on [7, p. 47], there is a bijection between transverse measures for $\mathcal{R}$ and ordinary measures on Y, so we choose a measure λ on Y, which we may assume to be a probability measure.
- By the von Neumann selection theorem ([12, Theorem A.9]), there exists a co-null standard Borel set $Z \subset Y$ and a Borel section $s : Z \to X$ such that $\pi \circ s = \mathrm{Id}_Z$. Replacing Y by Z and X by $\pi^{-1}(Z)$, we then define the Borel map $\iota : X \to \mathcal{R} : x \mapsto (x, s(\pi(x)))$ which is a section of $p_1|_{\mathcal{R}}$; observe that, for $\gamma = (y, x) \in \mathcal{R}$, we have $\gamma\iota(x) = \iota(y)$. We then define the transverse function ν as $\nu^x =: \delta_{\iota(x)}$ (the Dirac mass at $\iota(x)$, viewed as a probability measure on $\mathcal{R}$); we will use ν to check the conditions in Definition 2.2. Observe that $\nu^{s(y)} = \delta_{\iota(s(y))} = \delta_{(s(y),s(y))}$ for $y \in Y$.
- Set $\mu =: s_*\lambda$, form the measure $m = \int_X \nu^x \, \mathrm{d}\mu(x)$ on $\mathcal{R}$. Since
$$m = \int_Y \nu^{s(y)} \, \mathrm{d}\lambda(y) = \int_Y \delta_{(s(y),s(y))} \, \mathrm{d}\lambda(y),$$
the measure m is supported on the diagonal, hence invariant under the flip on $\mathcal{R}$.
- Finally, since $L^\infty([x]_{\mathcal{R}}, \nu^x)$ is one-dimensional, the state p_x is tautological: it is given by $p_x(\varphi) = \varphi(\iota(x))$ for $\varphi \in L^\infty([x]_{\mathcal{R}}, \nu^x)$. The invariance of the family $(p_x)_{x\in X}$ is then clear. Then for $f \in L^\infty(\mathcal{R}, m)$, the function $x \mapsto p_x(f) = f(\iota(x))$ is clearly μ-measurable. This concludes the proof. □

Let G be a locally compact second countable group acting continuously on a locally compact second countable space Y. The action of G on Y is *smooth* if the orbital equivalence relation $\mathcal{R}_G$ is smooth.

PROPOSITION 2.4. — *Let G act continuously on Y as above, and assume that G acts smoothly on Y. The following are then equivalent:*

(i) *The action $G \curvearrowright Y$ is topologically amenable.*
(ii) *All point stabilizers in G are amenable.*

Proof. — The implication (i) ⇒ (ii) follows immediately from Lemma 2.1. For the converse implication (ii) ⇒ (i): in view of the discussion following Definition 1.2, it is enough to prove that the action $G \curvearrowright Y$ is measurewise amenable. So fix a quasi-invariant measure μ on Y, we must prove that $G \curvearrowright (Y, \mu)$ is amenable in the sense of Zimmer. We appeal to a result by S. Adams, G. Elliott and T. Giordano ([1, Theorem 5.1]): *assume that the second countable locally compact group G acts measurably on a standard measure space (X, ν) with ν a quasi-invariant probability measure. If stabilizers are amenable ν-almost everywhere, and if the orbital equivalence relation $\mathcal{R}_G$ is amenable, then the action of G on (X, ν) is amenable in the sense of Zimmer.* In view of the Adams–Elliott–Giordano result, since stabilizers are assumed to be amenable, the result follows immediately from Lemma 2.3. □

As was already mentioned in Section 1, Jean Renault was able to generalize Proposition 2.4 as follows. *Let $\mathcal{G}$ be a second countable locally compact endowed with a Borel Haar system. If $\mathcal{G}$ is measurewise amenable, then the isotropy subgroup $\mathcal{G}(x)$ is amenable for every $x \in \mathcal{G}^{(0)}$. If $\mathcal{G}(x)$ is amenable for every $x \in \mathcal{G}^{(0)}$ and $\mathcal{G}$ is smooth, then $\mathcal{G}$ is measurewise amenable.* This result is presented as an appendix to the present paper.

3. Proof of Theorem 1.3

We will appeal to a deep result of A. Borel and J.-P. Serre ([5, Proposition 4.9 and Corollaire 6.4]; see also [12, Theorem 3.1.3]): all orbits of $G(K)$ in $V(K)$ are locally closed. By the Glimm–Effros theorem (see [12, Theorem 2.1.14]), having all orbits locally closed is equivalent to smoothness of the action, so we are in the assumptions of our Proposition 2.4, to the effect that topological amenability of $G(K) \curvearrowright V(K)$ is equivalent to amenability of stabilizers. Since solvable-by-compact groups are amenable, one implication in Theorem 1.3 becomes clear.

For the converse, assume that the action $G(K) \curvearrowright V(K)$ is amenable. Fix $y \in V(K)$, and denote by H the stabilizer of y in $G(K)$: by Lemma 2.1, the subgroup H is amenable, and we must prove that H is solvable-by-compact. Since H is the group of K-points of a linear algebraic group, factoring out the solvable radical we may assume that H is semi-simple and must prove that H is compact. For algebraic groups over local fields of characteristic 0, being compact is equivalent to being anisotropic. So assume that H is isotropic. Then by the Jacobson–Morozov theorem H contains a copy either of $SL_2(K)$ or of $PGL_2(K)$, contradicting amenability of H. □

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