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Appendix to "Amenable actions of real and p-adic algebraic groups" by Alain Valette (*)

Jean Renault (1)

Abstract. — As a complement to Alain Valette's article, this appendix gives an extension from group actions to groupoids of Proposition 2.4. Namely, it is shown that, for a smooth, second countable, locally compact groupoid endowed with a Borel Haar system, measurewise amenability is equivalent to amenability of the isotropy subgroups.

Résumé. — En complément à l'article d'Alain Valette, cet article généralise des actions de groupes aux groupoïdes la proposition 2.4. Spécifiquement, on montre qu'un groupoïde localement compact, à base dénombrable d'ouverts et lisse, est moyennable pour toute mesure quasi-invariante si et seulement si tous ses stabilisateurs sont moyennables.

Group actions on spaces (also called transformation groups) and equivalence relations, such as the orbital equivalence relation of a group action, are two important classes of groupoids which motivate the study of groupoids. We refer the reader to [1, 5] for definitions and constructions not given here. We simply recall that, given an action of a group G on a space X, one can consider the groupoid $\mathcal{G} = G \ltimes X$ (or $X \rtimes G$ if more convenient), called the semi-direct product of the action. By definition, the graph of an equivalence relation R on a set X is also a groupoid. We are interested in locally compact groupoids, such as $G \ltimes X$ when G and X are locally compact and the action is continuous, and measured groupoids. A measured groupoid is a triple $(\mathcal{G}, \lambda, \mu)$, where $\mathcal{G}$ is a Borel groupoid, $\lambda = (\lambda^x)_{x\in\mathcal{G}^{(0)}}$ is a Borel system of measures, λ^x being supported on $\mathcal{G}^x = r^{-1}(x)$, which is left invariant in the sense that $\gamma\lambda^{s(\gamma)} = \lambda^{r(\gamma)}$ for all γ in $\mathcal{G}$ and μ is a measure on the unit space $\mathcal{G}^{(0)}$ which is quasi-invariant in the sense that the measure $\mu\circ\lambda$ and its image

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Article proposé par Romain Tessera.

$(\mu \circ \lambda)^{-1}$ by the inverse map are equivalent measures on $\mathcal{G}$. A non-singular action of a locally compact group G on a measure space (X, μ) gives the measured groupoid $(X \rtimes G, \lambda, \mu)$, where λ^x is the product of the Dirac mass at x and a left Haar measure λ_G of G. Then, $\mu \circ \lambda = \mu \times \lambda_G$.

We are interested in two notions which have been extended to the framework of groupoids: smoothness and amenability. The Mackey–Glimm dichotomy, expressed first for continuous actions of second countable locally compact groups on second countable locally compact spaces by J. Glimm in [2], has been shown to hold for second countable locally compact groupoids by A. Ramsay in [4]. It gives the equivalence of a number of properties of a second countable locally compact groupoid $\mathcal{G}$ which characterise its smoothness (or equivalently the smoothness of its orbital equivalence relation). We retain from [4, Theorem 2.1] three of them, taking into account that the graph of the orbital equivalence relation is an F_σ subset of $\mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$:

(1) the orbits are locally closed;
(2) the ergodic quasi-invariant measures are transitive;
(3) the quotient space $\mathcal{G}^{(0)}/\mathcal{G}$ is a standard Borel space.

The classical notion of amenability of groups has been extended to non-singular actions of second countable locally compact groups and equivalence relations by R. Zimmer in [6]. It has been extended to measured groupoids and locally compact groupoids in [5]. The amenability in the sense of Zimmer of the action of the locally compact group G on the measured space (X, μ) is the amenability of the measured groupoid $(X \rtimes G, \lambda, \mu)$. The monograph [1] gives an account of these notions. An important property of amenability is that it is preserved under groupoid equivalence, both in the topological and in the measured settings ([1, Theorems 2.2.17 & 5.3.20]), a result which fits Mackey's virtual group point of view. This is an essential ingredient of the proof below. Generalising the definition given in the text for a Borel equivalence relation, a Borel groupoid $\mathcal{G}$ equipped with a Borel Haar system λ is said to be measurewise amenable if the measured groupoid $(\mathcal{G}, \lambda, \mu)$ is amenable for every quasi-invariant measure μ.

Proposition. — *Let $\mathcal{G}$ be a second countable locally compact groupoid endowed with a Borel Haar system λ. If $\mathcal{G}$ is measurewise amenable, then its isotropy subgroups $\mathcal{G}(x)$ are amenable for all $x \in \mathcal{G}^{(0)}$. If the isotropy subgroups $\mathcal{G}(x)$ are amenable for all $x \in \mathcal{G}^{(0)}$ and if $\mathcal{G}$ is smooth, then $\mathcal{G}$ is measurewise amenable.*

Proof. — The proof of both implications hinges on the groupoid equivalence of the isotropy subgroup $\mathcal{G}(x)$ and the semi-direct product $\mathcal{G} \ltimes (\mathcal{G}/\mathcal{G}(x))$.

Suppose that $\mathcal{G}$ is measurewise amenable. Let x be a unit. Since the embedding of $\mathcal{G}(x)$ into $\mathcal{G}$ is proper in the sense of [1, Definition 5.3.21], the measurewise amenability of $\mathcal{G}$ implies the amenability of $\mathcal{G}(x)$ ([1, Corollary 5.3.22]).

Suppose that $\mathcal{G}$ is smooth. Let μ be a quasi-invariant measure. Since the quotient space $\mathcal{G}^{(0)}/\mathcal{G}$ is a standard Borel space, we can disintegrate μ along the quotient map. We choose a pseudo-image $\underline{\mu}$ of μ and obtain $\mu = \int \mu_\omega \mathrm{d}\underline{\mu}(\omega)$. According to [1, Proposition 5.3.4], μ_ω is quasi-invariant for a.e. ω and the measured groupoid $(\mathcal{G}, \lambda, \mu)$ is amenable if and only if for a.e. ω, the measured groupoid $(\mathcal{G}, \lambda, \mu_\omega)$ is amenable. The measured groupoid $(\mathcal{G}, \lambda, \mu_\omega)$ is isomorphic to the transitive groupoid $(\mathcal{G}_{|\omega}, \lambda_{|\omega}, \mu_\omega)$, where $\mathcal{G}_{|\omega}$ is the reduction of $\mathcal{G}$ to the orbit ω. Since ω is locally closed, $\mathcal{G}_{|\omega}$ is locally compact. It is well known ([3, Example 2.2]) that the transitive second countable locally compact groupoid $\mathcal{G}_{|\omega}$ is topologically equivalent to any of the isotropy groups $\mathcal{G}(x)$, where x belongs to the orbit ω. If $\mathcal{G}(x)$ is amenable, according to [1, Theorem 2.2.17], $\mathcal{G}_{|\omega}$ is topologically amenable and the measured groupoid $(\mathcal{G}_{|\omega}, \lambda_{|\omega}, \mu_\omega)$ is amenable. Let us note that, in the transitive case, there is a unique invariant measure class and that R. Zimmer had already observed in [6] the equivalence of the amenability of a transitive group action and the amenability of any of its isotropy subgroups. Therefore, if $\mathcal{G}(x)$ is amenable for every $x \in G^{(0)}$, then the measured groupoid $(\mathcal{G}, \lambda, \mu_\omega)$ is amenable for a.e. ω. As recalled, this implies that $(\mathcal{G}, \lambda, \mu)$ is amenable. Since this holds for every quasi-invariant measure μ, $\mathcal{G}$ is measurewise amenable. □

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