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Geodesics and visual boundary of horospherical products ^(*)

TOM FERRAGUT ⁽¹⁾

ABSTRACT. — We study the geometry of *horospherical products* by providing a description of their distances, geodesics and visual boundary. These products contain both discrete and continuous examples, including Cayley graphs of lamplighter groups and solvable Lie groups of the form $\mathbb{R} \ltimes (N_1 \times N_2)$, where N_1 and N_2 are two simply connected, nilpotent Lie groups.

RÉSUMÉ. — Nous étudions la géométrie des *produits horosphériques* en fournissant une description de leurs distances, de leurs géodésiques et de leur bord visuel. Ces produits incluent à la fois des exemples discrets et continus, notamment les graphes de Cayley des groupes *lamplighter* ainsi que certains groupes de Lie résolubles de la forme $\mathbb{R} \ltimes (N_1 \times N_2)$, où N_1 et N_2 sont deux groupes de Lie nilpotents, simplement connexes.

1. Introduction

A *horospherical product* is a metric space constructed from two Gromov hyperbolic spaces X and Y , embedded in their Cartesian product $X \times Y$ and viewed as a diagonal subset of this product. Let $\beta_X : X \rightarrow \mathbb{R}$ and $\beta_Y : Y \rightarrow \mathbb{R}$ be two Busemann functions. The horospherical product of X and Y , denoted $X \bowtie Y$, is defined as the set of points in $X \times Y$ where the sum of the two Busemann functions equals zero, that is,

$$X \bowtie Y := \{(x, y) \in X \times Y \mid \beta_X(x) + \beta_Y(y) = 0\}.$$

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The level sets of the Busemann functions are called *horospheres*; intuitively, the horospherical product $X \bowtie Y$ can be seen as X crossed with an inverted copy of Y , parallel to these horospheres. We refer to the opposite of the chosen Busemann function as the *height function*.

Let N be a simply connected, nilpotent Lie group, and let A be a derivation of $\text{Lie}(N)$ whose eigenvalues have positive real parts. The group $\mathbb{R} \ltimes_A N$ is known as a *Heintze group* and is Gromov hyperbolic; these are the only examples of negatively curved Lie groups. Let X and Y be two Heintze groups. By selecting Busemann functions such that for $\forall (t, n) \in \mathbb{R} \ltimes_A N$, $\beta(t, n) = -t$, we obtain

$$(\mathbb{R} \ltimes_{A_1} N_1) \bowtie (\mathbb{R} \ltimes_{A_2} N_2) = \mathbb{R} \ltimes_{\text{Diag}(A_1, -A_2)} (N_1 \times N_2).$$

When $N = \mathbb{R}$, the corresponding Heintze group is the hyperbolic plane \mathbb{H}^2 . Moreover, the horospherical products in the case $N_1 = N_2 = \mathbb{R}$ yield the Sol geometry, one of the eight Thurston geometries. Other constructions, such as Diestel–Leader graphs and the Cayley 2-complexes of Baumslag Solitar groups $\text{BS}(1, n)$, arise as horospherical products of trees or hyperbolic planes. In [17, Section 2], these last three examples are thoroughly detailed and presented as horocyclic products of either regular trees or the hyperbolic plane \mathbb{H}^2 . Here, we adopt the term *horospherical product* instead of *horocyclic product*, as in higher dimensions the level sets defined by a Busemann function are horospheres rather than horocycles.

As suggested by Woess at the conclusion of [17], we explore a generalization of horospherical products. While the horospherical product construction can extend to more than two spaces (see [1] for a study of Brownian motion on a multiple horospherical product of trees), in this work, we restrict our attention to the case of two Gromov hyperbolic spaces.

A *Busemann space* is a metric space where the distance between any two geodesics is a convex function. A metric space X is said to be *geodesically complete* if every geodesic segment $\alpha : I \rightarrow X$ can be extended to a geodesic line $\hat{\alpha} : \mathbb{R} \rightarrow X$. For our study of the geometry of horospherical products, we assume that the component spaces X and Y are both proper, geodesically complete, Gromov hyperbolic, Busemann spaces. The Busemann property aligns naturally with the definition of horospherical products, as it ensures that the height functions of X and Y can be made exactly opposite (see Section 2.3 for further details). Furthermore, under the additional assumption of geodesic completeness of X and Y , we demonstrate that the horospherical product $X \bowtie Y$ is connected (see Lemma 3.11).

In the following section of this introduction, we present our main results, which apply under the assumptions that X and Y are two proper, geodesically complete, Gromov hyperbolic, Busemann spaces. These results contain the case where X and Y are solvable Lie groups of the form $\mathbb{R} \rtimes_A N$.

In [7, 8], Farb and Mosher used the horospherical product structure of treebolic spaces to prove a rigidity result for quasi-isometries of $\text{BS}(1, n)$. Similarly, in [5, 6], Eskin, Fisher, and Whyte established a comparable rigidity result for the Diestel–Leader graphs and Sol geometries, also utilizing their horospherical product structures.

In addition to being significant in their own right, the tools developed in this paper are used in [9] to study the quasi-isometry classification of the aforementioned horospherical products. More specifically, in [9] we generalize the results of Eskin, Fisher, and Whyte from [5], providing new quasi-isometric classifications for certain families of solvable Lie groups, and more.

There are various choices for defining a distance on $X \bowtie Y$. Here, we consider a family of length path metrics induced by distances on $X \times Y$ (see Definition 3.2). Specifically, we require that the distance on $X \bowtie Y$ comes from an *admissible* norm N on \mathbb{R}^2 (e.g. any ℓ_p norm). Our first result describes these distances.

THEOREM 1.1. — *Let d_{\bowtie} be an admissible distance on $X \bowtie Y$. Then there exists a constant M depending only on the metric spaces $(X \bowtie Y, d_{\bowtie})$ such that for all $p = (p^X, p^Y), q = (q^X, q^Y) \in X \bowtie Y$:*

$$|d_{\bowtie}(p, q) - (d^X(p^X, q^X) + d^Y(p^Y, q^Y) - |h(p) - h(q)|)| \leq M.$$

Hence, given two admissible distances d and d' , the horospherical products $(X \bowtie Y, d)$ and $(X \bowtie Y, d')$ are roughly isometric, meaning that they are $(1, c)$ -quasi-isometric for some constant $c \geq 0$. Le Donne, Pallier, and Xie demonstrated in [13] that, for the solvable groups $\mathbb{R} \rtimes_{\text{Diag}(A_1, -A_2)}(N_1 \times N_2)$, altering the left-invariant Riemannian metric results in the identity map being a rough similarity.

Theorem 1.1 is one of the tools we utilize in [9] to establish a geometric rigidity property of quasi-isometries within families of horospherical products. This property yields quasi-isometric invariants for these spaces, leading to a first result in the quasi-isometry classification of certain solvable Lie groups.

Throughout this paper, we also provide a coarse description of the geodesics and the visual boundary for a broad family of horospherical products.

We refer to a geodesic as *vertical* if the distance between any two of its points corresponds to the difference in their heights. Following our characterization of distances on horospherical products, we then describe the structure of these geodesic segments.

THEOREM 1.2. — *Let X and Y be two proper, geodesically complete, δ -hyperbolic, Busemann spaces and let d_{\bowtie} be an admissible distance on $X \bowtie Y$. Let $p = (p^X, p^Y)$ and $q = (q^X, q^Y)$ be two points of $X \bowtie Y$ and let α be a geodesic segment of $(X \bowtie Y, d_{\bowtie})$ linking p to q . There exists a constant M depending only on $(X \bowtie Y, d_{\bowtie})$, and there exist two vertical geodesics $V_1 = (V_1^X, V_1^Y)$ and $V_2 = (V_2^X, V_2^Y)$ such that:*

- (1) *If $h(p) \leq h(q) - M$ then α is in the M -neighbourhood of $V_1 \cup (V_1^X, V_2^Y) \cup V_2$;*
- (2) *If $h(p) \geq h(q) + M$ then α is in the M -neighbourhood of $V_1 \cup (V_2^X, V_1^Y) \cup V_2$;*
- (3) *If $|h(p) - h(q)| \leq M$ then at least one of the conclusions of (1) or (2) holds.*

Specifically V_1 and V_2 can be chosen such that p is close to V_1 and q is close to V_2 .

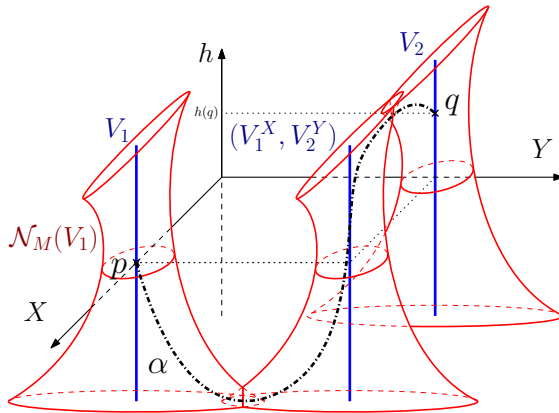


Figure 1.1. Shape of geodesic segments when $h(p) \leq h(q) - \kappa$ in $X \bowtie Y$. The neighbourhoods' shapes are distorted since when going upward, distances are contracted in the “direction” X and expanded in the “direction” Y .

An example is illustrated in Figure 1.1 for the case $h(p) \leq h(q) - \kappa$. Roughly speaking, Theorem 1.2 guarantees that any geodesic segment can

be decomposed into a concatenation of three vertical geodesics. This is analogous to the Gromov hyperbolic case, where a geodesic segment lies within a bounded neighborhood of two vertical geodesics. The structure of geodesics is already well understood in lamplighter groups, and in the case of Sol, this result recovers Troyanov’s description of global geodesics, up to an additive constant (see [15]).

The horospherical product between X and \mathbb{R} is isometric to X ; hence, given any vertical geodesic V^Y in Y , the set $X \bowtie V^Y$ forms an embedded copy of X within $X \bowtie Y$. Consequently, a geodesic line in $X \bowtie Y$ resembles either a geodesic of X or a geodesic of Y .

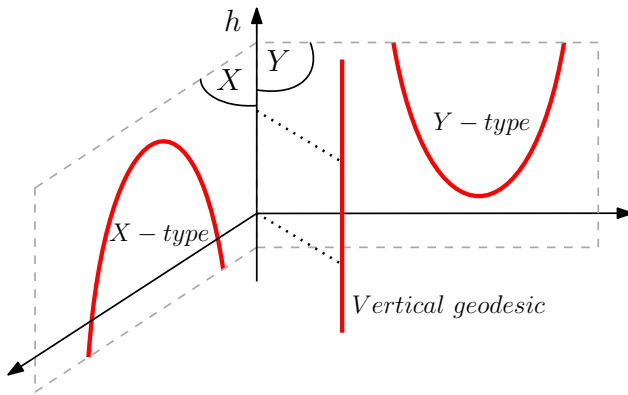


Figure 1.2. Different types of geodesics in $X \bowtie Y$.

COROLLARY 1.3. — *Let X and Y be two proper, geodesically complete, δ -hyperbolic, Busemann spaces. Then there exists $M \geq 0$ depending only on δ such that for all geodesic lines $\alpha : \mathbb{R} \rightarrow X \bowtie Y$ at least one of the two following statements holds.*

- (1) α is included in a constant M -neighbourhood of a geodesic contained in an embedded copy of X ;
- (2) α is included in a constant M -neighbourhood of a geodesic contained in an embedded copy of Y .

If a geodesic satisfies both conclusions, it is in the M -neighbourhood of a vertical geodesic of $X \bowtie Y$.

Let $o \in X \bowtie Y$. Two geodesic rays are considered equivalent if they remain at a finite distance from each other. The visual boundary of $X \bowtie Y$ relative to the base point o , denoted $\partial_o(X \bowtie Y)$, is the set of equivalence classes of geodesic rays starting from o . Based on our description of geodesic

segments, we find that for any geodesic ray k in $X \bowtie Y$, there exists a vertical geodesic ray at finite distance from k . Consequently, we classify all possible shapes of geodesic rays and provide a description of the visual boundary of $X \bowtie Y$ (pictured on Figure 1.3).

THEOREM 1.4. — *Let X and Y be two proper, geodesically complete, δ -hyperbolic, Busemann spaces. Let $(w^X, a^X) \in X \times \partial X$, $(w^Y, a^Y) \in Y \times \partial Y$ and let $X \bowtie Y$ be the horospherical product with respect to (w^X, a^X) and (w^Y, a^Y) . Then the visual boundary of $X \bowtie Y$ with respect to any point $o = (o^X, o^Y)$ can be decomposed as:*

$$\begin{aligned} \partial_o(X \bowtie Y) &= \left((\partial X \setminus \{a^X\}) \times \{a^Y\} \right) \cup \left(\{a^X\} \times (\partial Y \setminus \{a^Y\}) \right) \\ &= \left((\partial X \times \{a^Y\}) \cup (\{a^X\} \times \partial Y) \right) \setminus \{(a^X, a^Y)\}. \end{aligned}$$

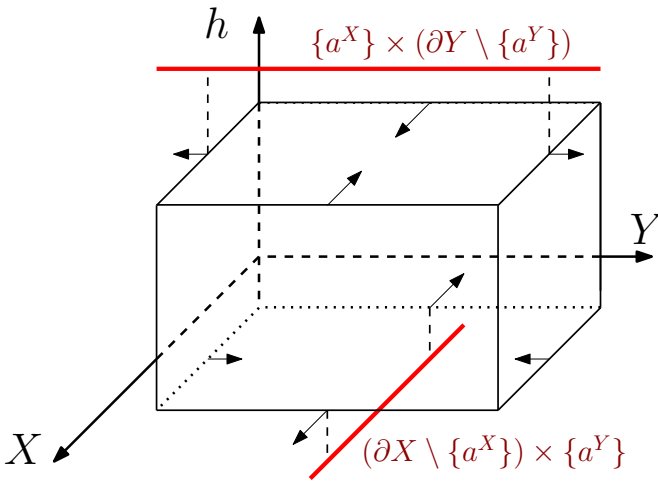


Figure 1.3. Depiction of $\partial_o(X \bowtie Y)$.

When $X := \mathbb{R} \times_{A_1} N_1$ and $Y := \mathbb{R} \times_{A_2} N_2$ we obtain that

$$\partial(\mathbb{R} \times_{\text{Diag}(A_1, -A_2)} (N_1 \times N_2)) = N_1 \times N_2.$$

In the case of Sol, the last result is similar to [15, Proposition 6.4]. However, unlike Troyanov in his work, we are focusing on *minimal* (globally-length-minimizing) geodesics and not on local ones. One can see that this visual boundary neither depends on the chosen admissible distance d nor on the base point o .

Framework

The paper is organized as follows:

- Section 2: We present the setting for our work, focusing on Gromov hyperbolic and Busemann spaces, which provide the foundation for constructing horospherical products.
- Section 3: We define horospherical products and illustrate the concept through several examples.
- Section 4: We establish a key result, Lemma 4.9, which provides an estimate on the length of paths avoiding horoballs in hyperbolic spaces. This lemma plays a central role in controlling distances on $X \bowtie Y$. We then obtain an estimate for these distances in Theorem 4.13.
- Section 5: We prove our main results. Theorem 1.1, which follows from Corollary 4.13, forms the basis for describing geodesic lines in Theorem 1.2. These results, in turn, equip us with the tools needed to prove Theorem 1.4.

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2. Context

The goal of this section is to present what is a Gromov hyperbolic, Busemann space and what are vertical geodesics in such a space. Let (H, d_H) be a proper, geodesic, metric space.

2.1. Gromov hyperbolic spaces

A geodesic line, respectively ray, segment, of H is the isometric image of a Euclidean line, respectively half Euclidean line, interval, in H . By slight

abuse, we may call geodesic, geodesic ray or geodesic segment, the map $\alpha : I \rightarrow H$ itself, which parametrises our given geodesic by arclength.

Let $\delta \geq 0$ be a non-negative number. Let x, y and z be three points of H . The geodesic triangle $[x, y] \cup [y, z] \cup [z, x]$ is called δ -slim if any of its sides is included in the δ -neighbourhood of the remaining two. The metric space H is called δ -hyperbolic if every geodesic triangle is δ -slim. A metric space H is called Gromov hyperbolic if there exists $\delta \geq 0$ such that H is a δ -hyperbolic space.

An important property of Gromov hyperbolic spaces is that they admit a nice compactification thanks to their *Gromov boundary*. We call two geodesic rays of H equivalent if their images are at finite Hausdorff distance. Let $w \in H$ be a base point. We define $\partial_w H$ the Gromov boundary of H as the set of families of equivalent rays starting from w . The boundary $\partial_w H$ does not depend on the base point w , hence we will simply denote it by ∂H . Both ∂H and $H \cup \partial H$, are compact endowed with the Hausdorff topology. For more details, see [11] or [3, Part III, Chapter H, p. 399].

Let us fix a point $a \in \partial H$ on the boundary. We call *vertical geodesic ray*, respectively *vertical geodesic line*, any geodesic ray in the equivalence class a , respectively with one of its rays in a . The study of these specific geodesic rays is central in this work.

2.2. Busemann spaces and Busemann functions

A metric space (H, d_H) is *Busemann* if and only if for every pair of geodesic segments parametrized by arclength $\gamma : [a, b] \rightarrow H$ and $\gamma' : [a', b'] \rightarrow H$, the following function is convex:

$$D_{\gamma, \gamma'} : [a, b] \times [a', b'] \longrightarrow H \\ (t, t') \longmapsto d_H(\gamma(t), \gamma'(t')).$$

More specifically, if one moves linearly from (t_1, t'_1) to (t_2, t'_2) along the geodesics γ and γ' , the value of $D_{\gamma, \gamma'}$ can only decrease and then increase. This property implies that H is uniquely geodesic, as two different geodesic segments cannot share the same endpoints.

It is a weaker assumption than being CAT(0) ([10, Theorem 1.3]). See [14, Chapters 8–12] for more details on Busemann spaces.

This convex assumption removes some technical difficulties in a significant number of proofs in this work. If H is a Busemann space in addition to being Gromov hyperbolic, for all $x \in H$ there exists a unique vertical geodesic ray, denoted by V_x , starting at H . In fact the distance between two vertical

geodesics starting at x is a convex and bounded function, hence decreasing and therefore constant equal to 0.

The construction of the *horospherical product* of two Gromov hyperbolic space X and Y requires the so called *Busemann functions*. Their definition is simplified by the Busemann assumption. Let us consider ∂X , the Gromov boundary of X (which, in this setting, is the same as the visual boundary). Both the boundary ∂X and $X \cup \partial X$, endowed with the natural Hausdorff topology, are compact. Then, given $a \in \partial X$ a point on the boundary, and $w \in X$ a base point, we define a Busemann function $\beta_{(a,w)}$ with respect to a and w . Let V_w be the unique vertical geodesic ray starting from w .

$$\forall x \in X, \beta_{(a,w)}(x) := \limsup_{t \rightarrow +\infty} (d(x, V_w(t)) - t).$$

This function computes the asymptotic delay a point $x \in X$ has in a race towards a against the vertical geodesic ray starting at w . The *horospheres* of X with respect to $(a, w) \in \partial X \times X$ are the level-sets of $\beta_{(a,w)}$. These horospheres depend on the previously chosen couple (a, w) of $\partial X \times X$.

2.3. Heights functions and vertical geodesics

In this section we fix $\delta \geq 0$, H a proper, geodesic, δ -hyperbolic space, $w \in H$ a base point and $a \in \partial H$ a point on the boundary of H . We call *height function*, denoted by h , the opposite of the Busemann function, $h := -\beta_{(a,w)}$.

Let us write of [11, Chapter 8, Proposition 2, p. 136] with our notations.

PROPOSITION 2.1 ([11, Chapter 8, p. 136]). — *Let H be a hyperbolic proper geodesic metric space. Let $a \in \partial H$ and $w \in H$, then:*

- (1) $\lim_{x \rightarrow a} h_{(a,w)}(x) = +\infty$,
- (2) $\lim_{x \rightarrow b} h_{(a,w)}(x) = -\infty, \forall b \in \partial H \setminus \{a\}$
- (3) $\forall x, y, z \in H, |\beta_a(x, y) + \beta_a(y, z) - \beta_a(x, z)| \leq 200\delta$.

Furthermore, a geodesic ray is in $a \in \partial H$ if and only if its height tends to $+\infty$.

COROLLARY 2.2. — *Let H be a hyperbolic proper geodesic metric space. Let $a \in \partial H$ and $w \in H$, and let $\alpha : [0, +\infty[\rightarrow H$ be a geodesic ray. The two following properties are equivalent:*

- (1) $\lim_{t \rightarrow +\infty} h_{(a,w)}(\alpha(t)) = +\infty$,
- (2) $\alpha([0, +\infty[) \in a$.

Proof. — As for any geodesic ray $\alpha : [0, +\infty[\rightarrow H$ there exists $b \in \partial H$ such that $\alpha([0, +\infty[) \in b$, this proposition is a particular case of Proposition 2.1. \square

An important property of the height function is to be Lipschitz.

PROPOSITION 2.3. — *Let $a \in \partial H$ and $w \in H$. The height function $h_a := -\beta_a(\cdot, w)$ is Lipschitz:*

$$\forall x, y \in H, \quad |h_{(a,w)}(x) - h_{(a,w)}(y)| \leq d(x, y).$$

Proof. — By using the triangle inequality we have for all $x, y \in H$:

$$\begin{aligned} -h_{(a,w)}(x) &= \beta_a(x, w) \\ &= \sup \left\{ \limsup_{t \rightarrow +\infty} (d(x, k(t)) - t) \mid k \text{ vertical rays starting at } w \right\} \\ &\leq d(x, y) \\ &\quad + \sup \left\{ \limsup_{t \rightarrow +\infty} (d(y, k(t)) - t) \mid k \text{ vertical rays starting at } w \right\} \\ &\leq d(x, y) + \beta_a(y, w) \leq d(x, y) - h_{(a,w)}(y). \end{aligned} \tag{2.1}$$

The result follows by exchanging the roles of x and y . \square

From now on, we fix a given $a \in \partial H$ and a given $w \in H$. Therefore we simply denote the height function by h instead of $h_{(a,w)}$.

PROPOSITION 2.4. — *Let α be a vertical geodesic of H . We have the following control on the height along α :*

$$\forall t_1, t_2 \in \mathbb{R}, \quad t_2 - t_1 - 200\delta \leq h(\alpha(t_2)) - h(\alpha(t_1)) \leq t_2 - t_1 + 200\delta.$$

Proof. — Let $t_1, t_2 \in \mathbb{R}$, then:

$$\begin{aligned} h(\alpha(t_2)) - h(\alpha(t_1)) &= \beta(\alpha(t_1), w) - \beta(\alpha(t_2), w) \\ &= \beta(\alpha(t_1), \alpha(t_2)) - \left(\beta(\alpha(t_2), w) - \beta(\alpha(t_1), w) + \beta(\alpha(t_1), \alpha(t_2)) \right). \end{aligned}$$

The third point of Proposition 2.1 applied to the last bracket gives:

$$\beta(\alpha(t_1), \alpha(t_2)) - 200\delta \leq h(\alpha(t_2)) - h(\alpha(t_1)) \leq \beta(\alpha(t_1), \alpha(t_2)) + 200\delta. \tag{2.2}$$

Since $t \mapsto \alpha(t + t_2)$ is a vertical geodesic starting at $\alpha(t_2)$ we have:

$$\begin{aligned} & \beta(\alpha(t_1), \alpha(t_2)) \\ &= \sup \left\{ \limsup_{t \rightarrow +\infty} (d(\alpha(t_1), k(t)) - t) \mid k \text{ vertical rays starting at } \alpha(t_2) \right\} \\ &\geq \limsup_{t \rightarrow +\infty} (d(\alpha(t_1), \alpha(t + t_2)) - t) \\ &\geq \limsup_{t \rightarrow +\infty} (|t + t_2 - t_1| - t) \geq t_2 - t_1, \text{ for } t \text{ large enough.} \end{aligned}$$

Using this last inequality in inequality (2.2) we get $t_2 - t_1 - 200\delta \leq h(\alpha(t_2)) - h(\alpha(t_1))$. The result follows by exchanging the roles of t_1 and t_2 . \square

Using Proposition 2.4 with $t_1 = 0$ and $t_2 = t$, the next corollary holds.

COROLLARY 2.5. — *Let α be a vertical geodesic parametrised by arclength and such that $h(\alpha(0)) = 0$. We have:*

$$\forall t \in \mathbb{R}, \quad |h(\alpha(t)) - t| \leq 200\delta.$$

From now on, H will be a proper, geodesic, Gromov hyperbolic, Busemann space. Hence the height function is convex along a vertical geodesic.

PROPERTY 2.6 (Papadopoulos [14, Proposition 12.1.5, p. 263]). — *Let $\delta \geq 0$ be a non negative number. Let H be a proper δ -hyperbolic, Busemann space. For every geodesic α , the function $t \mapsto -h(\alpha(t))$ is convex.*

The Busemann hypothesis implies that the height along geodesic behaves nicely. This means that we can drop the constant 200δ from Corollary 2.5. It is one of the main reasons why we require our spaces to be Busemann spaces.

PROPOSITION 2.7. — *Let H be a δ -hyperbolic and Busemann space and let $V : \mathbb{R} \rightarrow H$ be a path of H . Then V is a vertical geodesic if and only if $\exists c \in \mathbb{R}$ such that $\forall t \in \mathbb{R}, h(V(t)) = t + c$.*

Proof. — Let V be a vertical geodesic in H . By Property 2.6 we have that $t \mapsto -h(V(t))$ is convex. Furthermore, from Corollary 2.5, we get $|h(V(t)) - t| \leq 200\delta$. Thereby the bounded convex function $t \mapsto t - h(V(t))$ is constant. Then there exists a real number c such that $\forall t \in \mathbb{R}, h(V(t)) = t + c$.

We now assume that there exists a real number c such that $\forall t \in \mathbb{R}, h(V(t)) = t + c$. Therefore, for all real numbers t_1 and t_2 we have

$$d(V(t_1), V(t_2)) \geq \Delta h(V(t_1), V(t_2)) = |t_1 - t_2|.$$

By definition V is a connected path, hence $|t_1 - t_2| \geq d(V(t_1), V(t_2))$ which implies with the previous sentence that $|t_1 - t_2| = d(V(t_1), V(t_2))$, then V is

a geodesic. Furthermore $\lim_{t \rightarrow +\infty} h(V(t)) = +\infty$, which implies by definition that V is a vertical geodesic. \square

A metric space is called geodesically complete if all its geodesic segments can be prolonged into geodesic lines. In H is geodesically complete in addition to its other assumptions, then any point of H is included in a vertical geodesic line.

PROPERTY 2.8. — *Let H be a δ -hyperbolic Busemann geodesically complete space. Then for all $x \in H$ there exists a vertical geodesic $V_x : \mathbb{R} \rightarrow H$ such that V_x contains x .*

Proof. — Let us consider in this proof $w \in H$ and $a \in \partial H$, from which we constructed the height h of our space H . Then by definition we have $h_{(a,w)} = h$. Proposition 12.2.4 of [14] ensures the existence of a geodesic ray $R_x \in a$ starting at x . Furthermore as H is geodesically complete R_x can be prolonged into a geodesic $V_x : \mathbb{R} \rightarrow H$ such that $V_x([0; +\infty[) \in a$, hence V_x is a vertical geodesic. \square

3. Horospherical products

In this part we generalise the definition of horospherical product, as seen in [5] for two trees or two hyperbolic planes, to any pair of proper, geodesically complete, Gromov hyperbolic, Busemann spaces. We recall that given a proper, δ -hyperbolic space H with distinguished $a \in \partial H$ and $w \in H$, we defined the height function on H in Section 2.3 from the Busemann functions with respect to a and w .

3.1. Definitions

Let X and Y be two δ -hyperbolic spaces. We fix the base points $w_X \in X$, $w_Y \in Y$ and the directions in the boundaries $a_X \in \partial X$, $a_Y \in \partial Y$. We consider their heights functions h_X and h_Y respectively on X and Y .

DEFINITION 3.1 (Horospherical product). — *The horospherical product of X and Y , denoted by $X \bowtie Y = X \bowtie Y$ is*

$$X \bowtie Y := \{(p_X, p_Y) \in X \times Y \mid h_X(p_X) + h_Y(p_Y) = 0\}.$$

From now on, with a slight abuse of notation, we omit base points and fixed points at infinity in the construction of the horospherical product. The metric space $X \bowtie Y$ will refer to a horospherical product of two Gromov

hyperbolic Busemann spaces. To define a horospherical product in a broader setting (without any hyperbolicity assumption), it suffices to have a Busemann function on a given metric space.

We will define the distances on $X \bowtie Y$ as length path metrics induced by distances on $X \times Y$. A lot of natural distances on the Cartesian product $X \times Y$ come from norms on the vector space \mathbb{R}^2 . Let N be such a norm and let us denote $d_N := N(d_X, d_Y)$, which means that for all couples $(p_X, p_Y), (q_X, q_Y) \in X \times Y$ we have that $d_N((p_X, p_Y), (q_X, q_Y)) = N(d_X(p_X, q_X), d_Y(p_Y, q_Y))$. The length $l_N(\gamma)$ of a path $\gamma = (\gamma_X, \gamma_Y)$ in the metric space $(X \times Y, d_N)$ is defined by:

$$l_N(\gamma) = \sup_{\theta \in \Theta([t_1, t_2])} \left(\sum_{i=1}^{n_\theta-1} d_N(\gamma(\theta_i), \gamma(\theta_{i+1})) \right).$$

Where $\Theta([t_1, t_2])$ is the set of subdivisions of $[t_1, t_2]$. Then the N -path metrics on $X \bowtie Y$ is:

DEFINITION 3.2 (The N -path metrics on $X \bowtie Y$). — *Let N be a norm on the vector space \mathbb{R}^2 . The N -path metric on $X \bowtie Y$, denoted by d_{\bowtie} , is the length path metric induced by the distance $N(d_X, d_Y)$ on $X \times Y$. For all p and q in $X \bowtie Y$ we have:*

$$d_{\bowtie}(p, q) = \inf \{ l_N(\gamma) \mid \gamma \text{ path in } X \bowtie Y \text{ linking } p \text{ to } q \}. \quad (3.1)$$

Any norm N on \mathbb{R}^2 can be normalised such that $N(1, 1) = 1$. We call admissible any such norm which satisfies an additional condition.

DEFINITION 3.3 (Admissible norm). — *Let N be a norm on the vector space \mathbb{R}^2 such that $N(1, 1) = 1$. The norm N is called admissible if and only if for all real a and b we have:*

$$N(a, b) \geq \frac{a + b}{2}. \quad (3.2)$$

Since all norms are equivalent in \mathbb{R}^2 , there exists a constant $C_N \geq 1$ such that:

$$N(a, b) \leq C_N \frac{a + b}{2}. \quad (3.3)$$

As an example, any l_p norm with $p \geq 1$ is admissible.

PROPERTY 3.4. — *Let N be an admissible norm on the vector space \mathbb{R}^2 . Let $\gamma := (\gamma_X, \gamma_Y) \subset X \times Y$ be a connected path. Then we have:*

$$\frac{l_X(\gamma_X) + l_Y(\gamma_Y)}{2} \leq l_N(\gamma) \leq C_N \frac{l_X(\gamma_X) + l_Y(\gamma_Y)}{2}.$$

Proof. — Let $\gamma := (\gamma_X, \gamma_Y) : [t_1, t_2] \rightarrow X \times Y$ be a connected path and θ a subdivision of $[t_1, t_2]$, then by the definition of the length:

$$\begin{aligned} l_N(\gamma) &\geq \sum_{i=1}^{n_\theta-1} d_N(\gamma(\theta_i), \gamma(\theta_{i+1})) \\ &= \sum_{i=1}^{n_\theta-1} N\left(d_X(\gamma_X(\theta_i), \gamma_X(\theta_{i+1})), d_Y(\gamma_Y(\theta_i), \gamma_Y(\theta_{i+1}))\right) \\ &\geq \sum_{i=1}^{n_\theta-1} \frac{1}{2} \left(d_X(\gamma_X(\theta_i), \gamma_X(\theta_{i+1})) + d_Y(\gamma_Y(\theta_i), \gamma_Y(\theta_{i+1}))\right), \\ &\hspace{15em} \text{since } N \text{ is admissible.} \\ &\geq \frac{1}{2} \left(\sum_{i=1}^{n_\theta-1} d_X(\gamma_X(\theta_i), \gamma_X(\theta_{i+1})) + \sum_{i=1}^{n_\theta-1} d_Y(\gamma_Y(\theta_i), \gamma_Y(\theta_{i+1})) \right). \end{aligned}$$

Any couple of subdivision θ_1 and θ_2 can be merge into a subdivision θ that contains θ_1 and θ_2 . Furthermore the last inequality holds for any subdivision θ , hence by taking the supremum on all the subdivisions we have:

$$l_N(\gamma) \geq \frac{l_X(\gamma_X) + l_Y(\gamma_Y)}{2}.$$

Furthermore, we have that $\forall a, b \in \mathbb{R}, N(a, b) \leq C_N \frac{a+b}{2}$, hence:

$$\begin{aligned} &\sum_{i=1}^{n_\theta-1} d_N(\gamma(\theta_i), \gamma(\theta_{i+1})) \\ &\leq \frac{C_N}{2} \left(\sum_{i=1}^{n_\theta-1} d_X(\gamma_X(\theta_i), \gamma_X(\theta_{i+1})) + \sum_{i=1}^{n_\theta-1} d_Y(\gamma_Y(\theta_i), \gamma_Y(\theta_{i+1})) \right) \\ &\leq C_N \frac{l_X(\gamma_X) + l_Y(\gamma_Y)}{2}. \end{aligned}$$

Since last inequality holds for any subdivision θ , we have obtain

$$l_N(\gamma) \leq C_N \frac{l_X(\gamma_X) + l_Y(\gamma_Y)}{2}. \quad \square$$

The definition of height on X and Y is used to construct a height function on $X \bowtie Y$.

DEFINITION 3.5 (Height on $X \bowtie Y$). — *The height $h(p)$ of a point $p = (p_X, p_Y) \in X \bowtie Y$ is defined as $h(p) = h_X(p_X) = -h_Y(p_Y)$.*

On Gromov hyperbolic spaces we have that the distance between two points is greater than or equal to their height difference. The same occurs on horospherical products given with an admissible norm. Let x and y be two

points of $X \bowtie Y$, and let us denote $\Delta h(p, q) := |h(p) - h(q)|$ their height difference.

LEMMA 3.6. — *Let N be an admissible norm, and let d_{\bowtie} the distance on $X \bowtie Y$ induced by N . Then the height function is 1-Lipschitz with respect to the distance d_{\bowtie} , i.e.,*

$$\forall p, q \in X \bowtie Y, \quad d_{\bowtie}(p, q) \geq \Delta h(p, q). \quad (3.4)$$

Proof. — Since N is admissible we have:

$$\begin{aligned} d_{\bowtie}(p, q) &\geq \frac{d_X(p_X, q_X) + d_Y(p_Y, q_Y)}{2} \\ &\geq \frac{\Delta h(p_X, q_X) + \Delta h(p_Y, q_Y)}{2} = \Delta h(p_X, q_X) = \Delta h(p, q). \quad \square \end{aligned}$$

Following Proposition 2.7, we define a notion of vertical paths in a horospherical product.

DEFINITION 3.7 (Vertical paths in $X \bowtie Y$). — *Let $V : \mathbb{R} \rightarrow X \bowtie Y$ be a connected path. We say that V is vertical if and only if there exists a parametrisation by arclength of V such that $h(V(t)) = t$ for all t .*

Actually, a vertical path of a horospherical product is a geodesic.

LEMMA 3.8. — *Let N be an admissible norm. Let $V : \mathbb{R} \rightarrow X \bowtie Y$ be a vertical path. Then V is a geodesic of $(X \bowtie Y, d_{\bowtie})$.*

Proof. — Let $t_1, t_2 \in \mathbb{R}$. The path V is vertical therefore $\Delta h(V(t_1), V(t_2)) = |t_1 - t_2|$. Since V is connected and parametrised by arclength, we have that:

$$\begin{aligned} |t_1 - t_2| = l_N(V_{|[t_1, t_2]}) &\geq d_{\bowtie}(V(t_1), V(t_2)) \\ &\geq \Delta h(V(t_1), V(t_2)) = |t_1 - t_2|. \end{aligned}$$

Then $d_{\bowtie}(V(t_1), V(t_2)) = |t_1 - t_2|$, which ends the proof. □

Such geodesics are called vertical geodesics. Next proposition tells us that vertical geodesics of $X \bowtie Y$ are exactly couples of vertical geodesics of X and Y .

PROPOSITION 3.9. — *Let N be an admissible norm and let $V = (V_X, V_Y) : \mathbb{R} \rightarrow X \bowtie Y$ be a geodesic of $(X \bowtie Y, d_{\bowtie})$. The two following properties are equivalent:*

- (1) V is a vertical geodesic of $(X \bowtie Y, d_{\bowtie})$
- (2) V_X and V_Y are respectively vertical geodesics of X and Y .

Proof. — Let us first assume that V be a vertical geodesic, we have for all real t that $h(V_X(t)) = h(V(t)) = t$, hence $\forall t_1, t_2 \in \mathbb{R}$:

$$d_X(V_X(t_1), V_X(t_2)) \geq \Delta h(V_X(t_1), V_X(t_2)) = |t_1 - t_2|. \quad (3.5)$$

Similarly we have that $d_Y(V_Y(t_1), V_Y(t_2)) \geq |t_1 - t_2|$. Using that N is admissible and that V is a geodesic we have:

$$\begin{aligned} & d_X(V_X(t_1), V_X(t_2)) \\ &= 2 \frac{d_X(V_X(t_1), V_X(t_2)) + d_Y(V_Y(t_1), V_Y(t_2))}{2} - d_Y(V_Y(t_1), V_Y(t_2)) \\ &\leq 2d_{\bowtie}(V(t_1), V(t_2)) - |t_1 - t_2| = |t_1 - t_2|. \end{aligned}$$

Combine with inequality (3.5) we have that $d_X(V_X(t_1), V_X(t_2)) = |t_1 - t_2|$, hence V_X is a vertical geodesic of X . Similarly, V_Y is a vertical geodesic Y .

Let us assume that V_X and V_Y are vertical geodesics of X and Y . Let $t_1, t_2 \in \mathbb{R}$, we have:

$$\begin{aligned} & d_{\bowtie}(V(t_1), V(t_2)) \\ &= \sup_{\theta \in \Theta([t_1, t_2])} \left(\sum_{i=1}^{n_\theta-1} d_N(V(\theta_i), V(\theta_{i+1})) \right) \\ &= \sup_{\theta \in \Theta([t_1, t_2])} \left(\sum_{i=1}^{n_\theta-1} N \left(d_X(V_X(\theta_i), V_X(\theta_{i+1})), d_Y(V_Y(\theta_i), V_Y(\theta_{i+1})) \right) \right) \\ &= \sup_{\theta \in \Theta([t_1, t_2])} \left(\sum_{i=1}^{n_\theta-1} N \left(\Delta h(V_X(\theta_i), V_X(\theta_{i+1})), \Delta h(V_Y(\theta_i), V_Y(\theta_{i+1})) \right) \right) \\ &= \sup_{\theta \in \Theta([t_1, t_2])} \left(N(1, 1) \sum_{i=1}^{n_\theta-1} \Delta h(V_X(\theta_i), V_X(\theta_{i+1})) \right) \\ &= N(1, 1) \Delta h(V_X(t_1), V_X(t_2)) = |t_1 - t_2|, \text{ since } N(1, 1) = 1, \end{aligned}$$

where $\Theta([t_1, t_2])$ is the set of subdivision of $[t_1, t_2]$. Hence the proposition is proved. \square

This previous result is the main reason why we are working with distances which came from admissible norms.

DEFINITION 3.10. — *A geodesic ray of $X \bowtie Y$ is called vertical if it is a subset of a vertical geodesic.*

If X and Y are proper hyperbolic geodesically complete Busemann spaces, their horospherical product $X \bowtie Y$ is connected.

PROPERTY 3.11. — *Let X and Y be two proper, geodesically complete, δ -hyperbolic, Busemann spaces. Let $X \bowtie Y$ be their horospherical product. Then $X \bowtie Y$ is connected, furthermore*

$$\frac{1}{2}(d_X + d_Y) \leq d_{X \bowtie Y} \leq 2C_N(d_X + d_Y).$$

Proof. — Let $p = (p_X, p_Y)$ and $q = (q_X, q_Y)$ be two points of $X \bowtie Y$. From Property 2.8, there exists a vertical geodesic V_{p_Y} such that p_Y is in the image of V_{p_Y} , and there exists a vertical geodesic V_{q_X} such that q_X is in the image of V_{q_X} . Let q'_Y be the point of V_{p_Y} at height $h(q_Y)$. Let α_X be a geodesic of X linking p_X to q_X and let α'_Y be a geodesic of Y linking q'_Y to q_Y . We will connect x to y with a path composed with pieces of α_X , α'_Y , V_{p_Y} and V_{q_X} .

We first link (p_X, p_Y) to (q_X, q'_Y) with α_X and V_{p_Y} . It is possible since V_{p_Y} is parametrised by its height. More precisely we construct the following path c_1 :

$$\forall t \in [0, d(p_X, q_X)], \quad c_1(t) = \left(\alpha_X(t), V_{p_Y}(-h(\alpha_X(t))) \right).$$

Since V_{p_Y} is parametrised by its height, we have $h(V_{p_Y}(-h(\alpha_X(t)))) = -h(\alpha_X(t))$ which implies $c_1(t) \in X \bowtie Y$. Furthermore, using the fact that the height is 1-Lipschitz, we have $\forall t_1, t_2 \in [0, d(p_X, q_X)]$:

$$\begin{aligned} d_Y \left(V_{p_Y}(-h(\alpha_X(t_1))), V_{p_Y}(-h(\alpha_X(t_2))) \right) \\ = |h(\alpha_X(t_1)) - h(\alpha_X(t_2))| \leq d_X(\alpha_X(t_1), \alpha_X(t_2)). \end{aligned}$$

Hence $c_{1,Y} : t \mapsto V_{p_Y}(-h(\alpha_X(t)))$ is a connected path such that $l(c_{1,Y}) \leq l(\alpha_X) \leq d_X(p_X, q_X)$. Hence c_1 is a connected path linking (p_X, p_Y) to (q_X, q'_Y) . Using Property 3.4 on c_1 provides us with:

$$\begin{aligned} l_N(c_1) &\leq \frac{C_N}{2}(l(c_{1,Y}) + l(\alpha_X)) \leq C_N l(\alpha_X) \\ &\leq C_N d_X(p_X, q_X). \end{aligned}$$

We recall that by definition $q'_Y = V_{p_Y}(h(q_Y))$. We show similarly that $c_2 : t \mapsto (V_{q_X}(-h(\alpha'_Y(t))), \alpha'_Y(t))$ is a connected path linking (q_X, q'_Y) to (q_X, q_Y) such that:

$$\begin{aligned} l(c_2) &\leq C_N d_Y(q'_Y, q_Y) \leq C_N (d_Y(q'_Y, p_Y) + d_Y(p_Y, q_Y)) \\ &= C_N (\Delta h(p_Y, q_Y) + d_Y(p_Y, q_Y)), \end{aligned}$$

$$\text{since } q'_Y = V_{p_Y}(h(q_Y)) \leq 2C_N d_Y(p_Y, q_Y).$$

Hence, there exists a connected path $c = c_1 \cup c_2$ linking p to q such that:

$$\begin{aligned} l(c) &\leq C_N d_X(p_X, q_X) + 2C_N d_Y(p_Y, q_Y) \\ &\leq 2C_N (d_X(p_X, q_X) + d_Y(p_Y, q_Y)). \end{aligned} \tag{3.6} \quad \square$$

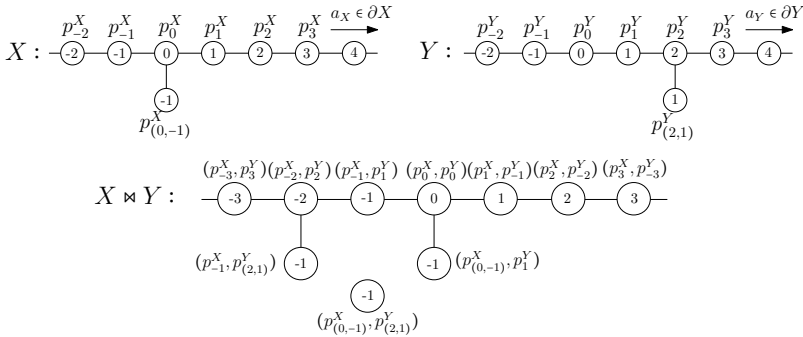


Figure 3.1. Example of horospherical product which is not connected. The number in a vertex is the height of that vertex.

However if the two components X and Y are not geodesically complete, $X \rtimes Y$ may not be connected.

Example 3.12. — Let X and Y be two graphs, constructed from an infinite line \mathbb{Z} (indexed by \mathbb{Z}) with an additional vertex glued on the 0 for X and on the -2 for Y . Their construction is illustrated in Figure 3.1. They are two 0-hyperbolic Busemann spaces which are not geodesically complete. Let $w_X \in X$ be the vertex indexed by 0 in X , and let $w_Y \in Y$ be the vertex indexed by -2 in Y . We choose them to be the base points of X and Y . Since ∂X and ∂Y contain two points each, we fix in both cases the point of the boundary a_X or a_Y to be the one that contains the geodesic ray indexed by \mathbb{N} . On Figure 3.1, we denoted the height of a vertex inside this one. Then the horospherical product $X \rtimes Y$ taken with the ℓ_1 path metric is not connected. Since some vertices of X and Y are not contained in a vertical geodesic, one may not be able to adapt its height correctly while constructing a path joining $(p_{-1}^X, p_{(2,1)}^Y)$ to $(p_{(0,-1)}^X, p_{(2,1)}^Y)$.

It is not clear that a horospherical product is still connected without the hypothesis that X and Y are Busemann spaces. In that case we would need a “coarse” definition of horospherical product. Indeed, the height along geodesics would not be smooth as in Proposition 2.7, therefore the condition requiring to have two exact opposite heights would not suits.

3.2. Examples

A *Heintze group* is a Lie group of the form $\mathbb{R} \rtimes_A N$ defined by the action on \mathbb{R} , $t \mapsto \exp(tA)$, with N a simply connected nilpotent Lie group and with

$A \in \text{Lie}(A)$ a derivation whose eigenvalues have positive real parts. Heintze proved in [12] that any simply connected, negatively curved Lie group is isomorphic to a Heintze group.

Moreover, a Busemann metric space is simply connected, hence any Gromov hyperbolic, Busemann Lie group is isomorphic to a Heintze group. Consequently, Heintze groups are natural candidates for the two components from which a horospherical product is constructed. In his paper [18], Xie classifies the subfamily of all negatively curved Lie groups $\mathbb{R} \ltimes \mathbb{R}^n$ up to quasi-isometry.

Let $H_1 := \mathbb{R} \ltimes_{A_1} N_1$ and $H_2 := \mathbb{R} \ltimes_{A_2} N_2$ be two Heintze groups, then $H_1 \bowtie H_2$ is isomorphic to $\mathbb{R} \ltimes_{\text{Diag}(A_1, -A_2)} (N_1 \times N_2)$, where $\text{Diag}(A_1, -A_2)$ is the block diagonal matrix containing A_1 and $-A_2$ on its diagonal. In fact, We have that $H_1 \times H_2$ is the group $\mathbb{R}^2 \ltimes_{(A_1, A_2)} (N_1 \times N_2)$ defined by the action on \mathbb{R}^2 , $(t_1, t_2) \mapsto (\exp(t_1 A_1), \exp(t_2 A_2))$. Let $(0, e_{N_1}) \in N_1$, $(0, e_{N_2}) \in N_2$ be the two base points, and let $t \mapsto (t, e_{N_1})$ and $t \mapsto (t, e_{N_2})$ be there respective vertical geodesic rays corresponding to the chosen Busemann functions. Then we have that for all $(t, n) \in H_i$, $h(t, n) = t$. Under this setting we have that

$$\begin{aligned} H_1 \bowtie H_2 &= \{(t_1, t_2, n_1, n_2) \in H_1 \times H_2 \mid t_1 = -t_2\} \\ &= \{(t, -t, n_1, n_2) \in H_1 \times H_2\}. \end{aligned}$$

Thanks to this characterisation, we show that $H_1 \bowtie H_2$ is a subgroup of $\mathbb{R}^2 \ltimes_{(A_1, A_2)} (N_1 \times N_2)$. Furthermore the following map is an isomorphism

$$\begin{aligned} H_1 \bowtie H_2 &\longrightarrow \mathbb{R} \ltimes_{\text{Diag}(A_1, -A_2)} (N_1 \times N_2) \\ (t, -t, n_1, n_2) &\longmapsto (t, n_1, n_2), \end{aligned}$$

where $\mathbb{R} \ltimes_{\text{Diag}(A_1, -A_2)} (N_1 \times N_2)$ is determined by the action $t \mapsto (\exp(tA_1), \exp(-tA_2))$. In fact, we have that

$$(\mathbb{R} \ltimes_{A_1} N_1) \bowtie (\mathbb{R} \ltimes_{A_2} N_2) \cong_{\text{iso}} \mathbb{R} \ltimes_{\text{Diag}(A_1, -A_2)} (N_1 \times N_2).$$

The Sol geometries are specific cases of such solvable Lie groups when $N_i = \mathbb{R}$ for $i \in \{1, 2\}$, and where the matrices A_i are positive reals. In this context, for $m > 0$ we have that $\mathbb{R} \ltimes_m \mathbb{R}$ is the Log model of a real hyperbolic plan, otherwise stated the Riemannian manifold with coordinates $(x, z) \in \mathbb{R}^2$ endowed with the Riemannian metric $ds^2 = e^{-2mz} dx^2 + dz^2$. Then $(\mathbb{R} \ltimes_m \mathbb{R}) \bowtie (\mathbb{R} \ltimes_n \mathbb{R}) = \mathbb{R} \ltimes_{\text{Diag}(m, -n)} \mathbb{R}^2$ is a Sol geometry, or also the Riemannian manifold with coordinates $(x_1, x_2, z) \in \mathbb{R}^3$ endowed with the Riemannian metric

$$ds^2 = e^{-2mz} dx_1^2 + e^{2nz} dx_2^2 + dz^2.$$

A first discrete example of horospherical product is the family of Diestel Leader graphs defined by $\text{DL}(n, m) = T_n \bowtie T_m$ with $n, m \geq 2$ and where T_n and T_m are regular trees. We see T_n and T_m as connected metric spaces

with the usual distance on them. By choosing half of the ℓ_1 path metric on $\text{DL}(n, m)$, this horospherical product becomes a graph with the natural distance on it. Indeed, the set of vertices of $\text{DL}(n, m)$ is then defined by the subset of couples of vertices of $T_n \times T_m$ included in $\text{DL}(n, m)$. In this horospherical product, two points (p_n, p_m) and (q_n, q_m) of $\text{DL}(n, m)$ are connected by an edge if and only if p_n and q_n are connected by an edge in T_n and if p_m and q_m are connected by an edge in T_m . Furthermore, when $n = m$, there is a one-to-one correspondence between $\text{DL}(n, n)$ and the Cayley graph of the lamplighter group $\mathbb{Z}_Y \wr \mathbb{Z}$, see [16] for further details.

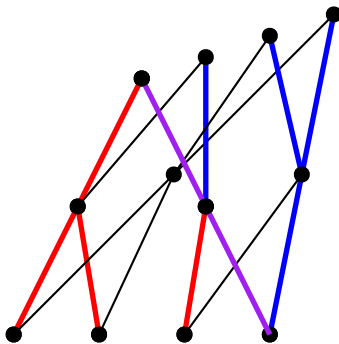


Figure 3.2. A portion of the graph $\mathbb{T}_3 \bowtie \mathbb{T}_3$.

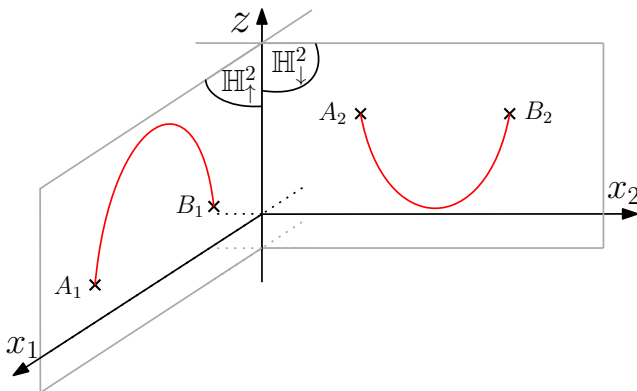


Figure 3.3. The Sol geometry and two geodesics of embedded copies of \mathbb{H}^2 .

Depending on the case, we either used the ℓ_1 path metric or the ℓ_2 path metric. However, we will see in Proposition 4.14 that it does not matter, up to an additive uniform constant.

The horospherical product of a hyperbolic plane and a regular tree has been studied as the 2-complex of Baumslag–Solitar groups in [2], they are called the treebolic spaces. The distance they choose on the treebolic spaces is similar to ours. In fact our Proposition 4.13 and their [2, Proposition 2.8 p. 9] tell us they are equal up to an additive constant. Rigidity results on the quasi-isometry classification of the treebolic spaces were brought up in [7, 8].

4. Estimates on the length of specific paths

4.1. Geodesics in Gromov hyperbolic Busemann spaces

This section focuses on length estimates in Gromov hyperbolic Busemann spaces. The central result is Proposition 4.9, which presents a lower bound on the length of a path staying between two horospheres. Before moving to the technical results of this section, let us introduce some notation.

Notation 4.1. — Unless otherwise specified, H will be a Gromov hyperbolic Busemann geodesically complete proper space. Let $\gamma : I \rightarrow H$ be a path. Let us denote the maximal height and the minimal height of this path as follows:

$$h^+(\gamma) = \sup_{t \in I} \{h(\gamma(t))\}; \quad h^-(\gamma) = \inf_{t \in I} \{h(\gamma(t))\}.$$

Let x and y be two points of H , we denote the height difference between them by:

$$\Delta h(x, y) = |h(x) - h(y)|.$$

We define the relative distance between two points x and y of H as:

$$d_r(x, y) = d(x, y) - \Delta h(x, y).$$

Let us denote V_x a vertical geodesic containing x . We will assume it to be parametrised by arclength. Thanks to Proposition 2.7 we choose a parametrisation by arclength such that $\forall t \in \mathbb{R}, h(V_x(t)) = t + 0$.

The relative distance between two points quantifies how far a point is from the nearest vertical geodesic containing the other point.

In the sequel we want to apply the slim triangles property on ideal triangles, hence we need the following result of [4].

PROPERTY 4.2 ([4, Proposition 2.2 p. 19]). — *Let a, b and c be three points of $X \cup \partial X$. Let α, β, γ be three geodesics of X linking respectively b to c , c to a , and a to b . Then every point of α is at distance less than 24δ from the union $\beta \cup \gamma$.*

Next lemma tells us that in order to connect two points, a geodesic needs to go sufficiently high. This height is controlled by the relative distance between these two points. (This Lemma finds further application in [9].)

LEMMA 4.3. — *Let H be a δ -hyperbolic and Busemann metric space, let x and y be two elements of H such that $h(x) \leq h(y)$, and let α be a geodesic linking x to y . Let us denote $z = \alpha(\Delta h(x, y) + \frac{1}{2}d_r(x, y))$, $x_1 := V_x(h(y) + \frac{1}{2}d_r(x, y))$ the point of V_x at height $h(y) + \frac{1}{2}d_r(x, y)$ and $y_1 := V_y(h(y) + \frac{1}{2}d_r(x, y))$ the point of V_y at the same height $h(y) + \frac{1}{2}d_r(x, y)$. Then we have:*

- (1) $h^+(\alpha) \geq h(y) + \frac{1}{2}d_r(x, y) - 96\delta$,
- (2) $d(z, x_1) \leq 144\delta$,
- (3) $d(z, y_1) \leq 144\delta$,
- (4) $d(x_1, y_1) \leq 288\delta$.

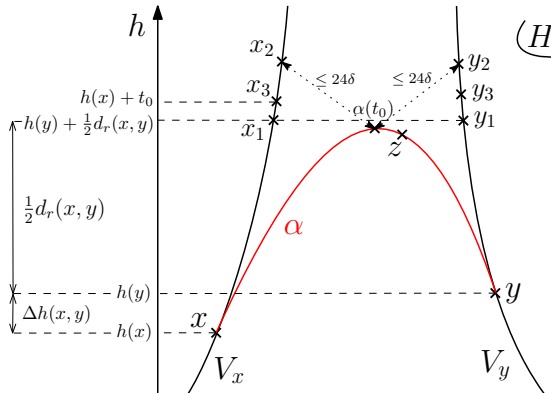


Figure 4.1. Proof of Lemma 4.3.

Proof. — Lemma 4.3 and its proof are illustrated in Figure 4.1. Following Property 4.2, the triple of geodesics α , V_x and V_y is a 24δ -slim triangle. Since the sets

$$\{t \in [0, d(x, y)] \mid d(\alpha(t), V_x) \leq 24\delta\}$$

and

$$\{t \in [0, d(x, y)] \mid d(\alpha(t), V_y) \leq 24\delta\}$$

are closed sets covering $[0, d(x, y)]$, their intersection is non empty. Hence there exists $t_0 \in [0, d(x, y)]$, $x_2 \in V_x$ and $y_2 \in V_y$ such that $d(\alpha(t_0), x_2) \leq 24\delta$

and $d(\alpha(t_0), y_2) \leq 24\delta$. Let us first prove that t_0 is close to $\Delta h(x, y) + \frac{1}{2}d_r(x, y)$. By the triangle inequality we have that:

$$|t_0 - d(x, x_2)| = |d(x, \alpha(t_0)) - d(x, x_2)| \leq d(x_2, \alpha(t_0)) \leq 24\delta.$$

Let us denote $x_3 := V_x(h(x) + t_0)$ the point of V_x at height $h(x) + t_0$, and $y_3 = V_y(h(y) + d(x, y) - t_0)$ the point of V_y at height $h(y) + d(x, y) - t_0$. Then by the triangle inequality:

$$\begin{aligned} d(\alpha(t_0), x_3) &\leq d(\alpha(t_0), x_2) + d(x_2, x_3) \\ &= d(\alpha(t_0), x_2) + |d(x, x_2) - d(x, x_3)| \\ &\leq d(\alpha(t_0), x_2) + |d(x, x_2) - t_0| \leq 48\delta. \end{aligned} \quad (4.1)$$

In the last inequality we used that $d(x, x_3) = t_0$, which holds by the definition of x_3 . We show in the same way that $d(\alpha(t_0), y_3) \leq 48\delta$. By the triangle inequality we have $d(x_3, y_3) \leq 96\delta$. As the height function is Lipschitz we have $\Delta h(x_3, y_3) \leq d(x_3, y_3) \leq 96\delta$, which provides us with:

$$\begin{aligned} \left| \frac{1}{2}d_r(x, y) + \Delta h(x, y) - t_0 \right| &= \frac{1}{2} |d_r(x, y) + \Delta h(x, y) + h(y) - h(x) - 2t_0| \\ &= \frac{1}{2} |h(y) + d(x, y) - t_0 - (h(x) + t_0)| \\ &= \frac{1}{2} \Delta h(x_3, y_3) \leq \frac{96\delta}{2} \leq 48\delta. \end{aligned} \quad (4.2)$$

In particular it gives us that $d(z, \alpha(t_0)) \leq 48\delta$. We are now ready to prove Lemma 4.3(1) using inequalities (4.1) and (4.2):

$$\begin{aligned} h^+(\alpha) &\geq h(\alpha(t_0)) \geq h(x_3) - \Delta h(\alpha(t_0), x_3) \geq h(x) + t_0 - 48\delta \\ &\geq h(x) + \frac{1}{2}d_r(x, y) + \Delta h(x, y) - 96\delta \\ &\geq h(y) + \frac{1}{2}d_r(x, y) - 96\delta, \quad \text{as we have } h(x) \leq h(y). \end{aligned}$$

The point (2) of our lemma is proved as follows:

$$\begin{aligned} d(z, x_1) &\leq d(z, \alpha(t_0)) + d(\alpha(t_0), x_1) \leq 48\delta + d(\alpha(t_0), x_3) + d(x_3, x_1) \\ &\leq 96\delta + \left| t_0 + h(x) - \left(\frac{1}{2}d_r(x, y) + h(y) \right) \right| \\ &= 96\delta + \left| t_0 - \left(\Delta h(x, y) + \frac{1}{2}d_r(x, y) \right) \right| \leq 144\delta. \end{aligned}$$

The proof of (3) is similar, and (4) is obtained from (2)–(3) by the triangle inequality. \square

The next lemma shows that in the case where $h(x) \leq h(y)$ a geodesic linking x to y is almost vertical until it reaches the height $h(y)$.

LEMMA 4.4. — *Let H be a δ -hyperbolic and Busemann space. Let x and y be two points of H such that $h(x) \leq h(y)$. We define $x' := V_x(h(y))$ to be the point of the vertical geodesic V_x at the same height as y . Then:*

$$|d_r(x, y) - d(x', y)| \leq 54\delta. \quad (4.3)$$

Proof. — Since H is δ -hyperbolic, the geodesic triangle $[x, y] \cup [y, x'] \cup [x', x]$ is δ -slim. Then there exists $p_1 \in [x, x']$, $p_2 \in [x', y]$ and $m \in [x, y]$ such that $d(p_1, m) \leq \delta$ and $d(p_2, m) \leq \delta$. Hence, $h^-([x', y]) - \delta \leq h(m) \leq h^+([x, x']) + \delta$. Let $R_{x'}$ and R_y be two vertical geodesic rays respectively contained in V_x and V_y and respectively starting at x' and y . Then Property 4.2 used on the ideal triangle $R_x \cup R_y \cup [x', y]$ implies that $h^-([x', y]) \geq h(y) - 24\delta$, therefore we have $h^+([x, x']) = h(y)$. Then $h(y) - 25\delta \leq h(m) \leq h(y) + \delta$ holds. It follows that m and x' are close to each other:

$$\begin{aligned} d(m, x') &\leq d(m, p_1) + d(p_1, x') \leq \delta + \Delta h(p_1, x') \\ &\leq \delta + \Delta h(p_1, m) + \Delta h(m, y) + \Delta h(y, x') \\ &\leq \delta + d(p_1, m) + 25\delta + 0 \leq 27\delta. \end{aligned} \quad (4.4)$$

Then we give an estimate on the distance between x and m :

$$|d(x, m) - \Delta h(x, y)| = |d(x, m) - d(x, x')| \leq d(m, x') \leq 27\delta. \quad (4.5)$$

However $d_r(x, y) = d(x, y) - \Delta h(x, y)$ and $d(x, y) = d(x, m) + d(m, y)$, therefore:

$$d_r(x, y) = d(x, m) + d(m, y) - \Delta h(x, y). \quad (4.6)$$

Combining inequalities (4.5) and (4.6) we have $|d_r(x, y) - d(m, y)| \leq 27\delta$. Then:

$$|d_r(x, y) - d(x', y)| \leq 27\delta + d(x', m) \leq 54\delta. \quad \square$$

We are now able to prove the estimates of the next section.

4.2. Length estimate of paths avoiding horospheres

Consider a path γ and a geodesic α sharing the same end-points in a proper, Gromov hyperbolic, Busemann space. We prove in this section that if the height of γ does not reach the maximal height of the geodesic α , then γ is much longer than α . Furthermore, its length increases exponentially with respect to the difference of maximal height between γ and α . To do so, we make use of [3, Part III, Chapter H, Proposition 1.6], which we recall here. Let us denote by $l(c)$ the length of a path c .

PROPOSITION 4.5 ([3]). — *Let X be a δ -hyperbolic geodesic space. Let c be a path in X . If $[p, q]$ is a geodesic segment connecting the endpoints of c , then for every $x \in [p, q]$:*

$$d(x, \text{im}(c)) \leq \delta |\log_2 l(c)| + 1.$$

This result implies that a path of X between p and q which avoids the ball of diameter $[p, q]$ has length greater than an exponential of the distance $d(p, q)$.

Our case is similar to the notion of *divergence* later introduced in [3]. The *divergence* between two paths γ_1 and γ_2 starting at $o \in X$ is a function whose value at r is the minimal length of a path connecting $c_1(r)$ and $c_2(r)$ by avoiding the ball of radius r centered on o . See [3, Definition 1.24, Chapter H, Part III] for further details.

From now on we will add as convention that $\delta \geq 1$. For all $\delta_1 \leq \delta_2$ a δ_1 -slim triangle is also δ_2 -slim, hence all δ_1 -hyperbolic spaces are δ_2 -hyperbolic spaces. That is why we can assume that all Gromov hyperbolic spaces are δ -hyperbolic with $\delta \geq 1$. It allows us to consider $\frac{1}{8}$ as a well defined term, we hence avoid the arising of separated cases in some of the proofs. We also use this assumption to simplify constants appearing in this document. The next result is a similar control on the length of path as Proposition 4.5, but we consider that the path is avoiding a horosphere instead of avoiding a ball in H .

LEMMA 4.6. — *Let $\delta \geq 1$ and H be a proper, geodesic, δ -hyperbolic, Busemann space. Let x and $y \in H$ and let V_x , respectively V_y , be a vertical geodesic containing x , respectively y . Let us consider $t_0 \geq \max(h(x), h(y))$ and let us denote $x_0 := V_x(t_0)$ and $y_0 := V_y(t_0)$, the respective points of V_x and V_y at the height t_0 . Assume that $d(x_0, y_0) > 768\delta$.*

Then for all connected path $\gamma : [0, T] \rightarrow H$ such that $\gamma(0) = x$, $\gamma(T) = y$ and $h^+(\gamma) \leq h(x_0)$ we have:

$$l(\gamma) \geq \Delta h(x, x_0) + \Delta h(y, y_0) + 2^{-386} 2^{\frac{1}{2\delta} d(x_0, y_0)} - 24\delta. \quad (4.7)$$

For trees (when $\delta = 0$) this lemma still makes sense. Indeed, if δ tends to 0 then the length of the path described in this lemma tends to infinity, which is consistent with the fact that such a path does not exist in trees. The proof would use the fact that in Proposition 4.5 we have $d(x, \text{im}(c)) = 0$ when $\delta = 0$ since 0-hyperbolic spaces are real trees.

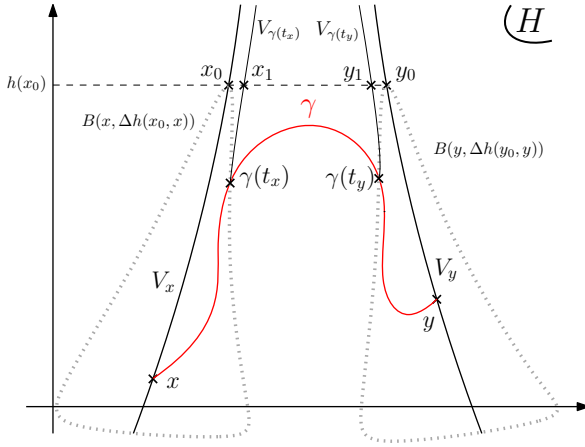


Figure 4.2. Proof of Lemma 4.6.

Proof. — One can follow the idea of the proof on Figure 4.2. We will consider γ to be parametrised by arclength. Let $B(x, \Delta h(x_0, x)) \subset H$ be the ball of radius $h(x_0) - h(x)$ centred on x , and let $m \in B(x, \Delta h(x_0, x))$ be a point in this ball. Then:

$$d_r(m, x) = d(m, x) - \Delta h(m, x) \leq \Delta h(x, x_0) - \Delta h(m, x) \leq \Delta h(x_0, m).$$

Let us first assume that $h(m) \geq h(x)$, then:

$$h(m) + \frac{d_r(m, x)}{2} \leq h(m) + \frac{\Delta h(x_0, m)}{2} \leq h(m) + \frac{h(x_0) - h(m)}{2} \quad (4.8)$$

$$= \frac{h(x_0)}{2} + \frac{h(m)}{2} \leq h(x_0). \quad (4.9)$$

By Lemma 4.3 we have:

$$d\left(V_x\left(h(m) + \frac{d_r(m, x)}{2}\right), V_m\left(h(m) + \frac{d_r(m, x)}{2}\right)\right) \leq 288\delta.$$

We now assume that $h(m) \leq h(x)$, then:

$$h(x) + \frac{d_r(x, m)}{2} \leq h(x) + \frac{d(x, m)}{2} \leq h(x) + \frac{\Delta h(x, x_0)}{2} \leq h(x_0).$$

Then Lemma 4.3 provides us with:

$$d\left(V_x\left(h(x) + \frac{d_r(m, x)}{2}\right), V_m\left(h(x) + \frac{d_r(m, x)}{2}\right)\right) \leq 288\delta.$$

Since H is a Busemann space, the function $t \rightarrow d(V_x(t), V_m(t))$ is convex. Furthermore $t \rightarrow d(V_x(t), V_m(t))$ is bounded on $[0; +\infty[$ as H is Gromov

hyperbolic, hence $t \rightarrow d(V_x(t), V_m(t))$ is a non increasing function. Therefore both cases $h(m) \leq h(x)$ and $h(x) \leq h(m)$ give us that:

$$d(x_0, V_m(h(x_0))) = d(V_x(h(x_0)), V_m(h(x_0))) \leq 288\delta. \quad (4.10)$$

In other words, all points of $B(x, \Delta h(x_0, x))$ belong to a vertical geodesic passing nearby x_0 . By the same reasoning we have $\forall n \in B(y, \Delta h(y_0, y))$:

$$d(y_0, V_n(h(y_0))) \leq 288\delta. \quad (4.11)$$

Then by the triangle inequality:

$$\begin{aligned} d(V_m(h(x_0)), V_n(h(y_0))) & \\ & \geq -d(x_0, V_m(h(x_0))) + d(x_0, y_0) - d(y_0, V_n(h(y_0))) \\ & \geq 768\delta - 288\delta - 288\delta \geq 192\delta. \end{aligned} \quad (4.12)$$

Specifically $d(V_m(h(x_0)), V_n(h(y_0))) = d(V_m(h(x_0)), V_n(h(x_0))) > 0$ which implies that $m \neq n$. Then $B(x, \Delta h(x_0, x)) \cap B(y, \Delta h(y_0, y)) = \emptyset$. By continuity of γ we deduce the existence of the two following times $t_x \leq t_y$ such that:

$$\begin{aligned} t_x &= \inf\{t \in [0, T] \mid d(\gamma(t), x) = \Delta h(x, x_0)\}, \\ t_y &= \sup\{t \in [0, T] \mid d(\gamma(t), y) = \Delta h(y, y_0)\}. \end{aligned}$$

In order to have a lower bound on the length of γ we will need to split this path into three parts:

$$\gamma = \gamma|_{[0, t_x]} \cup \gamma|_{[t_x, t_y]} \cup \gamma|_{[t_y, T]}.$$

As γ is parametrised by arclength and $d(\gamma(0), \gamma(t_x)) = \Delta h(x, x_0)$ we have that:

$$l(\gamma|_{[0, t_x]}) \geq \Delta h(x, x_0). \quad (4.13)$$

For similar reasons we also have:

$$l(\gamma|_{[t_y, T]}) \geq \Delta h(y, y_0). \quad (4.14)$$

We will now focus on proving a lower bound for the length of $\gamma|_{[t_x, t_y]}$.

We want to construct a path γ' joining $x_1 = V_{\gamma(t_x)}(h(x_0))$ to $y_1 = V_{\gamma(t_y)}(h(x_0))$, that stays below $h(x_0)$ and such that $\gamma|_{[t_x, t_y]}$ is contained in γ' .

Let $x_1 := V_{\gamma(t_x)}(h(x_0))$ and $y_1 := V_{\gamma(t_y)}(h(x_0))$. We construct γ' by gluing paths together:

$$\gamma' = \begin{cases} V_{\gamma(t_x)} & \text{from } x_1 \text{ to } \gamma(t_x), \\ \gamma & \text{from } \gamma(t_x) \text{ to } \gamma(t_y), \\ V_{\gamma(t_y)} & \text{from } \gamma(t_y) \text{ to } y_1. \end{cases}$$

Applying inequalities (4.10) and (4.11) used on $\gamma(t_x)$ and $\gamma(t_y)$ we get:

$$d(x_0, x_1) \leq 288\delta, \quad (4.15)$$

$$d(y_0, y_1) \leq 288\delta. \quad (4.16)$$

In order to apply Proposition 4.5 to γ' we need to check that there exists a point A of the geodesic segment $[x_1, y_1]$ such that $h(A) \geq h(x_0)$. Applying Lemma 4.3 to $[x_1, y_1]$ and since $h(x_1) = h(y_1)$ we get:

$$h^+([x_1, y_1]) \geq \frac{d_r(x_1, y_1)}{2} + h(x_0) - 96\delta = \frac{d(x_1, y_1)}{2} + h(x_0) - 96\delta.$$

Thanks to the triangle inequality and inequalities (4.15) and (4.16):

$$\begin{aligned} h^+([x_1, y_1]) &\geq \frac{d(y_0, x_0) - d(x_0, x_1) - d(y_0, y_1)}{2} + h(x_0) - 96\delta \\ &\geq \frac{d(x_0, y_0)}{2} + h(x_0) - 384\delta. \end{aligned}$$

Since by hypothesis $d(x_0, y_0) > 768\delta$, there exists a point A of $[x_1, y_1]$ exactly at the height:

$$h(A) = \frac{d(x_0, y_0)}{2} + h(x_0) - 384\delta.$$

We can then apply Proposition 4.5 to get:

$$\begin{aligned} \delta |\log_2(l(\gamma'))| + 1 &\geq d(A, \gamma') \geq \Delta h(A, x_0) \\ &\geq \frac{d(x_0, y_0)}{2} + h(x_0) - 384\delta - h(x_0) \\ &\geq \frac{d(x_0, y_0)}{2} - 384\delta. \end{aligned}$$

Since $\delta \geq 1$, last inequality implies that $l(\gamma') \geq 2^{-385} 2^{\frac{1}{2\delta} d(x_0, y_0)}$. Now we use this inequality to have a lower bound on the length of $\gamma|_{[t_x, t_y]}$:

$$\begin{aligned} l(\gamma|_{[t_x, t_y]}) &\geq l(\gamma') - \Delta h(\gamma(t_x), x_0) - \Delta h(\gamma(t_y), y_0) \\ &\geq 2^{-385} 2^{\frac{1}{2\delta} d(x_0, y_0)} - \Delta h(\gamma(t_x), x_0) - \Delta h(\gamma(t_y), y_0). \end{aligned} \quad (4.17)$$

We claim that $l(\gamma|_{[t_x, t_y]}) \geq \Delta h(\gamma(t_x), x_0) + \Delta h(\gamma(t_y), y_0) - 48\delta$, hence:

$$l(\gamma|_{[t_x, t_y]}) \geq 2^{-386} 2^{\frac{1}{2\delta} d(x_0, y_0)} - 24\delta, \quad (4.18)$$

which ends the proof by combining inequality (4.18) with inequalities (4.13) and (4.14).

Proof of the claim. — Inequality (4.12) with $m = \gamma(t_x)$ and $n = \gamma(t_y)$ gives $d(x_1, y_1) \geq 192\delta$. We want to prove that $h^+([\gamma(t_x), \gamma(t_y)]) \geq h(x_1) - 24\delta$. First, by Lemma 4.2 we have that $[\gamma(t_x), \gamma(t_y)] \cup V_{\gamma(t_x)} \cup V_{\gamma(t_y)}$ is

a 24δ -slim triangle. Then there exist three times t_0 , t_1 and t_2 such that $d(V_{\gamma(t_x)}(t_1), \gamma(t_0)) \leq 24\delta$ and such that $d(V_{\gamma(t_y)}(t_2), \gamma(t_0)) \leq 24\delta$. Then:

$$\begin{aligned} |t_1 - t_2| &= \Delta h(V_{\gamma(t_x)}(t_1), V_{\gamma(t_y)}(t_2)) \\ &\leq d(V_{\gamma(t_x)}(t_1), V_{\gamma(t_y)}(t_2)) \\ &\leq d(V_{\gamma(t_x)}(t_1), \gamma(t_0)) + d(\gamma(t_0), V_{\gamma(t_y)}(t_2)) \leq 48\delta. \end{aligned} \quad (4.19)$$

We will show by contradiction that either $t_1 = h(V_{\gamma(t_x)}(t_1)) \geq h(x_0)$ or $t_2 = h(V_{\gamma(t_y)}(t_2)) \geq h(x_0)$.

Assume that $t_1 < h(x_0)$ and $t_2 < h(x_0)$. Then by the triangle inequality:

$$\begin{aligned} &d(V_{\gamma(t_x)}(t_1), V_{\gamma(t_y)}(t_2)) \\ &\geq d(V_{\gamma(t_y)}(t_2), V_{\gamma(t_x)}(t_2)) - d(V_{\gamma(t_x)}(t_2), V_{\gamma(t_x)}(t_1)) \\ &\geq d(V_{\gamma(t_y)}(t_2), V_{\gamma(t_x)}(t_2)) - 48\delta, \quad \text{since } |t_1 - t_2| \leq 48\delta \text{ by eq. (4.19)}. \end{aligned}$$

As H is a Busemann space, the function $t \mapsto d(V_{\gamma(t_x)}(t), V_{\gamma(t_y)}(t))$ is non increasing (convex and bounded function). Furthermore, $h(x_0) \geq t_2$ hence:

$$\begin{aligned} 48\delta &\geq d(V_{\gamma(t_x)}(t_1), V_{\gamma(t_x)}(t_2)) \geq d(V_{\gamma(t_x)}(t_2), V_{\gamma(t_y)}(t_2)) - 48\delta \\ &\geq d(V_{\gamma(t_x)}(h(x_0)), V_{\gamma(t_y)}(h(x_0))) - 48\delta \geq d(x_1, y_1) - 48\delta \\ &\geq d(x_0, y_0) - d(x_0, x_1) - d(y_0, y_1) - 48\delta \\ &\geq d(x_0, y_0) - 624\delta, \quad \text{by inequalities (4.15)–(4.16),} \\ &\geq 49\delta, \quad \text{since } d(x_0, y_0) \geq 768\delta \text{ by assumption,} \end{aligned}$$

which is impossible. Therefore $t_1 \geq h(x_0)$ or $t_2 \geq h(x_0)$. We assume without loss of generality that $t_1 \geq h(x_0)$, then:

$$\Delta h(\gamma(t_0), V_{\gamma(t_x)}(t_1)) \leq d(\gamma(t_0), V_{\gamma(t_x)}(t_1)) \leq 24\delta,$$

which implies:

$$\begin{aligned} h^+([\gamma(t_x), \gamma(t_y)]) &\geq h(\gamma(t_0)) \\ &\geq h(V_{\gamma(t_x)}(t_1)) - \Delta h(\gamma(t_0), V_{\gamma(t_x)}(t_1)) \\ &\geq h(x_0) - 24\delta, \end{aligned}$$

and gives us:

$$\begin{aligned} l(\gamma|_{[t_x, t_y]}) &\geq h^+([\gamma(t_x), \gamma(t_y)]) - h(\gamma(t_x)) \\ &\quad + h^+([\gamma(t_x), \gamma(t_y)]) - h(\gamma(t_y)) \\ &\geq h(x_0) - 24\delta - h(\gamma(t_x)) + h(x_0) - 24\delta - h(\gamma(t_y)) \\ &\geq \Delta h(\gamma(t_x), x_0) + \Delta h(\gamma(t_y), y_0) - 48\delta. \end{aligned} \quad (4.20) \quad \square$$

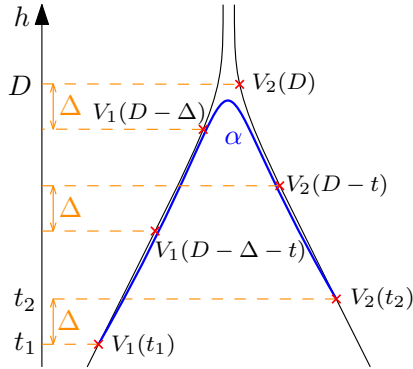


Figure 4.3. Proof of Lemma 4.7.

Next lemma shows that we are able to control the relative distance of a couple of points travelling along two vertical geodesics. We recall that for all $a, b \in H$, $d_r(a, b) = d(a, b) - \Delta h(a, b)$. (This lemma is also used in [9].)

LEMMA 4.7 (Backwards control). — *Let $\delta \geq 0$ and H be a proper, δ -hyperbolic, Busemann space. Let V_1 and V_2 be two vertical geodesics of H . Then for all couple of times (t_1, t_2) and for all $t \in [0, \frac{1}{2}d_r(V_1(t_1), V_2(t_2))]$:*

$$\left| d_r \left(V_1 \left(t_1 + \frac{1}{2}d_r(V_1(t_1), V_2(t_2)) - t \right), V_2 \left(t_2 + \frac{1}{2}d_r(V_1(t_1), V_2(t_2)) - t \right) \right) - 2t \right| \leq 288\delta.$$

Proof. — To simplify the computations, we use the following notations, $D := t_2 + \frac{1}{2}d_r(V_1(t_1), V_2(t_2))$ and $\Delta = |t_1 - t_2|$. The term Δ is the difference of height between $V_1(t_1)$ and $V_2(t_2)$ since vertical geodesics are parametrised by their height. Then we have to prove that $\forall t \in [0, \frac{1}{2}d_r(V_1(t_1), V_2(t_2))]$

$$\left| d_r(V_1(D - \Delta - t), V_2(D - t)) - 2t \right| \leq 288\delta.$$

We can assume without loss of generality that $t_1 \leq t_2$. Lemma 4.3 applied with $x = V_1(t_1)$ and with $y = V_2(t_2)$ gives us $d(V_1(D), V_2(D)) \leq 288\delta$. Furthermore, the relative distance is smaller than the distance, hence $d_r(V_1(D), V_2(D)) \leq 288\delta$. Now, if we move the two points backward

from $V_1(D - \Delta)$ and $V_2(D)$ along V_1 and V_2 , we have for $t \in [0, D]$:

$$\begin{aligned}
 d_r(V_1(D - \Delta - t), V_2(D - t)) & \\
 &= d(V_1(D - \Delta - t), V_2(D - t)) - \Delta \\
 &\leq d(V_1(D - \Delta - t), V_1(D - \Delta)) \\
 &\quad + d(V_1(D - \Delta), V_2(D)) + d(V_2(D), V_2(D - t)) - \Delta. \quad (4.21)
 \end{aligned}$$

Moreover, V_1 and V_2 are geodesics, therefore:

$$\begin{aligned}
 d_r(V_1(D - \Delta - t), V_2(D - t)) & \\
 &\leq t + d(V_1(D - \Delta), V_1(D)) + d(V_1(D), V_2(D)) + t - \Delta \\
 &\leq t + \Delta + 288\delta + t - \Delta \leq 2t + 288\delta. \quad (4.22)
 \end{aligned}$$

Let us consider a geodesic α between $V_1(t_1)$ and $V_2(t_2)$. Since H is a Busemann space, and thanks to Lemma 4.3 we have $d(V_1(D - \Delta - t), \alpha(D - \Delta - t_1 - t)) \leq 144\delta$ and $d(V_2(D - t), \alpha(D - t_1 + t)) \leq 144\delta$. Then the second part of our inequality follows:

$$\begin{aligned}
 d_r(V_1(D - \Delta - t), V_2(D - t)) & \\
 &= d(V_1(D - \Delta - t), V_2(D - t)) - \Delta \\
 &\geq d(\alpha(D - \Delta - t_1 - t), \alpha(D - t_1 + t)) \\
 &\quad - d(V_1(D - \Delta - t), \alpha(D - \Delta - t_1 - t)) \\
 &\quad - d(V_2(D - t), \alpha(D - t_1 + t)) - \Delta \quad (4.23) \\
 &\geq d(\alpha(D - \Delta - t_1 - t), \alpha(D - t_1 + t)) - 288\delta - \Delta \\
 &\geq 2t + \Delta - 288\delta - \Delta \geq 2t - 288\delta. \quad \square
 \end{aligned}$$

The next lemma is a slight generalisation of Lemma 4.6. The difference being that we control the length of a path with its maximal height instead of the distance between the projection of its extremities on a horosphere.

LEMMA 4.8. — *Let $\delta \geq 1$ and H be a proper, δ -hyperbolic, Busemann space. Let $x, y \in H$ such that $h(x) \leq h(y)$. Let α be a path connecting x to y with $h^+(\alpha) \leq h(y) + \frac{1}{2}d_r(x, y) - \Delta H$ and where ΔH is a positive number such that $\Delta H > 555\delta$. Then:*

$$l(\alpha) \geq d(x, y) + 2^{-530}2^{\frac{1}{5}\Delta H} - 2\Delta H - 24\delta.$$

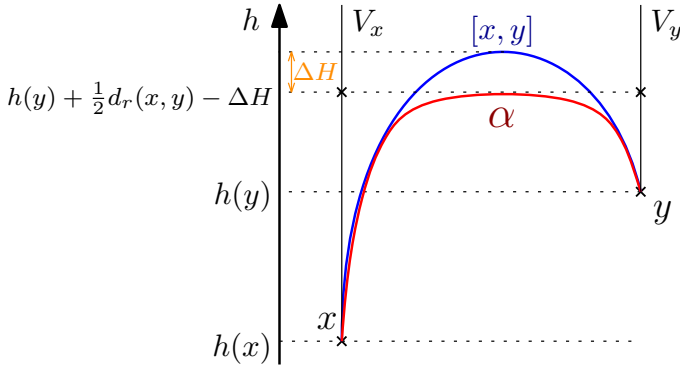


Figure 4.4. Proof of Lemma 4.8.

Proof. — This proof is illustrated in Figure 4.4. Since $h^+(\alpha) \geq h(y)$ we have that $\frac{1}{2}d_r(x, y) \geq \Delta H$. Applying Lemma 4.7 with $V_1 = V_x$, $V_2 = V_y$, $t_1 = h(x)$, $t_2 = h(y)$ and $t = \Delta H$ we have:

$$\left| d_r \left(V_x \left(h(x) + \frac{1}{2}d_r(x, y) - \Delta H \right), V_y \left(h(y) + \frac{1}{2}d_r(x, y) - \Delta H \right) \right) - 2\Delta H \right| \leq 288\delta.$$

Then we have:

$$d_r \left(V_x \left(h(x) + \frac{1}{2}d_r(x, y) - \Delta H \right), V_y \left(h(y) + \frac{1}{2}d_r(x, y) - \Delta H \right) \right) \geq 2\Delta H - 288\delta.$$

Furthermore, Lemma 4.4 applied on

$$V_x \left(h(x) + \frac{1}{2}d_r(x, y) - \Delta H \right) \quad \text{and} \quad V_y \left(h(y) + \frac{1}{2}d_r(x, y) - \Delta H \right)$$

gives (notice that the difference is that the height in the vertical geodesic V_x are evaluated on two different points):

$$\begin{aligned} & d_r \left(V_x \left(h(x) + \frac{1}{2}d_r(x, y) - \Delta H \right), V_y \left(h(y) + \frac{1}{2}d_r(x, y) - \Delta H \right) \right) \\ & \leq d \left(V_x \left(h(y) + \frac{1}{2}d_r(x, y) - \Delta H \right), V_y \left(h(y) + \frac{1}{2}d_r(x, y) - \Delta H \right) \right) + 54\delta. \end{aligned}$$

Then:

$$\begin{aligned} & d \left(V_x \left(h(y) + \frac{1}{2}d_r(x, y) - \Delta H \right), V_y \left(h(y) + \frac{1}{2}d_r(x, y) - \Delta H \right) \right) \\ & \geq 2\Delta H - 342\delta > 768\delta. \quad (4.24) \end{aligned}$$

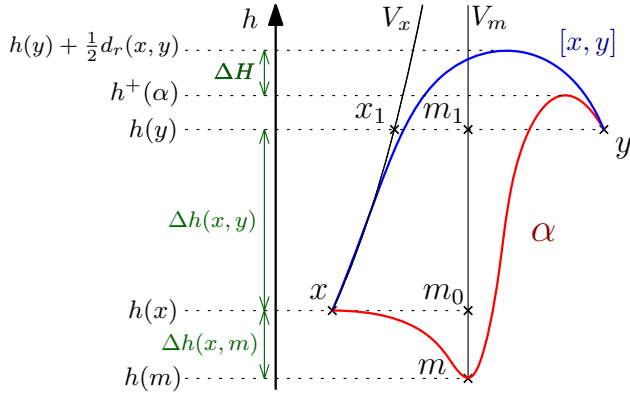


Figure 4.5. Proof of Proposition 4.9.

Let us denote $t_0 = h(y) + \frac{1}{2}d_r(x, y) - \Delta H$. Thanks to inequality (4.24) the hypothesis of Lemma 4.6 holds with $x_0 = V_x(h(y) + \frac{1}{2}d_r(x, y) - \Delta H)$ and $y_0 = V_y(h(y) + \frac{1}{2}d_r(x, y) - \Delta H)$. Applying this lemma on α provides:

$$\begin{aligned}
 l(\alpha) &\geq \Delta h(x, x_0) + \Delta h(y, y_0) + 2^{-386}2^{\frac{1}{28}d(x_0, y_0)} - 24\delta \\
 &\geq h(y) + \frac{1}{2}d_r(x, y) - \Delta H - h(x) + h(y) \\
 &\quad + \frac{1}{2}d_r(x, y) - \Delta H - h(y) + 2^{-386}2^{\frac{1}{28}d(x_0, y_0)} - 24\delta \\
 &\geq \Delta h(y, x) + d_r(y, x) - 2\Delta H + 2^{-386}2^{\frac{1}{28}d(x_0, y_0)} - 24\delta \\
 &\geq d(x, y) - 2\Delta H + 2^{-386}2^{\frac{1}{28}(2\Delta H - 288\delta)} - 24\delta, \text{ by equation (4.24).} \\
 &\geq d(x, y) + 2^{-530}2^{\frac{1}{8}\Delta H} - 2\Delta H - 24\delta. \quad \square
 \end{aligned}$$

This previous lemma tells us that a path needs to reach a sufficient height for its length not to increase to much. We give now a generalisation of Lemma 4.8, where the path reaches a given low height before going to its end point. This proposition will be the central result for the understanding of the geodesic shapes in a horospherical product.

PROPOSITION 4.9. — *Let $\delta \geq 1$ and H be a proper, δ -hyperbolic, Busemann space. Let $x, y, m \in H$ such that $h(m) \leq h(x) \leq h(y)$ and let $\alpha : [0, T] \rightarrow H$ be a path connecting x to y such that $h^-(\alpha) = h(m)$. With the notation $\Delta H = h(y) + \frac{1}{2}d_r(x, y) - h^+(\alpha)$ we have:*

$$l(\alpha) \geq 2\Delta h(x, m) + d(x, y) + 2^{-850}2^{\frac{1}{8}\Delta H} - 1 - \max(0, 2\Delta H) - 1700\delta.$$

Proof. — This proof is illustrated in Figure 4.5. We first assume that $\Delta H > 850\delta$, we postpone the other cases to the end of this proof. Let V_x and V_m be vertical geodesics respectively containing x and m . We call $x_1 = V_x(h(y))$ and $m_1 = V_m(h(y))$ the points of V_x and V_m at height $h(y)$. First, Lemma 4.4 provides $|d(x_1, y) - d_r(x, y)| \leq 54\delta$. Then we consider a geodesic triangle between the three points x_1, m_1 and y . Lemma 4.3 tells us that $h^+([x_1, y]) \geq h(y) + \frac{1}{2}d_r(x_1, y) - 96\delta \geq h(y) + \frac{1}{2}d_r(x, y) - 123\delta$. Since $[x_1, y]$ is included in the δ -neighbourhood of the two other sides of the geodesic triangle, one of the two following inequalities holds:

$$h^+([x_1, m_1]) \geq h(y) + \frac{1}{2}d_r(x, y) - 124\delta \quad (1)$$

$$h^+([m_1, y]) \geq h(y) + \frac{1}{2}d_r(x, y) - 124\delta. \quad (2)$$

We first assume (1) that $h^+([x_1, m_1]) \geq h(y) + \frac{1}{2}d_r(x, y) - 124\delta$, hence:

$$d(x_1, m_1) \geq d_r(x, y) - 248\delta. \quad (4.25)$$

Let us denote $m_0 = V_m(h(x))$ the point of V_m at height $h(x)$. By considering the 2δ -slim quadrilateral between the points x, x_1, m_0, m_1 we have that $[x_1, m_1]$ is in the 2δ -neighbourhood of $[x_1, x] \cup [x, m_0] \cup [m_0, m]$. Furthermore $d_r(x, y) \geq 2(h^+(\alpha) - h(y)) + 2\Delta H \geq 2\Delta H \geq 1700\delta$ by assumption, then $h^+([x_1, m_1]) \geq h(y) + \frac{1}{2}d_r(x, y) - 124\delta \geq h(y) + 726\delta$. Since $h^+([x_1, x]) = h^+([m_0, m_1]) = h(y)$ we have that $h^+([x, m_0]) \geq h^+([x_1, m_1]) - 2\delta \geq h(y) + 724\delta$. Moreover:

$$\begin{aligned} d_r(x, m_0) &= d(x, m_0) \geq h^+([x, m_0]) - h(x) \\ &\geq h(y) - h(x) + 724\delta \geq \Delta h(x, y) + 724\delta, \end{aligned}$$

which allows us to use Lemma 4.7 on V_x and V_m with $t = \frac{1}{2}d_r(x, m_0) - \Delta h(x, y) \geq 0$ and $t_1 = t_2 = h(x)$. It gives:

$$\left| d_r\left(V_x(h(x) + \Delta h(x, y)), V_m(h(x) + \Delta h(x, y))\right) - d_r(x, m_0) + 2\Delta h(x, y) \right| \leq 288\delta,$$

which implies in particular:

$$d_r\left(V_x(h(y)), V_m(h(y))\right) + 2\Delta h(x, y) - 288\delta \leq d_r(x, m_0). \quad (4.26)$$

Combining inequalities (4.25) and (4.26) we have $d(x, m_0) = d_r(x, m_0) \geq d_r(x, y) + 2\Delta h(x, y) - 536\delta$. Lemma 4.4 used on x and m then gives:

$$d_r(x, m) \geq d(x, m_0) - 54\delta \geq d_r(x, y) + 2\Delta h(x, y) - 590\delta. \quad (4.27)$$

Let us denote α_1 the part of α linking x to m and α_2 the part of α linking m to y . We have:

$$\begin{aligned}
 h^+(\alpha_1) &\leq h^+(\alpha) \leq h(y) + \frac{1}{2}d_r(x, y) - \Delta H \\
 &\leq h(x) + \Delta h(x, y) + \frac{1}{2}d_r(x, y) - \Delta H \\
 &\leq h(x) + \frac{1}{2}(2\Delta h(x, y) + d_r(x, y)) - \Delta H \\
 &\leq h(x) + \frac{1}{2}(d_r(x, m) + 590\delta) - \Delta H, \quad \text{by inequality (4.27).} \\
 &\leq h(x) + \frac{1}{2}d_r(x, m) + 295\delta - \Delta H \leq h(x) + \frac{1}{2}d_r(x, m) - \Delta H',
 \end{aligned}$$

with $\Delta H' = \Delta H - 295\delta$. By assumption $\Delta H > 850\delta$, hence $\Delta H' > 555\delta$ which allows us to apply Lemma 4.8 on α_1 . It follows:

$$\begin{aligned}
 l(\alpha_1) &\geq d(x, m) + 2^{-530}2^{\frac{1}{8}\Delta H'} - 2\Delta H' - 24\delta \\
 &\geq \Delta h(x, m) + d_r(x, m) + 2^{-825}2^{\frac{1}{8}\Delta H} - 2\Delta H - 614\delta, \\
 &\quad \text{since } \Delta H' = \Delta H - 295\delta. \\
 &\geq \Delta h(x, m) + d_r(x, y) - 590\delta + 2^{-825}2^{\frac{1}{8}\Delta H} - 2\Delta H - 614\delta, \\
 &\quad \text{by inequality (4.27),} \\
 &\geq \Delta h(x, m) + d_r(x, y) + 2^{-825}2^{\frac{1}{8}\Delta H} - 2\Delta H - 1204\delta.
 \end{aligned}$$

We use in the following inequalities that $l(\alpha_2) \geq d(m, y) \geq \Delta h(m, y)$, we have:

$$\begin{aligned}
 l(\alpha) &\geq l(\alpha_1) + l(\alpha_2) \\
 &\geq \Delta h(x, m) + d_r(x, y) + 2^{-825}2^{\frac{1}{8}\Delta H} - 2\Delta H - 1204\delta + \Delta h(m, y) \\
 &\geq 2\Delta h(x, m) + \Delta h(x, y) + d_r(x, y) + 2^{-825}2^{\frac{1}{8}\Delta H} - 2\Delta H - 1204\delta \\
 &\geq 2\Delta h(x, m) + d(x, y) + 2^{-825}2^{\frac{1}{8}\Delta H} - 2\Delta H - 1204\delta \\
 &\geq 2\Delta h(x, m) + d(x, y) + 2^{-850}2^{\frac{1}{8}\Delta H} - 1 - 2\Delta H - 1700\delta, \\
 &\geq 2\Delta h(x, m) + d(x, y) + 2^{-850}2^{\frac{1}{8}\Delta H} - 1 - \max(0, 2\Delta H) - 1700\delta
 \end{aligned}$$

since $\Delta H > 850\delta \geq 0$, which ends the proof for case (1) .

Now assume that (2) holds, which is $h^+([m_1, y]) \geq h(y) + \frac{1}{2}d_r(x, y) - 124\delta$. It implies $d(m_1, y) \geq d_r(x, y) - 248\delta$, then:

$$\begin{aligned}
 h^+(\alpha_2) &\leq h^+(\alpha) \leq h(y) + \frac{1}{2}d_r(x, y) - \Delta H \\
 &\leq h(y) + \frac{1}{2}d_r(m_1, y) + 124\delta - \Delta H
 \end{aligned}$$

$$\leq h(y) + \frac{1}{2}d_r(m_1, y) - \Delta H'',$$

with $\Delta H'' = \Delta H - 124\delta$. Lemma 4.4 provides us with:

$$d_r(m, y) \geq d(m_1, y) - 54\delta \geq d_r(x, y) - 302\delta. \quad (4.28)$$

Since $\Delta H > 850\delta$, we have $\Delta H'' > 726\delta$ which allows us to apply Lemma 4.8 on α_2 . It follows that:

$$\begin{aligned} l(\alpha_2) &\geq d(y, m) + 2^{-530}2^{\frac{1}{8}\Delta H''} - 2\Delta H'' - 24\delta \\ &\geq \Delta h(y, m) + d_r(y, m) + 2^{-654}2^{\frac{1}{8}\Delta H} - 2\Delta H - 272\delta, \\ &\quad \text{since } \Delta H'' = \Delta H - 124\delta. \\ &\geq \Delta h(y, m) + d_r(x, y) + 2^{-654}2^{\frac{1}{8}\Delta H} - 2\Delta H - 574\delta, \quad \text{by (4.26)}. \end{aligned}$$

Hence:

$$\begin{aligned} l(\alpha) &\geq l(\alpha_1) + l(\alpha_2) \\ &\geq \Delta h(x, m) + \Delta h(y, m) + d_r(x, y) + 2^{-654}2^{\frac{1}{8}\Delta H} - 2\Delta H - 574\delta \\ &\geq 2\Delta h(x, m) + \Delta h(y, x) + d_r(x, y) + 2^{-654}2^{\frac{1}{8}\Delta H} - 2\Delta H - 574\delta \\ &\geq 2\Delta h(x, m) + d(x, y) + 2^{-654}2^{\frac{1}{8}\Delta H} - 2\Delta H - 574\delta \\ &\geq 2\Delta h(x, m) + d(x, y) + 2^{-850}2^{\frac{1}{8}\Delta H} - 1 - \max(0, 2\Delta H) - 1700\delta. \end{aligned}$$

There remains to treat the case when $\Delta H \leq 850\delta$, where $\Delta H = h(y) + \frac{1}{2}d_r(x, y) - h^+(\alpha)$. Let n denote a point of α such that $h(n) = h^+(\alpha)$. If m comes before n , we have $l(\alpha) \geq d(x, m) + d(m, n) + d(n, y)$. Otherwise n comes before m and we have $l(\alpha) \geq d(x, n) + d(n, m) + d(m, y)$. Since $h(m) \leq h(x) \leq h(y) \leq h(n)$ we always have:

$$\begin{aligned} l(\alpha) &\geq \Delta h(x, m) + \Delta h(m, n) + \Delta h(n, y) \\ &\geq \Delta h(x, m) + \Delta h(m, x) + \Delta h(x, y) + \Delta h(y, n) + \Delta h(y, n) \\ &\geq 2\Delta h(x, m) + \Delta h(x, y) + 2(h^+(\alpha) - h(y)) \\ &\geq 2\Delta h(x, m) + \Delta h(x, y) + d_r(x, y) - 2\Delta H \\ &\geq 2\Delta h(m, x) + d(x, y) - 1700\delta. \end{aligned}$$

Furthermore $\Delta H \leq 850\delta$, then $2^{-850}2^{\frac{1}{8}\Delta H} \leq 1$. Therefore:

$$l(\alpha) \geq 2\Delta h(m, x) + d(x, y) + 2^{-850}2^{\frac{1}{8}\Delta H} - 1 - \max(0, 2\Delta H) - 1700\delta,$$

which ends the proof for the remaining case. \square

4.3. Length of geodesic segments in horospherical products

From now on, unless otherwise specified, X and Y will always be two proper, geodesically complete, δ -hyperbolic, Busemann spaces with $\delta \geq 1$, and N will always be an admissible norm. Let p and q be two points of $X \bowtie Y$, and let α be a geodesic of $X \bowtie Y$ connecting them. We first prove an upper bound on the length of α by computing the length of a path $\gamma \subset X \bowtie Y$ linking p to q

LEMMA 4.10. — *Let $p = (p_X, p_Y)$ and $q = (q_X, q_Y)$ be points of the horospherical product $X \bowtie Y$. There exists a path γ connecting p to q such that:*

$$l_N(\gamma) \leq d_r(p_Y, q_Y) + d_r(p_X, q_X) + \Delta h(p, q) + 1152\delta C_N.$$

Proof. — Without loss of generality, we assume $h(p) \leq h(q)$. One can follow the idea of the proof on Figure 4.6. We consider V_{p_X} and V_{q_X} two vertical geodesics of X containing p_X and q_X respectively. Similarly let V_{p_Y} and V_{q_Y} be two vertical geodesics of Y containing p_Y and q_Y respectively. We will use them to construct γ . Let A_1 be the point of the vertical geodesic $(V_{p_X}, V_{p_Y}) \subset X \bowtie Y$ at height $h(p) - \frac{1}{2}d_r(p_Y, q_Y)$ and A_2 be the point of the vertical geodesic $(V_{p_X}, V_{q_Y}) \subset X \bowtie Y$ at the same height $h(p) - \frac{1}{2}d_r(p_Y, q_Y)$. Let A_3 be the point of the vertical geodesic (V_{p_X}, V_{q_Y}) at height $h(q) + \frac{1}{2}d_r(p_X, q_X)$ and A_4 be the point of the vertical geodesic (V_{q_X}, V_{q_Y}) at the same height $h(q) + \frac{1}{2}d_r(p_X, q_X)$. Then $\gamma := \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \cup \gamma_5$ is constructed as follows:

- γ_1 is the part of (V_{p_X}, V_{p_Y}) linking p to A_1 .
- γ_2 is a geodesic linking A_1 to A_2 . Such a geodesic exists by Property 3.11.
- γ_3 is the part of (V_{p_X}, V_{q_Y}) linking A_2 to A_3 .
- γ_4 is a geodesic linking A_3 to A_4 . Such a geodesic exists by Property 3.11.
- γ_5 is the part of (V_{q_X}, V_{q_Y}) linking A_4 to q .

In fact A_1 and A_2 are close to each other. Indeed, the two points $A_1 = (A_{1,X}, A_{1,Y})$ and $A_2 = (A_{2,X}, A_{2,Y})$ are characterised by the two geodesics (V_{p_X}, V_{p_Y}) and (V_{p_X}, V_{q_Y}) . Then, because $-h(q) = Y(q_Y) \leq Y(p_Y)$, Lemma 4.3 applied on p_Y and q_Y in Y gives us $d_Y(A_{1,Y}, A_{2,Y}) \leq 288\delta$. Furthermore Property 3.11 provides us with $d_{\bowtie} \leq 2C_N(d_X + d_Y)$, however we have that $A_{1,X} = A_{2,X}$ hence:

$$d_{\bowtie}(A_1, A_2) \leq 576\delta C_N. \tag{4.29}$$

Lemma 4.3 applied on p_X and q_X provides similarly:

$$d_{\bowtie}(A_3, A_4) \leq 576\delta C_N, \tag{4.30}$$

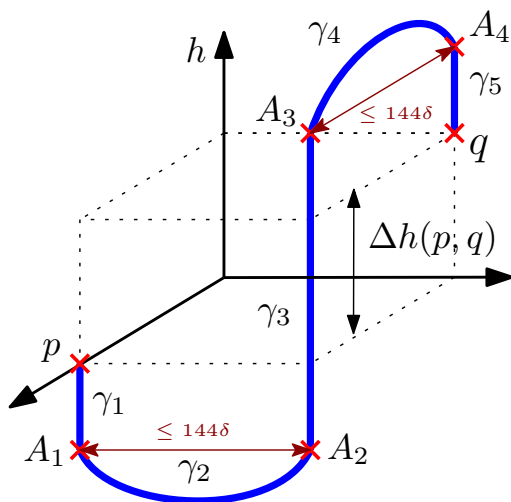


Figure 4.6. Construction of the path γ when $h(p) \leq h(q)$ for Lemma 4.10.

which gives us:

$$\begin{aligned} l_N(\gamma) &= l_N(\gamma_1) + l_N(\gamma_2) + l_N(\gamma_3) + l_N(\gamma_4) + l_N(\gamma_5) \\ &= d_{\bowtie}(p, A_1) + d_{\bowtie}(A_1, A_2) + d_{\bowtie}(A_2, A_3) + d_{\bowtie}(A_3, A_4) + d_{\bowtie}(A_4, q). \end{aligned}$$

Since γ_1 , γ_3 and γ_5 are vertical geodesics, we have:

$$\begin{aligned} &= \Delta h(p, A_1) + d_{\bowtie}(A_1, A_2) + \Delta h(A_2, A_3) + d_{\bowtie}(A_3, A_4) + \Delta h(A_4, q) \\ &= \frac{1}{2}d_r(p_Y, q_Y) + d_{\bowtie}(A_1, A_2) + \frac{1}{2}d_r(p_Y, q_Y) + \frac{1}{2}d_r(p_X, q_X) \\ &\quad + \Delta h(p, q) + d_{\bowtie}(A_3, A_4) + \frac{1}{2}d_r(p_X, q_X) \\ &\leq d_r(p_Y, q_Y) + d_r(p_X, q_X) + \Delta h(p, q) + 1152\delta C_N, \end{aligned}$$

by inequalities (4.29) and (4.30). □

We are aiming to use Proposition 4.9 on the two components $\alpha_X \subset X$ and $\alpha_Y \subset Y$ of α to obtain lower bounds on their lengths. We hence need the following lemma to ensure us that when α is a geodesic, the exponential term in the inequality of Proposition 4.9 will be small.

LEMMA 4.11. — *Let $C = 2853\delta C_N + 2^{851}$ and let $e : \mathbb{R} \rightarrow \mathbb{R}$ be a map defined by $\forall t \in \mathbb{R}, e(t) = \frac{1}{C}2^{C^{-1}t} - 2 \max(0, t)$. Then $\forall t \in \mathbb{R}$:*

- (1) $e(t) \geq -7C^2$
- (2) $(e(t) \leq 2853\delta C_N) \implies (t \leq 3C^2)$.

Proof. — For all time t , we have that $e(t) = \frac{1}{C}2^{C^{-1}t} - 2\max(0, t) \leq \frac{1}{C}2^{C^{-1}t} - 2t =: e_1(t)$. The derivative of e_1 is $e_1'(t) = \frac{\log(2)}{C^2}2^{C^{-1}t} - 2$, which is non negative $\forall t \geq C \log_2(\frac{2}{\log(2)}C^2)$ and non positive otherwise. Then $\forall t \in \mathbb{R}$:

$$\begin{aligned} e_1(t) &\geq e_1\left(\log_2\left(\frac{2}{\log(2)}C^2\right)\right) \geq \frac{2C}{\log(2)} - 2C \log_2\left(\frac{2}{\log(2)}C^2\right) \\ &\geq \frac{2C}{\log(2)} - 4C \log_2\left(\sqrt{\frac{2}{\log(2)}}C\right) \geq \frac{2C}{\log(2)} - 4\sqrt{\frac{2}{\log(2)}}C^2 \\ &\geq -4\sqrt{\frac{2}{\log(2)}}C^2 \geq -7C^2. \end{aligned}$$

Since $C \geq \frac{2}{\log(2)}$ we have $3C^2 \geq C \log_2(C^3) \geq C \log_2(\frac{2}{\log(2)}C^2)$, then e_1 is non decreasing on $[C \log_2(C^3); +\infty[$. We show that $e_1(3C^2) \geq 2853\delta C_N$:

$$\begin{aligned} e_1(3C^2) &\geq e_1(C \log_2(C^3)) = \frac{1}{C}2^{\frac{C \log_2(C^3)}{C}} - 2C \log_2(C^3) \\ &= C(C - 6 \log_2(C)). \end{aligned}$$

Since $C \geq 2^{851}$ we have $C - 6 \log_2(C) \geq 1$ and since $C \geq 2853\delta C_N$ we have that $e_1(3C^2) \geq C \times 1 \geq 2853\delta C_N$ which provides $\forall t \in [3C^2; +\infty[$ we have $e_1(t) \geq 2853\delta C_N$. Furthermore $\forall t \in \mathbb{R}^+$, $e_1(t) = e(t)$, hence $\forall t \in [3C^2; +\infty[$ we have $e(t) \geq 2853\delta C_N$ which implies Lemma 4.11 (2). \square

The following lemma provides us with a lower bound matching Lemma 4.10, and a first control on the heights a geodesic segment must reach.

LEMMA 4.12. — *Let $p = (p_X, p_Y)$ and $q = (q_X, q_Y)$ be two points of $X \bowtie Y$ such that $h(p) \leq h(q)$. Let $\alpha = (\alpha_X, \alpha_Y)$ be a geodesic segment of $X \bowtie Y$ linking p to q . Let $C_0 = (2853\delta C_N + 2^{851})^2$, we have:*

- (1) $l(\alpha) \geq \Delta h(p, q) + d_r(p_Y, q_Y) + d_r(p_X, q_X) - 15C_0$
- (2) $h^+(\alpha) \geq h(q) + \frac{1}{2}d_r(p_X, q_X) - 3C_0$
- (3) $h^-(\alpha) \leq h(p) - \frac{1}{2}d_r(p_Y, q_Y) + 3C_0$.

Proof. — Let us denote $\Delta H^+ = h(q) + \frac{1}{2}d_r(p_X, q_X) - h^+(\alpha)$ and $\Delta H^- = h^-(\alpha) - (h(p) - \frac{1}{2}d_r(p_Y, q_Y))$. Let m be a point of α at height $h^-(\alpha) = h(p) - \frac{1}{2}d_r(p_Y, q_Y) + \Delta H^-$, and n be a point of α at height $h^+(\alpha) = h(q) + \frac{1}{2}d_r(p_X, q_X) - \Delta H^+$. Then Proposition 4.9 used on α_X gives us:

$$\begin{aligned} l(\alpha_X) &\geq 2\Delta h(p_X, m_X) + d(p_X, q_X) + 2^{-850}2^{\frac{1}{3}\Delta H^+} \\ &\quad - 1 - 2\max(0, \Delta H^+) - 1700\delta \end{aligned}$$

$$\begin{aligned}
&\geq 2h(p_X) - 2\left(h(p_X) - \frac{1}{2}d_r(p_Y, q_Y) + \Delta H^-\right) \\
&\quad + d(p_X, q_X) + 2^{-850}2^{\frac{1}{3}\Delta H^+} - 1 - 2\max(0, \Delta H^+) - 1700\delta \\
&\geq d_r(p_Y, q_Y) + d_r(p_X, q_X) + \Delta h(p, q) + 2^{-850}2^{\frac{1}{3}\Delta H^+} - 1 \\
&\quad - 2\max(0, \Delta H^+) - 2\Delta H^- - 1700\delta.
\end{aligned}$$

Since $h(p_Y) \geq h(q_Y)$ and $h(n_Y) = h(q_Y) - \frac{1}{2}d_r(p_X, q_X) + \Delta H^+$, Proposition 4.9 used on α_Y provides similarly:

$$\begin{aligned}
l(\alpha_Y) &\geq d_r(p_X, q_X) + d_r(p_Y, q_Y) + \Delta h(p, q) \\
&\quad + 2^{-850}2^{\frac{1}{3}\Delta H^-} - 1 - 2\max(0, \Delta H^-) - 2\Delta H^+ - 1700\delta.
\end{aligned}$$

Hence by Property 3.4:

$$\begin{aligned}
l_N(\alpha) &\geq \frac{1}{2}(l(\alpha_X) + l(\alpha_Y)) \\
&\geq d_r(p_X, q_X) + d_r(p_Y, q_Y) + \Delta h(p, q) - 1700\delta + 2^{-851}2^{\frac{1}{3}\Delta H^-} \\
&\quad + 2^{-851}2^{\frac{1}{3}\Delta H^+} - 2\max(0, \Delta H^-) - 2\max(0, \Delta H^+) - 1. \quad (4.31)
\end{aligned}$$

Furthermore, we know by Lemma 4.10 that $l_N(\alpha) \leq \Delta h(p, q) + d_r(p_X, q_X) + d_r(p_Y, q_Y) + 1152\delta C_N$. Since $C_N \geq 1$ we have:

$$\begin{aligned}
2852\delta C_N &\geq 2^{-851}2^{\frac{1}{3}\Delta H^-} \\
&\quad - 2\max(0, \Delta H^-) + 2^{-851}2^{\frac{1}{3}\Delta H^+} - 2\max(0, \Delta H^+) - 1.
\end{aligned}$$

Let us denote $S := \max\{\Delta H^-, \Delta H^+\}$. Therefore we have

$$2^{-851}2^{\frac{1}{3}S} - 2\max(0, S) - 1 \leq 2852\delta C_N.$$

By assumption $\delta \geq 1$ hence $2^{-851}2^{\frac{1}{3}S} - 2\max(0, S) \leq 2853\delta C_N$. Furthermore, for $C = 2853\delta C_N + 2^{851}$, we have both $2^{-851} \geq \frac{1}{C}$ and $\frac{1}{\delta} \geq \frac{1}{C}$. Then we have $\frac{1}{C}2^{\frac{S}{\delta}} - 2\max(0, S) \leq 2853\delta C_N$. Lemma 4.11 provides $S \leq 3C^2 = 3C_0$ which implies (2) and (3) of Lemma 4.12. Lemma 4.11 also provides us with:

$$-14C_0 \leq 2^{-851}2^{\frac{1}{3}\Delta H^-} - 2\max(0, \Delta H^-) + 2^{-851}2^{\frac{1}{3}\Delta H^+} - 2\max(0, \Delta H^+).$$

Last inequality is a lower bound of the term we want to remove in inequality (4.31). Lemma 4.12(1) hence follows since $1700\delta + 1 \leq C_0$. \square

We recall that by definition:

$$\begin{aligned}
\forall p_X, q_X \in X, \quad d_r(p_X, q_X) &= d_X(p_X, q_X) - \Delta h(p_X, q_X) \\
\forall p_Y, q_Y \in Y, \quad d_r(p_Y, q_Y) &= d_Y(p_Y, q_Y) - \Delta h(p_Y, q_Y)
\end{aligned}$$

Hence combining Lemma 4.10 and 4.12 we get the following corollary.

COROLLARY 4.13. — *Let N be an admissible norm and let $C_0 = (2853\delta C_N + 2^{851})^2$. The length of a geodesic segment α connecting p to q in $(X \bowtie Y, d_{\bowtie})$ is controlled as follows:*

$$|l_N(\alpha) - (d_X(p_X, q_X) + d_Y(p_Y, q_Y) - \Delta h(p, q))| \leq 15C_0,$$

which gives us a control on the N -path metric, for all points p and q in $X \bowtie Y$ we have:

$$|d_{\bowtie}(p, q) - (d_X(p_X, q_X) + d_Y(p_Y, q_Y) - \Delta h(p, q))| \leq 15C_0.$$

This result is central as it shows that the shape of geodesics does not depend on the N -path metric chosen for the distance on the horospherical product.

COROLLARY 4.14. — *Let $r \geq 1$. For all p and q in $X \bowtie Y$ we have:*

$$|d_{\bowtie, \ell_r}(p, q) - d_{\bowtie, \ell_1}(p, q)| \leq 30(5706\delta + 2^{851})^2.$$

Proof. — The ℓ_r norm inequalities provide us with:

$$\sqrt[r]{d_X^r + d_Y^r} \leq d_X + d_Y \leq 2^{\frac{r-1}{r}} \sqrt[r]{d_X^r + d_Y^r}.$$

Hence we have $\frac{\sqrt[r]{2}}{2}(d_X + d_Y) \leq \sqrt[r]{d_X^r + d_Y^r} \leq d_X + d_Y$. Then the ℓ_r norms are admissible norms with $C_{\ell_r} \leq 2$, which ends the proof. \square

The next corollary tells us that changing this distance does not change the large scale geometry of $X \bowtie Y$.

COROLLARY 4.15. — *Let N_1 and N_2 be two admissible norms. Then the metric spaces $(X \bowtie Y, d_{\bowtie, N_1})$ and $(X \bowtie Y, d_{\bowtie, N_2})$ are roughly isometric.*

The control on the distances of Lemma 4.13 will help us understand the shape of geodesic segments and geodesic lines in a horospherical product.

5. Shapes of geodesics and visual boundary of $X \bowtie Y$

5.1. Shapes of geodesic segments

In this section we focus on the shape of geodesics. We recall that in all the following X and Y are assumed to be two proper, geodesically complete, δ -hyperbolic, Busemann spaces with $\delta \geq 1$, and N is assumed to be an admissible norm.

The next lemma gives a control on the maximal and minimal height of a geodesic segment in a horospherical product. It is similar to the traveling salesman problem, who needs to walk from x to y passing by m and n .

This result follows from the inequalities on maximal and minimal heights of Lemma 4.12 combined with Lemma 4.10.

LEMMA 5.1. — *Let $p = (p_X, p_Y)$ and $q = (q_X, q_Y)$ be two points of $X \bowtie Y$ such that $h(p) \leq h(q)$. Let N be an admissible norm and let $\alpha = (\alpha_X, \alpha_Y)$ be a geodesic of $(X \bowtie Y, d_{\bowtie})$ linking p to q . Let $C_0 = (2853\delta C_N + 2^{851})^2$, we have:*

- (1) $|h^-(\alpha) - (h(p) - \frac{1}{2}d_r(p_Y, q_Y))| \leq 4C_0$
- (2) $|h^+(\alpha) - (h(q) + \frac{1}{2}d_r(p_X, q_X))| \leq 4C_0$.

Proof. — Let us consider a point m of α such that $h(m) = h^-(\alpha)$ and a point n of α such that $h(n) = h^+(\alpha)$. Then m comes before n or n comes before m . In both cases, since $h(m) \leq h(p) \leq h(q) \leq h(n)$ and by Lemma 3.6 we have:

$$\begin{aligned} l_N(\alpha) &\geq \Delta h(p, q) + 2(h(p) - h^-(\alpha)) + 2(h^+(\alpha) - h(q)) \\ &\geq \Delta h(p, q) + 2(h(p) - h^-(\alpha)) + d_r(p_X, q_X) - 6C_0, \end{aligned}$$

by Lemma 4.12.

Furthermore Lemma 4.10 provides $l_N(\alpha) \leq \Delta h(p, q) + d_r(p_X, q_X) + d_r(p_Y, q_Y) + C_0$, hence:

$$\begin{aligned} \Delta h(p, q) + d_r(p_X, q_X) + d_r(p_Y, q_Y) + C_0 \\ \geq \Delta h(p, q) + 2(h(p) - h^-(\alpha)) + d_r(p_X, q_X) - 6C_0, \end{aligned}$$

which implies $(h(p) - \frac{1}{2}d_r(p_Y, q_Y)) - h^-(\alpha) \leq 4C_0$. In combination with the third point of Lemma 4.12 it proves the first point of our Lemma 5.1. The second point is proved similarly. \square

LEMMA 5.2. — *Let N be an admissible norm and let $C_0 = (2853\delta C_N + 2^{851})^2$. Let $p = (p_X, p_Y)$ and $q = (q_X, q_Y)$ be two points of $X \bowtie Y$. Let $\alpha = (\alpha_X, \alpha_Y)$ be a geodesic of $(X \bowtie Y, d_{\bowtie})$ linking p to q . Then there exist two points $a = (a_X, a_Y)$, $b = (b_X, b_Y)$ of α such that $h(a) = h(p)$, $h(b) = h(q)$ with the following properties:*

- (1) *If $h(p) \leq h(q) - 7C_0$ then:*
 - (a) $h^-(\alpha) = h^-([x, a])$ and $h^+(\alpha) = h^+([b, y])$,
 - (b) $|d_r(p_Y, a_Y) - d_r(p_Y, q_Y)| \leq 16C_0$ and $d_r(p_X, a_X) \leq 22C_0$,
 - (c) $|d_r(q_X, b_X) - d_r(p_X, q_X)| \leq 16C_0$ and $d_r(q_Y, b_Y) \leq 22C_0$,
 - (d) $|d_{\bowtie}(a, b) - \Delta h(a, b)| \leq 13C_0$.
- (2) *If $h(q) \leq h(p) - 7C_0$ then (1a), (1b), (1c) and (1d) hold by switching the roles of p and q and switching the roles of a and b .*
- (3) *If $|h(p) - h(q)| \leq 7C_0$ at least one of the two previous conclusions is satisfied.*

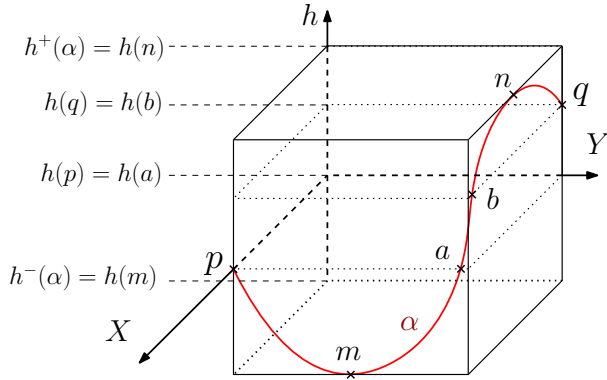


Figure 5.1. Notations of Lemma 5.2.

Lemma 5.2 is illustrated in Figure 5.1. Its notations will be used in all Section 5.

Proof. — Let us consider a point m of α such that $h(m) = h^-(\alpha)$ and a point n of α such that $h(n) = h^+(\alpha)$. We first assume that m comes before n in α oriented from p to q . Let us call a the first point between m and n at height $h(p)$ and b the last point between m and n at height $h(q)$. Property (1a) of Lemma 5.2 is then satisfied. Let us denote α_1 the part of α linking p to a , α_2 the part of α linking a to b and α_3 the part of α linking b to q . We have that m is a point of α_1 and that n is a point of α_3 . Inequalities (2)–(3) of Lemma 4.12 used on α_1 provide $l_N(\alpha_1) \geq d(p, m) + d(m, a) \geq 2\Delta h(p, m) \geq d_r(p_Y, q_Y) - 6C_0$ and similarly $l_N(\alpha_3) \geq d_r(p_X, q_X) - 6C_0$. Furthermore we have $l_N(\alpha_2) \geq \Delta h(p, q)$. Combining $l_N(\alpha_1) = l_N(\alpha) - l_N(\alpha_2) - l_N(\alpha_3)$ and Lemma 4.10 we have:

$$\begin{aligned}
 l_N(\alpha_1) &\leq \Delta h(p, q) + d_r(p_X, q_X) + d_r(p_Y, q_Y) \\
 &\quad + C_0 - \Delta h(p, q) - d_r(p_X, q_X) + 6C_0 \\
 &\leq d_r(p_Y, q_Y) + 7C_0.
 \end{aligned} \tag{5.1}$$

We have similarly that $l_N(\alpha_3) \leq d_r(p_X, q_X) + 7C_0$ and that $d_{\infty}(a, b) = l_N(\alpha_2) \leq \Delta h(p, q) + 13C_0$. It gives us $|d_{\infty}(a, b) - \Delta h(p, q)| \leq 13C_0$, point (1d) of Lemma 5.2. Furthermore, using Lemma 5.1 on α and α_1 provides:

$$\begin{aligned}
 \left| h^-(\alpha) - \left(h(p) - \frac{1}{2} d_r(p_Y, q_Y) \right) \right| &\leq 4C_0, \\
 \left| h^-(\alpha_1) - \left(h(p) - \frac{1}{2} d_r(p_Y, a_Y) \right) \right| &\leq 4C_0.
 \end{aligned}$$

Since $h^-(\alpha) = h^-(\alpha_1)$ we have:

$$|d_r(p_Y, a_Y) - d_r(p_Y, q_Y)| \leq 16C_0, \quad (5.2)$$

which is the first inequality of (1b). Using the first point of Lemma 4.12 on α_1 in combination with inequality (5.1) gives us:

$$\begin{aligned} d_r(p_Y, q_Y) + 7C_0 &\geq l_N(\alpha_1) \geq \Delta h(p, a) + d_r(p_X, a_X) + d_r(p_Y, a_Y) - 15C_0 \\ &\geq d_r(p_X, a_X) + d_r(p_Y, a_Y) - 15C_0 \\ &\geq d_r(p_X, a_X) + d_r(p_Y, q_Y) - 31C_0, \text{ by inequality (5.2).} \end{aligned}$$

Then $d_r(p_X, q_X) \leq 38C_0$ the second inequality of point (1b) holds. We prove similarly the inequality (1c) of Lemma 5.2. This ends the proof when m comes before n . If n comes before m , the proof is still working by orienting α from q to p hence switching the roles between p and q .

We will now prove that if $h(p) \leq h(q) - 7C_0$ then m comes before n on α oriented from p to q . Let us assume that $h(p) \leq h(q) - 7C_0$. We will proceed by contradiction, let us assume that n comes before m , using $h(m) \leq h(p) \leq h(q) \leq h(n)$ it implies:

$$\begin{aligned} l_N(\alpha) &\geq d_{\bowtie}(p, n) + d_{\bowtie}(n, m) + d_{\bowtie}(m, q) \\ &\geq \Delta h(p, n) + \Delta h(n, m) + \Delta h(m, q) \\ &\geq \Delta h(p, q) + \Delta h(q, n) + \Delta h(m, p) + \Delta h(p, q) \\ &\quad + \Delta h(q, n) + \Delta h(m, p) + \Delta h(p, q) \\ &\geq 2\Delta h(p, q) + \Delta h(p, q) + 2\Delta h(m, p) + 2\Delta h(q, n) \\ &\geq 14C_0 + \Delta h(p, q) + 2(h(p) - h^-(\alpha)) + 2(h^+(\alpha) - h(q)). \end{aligned}$$

However Lemma 4.12 applied on α provides $h^+(\alpha) \geq h(q) + \frac{1}{2}d_r(p_X, q_X) - 3C_0$ and $h^-(\alpha) \leq h(p) - \frac{1}{2}d_r(p_Y, q_Y) + 3C_0$. Then:

$$\begin{aligned} l_N(\alpha) &\geq 14C_0 + \Delta h(p, q) + d_r(p_X, q_X) + d_r(p_Y, q_Y) - 12C_0 \\ &\geq \Delta h(p, q) + d_r(p_X, q_X) + d_r(p_Y, q_Y) + 2C_0, \end{aligned}$$

which contradict Lemma 4.10. Hence, if $h(p) \leq h(q) - 7C_0$, the point m comes before the point n and by the first part of the proof, (1) holds. Similarly, if $h(q) \leq h(p) - 7C_0$ then n comes before m and then (2) holds. Otherwise when $|h(p) - h(q)| \leq 7C_0$ both cases could happened, then Lemma 5.2(1) or (2) hold. \square

This previous lemma essentially means that if p is sufficiently below q , the geodesic α first travels in a copy of Y in order to “lose” the relative distance between p_Y and q_Y , then it travels upward using a vertical geodesic from a to b until it can “lose” the relative distance between p_X and q_X by travelling in a copy of X . It looks like three successive geodesics of hyperbolic spaces, glued together. The idea is that the geodesic follows a shape similar

to the path γ we constructed in Lemma 4.10. The following theorem tells us that a geodesic segment is in the constant neighbourhood of three vertical geodesics. It is similar to the hyperbolic case, where a geodesic segment is in a constant neighbourhood of two vertical geodesics.

THEOREM 5.3. — *Let N be an admissible norm. Let $p = (p_X, p_Y)$ and $q = (q_X, q_Y)$ be two points of $X \bowtie Y$ and let α be a geodesic segment of $(X \bowtie Y, d_{\bowtie})$ linking p to q . Let $C_0 = (2853\delta C_N + 2^{851})^2$, there exist two vertical geodesics $V_1 = (V_{1,X}, V_{1,Y})$ and $V_2 = (V_{2,X}, V_{2,Y})$ such that:*

- (1) *If $h(p) \leq h(q) - 7C_0$ then α is in the $196C_0C_N$ -neighbourhood of $V_1 \cup (V_{1,X}, V_{2,Y}) \cup V_2$,*
- (2) *If $h(p) \geq h(q) + 7C_0$ then α is in the $196C_0C_N$ -neighbourhood of $V_1 \cup (V_{2,X}, V_{1,Y}) \cup V_2$,*
- (3) *If $|h(p) - h(q)| \leq 7C_0$ then at least one of the conclusions of 1. or 2. holds.*

Specifically V_1 and V_2 can be chosen such that p is close to V_1 and q is close to V_2 .

We refer again to Figure 1.1 that pictures the $M = 196C_0C_N$ -neighbourhood of such vertical geodesics when $h(p) \leq h(q) - 7C_0$. When $|h(p) - h(q)| \leq 7C_0$, there are two possible shapes for a geodesic segment. In some cases, two points can be linked by two different geodesics, one of type 1 and one of type 2.

Proof. — Let $m = (m_X, m_Y)$ be a point of α such that $h(m) = h^-(\alpha)$, and $n = (n_X, n_Y)$ be a point of α such that $h(n) = h^+(\alpha)$. Then by Lemma 5.1 we have:

$$\left| \Delta h(p, m) - \frac{1}{2} d_r(p_Y, q_Y) \right| \leq 4C_0. \quad (5.3)$$

We show similarly that:

$$\left| \Delta h(q, n) - \frac{1}{2} d_r(p_X, q_X) \right| \leq 4C_0. \quad (5.4)$$

(1). — In the first case we assume that $h(p) \leq h(q) - 7C_0$. With notations as in Lemma 5.2, and by inequality (5.1), we have that $l_N([p, a]) \leq d_r(p_Y, q_Y) + 7C_0$, hence:

$$\begin{aligned} l_N([p, m]) &= l_N([p, a]) - l_N([a, m]) \leq d_r(p_Y, q_Y) + 7C_0 - \Delta h(a, m) \\ &\leq \frac{1}{2} d_r(p_Y, q_Y) + 11C_0, \quad \text{since } \Delta h(p, m) = \Delta h(a, m). \end{aligned} \quad (5.5)$$

It follows from this inequality that:

$$\begin{aligned} d_X(p_X, m_X) &= 2d_{X \times Y}(p, m) - d_Y(p_Y, m_Y) \leq 2d_{\heartsuit}(p, m) - d_Y(p_Y, m_Y) \\ &\leq 2l_N([p, m]) - d_Y(p_Y, m_Y) \leq d_r(p_Y, q_Y) + 22C_0 - \Delta h(p, m) \\ &\leq \frac{1}{2}d_r(p_Y, q_Y) + 26C_0. \end{aligned}$$

Then:

$$\begin{aligned} d_r(p_X, m_X) &= d_X(p_X, m_X) - \Delta h(p, m) \\ &\leq \frac{1}{2}d_r(p_Y, q_Y) + 26C_0 - \Delta h(p, m) \leq 30C_0, \quad \text{by inequality (5.3).} \end{aligned}$$

Similarly $d_r(p_Y, m_Y) \leq 30C_0$. Let us consider the vertical geodesic V_{m_X} of X containing m_X , and the vertical geodesic V_{p_Y} of Y containing p_Y . Let us denote p'_X the point of V_{m_X} at the height $h(p)$. Since $d_r(p_X, m_X) \leq 30C_0$, Lemma 4.4 applied on p_X and m_X provides $d_X(p_X, p'_X) \leq 31C_0$. We will then consider two paths of X . The first one is $\alpha_{1,X} = [p_X, m_X]$, the part of α_X linking p_X to m_X . The second one is $[m_X, p'_X]$ a piece of vertical geodesic linking m_X to p'_X . We show that these two paths have close length. Using Property 3.4 with inequalities (5.3) and (5.5) provides us with:

$$\begin{aligned} l_X([p_X, m_X]) &\leq 2l_N([p, m]) - l_Y([p_Y, m_Y]) \\ &\leq 2\left(\frac{1}{2}d_r(p_Y, q_Y) + 11C_0\right) - \Delta h(p, m) \\ &\leq \Delta h(p, m) + 30C_0. \end{aligned}$$

Furthermore $l_X([p_X, m_X]) \geq \Delta h(p, m)$ and we know that $l_X([m_X, p'_X]) = \Delta h(p, m)$, hence:

$$|l_X([p_X, m_X]) - l_X([m_X, p'_X])| \leq 30C_0.$$

We already proved that their end points are also close to each other $d(p_X, p'_X) \leq 31C_0$. Since $\delta \leq C_0$, the property of hyperbolicity of X gives us that $\alpha_{1,X}$ is in the $(31+30+1)C_0 = 62C_0$ -neighbourhood of $[m_X, p'_X]$, a part of the vertical geodesic V_{m_X} . We show similarly that $\alpha_{1,Y}$ is in the $62C_0$ -neighbourhood of V_{p_Y} . Since N is an admissible norm, Property 3.11 gives us that α_1 is in the $124C_0C_N$ -neighbourhood of (V_{m_X}, V_{p_Y}) . We show similarly that α_3 , the portion of α linking n to q , is in the $124C_0C_N$ -neighbourhood of (V_{q_X}, V_{n_Y}) . We now focus on α_2 , the portion of α linking m to n . Let us denote $[m_X, n_X]$ the path $\alpha_{2,X}$ and $[m_Y, n_Y]$ the path $\alpha_{2,Y}$. Then Lemma 5.1 provides us with:

$$\left| \Delta h(m, n) - \left(\Delta h(p, q) + \frac{1}{2}d_r(p_Y, q_Y) + \frac{1}{2}d_r(p_X, q_X) \right) \right| \leq 8C_0. \quad (5.6)$$

However from Lemma 4.10 and since $1152\delta C_N \leq C_0$:

$$\begin{aligned}
 l_N(\alpha_2) &= l_N(\alpha) - l_N(\alpha_1) - l_N(\alpha_3) \\
 &\leq \Delta h(p, q) + d_r(p_X, q_X) + d_r(p_Y, q_Y) + C_0 - \Delta h(p, m) - \Delta h(n, q) \\
 &\leq \Delta h(p, q) + \frac{1}{2}d_r(p_X, q_X) + \frac{1}{2}d_r(p_Y, q_Y) + 9C_0, \\
 &\quad \text{by inequalities (5.3) and (5.4).}
 \end{aligned}$$

It follows from this inequality and the fact that N is admissible that:

$$\begin{aligned}
 d_X(m_X, n_X) &\leq 2l_N(\alpha_2) - d_Y(m_Y, n_Y) \\
 &\leq 2\Delta h(p, q) + d_r(p_X, q_X) + d_r(p_Y, q_Y) + 18C_0 - \Delta h(m, n) \\
 &\leq \Delta h(m, n) + 34C_0, \quad \text{by inequality (5.6).}
 \end{aligned}$$

Thus:

$$d_r(m_X, n_X) = d_X(m_X, n_X) - \Delta h(m, n) \leq 34C_0.$$

In the same way we have $d_r(m_Y, n_Y) \leq 34C_0$. Let us denote n'_X the point of V_{m_X} at the height $h(n_X)$. Since $d_r(p_X, m_X) \leq 34C_0$, Lemma 4.4 applied on m_X and n_X provides:

$$d_X(m_X, n'_X) \leq 35C_0. \quad (5.7)$$

Hence we have proved that $\alpha_{2,X}$ and $[m_X, n'_X]$ have their end points close to each other. Let us now prove that these paths have close lengths. We have that $l_X([m_X, n'_X]) = \Delta h(m, n)$, and from inequalities (5.3) and (5.4) we have:

$$\begin{aligned}
 l_X([m_X, n_X]) &\leq 2l_N(\alpha_{2,X}) - l_Y([m_Y, n_Y]) \\
 &= 2(l_N(\alpha) - l_N(\alpha_1) - l_N(\alpha_3)) - \Delta h(m, n) \\
 &\leq 2(5C_0 + \Delta h(p, q) + d_r(p_X, q_X) \\
 &\quad + d_r(p_Y, q_Y)\Delta h(p, m) - \Delta h(n, q)) - \Delta h(m, n) \\
 &\leq 2(\Delta h(p, q) + d_r(p_X, q_X) + d_r(p_Y, q_Y) \\
 &\quad - \Delta h(p, m) - \Delta h(n, q)) - \Delta h(m, n) \\
 &\leq 2(\Delta h(p, q) + \Delta h(p, m) + \Delta h(n, q) + 16C_0) \\
 &\quad - \Delta h(m, n) + 30C_0 \leq \Delta h(m, n) + 62C_0.
 \end{aligned}$$

As $l_X([m_X, n_X]) \geq \Delta h(m, n)$ we obtain:

$$|l_X([m_X, n_X]) - l_X([m_X, n'_X])| \leq 62C_0. \quad (5.8)$$

Then by similar arguments as for the path $\alpha_{1,X}$, inequalities (5.7)–(5.8) show that $\alpha_{2,X}$ is in the $(35 + 62 + 1)C_0 = 98C_0$ neighbourhood of V_{m_X} . Similarly we prove that $\alpha_{2,Y}$ is in the $98C_0$ neighbourhood of V_{n_Y} . Since N is an admissible norm, Property 3.11 gives us that α_2 is in the $196C_0C_N$ -neighbourhood of (V_{m_X}, V_{n_Y}) .

(2). — In the second case, we assume that $h(q) \leq h(p) - 7C_0$. Then by switching the role of p and q , Lemma 5.2 gives us the result identically.

(3). — In the third case, we assume that $|h(p) - h(q)| \leq 7C_0$. Then Lemma 5.2 tells us that one of the two previous situations prevail, which proves the result. \square

5.2. Coarse monotonicity

We will see that the following definition is related to being close to a vertical geodesic.

DEFINITION 5.4. — *Let C be a non negative number. A geodesic $\alpha : I \rightarrow X \bowtie Y$ of $X \bowtie Y$ is called C -coarsely increasing if $\forall t_1, t_2 \in I$:*

$$(t_2 > t_1 + C) \implies (h(\alpha(t_2)) > h(\alpha(t_1))).$$

The geodesic α is called C -coarsely decreasing if $\forall t_1, t_2 \in I$:

$$(t_2 > t_1 + C) \implies (h(\alpha(t_2)) < h(\alpha(t_1))).$$

The next lemma links the coarse monotonicity and the fact that a geodesic segment is close to vertical geodesics.

LEMMA 5.5. — *Let N be an admissible norm and let $C_0 = (2853\delta C_N + 2^{851})^2$. Let $p = (p_X, p_Y)$ and $q = (q_X, q_Y)$ be two points of $X \bowtie Y$ and let α be a geodesic segment of $(X \bowtie Y, d_{\bowtie})$ linking p to q . Let $m \in \alpha$ and $n \in \alpha$ be two points in $X \bowtie Y$ such that $h^-(\alpha) = h(m)$ and $h^+(\alpha) = h(n)$. We have:*

- (1) *If $h(p) \leq h(q) - 7C_0$, then α is $17C_0$ -coarsely decreasing on $[p, m]$ and $17C_0$ -coarsely increasing on $[m, n]$ and $17C_0$ -coarsely decreasing on $[n, q]$.*
- (2) *If $h(p) \geq h(q) + 7C_0$, then α is $17C_0$ -coarsely increasing on $[p, n]$ and $17C_0$ -coarsely decreasing on $[n, m]$ and $17C_0$ -coarsely increasing on $[m, q]$.*
- (3) *If $|h(p) - h(q)| \leq 7C_0$ then the conclusions of (1) or (2) holds.*

Proof. — Assume that $h(p) \leq h(q) - 7C_0$. Then from inequality (5.5) in the proof of Theorem 5.3, $l_N([p, m]) \leq \frac{1}{2}d_r(p_Y, q_Y) + 11C_0$. Furthermore Lemma 5.1 gives us that $|\Delta h(p, m) - \frac{1}{2}d_r(p_Y, q_Y)| \leq 4C_0$. Then:

$$l_N([p, m]) \leq \Delta h(p, m) + 15C_0. \tag{5.9}$$

We will proceed by contradiction, assume that $[p, m]$ is not $15C_0$ -coarsely decreasing, then there exists $i_1 \in \alpha$, $i_2 \in \alpha$ such that $h(i_1) = h(i_2)$ and

$l([i_1, i_2]) > 15C_0$. Hence:

$$\begin{aligned} l_N([p, m]) &\geq l_N([p, i_1]) + l_N([i_1, i_2]) + l_N([i_2, m]) \\ &\geq \Delta h(p, i_1) + l_N([i_1, i_2]) + \Delta h(i_2, m) \\ &> \Delta h(p, m) + 15C_0, \end{aligned}$$

which contradicts inequality (5.9). Then $[p, m]$ is $15C_0$ -coarsely decreasing. We show in a similar way that $[m, n]$ is $17C_0$ -coarsely increasing and that $[n, q]$ is $15C_0$ -coarsely decreasing. This proves Lemma 5.5 (1). Lemma 5.5 (2) is proved by switching the roles of p and q . We now assume $|h(p) - h(q)| \leq 7C_0$, as in the proof of Theorem 5.3 the inequality (5.5) or a corresponding inequality holds, which ends the proof of Lemma 5.5. \square

5.3. Shapes of geodesic rays and geodesic lines

In this section we are focusing on using the previous results to get information on the shapes of geodesic rays and geodesic lines. We first link the coarse monotonicity of a geodesic ray to the fact that it is close to a vertical geodesic. Let $\lambda \geq 1$ and $c \geq 0$, a (λ, c) -quasigeodesic of the metric space $(X \bowtie Y, d_{\bowtie})$ is the image of a function $\phi : \mathbb{R} \rightarrow X \bowtie Y$ verifying that $\forall t_1, t_2 \in \mathbb{R}$:

$$\frac{|t_1 - t_2|}{\lambda} - c \leq d_{\bowtie}(\phi(t_1), \phi(t_2)) \leq \lambda|t_1 - t_2| + c. \quad (5.10)$$

LEMMA 5.6. — *Let N be an admissible norm and let $C_0 = (2853\delta C_N + 2^{851})^2$. Let $\alpha = (\alpha_X, \alpha_Y)$ be a geodesic ray of $(X \bowtie Y, d_{\bowtie})$ and let K be a positive number such that α is K -coarsely monotone. Then α_X and α_Y are $(1, 26C_0 + 8K)$ -quasigeodesics.*

Proof. — Let t_1 and t_2 be two times. Let us denote $p = (p_X, p_Y) = \alpha(t_1)$ and $q = (q_X, q_Y) = \alpha(t_2)$. We apply Lemma 5.2 on the part of α linking p to q denoted by $[p, q]$. By K -coarse monotonicity of α we have that $d(p, a)_{X \bowtie Y, N} \leq K$ and $d_{\bowtie}(b, q) \leq K$. Hence using $d)$ of Lemma 5.2:

$$\begin{aligned} \Delta h(p, q) &\leq d_{\bowtie}(p, q) \leq d_{\bowtie}(p, a) + d_{\bowtie}(a, b) + d_{\bowtie}(b, q) \\ &\leq K + \Delta h(a, b) + 13C_0 + K \\ &\leq \Delta h(p, q) + \Delta h(p, a) + \Delta h(b, q) + 13C_0 + 2K \\ &\leq \Delta h(p, q) + 13C_0 + 4K. \end{aligned}$$

Furthermore, $d_X(p_X, q_X) \geq \Delta h(p_X, q_X) = \Delta h(p, q)$ and $d_Y(p_Y, q_Y) \geq \Delta h(p, q)$. Since N is an admissible norm we have:

$$\begin{aligned} \Delta h(p, q) &\leq d_X(p_X, q_X) = 2d_{X \times Y}(p, q) - d_Y(p_Y, q_Y) \\ &\leq 2d_{\bowtie}(p, q) - d_Y(p_Y, q_Y) \\ &\leq 2\Delta h(p, q) + 13C_0 + 4K - \Delta h(p, q) \\ &\leq \Delta h(p, q) + 13C_0 + 4K. \end{aligned}$$

Hence:

$$d_{\bowtie}(p, q) - 26C_0 - 8K \leq d_X(p_X, q_X) \leq d_{\bowtie}(p, q) + 26C_0 + 8K,$$

By definition we have $p_X = \alpha_X(t_1)$, $q_X = \alpha_X(t_2)$ and $d_{\bowtie}(p, q) = |t_1 - t_2|$. Then α_X is a $(1, 26C_0 + 8K)$ -quasigeodesic ray. We prove similarly that α_Y is a $(1, 26C_0 + 8K)$ -quasigeodesic ray. \square

We will now make use of the rigidity property of quasi-geodesics in Gromov hyperbolic spaces, presented in [4, Theorem 3.1 p. 41].

THEOREM 5.7 ([4]). — *Let H be a δ -hyperbolic geodesic space. If $f : \mathbb{R} \rightarrow H$ is a (λ, k) -quasi geodesic, then there exists a constant $\kappa > 0$ depending only on δ, λ and k such that the image of f is in the κ -neighbourhood of a geodesic in H .*

LEMMA 5.8. — *Let N be an admissible norm and let T_1 and T_2 be two real numbers. Let $\alpha = (\alpha_X, \alpha_Y) : [T_1, +\infty[\rightarrow X \bowtie Y$ be a geodesic ray of $(X \bowtie Y, d_{\bowtie})$. Let K be a positive number such that α is K -coarsely monotone. Then there exists a constant $\kappa > 0$ depending only on K, δ and N such that α is in the κ -neighbourhood of a vertical geodesic ray $V : [T_2; +\infty[\rightarrow X \bowtie Y$ and such that $d_{\bowtie}(\alpha(T_1), V(T_2)) \leq \kappa$.*

Proof. — We assume without loss of generality that $\lim_{t \rightarrow +\infty} h(\alpha(t)) = +\infty$. Let $C_0 = (2853\delta C_N + 2^{851})^2$, by Lemma 5.6, α_X is a $(1, 26C_0 + 8K)$ -quasi geodesic ray. Then Theorem 5.7 says there exists $\kappa_X > 0$ depending only on $26C_0 + 8K$ and δ such that α_X is in the κ_X -neighbourhood of a geodesic V_X . Since C_0 depends only on δ and N , κ_X depends only on K, δ and N . Then $\lim_{t \rightarrow +\infty} h(\alpha(t)) = +\infty$ gives us $\lim_{t \rightarrow +\infty} h(V_X(t)) = +\infty$ which implies that V_X is a vertical geodesic of X . We will now build the vertical geodesic we want in Y . We have $\lim_{t \rightarrow +\infty} h(\alpha_Y(t)) = -\infty$ and by Lemma 5.6:

$$\begin{aligned} \Delta h(\alpha_Y(t_1), \alpha_Y(t_2)) - 26C_0 - 8K \\ \leq d_Y(\alpha_Y(t_1), \alpha_Y(t_2)) \leq \Delta h(\alpha_Y(t_1), \alpha_Y(t_2)) + 26C_0 + 8K. \end{aligned}$$

Since Y is Busemann, there exists a vertical geodesic ray β starting at $\alpha_Y(T_1)$. Since β is parametrised by its height, $\alpha_Y \cup \beta$ is also a $(1, 26C_0 + 8K)$ -quasi geodesic, hence there exists κ_Y and V_Y depending only on K, δ and N

such that $\alpha_Y \cup \beta$ is in the κ_Y -neighbourhood of V_Y . Since $\lim_{t \rightarrow -\infty} h(V_Y(t)) = +\infty$, V_Y is a vertical geodesic of Y .

Furthermore, by Property 3.11, $d_{\bowtie} \leq 2C_N(d_X + d_Y)$, hence there exists κ depending only on K , δ and N such that α is in the κ -neighbourhood (for d_{\bowtie}) of (V_X, V_Y) , a vertical geodesic of $(X \bowtie Y, d_{\bowtie})$. Since $h(\alpha(t)) \geq h(\alpha(T_1)) - 26C_0 - 8K =: M$, α is in the κ -neighbourhood of $(V_X([M - \kappa; +\infty[), V_Y(]-\infty; -M + \kappa]))$ which is a vertical geodesic ray.

We will now show that the starting points of α and V are close to each other. Let us denote T'_1 a time such that $d_{\bowtie}(\alpha(T'_1), V(T'_1)) \leq \kappa$, then $\Delta h(\alpha(T_1), V(T'_1)) \leq \kappa$, hence $|T'_1 - M| \leq 26C_0 + 8K + \kappa$. Then by the triangle inequality:

$$\begin{aligned} d_{\bowtie}(\alpha(T_1), V(M - \kappa)) &\leq d_{\bowtie}(\alpha(T_1), V(T'_1)) + d_{\bowtie}(V(T'_1), V(M - \kappa)) \\ &\leq \kappa + 26C_0 + 8K + \kappa + \kappa = 26C_0 + 8K + 3\kappa \end{aligned}$$

Let us denote $\kappa' := 26C_0 + 8K + 3\kappa \geq \kappa$ and $T_2 := M - \kappa$. Hence $\alpha : [T_1; +\infty[\rightarrow X \bowtie Y$ is in the κ' -neighbourhood of a vertical geodesic ray $V : [T_2; +\infty[\rightarrow X \bowtie Y$, we have $d_{\bowtie}(\alpha(T_1), V(T_2)) \leq \kappa'$ and κ' depends only on δ and K . \square

LEMMA 5.9. — *Let N be an admissible norm and let $\alpha : \mathbb{R}^+ \rightarrow X \bowtie Y$ be a geodesic ray of $(X \bowtie Y, d_{\bowtie})$. Then α changes its $17C_0$ -coarse monotonicity at most once.*

Proof. — Let $\alpha : \mathbb{R}^+ \rightarrow X \bowtie Y$ be a geodesic ray. Thanks to Lemma 5.5 α changes at most twice of $17C_0$ -coarse monotonicity. Indeed, assume it changes three times, applying Lemma 5.5 on the geodesic segment which includes these three times provides a contradiction. We will show in the following that it actually only changes once.

Assume α changes twice of $17C_0$ -coarse monotonicity. Then α must be first $17C_0$ -coarsely increasing or $17C_0$ -coarsely decreasing. We assume without loss of generality that α is first $17C_0$ -coarsely decreasing. Then there exist $t_1, t_2, t_3 \in \mathbb{R}$ such that α is $17C_0$ -coarsely decreasing on $[\alpha(t_1), \alpha(t_2)]$ then $17C_0$ -coarsely increasing on $[\alpha(t_2), \alpha(t_3)]$ then $17C_0$ -coarsely decreasing on $[\alpha(t_3), \alpha(+\infty)[$.

Hence Lemma 5.8 applied on $[\alpha(t_3), \alpha(+\infty)[$ implies that there exists $\kappa > 0$ depending only on δ (since the constant of coarse monotonicity depends only on δ) and a vertical geodesic ray $V = (V_X, V_Y)$ such that $[\alpha(t_3), \alpha(+\infty)[$ is in the κ -neighbourhood of V . Since $h^+([\alpha(t_3), \alpha(+\infty)]) < +\infty$, we have that $\lim_{t \rightarrow +\infty} h(\alpha(t)) = -\infty$, hence there exists $t_4 \geq t_3$ such that $h(\alpha(t_4)) \leq h(\alpha(t_1)) - 7C_0$. Then Lemma 5.5 tells us that α is first $17C_0$ -coarsely increasing, which contradicts what we assumed. \square

We have classified the possible shapes of geodesic rays. Since geodesic lines are constructed from two geodesic rays glued together, we will be able to classify their shapes too.

DEFINITION 5.10. — *Let N be an admissible norm and let $\alpha = (\alpha_X, \alpha_Y) : \mathbb{R} \rightarrow X \bowtie Y$ be a path of $(X \bowtie Y, d_{\bowtie})$. Let $\kappa \geq 0$.*

- (1) α is called *X-type* at scale κ if and only if:
 - (a) α_X is in a κ -neighbourhood of a geodesic of X ,
 - (b) α_Y is in a κ -neighbourhood of a vertical geodesic of Y .
- (2) α is called *Y-type* at scale κ if and only if:
 - (a) α_Y is in a κ -neighbourhood of a geodesic of Y ,
 - (b) α_X is in a κ -neighbourhood of a vertical geodesic of X .

The X-type paths follow geodesics of X , meaning that they are close to a geodesic in a copy of X inside $X \bowtie Y$. The Y-type paths follow geodesics of Y .

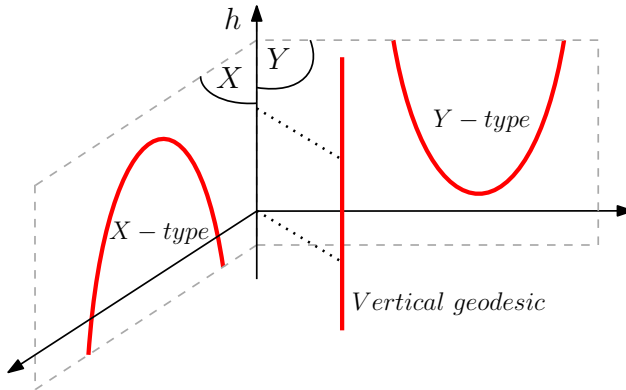


Figure 5.2. Different type of geodesics in $X \bowtie Y$.

Remark 5.11. — In a horospherical product, being close to a vertical geodesic is equivalent to be both X-type and Y-type.

THEOREM 5.12. — *Let N be an admissible norm. There exists $\kappa \geq 0$ depending only on δ and N such that for any $\alpha : \mathbb{R} \rightarrow X \bowtie Y$ geodesic of $(X \bowtie Y, d_{\bowtie})$ at least one of the two following statements holds.*

- (1) α is a X-type geodesic at scale κ of $(X \bowtie Y, d_{\bowtie})$,
- (2) α is a Y-type geodesic at scale κ of $(X \bowtie Y, d_{\bowtie})$.

Proof. — It follows from Lemma 5.9 that α changes its coarse monotonicity at most once. Otherwise there would exist a geodesic ray included in α

that changes at least two times of coarse monotonicity. We cut α in two coarsely monotone geodesic rays $\alpha_1 : [0, +\infty[\rightarrow X \bowtie Y$ and $\alpha_2 : [0, +\infty[\rightarrow X \bowtie Y$ such that up to a parametrisation $\alpha_1(0) = \alpha_2(0)$ and $\alpha_1 \cup \alpha_2 = \alpha$. By Lemma 5.8 there exists κ_1 and κ_2 depending only on δ such that α_1 is in the κ_1 -neighbourhood of a vertical geodesic ray $V_1 = (V_{1,X}, V_{1,Y}) : [0; +\infty[\rightarrow X \bowtie Y$ and such that α_2 is in the κ_2 -neighbourhood of a vertical geodesic ray $V_2 = (V_{2,X}, V_{2,Y}) : [0; +\infty[\rightarrow X \bowtie Y$. This lemma also gives us $d_{\bowtie}(\alpha_1(0), V_1(0)) \leq \kappa_1$ and $d_{\bowtie}(\alpha_2(0), V_2(0)) \leq \kappa_2$.

Assume that $\lim_{t \rightarrow +\infty} h(V_{1,X}(t)) = \lim_{t \rightarrow +\infty} h(V_{2,X}(t)) = +\infty$, then they are both vertical rays hence are close to a common vertical geodesic ray. Furthermore $\lim_{t \rightarrow +\infty} h(V_{1,Y}(t)) = \lim_{t \rightarrow +\infty} h(V_{2,Y}(t)) = -\infty$ in that case. Let W_Y be the non continuous path of Y defined as follows.

$$W_Y(t) = \begin{cases} V_{1,Y}(-t) & \forall t \in]-\infty; 0] \\ V_{2,Y}(t) & \forall t \in]0; +\infty[\end{cases}$$

We now prove that $W_Y : \mathbb{R} \rightarrow Y$ is a quasigeodesic of Y . Let t_1 and t_2 be two real numbers. Since $V_{1,Y}$ and $V_{2,Y}$ are geodesics, $d_Y(W_Y(t_1), W_Y(t_2)) = |t_1 - t_2|$ if t_1 and t_2 are both non positive or both positive. Thereby we can assume without loss of generality that t_1 is non positive and that t_2 is positive. We also assume without loss of generality that $|t_1| \geq |t_2|$. The quasi-isometric upper bound is given by:

$$\begin{aligned} d_Y(W_Y(t_1), W_Y(t_2)) &= d_Y(V_{1,Y}(-t_1), V_{2,Y}(t_2)) \\ &\leq d_Y(V_{1,Y}(-t_1), V_{1,Y}(0)) \\ &\quad + d_Y(V_{1,Y}(0), V_{2,Y}(0)) + d_Y(V_{2,Y}(0), V_{2,Y}(t_2)) \\ &\leq |t_1| + \kappa_1 + \kappa_2 + |t_2| \\ &\leq |t_1 - t_2| + \kappa_1 + \kappa_2, \\ &\quad \text{since } t_1 \text{ and } t_2 \text{ have different signs.} \end{aligned}$$

It remains to prove the lower bound of the quasi-geodesic definition on W_Y .

$$\begin{aligned} d_Y(W_Y(t_1), W_Y(t_2)) &= d_Y(V_{1,Y}(-t_1), V_{2,Y}(t_2)) \\ &\geq \frac{1}{2C_N} d_{\bowtie}(V_1(-t_1), V_2(t_2)) - d_X(V_{1,X}(-t_1), V_{2,X}(t_2)) \\ &\geq \frac{1}{2C_N} d_{\bowtie}(\alpha(t_1), \alpha(t_2)) - \frac{\kappa_1 + \kappa_2}{C_N} - d_X(V_{1,X}(-t_1), V_{2,X}(t_2)). \end{aligned} \quad (5.11)$$

The Busemann assumption on X provides us with:

$$d_X(V_{1,X}(-t_1), V_{2,X}(-t_1)) \leq d_X(V_{1,X}(0), V_{2,X}(0)) \leq \kappa_1 + \kappa_2.$$

Since α is a geodesic and by using the triangle inequality on (5.11) we have:

$$\begin{aligned} d_Y(W_Y(t_1), W_Y(t_2)) &\geq \frac{|t_1 - t_2|}{2C_N} - d_X(V_{1,X}(-t_1), V_{2,X}(-t_1)) \\ &\quad - d_X(V_{2,X}(-t_1), V_{2,X}(t_2)) - \frac{\kappa_1 + \kappa_2}{C_N} \\ &\geq \frac{|t_1 - t_2|}{2C_N} - \Delta h(V_{2,Y}(-t_1), V_{2,Y}(t_2)) - \left(\frac{1}{C_N} + 1\right)(\kappa_1 + \kappa_2). \end{aligned}$$

Assume that $\Delta h(V_{2,Y}(-t_1), V_{2,Y}(t_2)) \leq \frac{|t_1 - t_2|}{4C_N}$, then:

$$d_Y(W_Y(t_1), W_Y(t_2)) \geq \frac{|t_1 - t_2|}{4C_N} - \left(\frac{1}{C_N} + 1\right)(\kappa_1 + \kappa_2).$$

Hence W_Y is a $(\frac{1}{4C_N}, (\frac{1}{C_N} + 1)(\kappa_1 + \kappa_2))$ quasi-geodesic, which was the remaining case. Since κ_1 and κ_2 depend only on δ and N , there exists a constant κ' depending only on δ and N such that $V_{1,Y} \cup V_{2,Y}$ is in the κ' -neighbourhood of a geodesic of Y . The geodesic α is a Y -type geodesic in this case.

Assume $\lim_{t \rightarrow +\infty} h(V_{1,X}(t)) = \lim_{t \rightarrow +\infty} h(V_{2,X}(t)) = -\infty$, we prove similarly that α is a X -type geodesic. \square

If a geodesic is both X -type at scale κ and Y -type at scale κ , then it is in a κ -neighbourhood of a vertical geodesic of $X \bowtie Y$.

5.4. Visual boundary of $X \bowtie Y$

We will now look at the visual boundary of our horospherical products. This notion is described for the Sol geometry in the work of Troyanov [15] through the objects called geodesic horizons. We extend one of the definitions presented in [15, p. 4] for horospherical products.

We will use a result of [14] to describe the visual boundary of horospherical products.

PROPERTY 5.13 ([14, Property 10.1.7 p. 234]). — *Let X be a proper Busemann space, let q be a point in X and let $r : [0, +\infty[\rightarrow X$ be a geodesic ray. Then, there exists a unique geodesic ray r' starting at q that is asymptotic to r .*

THEOREM 5.14. — *Let N be an admissible norm. We fix base points and directions $(w_X, a_X) \in X \times \partial X$, $(w_Y, a_Y) \in Y \times \partial Y$. Let $X \bowtie Y$ be the horospherical product with respect to (w_X, a_X) and (w_Y, a_Y) . Then the*

visual boundary of $(X \bowtie Y, d_{\bowtie})$ with respect to a base point $o = (o_X, o_Y)$ is given by:

$$\begin{aligned} \partial_o(X \bowtie Y) &= \left((\partial X \setminus \{a_X\}) \times \{a_Y\} \right) \cup \left(\{a_X\} \times (\partial Y \setminus \{a_Y\}) \right) \\ &= \left((\partial X \times \{a_Y\}) \cup (\{a_X\} \times \partial Y) \right) \setminus \{(a_X, a_Y)\}. \end{aligned}$$

The fact that (a_X, a_Y) is not allowed as a direction in $X \bowtie Y$ is understandable since both heights in X and Y would tend to $+\infty$, which is impossible by the definition of $X \bowtie Y$.

Proof. — Let α be a geodesic ray. Lemma 5.9 implies that there exists $t_0 \in \mathbb{R}$ such that α is coarsely monotone on $[t_0, +\infty[$. Then Lemma 5.8 tells us that $\alpha([t_0, +\infty[)$ is at finite Hausdorff distance from a vertical geodesic ray $V = (V_X, V_Y)$, hence α is also at finite Hausdorff distance from V .

Since X is Busemann and proper, Property 5.13 ensure us there exists V'_X a vertical geodesic ray such that V_X and V'_X are at finite Hausdorff distance with $V'_X(0) = o_X$. Similarly, there exists V'_Y a vertical geodesic ray of Y with $V'_Y(0) = o_Y$ such that V_Y and V'_Y are at finite Hausdorff distance.

Furthermore, there is at least one vertical geodesic ray $V' = (V'_Y, V'_X)$ in every asymptotic equivalence class of geodesic rays, hence $\partial_o X \bowtie Y$ is the set of asymptotic equivalence classes of vertical geodesic rays starting at o . Therefore, an asymptotic equivalence class can be identified by the couple of directions of a vertical geodesic ray. Then $\partial_o X \bowtie Y$ can be identified to:

$$\left((\partial X \setminus \{a_X\}) \times \{a_Y\} \right) \cup \left(\{a_X\} \times (\partial Y \setminus \{a_Y\}) \right).$$

the union between downward directions and upward directions, which proves Theorem 5.14. \square

Example 5.15. — In the case of Sol, X and Y are hyperbolic planes \mathbb{H}_2 , hence their boundaries are $\partial X = \partial \mathbb{H}_2 = S^1$ and $\partial Y = S^1$. Then $\partial_o \text{Sol}$ can be identified to the following set:

$$(S^1 \setminus \{a_X\}) \times \{a_Y\} \cup \{a_X\} \times (S^1 \setminus \{a_Y\}). \quad (5.12)$$

It can be seen as two lines at infinity, one upward $\{a_X\} \times (S^1 \setminus \{a_Y\})$ and the other one downward $(S^1 \setminus \{a_X\}) \times \{a_Y\}$.

It is similar to [15, Proposition 6.4].

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