Annales de la Faculté des Sciences de Toulouse
MATHÉMATIQUES

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Tome XXXV, n° 2 (2026), p. 459–466.

https://doi.org/10.5802/afst.1853

Publication membre du centre
Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2258-7519

Annales de la faculté des sciences de Toulouse
Volume XXXV, n° 2, 2026
pp. 459-466

Denseness results for zeros and roots of unity in character tables [(*)]

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Abstract. — For any irreducible character χ of a finite group G, let $\theta(\chi)$ denote the proportion of elements $g \in G$ for which $\chi(g)$ is either zero or a root of unity. Then for any $L \in [\frac{1}{2}, 1]$ and any $\epsilon > 0$, there exists an irreducible character χ of a finite group such that $|\theta(\chi) - L| < \epsilon$.

Résumé. — Pour tout caractère irréductible χ d'un groupe fini G, soit $\theta(\chi)$ la proportion des éléments $g \in G$ pour lesquels $\chi(g)$ est soit nulle, soit une racine de l'unité. Alors, pour tout $L \in [\frac{1}{2}, 1]$ et tout $\epsilon > 0$, il existe un caractère irréductible χ d'un groupe fini tel que $|\theta(\chi) - L| < \epsilon$.

1. Introduction

For any irreducible character χ of a finite group G, let

$$\theta(\chi) = \frac{|\{g \in G : \chi(g) \text{ is zero or a root of unity}\}|}{|G|},$$

so $\theta(\chi)$ is the proportion of elements $g \in G$ for which $\chi(g)$ is zero or a root of unity. Using a result of C. L. Siegel [8], J. G. Thompson [4, p. 46] showed that $\theta(\chi) > \frac{1}{3}$. We show that there are no gaps past $\frac{1}{2}$ in the following sense.

Theorem 1.1. — *For any $L \in [\frac{1}{2}, 1]$ and any $\epsilon > 0$, there exists an irreducible character χ of a finite group such that $|\theta(\chi) - L| < \epsilon$.*

(*) Reçu le 15 octobre 2024, accepté le 28 mai 2025.
Keywords: zeros, roots of unity, characters.
2020 *Mathematics Subject Classification:* 20C15.
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Article proposé par Jean-François Coulombel.

We also establish results for the following statistics. For any finite group G, let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G and let $\mathrm{Cl}(G)$ denote the set of conjugacy classes of G. For $\chi \in \mathrm{Irr}(G)$, let

$$z_I(\chi) = \frac{|\{g \in G : \chi(g) = 0\}|}{|G|}, \qquad z_{II}(\chi) = \frac{\left|\{g^G \in \mathrm{Cl}(G) : \chi(g) = 0\}\right|}{|\mathrm{Cl}(G)|},$$

$$u_I(\chi) = \frac{|\{g \in G : |\chi(g)| = 1\}|}{|G|}, \qquad u_{II}(\chi) = \frac{\left|\{g^G \in \mathrm{Cl}(G) : |\chi(g)| = 1\}\right|}{|\mathrm{Cl}(G)|},$$

and

$$\theta_{II}(\chi) = \frac{\left|\{g^G \in \mathrm{Cl}(G) : \chi(g) \text{ is zero or a root of unity}\}\right|}{|\mathrm{Cl}(G)|}.$$

Since $|\chi(g)| = 1$ if and only if $\chi(g)$ is a root of unity, we have $\theta = z_I + u_I$ and $\theta_{II} = z_{II} + u_{II}$.

Theorem 1.2. — *Let $f \in \{z_I, z_{II}, u_I, u_{II}, \theta_{II}\}$. Then for any $L \in [0,1]$ and any $\epsilon > 0$, there exists an irreducible character χ of a finite group such that $|f(\chi) - L| < \epsilon$.*

The parts of Theorem 1.2 concerning $z_I(\chi)$ and $z_{II}(\chi)$ are local analogues of results established by the author in [6]. For any finite group G, let

$$z_I(G) = \mathbb{E}_{\chi\in\mathrm{Irr}(G)}(z_I(\chi)) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) = 0\}|}{|\mathrm{Irr}(G) \times G|}$$

and

$$z_{II}(G) = \mathbb{E}_{\chi\in\mathrm{Irr}(G)}(z_{II}(\chi)) = \frac{\left|\{(\chi, g^G) \in \mathrm{Irr}(G) \times \mathrm{Cl}(G) : \chi(g) = 0\}\right|}{|\mathrm{Irr}(G) \times \mathrm{Cl}(G)|},$$

so $z_{II}(G)$ is the fraction of the character table of G that is covered by zeros. The main result of [6] is as follows.

Theorem. — *The sets $\{z_I(G) : |G| < \infty\}$ and $\{z_{II}(G) : |G| < \infty\}$ are dense subsets of the interval $[0,1]$.*

For any finite group G, let

$$\theta(G) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) \text{ is zero or a root of unity}\}|}{|\mathrm{Irr}(G) \times G|},$$

$$\theta_{II}(G) = \frac{|\{(\chi, g^G) \in \mathrm{Irr}(G) \times \mathrm{Cl}(G) : \chi(g) \text{ is zero or a root of unity}\}|}{|\mathrm{Irr}(G) \times \mathrm{Cl}(G)|},$$

$$u_I(G) = \frac{|\{(\chi, g) \in \mathrm{Irr}(G) \times G : \chi(g) \text{ is a root of unity}\}|}{|\mathrm{Irr}(G) \times G|},$$

$$u_{II}(G) = \frac{|\{(\chi, g^G) \in \mathrm{Irr}(G) \times \mathrm{Cl}(G) : \chi(g) \text{ is a root of unity}\}|}{|\mathrm{Irr}(G) \times \mathrm{Cl}(G)|}.$$

The global analogues of Theorems 1.1 and 1.2 are as follows.

Theorem 1.3. — *For any* $L \in \left[\frac{1}{2}, 1\right]$ *and any* $\epsilon > 0$, *there exists a finite group* G *such that* $|\theta(G) - L| < \epsilon$.

Theorem 1.4. — *Let* $f \in \{z_I, z_{II}, u_I, u_{II}, \theta_{II}\}$. *Then for any* $L \in [0, 1]$ *and any* $\epsilon > 0$, *there exists a finite group* G *such that* $|f(G) - L| < \epsilon$.

2. Preliminaries

Our constructions make use of extraspecial groups, dihedral groups, projective special linear groups, and the following important consequence of [7, Theorem 8].

Theorem 2.1. — *If* χ *is an irreducible character of a finite nilpotent group* G *and if* $g \in G$, *then* $\chi(g)$ *is a root of unity if and only if* $\chi(1) = 1$.

For any algebraic integer α that belongs to a cyclotomic field $\mathbb{Q}(\zeta)$, let

$$\begin{aligned}\mathfrak{m}(\alpha) &= \frac{1}{[\mathbb{Q}(|\alpha|^2) : \mathbb{Q}]} \operatorname{Tr}_{\mathbb{Q}(|\alpha|^2)/\mathbb{Q}}(|\alpha|^2) \\ &= \frac{1}{|\operatorname{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})|} \sum_{\sigma \in \operatorname{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} |\sigma(\alpha)|^2.\end{aligned}$$

The following lemma is well known. Cf. [1, p. 113].

Lemma 2.2. — *Let* α *be an algebraic integer that belongs to a cyclotomic field. Then the following are equivalent.*

(1) α *is a root of unity.*
(2) $|\alpha| = 1$.
(3) $\mathfrak{m}(\alpha) = 1$.

Proof of Lemma 2.2. — Let ζ be a root of unity such that $\alpha \in \mathbb{Q}(\zeta)$. Let $\mathcal{G} = \operatorname{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$. Since $\mathcal{G}$ is abelian and contains the restriction of complex conjugation, $|\alpha| = 1$ if and only if $|\sigma(\alpha)|^2 = 1$ for all $\sigma \in \mathcal{G}$. By a theorem of Kronecker, $|\sigma(\alpha)|^2 = 1$ for all $\sigma \in \mathcal{G}$ if and only if α is a root of unity.

For the equivalence of the second and third statements, we may assume that $\alpha \neq 0$, since $\mathfrak{m}(\alpha) = 0$ if and only if $\alpha = 0$. Then by the inequality of the arithmetic and geometric means, we have

$$\mathfrak{m}(\alpha) \geqslant \Big(\prod_{\sigma \in \mathcal{G}} |\sigma(\alpha)|^2\Big)^{1/|\mathcal{G}|} \geqslant 1,$$

since the product $\prod_{\sigma \in \mathcal{G}} |\sigma(\alpha)|^2$ is a nonzero algebraic integer in $\mathbb{Q}(\zeta)$ that is invariant under $\mathcal{G}$ and is therefore a positive rational integer. Moreover, equality holds in both places if and only if $|\sigma(\alpha)|^2 = 1$ for all $\sigma \in \mathcal{G}$, which again is equivalent to $|\alpha| = 1$. $\square$

Proof of Theorem 2.1. — Let χ be an irreducible character of a finite nilpotent group G, and let $g \in G$. Then by [7, Theorem 8],

$$\chi(g) = 0 \quad \text{or} \quad \mathfrak{m}(\chi(g)) \geqslant 2^{|\{\text{primes dividing } \chi(1)\}|}.$$

Hence $\chi(g)$ is a root of unity if and only if $\chi(1) = 1$. $\square$

Our constructions in Section 3 will make use of direct products of certain groups. For the purposes of this paper, for any non-negative integer n and any irreducible character χ of a finite group G, let χ^n be shorthand for the irreducible character $\chi\times\chi\times\ldots\times\chi$ of the direct product $G^n = G\times G\times\ldots\times G$, where χ^0 is the principal character of the trivial group G^0.

Lemma 2.3. — *Let $u \in \{u_I, u_{II}\}$. If G and H are finite nilpotent groups and $\chi\times\psi \in \mathrm{Irr}(G\times H)$, then $u(G\times H) = u(G)u(H)$ and $u(\chi\times\psi) = u(\chi)u(\psi)$.*

Proof of Lemma 2.3. — Let $g \in G$ and $h \in H$. By Theorem 2.1, since $\chi\times\psi$ is an irreducible character of a finite nilpotent group, the value $\chi(g)\psi(h)$ is a root of unity if and only if the degree $\chi(1)\psi(1)$ is equal to 1, which, by Theorem 2.1, is equivalent to $\chi(g)$ and $\psi(h)$ being roots of unity. $\square$

Lemma 2.4. — *Let $z \in \{z_I, z_{II}\}$. Let G be a finite nonabelian group and let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) > 1$. Then*

(1) *$z(G^1), z(G^2), \ldots$ tends to 1 with steps of size less than $z(G)$,*
(2) *$z(\chi^1), z(\chi^2), \ldots$ tends to 1 with steps of size less than $z(\chi)$.*

Proof of Lemma 2.4. — Let $x \in \{G, \chi\}$. Then

$$z(x^{k+1}) = z(x^k) + (1 - z(x^k))z(x), \quad k = 1, 2, \ldots, \tag{2.1}$$

so the sequence $z(x^k)$ $(k = 1, 2, \ldots)$ is increasing, bounded, and thus convergent with limit ℓ satisfying $\ell = \ell + (1-\ell)z(x)$. By a theorem of Burnside, $z(x) \neq 0$, so $\ell = 1$. By (2.1), $|z(x^{k+1}) - z(x^k)| < z(x)$ for $k = 1, 2, \ldots$. $\square$

For each prime power q, let $\chi_q = \pi_q - 1$, where π_q is the permutation character of the group $L_2(q) = \mathrm{PSL}_2(q)$ acting faithfully in the natural way on the points of the projective line over $\mathbb{F}_q$. Recall that if q is even, then $L_2(q)$ has exactly $q^3 - q$ elements and $q + 1$ conjugacy classes.

Lemma 2.5. — *Let q be a prime power. Then χ_q is the unique irreducible character of $L_2(q)$ such that $\chi_q(1) = q$, and it has the following properties.*

(1) *χ_q vanishes on exactly $\frac{2}{\gcd(2,q)}$ conjugacy classes, each of size $\frac{|L_2(q)|}{q}$.*
(2) *If $\chi_q(g) \neq 0$ and $g \neq 1$, then $\chi_q(g) = \pm 1$.*

Proof of Lemma 2.5. — This follows from [3, Chapter XI, Lemma 5.2 and Theorems 5.5–5.7] and the character table of $L_2(q)$ [5, pp. 402–403]. ☐

For each positive integer n, let H_n denote a fixed extraspecial group of order 2^{2n+1}. In general, for any positive integer n and prime p, if E is an extraspecial group of order p^{2n+1}, then the center $Z(E)$ is cyclic of order p and the irreducible characters of E are well known.

Lemma 2.6. — *Let E be an extraspecial group of order p^{2n+1} for some prime p and positive integer n. Then E has exactly $p^{2n}+p-1$ irreducible characters. Exactly $p-1$ irreducible characters of E have degree greater than 1, and they are $\frac{1}{p^n}\lambda^E$ for $\lambda \in \mathrm{Irr}(Z(E)) - \{1\}$. In particular, for each $\chi \in \mathrm{Irr}(E)$ with $\chi(1) > 1$, the zeros of χ are the elements $g \in E - Z(E)$.*

Proof of Lemma 2.6. — See [2, Kapitel V, Satz 16.14]. ☐

Lemma 2.7. — *If E is an extraspecial group of order p^{2n+1} for some prime p and positive integer n, then*

$$u_I(E) = \frac{p^{2n}}{p^{2n}+p-1}, \qquad z_I(E) = \frac{(p-1)\big(p^{2n+1}-p\big)}{(p^{2n}+p-1)p^{2n+1}}, \tag{2.2}$$

$$u_{II}(E) = \frac{p^{2n}}{p^{2n}+p-1}, \qquad z_{II}(E) = \frac{(p-1)(p^{2n}-1)}{(p^{2n}+p-1)^2}. \tag{2.3}$$

Proof of Lemma 2.7. — This follows from Lemma 2.6. See also [6]. ☐

For each positive integer n, let $G_n = \langle s, t \mid s^2 = t^{2^n} = (st)^2 = 1\rangle$, so G_n is the nilpotent dihedral group of order 2^{n+1} with $2^{n-1}+3$ conjugacy classes. If $n = 1$, then G_n is the Klein 4-group. If $n \geqslant 2$, then 1 and $t^{2^{n-1}}$ make up the center, each noncentral element of G_n lying in $\langle t\rangle$ is conjugate only to itself and its distinct inverse, and the complement of $\langle t\rangle$ breaks into 2 classes of size 2^{n-1}. Recall [2] that G_n has exactly 4 characters λ_i of degree 1 and $2^{n-1}-1$ irreducible characters $\eta_1, \eta_2, \dots, \eta_{2^{n-1}-1}$ of degree greater than 1 given by

$$\eta_h(t^k) = 2\cos\frac{\pi hk}{2^{n-1}}, \qquad \eta_h(st^k) = 0. \tag{2.4}$$

Let $\nu_p(n)$ denote the p-adic valuation of an integer n.

Lemma 2.8. — *Let n and h be positive integers such that $0 < h < 2^{n-1}$, and let η_h be the irreducible character of G_n given by (2.4).*

(1) *$\eta_h(g) = 0$ for exactly $2^{\nu_2(h)+1} + 2^n$ elements $g \in G_n$.*
(2) *η_h vanishes on exactly $2^{\nu_2(h)} + 2$ conjugacy classes of G_n.*
(3) *The number of pairs $(\chi, g) \in \mathrm{Irr}(G_n) \times G_n$ that satisfy $\chi(g) = 0$ is equal to $2^{n-1}(n-1) + 2^{2n-1} - 2^n$.*
(4) *The number of zeros in the character table of G_n is equal to*

$$2^{n-2}(n-1) + 2^n - 2.$$

Proof of Lemma 2.8. — The integers k that satisfy $\eta_h(t^k) = \cos\frac{\pi hk}{2^{n-1}} = 0$ and $0 \leqslant k < 2^n$ are the solutions of $2hk = 2^{n-1} \bmod 2^n$ with $0 \leqslant k < 2^n$, which are

$$2^{n-\nu_2(h)-2}(1+2j), \qquad 0 \leqslant j < 2^{\nu_2(h)+1}. \tag{2.5}$$

Hence η_h has exactly $2^{\nu_2(h)+1}$ zeros in $\langle t\rangle$. Since all elements of $s\langle t\rangle$ are also zeros of η_h, we get a total of $2^{\nu_2(h)+1} + 2^n$ zeros. For (2), since $t^{2^{n-1}}$ is not a zero of η_h, each zero of η_h that lies in $\langle t\rangle$ belongs to a class of size 2, while the zeros of η_h in $s\langle t\rangle$ split into two classes. For (3), the number of integers $0 < \ell < 2^{n-1}$ with $\nu_2(\ell) = k$ $(0 \leqslant k \leqslant n-2)$ is 2^{n-k-2}, so by (1), the number of pairs $(\chi, g) \in \mathrm{Irr}(G_n) \times G_n$ with $\chi(g) = 0$ is $\sum_{k=0}^{n-2} 2^{n-k-2}(2^{k+1} + 2^n)$. Similarly, by (2), the number of pairs $(\chi, g^{G_n}) \in \mathrm{Irr}(G_n) \times \mathrm{Cl}(G_n)$ with $\chi(g) = 0$ is $\sum_{k=0}^{n-2} 2^{n-k-2}(2^k + 2)$. ☐

Lemma 2.9. — *For any positive integer n,*

$$u_I(G_n) = \frac{4}{2^{n-1}+3}, \qquad z_I(G_n) = \frac{1}{2}\frac{2^n+n-3}{2^n+6}, \tag{2.6}$$

$$u_{II}(G_n) = \frac{4}{2^{n-1}+3}, \qquad z_{II}(G_n) = \frac{2^{n-2}(n+3)-2}{(2^{n-1}+3)^2}. \tag{2.7}$$

Proof of Lemma 2.9. — This follows from Theorem 2.1, Lemma 2.8, and the fact that G_n has exactly 4 characters of degree 1. ☐

3. Proofs of Theorems 1.1–1.4

Proof of Theorem 1.1. — Let $\epsilon > 0$. Let $n \geqslant 2$ be an integer such that $2^n > \frac{1}{\epsilon}$. Let γ be the irreducible character η_1 of the dihedral group G_n given by (2.4) with $h = 1$. By Lemma 2.8 (1), γ has exactly $2^n + 2$ zeros, so

$$z_I(\gamma) = \frac{1}{2} + \frac{1}{2^n} < \frac{1}{2} + \epsilon.$$

By Theorem 2.1, since $\gamma(1) > 1$ and G_n is nilpotent, $u_I(\gamma) = 0$.

Let $q = 2^r$ for some positive integer r such that $q > \frac{1}{\epsilon}$. By Lemma 2.5 (1), $z_I(\chi_q) = \frac{1}{q} < \epsilon$. By Lemma 2.5 (2) and the fact that $u_I(\gamma) = 0$, for each non-negative integer k, we have that $u_I(\gamma \times \chi_q^k) = 0$ and hence

$$\theta(\gamma \times \chi_q^{k+1}) = z_I(\gamma \times \chi_q^{k+1}) = z_I(\gamma \times \chi_q^k) + (1 - z_I(\gamma \times \chi_q^k))z_I(\chi_q).$$

It follows that the sequence $\theta(\gamma \times \chi_q^k)$ is increasing, bounded, and therefore convergent with limit ℓ satisfying $\ell = \ell + (1-\ell)z_I(\chi_q)$, which implies $\ell = 1$. Since $z_I(\gamma) < \frac{1}{2} + \epsilon$ and $z_I(\chi_q) < \epsilon$, the sequence $\theta(\gamma \times \chi_q^k)$ $(k = 0, 1, 2, \dots)$ starts out less than $\frac{1}{2} + \epsilon$ and tends to 1 with steps of size less than ϵ. Hence, for any $L \in [\frac{1}{2}, 1]$, there exists a k such that $|\theta(\gamma \times \chi_q^k) - L| < \epsilon$. ☐

Proof of Theorem 1.3. — Let $\epsilon > 0$. By Lemma 2.9, the nilpotent dihedral groups G_n satisfy

$$u_I(G_n) = \frac{4}{2^{n-1}+3}, \qquad z_I(G_n) = \frac{1}{2}\frac{2^n+n-3}{2^n+6}.$$

Let l be a positive integer such that

$$u_I(G_l) < \frac{\epsilon}{2}, \qquad \frac{1}{2} < z_I(G_l) < \frac{1}{2} + \frac{\epsilon}{2}. \tag{3.1}$$

By Lemma 2.7, the extraspecial groups H_n of order 2^{2n+1} satisfy

$$u_I(H_n) = \frac{2^{2n}}{2^{2n}+1}, \qquad z_I(H_n) = \frac{2^{2n+1}-2}{2^{4n+1}+2^{2n+1}}.$$

Let m be a positive integer such that $z_I(H_m) < \epsilon$.

By Lemma 2.3, for $k = 0, 1, 2, \dots$, we have $u_I(G_l \times H_m^k) = u_I(G_l)u_I(H_m)^k$ and therefore

$$\theta(G_l \times H_m^k) = u_I(G_l)u_I(H_m)^k + z_I(G_l) + (1 - z_I(G_l))z_I(H_m^k). \tag{3.2}$$

By Lemma 2.4, the sequence $z_I(H_m^k)$ $(k = 0, 1, 2, \dots)$ tends to 1 with steps of size less than ϵ. Since $0 < u_I(H_m) < 1$, the sequence $u_I(H_m)^k$ tends to 0. It follows by (3.1) and (3.2) that the sequence $\theta(G_l \times H_m^k)$ $(k = 0, 1, 2, \dots)$ starts out less than $\frac{1}{2} + \frac{\epsilon}{2}$ and tends to 1 with steps of size less than ϵ. Hence, for any $L \in [\frac{1}{2}, 1]$, there exists a k such that $|\theta(G_l \times H_m^k) - L| < \epsilon$. □

Proof of Theorem 1.2. — Let $\epsilon > 0$. Let $z \in \{z_I, z_{II}\}$. Put $q = 2^r$ for some positive integer r such that $q > \frac{1}{\epsilon}$. By Lemma 2.5 (1), we have $z_I(\chi_q) = \frac{1}{q}$ and $z_{II}(\chi_q) = \frac{1}{q+1}$, so $z(\chi_q) < \epsilon$. Therefore, by Lemma 2.4, the sequence $z(\chi_q^n)$ $(n = 1, 2, \dots)$ starts out less than ϵ and tends to 1 with steps of size less than ϵ. Hence, for any $L \in [0, 1]$, there exists an n such that $|z(\chi_q^n) - L| < \epsilon$.

Let $u \in \{u_I, u_{II}\}$. Put $q = 2^s$ for some positive integer s such that $q > \frac{2}{\epsilon}$. By Lemma 2.5, for each positive integer n, $u(\chi_q^n) = u(\chi_q)^n = (1 - \Delta_q)^n$, where $\Delta_q = \frac{1}{q} + \frac{1}{|L_2(q)|} < \frac{2}{q}$ if $u = u_I$, and $\Delta_q = \frac{2}{q+1}$ if $u = u_{II}$. So $0 < \Delta_q < \epsilon$. The sequence $u(\chi_q^n)$ $(n = 1, 2, \dots)$ therefore starts out greater than $1 - \epsilon$ and tends monotonically to 0 with steps of size less than ϵ. Hence, for any $L \in [0, 1]$, there exists an n such that $|u(\chi_q^n) - L| < \epsilon$.

For θ_{II}, choose an integer $n \geqslant 2$ such that $2^{n-1} > \frac{3}{\epsilon}$. Let γ be the irreducible character η_1 of the dihedral group G_n given by (2.4) with $h = 1$. By Theorem 2.1, $u_{II}(\gamma) = 0$, so $\theta_{II}(\gamma^k) = z_{II}(\gamma^k)$ for each positive integer k. By Lemma 2.8 (2), γ vanishes on exactly 3 classes, so $z_{II}(\gamma) = \frac{3}{2^{n-1}+3} < \epsilon$. Therefore, by Lemma 2.4, the sequence $\theta_{II}(\gamma^k)$ $(k = 1, 2, \dots)$ starts out less than ϵ and tends to 1 with steps of size less than ϵ. Hence, for any $L \in [0, 1]$, there exists a k such that $|\theta_{II}(\gamma^k) - L| < \epsilon$. □

Proof of Theorem 1.4. — Let $\epsilon > 0$. Consider first θ_{II}. By Lemma 2.9, the nilpotent dihedral groups G_n satisfy

$$u_{II}(G_n) = \frac{4}{2^{n-1}+3}, \qquad z_{II}(G_n) = \frac{2^{n-2}(n+3)-2}{(2^{n-1}+3)^2}.$$

Let $l \geqslant 2$ be an integer such that $u_{II}(G_l) < \frac{\epsilon}{2}$ and $z_{II}(G_l) < \frac{\epsilon}{2}$. By Lemma 2.3, $u_{II}(G_l^k) = u_{II}(G_l)^k$ for each positive integer k, so the sequence $u_{II}(G_l^k)$ $(k = 1, 2, \dots)$ starts out less than $\frac{\epsilon}{2}$ and tends monotonically to 0. By Lemma 2.4, the sequence $z_{II}(G_l^k)$ $(k = 1, 2, \dots)$ starts out less than $\frac{\epsilon}{2}$ and tends to 1 with steps of size less than $\frac{\epsilon}{2}$. The sequence $\theta_{II}(G_l^k)$ $(k = 1, 2, \dots)$ therefore starts out less than ϵ and tends to 1 with steps of size less than ϵ. Hence, for any $L \in [0, 1]$, there exists a k such that $|\theta_{II}(G_l^k) - L| < \epsilon$.

Let $u \in \{u_I, u_{II}\}$. By Lemma 2.7, we have that the extraspecial groups H_n of order 2^{2n+1} satisfy $u(H_n) = \frac{2^{2n}}{2^{2n}+1}$. Let m be a positive integer such that $|u(H_m) - 1| < \epsilon$. By Lemma 2.3, we have $u(H_m^k) = u(H_m)^k$ for each positive integer k. Therefore, the sequence $u(H_m^k)$ $(k = 1, 2, \dots)$ starts out less than ϵ away from 1 and tends to 0 with steps of size less than ϵ. Hence, for any $L \in [0, 1]$, there exists a k such that $|u(H_m^k) - L| < \epsilon$.

Let $z \in \{z_I, z_{II}\}$. Let r be a positive integer such that $z(H_r) < \epsilon$, which is possible by Lemma 2.7. By Lemma 2.4, the sequence $z(H_r^k)$ $(k = 1, 2, \dots)$ starts out less than ϵ and tends to 1 with steps of size less than ϵ. Hence, for any $L \in [0, 1]$, there exists a k such that $|z(H_r^k) - L| < \epsilon$. (Cf. [6].) ☐

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