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Global Lipschitz and Sobolev estimates for the Monge–Ampère eigenfunctions of general bounded convex domains (*)

Nam Q. Le (1)

Abstract. — We show that the Monge–Ampère eigenfunctions of general bounded convex domains are globally Lipschitz. The same result holds for convex solutions to degenerate Monge–Ampère equations of the form $\det D^2u = M|u|^p$ with zero boundary condition on general bounded convex domains in $\mathbb{R}^n$ within the sharp threshold $p > n-2$. As a consequence, we obtain global $W^{2,1}$ estimates for these solutions.

Résumé. — Nous montrons que les fonctions propres de l'opérateur de Monge–Ampère des domaines convexes bornés généraux sont globalement lipschitziennes. Le même résultat est valable pour les solutions convexes des équations de Monge–Ampère dégénérées de la forme $\det D^2u = M|u|^p$ avec condition aux limites nulle sur les domaines convexes bornés généraux de $\mathbb{R}^n$ dans le seuil précis $p > n-2$. En conséquence, nous obtenons des estimations globales dans $W^{2,1}$ pour ces solutions.

1. Introduction and statement of the main results

1.1. Motivations

We are interested in obtaining global Lipschitz and Sobolev regularity for nonzero convex Aleksandrov solutions $u \in C(\overline{\Omega})$ to the degenerate Monge–Ampère equation

$$\det D^2u = M|u|^p \quad \text{in} \quad \Omega, \qquad\qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.1}$$

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where $M > 0$, $p > n - 2$, and $\Omega \subset \mathbb{R}^n$ ($n \geqslant 2$) is a general bounded convex domain that is not assumed to be smooth nor strictly convex. The case $p = n$ corresponds to the Monge–Ampère eigenvalue problem. When $p \neq n$, by multiplying u with a suitable positive constant, we can always assume $M = 1$. For $0 < p \neq n$, the existence of a nonzero convex Aleksandrov solution to (1.1) is guaranteed by [10, Theorem 4.2] and it is unique when $p < n$.

Regarding interior regularity, solutions to (1.1) satisfy $u \in C^\infty(\Omega)$; see, for example, [10, Proposition 2.8] and [14, Proposition 6.36]. On smooth and uniformly convex domains, the global smoothness of solutions to (1.1) is by now well understood, thanks to [8] in two dimensions and [16, 20] in all dimensions; see also [9] for the global analyticity.

The eigenvalue problem for the Monge–Ampère operator $\det D^2 u$ on uniformly convex domains Ω in $\mathbb{R}^n$ ($n \geqslant 2$) with smooth boundary was first investigated by Lions [17]. He showed that there exist a unique positive constant $\lambda = \lambda(\Omega)$ and a unique (up to positive multiplicative constants) nonzero convex function $u \in C^{1,1}(\overline{\Omega}) \cap C^\infty(\Omega)$ solving the Monge–Ampère eigenvalue problem:

$$\begin{cases} \det D^2 u = \lambda |u|^n & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The constant $\lambda(\Omega)$ is called the Monge–Ampère eigenvalue of Ω. The nonzero convex solutions to (1.2) are called the Monge–Ampère eigenfunctions of Ω.

A variational characterization of $\lambda(\Omega)$ was found by Tso [21] who discovered that for uniformly convex domains Ω with sufficiently smooth boundaries, the following formula holds:

$$\lambda(\Omega) \ = \ \inf \left\{ \frac{\int_\Omega |u| \det D^2 u \,\mathrm{d}x}{\int_\Omega |u|^{n+1} \,\mathrm{d}x} \ : \ \begin{array}{l} u \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega) \setminus \{0\}, \\ u \text{ is convex in } \Omega, \ u = 0 \text{ on } \partial\Omega \end{array} \right\}. \tag{1.3}$$

For general bounded convex domains $\Omega \subset \mathbb{R}^n$, the Monge–Ampère eigenvalue problem (1.2) was studied in [10] where the existence, uniqueness and variational characterization of the Monge–Ampère eigenvalue, and the uniqueness of the convex Monge–Ampère eigenfunctions were obtained. In particular, there is a unique positive constant $\lambda = \lambda[\Omega]$ for problem (1.2) to have a nonzero convex solution $u \in C(\overline{\Omega})$, and the Monge–Ampère eigenvalue $\lambda[\Omega]$ is characterized by

$$\lambda[\Omega] \;=\; \inf\left\{ \frac{\int_\Omega |u| \det D^2u\,\mathrm{d}x}{\int_\Omega |u|^{n+1}\,\mathrm{d}x} : \begin{array}{l} u \in C(\overline{\Omega}) \setminus \{0\}, \\ u \text{ is convex in } \Omega,\; u = 0 \text{ on } \partial\Omega \end{array} \right\}. \tag{1.4}$$

Throughout, by an abuse of notation, we use $\det D^2u\,\mathrm{d}x$ to denote the Monge–Ampère measure associated with a convex function u on Ω (for which we refer to [14, Chapter 3] for more details). When Ω is uniformly convex with smooth boundary, by uniqueness, $\lambda(\Omega)$ defined by (1.3) and $\lambda[\Omega]$ defined by (1.4) must be equal, though the set of competitors for $\lambda(\Omega)$ is strictly contained in those for $\lambda[\Omega]$. We chose the bracket notation $\lambda[\Omega]$ in (1.4) for the Monge–Ampère eigenvalue of a general bounded convex domain Ω to emphasize that the boundary $\partial\Omega$ might have flat parts or corners.

It was shown in [10, Theorem 1.1] that the infimum in (1.4) is achieved by a nonzero convex function $u \in C^{0,\beta}(\overline{\Omega}) \cap C^\infty(\Omega)$ for all $\beta \in (0,1)$ and the pair $(u, \lambda[\Omega])$ solves (1.2). This makes one wonder if one can take $\beta = 1$ so as to extend Tso's characterization of the Monge–Ampère eigenvalue from uniformly convex domains with smooth boundaries to general bounded convex domains. This boils down to the question of whether the Monge–Ampère eigenfunctions of general bounded convex domains are globally Lipschitz. We will answer positively this question in Theorem 1.1. Due to the degeneracy of (1.2) or more general equation (1.1) near the boundary, the global Lipschitz regularity is a very nontrivial issue, as reviewed below.

- When $p < n-2$, [13, Theorem 1.1] shows that the unique nonzero convex solution u to (1.1) has its gradient blowing up near any *flat part of the boundary*. However, for $0 < p < n-2$, [12, Proposition 1] shows that for all $\beta \in (0, \frac{2}{n-p})$, one has

$$|u(x)| \leqslant C(n,p,\beta,\Omega)[\mathrm{dist}(x,\partial\Omega)]^\beta \quad \forall\, x \in \Omega. \tag{1.5}$$

- When $n = 2$ and $p = n-2 = 0$, the explicit solution to (1.1) on a planar triangle (see, for example, [13, Section 2.2] and [14, Example 3.32]) shows that u is only log-Lipschitz. The optimal boundary behavior for the nonzero convex solution to (1.1) in the case $p = n-2 > 0$ has not been addressed in the literature.
- When $p > n-2$, [13, Corollary 1.5] shows that convex solutions to (1.1) are globally log-Lipschitz:

$$|u(x)| \leqslant C(\Omega,n,p)\,\mathrm{dist}(x,\partial\Omega)(1 + |\log \mathrm{dist}(x,\partial\Omega)|)\|u\|_{L^\infty(\Omega)} \quad \text{for all } x \in \Omega. \tag{1.6}$$

1.2. Main results

Our first main theorem removes the logarithmic term in (1.6), thereby establishing the global Lipschitz regularity for convex solutions to (1.1) when $p > n-2$.

Theorem 1.1 (Global Lipschitz estimates for the Monge–Ampère eigenfunctions). — *Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a bounded convex domain. Let $p > n-2$ and let $u \in C(\overline{\Omega})$ be a nonzero convex function.*

(i) *Assume that u solves the degenerate Monge–Ampère equation*

$$\begin{cases} \det D^2 u = M|u|^p & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $M > 0$. Then u is globally Lipschitz and we have the estimate

$$|u(x)| \leqslant C(|\Omega|, \operatorname{diam}(\Omega), n, p, M) \operatorname{dist}(x, \partial\Omega) \|u\|_{L^\infty(\Omega)} \quad \text{for all } x \in \Omega.$$

(ii) *Assume that u satisfies in the sense of Aleksandrov*

$$\det D^2 u \leqslant M[\operatorname{dist}(\,\cdot\,, \partial\Omega)]^p \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega,$$

where $M > 0$. Then u is globally Lipschitz and we have the estimate

$$|u(x)| \leqslant C(\operatorname{diam}(\Omega), n, p, M) \operatorname{dist}(x, \partial\Omega) \quad \text{for all } x \in \Omega.$$

Note that the estimates in Theorem 1.1 give the global Lipschitz property of u due to its convexity. By [10, Lemma 3.1 (iii)], for u satisfying Theorem 1.1 (i), we have the estimates

$$c(n,p)|\Omega|^{-2} \leqslant M\|u\|^{p-n}_{L^\infty(\Omega)} \leqslant C(n,p)|\Omega|^{-2}. \tag{1.7}$$

Thus, the appearance of $\|u\|_{L^\infty(\Omega)}$ is only necessary in the case of the Monge–Ampère eigenvalue problem.

We quickly mention some implications of Theorem 1.1.

As discussed earlier, an immediate consequence of Theorem 1.1 is the following characterization of the Monge–Ampère eigenvalue $\lambda[\Omega]$ defined in (1.4).

Corollary 1.2. — *Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a bounded convex domain. Then the Monge–Ampère eigenvalue $\lambda[\Omega]$ of Ω has also the following characterization:*

$$\lambda[\Omega] = \inf\left\{ \frac{\int_\Omega |u| \det D^2 u \, \mathrm{d}x}{\int_\Omega |u|^{n+1} \, \mathrm{d}x} : \begin{array}{l} u \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega) \setminus \{0\}, \\ u \text{ is convex in } \Omega,\ u = 0 \text{ on } \partial\Omega \end{array} \right\}.$$

The global Lipschitz estimates in Theorem 1.1 (i) give global $W^{2,1}$ estimates for the solutions. We use the Hilbert–Schmidt norm for matrices in this note.

Corollary 1.3 (Global $W^{2,1}$ estimates for the Monge–Ampère eigenfunctions). — *Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a bounded convex domain. Let $p > n-2$ and let $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ be a nonzero convex function solving the degenerate Monge–Ampère equation*

$$\begin{cases} \det D^2 u = M|u|^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $M > 0$. Then $D^2u \in L^1(\Omega)$ with the estimate

$$\int_\Omega \|D^2 u\| \,\mathrm{d}x \leqslant C(|\Omega|, \operatorname{diam}(\Omega), n, p, M)\|u\|_{L^\infty(\Omega)}.$$

The conclusion of Corollary 1.3 is false when $p < n-2$ and Ω is a rectangular box in $\mathbb{R}^n$, as [15, Proposition 3.1] shows that in this situation, $D^2u \notin L^{\frac{n-p}{2(n-p)-2}+\varepsilon}(\Omega)$ for any $\varepsilon > 0$, so $D^2u \notin L^1(\Omega)$.

Though simple, the proof of Corollary 1.3 uses interior C^2 regularity for solutions to our degenerate Monge–Ampère equation. This interior second-order regularity or even interior $W^{2,1}$ regularity as in the nondegenerate case [19, 5] is not known for the convex functions in Theorem 1.1 (ii) so it is not clear if global $W^{2,1}$ estimates are possible for them.

Another consequence of Theorem 1.1 is the eventual global Lipschitz regularity for an iterative scheme for the Monge–Ampère eigenvalue problem with general initial data.

Corollary 1.4. — *Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a bounded and convex domain. For a convex function u on Ω, we define its Rayleigh quotient by*

$$R(u) = \frac{\int_\Omega |u| \det D^2 u \,\mathrm{d}x}{\int_\Omega |u|^{n+1}\,\mathrm{d}x}.$$

Let $u_0 \in C(\Omega)$ be a nonzero convex function on Ω with $0 < R(u_0) < \infty$. For $k \geqslant 0$, define the sequence $u_{k+1} \in C(\overline{\Omega})$ to be the convex Aleksandrov solutions of the Dirichlet problem

$$\begin{cases} \det D^2 u_{k+1} = R(u_k)|u_k|^n & \text{in } \Omega, \\ u_{k+1} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Then $u_k \in C^{2k-2}(\Omega) \cap C^{0,1}(\overline{\Omega})$ for all $k \geqslant (n+4)/2$.

The iterative scheme (1.8) was first introduced by Abedin and Kitagawa [1] to solve the Monge–Ampère eigenvalue problem (1.2) for a large class of convex initial data u_0 satisfying $\det D^2u_0 \geqslant c_0 > 0$. The sequence $\{u_k\}_{k=1}^{\infty}$ is obtained by repeatedly inverting the Monge–Ampère operator with Dirichlet boundary condition. The iterative scheme (1.8) was showed in [11, Theorem 1.4] to converge to problem (1.2) for all convex initial data having finite and nonzero Rayleigh quotient. We refer to [18] for numerical analysis of the scheme (1.8).

The threshold $p > n-2$ in Theorem 1.1 is sharp in all dimensions as global Lipschitz estimates are not possible in general for (1.1) when $p \leqslant n-2$. The case $p < n-2$ was treated in [13, Theorem 1.1]. Our next theorem shows that when $p = n-2$, the nonzero convex solution of (1.1) can have log-Lipschitz type behavior near the flat part of the boundary.

Theorem 1.5 (Infinite boundary gradient for degenerate Monge–Ampère equations). — *Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a bounded convex domain. Let $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ be the nonzero convex solution to*

$$\begin{cases} \det D^2u = |u|^{n-2} & \text{in } \Omega,\\ \qquad u = 0 & \text{on } \partial\Omega. \end{cases}$$

(i) *We have the estimate*

$$|u(z)| \leqslant C(\mathrm{diam}(\Omega), n)\,\mathrm{dist}(z,\partial\Omega)\big(1 + |\log \mathrm{dist}(z,\partial\Omega)|^{\frac{n}{2}}\big) \quad \forall\, z \in \Omega.$$

(ii) *Assume that there is a nonempty closed subset Γ of the boundary $\partial\Omega$ that lies on a hyperplane. Then, for $x \in \Omega$ sufficiently close to the interior of Γ, we have*

$$|u(x)| \geqslant c(n,\Omega,\Gamma)\,\mathrm{dist}(x,\partial\Omega)|\log \mathrm{dist}(x,\partial\Omega)|^{\frac{1}{n}}.$$

It would be interesting to bridge the gap between the exponents $n/2$ and $1/n$ in the upper bound and lower bound in Theorem 1.5.

1.3. On the proofs

We say a few words about the proof of Theorem 1.1 in Section 3. A key ingredient is the construction in Lemma 3.1 of Lipschitz convex subsolutions v to degenerate equations of the form $\det D^2v \geqslant |v|^p$ where $p > n-2$ and v is nonpositive on the boundary. If $\det D^2u = M|u|^p$ and $p \geqslant n$, then $\det D^2u \leqslant M\|u\|^{p-n+1}_{L^\infty(\Omega)}|u|^{n-1}$, so we can reduce the exponent p to be below n, the critical exponent. Our interest here is the range $p \in (n-2, n)$. From Lemma 3.1, we obtain a direct proof of Theorem 1.1 based on a comparison principle for degenerate subcritical Monge–Ampère equations $\det D^2w = |w|^q$ $(0 \leqslant q < n)$ in Lemma 2.1. This lemma allows us to improve the exponent β in (1.5) to $2/(n-p)$ when $p \in (0, n-2)$, avoiding the iteration argument in [12] using C^α $(0<\alpha<1)$ convex subsolutions of the form

$$w_\alpha(x) = x_n^\alpha\big(|x'|^2 - C_\alpha\big), \quad C_\alpha > 0$$

on the upper half space $\{x_n > 0\}$. These functions were motivated by [3, Lemma 1] where Caffarelli considered the case $\alpha = 2/n$ when $n \geqslant 3$ and $\alpha \in (0,1)$ when $n = 2$.

One cannot take $\alpha \geqslant 1$ in w_α due to the convexity requirement; see (2.6). Our crucial observation from (2.6) is that we can actually take $\alpha > 1$ as long as we change the sign in front of C_α, that is, we can look for convex subsolutions of the form $x_n^a(|x'|^2+A)$ where $a>1$ and $A>0$. The only issue with this ansatz is that it is always positive. However, this can be handled by subtracting from it a large multiple of x_n to obtain desired globally Lipschitz convex subsolutions

$$v(x) = x_n^a\big(|x'|^2 + A\big) - Bx_n$$

to degenerate equations of the form $\det D^2 v \geqslant |v|^p$ where $p > n-2$ and v is nonpositive on the boundary. Though simple, this final ansatz has escaped our attention up to now. Finally, the Lipschitz subsolutions allow us to obtain optimal boundary estimates for the Abreu's equation [2] with degenerate boundary data.

Throughout, we denote $x \in \mathbb{R}^n$ by $x = (x', x_n)$ and $\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$. We use $\mathrm{dist}(\,\cdot\,, E)$ to denote the distance function to a closed set $E \subset \mathbb{R}^n$, $\mathrm{diam}(\Omega)$ for the diameter of a set $\Omega \subset \mathbb{R}^n$, and $|\Omega|$ for the n-dimensional Lebesgue measure of an open set $\Omega \subset \mathbb{R}^n$. We use $C = C(*, \ldots, \star)$ to denote a positive constant C depending on the quantities $*, \ldots, \star$ in the parentheses. In general, C can be computed explicitly, and its value may change from line to line in a given context.

Outline of the paper

The rest of this note is organized as follows. We prove a comparison principle for degenerate subcritical Monge–Ampère equations in Section 2. The construction of globally Lipschitz subsolutions and applications including the proofs of Theorem 1.1 and Corollaries 1.3 and 1.4 will be given in Section 3. We prove Theorem 1.5 in Section 4.

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2. A comparison principle for degenerate subcritical Monge–Ampère equations

The following comparison principle for degenerate subcritical Monge–Ampère equations is motivated from a uniqueness result in Tso [21, Proposition 4.1]. The supersolution here is not required to be convex.

Lemma 2.1 (Comparison principle for degenerate subcritical Monge–Ampère equations). — *Let $p \in [0, n)$, $\delta \in [0, \infty)$, and Ω be a bounded convex domain in $\mathbb{R}^n$. Let $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ where*

- *v is convex in Ω, $v < 0$ in Ω, $v \leqslant 0$ on $\partial\Omega$, and v is a subsolution in the sense that*

$$\det D^2 v \geqslant (|v| + \delta)^p \quad \text{in } \Omega, \tag{2.1}$$

- *$u \geqslant 0$ on $\partial\Omega$, and u is a supersolution in the sense that*

$$\det D^2 u \leqslant (|u| + \delta)^p \quad \text{in } \Omega. \tag{2.2}$$

Then

$$u \geqslant v \quad \text{in } \Omega.$$

Consequently, if instead of (2.1)–(2.2), we have

$$\det D^2 v \geqslant K|v|^p \quad \text{and} \quad \det D^2 u \leqslant L|u|^p \quad \text{in } \Omega, \tag{2.3}$$

where $K, L > 0$, then

$$-u \leqslant (K/L)^{\frac{1}{p-n}} |v| \quad \text{in } \Omega.$$

Proof. — We first prove $u \geqslant v$ in Ω when (2.1)–(2.2) hold. Suppose that $v - u$ is positive somewhere in Ω. Note that $v - u \leqslant 0$ on $\partial\Omega$. By translation of coordinates, we can assume that $0 \in \Omega$ satisfies $v(0) - u(0) = \max_{\overline{\Omega}}(v - u) = \tau > 0$. For $1 < \gamma \leqslant 2$, consider for $x \in \Omega$,

$$v_\gamma(x) = v(x/\gamma), \quad \text{and} \quad \eta_\gamma(x) = u(x)/v_\gamma(x).$$

If $\operatorname{dist}(x, \partial\Omega) \to 0$, then $\limsup u(x) \geqslant 0$, so $\liminf \eta_\gamma(x) \leqslant 0$. Since $v < 0$ in Ω, we have

$$\eta_\gamma(0) = u(0)/v(0) = [v(0) - \tau]/v(0) \geqslant 1 + \varepsilon \quad \text{for some } \varepsilon > 0.$$

Therefore, the function η_γ attains its maximum value in $\overline{\Omega}$ at $x_\gamma \in \Omega$ with $\eta_\gamma(x_\gamma) \geqslant 1 + \varepsilon$. At $x = x_\gamma$, we have $D\eta_\gamma(x_\gamma) = 0$, $D^2\eta_\gamma(x_\gamma) \leqslant 0$, and $v_\gamma(x_\gamma) < 0$, so we can compute

$$D^2 u(x_\gamma) = \eta_\gamma(x_\gamma) D^2 v_\gamma(x_\gamma) + D^2\eta_\gamma(x_\gamma) v_\gamma(x_\gamma) \geqslant \eta_\gamma(x_\gamma) D^2 v_\gamma(x_\gamma),$$

in the sense of positive definite matrices. Hence, the convexity of v_λ and (2.1)–(2.2) give

$$\begin{aligned}(|u(x_\gamma)|+\delta)^p \geqslant \det D^2u(x_\gamma) &\geqslant [\eta_\gamma(x_\gamma)]^n \det D^2 v_\gamma(x_\gamma)\\ &\geqslant [\eta_\gamma(x_\gamma)]^n\gamma^{-2n}(|v(x_\gamma/\gamma)|+\delta)^p.\end{aligned} \tag{2.4}$$

Using $\eta_\gamma(x_\gamma)\geqslant 1+\varepsilon>1$ and $\delta\geqslant 0$, we find

$$|u(x_\gamma)|+\delta = \eta_\gamma(x_\gamma)|v(x_\gamma/\gamma)|+\delta \leqslant \eta_\gamma(x_\gamma)(|v(x_\gamma/\gamma)|+\delta). \tag{2.5}$$

We deduce from the estimates (2.4)–(2.5) that

$$1\geqslant [\eta_\gamma(x_\gamma)]^{n-p}\gamma^{-2n}\geqslant (1+\varepsilon)^{n-p}\gamma^{-2n}.$$

Letting $\gamma\searrow 1$, using $p<n$ and $\varepsilon>0$, we obtain a contradiction. Therefore, $u\geqslant v$ in Ω.

For the consequence, we rescale and take advantage of the subcriticality. Assume now that instead of (2.1)–(2.2), we have (2.3). Let $s_K = K^{\frac{1}{p-n}}$ and $s_L = L^{\frac{1}{p-n}}$. Then

$$\det D^2(s_K v)\geqslant |s_K v|^p \quad\text{and}\quad \det D^2(s_L u)\leqslant |s_L u|^p \quad \text{in } \Omega.$$

Thus, as above for the case $\delta=0$, we have $s_L u\geqslant s_K v$ in Ω. Hence, $-u\leqslant (K/L)^{\frac{1}{p-n}}|v|$ in Ω, completing the proof of Lemma 2.1. □

As an application of Lemma 2.1, we improve upon (1.5) where β is allowed to be $2/(n-p)$.

Proposition 2.2. — *Let $\Omega\subset\mathbb{R}^n$ $(n\geqslant 3)$ be a bounded convex domain. Let $p\in(0,n-2)$ and $M>0$. Let $u\in C(\overline{\Omega})$ be the nonzero convex solution to*

$$\det D^2 u = M|u|^p \ \ \textit{in } \Omega, \qquad\qquad u=0 \ \ \textit{on } \partial\Omega.$$

Then

$$|u(z)|\leqslant C(n,p,M,\operatorname{diam}(\Omega))[\operatorname{dist}(z,\partial\Omega)]^{\frac{2}{n-p}} \quad \forall\, z\in\Omega.$$

Proof. — Note that $u<0$ in Ω. By [10, Proposition 2.8], $u\in C^\infty(\Omega)$. Let $z=(z',z_n)$ be an arbitrary point in Ω. By translation and rotation of coordinates, we can assume that: $0\in\partial\Omega$, $\Omega\subset\mathbb{R}^n_+$, the x_n-axis points inward Ω, z lies on the x_n-axis so $z=(0,z_n)$, and $z_n=\operatorname{dist}(z,\partial\Omega)$. Let $D:=\operatorname{diam}(\Omega)$. Consider for $\alpha=\frac{2}{n-p}\in[\frac{2}{n},1)$

$$w_\alpha(x) = x_n^\alpha\big(|x'|^2 - C_\alpha\big) \quad\text{where}\quad C_\alpha = \big(1+2D^2\big)/[\alpha(1-\alpha)].$$

Then, we find from [12, Lemma 2.2] that w_α is convex in Ω, $w_\alpha<0$ in Ω, $w_\alpha\leqslant 0$ on $\partial\Omega$, and

$$\begin{aligned}\det D^2 w_\alpha &= 2^{n-1}x_n^{n\alpha-2}\big[\alpha(1-\alpha)C_\alpha - (\alpha^2+\alpha)|x'|^2\big]\\ &\geqslant x_n^{n\alpha-2}\geqslant C_\alpha^{-\frac{n\alpha-2}{\alpha}}|w_\alpha|^{\frac{n\alpha-2}{\alpha}} \quad \text{in } \Omega.\end{aligned} \tag{2.6}$$

The choice of α gives

$$\det D^2 w_\alpha \geqslant C_\alpha^{-p} |w_\alpha|^p \quad \text{in } \Omega.$$

As a consequence of the comparison principle in Lemma 2.1, we have

$$\begin{aligned}|u(z)| &\leqslant C(M,\alpha,n,p,D)|w_\alpha(z)| \\ &\leqslant C(M,n,p,D) z_n^\alpha = C(n,p,M,D)[\mathrm{dist}(z,\partial\Omega)]^{\frac{2}{n-p}}.\end{aligned}$$

This proves Proposition 2.2. □

3. Globally Lipschitz subsolutions and applications

In this section, we prove Theorem 1.1 and Corollaries 1.3 and 1.4.

We first construct globally Lipschitz convex subsolutions to the Monge–Ampère equations $\det D^2 v = C|v|^p$ and $\det D^2 v = C[\mathrm{dist}(\cdot, \partial\Omega)]^p$ where $p > n-2$ with nonpositive boundary values.

Lemma 3.1 (Globally Lipschitz convex subsolutions). — *Let $\Omega \subset \mathbb{R}^n_+$ $(n \geqslant 2)$ be a bounded convex domain with $0 \in \partial\Omega$. Let $D = \mathrm{diam}(\Omega)$. For $a > 1$, let*

$$v(x) = v_{a,D}(x) := x_n^a \big(|x'|^2 + A\big) - Bx_n, \qquad \text{for } x = (x', x_n) \in \overline{\Omega},$$

where

$$A = A(a,D) := \frac{1 + (a+1)D^2}{a-1}, \qquad B = B(a,D) := D^{a-1}\big(A + D^2\big).$$

Then $v \in C^{0,1}(\overline{\Omega}) \cap C^\infty(\Omega)$, v is convex in Ω, $v \leqslant 0$ in $\overline{\Omega}$, and

$$\det D^2 v(x) \geqslant 2^{n-1} a x_n^{na-2} \geqslant \frac{2^{n-1}a}{B^{na-2}} |v|^{na-2} \quad \text{in } \Omega.$$

Proof. — It is clear that $v \in C^{0,1}(\overline{\Omega}) \cap C^\infty(\Omega)$. We have

$$D^2 v(x) = \begin{pmatrix} 2x_n^a & 0 & \dots & 0 & 2ax_1x_n^{a-1} \\ 0 & 2x_n^a & \dots & 0 & 2ax_2x_n^{a-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2x_n^a & 2ax_{n-1}x_n^{a-1} \\ 2ax_1x_n^{a-1} & 2ax_2x_n^{a-1} & \dots & 2ax_{n-1}x_n^{a-1} & M \end{pmatrix},$$

where

$$M := a(a-1)x_n^{a-2}\big(|x'|^2 + A\big).$$

Note that the first $(n-1)$-leading principal minors of D^2v are all positive in Ω. By induction, we find that

$$\begin{aligned}\det D^2v(x) &= 2^{n-1}a(a-1)x_n^{na-2}\big(|x'|^2+A\big) - 2^n a^2|x'|^2x_n^{na-2}\\ &= 2^{n-1}ax_n^{na-2}\big[A(a-1)-(a+1)|x'|^2\big].\end{aligned} \tag{3.1}$$

With the choices of A and B, we have v is convex in Ω, and $v \leqslant 0$ in $\overline{\Omega}$. Clearly,

$$A(a-1)-(a+1)|x'|^2 \geqslant A(a-1)-(a+1)D^2 \geqslant 1 \quad \text{in } \Omega,$$

so (3.1) gives

$$\det D^2v(x) \geqslant 2^{n-1}ax_n^{na-2} \geqslant \frac{2^{n-1}a}{B^{na-2}}|v|^{na-2} \quad \text{in } \Omega.$$

Lemma 3.1 is proved. □

Proof of Theorem 1.1. — Let $z=(z',z_n)$ be an arbitrary point in Ω. By translation and rotation of coordinates, we can assume that: $0\in\partial\Omega$, $\Omega\subset\mathbb{R}^n_+$, the x_n-axis points inward Ω, z lies on the x_n-axis so $z=(0,z_n)$, and $z_n = \operatorname{dist}(z,\partial\Omega)$.

(i). — Note that the function u in this part satisfies $u\in C^\infty(\Omega)$ by [10, Proposition 2.8]. We need to show that

$$\begin{aligned}|u(z)| &\leqslant C(p,n,M,|\Omega|,\operatorname{diam}(\Omega))\operatorname{dist}(z,\partial\Omega)\\ &= C(p,n,M,|\Omega|,\operatorname{diam}(\Omega))z_n.\end{aligned} \tag{3.2}$$

First, we consider the case $n-2<p<n$. Let

$$a=\frac{2+p}{n}>1, \quad D=\operatorname{diam}(\Omega), \quad v=v_{a,D}, \tag{3.3}$$

where $v_{a,D}(x)=x_n^a(|x'|^2+A)-Bx_n$ is the convex function in Lemma 3.1. Recall that

$$\det D^2v \geqslant c(a,n,D)|v|^{na-2} = c(a,n,D)|v|^p, \quad \det D^2u\leqslant M|u|^p \quad \text{in } \Omega;$$

while

$$v\leqslant u=0 \quad \text{on } \partial\Omega.$$

From the consequence of the comparison principle in Lemma 2.1, we have

$$|u|\leqslant C(a,D,n,p,M)|v_{a,D}| \quad \text{in } \Omega.$$

Since $|v(z)|=|v_{a,D}(z)|\leqslant C(a,D)z_n$, we obtain (3.2).

Now, we consider the case $p\geqslant n$. If $p=n$, then by homogeneity, we can assume that $\|u\|_{L^\infty(\Omega)}=1$. If $p>n$, we use (1.7) to estimate $\|u\|_{L^\infty(\Omega)}$. In all cases, we have

$$\det D^2u = M|u|^p \leqslant M\|u\|^{p-n+1}_{L^\infty(\Omega)}|u|^{n-1} \leqslant C(M,n,p,|\Omega|)|u|^{n-1} \quad \text{in } \Omega,$$

and (3.2) follows from the case $p = n - 1$. Part (i) is proved.

(ii). — We need to show that

$$|u(z)| \leqslant C(p, n, M, \mathrm{diam}(\Omega))\,\mathrm{dist}(z, \partial\Omega) = C(p, n, M, \mathrm{diam}(\Omega)) z_n. \quad (3.4)$$

Let

$$a = (2 + p)/n > 1, \quad D = \mathrm{diam}(\Omega),$$

and

$$v(x) = v_{a,D}(x) = x_n^a \left(|x'|^2 + A\right) - Bx_n$$

be the convex function in Lemma 3.1. Let $s = (M2^{1-n}/a)^{1/n}$. From Lemma 3.1, we have

$$\det D^2(sv) = s^n \det D^2 v \geqslant s^n 2^{n-1} a x_n^{na-2} = M x_n^p.$$

Thus

$$\det D^2(sv) \geqslant M[\mathrm{dist}(\,\cdot\,, \partial\Omega)]^p \geqslant \det D^2 u \quad \text{in } \Omega.$$

Note that $sv \leqslant u = 0$ on $\partial\Omega$. By the comparison principle for the Monge–Ampère equation (see [14, Theorem 3.21]), we have

$$sv \leqslant u \quad \text{in } \Omega.$$

In particular, at $z = (0, z_n)$, we have

$$|u(z) \leqslant |sv(z)| \leqslant sB(a, D) z_n.$$

This implies (3.4), completing the proof of Theorem 1.1. □

Proof of Corollary 1.3. — By the global Lipschitz estimates in Theorem 1.1 (i) and the convexity of u, we have

$$\|Du\|_{L^\infty(\Omega)} \leqslant C(|\Omega|, \mathrm{diam}(\Omega), n, p, M) \|u\|_{L^\infty(\Omega)}. \quad (3.5)$$

The convexity of u also implies that

$$\left\|D^2 u\right\| \leqslant \Delta u \quad \text{in } \Omega.$$

Let $\{\Omega_m\}_{m=1}^\infty \subset \Omega$ be a sequence of smooth convex domains that converges to Ω in the Hausdorff distance. Let $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

For each m, integrating by parts gives

$$\int_{\Omega_m} \left\|D^2 u\right\| \mathrm{d}x \leqslant \int_{\Omega_m} \Delta u \,\mathrm{d}x = \int_{\partial\Omega_m} \frac{\partial u}{\partial \nu} \,\mathrm{d}\mathcal{H}^{n-1} \leqslant \mathcal{H}^{n-1}(\partial\Omega_m) \|Du\|_{L^\infty(\Omega)},$$

where ν is the outer normal unit vector field on $\partial\Omega_m$.

From (3.5), we easily obtain for all m

$$\int_{\Omega_m} \|D^2 u\| \,\mathrm{d}x \leqslant C(|\Omega|, \mathrm{diam}(\Omega), n, p, M) \|u\|_{L^\infty(\Omega)}.$$

Now, we let $m \to \infty$. The monotone convergence theorem then gives $D^2 u \in L^1(\Omega)$ with estimate for $\|D^2 u\|_{L^1(\Omega)}$ stated in the corollary. $\square$

Proof of Corollary 1.4. — The interior regularity of the scheme $u_k \in C^{2k-2}(\Omega)$ for $k \geqslant 2$ follows from [11, Proposition 3.1]. In view of Step 1 in the proof of [11, Theorem 1.4], the sequence $\{R(u_k)\}_{k=0}^{\infty}$ is bounded:

$$R(u_k) \leqslant C(u_0, n, \Omega) \quad \forall\, k \geqslant 0. \tag{3.6}$$

The finiteness of $R(u_0)$ implies that of $\|u_0\|_{L^{n+1}(\Omega)}$. Hence

$$\int_\Omega \det D^2 u_1 \,\mathrm{d}x = R(u_0) \int_\Omega |u_0|^n \,\mathrm{d}x \leqslant C(u_0, n, \Omega).$$

As a consequence of the Aleksandrov estimate (see [14, Theorem 3.12]), we have

$$|u_1(x)| \leqslant C(u_0, n, \Omega)[\mathrm{dist}(x, \partial\Omega)]^{\frac{1}{n}} \quad \forall\, x \in \Omega. \tag{3.7}$$

Step 1. — We will show by induction that if $1 \leqslant k \leqslant n/2$, then

$$|u_k(x)| \leqslant C(u_0, n, \Omega)[\mathrm{dist}(x, \partial\Omega)]^{\frac{2k-1}{n}} \quad \forall\, x \in \Omega. \tag{3.8}$$

When $k = 1$, (3.8) is exactly (3.7). Assume (3.8) holds for $1 \leqslant k \leqslant (n-2)/2$ (if $n \leqslant 3$, then we are done). We prove it for $k+1$. Indeed, recalling (3.6)–(3.8), we have

$$\det D^2 u_{k+1} = R(u_k)|u_k|^n \leqslant C_k(u_0, n, \Omega)[\mathrm{dist}(\,\cdot\,, \partial\Omega)]^{2k-1}. \tag{3.9}$$

Let $z = (z', z_n)$ be an arbitrary point in Ω. By translation and rotation of coordinates, we can assume that: $0 \in \partial\Omega$, $\Omega \subset \mathbb{R}^n_+$, the x_n-axis points inward Ω, z lies on the x_n-axis so $z = (0, z_n)$, and $z_n = \mathrm{dist}(z, \partial\Omega)$. Let $D := \mathrm{diam}(\Omega)$. Consider for $\alpha = \frac{2k+1}{n} \in [\frac{3}{n}, 1)$

$$w_\alpha(x) = x_n^\alpha(|x'|^2 - C_\alpha) \quad \text{where} \quad C_\alpha = \big(1 + 2D^2\big)/[\alpha(1-\alpha)].$$

Then, w_α is convex in Ω, $w_\alpha \leqslant 0$ on $\partial\Omega$, and recalling (2.6), we have

$$\det D^2 w_\alpha \geqslant x_n^{n\alpha - 2} \geqslant [\mathrm{dist}(\,\cdot\,, \partial\Omega)]^{2k-1} \quad \text{in } \Omega.$$

It follows from (3.9) that

$$\det D^2 (C_k^{1/n} w_\alpha) \geqslant C_k[\mathrm{dist}(\,\cdot\,, \partial\Omega)]^{2k-1} \geqslant \det D^2 u_{k+1} \quad \text{in } \Omega.$$

The comparison principle for the Monge–Ampère equation (see [14, Theorem 3.21]) gives

$$C_k^{1/n} w_\alpha \leqslant u_{k+1} \quad \text{in } \Omega.$$

In particular, at $z = (0, z_n)$, we have

$$|u_{k+1}(z)| \leqslant C_k^{1/n}|w_\alpha(z)| \leqslant C(u_0, n, \Omega) z_n^{\frac{2k+1}{n}} = C(u_0, n, \Omega)[\mathrm{dist}(z, \partial\Omega)]^{\frac{2k+1}{n}}.$$

The arbitrariness of z proves (3.8) for $k+1$. Therefore, (3.8) holds for all $1 \leqslant k \leqslant n/2$.

Step 2. — We now deduce the corollary from (3.8). Indeed, for $m := \lfloor n/2 \rfloor$, using (3.6)–(3.8), we find

$$\begin{aligned}\det D^2 u_{m+1} = R(u_m)|u_m|^n &\leqslant C[\mathrm{dist}(\,\cdot\,, \partial\Omega)]^{2m-1}\\ &\leqslant C(u_0, n, \Omega)[\mathrm{dist}(\,\cdot\,, \partial\Omega)]^{n-5/2}.\end{aligned}$$

The proof of (3.8) in fact shows that if $\det D^2 u_k \leqslant C[\mathrm{dist}(\,\cdot\,, \partial\Omega)]^{\alpha}$ where $\alpha < n-2$, then the modulus of u_k grows less than $C[\mathrm{dist}(\,\cdot\,, \partial\Omega)]^{\frac{2+\alpha}{n}}$. Thus, we have

$$|u_{m+1}| \leqslant C(u_0, n, \Omega)[\mathrm{dist}(\,\cdot\,, \partial\Omega)]^{\frac{n-1/2}{n}} \leqslant C(u_0, n, \Omega)[\mathrm{dist}(\,\cdot\,, \partial\Omega)]^{\frac{n-1}{n}} \quad \text{in } \Omega.$$

Therefore

$$\det D^2 u_{m+2} = R(u_{m+1})|u_{m+1}|^n \leqslant C[\mathrm{dist}(\,\cdot\,, \partial\Omega)]^{n-1}.$$

By Theorem 1.1 (ii) for $p = n-1$, we have $u_{m+2} \in C^{0,1}(\overline{\Omega})$ with the estimate

$$|u_{m+2}| \leqslant C(u_0, n, \Omega)\,\mathrm{dist}(\,\cdot\,, \partial\Omega) \quad \text{in } \Omega.$$

Using (1.8) and Theorem 1.1 (ii) repeatedly, we obtain $u_k \in C^{0,1}(\overline{\Omega})$ for all $k \geqslant m+2 = \lfloor (n+4)/2 \rfloor$. Corollary 1.4 is proved. □

We indicate an application of the Lipschitz subsolution in Lemma 3.1 in obtaining optimal boundary estimates for the Abreu's equation [2] with degenerate boundary data which arises in the study of the existence of constant scalar curvature Kähler metrics for toric varieties [6, 7]; see also [4] for related Abreu's equation with degenerate boundary data.

Theorem 3.2 (Boundary Lipschitz estimates for the inverse of the Hessian determinant of Abreu's equation with degenerate boundary data). — *Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a bounded convex domain. Let $u \in C^4(\Omega)$ be a locally uniformly convex solution of*

$$\begin{cases} U^{ij} D_{ij} w = -f & \text{in } \Omega,\\ \qquad w = \left[\det D^2 u\right]^{-1} & \text{in } \Omega,\\ \qquad w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

where $U = (U^{ij}) = (\det D^2 u)(D^2 u)^{-1}$ is the cofactor matrix of the Hessian matrix $D^2 u$, $f \in L^\infty(\Omega)$ with $f^+ > 0$, and $w \in C(\overline{\Omega})$. Then, we have the estimate

$$\det D^2 u \geqslant c(n, \mathrm{diam}(\Omega)) \left\| f^+ \right\|^{-n}_{L^\infty(\Omega)} [\mathrm{dist}(\,\cdot\,, \partial\Omega)]^{-1} \quad \text{in } \Omega. \tag{3.11}$$

We refer the readers to [7, Section 2] for a discussion of the optimality of (3.11) when Ω is a polytope.

Proof. — Note that for $\gamma > 0$, $\widetilde{u} = \gamma u$ satisfies

$$\widetilde{U}^{ij} D_{ij} \widetilde{w} = -\gamma^{-1} f, \tag{3.12}$$

where $\widetilde{U} = (\widetilde{U}^{ij})$ is the cofactor matrix of $D^2\widetilde{u}$, and $\widetilde{w} = (\det D^2\widetilde{u})^{-1}$. Thus, by considering $\|f^+\|_{L^\infty(\Omega)} u$ instead of u, it suffices to prove the theorem under the assumption that

$$\|f^+\|_{L^\infty(\Omega)} = 1.$$

Let $D := \operatorname{diam}(\Omega)$. By [13, Theorem 1.6], we already have

$$\det D^2 u(x) \geqslant c_1(n, D)[\operatorname{dist}(x, \partial\Omega)]^{-\frac{n-1}{n}} \quad \text{for all } x \in \Omega. \tag{3.13}$$

Let $z = (z', z_n)$ be an arbitrary point in Ω. We prove

$$\det D^2 u(z) \geqslant c(n, D)[\operatorname{dist}(z, \partial\Omega)]^{-1}. \tag{3.14}$$

By translation and rotation of coordinates, we can assume that: $0 \in \partial\Omega$, $\Omega \subset \mathbb{R}^n_+$, the x_n-axis points inward Ω, z lies on the x_n-axis so $z = (0, z_n)$, and $z_n = \operatorname{dist}(z, \partial\Omega)$.

To prove (3.14), we will compare w with $-C(a, D)v_{a,D}$ where

$$a = \frac{2}{n} + \frac{(n-1)^2}{n^2} > 1,$$

$v_{a,D}$ is the convex function in Lemma 3.1, and $C(a, D) > 0$ is to be chosen. For this, we use the fact that for $n \times n$ positive definite matrices A and B,

$$\operatorname{trace}(AB) \geqslant n(\det A)^{1/n}(\det B)^{1/n}.$$

Then, using (3.13) together with

$$\det(U^{ij}) = \left(\det D^2 u\right)^{n-1} \quad \text{and} \quad \det D^2 v_{a,D} \geqslant 2^{n-1} a [\operatorname{dist}(x, \partial\Omega)]^{na-2},$$

we find that

$$\begin{aligned} U^{ij} D_{ij}(-C v_{a,D}) &\leqslant -Cn(\det D^2 u)^{\frac{n-1}{n}} \left(\det D^2 v_{a,D}\right)^{\frac{1}{n}} \\ &\leqslant -C c_2(n, D)[\operatorname{dist}(x, \partial\Omega)]^{-\frac{(n-1)^2}{n^2}} [\operatorname{dist}(x, \partial\Omega)]^{\frac{na-2}{n}} \\ &= -C c_2(n, D) < -1 \leqslant -f = U^{ij} D_{ij} w, \end{aligned}$$

if $C = C(n, D)$ is large.

Since

$$-U^{ij} D_{ij}(w + C v_{a,D}) < 0 \ \text{ in } \ \Omega, \quad \text{and} \quad w + C v_{a,D} \leqslant 0 \ \text{ on } \partial\Omega,$$

the classical comparison principle implies

$$w + C(a, D) v_{a,D} \leqslant 0 \ \text{ in } \Omega.$$

Therefore, at $z = (0, z_n)$, we have

$$w(z) \leqslant -C(n,D)v_{a,D}(z) \leqslant C(n,D)z_n = C(n,D)\,\mathrm{dist}(z,\partial\Omega).$$

This gives (3.14) because $\det D^2u(z) = [w(z)]^{-1}$. Theorem 3.2 is proved. □

4. Infinite boundary gradient

In this section, we prove Theorem 1.5 using suitable subsolutions and supersolutions of the Monge–Ampère equation $\det D^2 u = |u|^{n-2}$.

First, we construct subsolutions with $\mathrm{dist}(\,\cdot\,,\partial\Omega)(1+|\log \mathrm{dist}(\,\cdot\,,\partial\Omega)|^{n/2})$ growth.

Lemma 4.1 (Subsolutions). — *Assume that* $0 \in \partial\Omega$ *and* $\Omega \subset \mathbb{R}^n_+ (n \geqslant 2)$. *Let* $D := \mathrm{diam}(\Omega)$ *and define for* $x \in \overline{\Omega}$

$$w(x) := \frac{x_n}{e^{2n}D}\left|\log\frac{x_n}{e^{2n}D}\right|^{\frac{n-2}{2}}\left(|x'|^2 - D^2\right) - \left(1+4D^2\right)\frac{x_n}{e^{2n}D}\left|\log\frac{x_n}{e^{2n}D}\right|^{\frac{n}{2}}.$$

Then $w \in C^\infty(\Omega)$, w *is convex in* Ω *with* $w \leqslant 0$ *on* $\partial\Omega$, *and there exists a constant* $c = c(n, \mathrm{diam}(\Omega)) > 0$ *such that*

$$\det D^2 w \geqslant c|w|^{n-2} \quad \text{in } \Omega.$$

Proof. — For $\alpha \in [0,n)$ and $t \in (0, e^{-2n})$, let

$$f_\alpha(t) := t(-\log t)^\alpha.$$

Then

$$f'_\alpha(t) = (-\log t)^\alpha - \alpha(-\log t)^{\alpha-1},$$

and

$$f''_\alpha(t) = -\frac{\alpha}{t}(-\log t)^{\alpha-1} - \frac{\alpha(1-\alpha)}{t}(-\log t)^{\alpha-2}.$$

Observe that for all $t \in (0,e^{-2n})$ and $\alpha \in [0,n)$, we have

$$0 < f_\alpha(t), \quad 0 \leqslant f'_\alpha(t) \leqslant |\log t|^\alpha, \quad \frac{\alpha|\log t|^{\alpha-1}}{2t} < -f''_\alpha(t).$$

Let

$$E = 1+4D^2, \quad s = e^{2n}D, \quad y' = x'/s, \quad y_n = x_n/s \quad \text{for } x = (x',x_n) \in \Omega.$$

Consider

$$\begin{aligned} u_\alpha(x) &:= f_\alpha(x_n/s)\big(|x'|^2 - D^2\big) - Ef_{\alpha+1}(x_n/s) \\ &\equiv f_\alpha(y_n)\big(|x'|^2 - D^2\big) - Ef_{\alpha+1}(y_n). \end{aligned}$$

Then, $y_n \in (0, e^{-2n})$ and

$$|u_\alpha(x)| \leqslant 2Ey_n|\log y_n|^{\alpha+1} \quad \text{in } \Omega. \tag{4.1}$$

We have

$$D^2u_\alpha(x) = \begin{pmatrix} 2f_\alpha(y_n) & 0 & \dots & 0 & 2y_1 f'_\alpha(y_n) \\ 0 & 2f_\alpha(y_n) & \dots & 0 & 2y_2 f'_\alpha(y_n) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2f_\alpha(y_n) & 2y_{n-1} f'_\alpha(y_n) \\ 2y_1 f'_\alpha(y_n) & 2y_2 f'_\alpha(y_n) & \dots & 2y_{n-1} f'_\alpha(y_n) & K \end{pmatrix},$$

where

$$K := \frac{f''_\alpha(y_n)\big(|x'|^2 - D^2\big) - E f''_{\alpha+1}(y_n)}{s^2}.$$

Note that the first $(n-1)$-leading principal minors of $D^2u_{(\alpha)}$ are all positive in Ω. We can compute by induction that

$$\begin{aligned}
&\det D^2u_\alpha(x) \\
&\quad = 2^{n-1}s^{-2}[f_\alpha(y_n)]^{n-1}\big[f''_\alpha(y_n)\big(|x'|^2 - D^2\big) - E f''_{\alpha+1}(y_n)\big] \\
&\qquad - 2^n s^{-2}|x'|^2[f'_\alpha(y_n)]^2[f_\alpha(y_n)]^{n-2} \\
&\quad \geqslant -2^{n-1}s^{-2}E[f_\alpha(y_n)]^{n-1} f''_{\alpha+1}(y_n) - 2^n s^{-2}D^2|\log y_n|^{2\alpha}[f_\alpha(y_n)]^{n-2} \\
&\quad \geqslant 2^{n-1}s^{-2}E[y_n|\log y_n|^\alpha]^{n-1}\frac{\alpha+1}{2y_n}|\log y_n|^\alpha \\
&\qquad - 2^n s^{-2}D^2|\log y_n|^{2\alpha}[y_n|\log y_n|^\alpha]^{n-2} \\
&\quad = 2^{n-2}s^{-2}y_n^{n-2}|\log y_n|^{n\alpha}\big[(1+\alpha)E - 4D^2\big] \\
&\quad \geqslant 2^{n-2}s^{-2}y_n^{n-2}|\log y_n|^{n\alpha}.
\end{aligned}$$

Clearly, $u_\alpha \in C^\infty(\Omega)$, u_α is convex in Ω with $u_\alpha \leqslant 0$ on $\partial\Omega$. Moreover, when

$$(\alpha+1)(n-2) = n\alpha, \quad \text{or} \quad \alpha = (n-2)/2,$$

we obtain from the preceding inequalities and (4.1) that

$$\begin{aligned}
\det D^2u_{(n-2)/2} &\geqslant 2^{n-2}s^{-2}\big(y_n|\log y_n|^{n/2}\big)^{n-2} \\
&\geqslant c(n,\operatorname{diam}(\Omega))\big|u_{(n-2)/2}\big|^{n-2} \quad \text{in } \Omega,
\end{aligned}$$

for some constant $c(n,\operatorname{diam}(\Omega)) > 0$.

Since $w = u_{(n-2)/2}$, Lemma 4.1 is proved. □

Next, we construct supersolutions with infinite boundary gradient. Note that these supersolutions are not necessarily convex.

Lemma 4.2 (Supersolutions with infinite boundary gradient).
Assume $0 < x_n < e^{-1}$ *for all* $x \in \Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$*. Let*

$$v(x) = x_n\Big[(-\log x_n)^{1/n} - 1\Big]\big(|x'|^2 - 1\big), \qquad x = (x', x_n) \in \Omega.$$

Then $v \in C^\infty(\Omega)$ *and*

$$\det D^2 v(x) \leqslant 2^n x_n^{n-2} \quad \textit{in } \Omega.$$

Proof. — For $\alpha \in (0,1)$ and $t \in (0, e^{-1})$, let

$$g_\alpha(t) = t(-\log t)^\alpha - t.$$

Note that for all $t \in (0, e^{-1})$ and $\alpha \in (0,1)$, we have

$$0 < g_\alpha(t) < t|\log t|^\alpha, \qquad 0 < -g''_\alpha(t) \leqslant \frac{2\alpha|\log t|^{\alpha-1}}{t}.$$

Let

$$v_\alpha(x) := g_\alpha(x_n)(|x'|^2 - 1) \quad \text{for } x \in \Omega.$$

Then, we can compute as in Lemma 4.1 that

$$\begin{aligned} \det D^2 v_\alpha(x) &= 2^{n-1}[g_\alpha(x_n)]^{n-1} g''_\alpha(x_n)\big(|x'|^2 - 1\big) \\ &\quad - 2^n |x'|^2 [g'_\alpha(x_n)]^2 [g_\alpha(x_n)]^{n-2} \\ &\leqslant -2^{n-1}[g_\alpha(x_n)]^{n-1} g''_\alpha(x_n) \\ &\leqslant 2^n \alpha x_n^{n-2} |\log x_n|^{n\alpha-1}. \end{aligned} \tag{4.2}$$

Thus, choosing $\alpha = 1/n$, we have $v = v_{1/n} \in C^\infty(\Omega)$, and $\det D^2 v(x) \leqslant 2^n x_n^{n-2}$ in Ω. Lemma 4.2 is proved. □

Proof of Theorem 1.5.

(i). — Let $z = (z', z_n)$ be an arbitrary point in Ω. By translation and rotation of coordinates, we can assume that: $0 \in \partial\Omega$, $\Omega \subset \mathbb{R}^n_+$, the x_n-axis points inward Ω, z lies on the x_n-axis so $z = (0, z_n)$, and $z_n = \operatorname{dist}(z, \partial\Omega)$. Let $D = \operatorname{diam}(\Omega)$. We need to show that

$$|u(z)| \leqslant C(n, D) z_n \big(1 + |\log z_n|^{\frac{n}{2}}\big). \tag{4.3}$$

Let w be the convex function in Lemma 4.1. Recall that

$$\det D^2 w \geqslant c(n, D)|w|^{n-2}, \qquad \det D^2 u = |u|^{n-2} \quad \text{in } \Omega.$$

From the consequence of the comparison principle in Lemma 2.1, we have

$$|u| \leqslant C(n, D)|w| \quad \text{in } \Omega.$$

Since

$$|w(z)| \leqslant C(n,D) z_n \left|\log \frac{z_n}{e^{2n}D}\right|^{n/2},$$

we obtain (4.3).

(ii). — Fix $z \in \Omega$ being close to Γ. By translating and rotating coordinates, we can assume that for some $s = s(n,\Omega,\Gamma) \in (0, e^{-1})$,

$$z = (0, z_n) \in \Omega_s := \{(x', x_n) : |x'| < s,\ 0 < x_n < s\} \subset \Omega,$$
$$\{(x', 0) : |x'| \leqslant s\} \subset \Gamma \subset \partial\Omega, \quad \text{and} \quad \operatorname{dist}(x, \partial\Omega) = x_n \quad \text{for } x \in \Omega_s.$$

Let $g(t) = t(-\log t)^{1/n} - t$ and

$$\overline{w}(x) = g(x_n/(es))\big(|x'|^2 - s^2\big) \quad \text{in } \Omega_s.$$

Then, $\overline{w} \in C^\infty(\Omega_s)$. As in (4.2), we have

$$\begin{aligned}\det D^2 \overline{w} &\leqslant -2^{n-1} e^{-2} [g(x_n/(es))]^{n-1} g''(x_n/(es)) \\ &\leqslant 2^n (es)^{2-n} x_n^{n-2} = 2^n (es)^{2-n} [\operatorname{dist}(x, \partial\Omega)]^{n-2} \quad \text{in } \Omega_s.\end{aligned}$$

On the other hand, the convexity of u and (1.7) give

$$|u(x)| \geqslant \frac{\operatorname{dist}(x,\partial\Omega)}{\operatorname{diam}(\Omega)} \|u\|_{L^\infty(\Omega)} \geqslant c(n,\Omega) \operatorname{dist}(x, \partial\Omega).$$

Thus, for some $R = R(n, s, \Omega) > 0$, we have

$$\det D^2(Ru) = R^n |u|^{n-2} > \det D^2 \overline{w} \quad \text{in } \Omega_s. \tag{4.4}$$

Observe that

$$Ru \leqslant \overline{w} \quad \text{in } \overline{\Omega_s}. \tag{4.5}$$

Indeed, since $\overline{w} = 0$ on $\partial\Omega_s$, this holds on $\partial\Omega_s$. Suppose otherwise that $Ru - \overline{w}$ attains a positive maximum at $\bar{x} \in \Omega_s$. Then, $D^2\overline{w}(\bar{x}) \geqslant RD^2u(\bar{x})$, as symmetric matrices. Now, the convexity of u gives $RD^2u(\bar{x}) \geqslant 0$ and $\det D^2\overline{w}(\bar{x}) \geqslant \det D^2(Ru)(\bar{x})$, which contradicts (4.4).

Using (4.5) at $z = (0, z_n) \in \Omega_s$ where $0 < z_n$ is small (due to z being close to Γ), we have

$$\begin{aligned}|u(z)| \geqslant |\overline{w}(z)|/R &\geqslant c_1(n,\Gamma,\Omega) \operatorname{dist}(z,\partial\Omega)\left(\left|\log \frac{\operatorname{dist}(z,\partial\Omega)}{es}\right|^{\frac{1}{n}} - 1\right) \\ &\geqslant c(n,\Gamma,\Omega) \operatorname{dist}(z,\partial\Omega) |\log \operatorname{dist}(z,\partial\Omega)|^{\frac{1}{n}}.\end{aligned}$$

Theorem 1.5 is proved. □

Remark 4.3. — The conclusion of Theorem 1.5 (ii) still holds if $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a convex function satisfying

$$\det D^2 u \geqslant [\operatorname{dist}(\,\cdot\,, \partial\Omega)]^{n-2} \ \text{ in } \Omega, \qquad u = 0 \ \text{ on } \partial\Omega.$$

Bibliography

[1] F. ABEDIN & J. KITAGAWA, "Inverse iteration for the Monge-Ampère eigenvalue problem", *Proc. Am. Math. Soc.* **148** (2020), no. 11, p. 4875-4886.

[2] M. ABREU, "Kähler geometry of toric varieties and extremal metrics", *Int. J. Math.* **9** (1998), no. 6, p. 641-651.

[3] L. A. CAFFARELLI, "A localization property of viscosity solutions to the Monge–Ampère equation and their strict convexity", *Ann. Math. (2)* **131** (1990), no. 1, p. 129-134.

[4] B. CHEN, A.-M. LI & L. SHENG, "The Abreu equation with degenerated boundary conditions", *J. Differ. Equations* **252** (2012), no. 10, p. 5235-5259.

[5] G. DE PHILIPPIS, A. FIGALLI & O. V. SAVIN, "A note on interior $W^{2,1+\varepsilon}$ estimates for the Monge–Ampère equation", *Math. Ann.* **357** (2013), no. 1, p. 11-22.

[6] S. K. DONALDSON, "Scalar curvature and stability of toric varieties", *J. Differ. Geom.* **62** (2002), no. 2, p. 289-349.

[7] ———, "Interior estimates for solutions of Abreu's equation", *Collect. Math.* **56** (2005), no. 2, p. 103-142.

[8] J. HONG, G. HUANG & W. WANG, "Existence of global smooth solutions to Dirichlet problem for degenerate elliptic Monge–Ampère equations", *Commun. Partial Differ. Equations* **36** (2011), no. 4, p. 635-656.

[9] G. HUANG & Y. LÜ, "Analyticity of the solutions to degenerate Monge–Ampère equations", *J. Differ. Equations* **376** (2023), p. 633-654.

[10] N. Q. LE, "The eigenvalue problem for the Monge–Ampère operator on general bounded convex domains", *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)* **18** (2018), no. 4, p. 1519-1559.

[11] ———, "Convergence of an iterative scheme for the Monge–Ampère eigenvalue problem with general initial data", *Adv. Cont. Discr. Mod.* **2026** (2020), article no. 1 (14 pages).

[12] ———, "Optimal boundary regularity for some singular Monge–Ampère equations on bounded convex domains", *Discrete Contin. Dyn. Syst.* **42** (2022), no. 5, p. 2199-2214.

[13] ———, "Remarks on sharp boundary estimates for singular and degenerate Monge–Ampère equations", *Commun. Pure Appl. Anal.* **22** (2023), no. 5, p. 1701-1720.

[14] ———, *Analysis of Monge–Ampère equations*, Graduate Studies in Mathematics, vol. 240, American Mathematical Society, 2024, xx+576 pages.

[15] ———, "On global $W^{2,\delta}$ estimates for the Monge–Ampère equation on general bounded convex domains", *Partial Differ. Equ. Appl.* **5** (2024), no. 4, article no. 21 (10 pages).

[16] N. Q. LE & O. V. SAVIN, "Schauder estimates for degenerate Monge–Ampère equations and smoothness of the eigenfunctions", *Invent. Math.* **207** (2017), no. 1, p. 389-423.

[17] P.-L. LIONS, "Two remarks on Monge–Ampère equations", *Ann. Mat. Pura Appl. (4)* **142** (1985), p. 263-275.

[18] H. LIU, S. LEUNG & J. QIAN, "An Efficient Operator-Splitting Method for the Eigenvalue Problem of the Monge–Ampère Equation", *Commun. Optim. Theory* **2022** (2022), article no. 7 (22 pages).

[19] G. DE PHILIPPIS & A. FIGALLI, "$W^{2,1}$ regularity for solutions of the Monge–Ampère equation", *Invent. Math.* **192** (2013), no. 1, p. 55-69.

[20] O. V. SAVIN, "A localization theorem and boundary regularity for a class of degenerate Monge–Ampère equations", *J. Differ. Equations* **256** (2014), no. 2, p. 327-388.

[21] K. Tso, "On a real Monge–Ampère functional", *Invent. Math.* **101** (1990), no. 2, p. 425-448.