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The Path-product in Morse Homology with differential graded coefficients ^(*)

ROBIN RIEGEL ⁽¹⁾

ABSTRACT. — We will use the tools developed in the article [5] to give a Morse-theoretical description of a string topology product on the homology of the space of paths in a manifold Y with endpoints in a submanifold X and a module structure on this homology over the Chas–Sullivan ring of Y . These operations have been defined and studied by Stegemeyer in [6] in singular homology.

RÉSUMÉ. — Nous utilisons les outils développés dans l'article [5] pour donner une description en théorie de Morse d'une opération de topologie des cordes sur l'homologie de l'espace des chemins dans une variété Y à extrémités dans une sous-variétés X et une structure de module de cette homologie sur l'anneau de Chas–Sullivan de Y . Ces opérations ont été définies et étudiées par Stegemeyer in [6] en homologie singulière.

1. Introduction

Let $(Y^k, \star_Y)$ be a pointed, oriented, closed, and connected manifold and let $(X^n, \star_X)$ be a pointed, oriented, closed, and connected submanifold of Y . The goal of this article is to use Morse Homology with differential graded coefficients defined and studied by Barraud–Damian–Humilière–Oancea in [2] and the tools developed in [5] in order to give a Morse model for the *Path-product*

$$\Lambda : H_i(\mathcal{P}_X Y) \otimes H_j(\mathcal{P}_X Y) \longrightarrow H_{i+j-n}(\mathcal{P}_X Y)$$

defined by Stegemeyer in [6], where

$$\mathcal{P}_X Y = \{\gamma : [0, a] \longrightarrow Y \text{ continuous, } \gamma(0), \gamma(a) \in X\}.$$

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Article proposé par Francesco Costantino.

We start by a brief introduction of the main tools that we will use in this article.

1.1. Context

The authors of [2] construct *Morse complex of X with coefficients in a differential graded (DG) module $(\mathcal{F}_*, \partial_{\mathcal{F}})$* over the differential graded algebra (DGA) $R_* = C_*(\Omega X)$ of the cubical complex of the space of loops in X based at $\star$. This is also referred to as an *enriched Morse complex*. Given a Morse function $f : X \rightarrow \mathbb{R}$ and ξ , a pseudo-gradient adapted to f , this complex writes

$$C_*(X, \mathcal{F}_*) = \mathcal{F}_* \otimes \mathbb{Z} \text{Crit}(f),$$

where the differential is twisted by a family of chains called, in reference to the seminal paper [1], *Barraud–Cornea twisting cocycle*

$$\{m_{x,y} \in C_{|x|-|y|-1}(\Omega X), x, y \in \text{Crit}(f)\}.$$

The cocycle $(m_{x,y})_{x,y}$ is obtained by evaluating in ΩX representatives of the fundamental classes of the moduli spaces of Morse trajectories $\bar{\mathcal{L}}(x, y)$. The twisted differential is given by

$$\partial(\alpha \otimes x) = \partial_{\mathcal{F}} \alpha \otimes x + (-1)^{|\alpha|} \sum_y \alpha \cdot m_{x,y} \otimes y.$$

In [5], we make use of their construction and, in particular, of one of their main results, the Fibration Theorem 1.1, in order to build a Morse theoretic Chas–Sullivan product

$$\text{CS}_{DG} : H_*(E)^{\otimes 2} \longrightarrow H_*(E),$$

for any (Hurewicz) fibration $E \xrightarrow{\pi} X$ endowed with a morphism of fibrations $m : E \times_{\pi \times \pi} E \rightarrow E$.

Given a fibration $E \xrightarrow{\pi} X$, a *transitive lifting function* $\Phi : E \times_{\pi \times \text{ev}_0} \mathcal{P}X \rightarrow E$ is a map that lifts all paths in X and respects concatenation in the sense that lifting a concatenation of two paths is the same as lifting them one after the other : $\Phi(\Phi(e, \gamma), \delta) = \Phi(e, \gamma \# \delta)$. Let $F = \pi^{-1}(\star_X)$. The restriction $\Phi : F \times \Omega X \rightarrow F$ induces a $C_*(\Omega X)$ -module structure on the cubical complex $C_*(F)$.

FIBRATION THEOREM 1.1 ([2, Theorem 7.2.1]). — *Let $E \xrightarrow{\pi} X$ be a fibration with model fiber $F = \pi^{-1}(\star)$ and equip $C_*(F)$ with the $C_*(\Omega X)$ -module structure induced by a transitive lifting function $\Phi : E \times_{\pi \times \text{ev}_0} \mathcal{P}X \rightarrow E$ associated with this fibration. Then, there exists a quasi-isomorphism denoted*

$$\Psi_E : C_*(X, C_*(F)) \longrightarrow C_*(E).$$

DG CHAS–SULLIVAN PRODUCT 1.2 ([5, Theorem 7.1]). — *Let $E \xrightarrow{\pi} X$ be a fibration endowed with a morphism of fibrations $m : E \times_{\pi} E \rightarrow E$. Let $F = \pi^{-1}(\star)$ and endow $\mathcal{F} = C_*(F)$ with a DG right $C_*(\Omega X)$ -module structure induced by a transitive lifting function. There exists a degree $-n$ product*

$$\text{CS}_{DG} : H_*(X, \mathcal{F})^{\otimes 2} \longrightarrow H_*(X, \mathcal{F})$$

that is associative if m_ is associative in homology, graded commutative if m_* is commutative in homology, admits a neutral element if there exists a unit section $s : X \rightarrow E$ and this product satisfies the functoriality and spectral sequence properties.*

Moreover, this product corresponds in homology, via the Fibration Theorem 1.1, to the product $\mu_ : H_i(E) \otimes H_j(E) \rightarrow H_{i+j-n}(E)$ defined by Gruher–Salvatore in [3]. In particular, if the fibration is the loop-loop fibration $\Omega X \hookrightarrow \mathcal{L}X \rightarrow X$, then CS_{DG} corresponds to the Chas–Sullivan product.*

The tools and techniques used to build such a product also enable to build a Morse model

$$\begin{aligned} \text{PP}_{DG} : H_i(X \times X, C_*(\Omega Y)) \otimes H_j(X \times X, C_*(\Omega Y)) \\ \longrightarrow H_{i+j-n}(X \times X, C_*(\Omega Y)), \end{aligned}$$

for the Path-product

$$\Lambda : H_i(\mathcal{P}_X Y) \otimes H_j(\mathcal{P}_X Y) \longrightarrow H_{i+j-n}(\mathcal{P}_X Y),$$

and a module structure

$$\cdot_M : H_i(Y, C_*(\Omega Y)) \otimes H_j(X \times X, C_*(\Omega Y)) \longrightarrow H_{i+j-k}(X \times X, C_*(\Omega Y)),$$

where $H_*(Y, C_*(\Omega Y))$ is endowed with the ring structure CS_{DG} .

1.2. The Path-product

Let Y be a topological space, let X^n be a pointed, oriented, closed, and connected manifold and let $f : X \rightarrow Y$ be a continuous map. Let $\mathcal{P}Y = \{\gamma : [0, a] \rightarrow Y, \text{ continuous}\}$ be the space of Moore paths. Consider the pullback fibration

$$\Omega Y \hookrightarrow \mathcal{P}_{X,f} Y \xrightarrow{(\text{ev}_0, \text{ev}_1)} X^2$$

of the fibration $\Omega Y \hookrightarrow \mathcal{P}Y \xrightarrow{(\text{ev}_0, \text{ev}_1)} Y^2$ by $f \times f : X^2 \rightarrow Y^2$. The Fibration Theorem [2, Theorem 7.2.1] implies that

$$H_*(X^2, (f \times f)^* C_*(\Omega Y)) \cong H_*(\mathcal{P}_{X,f} Y).$$

For readability, we will often denote $H_*(X^2, C_*(\Omega Y); f)$ this homology or $H_*(X^2, C_*(\Omega Y))$ when f is implied. We will use similar notations for the complexes of chains. The construction by Stegemeyer in [6] of a degree $-n$ product

$$\Lambda : H_*(\mathcal{P}_{X,f}Y)^{\otimes 2} \longrightarrow H_*(\mathcal{P}_{X,f}Y)$$

relies on the idea of “intersecting on the base” to obtain concatenable paths and concatenating. This idea is similar to the construction of the Chas–Sullivan product but does not fall under the category of products defined in [5, Theorem 7.1] since the intersection does not occur along the diagonal $\Delta : X^2 \rightarrow X^4$ but rather along $D : X^3 \rightarrow X^4$, $D(a, b, c) = (a, b, b, c)$. In other words, the concatenation of interest is

$$m : \mathcal{P}_{X,f}Y \times_{\text{ev}_1} \mathcal{P}_{X,f}Y \longrightarrow \mathcal{P}_{X,f}Y$$

and not

$$\mathcal{P}_{X,f}Y \times_{(\text{ev}_0, \text{ev}_1)} \mathcal{P}_{X,f}Y \longrightarrow \mathcal{P}_{X,f}Y.$$

The concatenation m is not a morphism of fibrations in the sense defined in [5, Section 5] since $\mathcal{P}_{X,f}Y \times_{\text{ev}_1} \mathcal{P}_{X,f}Y$ is seen as the total space of a fibration over X^3 and $\mathcal{P}_{X,f}Y$ is seen as the total space of a fibration over X^2 . Therefore, in order to define a Morse equivalent of this map, we generalize [5, Theorem 5.8] for a broader class of morphisms that we define, i.e. *relative morphism of fibrations*.

DEFINITION 1.3. — *Let Z_0 and Z_1 be topological spaces and let $E_0 \xrightarrow{\pi_0} Z_0$ and $E_1 \xrightarrow{\pi_1} Z_1$ be fibrations. Let $g : Z_0 \rightarrow Z_1$ be a continuous map. We say that $\varphi : E_0 \rightarrow E_1$ is a morphism of fibrations relative to g if $\pi_1 \circ \varphi = g \circ \pi_0$.*

$$\begin{array}{ccc} E_0 & \xrightarrow{\varphi} & E_1 \\ \pi_0 \downarrow & & \downarrow \pi_1 \\ Z_0 & \xrightarrow{g} & Z_1. \end{array}$$

The following result generalizes [5, Thm 5.8] using [2, Prop. 9.8.1].

THEOREM 1.4 (Theorem 2.3). — *Let Z_0 and Z_1 be two pointed, closed, and connected manifolds. Let $F_0 \hookrightarrow E_0 \rightarrow Z_0$ and $F_1 \hookrightarrow E_1 \rightarrow Z_1$ be two fibrations. If $\varphi : E_0 \rightarrow E_1$ is a morphism of fibrations relative to a continuous map $g : Z_0 \rightarrow Z_1$, then there exists a morphism of complexes $\tilde{\varphi} : C_*(Z_0, C_*(F_0)) \rightarrow C_*(Z_0, g^*C_*(F_1))$ such that the following diagram*

commutes up to chain homotopy

$$\begin{array}{ccc}
 C_*(Z_0, C_*(F_0)) & \xrightarrow{g_* \circ \tilde{\varphi}} & C_*(Z_1, C_*(F_1)) \\
 \downarrow \Psi_0 & & \downarrow \Psi_1 \\
 C_*(E_0) & \xrightarrow{\varphi_*} & C_*(E_1),
 \end{array}$$

where the morphisms Ψ_0 and Ψ_1 are the quasi-isomorphisms given by the Fibration Theorem [2, Theorem 7.2.1].

The concatenation $m : \mathcal{P}_{X,f}Y_{\text{ev}_1} \times_{\text{ev}_0} \mathcal{P}_{X,f}Y \rightarrow \mathcal{P}_{X,f}Y$ is a morphism of fibrations relative to $p : X^3 \rightarrow X^2$, $p(a, b, c) = (a, c)$ and we will prove the following:

THEOREM 1.5 (Theorem 2.4). — *The product*

$$\text{PP}_{DG} : H_*(X^2, C_*(\Omega Y))^{\otimes 2} \longrightarrow H_*(X^2, C_*(\Omega Y))$$

of degree $-n$ defined by the composition

$$\begin{aligned}
 H_i(X^2, C_*(\Omega Y)) \otimes H_j(X^2, C_*(\Omega Y)) & \xrightarrow{(-1)^{nj}K} H_{i+j}(X^4, C_*(\Omega Y^2)) \\
 & \xrightarrow{D_!} H_{i+j-n}(X^3, D^*C_*(\Omega Y^2)) \\
 & \xrightarrow{\tilde{m}} H_{i+j-n}(X^3, p^*C_*(\Omega Y)) \\
 & \xrightarrow{p_*} H_{i+j-n}(X^2, C_*(\Omega Y))
 \end{aligned}$$

is associative, admits a neutral element and corresponds, via the Fibration Theorem 1.1 to the product

$$\Lambda : H_i(\mathcal{P}_{X,f}Y) \otimes H_j(\mathcal{P}_{X,f}Y) \longrightarrow H_{i+j-n}(\mathcal{P}_{X,f}Y)$$

defined by Stegemeyer in [6].

1.3. Homotopy invariance and module structure over the Chas–Sullivan ring

We can reprove several results from [6] in this setting. In particular, if Y^k is a pointed, oriented, closed, and connected manifold, we prove that $H_*(X^2, C_*(\Omega Y))$ has a module structure over $H_*(Y, C_*(\Omega Y))$ endowed with the ring structure CS_{DG} .

PROPOSITION 1.6 (Proposition 4.1). — *There exists a pairing*

$$\cdot_M : H_i(Y, C_*(\Omega Y)) \otimes H_j(X^2, C_*(\Omega Y)) \longrightarrow H_{i+j-k}(X^2, C_*(\Omega Y))$$

that induces a left $H_*(Y, C_*(\Omega Y))$ -module structure on $H_*(X^2, C_*(\Omega Y))$ with the additional property that if $\gamma \in H_*(Y, C_*(\Omega Y))$ and $\tau, \tau' \in H_*(X^2, C_*(\Omega Y))$, then

$$\gamma \cdot_M \text{PP}_{DG}(\tau \otimes \tau') = \text{PP}_{DG}((\gamma \cdot_M \tau) \otimes \tau').$$

We also prove that the ring and module structures only depend on the homotopy type of $f : X \rightarrow Y$

PROPOSITION 1.7 (Propositions 5.2–5.3). — *If $g : X \rightarrow Y$ is homotopic to f , then there exists a chain homotopy equivalence $\tilde{\Psi}_{f,g} : C_*(X, C_*(\Omega Y); f) \rightarrow C_*(X, C_*(\Omega Y); g)$ that induces an isomorphism*

$$\tilde{\Psi}_{f,g} : H_*(X, C_*(\Omega Y); f) \xrightarrow{\sim} H_*(X, C_*(\Omega Y); g)$$

of rings and modules.

And independent of the homotopy equivalence class of the pair (Y, X) when $X \subset Y$.

PROPOSITION 1.8 (Propositions 5.4–5.5). — *If $\varphi : (Y_1, X_1) \rightarrow (Y_2, X_2)$ is a homotopy equivalence of pairs, then there exists an isomorphism of rings*

$$\varphi_{\#} : H_*((X_2)^2, C_*(\Omega Y_2)) \longrightarrow H_*((X_1)^2, C_*(\Omega Y_1))$$

that satisfies the equation

$$\begin{aligned} \forall \gamma \in H_*(Y_2, C_*(\Omega Y_2)), \quad \forall \tau \in H_*((X_2)^2, C_*(\Omega Y_2)), \\ \varphi_{\#}(\gamma \cdot_M \tau) = \varphi_{\#}^{\mathcal{L}}(\gamma) \cdot_M \varphi_{\#}(\tau), \end{aligned}$$

where $\varphi_{\#}^{\mathcal{L}} = \widetilde{\mathcal{L}\varphi}^{-1} \circ \varphi_{\#} : H_*(Y_2, C_*(\Omega Y_2)) \rightarrow H_*(Y_1, C_*(\Omega Y_1))$ is the ring isomorphism induced by the homotopy equivalence $\varphi : Y_1 \rightarrow Y_2$ (see [5, Proposition 7.13]).

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2. Definition

Let Y be a topological space, $(X, \star)$ be a pointed, oriented, closed, and connected manifold and $f : X \rightarrow Y$ be a continuous map. Denote $\star_Y = f(\star_X)$. Let

$$\mathcal{P}Y = \{(\gamma, a), a \in [0, +\infty), \gamma : [0, a] \rightarrow Y \text{ continuous}\}$$

be the Moore space of paths in Y . For $(\gamma, a) \in \mathcal{P}Y$, we will denote $\gamma \in \mathcal{P}Y$ when the mention of the non-negative real number a is not useful. This space is naturally endowed with a fibration

$$\Omega Y \hookrightarrow \mathcal{P}Y \xrightarrow{\text{ev}_0, \text{ev}_1} Y^2$$

given by the evaluation at the basepoint and endpoint. Consider its pullback by $f \times f : X^2 \rightarrow Y^2$,

$$\Omega Y \hookrightarrow \mathcal{P}_{X,f}Y \xrightarrow{\text{ev}} X^2,$$

where

$$\mathcal{P}_{X,f}Y = \{(x_0, x_1, (\alpha, a)) \in X^2 \times \mathcal{P}Y, \alpha(0) = f(x_0), \alpha(a) = f(x_1)\}$$

and the evaluation is given by

$$\text{ev}(x_0, x_1, \alpha) := (\text{ev}_0(x_0, x_1, \alpha), \text{ev}_1(x_0, x_1, \alpha)) := (x_0, x_1).$$

We will often see an element $(x_0, x_1, \gamma) \in \mathcal{P}_{X,f}Y$ as $\gamma \in \mathcal{P}Y$ in the fiber $\text{ev}^{-1}(x_0, x_1)$ and therefore write $\gamma \in \mathcal{P}_{X,f}Y$ when it is not necessary to precise the fiber in which γ belong. We will consider the following *transitive lifting function*

$$\Phi_{\text{ev}} : \mathcal{P}_{X,f}Y \times_{\text{ev} \times \text{ev}_0} \mathcal{P}X^2 \rightarrow \mathcal{P}_{X,f}Y, \Phi_{\text{ev}}(\gamma, (\tau_1, \tau_2)) = (f \circ \tau_1^{-1})\gamma(f \circ \tau_2)$$

for this fibration. A degree $-n$ product

$$\Lambda : H_*(\mathcal{P}_{X,f}Y)^{\otimes 2} \rightarrow H_*(\mathcal{P}_{X,f}Y)$$

has been defined and studied by [6] if Y is a closed manifold. This product, as the Chas–Sullivan product, is defined by intersecting on a space where the paths are concatenable and then concatenate.

$$\begin{aligned} \Lambda : H_i(\mathcal{P}_{X,f}Y) \otimes H_j(\mathcal{P}_{X,f}Y) &\longrightarrow H_{i+j}(\mathcal{P}_{X,f}Y \times \mathcal{P}_{X,f}Y) \\ &\xrightarrow{\text{inters.}} H_{i+j-n}(\mathcal{P}_{X,f}Y \times_{\text{ev}_1 \times \text{ev}_0} \mathcal{P}_{X,f}Y) \\ &\xrightarrow{\text{concat.}} H_{i+j-n}(\mathcal{P}_{X,f}Y). \end{aligned}$$

The Fibration Theorem [2, Theorem 7.2.1] states that

$$H_*(\mathcal{P}_{X,f}Y) \cong H_*(X^2, (f \times f)^*C_*(\Omega Y)),$$

where the $C_*(\Omega X^2)$ module structure on $(f \times f)^*C_*(\Omega Y)$ is given by the transitive lifting function Φ_{ev} . To abbreviate the notations, we will denote

this homology $H_*(X^2, C_*(\Omega Y); f)$ or just $H_*(X^2, C_*(\Omega Y))$ when f is implied.

DEFINITION 2.1. — *Let Y be a topological space, X be a n -dimensional closed, oriented, path-connected and pointed manifold and $f : X \rightarrow Y$ be a continuous map. We define a degree $-n$ product*

$$\text{PP}_{DG} : H_*(X^2, C_*(\Omega Y))^{\otimes 2} \longrightarrow H_*(X^2, C_*(\Omega Y))$$

by the composition of the four maps

$$\begin{aligned} & H_i(X^2, C_*(\Omega Y)) \otimes H_j(X^2, C_*(\Omega Y)) \\ & \xrightarrow{(-1)^{nj}K} H_{i+j}(X^4, C_*(\Omega Y^2)) \simeq H_{i+j}(\mathcal{P}_{X,f}Y \times \mathcal{P}_{X,f}Y) \\ & \xrightarrow{D!} H_{i+j-n}(X^3, D^*C_*(\Omega Y^2)) \simeq H_{i+j-n}(\mathcal{P}_{X,f}Y \times_{\text{ev}_1 \times \text{ev}_0} \mathcal{P}_{X,f}Y) \\ & \xrightarrow{\tilde{m}} H_{i+j-n}(X^3, p^*C_*(\Omega Y)) \simeq H_{i+j-n}(p^*\mathcal{P}_{X,f}Y) \\ & \xrightarrow{p_*} H_{i+j-n}(X^2, C_*(\Omega Y)) \simeq H_{i+j-n}(\mathcal{P}_{X,f}Y), \end{aligned}$$

where:

- $K : H_i(X^2, C_*(\Omega Y)) \otimes H_j(X^2, C_*(\Omega Y)) \rightarrow H_{i+j}(X^4, C_*(\Omega Y^2))$ is the topological DG Künneth map defined in [5, Theorem 6.14],
- $D! : H_{i+j}(X^4, C_*(\Omega Y^2)) \rightarrow H_{i+j-n}(X^3, D^*C_*(\Omega Y^2))$ is the shriek map of

$$D : X^3 \longrightarrow X^4, \quad D(a, b, c) = (a, b, b, c)$$

and

$$p_* : H_{i+j-n}(X^3, p^*C_*(\Omega Y)) \longrightarrow H_{i+j-n}(X^2, C_*(\Omega Y))$$

is the direct map of $p : X^3 \rightarrow X^2$, $p(a, b, c) = (a, c)$ as defined in [2, Sections 9–10],

- $\tilde{m} : H_{i+j-n}(X^3, \Delta^*C_*(\Omega Y^2)) \rightarrow H_{i+j-n}(X^3, p^*C_*(\Omega Y))$ is the morphism associated with the morphism of fibrations given by the concatenation $m : \mathcal{P}_{X,f}Y \times_{\text{ev}_1 \times \text{ev}_0} \mathcal{P}_{X,f}Y \rightarrow p^*\mathcal{P}_{X,f}Y$, (see [5, Theorem 5.8]).

Remark 2.2. — Intuitively, given two chains σ, τ in $\mathcal{P}_{X,f}Y$, this product will intersect $\text{ev}_{1,*}(\sigma)$ with $\text{ev}_{0,*}(\tau)$ on X , concatenate on the intersection and forget the intersection points in which it concatenated.

We can remark that since the intersection is not along the diagonal $\Delta : X^2 \rightarrow X^4$, this product does not fall under the category of products studied in [5]. Nonetheless, the tools developed there enable to define and study it.

The reader may wonder why it takes two maps $\tilde{m} : H_*(X^3, p^*C_*(\Omega Y))$ and $p_* : H_*(X^2, C_*(\Omega Y))$ to translate in Morse theoretic terms the concatenation $m : H_*(\mathcal{P}_{X,f}Y_{\text{ev}_1} \times_{\text{ev}_0} \mathcal{P}_{X,f}Y) \rightarrow H_*(\mathcal{P}_{X,f})$. Here the fiber product $\mathcal{P}_{X,f}Y_{\text{ev}_1} \times_{\text{ev}_0} \mathcal{P}_{X,f}Y$ is seen as the total space of the fibration

$$\begin{aligned} \mathcal{P}_{X,f}Y_{\text{ev}_1} \times_{\text{ev}_0} \mathcal{P}_{X,f}Y &\longrightarrow X^3 \\ (\gamma_1, \gamma_2) &\longmapsto (\text{ev}_0(\gamma), \text{ev}_1(\gamma_1), \text{ev}_1(\gamma_2)), \end{aligned}$$

while $\mathcal{P}_{X,f}Y$ is a fibration over X^2 . Since the bases are not the same, the concatenation m is not a morphism of fibrations in the sense given in [5, Theorem 5.8].

Using [2, Proposition 9.8.1], we obtain a direct generalization of [5, Theorem 5.8] for relative morphism of fibrations (see Definition 1.3).

THEOREM 2.3. — *Let $g : Z_0 \rightarrow Z_1$ be continuous map between oriented, closed, and connected manifolds each endowed with a fibration $F_0 \hookrightarrow E_0 \xrightarrow{\pi_0} Z_0$ and $F_1 \hookrightarrow E_1 \xrightarrow{\pi_1} Z_1$. Let $\varphi : E_0 \rightarrow E_1$ be a morphism of fibrations relative to g . Then $\varphi_* : C_*(E_0) \rightarrow C_*(E_1)$, corresponds, via the Fibration Theorem 1.1, up to chain homotopy equivalence, to the composition*

$$g_* \circ \tilde{\varphi}_0 : C_*(Z_0, C_*(F_0)) \longrightarrow C_*(Z_0, g^*C_*(F_1)) \longrightarrow C_*(Z_1, C_*(F_1)).$$

In other words, the following diagram is commutative up to chain homotopy:

$$\begin{array}{ccccc} C_*(Z_0, C_*(F_0)) & \xrightarrow{\tilde{\varphi}_0} & C_*(Z_0, g^*C_*(F_1)) & \xrightarrow{g_*} & C_*(Z_1, C_*(F_1)) \\ \downarrow \Psi_{E_0} & & & & \downarrow \Psi_{E_1} \\ C_*(E_0) & \xrightarrow{\varphi_*} & & & C_*(E_1). \end{array}$$

Proof. — The map φ factors through

$$\begin{aligned} \varphi : E_0 &\xrightarrow{\varphi_0} g^*E_1 \longrightarrow E_1 \\ e &\longmapsto (\pi_0(e), \varphi(e)) \longmapsto \varphi(e), \end{aligned}$$

where $\varphi_0 : E_0 \rightarrow g^*E_1$ is a morphism of fibrations over Z_0 .

$$\begin{array}{ccccc} & & \varphi & & \\ & \searrow & & \nearrow & \\ E_0 & \xrightarrow{\quad} & g^*E_1 & \xrightarrow{\quad} & E_1 \\ \pi_0 \downarrow & & \downarrow pr_1 & & \downarrow \pi_1 \\ Z_0 & \xlongequal{\quad} & Z_0 & \xrightarrow{g} & Z_1. \end{array}$$

Therefore, using [5, Theorem 5.8] and [2, Proposition 9.8.1] we obtain that the following diagram is commutative up to chain homotopy:

$$\begin{array}{ccc}
 C_*(Z_0, C_*(F_0)) & \xrightarrow{\tilde{\varphi}_0} & C_*(Z_0, h^*C_*(F_1)) \xrightarrow{h_*} C_*(Z_1, C_*(F_1)) \\
 \downarrow \Psi_{E_0} & & \downarrow \Psi_{E_1} \\
 C_*(E_0) & \xrightarrow{\varphi_*} & C_*(E_1).
 \end{array} \quad \square$$

Using very similar techniques as those used in [5, Section 7] to prove [5, Theorem 7.1], we will prove:

THEOREM 2.4 (Theorem 1.5). — *The product PP_{DG} is associative admits a neutral element and corresponds to the product*

$$\Lambda : H_i(\mathcal{P}_{X,f}Y) \otimes H_j(\mathcal{P}_{X,f}Y) \longrightarrow H_{i+j-n}(\mathcal{P}_{X,f}Y)$$

defined by Stegemeyer [6].

3. Proof of Theorem 1.5

This product uses both direct maps and shriek maps. Therefore, we need to understand how these maps interact with each other. The only pre-existing result exploring this question is the following:

PROPOSITION 3.1 ([2, Proposition 10.6.1]). — *Let $\varphi : X \rightarrow Y$ be a map of degree $d \in \mathbb{Z}$ between closed oriented manifolds of equal dimensions. Let $\mathcal{F}$ be a DG local system on Y . Then*

$$\varphi_*\varphi_! = d \cdot \text{Id} : H_*(Y, \mathcal{F}) \longrightarrow H_*(Y, \mathcal{F}).$$

In particular, if $d = \pm 1$, the map $\varphi_!$ is injective and the map φ_ is surjective.*

We will use this result to prove the following lemma:

LEMMA 3.2. — *Let $i_X : X^n \hookrightarrow W^{n+k}$ and $i_Y : Y^m \hookrightarrow Z^{m+k}$ be two k -codimensional embeddings of oriented, closed, and connected manifolds. Let $\mathcal{F}$ be a DG local system on Z and Ξ_X, Ξ_Y, Ξ_Z and Ξ_W be sets of DG Morse data. Let $\varphi : W \rightarrow Z$ and $\psi : X \rightarrow Y$ such that*

$$\varphi \circ i_X = i_Y \circ \psi : X \longrightarrow Z \quad \text{and} \quad \varphi^{-1}(i_Y(Y)) = i_X(X).$$

Assume that the normal bundle N_X of i_X and the normal bundle N_Y of i_Y are isomorphic and denote $\varepsilon \in \{-1, 1\}$ such that

$$\text{Or } N_X = \varepsilon \cdot \text{Or } N_Y.$$

Then, the maps

$$\varepsilon \cdot (\psi_* \circ i_{X,!}) \quad \text{and} \quad i_{Y,!} \circ \varphi_* : C_*(W, \Xi_W, \varphi^* \mathcal{F}) \longrightarrow C_{*-k}(Y, \Xi_Y, i_Y^* \mathcal{F}),$$

are chain homotopic.

Proof of Lemma 3.2.

Step 1. — First assume that $X \subset W$ and $Y \subset Z$ are submanifolds. Then, $\psi = \varphi|_X : X \rightarrow Y$ and $\varphi^{-1}(Y) = X$. It suffices to prove the result for well-chosen data Ξ_X, Ξ_Y, Ξ_Z and Ξ_W , since the continuation maps are chain homotopy equivalence. Recall that since X, Y, Z and W are oriented manifolds, the orientations of the normal bundles are given by

$$(\text{Or } Y, \text{Or } N_Y) = \text{Or } Z,$$

and

$$(\text{Or } X, \text{Or } N_X) = \text{Or } W.$$

Choose Ξ_X and let Ξ_W be the set of DG Morse data described in [2, Section 9.2] given by considering a tubular neighborhood $U \subset^{\text{iu}} W$ of X and extending the Morse–Smale pair $(f, \xi) \in \Xi_X$ in $(f_U, \xi_U) \in \Xi_U$ by taking the pseudo-gradient ξ_U to point outwards U . Orient the unstable manifolds by

$$\text{Or } \overline{W}_U^u(x) = \left(\text{Or } \overline{W}_X^u(x), \text{Or } N_X \right).$$

With such data,

$$C_*(X, \Xi_X, i_X^* \varphi^* \mathcal{F}) = C_{*+k}(U, \Xi_U, i_U^* \varphi^* \mathcal{F}).$$

Then, extend (f_U, ξ_U) to $(f_W, \xi_W) \in \Xi_W$. The shriek map of the inclusion is then given by

$$\begin{aligned} i_{X,!} : \quad C_*(W, \Xi_W, \varphi^* \mathcal{F}) &\longrightarrow C_{*-k}(X, \Xi_X, i_X^* \varphi^* \mathcal{F}) \\ \alpha \otimes w &\longmapsto \begin{cases} \alpha \otimes w & \text{if } w \in X \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Apply the same procedure from a set of DG Morse data Ξ_Y to obtain Ξ_Z such that

$$\begin{aligned} i_{Y,!} : \quad C_*(Z, \Xi_Z, \mathcal{F}) &\longrightarrow C_{*-k}(Y, \Xi_Y, i_Y^* \mathcal{F}) \\ \alpha \otimes z &\longmapsto \begin{cases} \alpha \otimes z & \text{if } z \in Y \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Denote $f_Z : Z \rightarrow \mathbb{R}$ and $f_Y : Y \rightarrow \mathbb{R}$ the Morse functions in the data sets Ξ_Z and Ξ_Y respectively. With these data, the composition $\varphi|_{X,*} \circ i_{X,!}$ is given by

$$\begin{aligned} \varphi|_{X,*} \circ i_{X,!} : \quad C_*(W, \Xi_W, \varphi^* \mathcal{F}) &\longrightarrow C_{*-k}(Y, \Xi_Y, i_Y^* \mathcal{F}) \\ \alpha \otimes w &\longmapsto \begin{cases} \sum_y \alpha \cdot \nu_{w,y}^{\varphi|_X} \otimes y & \text{if } w \in X \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $\nu_{w,y}^{\varphi|_X} \in C_{|w|-|y|}(\Omega Y)$ is the evaluation onto ΩY of a chain representative system of the space of trajectories

$$\overline{\mathcal{M}}^{\varphi|_X}(w, y) = \overline{W}_{f_X}^u(w) \cap \varphi|_X^{-1}(\overline{W}_{f_Y}^s(y)).$$

The composition $i_{Y,!} \circ \varphi_*$ is given by

$$\begin{aligned} i_{Y,!} \circ \varphi_* : \quad C_*(W, \Xi_W, \varphi^* \mathcal{F}) &\longrightarrow C_{*-k}(Y, \Xi_Y, i_Y^* \mathcal{F}) \\ \alpha \otimes w &\longmapsto \begin{cases} \sum_{y \in \text{Crit}(f_Y)} \alpha \cdot \nu_{w,y}^{\varphi} \otimes y & \text{if } w \in X \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Indeed, for all $w \in \text{Crit}(f_W)$,

$$\varphi_*(\alpha \otimes w) = \sum_{z \in \text{Crit}(f_Z)} \alpha \cdot \nu_{w,z}^{\varphi} \otimes z,$$

and there are three cases:

- (1) If $z \notin Y$, then $i_{Y,!}(\alpha \cdot \nu_{w,z}^{\varphi} \otimes z) = 0$.
- (2) If $w \notin X$ and $z \in Y$, then $\overline{W}_{f_W}^u(w) \cap X = \emptyset$ since the pseudo-gradient ξ_W points outwards U and therefore, a trajectory that begins outside of U cannot reach U . Moreover, for the same reason, $\overline{W}_{f_Z}^s(z) = \overline{W}_{f_Y}^s(z) \subset Y$ and since we assume that $\varphi^{-1}(Y) = X$, it follows that $\varphi^{-1}(\overline{W}_{f_Z}^s(z)) \subset X$. In particular $\overline{\mathcal{M}}^{\varphi}(w, z) = \overline{W}_{f_W}^u(w) \cap \varphi^{-1}(\overline{W}_{f_Z}^s(z)) = \emptyset$ and $\nu_{w,z}^{\varphi} = 0$.
- (3) If $w \in X$ and $z \in Y$, then $i_{Y,!}(\alpha \cdot \nu_{w,z}^{\varphi} \otimes z) = \alpha \cdot \nu_{w,z}^{\varphi} \otimes z$.

To conclude this first step, it suffices to prove that

$$\begin{aligned} \forall w \in \text{Crit}_i(f_X) \subset \text{Crit}_{i+k}(f_W), \quad \forall z \in \text{Crit}_j(f_Y) \subset \text{Crit}_{j+k}(f_Z), \\ \overline{\mathcal{M}}^{\varphi|_X}(w, z) = \overline{\mathcal{M}}^{\varphi}(w, z) \end{aligned}$$

and compare their orientations.

Let

$$a \in \overline{\mathcal{M}}^{\varphi|_X}(w, z) = \overline{W}_{f_X}^u(w) \cap \varphi|_X^{-1}(\overline{W}_{f_Y}^s(z)).$$

Then,

$$a \in \overline{W}_{f_W}^u(w) \cap \varphi^{-1}\left(\overline{W}_{f_Z}^s(z)\right) = \overline{\mathcal{M}}^\varphi(w, z)$$

since

$$\overline{W}_{f_X}^u(w) \subset \overline{W}_{f_W}^u(w) \quad \text{and} \quad \overline{W}_{f_Y}^s(z) = \overline{W}_{f_Z}^s(z).$$

Therefore

$$\overline{\mathcal{M}}^{\varphi|_X}(w, z) \subset \overline{\mathcal{M}}^\varphi(w, z).$$

Let $a \in \overline{\mathcal{M}}^\varphi(w, z)$. In particular

$$a \in \varphi^{-1}\left(\overline{W}_{f_Z}^s(z)\right) = \varphi^{-1}\left(\overline{W}_{f_Y}^s(z)\right) \subset \varphi^{-1}(Y) = X.$$

Therefore,

$$a \in \left(\overline{W}_{f_W}^u(w) \cap X\right) \cap \varphi_{|X}^{-1}\left(\overline{W}_{f_Y}^s(z)\right) = \overline{W}_{f_X}^u(w) \cap \varphi_{|X}^{-1}\left(\overline{W}_{f_Y}^s(z)\right)$$

and

$$\overline{\mathcal{M}}^\varphi(w, z) \subset \overline{\mathcal{M}}^{\varphi|_X}(w, z).$$

It follows that $\overline{\mathcal{M}}^\varphi(w, z) = \overline{\mathcal{M}}^{\varphi|_X}(w, z)$ and we now compare their orientations.

$$\left(\text{Or } \overline{\mathcal{M}}^\varphi(w, z), \text{Or } \overline{W}_Z^u(z)\right) = \text{Or } \overline{W}_W^u(w).$$

$$\begin{aligned} \left(\text{Or } \overline{\mathcal{M}}^{\varphi|_X}(w, z), \text{Or } \overline{W}_Z^u(z)\right) &= \left(\text{Or } \overline{\mathcal{M}}^{\varphi|_X}(w, z), \text{Or } \overline{W}_Y^u(z), \text{Or } N_Y\right) \\ &= \left(\text{Or } \overline{W}_X^u(w), \text{Or } N_Y\right) \\ &= \varepsilon \left(\text{Or } \overline{W}_X^u(w), \text{Or } N_X\right) \\ &= \varepsilon \cdot \text{Or } \overline{W}_W^u(w). \end{aligned}$$

Therefore, with these data, for all $w \in \text{Crit}(f_X) \subset \text{Crit}(f_W)$ and $y \in \text{Crit}(f_Y) \subset \text{Crit}(f_Z)$,

$$\nu_{w,y}^{\varphi|_X} = \varepsilon \cdot \nu_{w,z}^\varphi$$

and

$$\varepsilon \cdot (\varphi_{|_{X,*}} \circ i_{X,!}) = i_Y \circ \psi.$$

Step 2. — In the general case, we consider $i_X : X \rightarrow W$ as the composition of the orientation-preserving diffeomorphism $i_X : X \rightarrow i_X(X)$ with the inclusion $i : i_X(X) \rightarrow W$. Denote $j : i_Y(Y) \rightarrow Z$ the inclusion.

Consider the following diagram:

$$\begin{array}{ccccc}
 C_{*-k}(X, \Xi_X, i_X^* \varphi^* \mathcal{F}) & \xleftarrow[i_{X,*}^{-1} = i_{X,!}]{} & C_{*-k}(i_X(X), i_X^* \Xi_X, i_X^* \varphi^* \mathcal{F}) & \xleftarrow[i_!]{\varphi|_{i_X(X),*}} & C_*(W, \Xi_W, \varphi^* \mathcal{F}) \\
 \downarrow \psi_* & & \downarrow & & \downarrow \varphi_* \\
 C_{*-k}(Y, \Xi_Y, i_Y^* \mathcal{F}) & \xleftarrow[i_{Y,*}^{-1} = i_{Y,!}]{} & C_{*-k}(i_Y(Y), i_Y^* \Xi_Y, j^* \mathcal{F}) & \xleftarrow[j_!]{\varphi|_{i_X(X),*}} & C_*(Z, \Xi_Z, \mathcal{F}).
 \end{array}$$

We proved in the first step that the right square commutes up to chain homotopy up to the sign ε . The fact that $i_{X,!} = i_{X,*}^{-1}$ and $i_{X,!} = i_{X,*}^{-1}$ is a direct consequence of [2, Proposition 10.6.1] stated above. The left square commutes up to chain homotopy by the composition property of direct maps. $\square$

We now prove that the product PP_{DG} is associative and admits a neutral element.

PROPOSITION 3.3. — *The product PP_{DG} is associative at the homology level.*

Proof. — We denote $\mathcal{F}^k = C_*(\Omega Y^k)$, $\mathcal{F} = \mathcal{F}^1$ and $H_* = H_*(X^2, \mathcal{F})$. Consider the diagram

$$\begin{array}{ccccccc}
 H_*^{\otimes 3} & \xrightarrow{K \otimes 1} & H_*(X^4, \mathcal{F}^2) \otimes H_* & \xrightarrow{D_1 \otimes 1} & H_*(X^3, D^* \mathcal{F}^2) \otimes H_* & \xrightarrow{\bar{m} \otimes 1} & H_*(X^3, p^* \mathcal{F}^2) \otimes H_* & \xrightarrow{p_* \otimes 1} & H_*^{\otimes 2} \\
 \downarrow 1 \otimes K & & \downarrow K & & \downarrow K & & \downarrow K & & \downarrow K \\
 H_* \otimes H_*(X^4, \mathcal{F}^2) & \xrightarrow{K} & H_*(X^6, \mathcal{F}^3) & \xrightarrow{(D \times 1)_!} & H_*(X^5, D^* \mathcal{F}^2 \times \mathcal{F}) & \xrightarrow{\widetilde{m} \times 1} & H_*(X^5, p^* \mathcal{F}^2 \times \mathcal{F}) & \xrightarrow{(p \times 1)_*} & H_*(X^4, \mathcal{F}^2) \\
 \downarrow 1 \otimes D_1 & & \downarrow (1 \times D)_! & & \downarrow \Delta_{R,!} & & \downarrow \Delta_{R,!} & & \downarrow D_1 \\
 H_* \otimes H_*(X^3, D^* \mathcal{F}^2) & \xrightarrow{K} & H_*(X^5, \mathcal{F} \times D^* \mathcal{F}^2) & \xrightarrow{\Delta_{L,!}} & H_*(X^4, \Delta_2^* \mathcal{F}^3) & \xrightarrow{\widetilde{m} \times 1} & H_*(X^4, \Delta_2^* \mathcal{F}^2) & \xrightarrow{p_{2,*}} & H_*(X^3, \Delta^* \mathcal{F}^3) \\
 \downarrow 1 \otimes \bar{m} & & \downarrow \widetilde{1 \times m} & & \downarrow \widetilde{1 \times m} & & \downarrow \bar{m} & & \downarrow \bar{m} \\
 H_* \otimes H_*(X^3, p^* \mathcal{F}^2) & \xrightarrow{K} & H_*(X^5, \mathcal{F} \times p^* \mathcal{F}^2) & \xrightarrow{\Delta_{L,!}} & H_*(X^4, p_3^* D^* \mathcal{F}^2) & \xrightarrow{\bar{m}} & H_*(X^4, p_{1,4}^* \mathcal{F}^2) & \xrightarrow{p_{2,*}} & H_*(X^3, p^* \mathcal{F}) \\
 \downarrow 1 \otimes p_* & & \downarrow (1 \times p)_* & & \downarrow p_{3,*} & & \downarrow p_{3,*} & & \downarrow p_* \\
 H_*^{\otimes 2} & \xrightarrow{K} & H_*(X^4, \mathcal{F}^2) & \xrightarrow{D_1} & H_*(X^3, D^* \mathcal{F}^2) & \xrightarrow{\bar{m}} & H_*(X^3, p^* C_*(\Omega Y)) & \xrightarrow{p_*} & H_*
 \end{array}$$

where the embeddings $\Delta_R, \Delta_L : X^4 \rightarrow X^5$ and $\Delta_2 := (D \times 1) \circ \Delta_R : X^4 \rightarrow X^6$ are defined by

$$\Delta_L = \text{Id} \times \Delta \times \text{Id}_{X^2}, \quad \Delta_R = \text{Id}_{X^2} \times \Delta \times \text{Id} \quad \text{and} \quad \Delta_2 = \text{Id} \times \Delta \times \Delta \times \text{Id},$$

and the projections $p_3, p_2 : X^4 \rightarrow X^3$ and $p_2 : X^4 \rightarrow X^2$ are defined by

$$p_3(a, b, c, d) = (a, b, d), \quad p_2(a, b, c, d) = (a, c, d) \quad \text{and} \quad p_{1,4} := p \circ p_2 = p \circ p_3.$$

The square (4) commutes. — This is a consequence of Lemma 3.2. Indeed, since $D \circ p_2 = (p \times 1) \circ \Delta_R : X^4 \rightarrow X^4$, $(a, b, c, d) \mapsto (a, c, c, d)$ and we have the equality

$$(p \times 1)^{-1}(D(X^3)) = \{(a, b, c, c, d) \in X^5\} = \Delta_R(X^4).$$

We now compare the orientations of the normal bundle N_{Δ_R} of $\Delta_R(X^4) \subset X^5$ with the normal bundle N_D of $D(X^3) \subset X^4$. Choose the orientation

$\text{Or } N_\Delta = \text{Or } X$ of the normal bundle N_Δ of $\Delta(X) \subset X^2$. Following [4, Remark 1.1], this induces

$$(\text{Or } T\Delta(X), \text{Or } N_\Delta) = \text{Or } T(X \times X)|_{\Delta(X)}.$$

Therefore

$$\begin{aligned} (\text{Or } T\Delta_{R,*}(X^4), \text{Or } N_\Delta) \\ &= (\text{Or } TX, \text{Or } TX, \text{Or } T\Delta(X), \text{Or } TX, \text{Or } N_\Delta) \\ &= (-1)^n (\text{Or } TX, \text{Or } TX, \text{Or } T(X \times X)|_{\Delta(X)}, \text{Or } TX) \\ &= (-1)^n \text{Or } T(X^5)|_{\Delta_R(X^4)}. \end{aligned}$$

and it follows that $\text{Or } N_{\Delta_R} = (-1)^n \text{Or } N_\Delta = (-1)^n \text{Or } X$. We prove using the same arguments that $\text{Or } N_D = (-1)^n \text{Or } X = \text{Or } N_{\Delta_R}$ and that the mentioned square commutes.

The square (3) commutes up to the sign $(-1)^n$. — We use the same techniques as for square (4). We first check that $(1 \times p) \circ \Delta_L = D \circ p_3 : X^4 \rightarrow X^4$, $(a, b, c, d) \mapsto (a, b, b, d)$ and

$$(1 \times p)^{-1}(D(X^3)) = \{(a, b, b, c, d) \in X^5\} = \Delta_L(X^4).$$

The normal bundles $N_{\Delta_L} \cong TX \cong N_D$ are isomorphic and their orientations differ by $(-1)^n$.

The squares (1)–(2) commute up to the sign $(-1)^{nk}$ and $(-1)^{ni}$ respectively. — This is a direct consequence of [5, Lemma 6.17].

All the other squares are commutative by compatibility properties. Therefore,

$$\begin{aligned} p_* \circ \tilde{m} \circ \Delta_! \circ K \circ (p_* \otimes 1) \circ (\tilde{m} \otimes 1) \circ (\Delta_! \otimes 1) \circ (K \otimes 1) \\ &= (-1)^{n(n-k)} (-1)^{ni} p_* \circ \tilde{m} \circ \Delta_! \circ K \circ (1 \otimes p_*) \\ &\quad \circ (1 \otimes \tilde{m}) \circ (1 \otimes \Delta_!) \circ (1 \otimes K). \end{aligned}$$

Let $\gamma \in H_i(X^2, C_*(\Omega Y))$, $\tau \in H_j(X^2, C_*(\Omega Y))$ and $\delta \in H_k(X^2, C_*(\Omega Y))$.

$$\begin{aligned} \text{PP}_{DG}(\gamma \otimes \text{PP}_{DG}(\tau \otimes \delta)) \\ &= (-1)^{n(n-k)} (-1)^{n(n-j-k+n)} (-1)^{ni} p_* \circ \tilde{m} \circ \Delta_! \circ K \\ &\quad \circ (1 \otimes p_*) \circ (1 \otimes \tilde{m}) \circ (1 \otimes \Delta_!) \circ (1 \otimes K)(\gamma \otimes \tau \otimes \delta) \end{aligned}$$

and

$$\begin{aligned} & \text{PP}_{DG}(\text{PP}_{DG}(\gamma \otimes \tau) \otimes \delta) \\ &= (-1)^{n(n-k)} (-1)^{n(n-j)} p_* \circ \tilde{m} \circ \Delta_! \circ K \circ (p_* \otimes 1) \\ & \quad \circ (\tilde{m} \otimes 1) \circ (\Delta_! \otimes 1) \circ (K \otimes 1)(\gamma \otimes \tau \otimes \delta). \end{aligned}$$

We conclude

$$\text{PP}_{DG}(\gamma \otimes \text{PP}_{DG}(\tau \otimes \delta)) = \text{PP}_{DG}(\text{PP}_{DG}(\gamma \otimes \tau) \otimes \delta). \quad \square$$

Consider the inclusion of constant paths $s : X \rightarrow \mathcal{P}_{X,f} Y, s(a) = (a, a, c_{f(a)})$, where we denoted $c_{f(a)}$ the constant path at $f(a) \in Y$. This is a morphism of fibrations relative to $\Delta : X \rightarrow X^2$.

PROPOSITION 3.4. — *The product PP_{DG} admits the neutral element*

$$\Delta_* \tilde{s}([X]) \in H_n(X^2, C_*(\Omega Y))$$

denoted 1_{PP} .

Proof. — Let $J = \text{Id} \times \Delta : X^2 \rightarrow X^3$, $(a, b) \mapsto (a, b, b)$. Denote $\mathcal{F}^i = C_*(\Omega Y^i)$ and $\sigma : \mathcal{P}_{X,f} \text{ev}_1 \times_{\text{Id}} X \rightarrow \mathcal{P}_{X,f}$, the morphism of fibrations $\sigma(\gamma, \text{ev}_1(\gamma)) = \gamma$ over X^2 . Consider the following diagram:

$$\begin{array}{ccccc} H_*(X^2, \mathcal{F}) \otimes H_n(X, \mathbb{Z}) & \xrightarrow{1 \otimes \tilde{s}} & H_*(X^2, \mathcal{F}) \otimes H_n(X, \Delta^* \mathcal{F}) & \xrightarrow{1 \otimes \Delta_*} & H_*(X^2, \mathcal{F}) \otimes H_n(X^2, \mathcal{F}) \\ \downarrow K & & \downarrow K & & \downarrow K \\ H_{*+n}(X^3, C_*(\Omega Y \times \{\star\})) & \xrightarrow{\widetilde{1 \times s}} & H_{*+n}(X^3, (1 \times \Delta)^* \mathcal{F}^2) & \xrightarrow{(1 \times \Delta)_*} & H_{*+n}(X^4, \mathcal{F}^2) \\ \downarrow J_! & & \downarrow J_! & & \downarrow D_! \\ H_*(X^2, J^* C_*(\Omega Y \times \{\star\})) & \xrightarrow{J^* \widetilde{1 \times s}} & H_*(X^2, J^*(1 \times \Delta)^* \mathcal{F}^2) & \xrightarrow{J_*} & H_*(X^3, D^* \mathcal{F}^2) \\ & \searrow \tilde{\sigma} & \downarrow J^* \tilde{m} & & \downarrow \tilde{m} \\ & & H_*(X^2, \mathcal{F}) & \xrightarrow{J_*} & H_*(X^3, p^* \mathcal{F}) \\ & & & \searrow & \downarrow p_* \\ & & & & H_*(X^2, \mathcal{F}). \end{array}$$

Remark that $p_* \circ J_* = (p \circ J)_* = \text{Id}_* : H_*(X^2, \mathcal{F}) \rightarrow H_*(X^2, \mathcal{F})$. The map

$$\begin{aligned} m \circ (1 \times s) : \mathcal{P}_{X,f} Y \text{ev}_1 \times_{\text{Id}} X &\longrightarrow \mathcal{P}_{X,f} Y \text{ev}_1 \times_{\text{ev}} \Delta^* \mathcal{P}_{X,f} Y \longrightarrow \mathcal{P}_{X,f} Y \\ m \circ (1 \times s)(\gamma, \text{ev}_1(\gamma)) &= m(\gamma, c_{\text{ev}_1(\gamma)}) = \gamma \end{aligned}$$

is equal to σ , where we denoted $c_{\text{ev}_1(\gamma)}$ the constant path at $\text{ev}_1(\gamma)$. Therefore,

$$J^* \tilde{m} \circ J^* \widetilde{1 \times s} = \tilde{\sigma}.$$

Since,

$$(1 \times \Delta) \circ J = D \circ J : X^2 \longrightarrow X^4, (a, b) \longmapsto (a, b, b, b)$$

and

$$(1 \times \Delta)^{-1}(D(X^3)) = \{(a, b, b), (a, b) \in X^2\} = J(X^2),$$

it follows from Lemma 3.2 that

$$D_! \circ (1 \times \Delta)_* = (-1)^n J_* \circ J_!.$$

Indeed, using the same arguments as in the previous proof, we prove that $\text{Or } N_J = \text{Or } X$ and $\text{Or } N_D = (-1)^n \text{Or } X$. Therefore, this diagram commutes up to the sign $(-1)^n$.

Let $h : X \rightarrow \mathbb{R}$ be a Morse function with a unique local maximum $x_{\max} \in \text{Crit}_n(h)$ such that

$$x_{\max} = \star \otimes x_{\max} = [X] \in H_n(X, C_*(\{\star\})) = H_n(X, \mathbb{Z}).$$

Let ξ be a pseudo-gradient associated with h . It suffices to check that

$$\tilde{\sigma} \circ J_! \circ K(\cdot \otimes (\star \otimes x_{\max})) = \text{Id} : H_*(X^2, \mathcal{F}) \longrightarrow H_*(X^2, \mathcal{F}).$$

Let $v_0, v_1 : X \rightarrow \mathbb{R}$ be Morse functions and ξ_0, ξ_1 be associated pseudo-gradients. Denote $v = v_1 + v_2$ and $\xi_2 = \xi_0 + \xi_1$. Consider $\Xi_2 = (v, \xi_2, \dots)$ be a set of DG Morse data on X^2 and $\Xi_3 = (v + h, \xi_2 + \xi, \dots)$ be a set of Morse data on X^3 .

Let $\alpha \otimes (x, y) \in C_*(X^2, \Xi_2, C_*(\Omega Y))$ and consider

$$J_! : C_*(X^3, \Xi_3, C_*(\Omega Y \times \{\star\})) \longrightarrow C_{*-n}(X^2, \Xi_2, J^* C_*(\Omega Y \times \{\star\})).$$

Notice that

$$\begin{aligned} \overline{\mathcal{M}}^{J_!}((x, y, x_{\max}), (z, w)) &= \overline{W}^s(z, w) \cap J^{-1}\left(\overline{W}^u(x, y, x_{\max})\right) \\ &= \overline{W}^s(z, w) \cap \overline{W}^u(x, y) \\ &= \overline{\mathcal{M}}^{\text{Id}_!}((x, y), (z, w)). \end{aligned}$$

Since $\text{Or } \overline{W}^s(x, y, x_{\max}) = \text{Or } \overline{W}^s(x, y)$, these manifolds have the same orientation

$$\left(\text{Or } \overline{W}^s(x, y), \text{Or } \overline{\mathcal{M}}^{\text{Id}_!}((x, y), (z, w)) \right) = \text{Or } \overline{W}^s(z, w)$$

and

$$\left(\text{Or } \overline{W}^s(x, y), \text{Or } \overline{\mathcal{M}}^{J_!}((x, y, x_{\max}), (z, w)) \right) = \text{Or } \overline{W}^s(z, w).$$

Hence, $J_!((\alpha, \star) \otimes (x, y, x_{\max})) = \text{Id}_!((\alpha, \star) \otimes (x, y))$.

Since $\sigma_* : J^*C_*(\Omega Y \times \{\star\}) \rightarrow C_*(\Omega Y)$ is a morphism of $C_*(\Omega X^2)$ -modules, $\tilde{\sigma}((\alpha, \star) \otimes (x, y)) = \alpha \otimes (x, y)$, then,

$$\tilde{\sigma} \circ J_! \circ K(\cdot \otimes (\star \otimes x_{\max})) = \text{Id} : H_*(X^2, C_*(\Omega Y)) \longrightarrow H_*(X^2, C_*(\Omega Y)).$$

It follows that

$$\text{PP}_{DG}(\tau \otimes \Delta_* \tilde{s}(\star \otimes x_{\max})) = (-1)^{n^2} (-1)^n \tilde{\sigma} \circ J_! \circ K(\tau \otimes (\star \otimes x_{\max})) = \tau$$

for all $\tau \in H_*(X^2, C_*(\Omega Y))$. With similar arguments, we prove that

$$\text{PP}_{DG}(\Delta_* \tilde{s}(\star \otimes x_{\max}) \otimes \tau) = \tau$$

for all $\tau \in H_*(X^2, C_*(\Omega Y))$. $\square$

We conclude by proving that this product computes the product defined by Stegemeyer in [6].

PROPOSITION 3.5. — *The products*

$$\Lambda : H_i(\mathcal{P}_{X,f}Y) \otimes H_j(\mathcal{P}_{X,f}Y) \longrightarrow H_{i+j-n}(\mathcal{P}_{X,f}Y)$$

and

$$\text{PP}_{DG} : H_i(X^2, C_*(\Omega Y)) \otimes H_j(X^2, C_*(\Omega Y)) \longrightarrow H_{i+j-n}(X^2, C_*(\Omega Y))$$

correspond via the Fibration Theorem 1.1.

Proof. — This statement amounts to the commutativity of the following diagram.

$$\begin{array}{ccc} H_i(\mathcal{P}_{X,f}Y) \otimes H_j(\mathcal{P}_{X,f}Y) & \xleftarrow{\sim \Psi \otimes \Psi} & H_i(X^2, C_*(\Omega Y)) \otimes H_j(X^2, C_*(\Omega Y)) \\ \downarrow (-1)^{n-ni} \times & & \downarrow (-1)^{nj} K \\ H_{i+j}(\mathcal{P}_{X,f}Y \times \mathcal{P}_{X,f}Y) & \xleftarrow{\sim} & H_{i+j}(X^4, C_*(\Omega Y^2)) \\ \downarrow R_{Cf,*} \circ \tau_{Cf} \cap & & \downarrow D_! \\ H_{i+j-n}(\mathcal{P}_{X,f}Y \times_{\text{ev}_1} \times_{\text{ev}_0} \mathcal{P}_{X,f}Y) & \xleftarrow{\sim} & H_{i+j-n}(X^3, D^*C_*(\Omega Y^2)) \\ \downarrow m_* & & \downarrow \tilde{m} \\ & & H_{i+j-n}(X^3, p^*C_*(\Omega Y)) \\ & & \downarrow p_* \\ H_{i+j-n}(\mathcal{P}_{X,f}Y) & \xleftarrow{\sim} & H_{i+j-n}(X^2, C_*(\Omega Y)). \end{array}$$

We know from Theorem 2.3 that the bottom square commutes. The top square commutes up to the sign $(-1)^{n-n(i+j)}$ and we now prove that the second square commutes up to the same sign.

For this purpose, we will now compare our orientation conventions to the one used in [4, 6].

Denote $U_\Delta \subset X^2$ a tubular neighborhood of the diagonal ΔX in X^2 and denote $\tau_\Delta \in C^n(U_\Delta, U_\Delta \setminus \Delta X)$ a Thom class of the normal bundle $N_\Delta \cong TX$ such that

$$\tau_\Delta \cap [X^2] = \Delta_*([X]).$$

This yields the “singular” orientation

$$(\text{Or } N_\Delta, \text{Or } T(\Delta X)) = \text{Or } T(X \times X)|_{\Delta X}$$

which induces the orientation $\text{Or } N_\Delta = (-1)^n \text{Or } X$.

Denote

$$U_{C^f} = (\text{ev}_1 \times \text{ev}_0)^{-1} U_\Delta \subset \mathcal{P}_{X,f} Y \times \mathcal{P}_{X,f} Y,$$

and define the Thom class

$$\tau_{C^f} = (\text{ev}_1 \times \text{ev}_0)^*[\tau_\Delta] \in H^n(U_{C^f}, U_{C^f} \setminus (\text{ev}_1 \times \text{ev}_0)^{-1}(\Delta X)).$$

Consider $p_{2,3} : X^4 \rightarrow X^2$, $p_{2,3}(a, b, c, d) = (b, c)$ and $U_D = p_{2,3}^{-1}(U_\Delta)$ a tubular neighborhood of $D(X^3) \subset X^4$. It is proven in [5, Theorem 3.2] that the diagram

$$\begin{array}{ccc} H_{i+j}(\mathcal{P}_{X,f} Y \times \mathcal{P}_{X,f} Y) & \xleftarrow{\sim} & H_{i+j}(X^4, C_*(\Omega Y^2)) \\ \downarrow R_{C^f, *} \circ (\text{ev} \times \text{ev})^* \tau_D^M \cap & & \downarrow D_! \\ H_{i+j-n}(\mathcal{P}_{X,f} Y \times_{\text{ev}_1 \times \text{ev}_0} \mathcal{P}_{X,f} Y) & \xleftarrow{\sim} & H_{i+j-n}(X^3, D^* C_*(\Omega Y^2)) \end{array}$$

commutes up to the sign $(-1)^{n(i+j)+n}$ where $[\tau_D^M] \in H^n(U_D, U_D \setminus D(X^3))$ is the Thom class associated with the normal bundle $N_D \cong TX$ of the tubular neighborhood U_D with the “Morse” orientation convention

$$(\text{Or } D(X^3), \text{Or } N_D) = \text{Or } X^4.$$

This yields the following equation.

$$[\tau_D^M] \cap [X^4] = (-1)^n D_*([X^3]).$$

It remains to prove that

$$\begin{aligned} (\text{ev} \times \text{ev})^*[\tau_D^M] &= (\text{ev}_1 \times \text{ev}_0)^*[\tau_\Delta] \\ &= [\tau_{C^f}] \in H^n(U_{C^f}, U_{C^f} \setminus (\text{ev} \times \text{ev})^{-1}(DX^3)). \end{aligned}$$

Consider $[\tau_D] = p_{2,3}^*[\tau_\Delta] \in H^n(U_D, U_D \setminus DX^3)$. Since $p_{2,3} \circ (\text{ev} \times \text{ev}) = \text{ev}_1 \times \text{ev}_0 : \mathcal{P}_{X,f} Y \times \mathcal{P}_{X,f} Y \rightarrow X^2$, it follows

$$(\text{ev} \times \text{ev})^*[\tau_D] = (\text{ev}_1 \times \text{ev}_0)^*[\tau_\Delta] = [\tau_{C^f}].$$

We now prove that $\tau_D = \tau_D^M$ by showing that

$$[\tau_D] \cap [X^4] = (-1)^n D_*([X^3]).$$

Let $\sigma_1 \in C^n(X)$ be a representative of the fundamental class $[X]$ and $\tau_D \in C^n(U_D, U_D \setminus DX^3)$ be a representative of $[\tau_D]$. We will denote $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4$.

$$\begin{aligned} & \tau_D \cap (\sigma_1 \times \sigma_2 \times \sigma_3 \times \sigma_4) \\ &= \sum_{\substack{J \sqcup K = \{1, \dots, 4n\} \\ |J|=n}} (-1)^{\text{sgnP}(J, K)} \tau_D(\sigma_1 \times \sigma_2 \times \sigma_3 \times \sigma_4)|_{I^J \times \{0\}^K} \\ & \quad (\sigma_1 \times \sigma_2 \times \sigma_3 \times \sigma_4)|_{\{1\}^J \times I^K} \\ &\stackrel{(1)}{=} \sum_{\substack{J \sqcup K = \{1, \dots, 4n\} \\ J \subset \{n+1, \dots, 3n\} \\ |J|=n}} (-1)^{\text{sgnP}(J, K)} \tau_\Delta(\sigma_2 \times \sigma_3)|_{I^J \times \{0\}^K} \\ & \quad (\sigma_1 \times \sigma_2 \times \sigma_3 \times \sigma_4)|_{\{1\}^J \times I^K} \\ &\stackrel{(2)}{=} \sum_{\substack{J' \sqcup K' = \{1, \dots, 2n\} \\ |J'|=n}} (-1)^{\text{sgnP}(J', K')} \tau_\Delta(\sigma_2 \times \sigma_3)|_{I^{J'} \times \{0\}^{K'}} \\ & \quad (\sigma_1 \times (\sigma_2 \times \sigma_3))|_{\{1\}^{J'} \times I^{K'} \times \sigma_4} \\ &= (\text{Id} \times \tau_\Delta \times \text{Id}) \cap (\sigma_1 \times \sigma_2 \times \sigma_3 \times \sigma_4). \end{aligned}$$

The equality (1) is true since the chain $p_{2,3,*}(\sigma_1 \times \sigma_2 \times \sigma_3 \times \sigma_4)|_{I^J \times \{0\}^K} = (\sigma_2 \times \sigma_3)|_{I^J \times \{0\}^K}$ is degenerate and therefore equal to 0 if $J \not\subset \{n+1, \dots, 3n\}$.

The equality (2) comes from the fact that the transformation $J \xrightarrow{\nu} J'$ is performed by the permutation

$$\nu = \begin{pmatrix} 0 & 0 & I_n & 0 \\ I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix}$$

that has signature 1. Since $J \subset \{n+1, \dots, 3n\}$, then $\nu J \subset \{1, \dots, 2n\}$ and we consider K' to be the complement of $J' = \nu J$ in $\{1, \dots, 2n\}$. It follows that

$$\begin{aligned} [\tau_D] \cap [X^4] &= [\text{Id} \times \tau_\Delta \times \text{Id}] \cap [X^4] \\ &= (-1)^n (\text{Id} \times \Delta \times \text{Id})_*[X^3] = (-1)^n D_*[X^3]. \end{aligned}$$

Therefore $[\tau_D] = [\tau_D^M]$ and $(\text{ev} \times \text{ev})^*[\tau_D^M] = [\tau_{Cf}]$. $\square$

4. Compatibility with the Chas–Sullivan product

In this section we suppose that Y^k is a closed, oriented, connected manifold of dimension k and we endow $H_*(Y, C_*(\Omega Y))$ with the ring structure

induced by the product

$$\mathrm{CS}_{DG} : H_i(Y, C_*(\Omega Y)) \otimes H_j(Y, C_*(\Omega Y)) \longrightarrow H_{i+j-n}(Y, C_*(\Omega Y)).$$

Let $f : X \rightarrow Y$ be a continuous map and let $I := I_f : X^2 \rightarrow Y \times X^2$, $I(x, x') = (f(x), x, x')$. Define

$$\begin{aligned} \cdot_M : H_i(Y, C_*(\Omega Y)) \otimes H_j(X^2, C_*(\Omega Y)) \\ \xrightarrow{(-1)^{kj}K} H_{i+j-k}(Y \times X^2, C_*(\Omega Y^2)) \\ \xrightarrow{I_!} H_{i+j-k}(X^2, I^*C_*(\Omega Y^2)) \\ \xrightarrow{\tilde{m}} H_{i+j-k}(X^2, C_*(\Omega Y)), \end{aligned}$$

where $m : \mathcal{L}Y_{\mathrm{ev} \times \mathrm{ev}0} \mathcal{P}_{X,f}Y \rightarrow \mathcal{P}_{X,f}Y$ is the morphism of fibrations over X^2 induced by the concatenation.

PROPOSITION 4.1. — *The pairing*

$$\cdot_M : H_i(Y, C_*(\Omega Y)) \otimes H_j(X^2, C_*(\Omega Y)) \longrightarrow H_{i+j}(X^2, C_*(\Omega Y))$$

induces a left $H_*(Y, C_*(\Omega Y))$ -module structure on $H_*(X^2, C_*(\Omega Y))$ with the additional property that if $\gamma \in H_*(Y, C_*(\Omega Y))$ and $\tau, \tau' \in H_*(X^2, C_*(\Omega Y))$, then

$$\gamma \cdot_M \mathrm{PP}_{DG}(\tau \otimes \tau') = \mathrm{PP}_{DG}((\gamma \cdot_M \tau) \otimes \tau').$$

COROLLARY 4.2. — *This module structure induces a morphism of rings*

$$\begin{aligned} H_i(Y, C_*(\Omega Y)) &\longrightarrow H_{i+n-k}(X^2, C_*(\Omega Y)) \\ \gamma &\longmapsto \gamma \cdot_M 1_{PP}. \end{aligned}$$

Proof of Proposition 4.1. — We now prove that $\cdot_M$ is indeed a module structure, i.e. for all $\gamma, \gamma' \in H_*(Y, \Omega Y)$ and $\tau \in H_*(X^2, \Omega Y)$

$$\mathrm{CS}_{DG}(\gamma \otimes \gamma') \cdot_M \tau = \gamma \cdot_M (\gamma' \cdot_M \tau).$$

Denote $\mathcal{F}^i = C_*(\Omega Y^i)$ for any $i \in \mathbb{N}^*$ and $\mathcal{F} = \mathcal{F}^1 = C_*(\Omega Y)$. Let $\gamma, \gamma' \in H_*(Y, \mathcal{F})$ and $\tau \in H_*(X^2, \mathcal{F})$.

$$\begin{aligned} \mathrm{CS}_{DG}(\gamma \otimes \gamma') \cdot_M \tau &= (-1)^{k+k|\gamma'|} (-1)^{k|\tau|} \tilde{m} \circ I_! \circ K \circ (\tilde{m} \otimes \mathrm{Id}) \\ &\quad \circ (\Delta_! \otimes \mathrm{Id}) \circ (K \otimes \mathrm{Id})(\gamma \otimes \gamma' \otimes \tau). \end{aligned}$$

$$\begin{aligned} \gamma \cdot_M (\gamma' \cdot_M \tau) &= (-1)^{k|\tau|} (-1)^{k(|\gamma'|+|\tau|-k)} (-1)^{k|\gamma|} \tilde{m} \circ I_! \circ K \circ (\mathrm{Id} \otimes \tilde{m}) \\ &\quad \circ (\mathrm{Id} \otimes I_!) \circ (\mathrm{Id} \otimes K)(\gamma \otimes \gamma' \otimes \tau). \end{aligned}$$

The equality $\text{CS}_{DG}(\gamma \otimes \gamma') \cdot_M \tau = \gamma \cdot_M (\gamma' \cdot_M \tau)$ is hence equivalent to the commutativity of the following diagram up to the sign $(-1)^{k|\tau|+k|\gamma|}$. For readability, we denote $H_* := H_*(X^2, \mathcal{F})$ in the following diagrams.

$$\begin{array}{ccccccc}
 H_*(Y, \mathcal{F})^{\otimes 2} \otimes H_* & \xrightarrow{K \otimes 1} & H_*(Y^2, \mathcal{F}^2) \otimes H_* & \xrightarrow{\Delta_! \otimes 1} & H_*(Y, \Delta^* \mathcal{F}^2) \otimes H_* & \xrightarrow{\tilde{m} \otimes 1} & H_*(Y, \mathcal{F}) \otimes H_* \\
 \downarrow 1 \otimes K & & \downarrow K & & \downarrow K & & \downarrow K \\
 H_*(Y, \mathcal{F}) \otimes H_*(Y \times X^2, \mathcal{F}^2) & \xrightarrow{K} & H_*(Y^2 \times X^2, \mathcal{F}^3) & \xrightarrow{(\Delta \times 1)_!} & H_*(Y \times X^2, (\Delta \times 1)^* \mathcal{F}^3) & \xrightarrow{\widetilde{m \times 1}} & H_*(Y \times X^2, \mathcal{F}^2) \\
 \downarrow 1 \otimes I_! & & \downarrow (1 \times I)_! & & \downarrow I_! & & \downarrow I_! \\
 H_*(Y, \mathcal{F}) \otimes H_*(X^2, I^* \mathcal{F}^2) & \xrightarrow{K} & H_*(Y \times X^2, (1 \times I)^* \mathcal{F}^3) & \xrightarrow{I_!} & H_*(X^2, I^* (\Delta \times 1)^* \mathcal{F}^3) & \xrightarrow{I^* \widetilde{m \times 1}} & H_*(X^2, I^* \mathcal{F}^2) \\
 \downarrow 1 \otimes \tilde{m} & & \downarrow \widetilde{1 \times m} & & \downarrow I^* \widetilde{1 \times m} & & \downarrow \tilde{m} \\
 H_*(Y, \mathcal{F}) \otimes H_* & \xrightarrow{K} & H_*(Y \times X^2, \mathcal{F}^2) & \xrightarrow{I_!} & H_*(X^2, I^* \mathcal{F}^2) & \xrightarrow{\tilde{m}} & H_*
 \end{array}$$

Indeed, it is shown in [5, Lemma 6.17] the squares (1) and (2) commute up to the sign $(-1)^{k|\tau|}$ and $(-1)^{k|\gamma|}$ respectively. We prove that

$$\gamma \cdot_M \text{PP}_{DG}(\tau \otimes \tau') = \text{PP}_{DG}((\gamma \cdot_M \tau) \otimes \tau')$$

using similar arguments. This equality is equivalent to the commutativity of the following diagram up to the sign $(-1)^{k|\tau'|}(-1)^{kn}(-1)^{n|\gamma|}$.

$$\begin{array}{ccccccc}
 H_*(Y, \mathcal{F}) \otimes H_*^{\otimes 2} & \xrightarrow{K \otimes 1} & H_*(Y \times X^2, \mathcal{F}^2) \otimes H_* & \xrightarrow{I_! \otimes 1} & H_*(X^2, I^* \mathcal{F}^2) \otimes H_* & \xrightarrow{\tilde{m} \otimes 1} & H_*^{\otimes 2} \\
 \downarrow 1 \otimes K & & \downarrow K & & \downarrow K & & \downarrow K \\
 H_*(Y, \mathcal{F}) \otimes H_*(X^4, \mathcal{F}^2) & \xrightarrow{K} & H_*(Y \times X^4, \mathcal{F}^3) & \xrightarrow{(I \times 1)_!} & H_*(X^4, (I \times 1)^* \mathcal{F}^3) & \xrightarrow{\widetilde{m \times 1}} & H_*(X^4, \mathcal{F}^2) \\
 \downarrow 1 \otimes D_! & & \downarrow (1 \times D)_! & & \downarrow D_! & & \downarrow D_! \\
 H_*(Y, \mathcal{F}) \otimes H_*(X^3, D^* \mathcal{F}^2) & \xrightarrow{K} & H_*(Y \times X^3, (1 \times D)^* \mathcal{F}^3) & \xrightarrow{I_!} & H_*(X^3, (I \times 1)^* D^* \mathcal{F}^3) & \xrightarrow{D^* \widetilde{m \times 1}} & H_*(X^3, D^* \mathcal{F}^2) \\
 \downarrow 1 \otimes \tilde{m} & & \downarrow \widetilde{1 \times m} & & \downarrow \widetilde{1 \times m} & & \downarrow \tilde{m} \\
 H_*(Y, \mathcal{F}) \otimes H_*(X^3, p^* \mathcal{F}) & \xrightarrow{K} & H_*(Y \times X^3, (1 \times p)^* \mathcal{F}^3) & \xrightarrow{(I \times 1)_!} & H_*(X^3, p^* I^* \mathcal{F}^2) & \xrightarrow{p^* \tilde{m}} & H_*(X^3, p^* \mathcal{F}) \\
 \downarrow 1 \otimes p_* & & \downarrow (1 \times p)_* & & \downarrow p_* & & \downarrow p_* \\
 H_*(Y, \mathcal{F}) \otimes H_* & \xrightarrow{K} & H_*(Y \times X^2, \mathcal{F}^2) & \xrightarrow{I_!} & H_*(X^2, I^* \mathcal{F}^2) & \xrightarrow{\tilde{m}} & H_*
 \end{array}$$

It is shown in [5, Lemma 6.17] that the squares (1) and (2) commute up to the sign $(-1)^{k|\tau'|}$ and $(-1)^{k|\gamma|}$ respectively. We now prove that square (3) commutes up to the sign $(-1)^{kn}$ using Lemma 3.2.

First, we check that $(\text{Id} \times p) \circ (I \times \text{Id}) = I \circ p : X^3 \rightarrow Y \times X^2$, $(a, b, c) \mapsto (f(a), a, c)$ and

$$(\text{Id} \times p)^{-1}(I(X^2)) = \{(f(a), a, b, c), (a, b, c) \in X^3\} = (I \times \text{Id})(X^3).$$

The normal bundles $N_I \cong TY \cong N_{I \times \text{Id}}$ are isomorphic and their orientations are given by the rules

$$(\text{Or } X^2, \text{Or } N_I) = (\text{Or } Y, \text{Or } X^2)$$

and

$$(\text{Or } X^3, \text{Or } N_{I \times \text{Id}}) = (\text{Or } Y, \text{Or } X^3).$$

We compute

$$\begin{aligned} (\text{Or } X^3, \text{Or } N_I) &= (-1)^{nk} (\text{Or } X^2, \text{Or } N_I, \text{Or } X) \\ &= (-1)^{nk} (\text{Or } Y, \text{Or } X^2, \text{Or } X) \\ &= (-1)^{nk} (\text{Or } Y, \text{Or } X^3). \end{aligned}$$

Therefore $\text{Or } N_I = (-1)^{nk} \text{Or } N_{I \times \text{Id}}$. It follows that the square (3) commutes up to the sign $(-1)^{nk}$. $\square$

5. Homotopy invariance

We prove here that the ring and module structure on $H_*(X^2, C_*(\Omega Y))$ only depends on the homotopy type of $f : X \rightarrow Y$. More precisely, if $g : X \rightarrow Y$ is a continuous map homotopic to f , there exists a chain homotopy equivalence

$$\underbrace{C_*(X^2, \Xi, (f \times f)^* C_*(\Omega Y))}_{\simeq C_*(\mathcal{P}_{X,f} Y)} \longrightarrow \underbrace{C_*(X^2, \Xi, (g \times g)^* C_*(\Omega Y))}_{\simeq C_*(\mathcal{P}_{X,g} Y)}$$

that induces an isomorphism of rings and modules at the homology level. We recall the notations

$$C_*(X, \Xi, C_*(\Omega Y); f) := C_*(X^2, \Xi, (f \times f)^* C_*(\Omega Y))$$

and

$$C_*(X, \Xi, C_*(\Omega Y); g) := C_*(X^2, \Xi, (g \times g)^* C_*(\Omega Y)).$$

We will also prove that if $X_1 \subset Y_1$ and $X_2 \subset Y_2$ are manifolds included in topological spaces Y_1, Y_2 such that there exists a homotopy equivalence of pairs $\varphi : (Y_1, X_1) \rightarrow (Y_2, X_2)$, then there exists an isomorphism of rings

$$H_*((X_1)^2, C_*(\Omega Y_1)) \xrightarrow{\sim} H_*((X_2)^2, C_*(\Omega Y_2)).$$

5.1. DG Identification morphism in the case of fibrations

Let $(X, \star)$ and $(Y, \star_Y)$ be pointed, oriented, closed, and connected manifolds. Let $F \hookrightarrow E \xrightarrow{\pi} Y$ be a fibration and $\Phi : E \times_{ev_0} \mathcal{P}Y \rightarrow E$ be a transitive lifting function associated with this fibration. Let $\varphi_0, \varphi_1 : X \rightarrow Y$ be two homotopic maps such that $\varphi_0(\star) = \varphi_1(\star) = \star_Y$. Let $\varphi : X \times [0, 1] \rightarrow Y$ be a homotopy between $\varphi_0 = \varphi(\cdot, 0)$ and $\varphi_1 = \varphi(\cdot, 1)$. Denote

$$\begin{aligned} X &\longrightarrow \mathcal{P}Y \\ \eta_\varphi : x &\longmapsto (s \mapsto \varphi(x, s)) \end{aligned}$$

the map that sends $x \in X$ to the path going from $\varphi_0(x)$ to $\varphi_1(x)$ defined by the homotopy φ .

PROPOSITION 5.1. — *Let $\Xi = (f, \xi, \dots)$ be a set of DG Morse data on X . The morphism of fibrations*

$$\Psi^\varphi : \begin{array}{l} \varphi_0^* E \longrightarrow \varphi_1^* E \\ (x, e) \longmapsto (x, \Phi(e, \eta_\varphi(x))) \end{array}$$

induces a chain homotopy equivalence

$$\tilde{\Psi}^\varphi : C_*(X, \Xi, \varphi_0^* C_*(F)) \longrightarrow C_*(X, \Xi, \varphi_1^* C_*(F))$$

that satisfies the same properties as the Identification morphism defined in [2, Proposition 8.2.1]. More precisely, it holds that:

- (1) *If $\varphi = \text{Id}$ is the constant homotopy at φ_0 , then $\tilde{\Psi}^{\text{Id}} = \text{Id}$.*
- (2) *If two homotopies φ and φ' are homotopic with fixed endpoints, then $\tilde{\Psi}^\varphi$ and $\tilde{\Psi}^{\varphi'}$ are chain homotopic.*
- (3) *If φ_{01} is a homotopy between φ_0 and φ_1 and φ_{12} is a homotopy between φ_1 and φ_2 , then denoting φ_{02} the concatenation of φ_{01} and φ_{12} we have that $\tilde{\Psi}^{\varphi_{12}} \circ \tilde{\Psi}^{\varphi_{01}}$ and $\tilde{\Psi}^{\varphi_{02}}$ are homotopic. Therefore,*

$$\tilde{\Psi}^{\varphi_{12}} \circ \tilde{\Psi}^{\varphi_{01}} = \tilde{\Psi}^{\varphi_{02}}$$

in homology. In particular $\tilde{\Psi}^\varphi$ is always a chain homotopy equivalence of chain homotopy inverse $\tilde{\Psi}^{\bar{\varphi}}$.

- (4) *The maps $\tilde{\Psi}^\varphi \circ \varphi_{0,!}$ and $\varphi_{1,!}$ are homotopic.*
- (5) *The maps $\varphi_{1,*} \circ \tilde{\Psi}^\varphi$ and $\varphi_{0,*}$ are homotopic.*

Proof of Proposition 5.1.

Proof of (1). — If $b = \eta_{\text{Id}}(x)$ is the constant path at $\varphi_0(x) = \pi(e)$, then $\Phi(e, b) = e$. It follows that $\Psi^{\text{Id}} = \text{Id}$ and $\tilde{\Psi}^{\text{Id}} = \text{Id}$.

Proof of (2). — Let $H : X \times [0, 1]^2 \rightarrow Y$ be a homotopy with fixed endpoints between $H(\cdot, \cdot, 0) = \varphi$ and $H(\cdot, \cdot, 1) = \varphi'$. For $s \in [0, 1]$, denote $H(\cdot, \cdot, s) =: H_s$. Consider the family of morphism of fibrations

$$\Psi^{H_s} : \begin{array}{l} \varphi_0^* E \longrightarrow \varphi_1^* E \\ (x, e) \longmapsto (x, \Phi(e, \eta_{H_s}(x))) \end{array}$$

for $s \in [0, 1]$. This is well-defined since $H_s(\cdot, 0) = \varphi_0 = \varphi'_0$ and $H_s(\cdot, 1) = \varphi_1 = \varphi'_1$ for all $s \in [0, 1]$. For each $s \in [0, 1]$, the morphism of fibrations Ψ^{H_s}

induces the morphism of complexes

$$\begin{aligned} \tilde{\Psi}^{H_s} &= \sum_n \left(\Psi_{n+1}^{H_s} \otimes 1 \right) \tilde{\mathbf{m}}^n : C_k(X, \Xi, \varphi_0^* C_*(F)) \\ &\longrightarrow \bigoplus_{i+j=k} C_i(F) \otimes \mathbb{Z} \text{Crit}_j(f). \end{aligned}$$

Therefore, the map

$$\begin{aligned} \tilde{\Psi}^H : C_k(X, \Xi, \varphi_0^* C_*(F)) &\longrightarrow \bigoplus_{i+j=k} C_{i+1}(F) \otimes \mathbb{Z} \text{Crit}_j(f) \\ \alpha \otimes x &\longmapsto s \mapsto \sum_n \left(\Psi_{n+1}^{H_s} \otimes 1 \right) \tilde{\mathbf{m}}^n(\alpha \otimes x) \end{aligned}$$

satisfies

$$\partial \tilde{\Psi}^H + \tilde{\Psi}^H \partial = \tilde{\Psi}^{\varphi'} - \tilde{\Psi}^{\varphi}.$$

Proof of (3): This is a direct application of [5, Corollary 5.18]. Indeed, since

$$\begin{aligned} \Psi^{\varphi_{12}} \circ \Psi^{\varphi_{01}}(x, e) &= (x, \Phi(\Phi(e, \eta_{\varphi_{01}}(x)), \eta_{\varphi_{12}}(x))) \\ &= (x, \Phi(e, \eta_{\varphi_{01}}(x) \# \eta_{\varphi_{12}}(x))) \\ &= (x, \Phi(e, \eta_{\varphi_{02}}(x))) \\ &= \Psi^{\varphi_{02}}(x, e), \end{aligned}$$

it follows that $\tilde{\Psi}^{\varphi_{12}} \circ \tilde{\Psi}^{\varphi_{01}} = \tilde{\Psi}^{\varphi_{02}}$.

Proof of (4). — For $s \in [0, 1]$, define

$$\begin{aligned} \Psi^{s1} : \varphi_s^* E &\longrightarrow \varphi_1^* E \\ (x, e) &\longmapsto (x, \Phi(e, \eta_{\varphi}(x)|_{[s, 1]})) \end{aligned}$$

and consider

$$\begin{aligned} \kappa : C_*(Y, \Xi_Y, C_*(F)) &\longrightarrow C_{*-k+1}(X, \Xi, \varphi_1^* C_*(F)) \\ \alpha \otimes x &\longmapsto s \mapsto \tilde{\Psi}^{s1} \circ \varphi_{s,!} \end{aligned}$$

Then,

$$\partial \kappa + \kappa \partial = \varphi_{1,!} - \tilde{\Psi}^{\varphi} \circ \varphi_{0,!}.$$

Proof of (5). — It's the same considering

$$\begin{aligned} \kappa' : C_*(X, \Xi, \varphi_1^* C_*(F)) &\longrightarrow C_{*+1}(Y, \Xi_Y, C_*(F)) \\ \beta \otimes y &\longmapsto s \mapsto \varphi_{s,*} \circ \tilde{\Psi}^{0s}, \end{aligned}$$

where

$$\begin{aligned} \Psi^{0s} : \varphi_0^* E &\longrightarrow \varphi_s^* E \\ (x, e) &\longmapsto (x, \Phi(e, \eta_{\varphi}(x)|_{[0, s]})). \end{aligned}$$

□

5.2. Invariance with regard to the homotopy type of $f : X \rightarrow Y$

Denote $\varphi_{f,g} : X \times [0, 1] \rightarrow Y$ a homotopy between $\varphi_{f,g}(\cdot, 0) = f$ and $\varphi_{f,g}(\cdot, 1) = g$. Denote

$$\varphi_{f,g}^2 : X^2 \times [0, 1] \longrightarrow Y^2, \quad \varphi_{f,g}^2(a, b, t) = (\varphi_{f,g}(a, t), \varphi_{f,g}(b, t))$$

and $\varphi_{g,f}^2 = \overline{\varphi_{f,g}^2} : X^2 \times [0, 1] \rightarrow Y^2$. For readability, if $a \in X$, we will use the notation

$$\eta(a) = (s \mapsto \varphi_{f,g}(a, s)) \in \mathcal{P}_{f(a) \rightarrow g(a)} Y.$$

Proposition 5.1 states that the morphism of fibrations

$$\begin{aligned} \Psi_{f,g} := \Psi^{\varphi_{f,g}^2} : \quad \mathcal{P}_{X,f} Y &\longrightarrow \mathcal{P}_{X,g} Y \\ (x_0, x_1, \gamma) &\longmapsto (x_0, x_1, \eta(x_0)^{-1} \gamma \eta(x_1)) \end{aligned}$$

induces the chain homotopy equivalence

$$\tilde{\Psi}_{f,g} : C_*(X^2, \Xi, C_*(\Omega Y); f) \longrightarrow C_*(X^2, \Xi, C_*(\Omega Y); g)$$

with chain homotopy inverse

$$\tilde{\Psi}_{g,f} := \widetilde{\Psi^{\varphi_{g,f}^2}} : C_*(X^2, \Xi, C_*(\Omega Y); g) \longrightarrow C_*(X^2, \Xi, C_*(\Omega Y); f).$$

PROPOSITION 5.2. — *The isomorphism*

$$\tilde{\Psi}_{f,g} : H_*(X^2, C_*(\Omega Y); f) \xrightarrow{\sim} H_*(X^2, C_*(\Omega Y); g)$$

is an isomorphism of rings.

Proof. — To prove that it is an isomorphism of rings, we show that the following diagram commutes:

$$\begin{array}{ccc} H_*(X^2, C_*(\Omega Y); f)^{\otimes 2} & \xrightarrow{\tilde{\Psi}_{f,g}^{\otimes 2}} & H_*(X^2, C_*(\Omega Y); g)^{\otimes 2} \\ \downarrow K & & \downarrow K \\ H_*(X^4, C_*(\Omega Y^2); f) & \xrightarrow{\widetilde{\Psi_{f,g} \times \Psi_{f,g}}} & H_*(X^4, C_*(\Omega Y^2); g) \\ \downarrow D_! & & \downarrow D_! \\ H_*(X^3, D^* C_*(\Omega Y^2); f) & \xrightarrow{D^* \widetilde{\Psi_{f,g} \times \Psi_{f,g}}} & H_*(X^3, D^* C_*(\Omega Y^2); g) \\ \downarrow \tilde{m} & & \downarrow \tilde{m} \\ H_*(X^3, p^* C_*(\Omega Y); f) & \xrightarrow{p^* \tilde{\Psi}_{f,g}} & H_*(X^3, p^* C_*(\Omega Y); g) \\ \downarrow p_* & & \downarrow p_* \\ H_*(X^2, C_*(\Omega Y), f) & \xrightarrow{\tilde{\Psi}_{f,g}} & H_*(X^2, C_*(\Omega Y), g). \end{array}$$

All the squares commute by basic properties about the maps induced by morphisms of fibrations. We only have to check that, in homology,

$$m_* \circ (\Psi_{f,g} \times \Psi_{f,g})_* = \Psi_{f,g,*} \circ m_* : H_*(\mathcal{P}_{X,f} \times_{ev_1 \times ev_0} \mathcal{P}_{X,f}) \longrightarrow H_*(\mathcal{P}_{X,g}).$$

Indeed, the maps

$$\begin{aligned} m \circ \Psi_{f,g} \times \Psi_{f,g} : \\ \mathcal{P}_{X,f} \times_{ev_1 \times ev_0} \mathcal{P}_{X,f} &\longrightarrow \mathcal{P}_{X,g} \\ (x_0, x_1, x_2, \gamma_1, \gamma_2) &\longmapsto (x_0, x_2, \eta(x_0)^{-1} \gamma_1 \eta(x_1) \eta(x_1)^{-1} \gamma_2 \eta(x_2)) \end{aligned}$$

and

$$\begin{aligned} \Psi_{f,g} \circ m : \mathcal{P}_{X,f} \times_{ev_1 \times ev_0} \mathcal{P}_{X,f} &\longrightarrow \mathcal{P}_{X,g} \\ (x_0, x_1, x_2, \gamma_1, \gamma_2) &\longmapsto (x_0, x_2, \eta(x_0)^{-1} \gamma_1 \gamma_2 \eta(x_2)), \end{aligned}$$

are homotopic. This concludes that

$$\tilde{\Psi}_{f,g} : H_*(X^2, C_*(\Omega Y); f) \xrightarrow{\sim} H_*(X^2, C_*(\Omega Y); g)$$

is an isomorphism of rings. □

PROPOSITION 5.3. — *The isomorphism*

$$\tilde{\Psi}_{f,g} : H_*(X^2, C_*(\Omega Y); f) \xrightarrow{\sim} H_*(X^2, C_*(\Omega Y); g)$$

is an isomorphism of $H_(Y, C_*(\Omega Y))$ -modules.*

Proof. — We prove that the following diagram commutes:

$$\begin{array}{ccc} H_*(Y, C_*(\Omega Y)) \otimes H_*(X^2, C_*(\Omega Y); f) & \xrightarrow{1 \otimes \tilde{\Psi}_{f,g}} & H_*(Y, C_*(\Omega Y)) \otimes H_*(X^2, C_*(\Omega Y); g) \\ \downarrow K & & \downarrow K \\ H_*(Y \times X^2, C_*(\Omega Y^2); f) & \xrightarrow{1 \times \tilde{\Psi}_{f,g}} & H_*(Y \times X^2, C_*(\Omega Y^2); g) \\ \downarrow I_{f,!} & & \downarrow I_{f,!} \\ H_*(X^2, I_f^* C_*(\Omega Y^2); f) & \xrightarrow{I_f^* 1 \times \tilde{\Psi}_{f,g}} & H_*(X^2, I_f^* C_*(\Omega Y^2); g) \\ \downarrow \tilde{m} & & \downarrow \tilde{\Psi}^H \\ H_*(X^2, C_*(\Omega Y); f) & \xrightarrow{\tilde{\Psi}_{f,g}} & H_*(X^2, C_*(\Omega Y); g) \end{array}$$

$\curvearrowright I_{g,!}$

where $H : X^2 \times [0, 1] \rightarrow Y \times X^2$, $H(a, b, t) = (\varphi_{f,g}(a, t), a, b)$ is a homotopy between $I_f : X^2 \rightarrow Y \times X^2$, $I_f(a, b) = (f(a), a, b)$ and $I_g : X^2 \rightarrow Y \times X^2$,

$I_g(a, b) = (g(a), a, b)$. Therefore

$$\begin{aligned} \Psi^H : \mathcal{L}Y_{\text{ev} \times f(\text{ev}_0)} \mathcal{P}_{X,g} Y &\longrightarrow \mathcal{L}Y_{\text{ev} \times g(\text{ev}_0)} \mathcal{P}_{X,g} Y \\ (\gamma, (x_0, x_1, \tau)) &\longmapsto (\eta(x_0)^{-1} \gamma \eta(x_0), (x_0, x_1, \tau)). \end{aligned}$$

It follows from compatibility properties between morphisms of fibrations, shriek maps and K that the two squares on the top commute. Proposition 5.1(4) states that $\tilde{\Psi}^H \circ I_{f,!} = I_{g,!}$. Since

$$\begin{aligned} \mathcal{L}Y_{\text{ev} \times f(\text{ev}_0)} \mathcal{P}_{X,f} Y &\longrightarrow \mathcal{P}_{X,g} Y \\ (\gamma, (x_0, x_1, \tau)) &\longmapsto (x_0, x_1, \eta(x_0)^{-1} \gamma \eta(x_0) \eta(x_0)^{-1} \tau \eta(x_1)) \end{aligned}$$

is homotopic to

$$\begin{aligned} \Psi_{f,g} \circ m : \mathcal{L}Y_{\text{ev} \times f(\text{ev}_0)} \mathcal{P}_{X,f} Y &\longrightarrow \mathcal{P}_{X,f} Y \longrightarrow \mathcal{P}_{X,g} Y \\ \Psi_{f,g} \circ m(\gamma, (x_0, x_1, \tau)) &= (x_0, x_1, \eta(x_0)^{-1} \gamma \tau \eta(x_1)), \end{aligned}$$

it follows that $\tilde{m} \circ \tilde{\Psi}^H \circ (1 \times \widetilde{\Psi_{f,g}}) = \tilde{\Psi}_{f,g} \circ \tilde{m} : H_*(X^2, I_f^* C_*(\Omega Y^2); f) \rightarrow H_*(X^2, C_*(\Omega Y); g)$.

This concludes that the diagram commutes and that $\tilde{\Psi}_{f,g} : H_*(X^2, C_*(\Omega Y); f) \xrightarrow{\sim} H_*(X^2, C_*(\Omega Y); g)$ is an isomorphism of $H_*(Y, C_*(\Omega Y))$ -modules. $\square$

5.3. Homotopy invariance of the product

Let Y_1, Y_2 be two topological spaces and $X_1 \xrightarrow{i_1} Y_1$, $X_2 \xrightarrow{i_2} Y_2$ be two pointed, oriented, closed, and connected manifolds. Assume that there exists a homotopy equivalence of pairs $\varphi : (Y_1, X_1) \rightarrow (Y_2, X_2)$ and denote $\psi : (Y_2, X_2) \rightarrow (Y_1, X_1)$ a homotopy inverse of φ . Denote $n = \dim(X_1) = \dim(X_2)$ and $d = \deg(\varphi|_{X_1}) \in \{-1, 1\}$.

We will denote $\mathcal{P}\varphi : \mathcal{P}_{X_1} Y_1 \rightarrow \varphi^{2,*} \mathcal{P}_{X_2} Y_2$ the morphism of fibrations over $(X_1)^2$ defined by

$$\mathcal{P}\varphi(x_0, x_1, \gamma) = (x_0, x_1, \varphi \circ \gamma).$$

The map $\mathcal{P}\varphi$ is a homotopy equivalence and therefore induces an isomorphism

$$\widetilde{\mathcal{P}\varphi} : H_*((X_1)^2, C_*(\Omega Y_1)) \longrightarrow H_*((X_1)^2, \varphi^{2,*} C_*(\Omega Y_2)).$$

PROPOSITION 5.4. — *The map*

$$\varphi_{\#} = d \cdot (\widetilde{\mathcal{P}\varphi})^{-1} \circ (\varphi^2)_! : H_*((X_2)^2, C_*(\Omega Y_2)) \longrightarrow H_*((X_1)^2, C_*(\Omega Y_1))$$

is an isomorphism of rings.

Proof. — It suffices to prove that the following diagram commutes up to the sign d .

$$\begin{array}{ccccc}
 H_*((X_2)^2, C_*(\Omega Y_2))^{\otimes 2} & \xrightarrow{(\varphi^2)_!^{\otimes 2}} & H_*((X_1)^2, \varphi^{2,*} C_*(\Omega Y_2))^{\otimes 2} & \xleftarrow{\widetilde{\mathcal{P}\varphi}^{\otimes 2}} & H_*((X_1)^2, C_*(\Omega Y_1)) \\
 \downarrow K & & \downarrow K & & \downarrow K \\
 H_*((X_2)^4, C_*(\Omega Y_2^2)) & \xrightarrow{(\varphi^4)_!} & H_*((X_1)^4, \varphi^{4,*} C_*(\Omega Y_2^2)) & \xleftarrow{\widetilde{\mathcal{P}\varphi \times \mathcal{P}\varphi}} & H_*((X_1)^4, C_*(\Omega Y_1^2)) \\
 \downarrow D_{2,!} & & \downarrow D_{1,!} & & \downarrow D_{1,!} \\
 H_*((X_2)^3, D_2^* C_*(\Omega Y_2^2)) & \xrightarrow{(\varphi^3)_!} & H_*((X_1)^3, \varphi^{3,*} D_1^* C_*(\Omega Y_2^2)) & \xleftarrow{D_1^* \widetilde{\mathcal{P}\varphi \times \mathcal{P}\varphi}} & H_*((X_1)^3, D_1^* C_*(\Omega Y_1^2)) \\
 \downarrow \tilde{m}_2 & & \downarrow \varphi^{3,*} \tilde{m}_2 & & \downarrow \tilde{m}_1 \\
 H_*((X_2)^3, p_2^* C_*(\Omega Y_2)) & \xrightarrow{(\varphi^3)_!} & H_*((X_1)^3, p_1^* \varphi^{2,*} C_*(\Omega X_2)) & \xleftarrow{p_1^* \widetilde{\mathcal{P}\varphi}} & H_*((X_1)^3, p_1^* C_*(\Omega Y_1)) \\
 \downarrow p_{2,*} & (*) & \downarrow p_{1,*} & & \downarrow p_{1,*} \\
 H_*((X_2)^2, C_*(\Omega Y_1)) & \xrightarrow{(\varphi^2)_!} & H_*((X_1)^2, \varphi^{2,*} C_*(\Omega Y_2)) & \xleftarrow{\widetilde{\mathcal{P}\varphi}} & H_*((X_1)^2, C_*(\Omega Y_1)).
 \end{array}$$

The square $(*)$ commutes up to the sign d since $(\varphi^3)_! = d^3(\varphi^3)_*^{-1} = d(\varphi^3)_*^{-1}$ and $(\varphi^2)_! = d^2(\varphi^2)_*^{-1} = (\varphi^2)_*^{-1}$. $\square$

PROPOSITION 5.5. — *The isomorphism $\varphi_{\#} : H_*((X_2)^2, C_*(\Omega Y_2)) \rightarrow H_*((X_1)^2, C_*(\Omega Y_1))$ satisfies the equation*

$$\begin{aligned}
 \forall \gamma \in H_*(Y_2, C_*(\Omega Y_2)), \quad \forall \tau \in H_*((X_2)^2, C_*(\Omega Y_2)), \\
 \varphi_{\#}(\gamma \cdot_M \tau) = \varphi_{\#}^{\mathcal{L}}(\gamma) \cdot_M \varphi_{\#}(\tau),
 \end{aligned}$$

where $\varphi_{\#}^{\mathcal{L}} = \widetilde{\mathcal{L}\varphi}^{-1} \circ \varphi_! : H_*(Y_2, C_*(\Omega Y_2)) \rightarrow H_*(Y_1, C_*(\Omega Y_1))$ is the ring isomorphism induced by the homotopy equivalence $\varphi : Y_1 \rightarrow Y_2$ (see [5, Proposition 7.13]).

Proof. — Denote $\mathcal{F}_i^j = C_*(\Omega Y_i^j)$ and $\mathcal{F}_i = \mathcal{F}_i^1$ for $i \in \{1, 2\}$ and $j \geq 0$. The following diagram commutes and concludes the proof.

$$\begin{array}{ccccc}
 H_*(Y_2, \mathcal{F}_2) \otimes H_*(X_2^2, \mathcal{F}_2) & \xrightarrow{\varphi_! \otimes \varphi_!^2} & H_*(Y_1, \varphi^* \mathcal{F}_2) \otimes H_*(X_1^2, \varphi^{2,*} \mathcal{F}_2) & \xleftarrow{\widetilde{\mathcal{L}\varphi \otimes \mathcal{P}\varphi}} & H_*(Y_1, \mathcal{F}_1) \otimes H_*(X_1^2, \mathcal{F}_1) \\
 \downarrow K & & \downarrow K & & \downarrow K \\
 H_*(Y_2 \times X_2^2, \mathcal{F}_2^2) & \xrightarrow{\varphi_!^3} & H_*(Y_1 \times X_1^2, \varphi^{3,*} \mathcal{F}_2^2) & \xleftarrow{\widetilde{\mathcal{L}\varphi \times \mathcal{P}\varphi}} & H_*(Y_1 \times X_1^2, \mathcal{F}_1^2) \\
 \downarrow I_{2,!} & & \downarrow I_{1,!} & & \downarrow I_{1,!} \\
 H_*(X_2^2, I_2^* \mathcal{F}_2^2) & \xrightarrow{\varphi_!^3} & H_*(X_1^2, I_1^* \varphi^{3,*} \mathcal{F}_1^2) & \xleftarrow{I_1^* \widetilde{\mathcal{L}\varphi \times \mathcal{P}\varphi}} & H_*(X_1^2, I_2^* \mathcal{F}_1^2) \\
 \downarrow \tilde{m} & & \downarrow \varphi^{2,*} \tilde{m} & & \downarrow \tilde{m} \\
 H_*(X_2^2, \mathcal{F}_2) & \xrightarrow{\varphi_!^2} & H_*(X_1^2, \varphi^{2,*} \mathcal{F}_2) & \xleftarrow{\widetilde{\mathcal{P}\varphi}} & H_*(X_1^2, \mathcal{F}_1).
 \end{array}$$

$\square$

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