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*Generalized amoebas for subvarieties of  $GL_n(\mathbb{C})$*

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Generalized amoebas for subvarieties of  $\mathrm{GL}_n(\mathbb{C})$  <sup>(\*)</sup>RÉMI DELLOQUE <sup>(1)</sup> AND KIUMARS KAVEH <sup>(2)</sup>


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**ABSTRACT.** — The paper addresses the problem of generalizing theory of amoebas from tropical geometry in algebraic torus to other reductive algebraic groups. The logarithm of singular values of a matrix has been suggested as an analogue of the logarithm map on the algebraic torus  $(\mathbb{C}^*)^n$  for the general linear group  $\mathrm{GL}_n(\mathbb{C})$ . We call the image of a subvariety under this logarithm map a matrix amoeba. In this paper, we establish the main properties of these matrix amoebas, in particular, extending the notion of Ronkin function to this setting. For hypersurfaces, we show how to describe the asymptotic directions of the matrix amoebas using a notion of Newton polytope and study the relationship between matrix amoebas and spherical tropical varieties. Our matrix amoebas should be considered as the Archimedean version of the spherical tropicalization of Tevelev–Vogiannou for the variety  $\mathrm{GL}_n(\mathbb{C})$  regarded as a spherical homogeneous space for the left-right action of  $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ .

**RÉSUMÉ.** — Cet article a pour but de généraliser la théorie des amibes (en géométrie tropicale) sur les tores algébriques à d'autres groupes algébriques réductifs. Le logarithme des valeurs singulières d'une matrice est proposé comme analogue de l'application logarithme sur un tore algébrique  $(\mathbb{C}^*)^n$  pour le groupe général linéaire  $\mathrm{GL}_n(\mathbb{C})$ . On appelle amibe de matrices l'image d'une sous-variété par cette application. Dans cet article, on montre les propriétés principales de ces amibes de matrices, en particulier en généralisant la notion de fonction de Ronkin à ce contexte. Pour les hypersurfaces, on montre comment décrire les directions asymptotiques des amibes de matrices grâce à une généralisation du polytope de Newton et on étudie la relation entre les amibes de matrices et les variétés tropicales sphériques. Ces amibes de matrices devraient être considérées comme la version archimédienne de la tropicalisation sphérique de Tevelev–Vogiannou pour la variété  $\mathrm{GL}_n(\mathbb{C})$  vue comme un espace homogène sphérique pour l'action à gauche et à droite de  $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ .

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## 1. Introduction

The logarithm map on the algebraic torus  $(\mathbb{C}^*)^n$  sends a point to the logarithm of absolute values of its coordinates. The image of a subvariety under the logarithm map is called its *amoeba*. The notion of amoeba was introduced by Gelfand, Kapranov and Zelevinsky ([6]) to study the asymptotic behavior at infinity of subvarieties in  $(\mathbb{C}^*)^n$ . An important question is whether the theory of amoebas and tropical geometry can be extended to subvarieties in other reductive algebraic groups and their homogeneous spaces. The present paper initiates this by answering some of the main questions in the case of subvarieties in  $\mathrm{GL}(n, \mathbb{C})$ .

In [10, Section 7] and [9], the logarithm of singular values of a matrix has been suggested as an analogue of the logarithm map on the algebraic torus  $(\mathbb{C}^*)^n$  for the general linear group  $\mathrm{GL}_n(\mathbb{C})$ . We call the image of a subvariety in  $\mathrm{GL}_n(\mathbb{C})$  under this logarithm map a *matrix amoeba*. In this paper we establish main properties of matrix amoebas and study their relationship with spherical tropical varieties. The ultimate ambitious goal is a theory of amoebas in arbitrary spherical homogeneous spaces for reductive algebraic groups.

We start with some background and motivations. From the point of view of algebraic geometry, tropical geometry is concerned with describing the “(exponential) behavior at infinity”, of subvarieties in  $(\mathbb{C}^*)^n$  where  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ . With componentwise multiplication,  $(\mathbb{C}^*)^n$  is an abelian group. It is usually referred to as an *algebraic torus* and is one of the basic examples of algebraic groups. A subvariety of  $(\mathbb{C}^*)^n$  is called a *very affine variety*. The behavior at infinity of a subvariety  $Y \subset (\mathbb{C}^*)^n$  is encoded in a union of convex polyhedral cones called the *tropical variety* of  $Y$ . There are (at least) two natural ways to define the tropical variety of a very affine variety: (1) using the formal Laurent series and *tropicalization map*, and (2) using the *logarithm map*.

### Tropicalization map (on torus)

Let  $\mathcal{K} = \mathbb{C}((t))$  be the field of formal Laurent series in one indeterminate  $t$ . Then the algebraic closure  $\bar{\mathcal{K}} = \mathbb{C}\{\{t\}\} = \bigcup_{k=1}^{\infty} \mathbb{C}((t^{1/k}))$  is the field of formal Puiseux series. The field  $\bar{\mathcal{K}}$  comes equipped with the *order of vanishing* valuation  $\mathrm{val} : (\bar{\mathcal{K}})^* = \bar{\mathcal{K}} \setminus \{0\} \rightarrow \mathbb{Q}$  defined as follows: for a Puiseux series  $f(t) = \sum_{i=m}^{\infty} a_i t^{i/k}$ , where  $a_m \neq 0$ , we put  $\mathrm{val}(f(t)) = m/k$ . The valuation  $\mathrm{val}$  gives rise to the *tropicalization map*  $\mathrm{trop}$  from  $(\bar{\mathcal{K}}^*)^n$  to  $\mathbb{Q}^n$ :

$$\mathrm{trop}(z_1(t), \dots, z_n(t)) = (\mathrm{val}(z_1(t)), \dots, \mathrm{val}(z_n(t))).$$

Let  $Y \subset (\mathbb{C}^*)^n$  be a subvariety with ideal  $I = I(Y) \subset \mathbb{C}[x_1^\pm, \dots, x_n^\pm]$ . Let  $Y(\bar{\mathcal{K}})$  denote the Puiseux series valued points on  $Y$ , that is,  $Y(\bar{\mathcal{K}}) = \{z(t) = (z_1(t), \dots, z_n(t)) \in (\bar{\mathcal{K}}^*)^n \mid \forall f \in I, f(z_1(t), \dots, z_n(t)) = 0\}$ . The tropical variety of  $Y$  is the closure (in  $\mathbb{R}^n$ ) of the image of  $Y(\bar{\mathcal{K}})$  under the map  $\text{trop}$ . One shows that the tropical variety of a subvariety always has the structure of a *fan* in  $\mathbb{R}^n$ , that is, it is a finite union of (strictly) convex rational polyhedral cones (see [11, Chapter 3]).

### Logarithm map (on torus)

The *logarithm map*  $\text{Log} : (\mathbb{C}^*)^n \rightarrow \mathbb{R}^n$  is defined by:

$$\text{Log}(z_1, \dots, z_n) = (\ln |z_1|, \dots, \ln |z_n|). \quad (1.1)$$

Clearly the inverse image of every point is an  $(\mathbb{S}^1)^n$ -orbit in  $(\mathbb{C}^*)^n$ . Here  $\mathbb{S}^1$  denotes the complex unit circle and

$$(\mathbb{S}^1)^n = \{(z_1, \dots, z_n) \mid |z_1| = \dots = |z_n| = 1\},$$

which is the maximal compact subgroup in  $(\mathbb{C}^*)^n$ .

For a subvariety  $Y \subset (\mathbb{C}^*)^n$ , its (*Archimedean*) *amoeba*  $\mathcal{A}(Y)$  is the image of  $Y$  in  $\mathbb{R}^n$  under the logarithm map  $\text{Log}$ . Amoebas were introduced in [6, Section 6.1], as a means to study the asymptotic behavior at infinity of subvarieties in  $(\mathbb{C}^*)^n$ . An amoeba goes to infinity along certain directions usually called its *tentacles* (and hence the name amoeba). The directions along which an amoeba goes to infinity in fact coincides with the tropical variety of  $Y$ . More precisely, we have the following fact that goes back to Bergman [2] (in a different form and before the notion of tropical variety was introduced):

As  $\rho \rightarrow 0^+$ , the rescaled amoeba  $\rho\mathcal{A}(Y)$  approaches  $-\text{trop}(Y)$ ,  
the negative of the tropical variety.

When  $Y$  is a hypersurface this is relatively easy to show and basically appears in [6, Section 6.1, Proposition 1.9]. Even though the statement that, for arbitrary  $Y$ , the amoeba approaches the tropical variety has been known as a folklore, a precise formulation and proof only appeared relatively recently in ([7, Theorem A]).

It is natural to ask whether tropical geometry and notions of tropicalization and logarithm map can be extended to other classes of varieties with group actions. To this end, it is natural to consider *spherical homogeneous spaces*  $G/H$  where  $G$  is a reductive algebraic group over  $\mathbb{C}$ . We recall that a  $G$ -variety is called *spherical* if a Borel subgroup (and hence all Borel subgroups) have an open (hence dense) orbit. The notion of tropicalization has

been extended to spherical homogeneous spaces in the work of Tevelev and Vogianou [16]. Suggestions for the notion of logarithm map on spherical homogeneous spaces appear in [1, 10]. In the case where the homogeneous space is  $\mathrm{GL}_n(\mathbb{C})$  the spherical logarithm map is the logarithm of singular values of a matrix [9]. Here we consider  $\mathrm{GL}_n(\mathbb{C})$  as a spherical homogeneous space for the left-right action of  $G = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ , thus identifying  $\mathrm{GL}_n(\mathbb{C})$  with  $(\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})) / \mathrm{GL}_n(\mathbb{C})_{\mathrm{diag}}$  where  $\mathrm{GL}_n(\mathbb{C})_{\mathrm{diag}} = \{(g, g) \mid g \in \mathrm{GL}_n(\mathbb{C})\}$ .

## Main results

For an  $n \times n$  matrix  $A$  we let  $\mathrm{sLog}(A)$  to be the collection of logarithms of singular values of  $A$  (see Definition 2.1). This defines the *spherical logarithm* map  $\mathrm{sLog} : \mathrm{GL}_n(\mathbb{C}) \rightarrow \mathbb{R}^n / \mathcal{S}_n$ , where  $\mathcal{S}_n$  is the group of permutations (symmetric group). We call the image of a subvariety  $Y \subset \mathrm{GL}_n(\mathbb{C})$  under the logarithm map  $\mathrm{sLog}$ , the *matrix amoeba* or *spherical amoeba* of  $Y$  and denote it by  $\mathrm{sA}(Y)$ .

In this paper, for any subvariety  $Y \subset \mathrm{GL}_n(\mathbb{C})$ , we show the following:

- The matrix amoeba  $\mathrm{sA}(Y)$  is closed.
- Each connected component of the complement of  $\mathrm{sA}(Y)$  is convex when  $Y$  is a hypersurface.
- We give an analogue of the notion of Ronkin function and show that it is affine on each connected component of the complement of  $\mathrm{sA}(Y)$  when  $Y$  is a hypersurface.
- For a regular function  $f \in \mathbb{C}[\mathrm{GL}_n]$  we consider its *spherical Newton polytope* (also called its *weight polytope*). When  $Y$  is a hypersurface given by an equation  $f = 0$ , we give a description of the asymptotic directions in  $\mathrm{sA}(Y)$  (in other words, the spherical tropical variety of  $Y$ ) in terms of the spherical Newton polytope of  $f$ .
- We show that the limit of the sets  $\rho \mathrm{sA}(Y)$ , as  $\rho \rightarrow 0^+$ , contains the spherical tropical variety of  $Y$  (in the sense of Tevelev–Vogianou). Moreover, it coincides with the spherical tropical variety when  $Y$  is a hypersurface.

*Remark 1.1.* — We point out that a very general construction of a logarithm map and amoeba appears in [3]. It is an interesting question to investigate the connection between our notion of logarithm map and the one in [3] (in the case of general linear group).

*Remark 1.2.* — We expect that the constructions, statements and proofs in the present note, with little change, extend to arbitrary connected reductive algebraic groups over  $\mathbb{C}$ .

Section 2 contains the definitions of matrix amoebas and some basic properties that will justify the definitions. Section 3 is a study of the geometric aspect of matrix amoebas of hypersurfaces. Section 4 is a small digression about representations of the general linear group and Newton polytopes. Section 5 generalises the Bergman theorem [2] that links amoebas to tropical varieties. Finally, Appendices A and B are dedicated respectively to notations and technical lemmas.

## 2. Definitions and elementary results

### 2.1. Definitions and results in the torus

We recall that the algebraic torus  $T = (\mathbb{C}^*)^n$  and the matrix group  $\mathrm{GL}_n(\mathbb{C})$  are affine varieties:

$$T \cong \{(z_1, \dots, z_n, w) \in \mathbb{C}^{n+1} \mid z_1, \dots, z_n w = 1\},$$

$$\mathrm{GL}_n(\mathbb{C}) \cong \left\{ ((a_{ij})_{1 \leq i, j \leq n}, z) \in \mathbb{C}^{n^2+1} \mid \det(a_{ij})z = 1 \right\}.$$

A subvariety of the algebraic torus is the set of all the common zeros of the functions of an ideal  $I$  of  $\mathbb{C}[T]$ , the ring of regular functions on  $T$ . In fact,  $\mathbb{C}[T] = \mathbb{C}[X^\pm] = \mathbb{C}[X_1^\pm, \dots, X_n^\pm]$  is the ring of Laurent polynomials with  $n$  indeterminates. The concept of amoeba of a very affine variety, that is, a subvariety of  $T$ , was introduced by Gelfand, Kapranov and Zelevinsky in [6]. The amoeba of a very affine variety  $Y = V(I)$  is defined to be its image under the Log map,

$$\mathrm{Log} : \begin{cases} T \longrightarrow \mathbb{R}^n \\ z \longmapsto (\ln|z_1|, \dots, \ln|z_n|). \end{cases}$$

We denote it by  $\mathcal{A}(Y)$  or  $\mathcal{A}(I)$ , or  $\mathcal{A}(f)$  when  $I = (f)$  is principal. It is known that an amoeba is a closed subset of  $\mathbb{R}^n$  and all the connected components of its complement are convex. The asymptotic directions along which an amoeba approaches infinity is a finite union of polyhedral cones which is the *tropical variety of  $Y$* . When  $I = (f)$  is a principal ideal of  $\mathbb{C}[X^\pm]$ , one can describe the tropical variety using the Newton polytope of  $f$ : the *support*  $S_f$  of  $f$  is the set of exponents  $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$  such that the coefficient of  $f$  at  $X^m = X_1^{m_1} \dots X_n^{m_n}$  is non zero and the *Newton polytope*  $\mathrm{Newt}(f)$  of  $f$  is the convex hull of its support. Bergman showed in [2] that the asymptotic directions on which  $\mathcal{A}(f)$  goes to infinity (that is, its tropical variety), coincides with the  $(n-1)$ -skeleton of the normal fan of  $\mathrm{Newt}(f)$ . For more details, see [11, Section 1.4].

## 2.2. Definitions in $\mathrm{GL}_n(\mathbb{C})$

To extend the notion of amoeba to other classes of varieties (in place of the torus) we need an extension of the notion of Log map. We note that the logarithm map  $\mathrm{Log} : (\mathbb{C}^*)^n \rightarrow \mathbb{R}^n$  is invariant by multiplication with elements of the compact torus  $(\mathbb{S}^1)^n$ . The compact torus is the maximal compact subgroup of  $(\mathbb{C}^*)^n$ . Similarly, the unitary group  $\mathrm{U}(n)$  is a maximal compact subgroup of  $\mathrm{GL}_n(\mathbb{C})$ . Recall from linear algebra that the *singular values decomposition* states that  $\mathrm{GL}_n(\mathbb{C}) = \mathrm{U}(n)D_n\mathrm{U}(n)$  where  $D_n$  is the subgroup of diagonal matrices with positive real entries. If we write  $A \in \mathrm{GL}_n(\mathbb{C})$  as  $A = PDQ$  where  $P, Q$  are unitary matrices and  $D$  is diagonal with positive diagonal entries, the diagonal entries of  $D$  are the *singular values* of  $A$ .

DEFINITION 2.1. — *Following [10] we define the matrix logarithm map (or spherical logarithm map) on  $\mathrm{GL}_n(\mathbb{C})$  as follows:*

$$\mathrm{sLog} : \begin{cases} \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathbb{R}^n / \mathcal{S}_n \\ A \longmapsto (\ln(\lambda_1), \dots, \ln(\lambda_n)), \end{cases}$$

where the  $\lambda_k$  are the singular values of  $A$ .

Here  $\mathcal{S}_n$  is the symmetric group (the group of permutation of  $\{1, \dots, n\}$ ) which acts on  $\mathbb{R}^n$  by permuting the coordinates.

We remark that the  $s$  in  $\mathrm{sLog}$  stands for *spherical*. This is because,  $\mathrm{GL}_n(\mathbb{C})$  with left-right action of  $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$  is an important example of a spherical homogeneous space.

By abuse of terminology and notation, we may identify subsets of  $\mathbb{R}^n / \mathcal{S}_n$  with subsets of  $\mathbb{R}^n$  invariant under permutations. Similarly, we identify functions on  $\mathbb{R}^n / \mathcal{S}_n$  with functions on  $\mathbb{R}^n$  that are invariant under permutations of the coordinates.

We note that  $|\mathrm{sLog}(A)| \rightarrow +\infty$  when at least one of the entries of  $A$  approaches infinity or when  $A$  approaches a non-invertible matrix. We will see in Section 5 other reasons that make of  $\mathrm{sLog}$  a good generalisation of  $\mathrm{Log}$ . Recall that the singular values of a matrix  $A$  are the square roots of the eigenvalues of the Hermitian non negative matrix (positive when  $A$  is invertible)  $AA^*$  (or  $A^*A$ ). Any matrix  $A$  can be written as  $UDV^*$  with  $U$  and  $V$  invertible and  $D$  non negative diagonal. Then, the diagonal coefficients of  $D$  are the singular values of  $A$ .

DEFINITION 2.2. — *When  $I \subset \mathbb{C}[\mathrm{GL}_n]$  (the ring of regular functions on  $\mathrm{GL}_n(\mathbb{C})$ ) is an ideal, the matrix amoeba of a matrix spherical variety  $Y = V(I)$  is  $\mathrm{sA}(Y) = \mathrm{sA}(I) = \mathrm{sLog}(Y) \subset \mathbb{R}^n / \mathcal{S}_n$ . We shall refer to  $\mathrm{sA}(I)$*

as matrix amoeba (or spherical amoeba) of  $Y$ . When there is no ambiguity we may simply refer to it as the amoeba of  $Y$ .

We will need a last definition that will help us to make the link between classical and matrix amoebas.

**DEFINITION 2.3.** — *Let  $f \in \mathbb{C}[\mathrm{GL}_n(\mathbb{C})]$  be a regular function and let  $A, B$  be invertible matrices. We define  $\Psi_{A,B}$  as follows:*

$$\Psi_{A,B}(f) : \begin{cases} T \longrightarrow \mathbb{C} \\ z \longmapsto f(A \operatorname{diag}(z) B^{-1}), \end{cases}$$

where  $\operatorname{diag}(z)$  is the diagonal matrix with coordinates of  $z$  as diagonal entries.  $\Psi_{A,B}$  is a  $\mathbb{C}$ -algebra homomorphism from  $\mathbb{C}[\mathrm{GL}_n(\mathbb{C})]$  to  $\mathbb{C}[X^\pm]$ .

We recall that  $\mathbb{C}[\mathrm{GL}_n(\mathbb{C})]$  is the ring of functions of the form  $\det^N f$  where  $f$  is a polynomial in the matrix entries and  $N \in \mathbb{Z}$ .

### 2.3. Elementary properties of matrix amoebas

We begin by showing that matrix amoebas are closed in  $\mathbb{R}^n/\mathcal{S}_n$  (with respect to the natural topology on it). Since all subvarieties are closed, it is enough to show that  $\operatorname{sLog}$  is a closed map.

**LEMMA 2.4.** — *For any  $A \in \mathrm{GL}_n(\mathbb{C})$  we have:*

$$\begin{aligned} |\operatorname{sLog}(A)|_\infty &= \frac{1}{2} \max\{\ln\|AA^*\|, \ln\|(AA^*)^{-1}\|\} \\ &= \max\{\ln\|A\|, \ln\|A^{-1}\|\} + O(1), \end{aligned}$$

where  $|\cdot|_\infty$  is the infinite norm and  $\|\cdot\|$  is the operator norm associated the Euclidean norm on  $\mathbb{C}^n$ .

*Proof.* — Let  $A \in \mathrm{GL}_n(\mathbb{C})$ . The matrix  $AA^*$  is a positive Hermitian matrix so it can be written as  $U \operatorname{diag}(\lambda) U^*$  with  $U$  unitary and  $\lambda = (\lambda_1 \geq \dots \geq \lambda_n > 0)$ . Moreover, being Hermitian, its norm is equal to its spectral radius  $\lambda_1$ . Also  $(AA^*)^{-1} = U \operatorname{diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1}) U^*$ . Thus the norm of  $(AA^*)^{-1}$  is  $\lambda_n^{-1}$  and we have

$$\frac{1}{2} \max\{\ln\|AA^*\|, \ln\|(AA^*)^{-1}\|\} = \frac{1}{2} \max\{\ln(\lambda_1), -\ln(\lambda_n)\}.$$

On the other hand, the singular values of  $A$  are the square roots of the eigenvalues of  $AA^*$ , namely,  $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}$ . We deduce that

$$\begin{aligned} |\mathrm{sLog}(A)|_\infty &= \max \left\{ \left| \ln(\sqrt{\lambda_1}) \right|, \dots, \left| \ln(\sqrt{\lambda_n}) \right| \right\} \\ &= \frac{1}{2} \max \{ \ln(\lambda_1), -\ln(\lambda_1), \dots, \ln(\lambda_n), -\ln(\lambda_n) \} \\ &= \frac{1}{2} \max \{ \ln(\lambda_1), -\ln(\lambda_n) \}, \end{aligned}$$

which proves the first equality.

The second equality comes from the fact that  $A \mapsto \sqrt{\|AA^*\|}$  is a norm (the spectral norm) and the equivalence of norms in finite dimensional vector spaces.  $\square$

**PROPOSITION 2.5.** — *sLog is continuous for the distance  $d : (x, y) \mapsto \min_{\sigma \in \mathcal{S}_n} \{\|\sigma \cdot x - y\|\}$  on  $\mathbb{R}^n / \mathcal{S}_n$ .*

*Proof.* — First of all, it is straightforward to see that  $d$  is a well-defined distance for which

$$\varphi : \begin{cases} \{x \in \mathbb{R}^n \mid x_1 \leq \dots \leq x_n\} \longrightarrow \mathbb{R}^n / \mathcal{S}_n \\ x \longmapsto \mathcal{S}_n \cdot x \end{cases}$$

is an isometry. If  $(A_m)_{m \in \mathbb{N}}$  is a sequence of invertible matrices that converges to some invertible matrix  $A$ . Then  $A_m A_m^* \xrightarrow{m \rightarrow +\infty} AA^*$  and thus the characteristic polynomial  $\chi_{A_m A_m^*}$  converges to the characteristic polynomial  $\chi_{AA^*}$ . By the continuity of the roots of a polynomial [14] we see that  $\mathrm{sLog}(A_m) \xrightarrow{m \rightarrow +\infty} \mathrm{sLog}(A)$ . That is, sLog is continuous.  $\square$

**PROPOSITION 2.6.** — *sLog is proper, hence closed.*

*Proof.* — Let  $K \subset \mathbb{R}^n / \mathcal{S}_n$  be compact. By Proposition 2.5,  $\mathrm{sLog}^{-1}(K)$  is closed, by continuity of sLog and closedness of  $K$ . By Lemma 2.4 and by boundedness of  $K$ , there are positive real numbers  $r, R$  such that for all matrices  $A \in \mathrm{sLog}^{-1}(K)$ ,  $r \leq \|A\| \leq R$ . In particular,  $\mathrm{sLog}^{-1}(K)$  is bounded in the vector space of matrices, hence it is compact. It proves that sLog is proper. Now, the space  $\mathbb{R}^n / \mathcal{S}_n$  is Hausdorff and locally compact, thus any proper map  $X \rightarrow \mathbb{R}^n / \mathcal{S}_n$  is closed with compact fibres. In particular, sLog is closed.

An other way to see the properness of sLog is to notice that it factorises as a continuous map  $\mathrm{GL}_n(\mathbb{C}) / (\mathrm{U}(n) \times \mathrm{U}(n)) \rightarrow \mathbb{R}^n / \mathcal{S}_n$  because the singular values of a matrix only depend on its equivalence classes modulo the left-right action of  $\mathrm{U}(n) \times \mathrm{U}(n)$ . This map is actually a homeomorphism and we deduce the properness of sLog by the compactness of  $\mathrm{U}(n) \times \mathrm{U}(n)$ .  $\square$

**COROLLARY 2.7.** — *Any matrix amoeba is closed.*

Next, we describe matrix amoebas in terms of classical amoebas (in the torus).

PROPOSITION 2.8. — *For any ideal  $I \subset \mathbb{C}[\mathrm{GL}_n]$  we have*

$$\mathrm{sA}(I) = \bigcup_{(U,V) \in \mathrm{U}(n) \times \mathrm{U}(n)} \mathcal{A}(\Psi_{U,V}(I)).$$

*Proof.*

⊂. — If  $x \in \mathrm{sA}(I)$ ,  $x$  can be written as  $(\ln(\lambda_1), \dots, \ln(\lambda_n))$  where the  $\lambda_k$  are the singular values of a matrix  $A \in V(I)$ . Therefore, we can write  $A$  as  $U \operatorname{diag}(\lambda) V^*$  with  $U$  and  $V$  unitary. Then for all  $f \in I$  we have:

$$0 = f(A) = f(U \operatorname{diag}(\lambda) V^*) = \Psi_{U,V}(f)(\lambda).$$

Thus  $\lambda \in V(\Psi_{U,V}(I))$  which shows that  $x = \operatorname{Log}(\lambda) \in \mathcal{A}(\Psi_{U,V}(I))$ .

⊃. — Let  $U$  and  $V$  be unitary matrices and  $x \in \mathcal{A}(\Psi_{U,V}(I))$ . We can write  $x$  as  $\operatorname{Log}(z) = (\ln|z_1|, \dots, \ln|z_n|)$  where  $z \in V(\Psi_{U,V}(I))$ . For all  $f \in I$ , we have  $0 = \Psi_{U,V}(f)(z) = f(U \operatorname{diag}(z) V^*)$ . Let  $A = U \operatorname{diag}(z) V^*$ . The singular values of  $A$  are the square roots of the eigenvalues of  $AA^* = U \operatorname{diag}(z) \overline{\operatorname{diag}(z)} U^* = U \operatorname{diag}(|z_1|^2, \dots, |z_n|^2) U^*$ , i.e.,  $|z_1|, \dots, |z_n|$ . As for all  $f \in I$ ,  $f(A) = 0$ , we deduce that  $(z_1, \dots, z_n)$  is in  $V(f)$  so  $x = \operatorname{Log}(z) \in \mathrm{sA}(I)$ .  $\square$

Even though the union in the above proposition is over an uncountable set, it is still useful. For example we can use it to prove the following.

PROPOSITION 2.9. — *For any  $f \in \mathbb{C}[\mathrm{GL}_n(\mathbb{C})]$ , the connected components of  $\mathrm{sA}(f)^C$  are convex.*

*Proof.* — By Proposition 2.8, we know that

$$\mathrm{sA}(f)^C = \bigcap_{(U,V) \in \mathrm{U}(n)} \mathcal{A}(\Psi_{U,V}(f))^C.$$

Let  $C$  be a connected component of  $\mathrm{sA}(f)^C$  and let  $x, y \in C$ . For all unitary matrices  $U, V$  we have  $C \subset \mathrm{sA}(f)^C \subset \mathcal{A}(\Psi_{U,V}(f))^C$ . This shows that  $x$  and  $y$  belong to the same connected component in  $\mathcal{A}(\Psi_{U,V}(f))^C$  which is convex. Therefore the line segment joining  $x$  and  $y$  lies in  $\mathcal{A}(\Psi_{U,V}(f))^C$ , for any unitary matrices  $U, V$ . It follows that this line segment lies in  $\mathrm{sA}(f)^C$  as required.  $\square$

### 3. Matrix amoebas of hypersurfaces and Ronkin function

In this section we study matrix amoebas for hypersurfaces. In particular, we generalise the notion of Ronkin function. It was, as its name suggests it,

introduced by Ronkin in [15]. It is a powerful tool to study the shape of amoebas of hypersurfaces. In particular, Passare and Rullgård used it in [13] to study the spine of the amoebas which gives an easy way to compute its global shape and its homology. We will see how to extend these to matrix amoebas.

### 3.1. Definitions and results for the torus $(\mathbb{C}^*)^n$

Let  $f$  be a Laurent polynomial. One defines the *Ronkin function* of  $f$  by:

$$R_f : \begin{cases} \mathbb{R}^n \longrightarrow \mathbb{R} \\ x \longmapsto \frac{1}{(2i\pi)^n} \int_{[0,2\pi]^n} \ln |f(\mathbf{e}^{x_1+i\theta_1}, \dots, \mathbf{e}^{x_n+i\theta_n})| \, d\theta_1, \dots, d\theta_n. \end{cases}$$

It can be rewritten as:

$$R_f(x) = \int_{\mathbb{T}^n} \ln |f(\mathbf{e}^{x_1} \lambda_1, \dots, \mathbf{e}^{x_n} \lambda_n)| \, d\mu(\lambda_1, \dots, \lambda_n)$$

where  $\mu$  is the unique probability Haar measure on the compact Lie group  $\mathbb{T}^n = (\mathbb{S}^1)^n$  where  $\mathbb{S}^1$  denotes the unit circle. An important property of the Ronkin function is that it is convex on  $\mathbb{R}^n$ , affine on every connected component of  $\mathcal{A}(f)^C$ , and conversely, if  $x \in \mathcal{A}(f)$ ,  $R_f$  is not affine on any open neighborhood of  $x$ . Moreover, consider the order function  $\nu_f$  given by (see [4]):

$$\nu_f : \begin{cases} \Gamma(\mathcal{A}(f)^C) \longrightarrow \mathbb{R}^n \\ C \longmapsto \nabla R_f(x) \text{ for some } x \in C \end{cases}$$

Then  $\nu_f$  is injective and we have  $\text{Im}(\nu_f) \subset \text{Newt}(f) \cap \mathbb{Z}^n$  (recall that  $\text{Newt}(f)$  is the Newton polytope of  $f$  defined as the convex hull of exponents of monomials appearing in  $f$ ). Reciprocally, any vertex on  $\text{Newt}(f)$  (which is by definition in  $\mathbb{Z}^n$ ) belongs to  $\text{Im}(\nu_f)$ . We refer to [4, 13] for several interesting results in this regard. The vector  $\nu_f(C)$  is called the *order of the connected component*  $C \subset \mathcal{A}(f)^C$ .

### 3.2. Definitions for $\text{GL}_n(\mathbb{C})$

From now on, unless otherwise stated,  $f$  is a non zero element of  $\mathbb{C}[\text{GL}_n(\mathbb{C})]$ . As the unitary group  $\text{U}(n)$  is a compact Lie group, there is a unique probability measure  $\mu$  (the Haar measure) that is invariant under left-right multiplication.

DEFINITION 3.1. — We define the Ronkin function of  $f$  by:

$$R_f : \begin{cases} \mathbb{R}^n \longrightarrow \mathbb{R} \\ x \longmapsto \int_{U(n)} \int_{U(n)} \ln |f(U \exp(\text{diag}(x)) V^*)| d\mu(U) d\mu(V) \end{cases}$$

Note that the set of all the  $U \exp(\text{diag}(x)) V^*$  is  $\text{sLog}^{-1}(x)$  which is compact by properness of  $\text{sLog}$ . In [4, 15], the Ronkin function is well-defined for any complex analytic function using the fact that  $\ln |f|$  is locally integrable whenever  $f$  is analytic and not identically zero. As this property is local, it is also true on any Riemannian analytic manifold.  $U(n)^2$  is Riemannian analytic and compact, thus  $R_f$  is well-defined whenever  $f$  is analytic and not identically zero (and in particular when it is a non zero polynomial).

We will also need to look at the coefficients of the Laurent polynomials of the form  $\Psi_{A,B}(f)$ .

DEFINITION 3.2. — Let  $Q_m \in \mathbb{C}[\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})]$  be defined by:

$$f(A \text{diag}(z) B^{-1}) = \sum_{m \in \mathbb{Z}^n} Q_m(A, B) z^m,$$

for any  $z \in T$  and invertible matrices  $A$  and  $B$ .

The regular functions  $Q_m$  will be important in the study of matrix amoebas. We define the support of  $f$  and its matrix Newton polytope using the  $Q_m$ .

DEFINITION 3.3. — The support  $S_f$  of  $f \in \text{GL}_n(\mathbb{C})$  is the set of  $m \in \mathbb{Z}^n$  such that  $Q_m$  is not identically zero. The matrix Newton polytope of  $f$ ,  $\text{sNewt}(f)$  is the convex hull of  $S_f$ .

The support  $S_f$  (respectively the polytope  $\text{sNewt}(f)$ ) coincides with  $S_{\Psi_{U,V}(f)}$  (respectively  $\text{sNewt}(\Psi_{U,V}(f))$ ), for generic choices of unitary matrices  $U$  and  $V$ . More precisely, we have the following.

PROPOSITION 3.4. — For almost every pair  $(U, V)$  of unitary matrices (with respect to the Haar measure on  $U(n) \times U(n)$ ), we have  $S_f = S_{\Psi_{U,V}(f)}$ .

*Proof.* — The claim follows from Lemma B.1 applied to the  $Q_m$ . □

### 3.3. Some properties of the Ronkin function

The following expresses the matrix Ronkin function in terms of the classical Ronkin functions. It will be useful as it allows us to reduce statements about the matrix Ronkin function to those of classical Ronkin function.

PROPOSITION 3.5. — *For all  $f \in \mathbb{C}[\mathrm{GL}_n]$ , for all  $x \in \mathbb{R}^n$ ,*

$$R_f(x) = \int_{\mathrm{U}(n) \times \mathrm{U}(n)} R_{\Psi_{U,V}(f)}(x) \, d\mu^2(U, V),$$

where  $d\mu^2$  denotes the Haar measure on  $\mathrm{U}(n) \times \mathrm{U}(n)$ .

*Proof.* — Take  $\Omega \in \mathrm{U}(n)$ . Substituting  $U$  by  $U\Omega$ , by the change of variable formula, we have:

$$R_f(x) = \int_{\mathrm{U}(n) \times \mathrm{U}(n)} \ln|f(U\Omega \exp(\mathrm{diag}(x)) V^*)| \, d\mu^2(U, V).$$

Noting that any diagonal matrix whose coefficients are in the unit circle  $\mathbb{S}^1$  is unitary, we can rewrite  $R_f$  as:

$$\begin{aligned} R_f(x) &= \int_{\mathrm{U}(n) \times \mathrm{U}(n)} \ln|f(U \exp(\mathrm{diag}(x)) V^*)| \, d\mu^2(U, V), \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \int_{\mathrm{U}(n) \times \mathrm{U}(n)} \ln|f(U \exp(\mathrm{diag}(\mathbf{i}\theta)) \exp(\mathrm{diag}(x)) V^*)| \, d\mu^2(U, V) \, d\theta, \\ &= \int_{\mathrm{U}(n) \times \mathrm{U}(n)} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \ln|f(U \exp(\mathrm{diag}(x + \mathbf{i}\theta)) V^*)| \, d\theta \, d\mu^2(U, V), \\ &= \int_{\mathrm{U}(n) \times \mathrm{U}(n)} R_{\Psi_{U,V}(f)}(x) \, d\mu^2(U, V). \end{aligned}$$

This finishes the proof. □

PROPOSITION 3.6.

- (a)  $R_f$  is convex.
- (b) For all  $f, g$ ,  $R_{fg} = R_f + R_g$ .
- (c)  $R_{\det}(x) = \mathbf{1} \cdot x$ , where  $\mathbf{1} = (1, \dots, 1)$ .

*Proof.* — The first part immediately follows from the Proposition 3.5 and the fact that the Ronkin function for Laurent polynomials is convex. The two last parts follow from simple computation. □

PROPOSITION 3.7. —  $R_f$  is invariant under permutations of the coordinates.

*Proof.* — Let  $\sigma \in \mathcal{S}_n$ . We denote by  $P_\sigma$  the permutation matrix associated with  $\sigma$ . It is in particular a unitary matrix. Recall that for all  $z \in \mathbb{C}^n$ ,  $\mathrm{diag}(\sigma \cdot z) = P_\sigma \mathrm{diag}(z) P_\sigma^*$ . By the change of variable  $U \mapsto UP_\sigma$  and  $V \mapsto VP_\sigma$ ,

$$R_f(\sigma \cdot x) = \int_{\mathrm{U}(n) \times \mathrm{U}(n)} \ln|U \exp(\mathrm{diag}(\sigma \cdot x)) V^*| \, d\mu^2(U, V)$$

$$\begin{aligned}
&= \int_{U(n) \times U(n)} \ln |UP_\sigma \exp(\text{diag}(x))P_\sigma^* V^*| d\mu^2(U, V) \\
&= \int_{U(n) \times U(n)} \ln |U \exp(\text{diag}(x))V^*| d\mu^2(U, V) \\
&= R_f(x),
\end{aligned}$$

which proves Proposition 3.7.  $\square$

Therefore,  $R_f$  can be seen as a function of  $\mathbb{R}^n/\mathcal{S}_n$ . The classical Ronkin function is important in the study of amoebas of Laurent polynomials because it contains the information about where the connected components of the complement of the amoeba are. Namely, the Ronkin function is affine on each connected component of complement of an amoeba. We have an analogues result for matrix amoebas.

**PROPOSITION 3.8.** — *The Ronkin function is affine on every connected component of  $\text{sA}(f)^C$ .*

*Proof.* — Let  $C \subset \mathcal{A}(f)^C$  be a connected component and let  $x \in C$ . It means that for all unitary matrices  $U, V$ ,  $x \notin \mathcal{A}(\Psi_{U,V}(f))$  so  $R_{\Psi_{U,V}(f)}$  is affine (thus smooth). It is clear that  $U, V \mapsto \nabla R_{\Psi_{U,V}(f)}(x)$  is continuous. Moreover, this function takes its values in  $\mathbb{Z}^n$  so it is actually constant. Let  $m(x) \in \mathbb{Z}^n$  its value. Still by an argument of discreteness/continuity, we deduce that  $m(x) = m$  is constant when  $x$  browses  $C$ . Therefore,  $R_f$  is affine over  $C$  and its gradient is  $m \in \mathbb{Z}^n$ .  $\square$

In the next subsection, we will see more advanced results that will give us information about the Ronkin function.

### 3.4. The order function and its image

Proposition 3.8 allows us to define the order function  $\nu_f$  for  $f \in \mathbb{C}[\text{GL}_n]$ .

**DEFINITION 3.9.** — *We define the order function by:*

$$\nu_f : \begin{cases} \Gamma(\text{sA}(f)^C) \longrightarrow \mathbb{Z}^n \\ C \longmapsto \nabla R_f(x) \text{ for some } x \in C. \end{cases}$$

We call  $\nu_f(C)$ , the order of the connected component  $C$ .

The proof of Proposition 3.8 tells us that for all unitary matrices  $U, V$  and for all  $x \in C \subset \text{sA}(f)^C$  with  $C$  connected,  $\nabla R_f(x) = \nabla R_{\Psi_{U,V}(f)}(x)$ , which will be very useful to determine the order of the connected components of  $\text{sA}(f)^C$ .

PROPOSITION 3.10. —  $\nu_f$  is injective.

*Proof.* — Let  $C_1$  and  $C_2$  be connected components of  $\text{s}\mathcal{A}(f)^C$  such that  $\nu_f(C_1) = \nu_f(C_2)$ . We call  $m$  this quantity. It implies that for all unitary matrices  $U, V$ ,  $\nu_{\Psi_{U,V}(f)}(C_1) = \nu_{\Psi_{U,V}(f)}(C_2)$ . As  $R_{\Psi_{U,V}(f)}$  is convex and its gradient is  $m$  over  $C_1$  and  $C_2$ , we deduce that  $\nabla R_{\Psi_{U,V}(f)} = m$  over  $C_3 = \text{Conv}(C_1 \cup C_2)$  which is open. Therefore, any point  $x \in C_3$  is in  $\mathcal{A}(\Psi_{U,V}(f))^C$  because  $R_{\Psi_{U,V}(f)}$  is affine around  $x$ . It is true for all  $U, V$  so  $C_3 \subset \text{s}\mathcal{A}(f)^C$ . We deduce that  $C_1 = C_2 = C_3$ .  $\nu_f$  is injective.  $\square$

PROPOSITION 3.11. — The image of  $\nu_f$  is included in

$$\bigcap_{(U,V) \in \text{U}(n) \times \text{U}(n)} \text{Newt}(\Psi_{U,V}(f)) \cap \mathbb{Z}^n.$$

*Proof.* — We proved that for all unitary  $U, V$ ,  $\nu_f(C) \in \text{Im}(\nu_{\Psi_{U,V}(f)}) \subset \text{Newt}(\Psi_{U,V}(f)) \cap \mathbb{Z}^n$  so  $\text{Im}(\nu_f)$  is included in

$$\bigcap_{(U,V) \in \text{U}(n) \times \text{U}(n)} \text{Newt}(\Psi_{U,V}(f)) \cap \mathbb{Z}^n. \quad \square$$

Knowing which of the  $Q_m$  vanish on  $\text{U}(n) \times \text{U}(n)$  is useful to eliminate quickly some points of  $S_f$  which are not in  $\text{Im}(\nu_f)$ . In fact, contrary to the case of Laurent polynomials where all the vertices  $v$  of the Newton polytope have an associated connected component of order  $v$ , at most two of the vertices of  $\text{sNewt}(f)$  can have an associated component. We want to use Lemma B.3 to prove that when  $m \notin \mathbb{Z}\mathbf{1}$  is a vertex,  $m \notin \text{Im}(\nu_f)$ . For this, we need some properties on the  $Q_m$ .

PROPOSITION 3.12. — For all  $m \in \mathbb{Z}^n$ , if  $\lambda \in T$ , for all invertible matrices  $A$  and  $B$ ,

$$Q_m(A \text{diag}(\lambda), B) = \lambda^m Q_m(A, B), \quad Q_m(A, B \text{diag}(\lambda)) = \lambda^{-m} Q_m(A, B)$$

and for all permutation matrix  $P$ ,  $Q_m(AP, BP) = Q_{Pm}(A, B)$ .

*Proof.* — For any invertible  $A, B$  and any  $z \in T$  and any  $\lambda \in T$ ,

$$\begin{aligned} \sum_{m \in \mathbb{Z}^n} Q_m(A \text{diag}(\lambda), B^{-1}) z^m &= f(A \text{diag}(\lambda) \text{diag}(z) B) \\ &= f(A \text{diag}(\lambda_1 z_1, \dots, \lambda_n z_n) B^{-1}) \\ &= \sum_{m \in \mathbb{Z}^n} Q_m(A, B) \lambda^m z^m, \end{aligned}$$

so by uniqueness of the  $Q_m$ , for every  $m \in \mathbb{Z}^n$ ,

$$Q_m(A \text{diag}(\lambda), B) = \lambda^m Q_m(A, B).$$

Same thing for  $B$ .

And if  $\sigma \in \mathcal{S}_n$ , let  $P_\sigma$  be the associated permutation matrix. We have,

$$\begin{aligned}
 \sum_{m \in \mathbb{Z}^n} Q_m(AP_\sigma, BP_\sigma) z^m &= f(AP_\sigma \operatorname{diag}(z) P_\sigma^{-1} B^{-1}) \\
 &= f(A \operatorname{diag}(\sigma \cdot z) B^{-1}) \\
 &= \sum_{m \in \mathbb{Z}^n} Q_m(A, B) (\sigma \cdot z)^m \\
 &= \sum_{m \in \mathbb{Z}^n} Q_{P_\sigma m}(A, B) (\sigma \cdot z)^{P_\sigma m} \\
 &= \sum_{m \in \mathbb{Z}^n} Q_{P_\sigma m}(A, B) z^m,
 \end{aligned}$$

so for every  $m \in \mathbb{Z}^n$ ,  $Q_m(AP_\sigma, BP_\sigma) = Q_{P_\sigma m}(A, B)$ . □

COROLLARY 3.13. —  $S_f$  and  $\operatorname{sNewt}(f)$  are permutation invariant.

### 3.5. Geometry of hypersurface matrix amoebas

In this section we study the shape of matrix amoebas of hypersurfaces, and in particular, the connected components of their complements and their maximal cones. Thanks to Proposition 3.10 ( $\nu_f$  is defined at Definition 3.9), we know that any connected component of  $\operatorname{sA}(f)^C$  is associated with exactly one point of  $\operatorname{sNewt}(f) \cap \mathbb{Z}^n$ . We will treat vertices of  $\operatorname{sNewt}(f)$  separately from the other points.

Recall that when  $\Delta \subset \mathbb{R}^n$  is a convex polytope and  $F$  is a face of  $\Delta$ , the normal cone associated to  $F$  is  $C_F(\Delta)$  (or  $C_F$  when there is no ambiguity) defined as  $\{w \in \mathbb{R}^n \mid \forall x \in F, \forall y \in \Delta, w \cdot x \leq w \cdot y\}$ , which is a cone (a rational one if  $\Delta$  is rational). See [11, Section 2.3] for more details about convex geometry. Notice that if  $F$  has dimension  $d$ ,  $C_F$  has dimension  $n - d$ .

PROPOSITION 3.14. — *For all connected component  $E \subset \operatorname{sA}(f)^C$ , if we set  $m = \nu_f(E) \in \operatorname{sNewt}(f) \cap \mathbb{Z}^n$  and  $F$  the smallest face of  $\operatorname{sNewt}(f)$  that contains  $m$  (i.e.  $F$  is the only face whose relative interior contains  $m$ ), then  $C_F$  is the recession cone of  $E$ , which means that,*

- (a)  $\forall x \in E, x + C_F \subset E$ .
- (b)  $\forall \text{ cone } C, C_F \subsetneq C \Rightarrow \forall x \in \mathbb{R}^n, x + C \not\subset E$ .

*Proof.* — To prove Proposition 3.14, we need its Laurent polynomial counterpart that can be found in [4, Proposition 2.6]. Let  $x \in E$  and let  $E_{U,V}$  be the connected component of  $\operatorname{A}(\Psi_{U,V}(f))^C$  where  $x$  belongs to. We know that for all  $U, V$ ,  $\nu_{\Psi_{U,V}(f)}(E_{U,V}) = \nu_f(E) = m$  (in particular,  $m$  belongs to

all the  $\text{Newt}(\Psi_{U,V}(f))$  and  $E$  is the intersection of the  $E_{U,V}$ . By [4, Proposition 2.6], for every unitary  $U, V$ , we have  $x + C_{F_{U,V}}(\text{Newt}(\Psi_{U,V}(f))) \subset E_{U,V}$  where  $F_{U,V}$  is the smallest face of  $\text{Newt}(\Psi_{U,V}(f))$  that contains  $m$ . As  $\text{Newt}(\Psi_{U,V}(f)) \subset \text{sNewt}(f)$ ,

$$C_F = C_F(\text{sNewt}(f)) \subset C_{F_{U,V}}(\text{Newt}(\Psi_{U,V}(f))),$$

we see that  $x + C_F \subset E_{U,V}$ . It is true for every  $U, V$  so  $x + C_F \subset E$ . Conversely, suppose  $C$  strictly contains  $C_F$ . Consider unitary  $U_0, V_0$  such that  $\text{Newt}(\Psi_{U,V}(f)) = \text{sNewt}(f)$  (we know there exists at least one by Proposition 4.12). We have, still by [4, Proposition 2.6], that for all  $x \in \mathbb{R}^n$ ,  $x + C \not\subset E_{U_0, V_0} \supset E$ . It proves Proposition 3.14.  $\square$

**COROLLARY 3.15.** — *It implies that the bounded components are exactly the ones whose order is an interior point of  $\text{sNewt}(f)$  because the only face  $F$  such that  $C_F$  is bounded is  $F = \text{sNewt}(f)$ .*

**PROPOSITION 3.16.** — *If  $v \in \text{sNewt}(f)$  is a vertex, there are two possibilities:*

- (a)  $v \in \mathbb{Z}\mathbf{1}$ : in this case,  $v \in \text{Im}(\nu_f)$ . Moreover,  $\mathbf{1}$  or  $-\mathbf{1}$  belongs in  $C_{\{v\}}$ . In particular,  $E$  is unbounded.
- (b)  $v \notin \mathbb{Z}\mathbf{1}$ : in this case,  $v \notin \text{Im}(\nu_f)$ .

*Proof.* — Assume  $v \in \mathbb{Z}\mathbf{1}$ . If there is another point  $m \in \text{sNewt}(f) \cap \mathbb{Z}^n$  such that  $v \cdot \mathbf{1} = m \cdot \mathbf{1}$ , then  $m \notin \mathbb{Z}\mathbf{1}$  (or it would be equal to  $v$ ) so the convex hull of  $\mathcal{S}_n \cdot m$  is an  $(n-1)$ -dimensional polytope contained in  $\text{sNewt}(f)$  (because the action of  $\mathcal{S}_n$  on  $\{\mathbf{1}\}^\perp$  is irreducible) that contains  $v$  because  $v = \sum_{\sigma \in \mathcal{S}_n} \frac{1}{n!} \sigma \cdot m$ . It implies that  $v$  is not a vertex of  $\text{sNewt}(f)$ , which contradicts our hypothesis. Therefore, for all  $m \in \text{sNewt}(f)$ ,  $m \neq v \Rightarrow m \cdot \mathbf{1} \neq m \cdot v$ . With the same kind of argument, we could prove that  $v \cdot \mathbf{1} - m \cdot \mathbf{1}$  is always positive or always negative when  $m$  browses  $\text{sNewt}(f) \cap \mathbb{Z}^n$ . We will assume without loss of generality that it is always positive. We set  $\delta > 0$  the minimum taken by the quantity  $v \cdot \mathbf{1} - m \cdot \mathbf{1}$ . We will need it later in the proof.

Now, decompose  $f$  as  $\sum_{d \in \mathbb{Z}} f_d$  where the  $f_d$  are homogeneous polynomials of degree  $d$ , all zero except a finite number of them. Let for all  $d$ , and for all  $A, B, z$ ,  $f_d(A \text{diag}(z)B^{-1}) = \sum_{m \in S_{f_d}} Q_{d,m}(A, B)z^m$ . For all  $\lambda \in \mathbb{C}$ , we have,

$$\begin{aligned} f_d(\lambda A \text{diag}(z)B^{-1}) &= \sum_{m \in S_{f_d}} \lambda^{m \cdot \mathbf{1}} Q_{d,m}(A, B)z^m \\ &= f_d(A \text{diag}(z)B^{-1}) \\ &= \lambda^d \sum_{m \in S_{f_d}} \lambda^d Q_{d,m}(A, B)z^m. \end{aligned}$$

Thus, for all  $m \in S_{f_d}$ ,  $m \cdot \mathbf{1} = d$ . It implies that  $S_f = \bigsqcup_{d \in \mathbb{N}} S_{f_d}$  and for all  $m \in S_f$ , the  $Q_m$  of  $f$  is the  $Q_m$  of  $f_{m \cdot \mathbf{1}}$ . In particular, if we set  $v = N\mathbf{1}$  with  $N \in \mathbb{Z}$  (because  $v \in \mathbb{Z}\mathbf{1}$ ), we have  $v \cdot \mathbf{1} = nN$  and  $f_{nN}(A \operatorname{diag}(z)B^{-1}) = Q_v(A, B)z^v$ . Notice that for all invertible diagonalisable matrix  $A = P \operatorname{diag}(z)P^{-1}$ , we have,

$$f_{nN}(A) = Q_v(P, P)z^v = f_{nN}(PP^{-1})z^v = f_{nN}(I_n)\det(A)^N.$$

By density, this equality remains true for any invertible matrix. In fact  $f_{nN} = \alpha \det^N$  for some  $\alpha \in \mathbb{C}^*$ . In particular,  $Q_v$  does not vanish on  $U(n) \times U(n)$ . Let  $(U, V) \in U(n) \times U(n)$  and  $x \in \mathbb{R}_+^*$ .

$$\begin{aligned} & |\Psi_{U,V}(f)(\mathbf{e}^x \mathbf{1}) - Q_v(U, V)(\mathbf{e}^x \mathbf{1})^v| \\ &= \left| \sum_{m \neq v} Q_m(U, V) \mathbf{e}^{xm \cdot \mathbf{1}} \right| \\ &\leq \sum_{m \neq v} \max_{U(n) \times U(n)} \{|Q_m|\} \mathbf{e}^{x(v \cdot \mathbf{1} - \delta)} \text{ by definition of } \delta. \\ &= \mathbf{e}^{-x\delta} \sum_{m \neq v} \max_{U(n) \times U(n)} \{|Q_m|\} \mathbf{e}^{xv \cdot \mathbf{1}} \\ &< |\alpha| \mathbf{e}^{xv \cdot \mathbf{1}} \text{ if } x \text{ is large enough.} \\ &= |Q_v(U, V)(\mathbf{e}^x \mathbf{1})^v|. \end{aligned}$$

By [4, Proposition 2.7] applied to  $\Psi_{U,V}$ , it implies that  $x\mathbf{1} \in \mathcal{A}(\Psi_{U,V}(f))^C$  and  $\nabla R_{\Psi_{U,V}(f)}(x\mathbf{1}) = v$  for every  $U, V$  so  $x\mathbf{1} \in s\mathcal{A}(f)^C$  and  $\nabla R_f(x\mathbf{1}) = v$ . Let  $E$  be the connected component where  $x$  belongs. We have by definition of  $\nu_f$  that  $\nu_f(E) = v$  so  $v \in \operatorname{Im}(\nu_f)$ . It is true for any  $x$  large enough so  $E$  is unbounded and contains a half-line included in  $\mathbb{R}\mathbf{1}$ . By Proposition 3.14, it means that  $\mathbf{1}$  or  $-\mathbf{1}$  belongs in  $C_{\{v\}}$ .

If  $v \notin \mathbb{Z}\mathbf{1}$ , by Proposition 3.12 and Lemma B.3 applied to the function  $A \mapsto \det(A)^N Q_v(A, I_n)$  for  $N$  large enough,  $Q_v$  vanishes on  $U(n) \times U(n)$ . As  $v$  is a vertex of  $s\operatorname{Newt}(f)$ , if we consider some  $U_0, V_0$  unitary such that  $Q_v(U_0, V_0) = 0$ , then  $v \notin \operatorname{Newt}(\Psi_{U_0, V_0}(f))$  so there is no connected component of order  $v$  in the complement of the amoeba of  $\Psi_{U_0, V_0}(f)$ . It implies that there is no connected component of order  $v$  in the complement of the spherical amoeba of  $f$ , which proves Proposition 3.16.  $\square$

*Remark 3.17.* — The above shows a difference between classical amoebas and matrix amoebas. For matrix amoebas, every vertex of the Newton polytope is in the image of the order map. In the case of matrices, there are at most two because there are at most two vertices on the line  $\mathbb{Z}\mathbf{1}$ .

*Remark 3.18.* — For a Laurent polynomial  $f$ , it is possible but rare to find a connected component of  $\mathcal{A}(f)^C$  of order  $m$  if the coefficient of  $f$  at the monomial  $x^m$  is 0. Nisse calls those coefficients “virtually non zero” and it has consequences as [12, Theorem 1.2]. Therefore, as each  $Q_m$  vanishes on  $U(n) \times U(n)$  when  $m \notin \mathbb{Z}\mathbf{1}$ , it seems possible that for all matrix polynomial  $f$ ,  $\text{Im}(\nu_f) \subset \mathbb{Z}\mathbf{1}$ . We have neither been able to prove it nor to find a counterexample.

CONJECTURE 3.19. — *The image of  $\nu_f$  is always included in  $\mathbb{Z}\mathbf{1}$ .*

### 3.6. Some examples

We will see three examples to illustrate this section. The  $\alpha_k$  in these examples denote non zero complex constants. They all are in dimension  $n = 2$  except the first example, which is in dimension  $n \geq 2$ . Notice that the support of  $f$  can easily be computed thus we won’t detail the computation of the  $S_f$ .

*Example 3.20.* —  $f \in \text{Mat}_n(\mathbb{C})^* \subset \mathbb{C}[\text{GL}_n]$  is a linear map.

We will see that in this case, the amoeba of  $f$  is the whole space  $\mathbb{R}^n$ . Indeed, we know that  $f(A)$  can be written as  $\text{tr}(MA)$  for some matrix  $M$ , by Riesz’s representation theorem. Let us use  $M$ ’s singular values decomposition  $M = U_0 D V_0^*$ . Let  $P$  be the matrix of a permutation which does not have a fixed point (it exists because  $n \geq 2$ ). We have for all  $z \in T$ ,

$$\Psi_{V_0 P, U_0}(f)(z) = \text{tr}(U_0 D V_0^* V_0 P \text{diag}(z) U_0^*) = \text{tr}(P \text{diag}(z) D) = 0.$$

We deduce that  $\Psi_{V_0 P, U_0}(f) = 0$  so its amoeba is  $\mathbb{R}^n$ . Therefore,  $\text{s}\mathcal{A}(f) = \mathbb{R}^n$ . Notice that it is different than for Laurent polynomial where the amoeba is the whole space if and only if the polynomial is identically zero.

*Example 3.21.* —  $n = 2$  and  $f : A \mapsto \alpha_1 a_{11}^2 + \alpha_2 \det(A)$ .

Here,  $S_f = \text{sNewt}(f) \cap \mathbb{Z}^2 = \{(2, 0), (1, 1), (0, 2)\}$ . Notice that it is contained in  $\{(1, 1)\}^\perp + (1, 1) = \{\mathbf{1}\}^\perp + \mathbf{1}$ . It is actually easy to verify that in general  $\text{sNewt}(f)$  is contained in an affine hyperplane of  $\mathbb{R}^n$  parallel to  $\{\mathbf{1}\}^\perp$  if and only if  $f$  is homogeneous. Here,  $f$  is indeed homogeneous of degree 2.  $(2, 0)$  and  $(0, 2)$  are vertices of  $\text{sNewt}(f)$  which are not in  $\mathbb{Z}\mathbf{1}$  so by Proposition 3.16, they do not have associated components. It implies that  $\text{s}\mathcal{A}(f)^C$  is empty or connected. By Proposition 3.14, if  $\text{s}\mathcal{A}(f)^C \neq \emptyset$ , the biggest cone contained in it is  $C_{\text{sNewt}(f)} = \mathbb{R}\mathbf{1}$  and it is open so it is of the form  $\{(a + b, -a + b) \mid |b| < r\}$  for some  $r > 0$ . Same thing with  $r = 0$  if  $\text{s}\mathcal{A}(f)^C = \emptyset$ . The fact that this set is empty or not depends on the  $\alpha_k$ .

PROPOSITION 3.22. —  $\text{sA}(f)^C = \emptyset$  if and only if  $|\alpha_2| \leq |\alpha_1|$ .

*Proof.* — In this case,  $\text{sA}(f)^C \neq \emptyset$  if and only if it contains 0, if and only if  $f$  vanishes on  $\text{U}(2)$ , if and only if  $f$  vanishes on  $\text{SU}(2)$  (by homogeneity). Let  $U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \text{SU}(2)$  with  $|a|^2 + |b|^2 = 1$ .  $f(U) = \alpha_1 a^2 + \alpha_2 = 0$  if and only if  $a$  is a square root of  $-\frac{\alpha_2}{\alpha_1}$ . It is possible to find such an  $a$  if and only if  $|\alpha_2| \leq |\alpha_1|$ , which proves the proposition.  $\square$

Notice that it provides a second example of non zero polynomial whose amoeba is the whole space (when  $|\alpha_2| \leq |\alpha_1|$ ). Figure 3.1 shows the Newton polytope of  $f$  and its amoeba in the case where  $|\alpha_2| > |\alpha_1|$ .

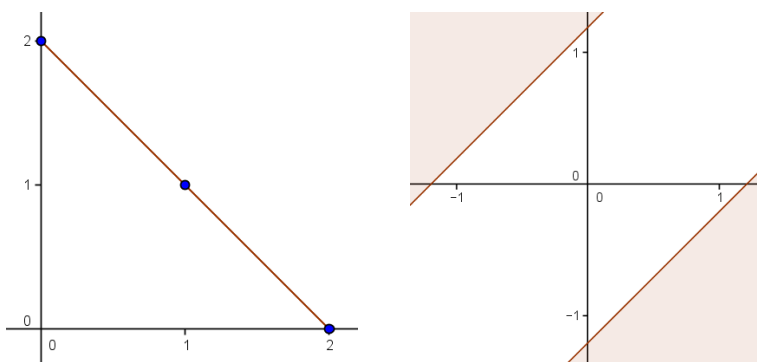


Figure 3.1. The Newton polytope (left) and the amoeba (right) of a polynomial of Example 3.21.

Example 3.23. —  $n = 2$  and  $f : A \mapsto \alpha_1 + \alpha_2 a_{11} + \alpha_3 \det(A) + \alpha_4 a_{22} \det(A)$ .

In this case,  $S_f = \text{sNewt}(f) \cap \mathbb{Z}^2 = \{(0, 0), (1, 0), (0, 1), (1, 1), (2, 1), (1, 2)\}$ . The points  $(1, 0)$ ,  $(0, 1)$ ,  $(2, 1)$  and  $(1, 2)$  all are vertices that do not belong to  $\mathbb{Z}(1, 1)$  so they do not have associated components by Proposition 3.16. The point  $(0, 0)$  is a vertex on  $\mathbb{Z}\mathbf{1}$  so it has an associated unbounded component  $C_0$  whose maximal cone is the quadrant  $\{(x, y) \in \mathbb{R}^2 \mid \max\{x, y\} \leq 0\}$  by Propositions 3.16 and 3.14. The point  $\mathbf{1}$  may or may not have an associated component in the  $\alpha_k$ . If it has one, it is bounded because  $\mathbf{1}$  is in the relative interior of  $\text{sNewt}(f)$ . Let  $C_1$  be this component, or  $C_1 = \emptyset$  if  $\mathbf{1} \notin \text{Im}(\nu_f)$ .

PROPOSITION 3.24. — If the polynomial  $|\alpha_4|t^3 - |\alpha_3|t^2 + |\alpha_2|t + |\alpha_1|$  takes negative values on  $\mathbb{R}_+^*$  (which means that  $|\alpha_3|$  is large enough compared to the others  $|\alpha_k|$ ),  $C_1 \neq \emptyset$ . Conversely, if  $|\alpha_3|^2 < 4|\alpha_2||\alpha_4|$ , then  $C_1 = \emptyset$ .

*Proof.* — Assume the first condition holds. Let  $r \in \mathbb{R}_+^*$  such that  $|\alpha_4|r^3 - |\alpha_3|r^2 + |\alpha_2|r + |\alpha_1| < 0$  and let  $x = \ln(r)$ . Let  $A$  be a matrix whose singular

values are both  $\mathbf{e}^x = r$ . Then  $A = rU$  where  $U \in \mathbf{U}(2)$  so,

$$|\alpha_3 \det(A)| - |\alpha_1 + \alpha_2 a_{11} + \alpha_4 a_{22} \det(A)| \geq |\alpha_3| r^2 - |\alpha_1| - |\alpha_2| r - |\alpha_4| r^3 > 0.$$

It implies that  $(x, x) \notin \text{s}\mathcal{A}(f)$  and by [4, Proposition 2.7] applied to each  $\Psi_{U,V}(f)$ , the order of the component containing  $(x, x)$  is  $(1, 1)$ .

Conversely, assume now that  $C_1 \neq \emptyset$ . Then, every  $\mathcal{A}(\Psi_{U,V}(f))^C$  has a connected component of order  $(1, 1)$ . It is, in particular, the case for  $g = \Psi_{I_2, I_2}(f)$ . For every  $(z_1, z_2) \in (\mathbb{C}^*)^2$ ,

$$g(z_1, z_2) = \alpha_1 + \alpha_2 z_1 + \alpha_3 z_1 z_2 + \alpha_4 z_1 z_2^2.$$

Therefore,  $S_g = \text{Newt}(g) \cap \mathbb{Z}^2 = \{(0, 0), (1, 0), (1, 1), (1, 2)\}$ . As  $\mathcal{A}(g)^C$  has a component of order  $(1, 1)$  and every other lattice point of  $\text{Newt}(g)$  are vertices, each lattice point of  $S_g$  has an associated connected component. The spine of  $\mathcal{A}(g)$  is the tropical curve of a tropical polynomial of the form  $\max\{c_{(0,0)}, c_{(1,0)} + x, c_{(1,1)} + x + y, c_{(1,2)} + x + 2y\}$  and by [13, Theorem 2] and [13, Theorem 3],  $c_{(1,1)}$  is given by,

$$c_{(1,1)} = \ln|\alpha_3| + \Re \left( \sum_{k \in K} \frac{(-k_3 - 1)!}{k_2! k_4!} (-1)^{k_3 - 1} \alpha_2^{k_2} \alpha_3^{k_3} \alpha_4^{k_4} \right),$$

where  $K = \{(k_2, k_3, k_4) \in \mathbb{Z}^3 \mid k_2 \geq 0, k_3 < 0, k_4 \geq 0, k_2 + k_3 + k_4 = 0, k_3 + 2k_4 = 0\}$ . We can parameterize the elements of  $K$  by

$$K = \{(q, -2q, q) \mid q \in \mathbb{N}^*\}$$

so,

$$c_{(1,1)} = \ln|\alpha_3| - \Re \left( \sum_{q \in \mathbb{N}^*} \frac{(2q - 1)!}{q!^2} \left( \frac{\alpha_2 \alpha_4}{\alpha_3^2} \right)^q \right),$$

and

$$\frac{(2q - 1)!}{q!^2} = \frac{\binom{2q}{q}}{2q} \underset{q \rightarrow +\infty}{\sim} \frac{4^q}{2\sqrt{\pi} q^{3/2}}$$

(this is a consequence of the Stirling formula) so this series has a convergence radius of  $\frac{1}{4}$ . It implies that  $|\alpha_3|^2 \geq 4|\alpha_2||\alpha_4|$ , which proves the Proposition 3.24.  $\square$

Figure 3.2 shows the Newton polytope of  $f$  and the shape of its amoeba in the case where  $C_1 \neq \emptyset$ . We are not able to compute spherical amoebas yet so this picture does not represent an approximation of the real amoeba of  $f$  for certain values of the  $\alpha_k$  but is only a representation of what its amoeba looks like (in particular, the picture and the real amoeba have the same homotopy).

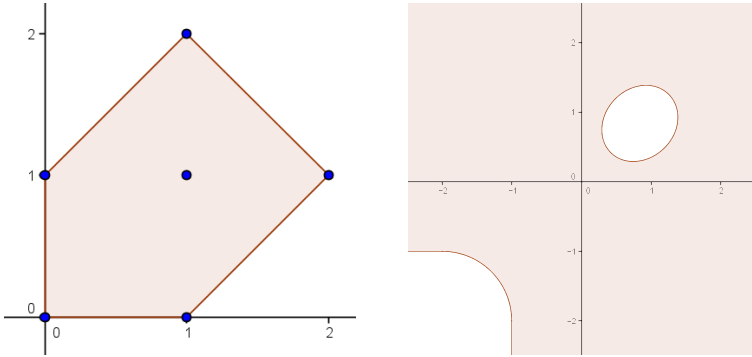


Figure 3.2. The Newton polytope (left) and the shape of the amoeba (right) of a polynomial of Example 3.23.

#### 4. Definition of matrix Newton polytope using representation theory

The purpose of this section is to show that our definition of the Newton polytope  $\text{sNewt}(f)$  of a polynomial  $f \in \mathbb{C}[\text{GL}_n]$  coincides with that of Kapranov [8] that we will call  $\Delta(f)$ . Let  $G$  be a reductive algebraic group (we will take  $G = \text{GL}_n(\mathbb{C})$ ). The group  $G^2 = G \times G$  acts transitively on  $G$  by left-right action and the stabilizer of 1 is  $H = \{(g, g) \mid g \in G\}$  thus  $G \cong G^2/H$  is a homogeneous space. Moreover, the Borel subgroup  $B \subset G^2$  acts on it with an open orbit, thus  $G$  is a spherical variety ( $B$  is the set of couples of invertible matrices  $(U, L)$  with  $U$  upper triangular and  $L$  lower triangular when  $G = \text{GL}_n(\mathbb{C})$ ).

By the Peter–Weyl theorem, we have that,

$$\mathbb{C}[G] = \bigoplus_{\lambda \in \Lambda} W_{\lambda},$$

where  $\Lambda$  is the Weyl chamber associated to  $G$  ( $\Lambda = \{x \in \mathbb{R}^n \mid x_1 \leq \dots \leq x_n\}$  when  $G = \text{GL}_n(\mathbb{C})$ ) and  $W_{\lambda}$  is the irreducible representation of  $G^2$  with highest  $B$ -weight  $\lambda$  by the highest weight theorem. Moreover, for each  $\lambda$ , there exists, up to a rescaling, a unique  $B$ -weight vector of weight  $\lambda$  (that belongs to  $W_{\lambda}$ ) and  $W_{\lambda} \cong V_{\lambda} \otimes V_{\lambda}^*$  as representations of  $G^2$  where  $V_{\lambda}$  is the irreducible representation of  $G$  of highest weight  $\lambda$ . More details about representation theory in [5].

Let  $W$  be the Weyl group of  $G$  ( $W = \mathcal{S}_n$  when  $G = \text{GL}_n(\mathbb{C})$ ). The moment polytope  $\Delta(f)$  of  $f$  is the convex hull of the  $W$ -orbits of the  $\lambda \in \Lambda$

such that the projection of  $f$  on  $W_\lambda$  parallel to  $\bigoplus_{\mu \neq \lambda} W_\mu$  is non zero in  $\mathbb{R}^n$ . It is in particular stable by  $W$ . Kapranov calls it the Newton polytope in [8].

This section is a digression about representation theory of  $\mathrm{GL}_n(\mathbb{C})$  that has for purpose to show that for any  $f \in \mathbb{C}[\mathrm{GL}_n]$ ,  $\mathrm{sNewt}(f) = \Delta(f)$ . This section is not linked to amoebas, neither tropical geometry, but this result supports the idea that most of the results of this article can be generalised to any reductive algebraic group.

#### 4.1. Some convex geometry

First of all, we need results about convex geometry.

**DEFINITION 4.1.** — *For every  $\lambda \in \Lambda = \{\lambda \in \mathbb{R}^n \mid \lambda_1 \leq \dots \leq \lambda_n\}$ , we define  $C(\lambda)$  as the convex hull of the  $\mathcal{S}_n$ -orbit of  $\lambda$  in  $\mathbb{R}^n$ . When  $(x, y) \in \Lambda^2$ , we say that  $x \preceq y$  if  $C(x) \subset C(y)$ .*

We want to show that  $\preceq$  is a (partial) order relation over  $\Lambda$ . It is obviously reflexive and transitive. Let us show that it is anti-symmetric.

For all  $K \subset \{1, \dots, n\}$ , we define the vector  $\mathbf{1}_K \in \mathbb{R}^n$  by  $(\mathbf{1}_K)_i = \mathbf{1}_K(i)$ . When  $1 \leq k \leq n$ , we also set  $\varpi_k = \mathbf{1}_{[n+1-k, n]} = (0, \dots, 0, 1, \dots, 1)$  with  $k$  ones.

**LEMMA 4.2.** — *For every  $x \in \Lambda$ ,  $C(x)$  is the set of all  $y \in \mathbb{R}^n$  such that for all  $K \subset \{1, \dots, n\}$ ,  $y \cdot \mathbf{1}_K \leq x \cdot \varpi_{|K|}$  with equality when  $K = \{1, \dots, n\}$ .*

*Proof.*

$\subset$ . — If  $y \in C(x)$ , then it can be written as  $\sum_{\sigma \in \mathcal{S}_n} \alpha_\sigma \sigma \cdot x$  where the  $\alpha_\sigma$  are non-negative real numbers whose sum equals 1. Therefore, for all  $K \subset \{1, \dots, n\}$ ,

$$y \cdot \mathbf{1}_K = \sum_{\sigma \in \mathcal{S}_n} \alpha_\sigma (\sigma \cdot x) \cdot \mathbf{1}_K \leq \sum_{\sigma \in \mathcal{S}_n} \alpha_\sigma x \cdot \varpi_{|K|} = x \cdot \varpi_{|K|}.$$

The inequality  $(\sigma \cdot x) \cdot \mathbf{1}_K \leq x \cdot \varpi_{|K|}$  comes from the fact that the coefficients of  $x$  increase. Moreover, it becomes an equality when  $K = \{1, \dots, n\}$ , which proves that  $y \cdot \mathbf{1} = x \cdot \mathbf{1}$ .

$\supset$ . — We will prove it by induction on  $n$ . It is trivial for  $n = 1$ . Let  $n \geq 2$  and  $y \in \mathbb{R}^n$  that verifies the hypothesis. Up to permuting its coordinates, we may assume that  $y \in \Lambda$ . By the hypothesis,  $y_n = y \cdot \mathbf{1}_{\{n\}} \leq x \cdot \varpi_1 = x_n$  and  $x_1 \leq \frac{1}{n} x \cdot \mathbf{1} = \frac{1}{n} y \cdot \mathbf{1} \leq y_n$ . We have  $x_1 \leq y_n \leq x_n$  so there

exists an integer  $1 \leq q \leq n$  such that  $x_q \leq y_n \leq x_{q+1}$ . Let  $\tau$  be the transposition between  $q$  and  $q+1$ . Let  $x' = (1-t)x + t\tau \cdot x$  for some  $0 \leq t \leq 1$ . As the support of  $\tau$  is  $\{q, q+1\}$ , for all  $k \notin \{q, q+1\}$ ,  $x'_k = x_k$ . Moreover,

$$\begin{aligned} t = 0 &\implies x'_q = x_q \leq y_n, \\ t = 1 &\implies x'_q = x_{q+1} \geq y_n. \end{aligned}$$

Therefore, for the right value of  $t$ ,  $x'_q = y_n$ . The sum of all the coefficients is stable by permutation so we have that  $x'_{q+1} = x_q + x_{q+1} - y_n$ . Now, let

$$\tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{q-1} \\ x_q + x_{q+1} - y_n \\ x_{q+2} \\ \vdots \\ x_n \end{pmatrix}, \quad \tilde{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}.$$

They are both vectors of  $\mathbb{R}^{n-1}$ . Let us show they verify the hypothesis of the lemma. They both belong in  $\Lambda$  (trivial) so it is enough to check that for all  $k$ ,  $\tilde{y} \cdot \varpi_k \leq \tilde{x} \cdot \varpi_k$  with equality when  $k = n-1$ . The equality when  $k = n-1$  is trivial. Now, let  $1 \leq k \leq n-1$ .

$$\begin{aligned} k \leq n-q-1 &\implies (\tilde{y} - \tilde{x}) \cdot \varpi_k = y_{n-k} + \cdots + y_{n-1} - x_{n+1-k} - \cdots - x_n \\ &= y \cdot \varpi_k + y_{n+1-k} - y_n - x \cdot \varpi_k \\ &\leq 0, \end{aligned}$$

$$\begin{aligned} k \geq n-q &\implies (\tilde{y} - \tilde{x}) \cdot \varpi_k = y_{n-k} + \cdots + y_{n-1} - x_{n-k} - \cdots - x_{q-1} \\ &\quad - (x_q + x_{q+1} - y_n) - x_{q+2} - \cdots - x_n \\ &= y \cdot \varpi_{k+1} - x \cdot \varpi_{k+1} \\ &\leq 0. \end{aligned}$$

By induction, it proves that  $\tilde{y} \in C(\tilde{x}) \Leftrightarrow \tilde{y} \preceq \tilde{x}$ . Notice that  $x' = (\tilde{x}, x_n) \preceq x$  by construction and  $y = (\tilde{y}, y_n) \preceq x'$  because  $\tilde{y} \preceq \tilde{x}$ . Finally, by transitivity,  $y \preceq x$ , which proves Lemma 4.2.  $\square$

PROPOSITION 4.3. —  $\preceq$  is anti-symmetric, thus an order relation over  $\Lambda$ .

*Proof.* — By Lemma 4.2, if  $x \preceq y$  and  $y \preceq x$  and if  $x$  and  $y$  both belong to  $\Lambda$ , then for all  $k$ ,  $x \cdot \varpi_k = y \cdot \varpi_k$ , which implies that  $x = y$ .  $\square$

PROPOSITION 4.4. — For every  $(x, y) \in \Lambda^2$  and for every  $\alpha \in \mathbb{R}$ ,  $C(\alpha x) = \alpha C(x)$  and  $C(x + y) = C(x) + C(y)$ .

*Proof.* — The first part is trivial, let us focus on the second one. First of all, the Minkowski sum of two convex sets is convex thus  $C(x) + C(y)$  is

convex. As it contains by definition every  $\sigma \cdot (x+y)$ , we have that  $C(x+y) \subset C(x) + C(y)$ . Let us show the reciprocal. As both sets are convex, it is enough to show that  $C(x+y)$  contains every  $\sigma \cdot x + \rho \cdot y$  where  $\sigma$  and  $\rho$  are permutations. Indeed, for all set  $K \subset \{1, \dots, n\}$ ,

$$(\sigma \cdot x + \rho \cdot y) \cdot \mathbf{1}_K = (\sigma \cdot x) \cdot \mathbf{1}_K + (\rho \cdot y) \cdot \mathbf{1}_K \leq (x+y) \cdot \varpi_K,$$

and  $(\sigma \cdot x + \rho \cdot y) \cdot \mathbf{1} = (x+y) \cdot \mathbf{1}$  so by the characterisation given by Lemma 4.2,  $\sigma \cdot x + \rho \cdot y \in C(x+y)$ .  $\square$

#### 4.2. The highest weight vector $v_\lambda$

The purpose of the subsection is to study the irreducible  $\mathrm{GL}_n(\mathbb{C})^2$ -representations  $W_\lambda$ . We won't be able to compute them explicitly but we can at least compute the only (up to a non zero scalar)  $B$ -weight vector  $v_\lambda \in \mathbb{C}[\mathrm{GL}_n]$  of weight  $\lambda \in \Lambda$ . We will focus particularly on the Newton polytopes  $\mathrm{sNewt}$  of the polynomials of  $W_\lambda \setminus \{0\}$ .

**DEFINITION 4.5.** — *When  $I$  and  $J$  are subsets of  $\{1, \dots, n\}$  of same cardinality, we define  $\det_{I,J}(A)$  as the determinant of the sub- $A$ -matrix of  $A$  where we only keep the lines indexed in  $I$  and the columns indexed in  $J$ . In particular,  $\det_{I,J}$  is a polynomial over  $\mathrm{Mat}_n(\mathbb{C})$ . We also define  $\det_k$  as  $\det_{[k+1-n,n],[k+1-n,n]}$ .*

**DEFINITION 4.6.** — *We define for every  $\lambda \in \Lambda$ ,*

$$v_\lambda = \prod_{k=1}^n \det_k^{\lambda_{n-k+1} - \lambda_{n-k}},$$

where  $\lambda_0 = 0$  by convention. As  $\lambda \in \Lambda$ , for every  $k \geq 1$ ,  $\lambda_k \leq \lambda_{k+1}$  so  $v_\lambda \in \mathbb{C}[\mathrm{GL}_n]$ . Notice that in particular,  $v_{\varpi_k} = \det_k$  for all  $k$ .

**PROPOSITION 4.7.** — *The vector  $v_\lambda$  is a  $B$ -weight vector of weight  $\lambda$  in  $\mathbb{C}[\mathrm{GL}_n]$  in the sense that for every  $(U, L) \in B$ , the Borel subgroup of  $\mathrm{GL}_n(\mathbb{C})^2$ ,  $(U, L) \cdot v_\lambda = \lambda(U)\lambda^*(L)v_\lambda$  where*

$$\lambda : U \mapsto \prod_{k=1}^n u_{kk}^{\lambda_k}, \quad \lambda^* : L \mapsto \prod_{k=1}^n l_{kk}^{-\lambda_k}.$$

are the associated weight and its dual.

*Proof.* — Let  $(U, L) \in B$  and  $X \in \mathrm{GL}_n(\mathbb{C})$ . Let us decompose them into blocs,

$$U = \begin{pmatrix} * & * \\ 0 & U' \end{pmatrix}, \quad L = \begin{pmatrix} * & 0 \\ * & L' \end{pmatrix}, \quad X = \begin{pmatrix} * & * \\ * & X' \end{pmatrix},$$

where  $L'$ ,  $U'$  and  $X'$  and  $k \times k$  matrices. We compute that

$$UXL^{-1} = \begin{pmatrix} * & * \\ * & U'X'(L')^{-1} \end{pmatrix}.$$

Therefore,  $\det_k(UXL^{-1}) = \det(U'X'(L')^{-1}) = \varpi_k(U)\varpi_k^*(L)\det_k(X)$  where  $\varpi_k \in \Lambda$  is the weight  $(0, \dots, 0, 1, \dots, 1)$  with  $k$  ones. By product,

$$v_\lambda(UXL^{-1}) = \lambda(U)\lambda^*(L)v_\lambda(X),$$

which proves Proposition 4.7.  $\square$

Now, let us determine the spherical Newton polytope of the  $v_\lambda$ .

PROPOSITION 4.8. —  $\text{sNewt}(v_\lambda) = C(\lambda)$  when  $\lambda = \varpi_k$  ( $1 \leq k \leq n$ ) or  $-\varpi_n$ .

*Proof.* — We have that  $v_{\varpi_k} = \det_k$ . This proposition is a direct consequence of the Binet–Cauchy formula,

$$\det_{I,J}(AB) = \sum_{|K|=k} \det_{I,K}(A)\det_{K,J}(B)$$

when  $A$  and  $B$  are matrices and  $|I| = |J| = k$ . In particular, with  $A \leftarrow A \text{diag}(z)$  and  $B \leftarrow B^{-1}$  when  $A$  and  $B$  are invertible and  $z$  is in the torus,

$$\det_k(A \text{diag}(z)B^{-1}) = \sum_{|K|=k} \det_{[k+1-n,n],K}(A)\det_{K,[k+1-n,n]}(B)z^{\mathbf{1}_K}.$$

Therefore,  $S_{\det_k} = \{\mathbf{1}_K \mid |K| = n\} = S_n \cdot \varpi_k$  so by definition,  $\text{sNewt}(v_{\varpi_k}) = C(\varpi_k)$ . For  $\lambda = -\varpi_n$ ,  $v_\lambda = \det^{-1}$  so  $\text{sNewt}(v_{-\varpi_n}) = \{-\mathbf{1}\} = C(-\varpi_n)$ .  $\square$

LEMMA 4.9. — If  $f, g$  are two matrix polynomials,  $\text{sNewt}(fg) = \text{sNewt}(f) + \text{sNewt}(g)$  (in the sense of Minkowski).

*Proof.* — It is well-known that this formula is true for classical Laurent polynomial and their Newton polytope. For any  $h \in \mathbb{C}[\text{GL}_n]$ , the set of  $(A, B) \in \text{GL}_n(\mathbb{C})^2$  such that  $\text{sNewt}(h) = \text{Newt}(\Psi_{A,B}(h))$  is Zariski open, thus dense. It implies that for generic invertible matrices  $A, B$ ,  $\text{sNewt}(f) = \text{Newt}(\Psi_{A,B}(f))$ ,  $\text{sNewt}(g) = \text{Newt}(\Psi_{A,B}(g))$  and  $\text{sNewt}(fg) = \text{Newt}(\Psi_{A,B}(fg))$  so,

$$\begin{aligned} \text{sNewt}(fg) &= \text{Newt}(\Psi_{A,B}(fg)) \\ &= \text{Newt}(\Psi_{A,B}(f)\Psi_{A,B}(g)) \\ &= \text{Newt}(\Psi_{A,B}(f)) + \text{Newt}(\Psi_{A,B}(g)) \\ &= \text{sNewt}(f) + \text{sNewt}(g). \end{aligned} \quad \square$$

PROPOSITION 4.10. — For every  $\lambda$ ,  $\text{sNewt}(v_\lambda) = C(\lambda)$ .

*Proof.* — Any  $\lambda \in \Lambda$  can be written as  $\lambda = \sum_{k=1}^n \alpha_k \varpi_k$  where the  $\alpha_k$  are non-negative integers except  $\alpha_n$  which is a relative integer. Notice that

$v_\lambda = \prod_{k=1}^n v_{\varpi_k}^{\alpha_k}$  so by Lemma 4.9, Proposition 4.8 and Proposition 4.4,

$$\text{sNewt}(v_\lambda) = \sum_{k=1}^n \alpha_k \text{sNewt}(v_{\varpi_k}) = \sum_{k=1}^n \alpha_k C(\varpi_k) = C(\lambda).$$

This remains true even when  $\alpha_n < 0$ . □

### 4.3. Proof of the equivalence

We now have enough tools to prove Proposition 4.12.

**PROPOSITION 4.11.** — *Every non zero polynomial  $f \in W_\lambda$  verifies  $S_f = S_{v_\lambda}$ , thus  $\text{sNewt}(f) = \text{sNewt}(v_\lambda) = C(\lambda)$ .*

*Proof.* — First of all, it is clear that the support of a matrix polynomial is stable by the action of  $\text{GL}_n(\mathbb{C})^2$  by definition. As  $W_\lambda = \text{Span}(\text{GL}_n(\mathbb{C}) \cdot v_\lambda)$ , any polynomial  $f \in W_\lambda$  verifies  $S_f \subset S_{v_\lambda}$  by sum. Now, consider some  $m \in S_{v_\lambda}$  and  $V = \{f \in W_\lambda \mid m \notin S_f\}$ . It is clearly a vector space because  $m \notin S_f \Leftrightarrow$  the  $Q_m$  of  $f$  is identically zero. Moreover,  $V$  is stable by the action of  $\text{GL}_n(\mathbb{C})^2$ , it is a representation of this group. But  $W_\lambda$  is irreducible so  $V = \{0\}$  or  $W_\lambda$ . However, as  $m \in S_{v_\lambda}$ ,  $v_\lambda \notin V$  thus  $V = \{0\}$ . It is true for any  $m$  so the proposition is proven. □

**PROPOSITION 4.12.** — *For all polynomial  $f \in \mathbb{C}[\text{GL}_n]$ ,  $\text{sNewt}(f) = \Delta(f)$ .*

*Proof.* — Let  $f \in \mathbb{C}[\text{GL}_n]$ . Write  $f = \sum_{\lambda \in A} f_\lambda$  where  $A \subset \Lambda$  is a finite set and for all  $\lambda$  in  $A$ ,  $f_\lambda \in W_\lambda \setminus \{0\}$ . Let us show that  $\text{sNewt}(f) = \Delta(f)$ . As  $\text{sNewt}(f) = \text{Conv}(S_f)$  and  $\Delta(f) = \text{Conv}(\mathcal{S}_n \cdot A)$  and they are both stable by permutation, it is enough to show that  $S_f \subset \text{Conv}(A)$  and  $A \subset \text{Conv}(S_f)$ .

$\subset$ . — If  $\lambda \in S_f$ . It means that  $Q_\lambda^{(f)} \neq 0$  so  $Q_\lambda^{(f_\mu)} \neq 0$  for at least one  $\mu \in A$ . It means that  $\lambda \in \text{sNewt}(f_\mu) = C(\mu)$  by Proposition 4.11.  $\lambda \in C(\mu) \subset \text{Conv}(A) = \Delta(f)$ .

$\supset$ . — If  $\lambda \in A$ , let  $\lambda_M \in A$  such that  $\lambda \preccurlyeq \lambda_M$  and  $\lambda_M$  is a maximal element of  $A$ . Such a  $\lambda_M$  exists because  $A$  is finite. Let us show that  $\lambda_M \in S_f$ . Indeed,  $Q_{\lambda_M}^{(f_{\lambda_M})} \neq 0$  and for all  $\mu \in A \setminus \{\lambda_M\}$ ,  $\lambda_M \not\preccurlyeq \mu$  because  $\lambda_M$  is a maximal element of  $A$ , thus  $\lambda_M \notin C(\mu) = \text{sNewt}(f_\mu)$  so  $Q_{\lambda_M}^{(f_\mu)} = 0$ . It implies that  $Q_{\lambda_M}^{(f)} = Q_{\lambda_M}^{(f_{\lambda_M})} \neq 0$ .  $\lambda_M \in S_f$ . It proves that  $\lambda \in C(\lambda_M) \subset \text{Conv}(S_f) = \text{sNewt}(f)$ . □

## 5. Tropical geometry

The goal of this section is to generalize a theorem of Bergman [2] that makes a link between the tropical variety of an ideal and its amoeba. We will also recall the definition of the tropical variety of a matrix spherical variety  $Y \subset \mathrm{GL}_n(\mathbb{C})$  which is a particular case of the definition given by Tevelev and Vogliannou [16].

### 5.1. Definitions and theorems in $(K^*)^n$

Let  $K$  be an algebraically closed field of characteristic 0 endowed with a non trivial valuation that is trivial on  $\mathbb{Q}$ . Let  $I$  be a proper ideal of the Laurent polynomial ring  $K[X^\pm] = K[X_1^\pm, \dots, X_n^\pm]$ . We define the algebraic variety  $Y = Y(K)$  associated to  $I$  as the set of non zero vectors  $z \in (K^*)^n$  where all the polynomials of  $I$  vanish. For every polynomial in  $K[X^\pm]$ , we define its tropical version as

$$\mathrm{trop}\left(\sum_{m \in \mathbb{Z}^n} a_m X^m\right) : x \longmapsto \min\{\mathrm{val}(a_m) + m \cdot x \mid m \in \mathbb{Z}^n\},$$

which is a affine by part convex function from  $\mathbb{R}^n$  to  $\mathbb{R}$ . Notice that the min convention has been used but a similar version with a max also exists. *Introduction to Tropical Geometry*, by MacLagan and Sturmfel [11] is a good reference for tropical geometry. We define the tropical hypersurface of any tropical polynomial  $P$  as the set of points where the minimum is reached at least twice, i.e. the set where it is not differentiable. We call it  $V(P)$ . When  $Y = V(I) \subset (K^*)^n$  is a very affine variety, we have the fundamental theorem of tropical algebra,

$$\bigcap_{f \in I} V(\mathrm{trop}(f)) = \overline{\{(\mathrm{val}(z_1), \dots, \mathrm{val}(z_n)) \mid z \in Y\}}.$$

We call this set  $\mathrm{trop}(Y)$ . Notice that it only depends on  $Y$  and not on the choice of  $I$ . There is a third definition using initial ideals (see [11, Theorem 3.2.3]). Assume now that  $K = \bar{\mathbb{C}}$  is the set of complex Puiseux series, endowed with a valuation,

$$\mathrm{val} : \sum_{m \in S} a_m t^m \longmapsto \min(S),$$

where  $S$  is non empty and the  $a_m$  are non zero complex numbers, and  $\mathrm{val}(0) = +\infty$ . This field is algebraically closed of characteristic 0 and the valuation is non trivial, but is trivial over  $\mathbb{Q}$ . Therefore, the previous theorem holds. When  $I \subset \mathbb{C}[X^\pm]$ , let  $Y = Y(\mathbb{C})$  be the variety associated to  $I$  and

$Y(\bar{K}) \subset (\bar{K}^*)^n$  be the variety associated with the ideal of  $\bar{K}[X^\pm]$  generated by the elements of  $I$ . We have the Bergman's theorem,

$$\rho\mathcal{A}(Y) \xrightarrow{\rho \rightarrow 0^+} -\text{trop}(Y) \text{ in the sense of Kuratowski,}$$

and  $\text{trop}(Y)$  is a finite union of polyhedral cones. Moreover, when  $I = (f)$  is principal,  $-\text{trop}(Y)$  is the normal cone of the Newton polytope of  $f$ . See [11, Section 2.3] for an introduction to convex geometry and the definition of the normal cone of a convex polytope. Let us extend it to matrices.

## 5.2. Definitions in $\text{GL}_n(\mathbb{C})$

We will work exclusively on  $\mathbb{C}$  and  $\bar{K}$  since the notions of singular values and invariant factors are hardly generalisable to any field.

DEFINITION 5.1. — *Let  $\mathcal{R} = \{z(t) \in \bar{K} \mid \text{val}(z(t)) \geq 0\}$ .*

$\mathcal{R}$  is a ring and its invertible elements are  $\mathcal{R}^\times = \{z(t) \in \bar{K} \mid \text{val}(z(t)) = 0\} = \{z(t) \in \mathcal{R} \mid z(0) \neq 0\}$ . Notice that  $\mathfrak{m} = \mathcal{R} \setminus \mathcal{R}^\times$  is an ideal, so  $\mathcal{R}$  is local with maximal ideal  $\mathfrak{m}$  and the residue field of  $\bar{K}$  is  $\mathcal{R}/\mathfrak{m} = \mathbb{C}$ . But the most important is that  $\mathcal{R}$  is integral with  $\text{Frac}(\mathcal{R}) = \bar{K}$ . The natural way to extend the notion of coordinate-wise valuation of a vector of complex Puiseux series to matrices is to use the invariant factors given by the Smith normal form (we will see that it coincides with the definition given in [16]). However, Smith's theorem requires to work on a principal ideal domain and  $\mathcal{R}$  is not principal (it is not even Noetherian as  $\mathfrak{m}$  is not finitely generated). We need to extend Smith's theorem.

PROPOSITION 5.2 (Smith normal form). — *For any matrix of Puiseux series  $A(t) \in \text{Mat}_n(\bar{K})$ , there exist matrices  $(P(t), Q(t)) \in \text{GL}_n(\mathcal{R})$  and a diagonal matrix  $D(t)$  such that  $A(t) = P(t)D(t)Q^{-1}(t)$ . Moreover, the valuations of the diagonal elements of  $D(t)$  are unique up to permutation. We call them invariant factors.*

*Proof.* — Let  $A(t)$  be a Puiseux series matrix and  $G$  be the group generated by  $\{q \in \mathbb{Q} \mid t^q \text{ appears in } A(t)\}$ . Exponents in Puiseux series all have a common denominator and  $A(t)$  has a finite number of coefficients so  $G$  is discrete. Let  $K = \{\sum_{g \in G} a_g t^g \in \bar{K}\}$ .  $K$  is a field and the valuation inherited from the valuation on  $\bar{K}$  is discrete on  $K$  because  $\Gamma_{\text{val}|_K} = G$ . As  $K$  is a field with a discrete valuation, the ring  $R = \{z(t) \in K \mid \text{val}(z(t)) \geq 0\} \subset \mathcal{R}$  is a principal ideal domain. As all the coefficients of  $A(t)$  belong to  $K$ , we can apply the Smith normal form theorem. There exists  $(P(t), Q(t)) \in \text{GL}_n(R) \subset \text{GL}_n(\mathcal{R})$  and  $D(t)$  a diagonal matrix such that  $A(t) = P(t)D(t)Q^{-1}(t)$ .

In particular, the coefficients of  $P(t)$  and  $Q(t)$  are series whose exponents all are in  $G \cap \mathbb{R}_+$ .

If  $A(t) = P(t)D(t)Q^{-1}(t) = \tilde{P}(t)\tilde{D}(t)\tilde{Q}^{-1}(t)$  with  $\tilde{P}(t)$  and  $\tilde{Q}(t)$  invertible in  $\mathcal{R}$  and  $\tilde{D}(t)$  diagonal, we can use the uniqueness in the Smith normal form theorem in the field  $\tilde{K} = \{\sum_{g \in \tilde{G}} a_g t^g \in \tilde{K}\}$  where  $\tilde{G}$  is the group generated by the exponents of the coefficients of  $P(t)$ ,  $Q(t)$ ,  $D(t)$ ,  $\tilde{P}(t)$ ,  $\tilde{Q}(t)$  and  $\tilde{D}(t)$  to deduce that the valuation of the diagonal coefficients in  $D(t)$  are the same than in  $\tilde{D}(t)$  up to permutation. It proves Proposition 5.2.  $\square$

It allows us to define the matrix spherical valuation of a Puiseux series matrix  $A(t)$ ,

DEFINITION 5.3.

$$\text{sval} : \begin{cases} \text{GL}_n(\bar{K}) \longrightarrow \mathbb{Q}^n / \mathcal{S}_n, \\ A(t) = P(t) \text{diag}(z(t)) Q^{-1}(t) \longmapsto (\text{val}(z_1(t)), \dots, \text{val}(z_n(t))). \end{cases}$$

Now, let  $I$  be an ideal of the ring  $\mathbb{C}[\text{GL}_n]$ . We define the spherical variety  $Y(\mathbb{C})$  associated to  $I$  as the set of invertible matrices  $A$  where all the polynomials of  $I$  vanish and the spherical variety in Puiseux series  $Y(\bar{K})$  as the variety associated with the ideal generated by the elements of  $I$  in  $\bar{K}[\text{GL}_n]$ . When there is no ambiguity, we can use the notation  $Y$  to talk about  $Y(\mathbb{C})$  as well as  $Y(\bar{K})$ . There does not seem to be a good generalisation of the tropicalization of a matrix polynomial so we shall define the spherical tropical variety of  $Y$  thanks to the spherical valuation,

DEFINITION 5.4.

$$\text{strop}(Y) = \overline{\{\text{sval}(A(t)) \mid A(t) \in Y(\bar{K})\}},$$

where the set  $\text{strop}_{\mathbb{Q}}(Y) = \{\text{sval}(A(t)) \mid A(t) \in Y(\bar{K})\} \subset \mathbb{Q}^n / \mathcal{S}_n$  is mistaken by abuse with the set of  $x \in \mathbb{Q}^n$  such that the orbit of  $x$  under  $\mathcal{S}_n$  is in  $\text{strop}_{\mathbb{Q}}(Y)$ . We can make the link with classical tropical variety, and thus use if necessary the fundamental theorem of tropical algebra. Notice that this definition actually coincides with the more general definition of tropical varieties of a spherical variety by Tevelev and Voggiannou [16, Theorem 1.3].

PROPOSITION 5.5. — *Let  $I$  be a proper ideal of  $\mathbb{C}[\text{GL}_n]$ . For any matrices  $P(t), Q(t)$  that are invertible in  $\text{GL}_n(\mathcal{R})$ , and  $f \in \mathbb{C}[\text{GL}_n]$ , we extend the definition of  $\Psi$ ,*

$$\Psi_{P(t), Q(t)}(f) : \begin{cases} (\bar{K}^*)^n \longrightarrow \bar{K} \\ z(t) \longmapsto f(P(t) \text{diag}(z(t)) Q^{-1}(t)). \end{cases}$$

$\Psi_{P(t),Q(t)} : \bar{\mathcal{K}}[\mathrm{GL}_n] \rightarrow \bar{\mathcal{K}}[X^\pm]$  is a  $\bar{\mathcal{K}}$ -algebra homomorphism. Let  $Y_{P(t),Q(t)} = V(\Psi_{P(t),Q(t)}(I))$ . We have

$$\mathrm{strop}(Y) = \overline{\bigcup_{(P(t),Q(t)) \in \mathrm{GL}_n(\mathcal{R})} \mathrm{trop}(Y_{P(t),Q(t)})}$$

*Proof of Proposition 5.5.* — Let  $x \in \mathbb{Q}^n$ :

$$\begin{aligned} x &\in \mathrm{strop}_{\mathbb{Q}}(Y) \\ &\iff x = \mathrm{sval}(A(t)) \text{ for some } A(t) \in Y. \\ &\iff f(P(t) \operatorname{diag}(z(t))Q^{-1}(t)) = 0 \text{ for some } (P(t), Q(t)) \in \mathrm{GL}_n(\mathcal{R})^2 \\ &\quad \text{and for all } k, \operatorname{val}(z_k(t)) = x_k. \\ &\iff x \in \mathrm{trop}_{\mathbb{Q}}(Y_{P(t),Q(t)}) \text{ for some } (P(t), Q(t)) \in \mathrm{GL}_n(\mathcal{R})^2 \\ &\iff x \in \bigcup_{(P(t),Q(t)) \in \mathrm{GL}_n(\mathcal{R})} \mathrm{trop}_{\mathbb{Q}}(Y_{P(t),Q(t)}), \end{aligned}$$

which proves Proposition 5.5, by taking the closure.  $\square$

As for classical amoebas, we conjecture that for every spherical variety  $Y$ ,

CONJECTURE 5.6. —  $\rho\mathcal{A}(Y) \rightarrow_{\rho \rightarrow 0^+} -\mathrm{strop}(Y)$  in the sense of Kuratowski.

### 5.3. First inclusion

Recall the definition of Kuratowski limit: if  $(E_m)_{m \in \mathbb{N}}$  is a family of subsets of a topological space  $E$  (we can replace the discrete  $m \in \mathbb{N}$  by a continuous variable that converge, or diverges toward  $+\infty$  or  $-\infty$ ),

$$\begin{aligned} &\liminf_{m \rightarrow +\infty} E_m \\ &= \{x \in E \mid \forall U \text{ neighborhood of } x, \exists m_0 \in \mathbb{N}, \forall m \geq m_0, U \cap E_m \neq \emptyset\}, \\ &\limsup_{m \rightarrow +\infty} E_m \\ &= \{x \in E \mid \forall U \text{ neighborhood of } x, \forall m_0 \in \mathbb{N}, \exists m \geq m_0, U \cap E_m \neq \emptyset\}. \end{aligned}$$

In particular,  $\liminf_{m \rightarrow +\infty} E_m \subset \limsup_{m \rightarrow +\infty} E_m$  and when they are equal, we call  $\lim_{m \rightarrow +\infty} E_m$  the common limit. Therefore, given a variety  $Y$ , the conjecture is equivalent to

$$\limsup_{\rho \rightarrow 0^+} \rho\mathcal{A}(Y) \subset -\mathrm{strop}(Y) \subset \liminf_{\rho \rightarrow 0^+} \rho\mathcal{A}(Y).$$

First of all, let us show the second inclusion, which is the easiest one.

DEFINITION 5.7. — For any rational number  $q$ , define the truncation  $T_q$  by

$$T_q : \begin{cases} \bar{\mathcal{K}} \longrightarrow \bar{\mathcal{K}} \\ \sum_{m \in S} a_m t^m \longmapsto \sum_{\substack{m \in S \\ m \leq q}} a_m t^m \end{cases}$$

and extend it to  $\mathcal{Mat}_n(\bar{\mathcal{K}})$  by truncating each entry.

In both cases, it is a  $\mathbb{C}$ -linear map. It is clear that  $\mathcal{R}$  is stable under  $T_q$  and for all rational number  $q \geq 0$ ,  $(t^q)$  is an ideal such that for all  $a(t) \in \mathcal{R}$ ,  $T_q a(t) \equiv a(t) [t^q]$ . This remains true if we replace  $\mathcal{R}$  by  $\mathcal{Mat}_n(\mathcal{R})$ .

LEMMA 5.8. — For any matrices  $A(t) \in \mathrm{GL}_n(\bar{\mathcal{K}})$  and  $B(t) \in \mathcal{Mat}_n(\bar{\mathcal{K}})$ , if all the invariant factors of  $A(t)$  are less than or equal to all the invariant factors of  $B(t)$ , then  $B(t)A^{-1}(t)$  has its coefficients in  $\mathcal{R}$ .

Moreover, if we replace “less than or equal to” by “less than”,  $(A(t) + B(t))A^{-1}(t) \in \mathrm{GL}_n(\mathcal{R})$ .

*Proof.* — Let  $q$  be a rational number such that all the invariant factors of  $A(t)$  are less than or equal to  $q$  and all the invariant factors of  $B(t)$  are greater than or equal to  $q$ . Therefore,  $t^{-q}B(t) \in \mathcal{Mat}_n(\mathcal{R})$  and the invariant factors of  $A^{-1}(t)$  are opposite to the invariant factors of  $A(t)$  thus  $t^q A^{-1}(t) \in \mathcal{Mat}_n(\mathcal{R})$  so  $B(t)A^{-1}(t) = t^{-q}B(t)t^q A^{-1}(t)$  has coefficients in  $\mathcal{R}$ .

Now, if we replace “less than or equal to” by “less than”, by using the first part of Lemma 5.8 with  $A(t)$  and  $t^{-\varepsilon}B(t)$  for a small enough rational  $\varepsilon > 0$ , the matrix  $B(t)A^{-1}(t)$  is in the ideal  $(t^\varepsilon)$  of the ring  $\mathcal{Mat}_n(\mathcal{R})$ . Let  $S(t) = \sum_{m \in \mathbb{N}} (-1)^m (B(t)A^{-1}(t))^m$ . As  $B(t)A^{-1}(t) \in (t^\varepsilon)$ , this series converges in  $\mathcal{Mat}_n(\mathcal{R})$  and it is clear that  $S(t)(I_n + B(t)A^{-1}(t)) = I_n$ . It implies that  $I_n + B(t)A^{-1}(t) = (A(t) + B(t))A^{-1}(t)$  is invertible in  $\mathcal{R}$ .  $\square$

LEMMA 5.9. — For any  $A(t) \in \mathrm{GL}_n(\bar{\mathcal{K}})$ , for any  $q \in \mathbb{Q}$  greater than every invariant factor of  $A(t)$ ,  $T_q A(t)$  is invertible and  $\mathrm{sval}(T_q A(t)) = \mathrm{sval}(A(t))$ .

*Proof.* — If  $A(t) \in \mathcal{Mat}_n(\bar{\mathcal{K}})$ ,  $v_{\min}$  and  $v_{\max}$  be its smallest and the biggest invariant factors. We know that  $t^{-v_{\min}}A(t) \in \mathcal{Mat}_n(\mathcal{R})$  so we can write it as

$$t^{-v_{\min}}A(t) = P(t) \mathrm{diag}(t^{v_1 - v_{\min}}, \dots, t^{v_n - v_{\min}}) Q^{-1}(t),$$

where  $P(t)$  and  $Q(t)$  are both invertible in  $\mathcal{R}$  and  $\mathrm{sval}(A(t)) = (v_1, \dots, v_n)$ . Moreover, if  $q = v_{\max} - v_{\min} + \varepsilon$  for some positive rational  $\varepsilon$ ,

$$T_q \mathrm{diag}(t^{v_1 - v_{\min}}, \dots, t^{v_n - v_{\min}}) = \mathrm{diag}(t^{v_1 - v_{\min}}, \dots, t^{v_n - v_{\min}}),$$

so

$$\begin{aligned} t^{-v_{\min}} T_{q+v_{\min}} A(t) &= T_q(t^{-v_{\min}} A(t)) \\ &\equiv T_q P(t) \operatorname{diag}(t^{v_1-v_{\min}}, \dots, t^{v_n-v_{\min}}) T_q Q^{-1}(t) [t^q] \end{aligned}$$

and  $q + v_{\min} = v_{\max} + \varepsilon$  so by multiplying the previous equality by  $t^{v_{\min}}$ ,

$$T_{v_{\max}+\varepsilon} A(t) = T_q P(t) \operatorname{diag}(t^{v_1}, \dots, t^{v_n}) T_q Q^{-1}(t) + B(t)$$

where  $B(t) \in (t^{v_{\max}+\varepsilon})$ . Let  $\tilde{A}(t) = T_q P(t) \operatorname{diag}(v_1, \dots, v_n) T_q Q^{-1}(t)$ .  $q$  is positive by definition so we have that  $\det(T_q(P(t))(0)) = \det(P(0)) \in \mathbb{C}^*$  because  $P$  is invertible in  $\mathcal{R}$ . It implies that  $T_q P(t)$  is also invertible in  $\mathcal{R}$ . Same thing with  $Q^{-1}(t)$ .

Therefore,  $\tilde{A}(t)$  is invertible and its invariant factors are the  $v_k$  which are all dominated by  $v_{\max}$ , itself strictly dominated by all the invariant factors of  $B(t)$ . By Lemma 5.8,  $T_{v_{\max}+\varepsilon} A(t) = \tilde{A}(t) + B(t)$  and  $\tilde{A}(t)$  have the same image in  $\mathcal{R}$  up to isomorphism so they have the same invariant factors, which proves Lemma 5.9.  $\square$

Using the same reasoning with Lemma 5.8, we deduce that,

**COROLLARY 5.10.** — *For any  $(A(t), B(t)) \in \operatorname{GL}_n(\bar{K}) \times \operatorname{Mat}_n(\bar{K})$ , for any  $q \in \mathbb{Q}$  greater than every invariant factor of  $A(t)$ , if  $A(t) \equiv B(t)[t^q]$ ,  $B(t)$  is invertible and  $\operatorname{sval}(B(t)) = \operatorname{sval}(A(t))$ .*

Let  $(K, \operatorname{val})$  be a valued field and  $\Gamma \subset \mathbb{R}$  the image of  $\operatorname{val}$  (thus a subgroup of  $\mathbb{R}$ ). If  $Y \subset (K^*)^n$  is a very affine variety, we let

$$\operatorname{trop}_{\Gamma}(Y) = \{\operatorname{val}(z_1, \dots, z_n) \mid z \in Y\} \subset \Gamma^n,$$

be the set of points in  $\Gamma$  of the tropical variety of  $Y$  and  $\operatorname{trop}(Y) = \overline{\operatorname{trop}_{\Gamma}(Y)}$  its usual tropical variety.

**LEMMA 5.11.** — *Let  $F/K$  be a field extension with  $K$  and  $F$  both algebraically closed. Let  $I_F \subset F[X_1^{\pm}, \dots, X_n^{\pm}]$  and  $I_K = I_F K[X_1^{\pm}, \dots, X_n^{\pm}]$  which is an ideal of  $K[X_1^{\pm}, \dots, X_n^{\pm}]$ . If  $\operatorname{val} : K \rightarrow \mathbb{R}$  is a valuation, such that  $\operatorname{val}(K) = \operatorname{val}(F)$  (we call  $\Gamma$  this dense subgroup of  $\mathbb{R}$ ),  $\operatorname{trop}_{\Gamma}(V(I_F)) = \operatorname{trop}_{\Gamma}(V(I_K))$  and  $\operatorname{trop}(V(I_F)) = \operatorname{trop}(V(I_K))$ .*

*Proof.*

$\subset$ . — If  $x \in \operatorname{trop}_{\Gamma}(V(I_F))$ , for all  $1 \leq k \leq n$ ,  $x_k = \operatorname{val}(z_k)$  for some  $z \in V(I_F) \subset (F^*)^n$ .  $I_K$  is generated by  $I_F$  so  $z \in V(I_K)$  thus  $x \in \operatorname{trop}_{\Gamma}(V(I_K))$ .

$\supset$ . — If  $x \in \operatorname{trop}_{\Gamma}(V(I_K))$ , by the fundamental theorem of tropical geometry,  $x \in \Gamma \cap \bigcap_{f \in I_K} V(\operatorname{trop}(f))$ . As  $I_F \subset I_K$ ,

$$x \in \Gamma \cap \bigcap_{f \in I_F} V(\operatorname{trop}(f)) = \operatorname{trop}_{\Gamma}(V(I_F)).$$

We proved that

$$\text{trop}_F(V(I_F)) = \text{trop}_F(V(I_K)) \quad \text{thus} \quad \text{trop}(V(I_F)) = \text{trop}(V(I_K))$$

by taking the closure.  $\square$

Now, let us prove the last lemma we need for the first inclusion,

LEMMA 5.12. — *For any ideal  $I \subset \mathbb{C}[\text{GL}_n]$ , for any  $A(t) \in V(I)$ , there exists a  $B(t) \in \text{GL}_n(F) \cap V(I)$  that verifies  $\text{sval}(A(t)) = \text{sval}(B(t))$  where  $F$  is the set of Puiseux series which converge in  $]0, \varepsilon[$  for some  $\varepsilon > 0$ .*

*Proof.* — Let  $I$  be such an ideal and  $A(t) \in V(I)$ . Let  $q$  be a rational number greater than any invariant factor of  $A(t)$ . By Newton–Puiseux theorem,  $F$  is an algebraically closed field. Let

$$S = \{(i, j) \in \{1, \dots, n\}^2 \mid (A(t) - T_q A(t))_{ij} \neq 0\}$$

and for all polynomial  $f \in \mathbb{C}[\text{GL}_n]$ , we define  $g_f \in F[(X_{ij})_{(i,j) \in S}^\pm]$  by,

$$g_f : \begin{cases} (F^*)^S \longrightarrow F \\ (x_{ij}(t))_{(i,j) \in S} \longmapsto f(T_q A(t) + t^q X(t)) \end{cases}$$

where  $X_{ij}(t) = x_{ij}(t)$  if  $(i, j) \in S$  and equals 0 otherwise.

Let  $J_F$  be the ideal generated by the  $g_f$  for

$$f \in I \quad \text{and} \quad J_{\bar{K}} = J_F \bar{K} \left[ (X_{ij})_{(i,j) \in S}^\pm \right].$$

By Lemma 5.11,  $\text{trop}_{\mathbb{Q}}(J_F) = \text{trop}_{\mathbb{Q}}(J_{\bar{K}}) \subset \mathbb{Q}^S$ .

Let for all  $(i, j) \in S$ ,  $z_{ij}(t) = t^{-q}(A(t) - T_q A(t))_{ij} \neq 0$  and  $x_{ij} = \text{val}(z_{ij}(t)) > 0$  (they are positive by definition of  $T_q$ ). By definition of the  $g_f$  and  $J_{\bar{K}}$ , we have  $(z_{ij}(t)) \in V(J_{\bar{K}})$  because  $A(t) \in V(I)$ , thus  $(x_{ij}) \in \text{trop}(J_{\bar{K}}) = \text{trop}(J_F)$ .

Therefore, there exists Puiseux series  $(\tilde{z}_{ij}(t))_{(i,j) \in S}$  that converge in a neighborhood of  $0^+$  such that for all  $(i, j) \in S$ ,  $\text{val}(\tilde{z}_{ij}(t)) = x_{ij} > 0$  and which belong to  $V(J_F)$ . Let  $B(t) = T_q A(t) + t^q Z(t)$  where  $Z_{ij}(t) = \tilde{z}_{ij}(t)$  if  $(i, j) \in S$  and equals 0 otherwise. In particular,  $Z(t) \in \mathcal{M}\text{at}_n(\mathcal{R} \cap F)$ . By construction,  $B(t) \in \mathcal{M}\text{at}_n(F)$ ,  $B(t) \in V(I)$  and  $B(t) \equiv A(t) [t^q]$  thus  $B(t) \in \text{GL}_n(F)$  and  $\text{sval}(B(t)) = \text{sval}(A(t))$  by Corollary 5.10. It proves Lemma 5.12.  $\square$

And finally, the wanted inclusion,

PROPOSITION 5.13. —  $-\text{strop}_{\mathbb{Q}}(Y) \subset \liminf_{\rho \rightarrow 0^+} \rho \text{s}A(Y)$ .

*Proof.* — Let  $x \in -\text{strop}_{\mathbb{Q}}(Y)$ . It means that there exists  $A(t) \in \text{GL}_n(\bar{K})$  such that  $x = -\text{sval}(A(t))$  and for all  $f \in I$ ,  $f(A(t)) = 0$ . By Lemma 5.12, we may assume without loss of generality that  $A(t)$  converges on  $]0, \varepsilon[$  for

some  $\varepsilon > 0$ . We have [9, Theorem 1.1], which has been proven in  $\mathcal{K}$  but works the same way in  $\bar{\mathcal{K}}$ ,

$$\frac{\mathrm{sLog}(A(s))}{\ln(s)} \xrightarrow{s \rightarrow 0^+} \mathrm{sval}(A(t)),$$

so with  $\rho = -\frac{1}{\ln(s)}$ ,

$$\rho \mathrm{sLog}\left(A(\mathbf{e}^{-1/\rho})\right) \xrightarrow{\rho \rightarrow 0^+} x.$$

and all the  $\mathrm{sLog}(A(\mathbf{e}^{-1/\rho}))$  (for  $\rho > 0$  small enough) belong to  $\mathrm{sA}(Y)$ , which proves the inclusion.  $\square$

**COROLLARY 5.14.** —  $-\mathrm{strop}(Y) \subset \liminf_{\rho \rightarrow 0^+} \rho \mathrm{sA}(Y)$  because inferior and superior limits in the sense of Kuratowski are always closed.

## 5.4. The second inclusion when $I$ is principal

In this section, we consider  $I = (f)$  a principal ideal. It implies that the  $\Psi_{P(t), Q(t)}(I) = (\Psi_{P(t), Q(t)}(f))$  are also principal. Propositions 3.16 and 3.14 give an idea of the shape of the amoeba of  $f$  in function of its Newton polytope  $\mathrm{sNewt}(f)$ . It will be helpful in order to determine its spherical tropical variety.

**DEFINITION 5.15.** — If  $\mathrm{sNewt}(f) \cap \mathbb{Z}\mathbf{1} = \emptyset$ , we define  $C^- = C^+ = \emptyset$ . Otherwise, consider  $N^-$  (resp.  $N^+$ ) the smallest (resp. biggest) integer such that  $N^-\mathbf{1}$  (resp.  $N^+\mathbf{1}$ ) belongs to  $\mathrm{sNewt}(f)$ . We define  $C^-$  as  $C_{\{N^-\mathbf{1}\}}^\circ$  (resp.  $C^+$  as  $C_{\{N^+\mathbf{1}\}}^\circ$ ). Notice that if  $N^-\mathbf{1}$  (resp.  $N^+\mathbf{1}$ ) is not a vertex of  $\mathrm{sNewt}(f)$ ,  $C^- = \emptyset$  (resp.  $C^+ = \emptyset$ ).

**PROPOSITION 5.16.** —  $\limsup_{\rho \rightarrow 0^+} \rho \mathrm{sA}(f) \subset \mathbb{R}^n \setminus (C^- \cup C^+)$ .

*Proof.* — We need to prove that every point in  $C^-$  or in  $C^+$  does not belong to the limit. By symmetry, it is enough to prove it for  $C^+$ . It is trivial if  $C^+$  is empty. Assume now that  $\mathrm{sNewt}(f) \cap \mathbb{Z}\mathbf{1} \neq \emptyset$  and  $N^+\mathbf{1}$  is a vertex of  $\mathrm{sNewt}(f)$ . Let  $x \in C^+ = C_{\{N^+\mathbf{1}\}}^\circ$ . By Propositions 3.16 and 3.14, there exists a point  $y \in E$  where  $E \in \Gamma(\mathrm{sA}(f)^C)$  and  $\nu_f(E) = N^+\mathbf{1}$ , and  $y + C_{\{N^+\mathbf{1}\}} \subset E$ .

Consider  $r > 0$  such that  $B(x, r) \subset C_{\{N^+\mathbf{1}\}}$ . For all  $x' \in B(x, \frac{r}{2})$  and for all  $0 < \rho < \frac{r}{2\|y\|}$  (any  $\rho > 0$  if  $y = 0$ ),

$$x' - \rho y \in B\left(x', \frac{r}{2}\right) \subset B(x, r) \subset C_{\{N^+\mathbf{1}\}}.$$

As this set is a cone, it is stable by product by a positive real number so  $\frac{x'}{\rho} - y \in C_{\{N+1\}}$ , which means that  $x' \in \rho(y + C_{\{N+1\}}) \subset \rho s\mathcal{A}(f)^C$  for all  $\rho$  small enough. This is true for all  $x'$  in a neighborhood of  $x$  so  $x \notin \limsup_{\rho \rightarrow 0^+} \rho s\mathcal{A}(f)$ . It proves Proposition 5.16.  $\square$

We now need to prove that  $\mathbb{R}^n \setminus (C^- \cup C^+) \subset -\text{strop}(V(f))$  to get the wanted inclusion. The proof is a bit handmade and needs some lemmas.

LEMMA 5.17. — *Let  $p \geq 2$  and  $(f_k)_{1 \leq k \leq p}$  a family of continuous functions of a segment  $[a, b]$  such that there exists  $1 \leq i \neq j \leq p$  verifying for all  $1 \leq k \leq p$ ,  $f_i(a) \leq f_k(a)$  and  $f_j(b) \leq f_k(b)$ . Then, there exists  $a \leq c \leq b$  and  $1 \leq i' \neq j' \leq p$  such that for all  $1 \leq k \leq p$ ,  $f_{i'}(c) = f_{j'}(c) \leq f_k(c)$ .*

*Proof.* — Just apply the intermediate values theorem to the function  $s \mapsto f_i(s) - \min_{k \neq i} \{f_k(s)\}$ .  $\square$

LEMMA 5.18. — *Let  $p \geq 2$ ,  $(f_k)_{1 \leq k \leq p}$  be a collection of continuous functions on a segment  $[a, b]$  and  $(w_k)_{1 \leq k \leq p}$  a collection of distinct vectors in  $\mathbb{Z}^n$ . Let for all  $a \leq s \leq b$  the tropical polynomial  $T_s : x \mapsto \min_{1 \leq k \leq p} \{f_k(s) + w_k \cdot x\}$ . Let for all  $k$  and  $s$ ,  $M_k(s, x) = f_k(s) + w_k \cdot x$  be the monomials. If  $x \in \mathbb{R}^n$  is such that  $T_a(x) = M_i(a, x)$  and  $T_b(x) = M_j(b, x)$  for some  $1 \leq i \neq j \leq p$ , then  $x \in \overline{\bigcup_{s \in [a, b] \cap \mathbb{Q}} V(T_s)}$ .*

*Proof.* — According to previous Lemma 5.17 used with the functions  $(M_k(\cdot, x))$ , there exists a  $a \leq c \leq b$  such that  $x \in V(T_c)$ . Therefore, we can introduce  $K = \{1 \leq k \leq p \mid T_c(x) = M_k(c, x)\}$  which contains at least two elements. Let  $\varepsilon_0 = \min_{k \notin K} \{M_k(c, x) - T_c(x)\} > 0$ . Now, let  $\varepsilon > 0$  that we assume to be less than  $\varepsilon_0$  without loss of generality. As the  $M_k(\cdot, x)$  are continuous, there exists a  $\delta > 0$  such that for all  $s$ ,  $|s - c| \leq \delta \Rightarrow |M_k(s, x) - M_k(c, x)| \leq \frac{\varepsilon}{2}$ . Let  $q$  be a rational number such that  $|q - c| \leq \delta$ . Let  $i_1 \neq i_2$  be such that for all  $k \neq i_1$ ,  $M_{i_1}(q, x) \leq M_{i_2}(q, x) \leq M_k(q, x)$ . If  $i_1 \notin K$ , let  $k \in K$ ,

$$\begin{aligned} M_{i_1}(q, x) &\geq M_{i_1}(c, x) - \frac{\varepsilon}{2} \\ &> T_c(x) - \frac{\varepsilon}{2} + \varepsilon_0 \\ &= M_k(c, x) - \frac{\varepsilon}{2} + \varepsilon_0 \\ &\geq M_k(q, x) - \varepsilon + \varepsilon_0 \\ &\geq M_k(q, x), \end{aligned}$$

which is a contradiction. Therefore,  $i_1 \in K$ . If  $i_2 \notin K$ , we do the same reasoning but we choose  $k \in K \setminus \{i_1\}$  which exists because  $|K| \geq 2$ . It implies by the way that  $M_{i_1}(c, x) = M_{i_2}(c, x)$  so by triangular inequality and the definition of  $\delta$ ,  $M_{i_1}(q, x) - M_{i_2}(q, x) \leq \varepsilon$ . Recall that the  $w_k$  are distinct.

Let  $y = x + \alpha(w_{i_2} - w_{i_1})$  where  $\alpha = \frac{M_{i_1}(q, x) - M_{i_2}(q, x)}{|w_{i_1} - w_{i_2}|}$ .

$$M_{i_1}(q, y) - M_{i_2}(q, y) = M_{i_1}(q, x) - M_{i_2}(q, x) + (w_{i_1} - w_{i_2}) \cdot (y - x) = 0.$$

In particular, the minimum of the tropical polynomial  $T_q$  is reached for an index that is not  $i_1$  (it can be  $i_2$  or another index, it does not matter). Therefore, if we use again previous Lemma 5.17 with the  $s \mapsto M_k(q, x + s(w_{i_2} - w_{i_1}))$ , there exists a  $0 \leq \beta \leq \alpha$  such that  $z = x + \beta(w_{i_2} - w_{i_1}) \in V(T_q)$ . Moreover,

$$|z - x| \leq \alpha |w_{i_2} - w_{i_1}| = |M_{i_1}(q, x) - M_{i_2}(q, x)| \leq \varepsilon.$$

$q \in \mathbb{Q}$  and it is true for every  $\varepsilon$  in a neighborhood of  $0^+$  so

$$x \in \overline{\bigcup_{s \in [a, b] \cap \mathbb{Q}} V(T_s)}. \quad \square$$

Now, let us prove the desired inclusion.

**PROPOSITION 5.19.** — *If  $S_f \subset \mathbb{Z}\mathbf{1}$ ,  $\lim_{\rho \rightarrow 0^+} \rho s\mathcal{A}(f) = -\text{strop}(V(f)) = \mathbb{R}^n \setminus (C^- \cup C^+)$  and this set is  $\{\mathbf{1}\}^\perp$  if  $f$  is not invertible,  $\emptyset$  otherwise.*

*Proof.* — By an argument similar to the one in the proof of Proposition 3.16, we have for every  $A, B$ ,  $Q_{N1}(A, B) = \alpha_N \det(AB^{-1})$  so  $f = P \circ \det$  where  $P$  is a one variable Laurent polynomial. We compute easily that  $s\mathcal{A}(f) = \{\mathbf{1}\}^\perp + \mathcal{A}(P)$  and  $\mathcal{A}(P)$  is the finite set of the norms of the non zero roots of  $P$ .  $f$  being invertible is equivalent to  $|S_f| = 1$ , which is equivalent to  $P$  being a monomial. In that case  $\mathcal{A}(P) = \emptyset$  so

$$\lim_{\rho \rightarrow 0^+} \rho s\mathcal{A}(f) = -\text{strop}(V(f)) = \emptyset.$$

Otherwise,  $\mathcal{A}(P) \neq \emptyset$  so  $\lim_{\rho \rightarrow 0^+} \rho s\mathcal{A}(f) = \{\mathbf{1}\}^\perp$ . Let  $r$  be a non zero root of  $P$ . Notice that for all  $x \in \mathbb{Q}^n \cap \{\mathbf{1}\}^\perp$ ,  $f(\text{diag}(rt^{-x_1}, t^{-x_2}, \dots, t^{-x_n})) = P(r) = 0$  and

$$-\text{sval}(\text{diag}(rt^{-x_1}, t^{-x_2}, \dots, t^{-x_n})) = x.$$

We deduce that  $\{\mathbf{1}\}^\perp \subset -\text{strop}(V(f))$  and the reverse inclusion is given by Corollary 5.14.

When  $f$  is invertible,  $|S_f| = 1$  so  $C^- = C^+ = \mathbb{R}^n$ . If  $f$  is non invertible,  $|S_f| \geq 2$  so  $C^- = \{x \in \mathbb{R}^n \mid x \cdot \mathbf{1} < 0\}$  and  $C^+ = \{x \in \mathbb{R}^n \mid x \cdot \mathbf{1} > 0\}$ . In both cases,  $-\text{strop}(V(f)) = \mathbb{R}^n \setminus (C^- \cup C^+)$ .  $\square$

**PROPOSITION 5.20.** — *If  $S_f \not\subset \mathbb{Z}\mathbf{1}$  then  $\mathbb{R}^n \setminus (C^- \cup C^+) \subset -\text{strop}(V(f))$ .*

*Proof.* — Let  $x \in -\mathbb{R}^n \setminus (C^- \cup C^+)$ . Let  $m_0$  be a point in  $S_f$  that minimizes scalar product with  $x$ , i.e. for all  $m \in S_f$ ,  $m_0 \cdot x \leq m \cdot x$ . If there exists an  $m \in S_f \setminus \{m_0\}$  such that  $m_0 \cdot x = m \cdot x$ , we just have to consider unitary matrices  $U, V$  such that  $Q_{m_0}(U, V) \neq 0$  and  $Q_m(U, V) \neq 0$  and we verify easily

that  $x \in V(\text{trop}(\Psi_{U,V}(f)))$ , thus we will assume that for all  $m \in S_f \setminus \{m_0\}$ ,  $m_0 \cdot x < m \cdot x$ . As  $-x$  is not in  $C^- \cup C^+$  and  $m_0$  strictly maximizes scalar product with  $-x$  in  $S_f$ , we deduce that  $m_0$  is not in  $\{N^- \mathbf{1}, N^+ \mathbf{1}\}$ . And it is clear that  $m_0$  is not an interior point of the Newton polytope so  $m_0 \notin \mathbb{Z} \mathbf{1}$  ( $S_f \not\subset \mathbb{Z} \mathbf{1}$  so any point of  $\text{sNewt}(f) \cap \mathbb{Z} \mathbf{1} \setminus \{N^- \mathbf{1}, N^+ \mathbf{1}\}$  is an interior point of  $\text{sNewt}(f)$ ). It implies that  $Q_{m_0}$  vanishes at least once on  $U(n) \times U(n)$  by Lemma B.3. Consider  $U, V$  unitary matrices such that  $Q_{m_0}(U, V) = 0$ .

There are now two possibilities. If for all  $m \in S_f$ ,  $Q_m(U, V) = 0$ , then  $\Psi_{U,V}(f) = 0$  so we are in the trivial case where  $\text{sA}(f) = -\text{strop}(V(f)) = \mathbb{R}^n$ , which proves the inclusion. Else, there exists an  $m_1 \in S_f$  such that  $Q_{m_1}(U, V) \neq 0$ . We will use the characterisation of classical tropical varieties that uses the tropical polynomial. Given matrices  $P(t), Q(t)$  that are invertible in  $\mathcal{R}$ , we have

$$\begin{aligned} \Psi_{P(t), Q(t)}(f)(z(t)) &= \sum_{m \in S_f} Q_m(P(t), Q(t)) z^m(t) \\ \implies \text{trop}(\Psi_{P(t), Q(t)}(f))(y) &= \min_{m \in S_f} \{\text{val}(Q_m(P(t), Q(t))) + m \cdot y\}. \end{aligned}$$

We want to find some matrices  $(P(t), Q(t)) \in \text{GL}_n(\mathcal{R})^2$  that verify the two following conditions, so we can use Lemma 5.18,

- (1)  $0 < \text{val}(Q_{m_0}(P(t), Q(t))) < +\infty$ ,
- (2)  $\text{val}(Q_{m_1}(P(t), Q(t))) = 0$ .

Assume that those two conditions are verified and let us show that  $x \in \text{strop}(V(f))$ .  $P(t)$  and  $Q(t)$  are invertible in  $\mathcal{R}$  so all the  $Q_m(P(t), Q(t))$  have a non negative valuation. Let for all  $s > 0$  and for all  $m \in S_f$ , the tropical monomials  $M_m(s, y) = s \text{val}(Q_m(P(t), Q(t))) + m \cdot y$  which equal  $\text{val}(Q_m(P(t^s), Q(t^s))) + m \cdot y$  when  $s$  is rational. We have

$$\begin{aligned} T_s(x) &= \min_{m \in S_f} \{M_m(s, x)\} \\ &\xrightarrow{s \rightarrow 0^+} \min_{\substack{m \in S_f \\ Q_m(P(t), Q(t)) \neq 0}} \{m \cdot x\} = m_0 \cdot x < m \cdot x \text{ if } m \in S_f \setminus \{m_0\}, \end{aligned}$$

so for some  $a > 0$  small enough,  $T_a(x) = M_{m_0}(a, x)$ , and

$$\begin{aligned} M_{m_0}(s, x) - M_{m_1}(s, x) &= s \text{val}(Q_{m_0}(P(t), Q(t))) + m_0 \cdot x - s \text{val}(Q_{m_1}(P(t), Q(t))) - m_1 \cdot x \\ &= s \text{val}(Q_{m_0}(P(t), Q(t))) + (m_0 - m_1) \cdot x \xrightarrow{s \rightarrow +\infty} +\infty, \end{aligned}$$

so for  $b > 0$  large enough,  $M_{m_1}(b, x) < M_{m_0}(b, x)$  thus  $T_b(x) \neq M_{m_1}(b, x)$ .

The hypothesis of Lemma 5.18 are verified thus according to this lemma and Proposition 5.5,

$$x \in \overline{\bigcup_{s \in [a, b] \cap \mathbb{Q}} V(\text{trop}(\Psi_{P(t^s), Q^{-1}(t^s)}(f)))} \subset \text{strop}(V(f)).$$

Now, all we have to do is to find matrices  $P(t)$  and  $Q(t)$  that verify such conditions. Recall that there are matrices  $U, V$  such that  $Q_{m_1}(U, V) \neq Q_{m_0}(U, V) = 0$ . As  $U$  and  $V$  are invertible, any matrix of the form  $U + tA$  or  $V + tA$  with  $A \in \text{Mat}_n(\mathbb{C})$  are in  $\text{GL}_n(\mathcal{R})$ .  $Q_{m_0} \neq 0$  so there exists invertible matrices  $A$  and  $B$  such that  $Q_{m_0}(A, B) \neq 0$ . We choose  $P(t) = U + t(A - U)$  and  $Q(t) = V + t(B - V)$ .

*Condition (1).* — Recall first of all that for all  $m$ ,  $\text{val}(Q_m(P(t), Q(t))) \geq 0$ .  $Q_{m_0}(P(0), Q(0)) = Q_{m_0}(U, V) = 0$  by definition of  $U$  and  $V$  and

$$Q_{m_0}(P(1), Q(1)) = Q_{m_0}(A, B) \neq 0 \quad \text{so} \quad \text{val}(Q_{m_0}(P(t), Q(t)))$$

is neither 0 nor  $+\infty$ .

*Condition (2).* — It is a direct consequence of the fact that

$$Q_{m_1}(P(0), Q(0)) = Q_{m_1}(U, V) \neq 0.$$

We found  $P(t), Q(t)$  that satisfy the two wanted conditions, thus Proposition 5.20 is proven.  $\square$

**THEOREM 5.21.** — *When  $f \in \mathbb{C}[\text{GL}_n]$ , if  $Y = V(f)$ , then*

$$\lim_{\rho \rightarrow 0^+} \rho \, \text{sA}(Y) = -\text{strop}(Y) = \mathbb{R}^n \setminus (C^- \cup C^+).$$

*Proof.* — This is a consequence of Corollary 5.14 and Propositions 5.16, 5.19 and 5.20.  $\square$

Notice that the  $C^-$  and  $C^+$  only depend on the Newton polytope of  $f$ . Therefore, its tropical variety too.

*Example 5.22.* — Let  $n = 2$  and  $f$  any polynomial such that

$$\text{sNewt}(f) = \text{Conv}\{(-1, -1), (1, -2), (-2, 1), (4, 0), (0, 4), (4, 4)\},$$

as in Figure 5.1. For example, we can take  $f(A) = \frac{1}{\det(A)} + \mathbf{i} \frac{\text{tr}(A)^3}{\det(A)^2} - a_{11}a_{12}a_{21}a_{22} + 2\det(A)^4$ .

Here,  $N^-(1, 1) = (-1, -1)$  and  $N^+(1, 1) = (4, 4)$  are vertices of  $\text{Newt}(f)$ . The region  $C^-$  (resp.  $C^+$ ) is the interior of the normal cone of  $N^- \mathbf{1}$  (resp.  $N^+ \mathbf{1}$ ), cf. Figure 5.1.

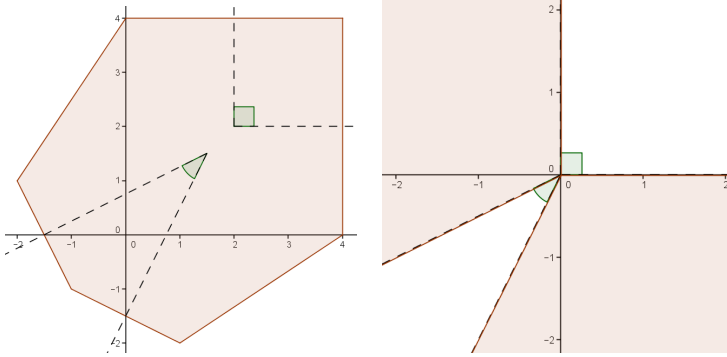


Figure 5.1. The tropical variety (right) of a polynomial with a given Newton polytope (left).

## Appendix A. Notations

- $\mathcal{M}_n(R)$  is the vector space of  $n \times n$  matrices with coefficients in the ring  $R$ .
- $\mathrm{GL}_n(R)$  is the group of  $n \times n$  invertible matrices.
- $\mathbb{C}[X^\pm]$  is the ring of Laurent polynomials with  $n$  indeterminates (when  $n$  is understood from the context).
- $\llbracket a, b \rrbracket$  is the set of integers between  $a$  and  $b$ .  $\mathcal{S}_n$  is the group of permutations of  $\{1, \dots, n\}$ .
- $K[Y]$  is the ring of regular functions on an algebraic variety  $Y$  over  $K$ .
- $\chi_A$  is the characteristic polynomial of a matrix  $A$ .
- $\Gamma(E)$  is the set of connected components of a topological space  $E$ .
- $\mathrm{Conv}(S)$  is the convex hull of a subset  $S$  of  $\mathbb{R}^n$ .
- When  $f = \sum_{m \in \mathbb{Z}^n} f_m X^m \in \mathbb{C}[X^\pm]$ ,  $S_f = \{m \in \mathbb{Z}^n \mid f_m \neq 0\}$  is the support of  $f$ .
- $\mathrm{Newt}(f) = \mathrm{Conv}(S_f)$  is the Newton polytope of  $f$ .
- When  $f(A \mathrm{diag}(z) B^{-1}) = \sum_{m \in \mathbb{Z}^n} Q_m(A, B) z^m$ ,  $S_f = \{m \in \mathbb{Z}^n \mid Q_m \neq 0\}$  is the support of  $f$ .
- $\mathrm{sNewt}(f) = \mathrm{Conv}(S_f)$  is its Newton polytope.
- $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$ .
- For  $K \subset \{1, \dots, n\}$ ,  $\mathbf{1}_K \in \mathbb{R}^n$  is the vector such that for all  $i$ ,  $(\mathbf{1}_K)_i = \mathbf{1}_K(i)$ .
- For  $1 \leq k \leq n$ ,  $\varpi_k = \mathbf{1}_{\llbracket k+1-n, n \rrbracket} = (0, \dots, 0, 1, \dots, 1)$  with  $k$  ones.
- $B = \{(U, L) \in \mathrm{GL}_n(\mathbb{C})^2 \mid U \text{ upper triangular and } L \text{ lower triangular}\}$  is the Borel subgroup of  $\mathrm{GL}_n(\mathbb{C})^2$ .

- $\Lambda = \{\lambda \in \mathbb{R}^n \mid \lambda_1 \leq \dots \leq \lambda_n\}$  is the set of dominant  $B$ -weights of  $\mathrm{GL}_n(\mathbb{C})$ .
- When  $\lambda \in \Lambda$ ,  $C(\lambda) = \mathrm{Conv}(\mathcal{S}_n \cdot \lambda)$ .
- $\Delta(f)$  is the moment polytope of  $f$  (cf Section 4).
- $V_\lambda$  is the irreducible  $\mathrm{GL}_n(\mathbb{C})$ -representation of highest weight  $\lambda \in \Lambda$ .
- $W_\lambda \cong V_\lambda \otimes V_\lambda^*$  is the irreducible  $\mathrm{GL}_n(\mathbb{C})^2$ -representation of highest weight  $\lambda \in \Lambda$ .
- $\bar{\mathcal{K}}$  is the field of complex Puiseux series.
- $\mathrm{val} : \sum_{q \in \mathbb{Q}} a_q t^q \mapsto \min\{q \in \mathbb{Q} \mid a_q \neq 0\}$  is its natural valuation.  $\mathrm{val}$  can also designate any valuation depending on the context.
- $\mathcal{R}$  is the ring of complex Puiseux series with a non negative valuation.
- $\mathrm{sval} : A(t) \mapsto$  the valuations of the invariant factors of  $A(t)$ , which is the spherical valuation of a Puiseux series matrix.
- $\Gamma_{\mathrm{val}}$  is the image of the valuation  $\mathrm{val}$ , it is a subgroup of  $\mathbb{R}$ .
- $\mathrm{strop}(Y) = \{\mathrm{sval}(A(t)) \mid A(t) \in Y(\bar{\mathcal{K}})\}$  is the spherical tropical variety of a spherical variety  $Y$ .
- $N^-$  (resp.  $N^+$ ) is the lowest (resp. largest) integer  $N$  such that  $N\mathbf{1} \in \mathrm{sNewt}(f)$  when it is well-defined.
- $C_F(\Delta)$  is the normal cone of the face  $F$  of the convex polytope  $\Delta$ .
- $C^-$  (resp.  $C^+$ ) is the interior of  $C_{\{N-1\}}(\mathrm{sNewt}(f))$  (resp.  $C_{\{N+1\}}(\mathrm{sNewt}(f))$ ), or the empty set when  $N^-$  (resp.  $N^+$ ) does not exist.

## Appendix B. Two technical lemmas

The purpose of this section is to show two useful technical lemmas which don't really fit into the global frame of the article.

**LEMMA B.1.** — *Let  $p \geq 1$  be an integer and  $P : \mathrm{Mat}_n(\mathbb{C})^p \rightarrow \mathbb{C}$  be a non zero polynomial with  $n^{2p}$  variables. Then the set of unitary zeros of  $P$ , i.e.  $Z = P^{-1}\{0\} \cap \mathrm{U}(n) \subset \mathrm{U}(n)$ , is included in a finite union of strict submanifolds of  $\mathrm{U}(n)$ . In particular, it has measure zero, with respect to the natural probability Haar measure  $\mu$  on  $\mathrm{U}(n)$  (equivalently, any non-measure zero subset of  $\mathrm{U}(n)$  is dense in  $\mathrm{GL}_n(\mathbb{C})$  for the Zariski topology).*

*Proof.* — Let us stratify  $P^{-1}\{0\} = V(P)$  as  $\bigsqcup_{i \in I} Y_i$  where each  $Y_i$  is a regular algebraic variety, hence a complex manifold. For all  $i$  and all  $U \in Y_i$ , the tangent space  $T_U Y_i$  is a complex vector space of dimension  $d = \dim(Y_i) \leq n^{2p} - 1$ , thus a real vector space of dimension  $2d$ . If moreover  $U \in \mathrm{U}(n)^p$ ,  $T_U \mathrm{U}(n)^p = (U^* T_{I_n} \mathrm{U}(n))^p = (\mathbf{i} U^* H(n))^p$  where  $H(n)$  is the real vector space of Hermitian  $n \times n$  matrices. In particular, this tangent space contains no

non zero complex vector subspace because  $\mathbf{i}H$  is never Hermitian when  $H$  is Hermitian unless  $H = 0$ .

Therefore, the real vector space  $T_U \mathbf{U}(n)^p \cap T_U Y_i$  is at most of dimension  $d$ . Recall that  $d \leq n^{2p} - 1$  and the real dimension of  $\mathbf{U}(n)$  as a real manifold is  $n^{2p}$ . Therefore,  $\mathbf{U}(n)^p \cap Y_i$  is contained in a real hypersurface of  $\mathbf{U}(n)^p$ , proving Lemma B.1.  $\square$

LEMMA B.2. — *If  $g : \mathbb{C}^* \rightarrow \mathbb{C}^*$  is a continuous function such that there are non zero complex numbers  $\alpha$  and  $\beta$  and integers  $m_1, m_2 \in \mathbb{Z}$  verifying for all  $\theta \in \mathbb{R}$ ,  $g(r\mathbf{e}^{\mathbf{i}\theta}) \sim_{r \rightarrow 0} \alpha r^{-m_2} \mathbf{e}^{\mathbf{i}\theta m_2}$  and  $g(r\mathbf{e}^{\mathbf{i}\theta}) \sim_{r \rightarrow +\infty} \beta r^{m_1} \mathbf{e}^{\mathbf{i}\theta m_1}$ , then  $m_1 = m_2$ .*

*Proof.* — Under those hypothesis, for some  $r_1, r_2 > 0$ , we have

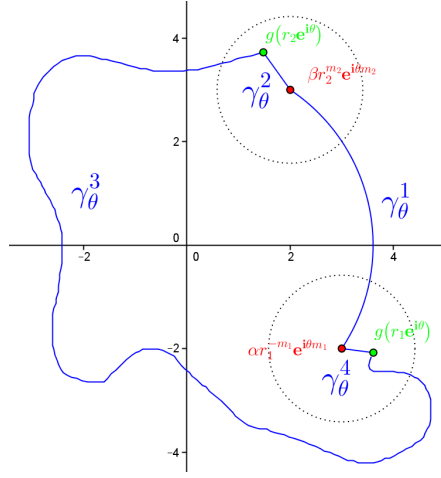
$$\begin{aligned} r \leq r_1 &\implies |g(r\mathbf{e}^{\mathbf{i}\theta}) - \alpha r^{-m_1} \mathbf{e}^{\mathbf{i}\theta m_1}| \leq \frac{1}{2} |\alpha| r^{-m_1} \\ r \geq r_2 &\implies |g(r\mathbf{e}^{\mathbf{i}\theta}) - \beta r^{-m_2} \mathbf{e}^{\mathbf{i}\theta m_2}| \leq \frac{1}{2} |\beta| r^{-m_2} \end{aligned}$$

As  $\mathbb{S}^1$  is compact, we can choose  $r_1$  and  $r_2$  uniformly with respect to  $\theta$ . Up to increasing  $r_2$  or decreasing  $r_1$ , we may assume that  $|\alpha| r_1^{-m_1} = |\beta| r_2^{-m_2}$ . We denote by  $\rho$  this quantity. Consider now the following paths from  $[0, 1]$  to  $\mathbb{C}^*$  for some  $\theta \in \mathbb{R}$ ,

$$\begin{aligned} \gamma_\theta^1 : t &\mapsto \rho \exp(\mathbf{i}((1-t)(\theta m_1 + \arg(\alpha)) + t(\theta m_2 + \arg(\beta)))) \\ &\implies \gamma_\theta^1(0) = \alpha r_1^{-m_1} \mathbf{e}^{\mathbf{i}\theta m_1}, \quad \gamma_\theta^1(1) = \beta r_2^{m_2} \mathbf{e}^{\mathbf{i}\theta m_2} \\ \gamma_\theta^2 : t &\mapsto (1-t)\beta r_2^{-m_2} \mathbf{e}^{\mathbf{i}\theta m_2} + t g(r_2 \mathbf{e}^{\mathbf{i}\theta}) \\ &\implies \gamma_\theta^2(0) = \beta r_2^{m_2} \mathbf{e}^{\mathbf{i}\theta m_2}, \quad \gamma_\theta^2(1) = g(r_2 \mathbf{e}^{\mathbf{i}\theta}) \\ \gamma_\theta^3 : t &\mapsto g(((1-t)r_2 + tr_1) \mathbf{e}^{\mathbf{i}\theta}) \\ &\implies \gamma_\theta^3(0) = g(r_2 \mathbf{e}^{\mathbf{i}\theta}), \quad \gamma_\theta^3(1) = g(r_1 \mathbf{e}^{\mathbf{i}\theta}) \\ \gamma_\theta^4 : t &\mapsto (1-t)g(r_1 \mathbf{e}^{\mathbf{i}\theta}) + t \alpha r_1^{m_1} \mathbf{e}^{\mathbf{i}\theta m_1} \\ &\implies \gamma_\theta^4(0) = g(r_1 \mathbf{e}^{\mathbf{i}\theta}), \quad \gamma_\theta^4(1) = \alpha r_1^{m_1} \mathbf{e}^{\mathbf{i}\theta m_1}. \end{aligned}$$

Let  $\gamma_\theta = (\gamma_\theta^1 * \gamma_\theta^2) * (\gamma_\theta^3 * \gamma_\theta^4)$ , which is a path from  $[0, 1]$  to  $\mathbb{C}^*$  because  $f$  does not vanish. Notice that  $\theta \mapsto \gamma_\theta$  is a homotopy so all the  $\gamma_\theta$  are homotopic the ones with the others. For any  $k \in \{2, 3, 4\}$ ,  $\theta \mapsto \gamma_\theta^k$  is  $2\pi$ -periodic so

$$\gamma_0 \sim \gamma_{2\pi} \implies \gamma_0^1 \sim \gamma_{2\pi}^1 \implies \text{Ind}_0(\gamma_{2\pi}^1 * \overline{\gamma_0^1}) = 0$$


 Figure B.1. Illustration of the paths  $\gamma_\theta^k$ .

And after noticing that for all  $t$ ,  $\gamma_{2\pi}^1(t) = e^{2i\pi(m_2-m_1)t}\gamma_0^1(t)$  we can compute that

$$\begin{aligned}
 & \text{Ind}_0\left(\gamma_{2\pi}^1 * \overline{\gamma_0^1}\right) \\
 &= \int_0^1 \frac{\gamma_{2\pi}^1{}'(t)}{\gamma_{2\pi}^1(t)} \frac{dt}{2i\pi} - \int_0^1 \frac{\gamma_0^1{}'(t)}{\gamma_0^1(t)} \frac{dt}{2i\pi} \\
 &= \int_0^1 \frac{2i\pi(m_2 - m_1)e^{2i\pi(m_2-m_1)t}\gamma_0^1(t) + e^{2i\pi(m_2-m_1)t}\gamma_0^1{}'(t)}{e^{2i\pi(m_2-m_1)t}\gamma_0^1(t)} \frac{dt}{2i\pi} \\
 &\quad - \int_0^1 \frac{\gamma_0^1{}'(t)}{\gamma_0^1(t)} \frac{dt}{2i\pi} \\
 &= m_2 - m_1.
 \end{aligned}$$

As this quantity vanishes, we deduce that  $m_1 = m_2$ . □

LEMMA B.3. — *When  $h : \text{Mat}_n(\mathbb{C}) \rightarrow \mathbb{C}$  is a homogeneous polynomial with homogeneity coefficient  $m \in \mathbb{N}^n$  regarding to the columns in the sense that for all  $A$ ,  $\lambda$ ,  $h(A \text{diag}(\lambda)) = \lambda^m h(A)$  and  $h$  does not vanish on  $U(n)$ , then  $m_1 = \dots = m_n$ .*

*Proof of Lemma B.3.*

*Case 1:  $n = 1$ . — Trivial.*

*Case 2:  $n = 2$ . — Let*

$$f : \begin{cases} \mathbb{C}^2 \longrightarrow \mathbb{C} \\ (x, y) \longmapsto h \begin{pmatrix} xy \\ 11 \end{pmatrix} \end{cases}$$

By homogeneity and because  $h$  does not vanish on  $U(2)$ ,  $h$  does not vanish on any invertible matrix  $A$  if its columns are orthonormal so for any  $z \in \mathbb{C}^*$ ,

$$f\left(z, -\frac{z}{|z|^2}\right) = h\begin{pmatrix} z & -\frac{z}{|z|^2} \\ 1 & 1 \end{pmatrix} \neq 0.$$

Moreover,  $h$  is a polynomial so  $f$  is and

$$f(x, y) = h\begin{pmatrix} x & y \\ 1 & 1 \end{pmatrix} = x^{m_1} y^{m_2} h\begin{pmatrix} 1 & 1 \\ \frac{1}{x} & \frac{1}{y} \end{pmatrix} \underset{|x|, |y| \rightarrow +\infty}{=} O(x^{m_1} y^{m_2}).$$

We deduce that  $f$  can be written as

$$f(x, y) = \sum_{p_1=0}^{m_1} \sum_{p_2=0}^{m_2} f_{p_1, p_2} x^{p_1} y^{p_2}.$$

Now, notice that

$$\begin{aligned} f\left(z, -\frac{z}{|z|^2}\right) &= h\begin{pmatrix} z & -\frac{z}{|z|^2} \\ 1 & 1 \end{pmatrix} \\ &= \left(\frac{z}{|z|^2}\right)^{m_2} h\begin{pmatrix} z & -1 \\ 1 & \bar{z} \end{pmatrix} \\ &\underset{z \rightarrow 0}{=} \frac{1}{\bar{z}^{m_2}} h\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + o\left(\frac{1}{|z|^{m_2}}\right). \end{aligned}$$

and

$$f\left(z, -\frac{z}{|z|^2}\right) = \sum_{p_1=0}^{m_1} \sum_{p_2=0}^{m_2} f_{p_1, p_2} z^{p_1} \frac{1}{\bar{z}^{p_2}} \underset{z \rightarrow 0}{=} f_{0, m_2} \frac{(-1)^{m_2}}{\bar{z}^{m_2}} + o\left(\frac{1}{|z|^{m_2}}\right).$$

so  $f_{0,m_2} = (-1)^{m_2} h \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \neq 0$  because this matrix is unitary. With the same kind of argument, we could show that  $f_{m_1,0} \neq 0$ . Let  $z = r\mathbf{e}^{i\theta} \in \mathbb{C}^*$ .

$$\begin{aligned} f\left(z, -\frac{z}{|z|^2}\right) &= f\left(r\mathbf{e}^{i\theta}, -\frac{\mathbf{e}^{i\theta}}{r}\right) \\ &= \sum_{p_1=0}^{m_1} \sum_{p_2=0}^{m_2} (-1)^{p_2} f_{p_1,p_2} r^{p_1-p_2} \mathbf{e}^{i\theta(p_1+p_2)} \\ &\underset{r \rightarrow 0}{\sim} (-1)^{m_2} f_{0,p_2} r^{-m_2} \mathbf{e}^{i\theta m_2} \end{aligned}$$

and

$$\underset{r \rightarrow +\infty}{\sim} f_{p_1,0} r^{m_1} \mathbf{e}^{i\theta m_1} \text{ using the same argument.}$$

Using Lemma B.2, we obtain that  $m_1 = m_2$ .

*Case 3:  $n \geq 3$ .* — Let  $1 \leq i \leq n-1$  and,

$$\tilde{h} : \begin{cases} \mathcal{M}_{\text{at}_2}(\mathbb{C}) \longrightarrow \mathbb{C} \\ A \longmapsto h \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_{n-i-1} \end{pmatrix} \end{cases}$$

$\tilde{h}$  verifies the hypothesis of Lemma B.3 with  $m_i, m_{i+1}$ . Using the case  $n = 2$ , we deduce that  $m_i = m_{i+1}$ . It is true for all  $i$  so  $m_1 = \dots = m_n$ .  $\square$

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