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Steady gradient Ricci solitons with nonnegative curvature away from a compact set ^(*)

ZIYI ZHAO ⁽¹⁾ AND XIAOHUA ZHU ⁽²⁾

ABSTRACT. — We analyse the blow-down solutions for n -dimensional ($n \geq 4$) noncompact κ -noncollapsed steady gradient Ricci solitons (M, g) with $\text{Rm} \geq 0$ and $\text{Ric} > 0$ away from a compact set of M . As one of main results, we prove that the $(n - 1)$ -dimensional compact split limit ancient Ricci flows of type I and type II cannot occur simultaneously from the blow-down of (M, g) . Consequently, we prove that (M, g) with $\text{Rm} \geq 0$ must be isometric to the Bryant Ricci soliton up to scaling, if there exists a sequence of normally rescaled Ricci flows of (M, g) , which converges subsequently to a family of shrinking quotient cylinders. The latter improves a previous result of Brendle.

RÉSUMÉ. — Nous étudions des solutions blow-down pour (M, g) soliton de Ricci gradient stable non-compact et κ -non effondré de dimension $n \geq 4$ à $\text{Rm} \geq 0$ et $\text{Ric} > 0$ en dehors d'un compact de M . L'une de nos principales conclusions est la preuve que, lors du blow-down de (M, g) , les flots de Ricci anciens limites compacts scindés de dimension $(n - 1)$ ne peuvent pas être simultanément de type I et de type II. Par conséquent, nous démontrons que si une suite de flots de Ricci normalement dilatés de (M, g) converge successivement vers une famille de cylindres quotients contractants, alors (M, g) à $\text{Rm} \geq 0$, doit être isométrique au soliton de Bryant à une dilatation près. Ce dernier résultat améliore un résultat antérieur dû à Brendle.

1. Introduction

Steady gradient Ricci soliton, as a singular model of type II of Ricci flow, has been extensively studied. In dimension 2, Hamilton [16, 20] proved that

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the cigar solution is the only $2d$ steady Ricci soliton up to scaling. In dimension 3, Perelman conjectured that the Bryant Ricci soliton is the only $3d$ κ -noncollapsed steady Ricci soliton up to scaling [28]. The conjecture has been proved by Brendle [6]. These two Ricci solitons are both rotationally symmetric. In dimension $n \geq 4$, besides the Bryant soliton [11], Lai recently constructed a family of $SO(n-1)$ -symmetric solutions with positive curvature operator [23]. Thus it is interesting to classify steady Ricci solitons under suitable conditions of symmetry and curvature.

To characterize the Bryant Ricci soliton, Brendle introduced the following notion [7].

DEFINITION 1.1. — *A complete steady gradient Ricci soliton $(M^n, g; f)$ of dimension n is called asymptotically cylindrical if the following conditions hold:*

- (i) *The scalar curvature satisfies $\frac{C_1}{\rho(x)} \leq R(x) \leq \frac{C_2}{\rho(x)}$ as $\rho(x) \gg 1$, where $\rho(x)$ is a distance function from a fixed point on M , and C_1, C_2 are two positive constants.*
- (ii) *Let p_i be an arbitrary sequence of marked points going to infinity. Consider the (normally) rescaled metrics*

$$g_{p_i}(t) = r_i^{-1} \phi_{r_i t}^*(g), \quad (1.1)$$

where $r_i R(p_i) = 1$ and ϕ_t is a family of transformations generated by $-\nabla f$, the flow $(M, g_{p_i}(t); p_i)$ converges in the Cheeger–Gromov sense to a family of shrinking cylinders $(\mathbb{S}^{n-1} \times \mathbb{R}, \bar{g}(t))$, $t \in (-\infty, 0]$. The metric $\bar{g}(t)$ is given by

$$\bar{g}(t) = 2(n-2)(1-t)g_{\mathbb{S}^{n-1}(1)} + ds^2, \quad (1.2)$$

where $\mathbb{S}^{n-1}(1)$ is the unit sphere of dimension $(n-1)$ in $\mathbb{R}^n$.

In [7], Brendle proved that any steady gradient Ricci soliton with nonnegative sectional curvature must be isometric to the Bryant Ricci soliton up to scaling if it is asymptotically cylindrical in the sense of Definition 1.1. Later, Deng–Zhu proved that the Brendle’s result still holds if one of the two conditions in Definition 1.1 is satisfied for κ -noncollapsed steady Ricci solitons with nonnegative curvature operator (cf. [19, Theorem 1.2], [18, Theorem 1.1] and [18, Lemma 6.5]). In a very recent paper [31], Zhao–Zhu showed that the condition of nonnegative sectional curvature in [7] can be weakened by assuming the nonnegative sectional curvature away from a compact set of M together with a P -curvature pinching condition.

It is a natural question to ask: is the above Brendle's result in [7] true if the condition (ii) in Definition 1.1⁽¹⁾ is satisfied by (at least) one sequence? At the present paper, we shall give a positive answer. Namely, we prove

THEOREM 1.2. — *Let (M^n, g) ($n \geq 4$) be a noncompact κ -noncollapsed steady gradient Ricci soliton with nonnegative curvature operator. Suppose that there exists a sequence of $p_i \in M$ ($\rightarrow \infty$) such that the rescaled Ricci flows $(M, g_{p_i}(t); p_i)$ of (M, g) converge subsequently to a family of shrinking quotient cylinders. Then (M, g) is isometric to the n -dimensional Bryant–Ricci soliton up to scaling.*

Let (M^n, g) be a complete noncompact κ -noncollapsed steady Ricci soliton with curvature operator $\text{Rm} \geq 0$ (sectional curvature $\text{Km} \geq 0$ for $n = 4$) and Ricci curvature $\text{Ric} > 0$ away from a compact set K of M . Let $p_i \rightarrow \infty$ be any sequence in M and $g_{p_i}(t)$ the rescaled flow of Ricci soliton g as in Definition 1.1. Then $(M, g_{p_i}(t); p_i)$ converges to a splitting flow in the Cheeger–Gromov sense,

$$\bar{g}(t) = h(t) + ds^2, \quad \text{on } N \times \mathbb{R}, \quad (1.3)$$

where $h(t)$ ($t \in (-\infty, 0]$) is an ancient κ -solution on an $(n-1)$ -dimensional N , see Proposition 2.2. For simplicity, we call $(N, h(t))$ a split limit ancient Ricci flow (of codimension one) arising from the blow-down of (M, g) .

To prove Theorem 1.2, we establish the following alternative principle.

THEOREM 1.3. — *Let (M^n, g) ($n \geq 4$) be a noncompact κ -noncollapsed steady Ricci soliton with $\text{Rm} \geq 0$ and $\text{Ric} > 0$ on $M \setminus K$. Suppose that there exists a noncompact split limit Ricci flow $(N, h(t))$ of codimension one arising from the blow-down of (M, g) as in (1.3). Then there is no any compact split limit Ricci flow of type I arising from the blow-down of (M, g) .*

Recall that a compact ancient κ -solution $(N, h(t))$ ($t \in (-\infty, 0]$) of type I means that it satisfies

$$\sup_{N \times (-\infty, 0]} (-t)|R(x, t)| < \infty.$$

Otherwise, it is called type II, i.e., it satisfies

$$\sup_{N \times (-\infty, 0]} (-t)|R(x, t)| = \infty.$$

⁽¹⁾ This condition is same as the uniqueness assumption of tangent flows of soliton Ricci flow $g(t) = \phi_t^*(g)$ at time $-\infty$ by a recent notion, see [3, 21], etc.

As a corollary of Theorem 1.3, we prove

COROLLARY 1.4. — *Let (M^n, g) ($n \geq 4$) be a noncompact κ -noncollapsed steady Ricci soliton as in Theorem 1.3. Then the $(n - 1)$ -dimensional compact split limit ancient Ricci flows of type I and type II cannot occur simultaneously from the blow-down of (M, g) .*

Thanks to a deep classification result for $3d$ compact κ -solutions proved by Brendle–Daskalopoulos–Sesum [9] (also see Theorem 6.1), we can improve Theorem 1.3 in $4d$ as follows.

THEOREM 1.5. — *Let (M, g) be a $4d$ complete noncompact κ -noncollapsed steady gradient Ricci soliton with $\text{Km} \geq 0$ and $\text{Ric} > 0$ away from a compact set K of M . Then either all $3d$ split limit Ricci flows $(N, h(t))$ arising from the blow-down of (M, g) are families of $3d$ shrinking round quotient spheres, or are $3d$ noncompact ancient κ -solutions.*

We note that the $3d$ noncompact ancient κ -solution (non-flat) has been recently classified by Brendle [8] (also see [4]). Namely, such a soliton is isometric to either a flow of $2d$ shrinking cylinders, or a flow of the Bryant Ricci soliton. On the other hand, by recent results of Chow–Deng–Ma [15, Theorem 1.3, Claim 6.4], both flows of the Bryant Ricci soliton and $2d$ shrinking cylinders will always arise in the second case in Theorem 1.5, since Theorem 1.5 in particular implies that (M, g) satisfies the canonical model condition in Chow–Deng–Ma’s paper. Thus we can actually classify the blow-down solutions for $4d$ noncompact κ -noncollapsed steady gradient Ricci solitons with $\text{Km} \geq 0$ and $\text{Ric} > 0$ away from a compact set K of M .

We also note that both cases can happen in Theorem 1.5 as seen in following examples. For any $2n \geq 4$ and each Z_k -group, Appleton [2] has constructed an example of noncompact κ -noncollapsed steady gradient Ricci soliton with $\text{Rm} > 0$ on $M \setminus K$, where all split ancient κ -solutions $h(t)$ form families of shrinking quotient spheres of dimension $(2n - 1)$ with Z_k -group. In each of Lai’s examples [23] of noncompact κ -noncollapsed steady gradient Ricci solitons with $\text{Rm} > 0$ on M , all split ancient κ -solutions $h(t)$ are noncompact.

We guess that an analogue of Theorem 1.5 holds for general dimensions (cf. Conjecture 6.4).⁽²⁾

By Theorem 1.5, we can improve Theorem 1.2 in $4d$ as follows.

⁽²⁾ This conjecture has been recently solved in a sequel of paper [32].

THEOREM 1.6. — *Let (M, g) be a 4d noncompact κ -noncollapsed steady gradient Ricci soliton with nonnegative sectional curvature. Suppose that there exists a 3d compact split limit Ricci flow $(N, h(t))$ as in (1.3) arising from the blow-down of (M, g) . Then (M, g) is isometric to the 4d Bryant–Ricci soliton up to scaling.*

Outline of the paper

The paper is organized as follows.

In Section 2, we prove a splitting result for any limit of normally rescaled Ricci flows sequence from a κ -noncollapsed steady gradient Ricci soliton (M, g) with $\text{Rm} \geq 0$ on $M \setminus K$, see Proposition 2.2.

In Section 3, we first get a decay estimate of curvature and then study the level set geometry of (M, g) by assuming the existence of compact split ancient κ -solution $(N, h(t))$, see Proposition 3.3, Proposition 3.8, etc.

In Section 4, we classify the split limit Ricci flow $(N, h(t))$, if all of them satisfy (3.1) see Proposition 4.1.

In Section 5, we prove our main results, Theorem 1.3, Theorem 1.2 and Corollary 1.4 by results obtained in Section 2–4.

In Section 6, we focus on a 4d steady gradient Ricci soliton to get a diameter estimate of $(N, h(0))$ for all 3d split Ricci flows $(N, h(t))$, see Proposition 6.3. We will modify the argument in the proof of Theorem 1.3 to prove Proposition 6.3. Then by Proposition 6.3 we prove Theorem 1.5 and Theorem 1.6.

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2. A splitting result for limit Ricci flows

A complete Riemannian metric g on M is called a gradient Ricci soliton if there exists a smooth function f (which is called a defining function) on M such that

$$R_{ij}(g) + \rho g_{ij} = \nabla_i \nabla_j f, \quad (2.1)$$

where $\rho \in \mathbb{R}$ is a constant. The gradient Ricci soliton is called expanding, steady and shrinking according to $\rho >, =, < 0$, respectively. These three types of Ricci solitons correspond to three different blow-up solutions of Ricci flow [20].

In the case of steady Ricci solitons, we can rewrite (2.1) as

$$2 \operatorname{Ric}(g) = L_X g, \quad (2.2)$$

where L_X is the Lie operator along the gradient vector field (VF) $X = \nabla f$ generalized by f . Let $\{\phi_t^*\}_{t \in (-\infty, \infty)}$ be a 1-ps of transformations generated by $-X$. Then $g(t) = \phi_t^*(g)$ ($t \in (-\infty, \infty)$) is a solution of Ricci flow. Namely, $g(t)$ satisfies

$$\frac{\partial g}{\partial t} = -2 \operatorname{Ric}(g), \quad g(0) = g. \quad (2.3)$$

For simplicity, we call $g(t)$ the soliton Ricci flow of (M, g) .

By (2.2), we have

$$\langle \nabla R, \nabla f \rangle = -2 \operatorname{Ric}(\nabla f, \nabla f), \quad (2.4)$$

where R is the scalar curvature of g . It follows

$$R + |\nabla f|^2 = \text{Const.}$$

Since R is always positive ([13, 30]), the above equation can be normalized by

$$R + |\nabla f|^2 = 1. \quad (2.5)$$

We recall that an ancient κ -solution is a κ -noncollapsed solution of Ricci flow (2.3) with $\operatorname{Rm} \geq 0$ defined for any $t \in (-\infty, 0]$. The following result is a version of Perelman's compactness theorem for higher dimensional ancient κ -solutions.

THEOREM 2.1. — *Let $(M^n, g_i(t); p_i)$ ($n \geq 3$) be any sequence of n -dimensional ancient κ -solutions with $R(p_i, 0) = 1$ on a noncompact manifold M . Then $(M, g_i(t); p_i)$ subsequently converges to a splitting flow $(N \times \mathbb{R}, \bar{g}(t); p_\infty)$ in the Cheeger–Gromov sense. Here*

$$\bar{g}(t) = h(t) + \mathrm{d}s^2, \quad (2.6)$$

and $(N, h(t))$ is an $(n-1)$ -dimensional ancient κ -solution for $t \in (-\infty, 0]$.

The convergence of $(M, g_i(t); p_i)$ comes from [17, Theorem 3.3]. The splitting property in (2.6) can also be obtained by Hamilton's argument [20, Lemma 22.2] with the help of Perelman's asymptotic volume ratio estimate for κ -solutions [22, Proposition 41.13]. In fact, for a sequence of rescaling Ricci flows arising from a steady Ricci soliton, we can improve Theorem 2.1 under a weaker curvature condition as follows.

PROPOSITION 2.2. — *Let (M^n, g) ($n \geq 3$) be a noncompact κ -noncollapsed steady gradient Ricci soliton with $\text{Rm} \geq 0$ away from K . Let $p_i \rightarrow \infty$ and $(M, g_{p_i}(t); p_i)$ a sequence of rescaled flows with $R_{p_i}(p_i, 0) = 1$ as in (1.1). Then $(M, g_{p_i}(t); p_i)$ subsequently converge to a splitting flow $(N \times \mathbb{R}, \bar{g}(t); p_\infty)$ as in Theorem 2.1. Moreover, for $n = 4$, $\text{Rm} \geq 0$ can be weakened to $\text{Km} \geq 0$ away from K .*

Proof. — Since $\text{Km} \geq 0$ on $M \setminus K$, we have the Harnack estimate by (2.4),

$$\frac{d}{dt}R(x, t) \geq 0, \quad \text{on } M \setminus K. \quad (2.7)$$

Then according to the proof of Theorem 2.1 (see [17, Lemmas 3.5–3.7] for details), for the convergence part in the proposition, we only need to show that the following asymptotic scalar curvature estimate,

$$\limsup_{x \rightarrow \infty} R(x)d^2(o, x) = \infty, \quad (2.8)$$

where $d(o, x)$ is a distance function from a fixed point $o \in M$. As a consequence, the rescaled flow $(M, g_{p_i}(t); p_i)$ has uniformly local curvature estimate, and therefore $(M, g_{p_i}(t); p_i)$ subsequently converges to a limit ancient κ -solution $(M_\infty, \bar{g}(t); p_\infty)$.

We note that (2.8) is true for any ancient κ -solution by Perelman's result of asymptotic zero volume ratio [22, 28] (cf. [17, Corollary 2.4]). In our case, we only have $\text{Rm} \geq 0$ away from K . We will use a different argument to prove (2.8) below.

On the contrary, we suppose that (2.8) is not true. Then there exists a constant $C > 0$, such that

$$R(x) \leq \frac{C}{d^2(o, x)} = o\left(\frac{1}{d(o, x)}\right). \quad (2.9)$$

In particular, the scalar curvature decays to zero uniformly. Due to a result in [15, Theorem 2.1], we know that there are two constants $c_1, c_2 > 0$ such that

$$c_1 d(0, x) \leq f(x) \leq c_2 d(o, x), \quad \text{as } d(o, x) \gg 1. \quad (2.10)$$

Thus by [19, Theorem 6.1] with the help of (2.9) and (2.10), we get

$$R(x) \geq \frac{C_0}{d(o, x)}, \quad \text{as } d(o, x) \gg 1,$$

for some constant C_0 . But this contradicts to (2.9). Hence (2.8) is true.

In the following, our goal is to show that $\bar{g}(t)$ is of form (2.6). First we prove the volume ratio estimate,

$$\text{AVR}(g) = \lim_{r \rightarrow \infty} \frac{\text{Vol}(B(p, r))}{r^n} = 0. \quad (2.11)$$

By (2.8), we can use the Hamilton's argument in [20, Lemma 22.2] to find sequences of points $q_i \rightarrow \infty$ and number $s_i > 0$ such that $\frac{s_i}{d(o, q_i)} \rightarrow 0$,

$$R(q_i)s_i^2 \rightarrow \infty, \quad (2.12)$$

and

$$R(x) \leq 2R(q_i), \quad \forall x \in B(q_i, s_i). \quad (2.13)$$

Consider a sequence of the rescaled flows $(M, g_{q_i}(t); q_i)$, $t \in (-s_i, 0]$, such that $R_{q_i}(q_i, 0) = 1$, where $R_{q_i}(\cdot, t)$ is the scalar curvature of $g_{q_i}(t)$. Then by (2.7), $R_{q_i}(x, t) \leq 2$ whenever $t \in (-s_i, 0]$ and $d_{q_i}(q_i, x) \leq R(q_i)^{\frac{1}{2}}s_i$, where $d_{q_i}(q_i, \cdot)$ is the distance function from q_i w.r.t $g_{q_i}(t)$. It follows that $(M, g_{q_i}(t); q_i)$ with $t \in (-s_i, 0]$ converges subsequently to a limit ancient κ -solution $(M_\infty, g_\infty(t); q_\infty)$. Moreover, by (2.12) and the curvature condition $\text{Km} \geq 0$ on $M \setminus K$, one can construct a geodesic line on $(M_\infty, g_\infty(0); q_\infty)$ (cf. [25, Theorem 5.35]). Thus, by the Cheeger–Gromoll splitting theorem, $(M_\infty, g_\infty(t); q_\infty)$ is in fact a splitting ancient flow $(N' \times \mathbb{R}, h'(t) + ds^2; q_\infty)$, where $(N', h'(t); q_\infty)$ is an $(n-1)$ -dimensional κ -noncollapsed ancient solution. Clearly, $(N', h'(0); q_\infty)$ can not be flat since $R_\infty(q_\infty, 0) = 1$, so $(M_\infty, g_\infty(t); q_\infty)$ is a non-flat ancient solution. Hence, by [22, Proposition 41.13], the asymptotic volume ratio of $(M_\infty, g_\infty(t); q_\infty)$ must be zero. This implies (2.11) by the monotonicity of volume since the (2.11) is invariant under the rescaling.

Next we let

$$r(p_i) = \sup \left\{ \rho \mid \text{Vol}(B(p_i, \rho)) \geq \frac{\omega}{2} \rho^n \right\}. \quad (2.14)$$

We prove

$$C_0^{-1}r(p_i) \leq R^{-\frac{1}{2}}(p_i) \leq C_0r(p_i). \quad (2.15)$$

In fact, for the first inequality in (2.15), by the volume comparison, there is $C_1(D) > 0$ for any $D > 0$ such that

$$\text{Vol}(B(x, r(p_i))) \geq C_1^{-1}r(p_i)^n, \quad \forall x \in B(p_i, Dr(p_i)).$$

Then by [17, Lemma 3.5], there is $C_0(D) > 0$ such that

$$R \leq C_0^2 r(p_i)^{-2}, \quad \forall x \in B\left(p_i, \frac{D}{2}r(p_i)\right). \quad (2.16)$$

Thus we only need to prove the second inequality of (2.15).

We use the above argument in the proof of (2.11). On the contrary, there is a sequence $p_i \rightarrow \infty$ (still denoted by $\{p_i\}$) such that

$$\lim_{i \rightarrow \infty} \frac{R^{-1/2}(p_i)}{r(p_i)} = \infty. \quad (2.17)$$

On the other hand, by (2.7) and (2.16), we have

$$R(x, t) \leq C_0 r(p_i)^{-2}, \quad \forall x \in B\left(p_i, \frac{D}{2} r(p_i)\right), \quad t \in \left(-\frac{D}{2}, 0\right].$$

Then the rescaled flow $(M, r(p_i)^{-2} g(r(p_i)^2 t); p_i)$ converges subsequently to a limit ancient solution $(M'_\infty, g'_\infty(t); p'_\infty)$. Note that $r(p_i) < \infty$ for each p_i by (2.11). Moreover, by the volume comparison, it follows

$$\lim_{i \rightarrow \infty} \frac{r(p_i)}{d(o, p_i)} = 0. \quad (2.18)$$

Hence, by (2.18) and the curvature condition $\text{Km} \geq 0$ on $M \setminus K$, one can construct a geodesic line on $(M'_\infty, g'_\infty(0); p'_\infty)$ (cf. [25, Theorem 5.35]), and so $(M'_\infty, g'_\infty(t); p'_\infty)$ is a splitting ancient flow $(\widehat{N} \times \mathbb{R}, \widehat{h}(t) + ds^2; p_\infty)$, where $\widehat{h}(t)$ is an $(n-1)$ dimensional ancient κ -solution. As a consequence, by (2.17), we have

$$R_\infty(p'_\infty, 0) = 0. \quad (2.19)$$

By the strong maximum principle and (2.19), $(N, \widehat{h}(0))$ is flat and so is $(M'_\infty, g'_\infty(0))$. Then by the injective radius estimate (cf. [17, Lemma 3.6]), one can show that $(M'_\infty, g'_\infty(0))$ must be isometric to the Euclidean space. In particular, $\text{Vol}(B_{g'_\infty(0)}(p'_\infty, 1)) = \omega$. But this is impossible by (2.14). Hence we finish the proof of (2.15).

At last, by (2.15) and (2.18), we have

$$\lim_{i \rightarrow \infty} \frac{R(p_i)^{-1}}{d^2(o, p_i)} = 0. \quad (2.20)$$

Then instead of the rescaled flow $(M, r(p_i)^{-2} g(r(p_i)^2 t); p_i)$ by $(M, g_{p_i}(t); p_i)$, the limit ancient solution $(M_\infty, \bar{g}(t); p_\infty)$ will split off a line as $(M'_\infty, g'_\infty(0); p'_\infty)$. Thus $\bar{g}(t)$ is of the form (2.6).

In the case of $n = 4$, we note that both split $3d$ κ -noncollapsed ancient flows $h'(t)$ and $\widehat{h}(t)$ in the above arguments are non-negatively curved when $\text{Km} \geq 0$ away from K . Thus both of $h'(t)$ and $\widehat{h}(t)$ are same as ancient κ -solutions. Hence the proofs above also work for the $4d$ steady Ricci solitons when the assumption $\text{Rm} \geq 0$ is replaced by $\text{Km} \geq 0$ away from K . $\square$

According to the proof in Proposition 2.2, we also get the following curvature comparison.

LEMMA 2.3. — *Let (M^n, g) be a complete noncompact κ -noncollapsed steady Ricci soliton as in Proposition 2.2. Let $\{p_i\} \rightarrow \infty$ be any sequence in (M^n, g) . Then for any $q_i \in B_{g_{p_i}}(p_i, D)$, there is $C_0(D) > 0$ such that*

$$C_0^{-1} R(p_i) \leq R(q_i) \leq C_0 R(p_i). \quad (2.21)$$

Proof. — We note that the rescaled flow $(M, g_{p_i}(t); p_i)$ converges to a splitting of ancient solution $(M_\infty, \bar{g}(t) = h(t) + ds^2; p_\infty)$. Then by (2.15) and (2.16), we get the second inequality of (2.21) immediately. Thus we only need to prove the first inequality.

By contradiction, there exists a sequence of points $q_i \in B_{g_{p_i}}(p_i, D)$ for some $D > 0$ such that

$$\frac{R(q_i)}{R(p_i)} \longrightarrow 0, \quad \text{as } i \longrightarrow \infty. \quad (2.22)$$

Then

$$R_{\bar{g}(0)}(q_\infty) = 0,$$

where q_∞ is a limit of $\{q_i\}$ from the convergence of $(M, g_{p_i}(t); p_i)$. By the strong maximum principle, it follows that $\bar{g}(0)$ is a flat metric, which contradicts to the fact $R_{\bar{g}(0)}(p_\infty) = 1$. Thus (2.21) holds. $\square$

3. Compact case of $(N, h(t))$

In this section, we assume that there exists a sequence of $p_i \rightarrow \infty$ on an n -dimensional steady Ricci soliton (M^n, g) such that the corresponding split ancient κ -solution $(N, h(t))$ of codimension one in Proposition 2.2 satisfies

$$\text{Diam}(N, h(0)) \leq C. \quad (3.1)$$

We will use the method in [18, 19] to prove several lemmas and propositions for level set geometry of (M, g) under the condition (3.1). The results will be often used in next sections.

Firstly we show there exists a decomposition of (M, g) in Proposition 2.2 if it satisfies (3.1).

LEMMA 3.1. — *Let (M, g) be the complete steady gradient Ricci soliton in Proposition 2.2. Suppose that there exists a sequence of $p_i \rightarrow \infty$ such that the split ancient κ -solution $(N, h(t))$ in Proposition 2.2 satisfies (3.1). Then for all $i \gg 1$, M can be divided into three parts*

$$M = B_g(p_i, 10CR(p_i)^{-\frac{1}{2}}) \cup K' \cup M', \quad (3.2)$$

where K' is a compact set containing K and $M' = M \setminus (B_g(p_i, 10CR(p_i)^{-\frac{1}{2}}) \cup K')$. Moreover, $M \setminus B_g(p_i, 10CR(p_i)^{-\frac{1}{2}})$ is disconnected.

Proof. — Since (M, g) satisfies $\text{Rm} \geq 0$ on $M \setminus K$, by [24, Section 1] (see also [12]), (M, g) has finite topological type. Then there exists a compact set $\widehat{K} \subset M$ such that M is homeomorphic to $\widehat{K}$. This implies that $M \setminus \widehat{K}$ is contractible. On the other hand, according to the proof in Proposition 2.2,

we have $B_g(p_i, 10CR(p_i)^{-\frac{1}{2}}) \subset M \setminus \widehat{K}$ for all $i \gg 1$. By the convergence, we know $B_g(p_i, 10CR(p_i)^{-\frac{1}{2}})$ are diffeomorphic to some open balls in $N \times \mathbb{R}$. Thus for each ball, there exists two boundaries denoted by Σ_1 and Σ_2 . Hence, we can find two points $q_1, q_2 \in M \setminus B_g(p_i, 10CR(p_i)^{-\frac{1}{2}})$, so that q_1, q_2 are close to Σ_1, Σ_2 respectively, and a path σ_1 which connects q_1, q_2 and passes through $B_g(p_i, 10CR(p_i)^{-\frac{1}{2}})$. On the contrary, if $M \setminus B_g(p_i, 10CR(p_i)^{-\frac{1}{2}})$ is connected, there exists another path σ_2 which connects q_1, q_2 such that $\sigma_2(s) \subset M \setminus B_g(p_i, 10CR(p_i)^{-\frac{1}{2}})$. By connecting σ_1 and σ_2 , we obtain a non-contractible circle in $M \setminus \widehat{K}$. But this is impossible since $M \setminus \widehat{K}$ is contractible. Therefore, $M \setminus B_g(p_i, 10CR(p_i)^{-\frac{1}{2}})$ must be disconnected.

By [26, Corollary 1.1], a complete noncompact steady gradient Ricci soliton is either connected at infinity (i.e., it has only one end), or it is a product of $\mathbb{R}$ with a compact Ricci flat manifold. The latter is impossible since (M, g) is not Ricci-flat. Thus $M \setminus B_g(p_i, 10CR(p_i)^{-\frac{1}{2}})$ has only one end. It follows that $M \setminus B_g(p_i, 10CR(p_i)^{-\frac{1}{2}})$ consists of one noncompact connected component M' and one compact set K' . Clearly, M' and K' satisfy (3.2). Moreover, K' will contain K when $d(o, p_i)$ is large. The proof of Lemma 3.1 is complete. $\square$

By Lemma 3.1, we prove that (M, g) has a convexity property in the sense of the geodesics.

LEMMA 3.2. — *Suppose that there exists a sequence of $p_i \rightarrow \infty$ such that the split $(n-1)$ -dimensional ancient κ -solution $(N, h(t))$ in Proposition 2.2 satisfies (3.1). Then there exists a compact set K' ($K \subset K'$) such that for $x_1, x_2 \in M \setminus K'$ the minimal geodesic curve $\sigma(s)$ connecting x_1 and x_2 , is contained in $M \setminus K$, where K is the compact set in Proposition 2.2.*

Proof. — By Lemma 3.1, we may choose a point $p \in \{p_i\}$ such that $B_g(p, 10CR(p)^{-\frac{1}{2}})$ divides M into three parts with a compact part K' containing K as follows,

$$M = B_g\left(p, 10CR(p)^{-\frac{1}{2}}\right) \cup K' \cup M',$$

where $B_g(p, 10CR(p)^{-\frac{1}{2}}) \cap K = \emptyset$ and $M' = M \setminus (B_g(p, 10CR(p)^{-\frac{1}{2}}) \cup K')$ is a noncompact set of M . Set

$$\widehat{K} = K' \cup B_g\left(p, 10CR(p)^{-\frac{1}{2}}\right).$$

We need to verify that $\widehat{K}$ is as required in Lemma 3.2.

On the contrary, there will exist two points $x_1, x_2 \in M \setminus \widehat{K}$ and another point $x \in \sigma(s) \cap K$, where $\sigma(s)$ is the minimal geodesic curve connecting x_1 and x_2 . By Lemma 3.1, the geodesic ball $B_g(p, 10CR(p)^{-\frac{1}{2}})$ separates both of x_1 and x_2 from x , and so $\sigma(s)$ will pass through $B_g(p, 10CR(p)^{-\frac{1}{2}})$

at least twice. Denote the first point by q_1 and the last point by q_2 in $B_g(p, 10CR(p)^{-\frac{1}{2}})$ respectively, which intersects with $\sigma(s)$. Let σ' be the part of $\sigma(s)$ between q_1 and q_2 . Thus by the triangle inequality, we have

$$\begin{aligned}
 d_g(q_1, q_2) &= \text{Length}(\sigma') \\
 &= d_g(q_1, x) + d_g(x, q_2) \\
 &\geq d_g(q_1, o) + d_g(o, q_2) - 2d_g(o, x) \\
 &\geq 2d_g(o, p) - d_g(q_1, p) - d_g(q_2, p) - 2C' \\
 &\geq 2d_g(o, p) - 20R(p)^{-\frac{1}{2}} - 2C'.
 \end{aligned} \tag{3.3}$$

On the other hand, by the estimate (2.20), we see that for any small δ it holds

$$R(p_i)^{-\frac{1}{2}} \leq \delta d_g(p_i, o), \tag{3.4}$$

as long as $i \gg 1$. By (3.3), it follows

$$d_g(q_1, q_2) \geq 2d_g(o, p) - 20\delta d_g(o, p) - 2C' \geq d_g(o, p).$$

However,

$$d_g(q_1, q_2) \leq d_g(q_1, p) + d_g(p, q_2) \leq 20R(p)^{-\frac{1}{2}} \leq 20\delta d_g(o, p) \leq \frac{1}{2}d_g(o, p).$$

Thus we get a contradiction! The Lemma 3.2 is proved. $\square$

3.1. Curvature decay estimate

By Lemmas 2.3 and 3.2, we prove

PROPOSITION 3.3. — *Let (M^n, g) be the steady Ricci soliton in Proposition 2.2 with $\text{Ric} > 0$ away from K . Suppose that there exists a sequence of $p_i \rightarrow \infty$ such that the split $(n-1)$ -dimensional ancient κ -solution $(N, h(t))$ in Proposition 2.2 satisfies (3.1). Then the curvature of (M^n, g) decays to zero uniformly. Namely,*

$$\lim_{x \rightarrow \infty} R(x) = 0. \tag{3.5}$$

Proof. — First we prove

$$\lim_{p_i \rightarrow \infty} R(p_i) = 0. \tag{3.6}$$

On the contrary, we assume that $R(p_i) \geq c$ for some constant $c > 0$. We consider a sequence of functions $f_{p_i} = f - f(p_i)$ on the Riemannian manifolds $(M, g_{p_i}(0); p_i)$. By (2.5), it is easy to see

$$|\nabla f_{p_i}|_{g_{p_i}} \leq c^{-\frac{1}{2}}.$$

Thus for any $D > 0$ it holds

$$|f_{p_i}(x)| \leq 2c^{-\frac{1}{2}}D, \quad \forall x \in B_{g_{p_i}}(p_i, D).$$

By the regularity of the Laplace equation,

$$\Delta_{g_{p_i}} f_{p_i} = R(g_{p_i(0)}),$$

f_i converges subsequently to a smooth function f_∞ on $N \times \mathbb{R}$ which satisfies the gradient steady Ricci soliton equation,

$$\text{Ric}(\bar{g}(0)) = \nabla^2 f_\infty.$$

Note that $\bar{g}(0) = h(0) + ds^2$ is a product metric. Hence $(N, h(0))$ is also a steady gradient Ricci soliton,

On the other hand, by the maximum principle, $(N, h(0); p_\infty)$ should be Ricci-flat. However, by the normalization of

$$R(g_{p_i}(0))(p_i) = 1,$$

$R(h(0))(p_\infty)$ is also 1. This is a contradiction! (3.6) is proved.

By (3.6) and Lemma 2.3, we get

$$\lim_{i \rightarrow \infty} \sup_{B_g(p_i, 10CR(p_i)^{-\frac{1}{2}})} R(x) = 0. \quad (3.7)$$

Next we use (3.7) to derive (3.5).

Recall that the set of equilibrium points of (M, g, f) is given by

$$S := \{x \mid |\nabla f|(x) = 0\}.$$

In general, S may be non-empty. But we have

Claim 1. — There are no equilibrium points away from a compact set $\widehat{K}$ of M which contains K' . Here K' is the set of M determined in Lemma 3.2.

If S is not empty and Claim 1 is not true, there will be two equilibrium points x_1 and x_2 and a compact set $\widehat{K}$ containing K' such that $x_1, x_2 \in M \setminus \widehat{K}$. Then by Lemma 3.2, there is a minimal geodesic curve $\sigma(s)$ connecting x_1 and x_2 such that $\sigma(0) = x_1$ and $\sigma(T) = x_2$ and $\sigma(s) \subset M \setminus K$. Note

$$\frac{d}{ds} (\langle \nabla f, \sigma' \rangle(\sigma(s))) = \nabla^2 f(\sigma', \sigma')(s) = \text{Ric}(\sigma', \sigma').$$

Thus we get

$$\begin{aligned} 0 &= \langle \nabla f, \sigma' \rangle(\sigma(t)) - \langle \nabla f, \sigma' \rangle(\sigma(0)) \\ &= \int_0^T \text{Ric}(\sigma', \sigma') ds > 0, \end{aligned}$$

which is a contradiction! Hence, Claim 1 is true.

By (2.20), we can choose a subsequence of $\{p_i\}$, still denoted by $\{p_i\}$ such that

$$B_g\left(p_i, 10CR(p_i)^{-\frac{1}{2}}\right) \cap B_g\left(p_j, 10CR(p_j)^{-\frac{1}{2}}\right) = \emptyset, \quad \forall i \neq j \gg 1. \quad (3.8)$$

Then by Lemma 3.1, as in (3.2), there are a compact set $\bar{K}$ and a sequence of compact set $\{K_i\}$ ($i \geq i_0$) of M such that $\hat{K} \subset \bar{K}$ and

$$\partial K_i \subset \partial B_g\left(p_i, 10CR(p_i)^{-\frac{1}{2}}\right) \cup \partial B_g\left(p_{i+1}, 10CR(p_{i+1})^{-\frac{1}{2}}\right),$$

and M is decomposed as

$$M = \bar{K} \bigcup_{i \geq i_0} \left(K_i \cup B_g\left(p_{i+1}, 10CR(p_{i+1})^{-\frac{1}{2}}\right) \right), \quad (3.9)$$

Claim 2. — For any $q_i \in K_i$, there exists $t_i > 0$ such that

$$q_i^{t_i} = \phi_{t_i}(q_i) \in B_g\left(p_i, 10R(p_i)^{-\frac{1}{2}}C\right) \cup B_g\left(p_{i+1}, 10R(p_{i+1})^{-\frac{1}{2}}C\right). \quad (3.10)$$

On the contrary, we see that $\phi_t(q_i) \subset K_i$ for all $t \geq 0$. Since K_i is compact, by Claim 1, there exists $c' > 0$ such that

$$\text{Ric} \geq c'g, \quad c'^{-1} \leq |\nabla f| \leq c'.$$

It follows

$$\begin{aligned} \frac{d}{dt} R(\phi_t(q_i)) &= -\langle \nabla R, \nabla f \rangle(\phi_t(q_i)) \\ &= 2 \text{Ric}(\nabla f, \nabla f) \geq 2c'^{-1} > 0, \quad \forall t \geq 0. \end{aligned} \quad (3.11)$$

As a consequence,

$$R(\phi_t(q_i)) \geq 2c'^{-1}t \longrightarrow \infty, \quad \text{as } t \longrightarrow \infty.$$

This is impossible since $R(\cdot)$ is uniformly bounded. Hence, Claim 2 is true.

By Claim 2 and (3.11), for any $q_i \in K_i$ we see

$$R(q_i) \leq R(q_i^{t_i}) \leq \max \left\{ R(x) \left| \begin{array}{l} x \in B_g\left(p_i, 10R(p_i)^{-\frac{1}{2}}C\right) \\ \cup B_g\left(p_{i+1}, 10R(p_{i+1})^{-\frac{1}{2}}C\right) \end{array} \right. \right\}.$$

Thus we get (3.5) from (3.7) and (3.9) immediately. $\square$

Remark 3.4. — The curvature of steady Ricci soliton in Proposition 3.3 decays uniformly to zero. Then $|\nabla f(x)| \rightarrow 1$ as $d(o, x) \rightarrow \infty$ by (2.5). Moreover, by [18, Lemma 2.2] (or [15, Theorem 2.1]), f satisfies (2.10). Hence, the integral curve $\gamma(s)$ generated by ∇f extends to the infinity as $s \rightarrow \infty$.

3.2. Geometry of level sets

By Proposition 3.3 and (2.5), there exists a point $p_0 \in M$ such that

$$R_{\max} = \sup_{p \in M} R(p) = R(p_0) = 1.$$

For any positive $c < 1$, we set

$$S(c) = \{p \in M \mid R(p) \geq R_{\max} - c\}.$$

Then $S(c)$ is a compact set. Furthermore, by Remark 3.4 there exists a constant c_0 such that $\widehat{K} \subset S(c_0)$ and $\nabla f \neq 0$ on $S(c_0) \setminus \widehat{K}$. Thus $\widehat{X} = \frac{\nabla f}{|\nabla f|}$ is well-defined on $S(c) \setminus \widehat{K}$ for any $c \geq c_0$. Moreover, we know $M \setminus S(c)$ is diffeomorphic to $\Sigma \times (A, \infty)$, where Σ is the levelset of f on $f(p_i)$ for some large i and $A > 0$ is a constant. Since (M, g) has only one end, Σ must be connected.

By [15, Lemmas 2.2–2.3], it is known that there exists a t_q such that $\phi_{t_q}(q) \in S(c_0)$ for any $q \in M \setminus S(c_0)$. Consequently, for any integral curve of $\widehat{X} = \frac{\nabla f}{|\nabla f|}$, $\Gamma(s) : [0, \infty) \rightarrow M$, we can reparametrize s such that $\Gamma(0) = p \in S(c_0) \setminus \widehat{K}$, and so $\Gamma(s) \subset M \setminus \widehat{K}$ is a smooth curve for any $s > 0$.

LEMMA 3.5. — *Let (M^n, g) be an n -dimensional steady soliton as in Proposition 3.3 and let $\Gamma(s)$ be any integral curve of $\widehat{X}$ with $\Gamma(0) = p \in S(c_0) \setminus \widehat{K}$. Then for any ϵ , there exists a uniform constant $C = C(\epsilon) > 0$ such that*

$$(1 - \epsilon)(s_2 - s_1) \leq d(\Gamma(s_2), \Gamma(s_1)) \leq (s_2 - s_1), \quad \forall s_2 > s_1 > C. \quad (3.12)$$

In particular,

$$(1 - \epsilon)s \leq d(\Gamma(s), p) \leq s, \quad \forall s > C. \quad (3.13)$$

Proof. — Firstly by Remark 3.4, we note that for any $\epsilon > 0$ there exists a compact set S' such that

$$|\nabla f|(x) > 1 - \epsilon, \quad \forall x \in M \setminus S'. \quad (3.14)$$

Moreover, (3.14) holds whenever $f(x) > L$. Since $\Gamma(s) \subset M \setminus \widehat{K}$, $|\nabla f|(\Gamma(s)) \geq c_0 > 0$ by (2.4) for all $s \geq 0$. It follows

$$f(\Gamma(s)) - f(\Gamma(0)) = \int_0^s \frac{d}{dt} f(\Gamma(t)) dt = \int_0^s |\nabla f|(\Gamma(t)) dt \geq cs.$$

Thus there exists a uniform constant $C = \frac{L}{c} + 1$ such that (3.14) holds as long as $s > C$.

Let $\gamma : [0, D] \rightarrow M$ be a minimal geodesic from $\Gamma(s_1)$ to $\Gamma(s_2)$, where $D = d(\Gamma(s_1), \Gamma(s_2))$. Then by

$$\frac{d}{dr} \langle \nabla f, \gamma'(r) \rangle = \nabla^2 f(\gamma'(r), \gamma'(r)) \geq 0,$$

we obtain

$$f(\Gamma(s_2)) - f(\Gamma(s_1)) = \int_0^D \langle \nabla f, \gamma'(r) \rangle dr \leq D \langle \nabla f, \gamma'(D) \rangle.$$

This implies

$$f(\Gamma(s_2)) - f(\Gamma(s_1)) \leq d(\Gamma(s_1), \Gamma(s_2)). \quad (3.15)$$

On the other hand, by (3.14), we have

$$\begin{aligned} f(\Gamma(s_2)) - f(\Gamma(s_1)) &= \int_{s_1}^{s_2} \langle \nabla f, \Gamma'(r) \rangle dr \\ &= \int_{s_1}^{s_2} |\nabla f|(\Gamma(r)) dr \geq (1 - \epsilon)(s_2 - s_1). \end{aligned} \quad (3.16)$$

Thus the first inequality in (3.12) follows from (3.15) and (3.16) immediately. Note that

$$d(\Gamma(s_1), \Gamma(s_2)) \leq \text{Length}(\Gamma(s))|_{s_1}^{s_2} = s_2 - s_1. \quad (3.17)$$

Hence, the second inequality in (3.12) also holds. (3.13) is a direct consequence of (3.12) by the triangle inequality. $\square$

As in Lemma 3.5, let $\Gamma_i(s)$ be an integral curve of $\hat{X}$ through p_i with $\Gamma_i(0) \in S(c_0)$ and $\Gamma_i(s_i) = p_i$. For any $D > 0$, we set

$$\hat{\Gamma}_i(s) = \Gamma_i\left(R(p_i)^{-\frac{1}{2}}s + s_i\right), \quad s \in [-D, D].$$

Then it is easy to see

$$\left| \frac{d\hat{\Gamma}_i(s)}{ds} \right|_{g_{p_i}(0)} = 1, \quad s \in [-D, D].$$

Thus $\hat{\Gamma}_i(s)$ is an integral curve of $\hat{X}_i = \frac{\nabla_i f}{|\nabla_i f|}$ through p_i , where ∇_i is the gradient operator w.r.t. the metric $(M, g_{p_i}(t); p_i)$.

With the help of Lemma 3.5, we prove that the splitting line obtained by Proposition 2.2 is actually a limit of a family of integral curves of $\hat{X}_i$ under the condition in Proposition 3.3.

LEMMA 3.6. — *Let (M^n, g) be the steady soliton in Proposition 3.3 and $(N \times \mathbb{R}, h(t) + ds^2; p_\infty)$ the splitting limit flow of $(M, g_{p_i}(t); p_i)$. Then $\hat{\Gamma}_i(s)$ converges locally to a geodesic line on $N \times \mathbb{R}$ w.r.t. the metric $(M, g_{p_i}(t); p_i)$.*

Proof. — Since $X_i = \nabla_i f$ is convergent w.r.t. the metrics $(M, g_{p_i}(t); p_i)$ (cf. [19, Lemma 4.6]), $\widehat{X}_i$ also converges subsequently to a VF $\widehat{X}_\infty$ on $(N \times \mathbb{R}, h(t) + ds^2; p_\infty)$. Thus $\widehat{\Gamma}_i(s)$ converges to an integral curve $\widehat{\Gamma}_\infty(s)$ of $\widehat{X}_\infty$ on $N \times \mathbb{R}$, where $s \in (-\infty, \infty)$. It remains to show that $\widehat{\Gamma}_\infty(s)$ is a line.

Since $p_i \rightarrow \infty$, we have $s_i \rightarrow \infty$. Then by (2.11) and (2.15), for any number $D > 0$, it holds

$$s_i - D R(p_i)^{-\frac{1}{2}} \longrightarrow \infty.$$

By applying (3.12) to each $\widehat{\Gamma}_i(s')$, we get

$$(1 - \epsilon)DR(p_i)^{-\frac{1}{2}} \leq d(\widehat{\Gamma}_i(-D), \widehat{\Gamma}_i(0)) \leq DR(p_i)^{-\frac{1}{2}}$$

and

$$(1 - \epsilon)DR(p_i)^{-\frac{1}{2}} \leq d(\widehat{\Gamma}_i(D), \widehat{\Gamma}_i(0)) \leq DR(p_i)^{-\frac{1}{2}}.$$

It follows

$$2(1 - \epsilon)DR(p_i)^{-\frac{1}{2}} \leq d(\widehat{\Gamma}_i(-D), \widehat{\Gamma}_i(D)) \leq 2DR(p_i)^{-\frac{1}{2}},$$

and consequently,

$$2(1 - \epsilon)D \leq d_{g_{p_i}}(\widehat{\Gamma}_i(-D), \widehat{\Gamma}_i(D)) \leq 2D.$$

Thus by taking the limit of $\widehat{\Gamma}_i(s)$ as well as $\epsilon \rightarrow 0$, we obtain

$$d_{g_\infty}(\widehat{\Gamma}_\infty(-D), \widehat{\Gamma}_\infty(D)) = 2D. \quad (3.18)$$

Note that $2D$ is the length of $\widehat{\Gamma}_\infty(s)$ between $\widehat{\Gamma}_\infty(-D)$ and $\widehat{\Gamma}_\infty(D)$. Hence, $\widehat{\Gamma}_\infty(s)$ must be a minimal geodesic connecting $\widehat{\Gamma}_\infty(-D)$ and $\widehat{\Gamma}_\infty(D)$. Since D is arbitrary, $\widehat{\Gamma}_\infty(s)$ can be extended to a geodesic line. $\square$

Now we begin to prove main results in this subsection.

LEMMA 3.7. — *Let (M^n, g) be the steady soliton in Proposition 3.3 and $(N \times \mathbb{R}, h(t) + ds^2; p_\infty)$ the splitting limit flow of $(M, g_{p_i}(t); p_i)$, which satisfies (3.1). Then $f^{-1}(f(p_i)) \subseteq B_{g_{p_i}}(p_i, 200C)$ when $i \gg 1$.*

Proof. — On the contrary, there will exist a $q'_i \in \partial B_{g_{p_i}}(p_i, 100C) \cap f^{-1}(f(p_i))$ and a minimal geodesic $\bar{\gamma}_i \subset f^{-1}(f(p_i))$ connecting p_i and q'_i w.r.t. the induced metric $\bar{g}_{p_i}$ on $f^{-1}(f(p_i))$ such that

$$\bar{\gamma}_i \subset B_{g_{p_i}}(p_i, 100C).$$

Then

$$\text{Length}_{\bar{g}_{p_i}}(\bar{\gamma}_i) \geq d_{g_{p_i}}(p_i, q'_i) = 100C. \quad (3.19)$$

On the other hand, according to the proofs in [19, Lemma 4.3, Proposition 4.5], the part $\Sigma_i = f^{-1}(f(p_i)) \cap B_{g_{p_i}}(p_i, 100C)$ of the level set $f^{-1}(f(p_i))$,

which contains $\bar{\gamma}_i$, converges subsequently to an $(n-1)$ -dimensional open manifold $(\Sigma_\infty, h'; p_\infty)$ w.r.t. the induced metric $\bar{g}_{p_i}$. As a consequence, the minimal geodesic $\bar{\gamma}_i$ converges subsequently to a minimal geodesic $\bar{\gamma}$ in Σ_∞ . Thus by (3.19), we get

$$\text{Length}_{h'}(\bar{\gamma}) \geq 100C. \quad (3.20)$$

Next we show that (Σ_∞, h') is an open set of $(N, h(0))$. Then it follows

$$\text{Diam}(N, h(0)) \geq \text{Diam}(\Sigma_\infty, h') \geq 100C,$$

which contradicts to (3.1). The Lemma 3.7 will be proved.

Let $\hat{X}_i = \frac{\nabla_i f}{|\nabla_i f|}$. By Proposition 3.3, (3.14) and Shi's estimates we can calculate that

$$\begin{aligned} & \sup_{B(p_i, 2D)_{g_{p_i}}} |\nabla_i \hat{X}_i|_{g_{p_i}} \\ &= \sup_{B(p_i, 2D)_{g_{p_i}}} R(p_i)^{-\frac{1}{2}} \left(\frac{|\text{Ric}|}{|\nabla f|} + \frac{|\text{Ric}(\nabla f, \nabla f)|}{|\nabla f|^3} \right) \leq CR(p_i)^{\frac{1}{2}} \longrightarrow 0, \end{aligned}$$

and

$$\sup_{B(p_i, 2D)_{g_{p_i}}} |\nabla_i^m \hat{X}_i|_{g_{p_i}} \leq C(m) \sup_{B(p_i, 2D)_{g_{p_i}}} |\nabla_i^{m-1} \text{Ric}(g_{p_i})|_{g_{p_i}} \leq C'.$$

Thus $\hat{X}_i$ converges subsequently to a parallel vector field $\hat{X}_\infty$ on $(N \times \mathbb{R}, h(t) + ds^2; p_\infty)$. Moreover,

$$\sup_{B(p_i, 2D)_{g_{p_i}}} |\hat{X}_i|_{g_{p_i}} = 1.$$

Hence, $\hat{X}_\infty$ is a non-trivial parallel vector field on $N \times \mathbb{R}$.

$\hat{X}_\infty$ is also perpendicular to (Σ_∞, h') . In fact, for any $V \in T\Sigma_\infty$ with $|V|_{h'} = 1$, by [19, Proposition 4.5], there is a sequence of $V_i \in T\Sigma_i$ such that $R(p_i)^{-\frac{1}{2}} V_i \rightarrow V$. Thus

$$h'(V, \hat{X}_\infty) = \lim_{i \rightarrow \infty} g_{p_i}(R(p_i)^{-\frac{1}{2}} V_i, \hat{X}_i) = \lim_{i \rightarrow \infty} g\left(V_i, \frac{\nabla f}{|\nabla f|}\right) = 0.$$

By Lemma 3.6, we have already known that $\hat{X}_\infty$ generates a geodesic line $\hat{\Gamma}_\infty$ through p_∞ on $N \times \mathbb{R}$. Note that $(N, h(0))$ is compact by (3.1). $\hat{X}_\infty$ must be tangent to the splitting line direction of $N \times \mathbb{R}$, and consequently, $(\Sigma_\infty, h'; p_\infty) \subset (N, h(0); p_\infty)$. Namely, (Σ_∞, h') is an open set of $(N, h(0))$. The proof of Lemma 3.7 is complete. $\square$

By Lemma 3.7, we prove

PROPOSITION 3.8. — *Let (M^n, g) be the steady Ricci soliton in Proposition 3.3 and $(N \times \mathbb{R}, h(t) + ds^2; p_\infty)$ the splitting limit flow of $(M, g_{p_i}(t); p_i)$, which satisfies (3.1). Then there exists $C_0(C) > 0$ such that for any $q_i \in f^{-1}(f(p_i))$ the splitting limit flow $(h'(t) + ds^2, N' \times \mathbb{R}; q_\infty)$ of rescaled flow $(M, g_{q_i}(t); q_i)$ satisfies*

$$\text{Diam}(h'(0)) \leq C_0. \quad (3.21)$$

Proof. — The convergence part comes from Proposition 2.2. We need to check (3.21). In fact, by Lemmas 3.7 and 2.3, there are $C_1, C_2 > 0$ such that for any $D > 0$ such that

$$B_{g_{q_i}}(q_i, D) \subset B_{g_{p_i}}(q_i, C_1 D) \subset B_{g_{p_i}}(p_i, C_1 D + C_2),$$

where $g_{q_i} = R(q_i)g$. Similarly, we have

$$B_{g_{q_i}}(q_i, D) \supset B_{g_{p_i}}(q_i, C_1^{-1} D) \supset B_{g_{p_i}}(p_i, C_1^{-1} D - 2C_2).$$

Then it is easy to see that the splitting Ricci flow $(h'(t) + ds^2, N' \times \mathbb{R}; q_\infty)$ of $(M, g_{q_i}(t); q_i)$ is isometric to $(h(t) + ds^2, N \times \mathbb{R}; p_\infty)$ up to scaling. As a consequence, we get

$$\text{Diam}(h'(0)) \leq (C_1 + 10) \text{Diam}(h(0)) \leq (C_1 + 10)C.$$

The Proposition 3.8 is proved. $\square$

Remark 3.9. — The arguments in the proofs of Lemma 3.7 and Proposition 3.8 also imply that both of N and N' are diffeomorphic to each level set $f^{-1}(f(p_i))$ when $i \gg 1$. In fact, the submanifolds $(f^{-1}(f(p_i)), \bar{g}_{p_i})$ and $(f^{-1}(f(p_i)), \bar{g}_{q_i})$ converge to $(N, h(0))$ and $(N', h'(0))$ in the Cheeger–Gromov sense, respectively.

4. A classification of split compact ancient solutions

In case that all split ancient κ -solution $(N, h(t))$ satisfies (3.1), we can classify $(N, h(t))$.

PROPOSITION 4.1. — *Let (M, g) be a steady gradient Ricci soliton as in Proposition 3.3. Suppose that there is a uniform constant C such that all split ancient κ -solution $(N, h(t))$ satisfies (3.1). Then every $h(t)$ must be an ancient κ -solution of type I.*

Compact ancient κ -solution of type I has been classified as follows (cf. [16, Theorem 7.34], [5], [27]).

LEMMA 4.2. — Suppose that $(N^{n-1}, h(t))$ of $(n-1)$ -dimension is a compact ancient κ -solution of type I with $\text{Rm} \geq 0$. Then

$$(N^{n-1}, h(t)) \cong (N_1, h_1(t)) \times \cdots \times (N_k, h_k(t)) \times (\widehat{N}_1, \widehat{h}_1(t)) \times \cdots \times (\widehat{N}_\ell, \widehat{h}_\ell(t)), \quad (4.1)$$

where each $(N_i, h_i(t))$ is a family of shrinking quotients of a closed symmetric space with nonnegative curvature operator, and each $(\widehat{N}_j, \widehat{h}_j(t))$ is a family of shrinking round quotient spheres.

To prove Proposition 4.1, we shall exclude the existence of compact κ -noncollapsed ancient solutions of type II. Our proof is based on the following two lemmas below,

LEMMA 4.3. — Let $(N^{n-1}, h(t))$ be an $(n-1)$ -dimensional compact κ -solution of type II. Fix $p \in N$, we consider $t_k \rightarrow -\infty$ and a sequence of points $x_k \in N$ such that $\ell(x_k, t_k) < \frac{n-1}{2}$, where ℓ denotes the reduced distance from $(p, 0)$. Then the rescaled manifold by dilating the manifold $(N, h(t_k))$ around the point x_k by the factor $\frac{1}{\sqrt{-t_k}}$ converges to a noncompact shrinking gradient Ricci soliton.

Proof. — According to a classification result of closed ancient κ -solutions with nonnegative curvature operator (cf. [16, Theorem 7.34], [5]), we have

$$(N, h(t)) \cong (N_1, h_1(t)) \times \cdots \times (N_k, h_k(t)) \times (\widehat{N}_1, \widehat{h}_1(t)) \times \cdots \times (\widehat{N}_\ell, \widehat{h}_\ell(t)), \quad (4.2)$$

where each $(N_i, h_i(t))$ is a family of shrinking quotients of an $(n-1)$ -dimensional closed symmetric space with nonnegative curvature operator, and each $(\widehat{N}_j, \widehat{h}_j(t))$ is an ancient flow with positive curvature operator on a quotient of $(n-1)$ -dimensional sphere. By Perelman's argument in [28, Section 11], the rescaled manifolds converge in the Cheeger–Gromov sense to a κ -noncollapsed shrinking gradient Ricci soliton $(N', h'(t))$ with nonnegative curvature operator. If the limit soliton $(N', h'(t))$ is compact, the limit $(N'_j, h'_j(t))$ of

$$(\widehat{N}_j, (-t_k)^{-1} \widehat{h}_j((-t_k)t); x_k)$$

is also compact with nonnegative curvature operator. Then by the convergence, N'_j is diffeomorphic to $\widehat{N}_j$, i.e., a quotient of sphere. Thus again by the classification of $(N'_j, h'_j(t))$ as in (4.2) for $(N, h(t))$, we know that $(N'_j, h'_j(t))$ is a shrinking Ricci soliton on a quotient sphere with positive curvature operator. By [10], it follows that $(N'_j, h'_j(t))$ is a family of shrinking round quotient spheres. As a consequence, the sectional curvature of $(\widehat{N}_j, \widehat{h}_j(\cdot, t_k))$ lies in the interval $[\frac{c_j - \epsilon_k}{-t_k}, \frac{c_j + \epsilon_k}{-t_k}]$, where $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$. Hence, by the

curvature pinching estimate in [10], each $(\widehat{N}_j, \widehat{h}_j(t))$ is also a family of shrinking round quotient spheres, and so $(N, h(t))$ is of type I, which contradicts to the type II condition. Therefore, $(N', h'(t))$ must be noncompact. The Lemma 4.3 is proved. $\square$

Lemma 4.3 can be regarded as a higher dimensional version of [1, Lemma 2.1].

LEMMA 4.4. — *Let $(N^{n-1}, h(t))$ be an $(n-1)$ -dimensional compact ancient κ -solution of type II. Then for any sequence $t_k \rightarrow -\infty$, it holds*

$$R_{\min}(t_k) \text{Diam}(h(t_k))^2 \longrightarrow \infty,$$

where $R_{\min}(t) = \min\{R(h(\cdot, t))\}$. Consequently,

$$\lim_{t \rightarrow -\infty} R_{\min}(t) \text{Diam}(h(t))^2 \longrightarrow \infty. \quad (4.3)$$

Proof. — By a result of Perelman [28, Section 11], for any sequence $t_k \rightarrow -\infty$, we can always find a sequence of points $x_k \in N$ such that $\ell(x_k, t_k) \leq \frac{n-1}{2}$ for each k . Then by Lemma 4.3, the rescaled Ricci flows $(N, (-t_k)^{-1}h((-t_k)t); x_k)$ converge to a noncompact shrinking Ricci soliton $(N', h'(t))(t < 0)$ with nonnegative curvature operator. It follows

$$\text{Diam}((-t_k)^{-1}h(t_k)) = (-t_k)^{-\frac{1}{2}} \text{Diam}(h(t_k)) \longrightarrow \infty. \quad (4.4)$$

Moreover, such a limit soliton is non-flat [22, Proposition 39.1] and so $R(h'(\cdot, t)) > 0$ for any $t < 0$ (cf. [13]). Thus there exists a uniform constant $\delta > 0$, such that $(-t_k)R(x_k, t_k) \geq \delta$ for all $k \gg 1$. Hence, by (4.4), we derive

$$\begin{aligned} R_{\max}(t_k) \text{Diam}(h(t_k))^2 &\geq R(x_k, t_k) \text{Diam}(h(t_k))^2 \\ &\geq \delta(-t_k)^{-1} \text{Diam}(h(t_k))^2 \longrightarrow \infty, \end{aligned} \quad (4.5)$$

where $R_{\max}(t) = \max_{q \in N}\{R(h(q, t))\}$.

Next we shall improve (4.5) to (4.3). Suppose that (4.3) fails. Then there exist a constant C and a sequence of times $t'_k \rightarrow -\infty$, such that

$$R_{\min}(t'_k) \text{Diam}(h(t'_k))^2 \leq C. \quad (4.6)$$

Choose p_k such that $R_{\min}(t'_k) = R(h(p_k, t'_k)) = \min_{q \in N}\{R(h(q, t'_k))\}$. By the compactness of ancient κ -solutions (cf. [17, Theorem 3.3], [14, Theorem 20.9]), rescaled flows $(N, h_k(t); p_k)$, where $h_k(t) = R(p_k, t'_k)h(t_k + R(p_k, t'_k)^{-1}t)$, converge subsequently to an ancient κ -solution $(N_\infty, h_\infty(t); p_\infty)$. By (4.6), we see that $(N_\infty, h_\infty(t); p_\infty)$ is a closed ancient κ -solution. Thus $R_\infty(\cdot, 0)$ is uniformly bounded on N_∞ .

On the other hand, by (4.5)–(4.6),

$$\frac{R(q_k, t'_k)}{R(p_k, t'_k)} \longrightarrow \infty, \quad (4.7)$$

where q_k is chosen as $R_{\max}(t'_k) = \max\{R(h(q_k, t'_k))\}$. Note

$$\lim_k \frac{R(q_k, t'_k)}{R(p_k, t'_k)} = R_\infty(q_\infty, 0),$$

where $q_\infty \in N_\infty$ is a limit of q_k . Thus $R_\infty(q_\infty, 0) = \infty$, which is impossible! The Lemma 4.4 is proved. $\square$

Remark 4.5. — By [9, Theorem 1.4], any 3d compact simply connected ancient κ -solution of type II coincides with the Perelman solution on S^3 . On the other hand, by an explicit computation for the Perelman solution [1, Theorem 1.3], the quantity $R(q, t) \text{Diam}(h(t))^2$ is strictly increasing as $t \rightarrow -\infty$ for any $q \in S^3$. Thus (4.3) holds for 3d compact ancient κ -solutions of type II. Hence, Lemma 4.4 can be regarded as a higher dimensional version of results in [1, 9].

We also need the following notion introduced by Perelman (cf. [29]).

DEFINITION 4.6. — *For any $\epsilon > 0$, we say a pointed Ricci flow $(M_1, g_1(t); p_1), t \in [-T, 0]$ (T may be the infinity), is ϵ -close to another pointed Ricci flow $(M_2, g_2(t); p_2), t \in [-T, 0]$, if there is a diffeomorphism onto its image $\bar{\phi} : B_{g_2(0)}(p_2, \epsilon^{-1}) \rightarrow M_1$, such that $\bar{\phi}(p_2) = p_1$ and $\|\bar{\phi}^* g_1(t) - g_2(t)\|_{C^{[-1]}} < \epsilon$ for all $t \in [-\min\{T, \epsilon^{-1}\}, 0]$, where the norms and derivatives are taken with respect to $g_2(0)$.*

By the above definition, we get from Proposition 2.2, immediately,

PROPOSITION 4.7. — *Let (M^n, g) be the steady Ricci soliton in Proposition 2.2. Then for any $\epsilon > 0$, there exists a compact set $D(\epsilon) > 0$, such that for any $p \in M \setminus D$, $(M, g_p(t); p)$ is ϵ -close to a splitting flow $(h_p(t) + ds^2; p)$, where $h_p(t)$ is an $(n-1)$ -dimensional ancient κ -solution.*

We note that for a given p and a number $\epsilon > 0$ the ϵ -close splitting flow $(h_p(t) + ds^2; p)$ may not be unique in Proposition 4.7. Due to [23], we introduce a function on $M \setminus D$ for each ϵ by

$$F_\epsilon(p) = \inf_{h_p} \{\text{Diam}(h_p(0)) \in (0, \infty)\}. \quad (4.8)$$

For simplicity, we always omit the subscript ϵ in the function $F_\epsilon(p)$ below.

Remark 4.8. — The function $F_\epsilon(p)$ is almost continuous on $M \setminus D$, at each point p of which $(M, g_p(t); p)$ is ϵ -close to a splitting flow $(h_p(t) + ds^2; p)$ with $\text{Diam}(h_p(0)) < \frac{2}{\epsilon}$. Precisely, there exists a neighborhood U around p such that

$$|F_\epsilon(p) - F_\epsilon(q)| \leq 2F_\epsilon(p)\sqrt{\epsilon} + 2\epsilon, \quad \forall q \in U. \quad (4.9)$$

In fact, we can choose an $(n-1)$ -dimensional ancient κ -solution $(N, \bar{h}_p(t))$ such that $F_\epsilon(p) = \text{Diam}(\bar{h}_p(0)) - \frac{\epsilon}{2}$ and $(M, g_p(t); p)$ is ϵ -close to the splitting

flow $(\bar{h}_p(t) + ds^2; p)$. Then there exists an open set U such that for any $q \in U$ it holds

$$\left\| \bar{\psi}^* g_q(t) - (\bar{h}_p(t) + ds^2) \right\|_{C^{[\epsilon^{-1}]}} < 2\epsilon, \quad \forall t \in [-\min\{T, \epsilon^{-1}\}, 0],$$

where $\bar{\psi}$ is a diffeomorphism between the underlying manifolds of flows $g_q(t)$ and $(\bar{h}_p(t) + ds^2)$. On the other hand, there is another $(n-1)$ -dimensional ancient κ -solution $(N', \bar{h}_q(t))$ such that $F_\epsilon(q) = \text{Diam}(\bar{h}_q(0)) - \frac{\epsilon}{2}$ and $(M, g_q(t); q)$ is ϵ -close to the flow $(\bar{h}_q(t) + ds^2; q)$. Thus there exists a diffeomorphism Φ from N to N' such that $|\Phi^* \bar{h}_p(0) - \bar{h}_q(0)| \leq 3\epsilon$. This will imply (4.9) immediately.

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. — If the proposition is false, there will exist a sequence of rescaled flows $(M, g_{p_i}(t), p_i)$, which converges subsequently to a splitting Ricci flow $(N \times \mathbb{R}, h(t) + ds^2; p_\infty)$, where $(N, h(t))$ is an $(n-1)$ -dimensional compact ancient κ -solution of type II. Choose $t_k \rightarrow -\infty$. by Lemma 4.4,

$$\lim_{k \rightarrow \infty} \text{Diam}(h(t_k)) R_h^{\frac{1}{2}}(p_\infty, t_k) \geq \lim_{k \rightarrow \infty} \text{Diam}(h(t_k)) R_{h, \min}^{\frac{1}{2}}(t_k) = \infty. \quad (4.10)$$

Thus there is $t_0 \in \{t_k\}$ for some k_0 such that

$$\text{Diam}(h(t_0)) R_h^{\frac{1}{2}}(p_\infty, t_0) > 100C. \quad (4.11)$$

Set $T = t_0 R^{-1}(p_i)$ and choose $\epsilon < -\frac{1}{100t_0}$. Then $(M, g_{p_i}(t); p_i)$ is ϵ -close to $(N \times \mathbb{R}, h(t) + ds^2; p_\infty)$ when $i \gg 1$. Thus by (4.11) and the convergence of $g_{p_i}(t)$, we get

$$\text{Diam}(\bar{g}(T)) R^{\frac{1}{2}}(p_i, T) = \text{Diam}(\bar{g}_{p_i}(t_0)) R_{p_i}^{\frac{1}{2}}(p_i, t_0) > 90C, \quad (4.12)$$

as long as $i \gg 1$, where $\bar{g}(T)$ is the induced metric of g on $f^{-1}(f(\phi_T(p_i)))$ and $\bar{g}_{p_i}(t)$ is the induced metric of $g_{p_i}(t)$ on $f^{-1}(f(p_i))$, respectively. It turns

$$\begin{aligned} \text{Diam}(R(\phi_T(p_i))\bar{g}(T)) &= \text{Diam}(\bar{g}(T)) R^{\frac{1}{2}}(\phi_T(p_i)) \\ &= \text{Diam}(\bar{g}(T)) R^{\frac{1}{2}}(p_i, T) > 90C, \quad i \gg 1. \end{aligned} \quad (4.13)$$

Let $(\tilde{N}, \tilde{h}(t))$ be an $(n-1)$ -dimensional split ancient flow as the limit of $(M, g(\phi_T(p_i))(t); \phi_T(p_i))$. Then by Remark 3.9, $(\tilde{N}, \tilde{h}(0))$ is a limit of submanifolds

$$(f^{-1}(f(\phi_T(p_i))), R(\phi_T(p_i))\bar{g}(T); \phi_T(p_i)).$$

Thus by (4.13), we get

$$\text{Diam}(\tilde{N}, \tilde{h}(0)) \geq 90C.$$

By the Definition 4.8, it follows

$$F(\phi_T(p_i)) > 50C, \quad i \gg 1.$$

So

$$\limsup_{i \rightarrow \infty} F(\phi_T(p_i)) \geq 50C. \quad (4.14)$$

However, by the condition of proposition, we see

$$\limsup_{p \rightarrow \infty} F(p) < 2C, \quad (4.15)$$

which contradicts to (4.14). Hence, $(N, h(t))$ must be of type I. By [27], we also get (4.1). The Proposition 4.1 is proved. $\square$

5. Proofs of Theorems 1.3 and 1.2

In this section, we prove Theorems 1.3 and 1.2 by the results obtained in Sections 2–4.

5.1. Proof of Theorem 1.3

We use the argument by contradiction. On the contrary, we suppose that there exists a sequence of rescaled Ricci flows $(M, g_{q_i}(t); q_i)$ ($q_i \rightarrow \infty$), which converges to a limit Ricci flow $(N' \times \mathbb{R}, h'(t) + ds^2; q_\infty)$, where $(N', h'(t))$ is a compact ancient κ -solution of type I. Then by Lemma 4.2, there exists a constant $C_0 > 0$ such that for any small $\epsilon > 0$ it holds,

$$F(q_i) = F_\epsilon(q_i) \leq C_0.$$

By Proposition 3.8, it follows

$$F(q) \leq C, \quad (5.1)$$

for all $q \in f^{-1}(f(q_i))$. We note that the constant C is uniform by the classification result, Lemma 4.2, i.e., it is independent of the sequence of normally rescaled Ricci flows with a compact ancient κ -solution limit of type I.

On the other hand, for the sequence of $(M, g_{p_i}(t); p_i)$ in Theorem 1.3, we can choose a point $p_{i_0} \in \{p_i\}$ such that

$$F(p_i) > 100C, \quad (5.2)$$

for all $i > i_0$. As in Section 3.2, we set $\hat{X} = \frac{\nabla f}{|\nabla f|}$ and $\Gamma(s)$ be an integral curve of $\hat{X}$ starting from p_{i_0} , i.e., $\Gamma(0) = p_{i_0}$. Then $\Gamma(s)$ tends to the infinity by Remark 3.4 and (2.10), since Proposition 3.3 holds. Hence, there is a point q'_i such that $q'_i = \Gamma(s_i)$ and $f(\Gamma(s_i)) = f(q_i)$. By (5.1), $F(q_i) \leq C$.

Set

$$s_i^1 = \inf \left\{ s \geq f(p_{i-1}) \left| \begin{array}{l} F(\Gamma(\mu)) \leq 10C \text{ for all } \mu \\ \text{with } f(\Gamma(\mu)) \in [f(\Gamma(s)), f(q_i)] \end{array} \right. \right\}$$

and

$$s_i^2 = \sup \left\{ s \geq f(q_i) \left| \begin{array}{l} F(\Gamma(\mu)) \leq 10C \text{ for all } \mu \\ \text{with } f(\Gamma(\mu)) \in [f(q_i), f(\Gamma(s))] \end{array} \right. \right\}.$$

Then by Remark 4.8, it is easy to see that $s_i^1 < s_i < s_i^2$. Let $p_i^1 = \Gamma(s_i^1)$ and $p_i^2 = \Gamma(s_i^2)$. We conclude: for $\epsilon \ll 1$, it holds (5.3)

$$f(p_i^1) < f(p_i^2); \quad (5.3.a)$$

$$f(p_i^1) < f(p_{i+1}^1), \quad f(p_i^2) < f(p_{i+1}^2); \quad (5.3.b)$$

$$f(p_i^1) < f(q_i) < f(p_i^2); \quad (5.3.c)$$

$$F(p_i^1), \quad F(p_i^2) \in \left[10C - \frac{1}{10}, 10C + \frac{1}{10} \right]; \quad (5.3.d)$$

$$F(p) \leq 10C, \quad \forall p \in \Gamma(s) \quad \text{with} \quad f(p_i^1) < f(p) < f(p_i^2). \quad (5.3.e)$$

See Figure 5.1.

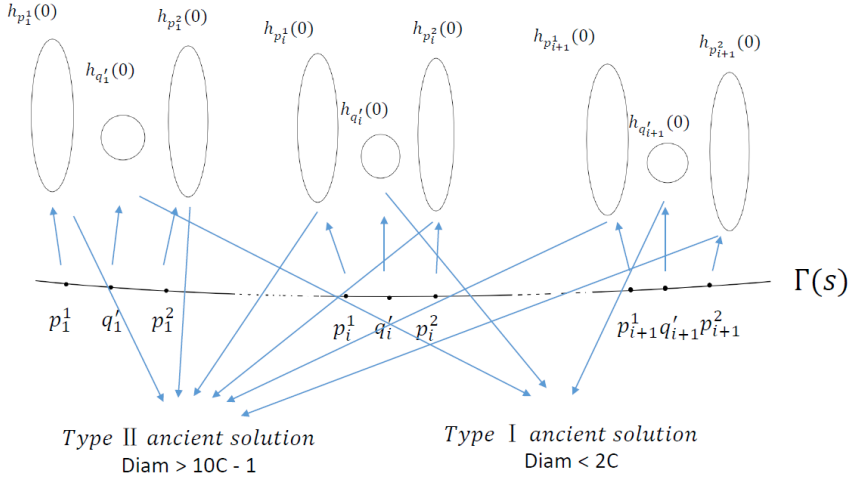


Figure 5.1.

We need only to check (5.3.d) for p_i^1 with same for p_i^2 . Suppose that $F_\epsilon(p_i^1) < 10C - \frac{1}{10}$. Then by Remark 4.8, there is $s < s_i^1$ such that $F_\epsilon(f(\Gamma(s))) < 10C - \frac{1}{20}$ whenever $\epsilon < (\frac{1}{100C})^2$. Thus by the definition of s_i^1 , we also get

$s \geq s_i^1$. But this is impossible! Hence $F_\epsilon(p_i^1) \geq 10C - \frac{1}{10}$. By the same argument, we can also get $F_\epsilon(p_i^1) \leq 10C + \frac{1}{10}$.

By Proposition 2.2 and the relation (5.3.d), $(M, g_{p_i^1}(t); p_i^1)$ converges sub-sequently to $(N_1 \times \mathbb{R}, h_1(t) + ds^2; p_\infty^1)$, where $h_1(t)$ satisfies

$$\text{Diam}(h_1(0)) \in [10C - 1, 10C + 1].$$

Thus, by (5.1), we see that $h_1(t)$ must be a compact ancient κ -solution of type II. By Lemma 4.4, it follows

$$R(p_\infty^1, t_k) \text{Diam}(h_1(t_k))^2 \longrightarrow \infty$$

for any $t_k \rightarrow -\infty$. Hence, there exists a k_0 such that

$$R(p_\infty^1, t_{k_0})^{1/2} \text{Diam}(h_1(t_{k_0})) > 100C. \quad (5.4)$$

We fix t_{k_0} so that $\epsilon^{-1} > -10t_{k_0}$ as long as $\epsilon \ll 1$.

Next we claim there exists a constant $A > 0$ such that

$$s_i^2 - s_i^1 < AR(p_i^1)^{-1} \quad (5.5)$$

for all $i \gg 1$.

Suppose that the above claim is not true. Then by taking a subsequence, we may assume

$$R(p_i^1)(s_i^2 - s_i^1) \longrightarrow \infty. \quad (5.6)$$

Recall that $\{\phi_t\}_{t \in (-\infty, \infty)}$ is the flow of $-\nabla f$ with ϕ_0 the identity and $(g(t), \Gamma(s))$ is isometric to $(g(0), \phi_t(\Gamma(s)))$, where $g(0) = g$. Then

$$\phi_t(\Gamma(s)) = \Gamma\left(s - \int_0^t |\nabla f|(\phi_\mu(\Gamma(s))) d\mu\right)$$

Let $T_{k_0, i} = t_{k_0} R(p_i^1)^{-1} = t_{k_0} R(\Gamma(s_i^1))^{-1}$ and

$$s'_i = s_i^1 - \int_0^{T_{k_0, i}} |\nabla f|(\phi_\mu(\Gamma(s_i^1))) d\mu. \quad (5.7)$$

It follows

$$s'_i - s_i^1 \leq -T_{k_0, i} = -t_{k_0} R(\Gamma(s_i^1))^{-1}. \quad (5.8)$$

By combining (5.6) and (5.8), we see that $s'_i \in [s_i^1, s_i^2]$ for $i \gg 1$. Hence, by the relation (5.3.e), we obtain

$$F(\Gamma(s'_i)) \leq 10C. \quad (5.9)$$

From the proof of Proposition 4.7, we know that for each $\Gamma(s_i^1)$, there exists an $(n-1)$ -dimensional compact ancient κ -solution $h_{\Gamma(s_i^1)}(t)$ such that

$$(M, R(\Gamma(s_i^1))g(R(\Gamma(s_i^1))^{-1}t); \Gamma(s_i^1)) \stackrel{\epsilon-\text{close}}{\sim} \left(N \times \mathbb{R}, h_{\Gamma(s_i^1)}(t) + ds^2; p_\infty^1 \right). \quad (5.10)$$

By the isometry, we also have

$$\begin{aligned} (M, R(\Gamma(s'_i))g; \Gamma(s'_i)) &\cong (M, R(\Gamma(s_i^1), T_{k_0, i})g(T_{k_0, i}); \Gamma(s_i^1)) \\ &\cong \left(M, \frac{R(\Gamma(s_i^1), T_{k_0, i})}{R(\Gamma(s_i^1))} R(\Gamma(s_i^1))g(T_{k_0, i}); \Gamma(s_i^1) \right). \end{aligned}$$

Since $R(\Gamma(s_i^1), T_{k_0, i}) \leq R(\Gamma(s_i^1))$ by the monotonicity of scalar curvature along $\Gamma(s)$, we get

$$(M, R(\Gamma(s'_i))g; \Gamma(s'_i)) \stackrel{\epsilon-\text{close}}{\sim} \left(N \times \mathbb{R}, \frac{R(\Gamma(s_i^1), T_{k_0, i})}{R(\Gamma(s_i^1))} h_{\Gamma(s_i^1)}(t_{k_0}) + ds^2; p_\infty^1 \right). \quad (5.11)$$

Note that there are also another $(n-1)$ -dimensional compact ancient κ -solutions $h_{\Gamma(s'_i)}(t)$ corresponding to the point $\Gamma(s'_i)$ as in (5.10) such that

$$(M, R(\Gamma(s'_i))g; \Gamma(s'_i)) \stackrel{\epsilon-\text{close}}{\sim} \left(h_{\Gamma(s'_i)}(0) + ds^2; p_\infty' \right). \quad (5.12)$$

Hence, combining (5.11) and (5.12), we derive

$$\begin{aligned} h_{\Gamma(s'_i)}(0) &\stackrel{\epsilon-\text{close}}{\sim} \frac{R(\Gamma(s_i^1), T_{k_0, i})}{R(\Gamma(s_i^1))} h_{\Gamma(s_i^1)}(t_{k_0}) \\ &\stackrel{\epsilon-\text{close}}{\sim} R_h(\Gamma(s_i^1), t_{k_0}) h_{\Gamma(s_i^1)}(t_{k_0}). \end{aligned} \quad (5.13)$$

As a consequence, there exists a constant $C_1 > 0$ such that

$$\left| \text{Diam}\left(R_h(\Gamma(s_i^1), t_{k_0})h_{\Gamma(s_i^1)}(t_{k_0})\right) - \text{Diam}(h_{\Gamma(s'_i)}(0)) \right| \leq C_1 \sqrt{\epsilon}. \quad (5.14)$$

On the other hand, by the fact

$$h_{\Gamma(s_i^1)}(t_{k_0}) \stackrel{\epsilon-\text{close}}{\sim} h_1(t_{k_0}), \quad \text{as } i \gg 1,$$

there exists a constant $C_2 > 0$ such that

$$|R_h(\Gamma(s_i^1), t_{k_0}) - R(p_\infty^1, t_{k_0})| \leq C_2 \epsilon, \quad \text{as } i \gg 1,$$

where R is the scalar curvature w.r.t. h_1 . Then

$$R_h(\Gamma(s_i^1), t_{k_0})h_{\Gamma(s_i^1)}(t_{k_0}) \stackrel{4C_2\epsilon-\text{close}}{\sim} R(p_\infty^1, t_{k_0})h_1(t_{k_0}).$$

Since $(N_1, h_1(t_{k_0}))$ is compact, we get

$$\left| \text{Diam}\left(R_h(\Gamma(s_i^1), t_{k_0})h_{\Gamma(s_i^1)}(t_{k_0})\right) - \text{Diam}\left(R(p_\infty^1, t_{k_0})h_1(t_{k_0})\right) \right| \leq C_3\sqrt{\epsilon}, \quad (5.15)$$

where $C_3 > 0$ is a uniform constant. Thus combining (5.14)–(5.15) with (5.4), we prove

$$\begin{aligned} F(\Gamma(s'_i)) &\geq \text{Diam}(h_{\Gamma(s'_i)}(0)) - C_4\sqrt{\epsilon} \\ &\geq \text{Diam}\left(R_h(\Gamma(s_i^1), t_{k_0})h_{\Gamma(s_i^1)}(t_{k_0})\right) - C_4\sqrt{\epsilon} - C_1\sqrt{\epsilon} \\ &\geq \text{Diam}(R(p_\infty^1, t_{k_0})h_1(t_{k_0})) - (C_1 + C_3 + C_4)\sqrt{\epsilon} \\ &> 98C, \end{aligned}$$

where $C_4 > 0$ is a uniform constant. But this is impossible by (5.9). Hence, (5.5) must be true.

Now we can finish the proof of Theorem 1.3. By (5.5) and the relation (5.3.c), we have

$$s_i^2 - s_i \leq s_i^2 - s_i^1 \leq AR(\Gamma(s_i^1))^{-1} \leq AR(\Gamma(s_i))^{-1}, \quad (5.16)$$

where the last inequality follows from the monotonicity of R along $\Gamma(s)$. Since $(M, g_{\Gamma(s_i)}(t); \Gamma(s_i))$ converges subsequently to the limit flow $(N' \times \mathbb{R}, h'(t) + ds^2; q_\infty)$, by the shrinking property of type I solution $(N', h'(t))$ in Lemma 4.2, we know

$$\text{Diam}(R_{h'}(q_\infty, t)h'(t)) \leq 2C, \quad \forall t \leq 0. \quad (5.17)$$

Then as in the proof of (5.13), we get

$$h_{\Gamma(s_i^2)}(0) \stackrel{\epsilon-\text{close}}{\sim} R_h(\Gamma(s_i), t'_i) h_{\Gamma(s_i)}(t'_i) \quad (5.18)$$

for some t'_i , where $-t'_i < 2A$ by (5.7) and (5.16). Thus, combining (5.17) and (5.18) and by the convergence of $(M, g_{\Gamma(s_i)}(t); \Gamma(s_i))$, we obtain

$$F(\Gamma(s_i^2)) \leq 3C, \quad \text{as } i \gg 1.$$

But this is impossible since by the relation (5.3.d) it holds

$$F(\Gamma(s_i^2)) = F(p_i^2) \geq 10C - \frac{1}{10}.$$

Therefore, we prove Theorem 1.3. □

Theorem 1.3 implies Corollary 1.4.

Proof of Corollary 1.4. — Suppose that there exist two splitting limit flows $(N_1 \times \mathbb{R}, h_1(t) + ds^2; p_\infty^1)$ and $(N_2 \times \mathbb{R}, h_2(t) + ds^2; p_\infty^2)$ of rescaled flows $(M, g_{p_i^1}(t); p_i^1)$ and $(M, g_{p_i^2}(t); p_i^2)$, respectively, such that $(N_1, h_1(t))$ is

a compact ancient κ -solution of type I, and $(N_2, h_2(t))$ is another compact ancient κ -solution of type II. We claim that

$$\lim_{\epsilon \rightarrow 0} \sup_{p \rightarrow \infty} F_\epsilon(p) = \infty. \quad (5.19)$$

On the contrary, there will be a uniform constant $C > 0$ such that all split ancient κ -solutions $(N^{n-1}, h(t))$ of $(n-1)$ -dimension satisfy (3.1). Then by Proposition 4.1, it follows that every split limit flow $h(t)$ must be an ancient κ -solution of type I. But this is impossible since $h_2(t)$ is a compact ancient κ -solution of type II. Thus the claim is true.

By (5.19), after taking a diagonal subsequence, it is easy to see that there is a sequence of pointed flows $(M, g_{q_i}(t); q_i)$, which converges subsequently to a splitting Ricci flow $(N' \times \mathbb{R}, h'(t) + ds^2; q_\infty)$ for some noncompact ancient κ -solution $(N', h'(t))$. Thus, by Theorem 1.3, there cannot exist any compact splitting limit flow of type I. But this is impossible since $h_1(t)$ is a compact ancient κ -solution of type I. This proves the corollary. $\square$

In the next subsection, we prove Theorem 1.2 by Theorem 1.3.

5.2. Proof of Theorem 1.2

By the assumption, the $(n-1)$ -dimensional split ancient flow $(N, h(t))$ of limit of $(M, g_{p_i}(t), p_i)$ is a family of shrinking round quotient spheres. Namely, $(N, h(0))$ is a quotient of round sphere, so it is of type I. We first show that (M, g) has positive Ricci curvature on M .

On the contrary, $\text{Ric}(g)$ is not strictly positive. We note that the scalar curvature $R(p_i)$ decaying to zero is still true in the proof of Proposition 3.3 without $\text{Ric}(g) > 0$ away from a compact set of M . Then as in the proof of [19, Lemma 4.6], we see that $X_i = R(p_i)^{-\frac{1}{2}} \nabla f \rightarrow X_\infty$ w.r.t. $(M, g_{p_i}(t); p_i)$, where X_∞ is a non-trivial parallel vector field. Thus according to the argument in the proof of [19, Theorem 1.3], the universal cover of $(N, h(t))$ must split off a flat factor $\mathbb{R}^d$ ($d \geq 1$). However, the universal cover of N is S^{n-1} . This is a contradiction! Hence, we conclude that $\text{Ric}(g) > 0$ on M .

Now we divide into two cases to prove Theorem 1.2.

Case 1: $\limsup_{p \rightarrow \infty} F_\epsilon(p) < C$ for any $\epsilon < 1$. — By Proposition 4.1 and Lemma 4.2, all $(n-1)$ -dimensional split ancient κ -solutions $(N', h'(t))$ are of type I, and each of them is one described in (4.1).

Case 2: $\limsup_{\epsilon \rightarrow 0} \limsup_{p \rightarrow \infty} F_\epsilon(p) = \infty$. — In this case, there will exist a sequence of pointed flows $(M, g_{q_i}(t); q_i)$, which converges subsequently

to a splitting Ricci flow $(N' \times \mathbb{R}, h'(t) + ds^2; q_\infty)$ for some noncompact ancient κ -solution $(N', h'(t))$. But this is impossible by Theorem 1.3 since we already have had a split ancient limit flow $(N, h(t))$ of type I. Thus Case 2 can be excluded. We note that in the case of $n = 4$, we can apply Theorem 1.5 to see that any $3d$ split ancient flow $(N', h'(t))$ of limit of $(M, g_{q_i}(t), q_i)$ is a family of shrinking round quotient spheres.

It remains to show that every split limit flow $(N', h'(t))$ in Case 1 is in fact a family of shrinking round spheres.

By Proposition 3.3, the scalar curvature of (M, g) decays to zero uniformly. Then (M, g) has unique equilibrium point o by the fact $\text{Ric}(g) > 0$. Thus the level set $\Sigma_r = \{f(x) = r\}$ is a closed manifold for any $r > 0$, and it is diffeomorphic to S^{n-1} (cf. [19, Lemma 2.1]).

On the other hand, as in the proof of Lemma 3.7, the level sets $(\Sigma_{f(q_i)}, \bar{g}_{q_i}; q_i)$ converge subsequently to $(N', h'(0); q_\infty)$ w.r.t. the induced metric $\bar{g}_{q_i}$ on $\Sigma_{f(q_i)}$ by g_{q_i} . Since each $\Sigma_{f(q_i)}$ is diffeomorphic to S^{n-1} , N' is also diffeomorphic to S^{n-1} . Thus $(N', h'(t))$ is a family of shrinking round spheres.

By the above argument, we see that Definition 1.1(ii) is satisfied. Thus by [18, Lemma 6.5], (M, g) is asymptotically cylindrical. It follows that (M, g) is isometric to the Bryant Ricci soliton up to scaling by [7]. Hence, the Theorem 1.2 is proved. $\square$

6. $4d$ steady gradient Ricci solitons

In this section, we prove Theorem 1.5 and Theorem 1.6 by improving Theorem 1.3 for $4d$ steady gradient Ricci solitons. We first recall recent works on compact $3d$ ancient κ -solitons by Angenent–Brendle–Daskalopoulos–Sesum and Brendle–Daskalopoulos–Sesum [1, 9], then we estimate the diameter of $(N, h(0))$ for any split limit flow $(N, h(t))$ of the rescaled flow sequence.

6.1. Perelman's ancient solution

As we know, Perelman's model of ancient solution is of type II, which is defined on S^3 with $Z_2 \times O(2)$ -symmetry for any $t \in (-\infty, 0)$ [29]. Fix $p_0 \in S^3$. We normalize the Perelman solution by

$$R_{\max}(-1) = R(p_0, -1) = 1. \quad (6.1)$$

For simplicity, we denote it by $(S^3, g_{\text{Pel}}(t); p_0)$, $t \in (-\infty, 0)$.

The asymptotic behavior of $(S^3, g_{\text{Pel}}(t); p_0)$ has been computed in [1] as follows,

$$\begin{aligned} \text{Diam}(g_{\text{Pel}}(t)) &\geq 2.1\sqrt{(-t)\log(-t)}, \\ R_{\max} &\leq 1.1\frac{\log(-t)}{-t}, \\ R_{\min} &\geq \frac{C}{-t}. \end{aligned} \tag{6.2}$$

Here $-t \geq L$ for some large $L > 10000C > 10000$. In particular, there exists a constant C_{Diam} such that $\text{Diam}(g_{\text{Pel}}(t)) \geq C_{\text{Diam}}$, when $-t \geq 2L$. Moreover,

$$\text{Diam}(g_{\text{Pel}}(t)) R^{\frac{1}{2}}(q, t)$$

is strictly increasing as $t \rightarrow -\infty$, and

$$\lim_{t \rightarrow -\infty} \text{Diam}(g_{\text{Pel}}(t)) R^{\frac{1}{2}}(q, t) = \infty, \quad \forall q \in S^3. \tag{6.3}$$

The above convergence is uniform. Usually, we call 3d ancient κ -solitons of type II on S^3 as Perelman (ancient) solutions.

The following classification result of 3d compact ancient κ -solutions of type II was proved in [9].

THEOREM 6.1. — *Any 3d compact simply connected ancient κ -solution of type II coincides with a reparametrization in space, a translation in time, and a parabolic rescaling of Perelman solution $(S^3, g_{\text{Pel}}(t); p_0)$.*

By Theorem 6.1, for any simply connected compact 3d κ -solution $(M, h(t); q)$ of type II and a point $q \in M$, there exist constant λ , a time T , $p \in S^3$ and a diffeomorphism Ψ from S^3 to M such that $\Psi(p) = q$ and

$$(\Psi^{-1}(M), \lambda\Psi^*(h(\lambda^{-1}t)), \Psi^{-1}(q)) = (S^3, g_{\text{Pel}}(t - T); p_0). \tag{6.4}$$

We note that the Perelman's solution $(S^3, g_{\text{Pel}}(t); p_0)$ is $Z_2 \times O(2)$ -symmetric. Then the isometric subgroup of $(S^3, g_{\text{Pel}}(t); p_0)$ must be $Z_2 \times G$, where G is a subgroup of $O(2)$. Thus G fixes the minimal geodesic connecting the two tips of the Perelman solution. It follows that any quotient of Perelman solution, which is also an ancient κ -solutions, satisfies (6.2). Hence, by the classification Theorem 6.1, we get

PROPOSITION 6.2. — *Let $(M, h(t))$ be a 3d compact ancient κ -solutions of type II and $p \in M$, which satisfies*

$$R(p, 0) = 1 \quad (6.5)$$

and

$$\text{Diam}(h(0)) = \frac{2C'}{3} > 10C_{\text{Diam}}. \quad (6.6)$$

Then for any $q \in M$, $\text{Diam}(h(t))R^{\frac{1}{2}}(q, t)$ is strictly decreasing for $t \leq 0$. Moreover, there exists a $T(C')$ such that

$$\text{Diam}(h(T(C'))R^{\frac{1}{2}}(q, T(C'))) = 2C'. \quad (6.7)$$

By Theorem 6.1 and Proposition 6.2, we are able to classify the split ancient κ -solutions of dimension 3 when the 4d noncompact κ -noncollapsed steady Ricci soliton in Theorem 1.5 admits a split noncompact ancient κ -solution $(N, h(t))$.

6.2. An estimate of (4.8)

In this subsection we estimate (4.8) by Proposition 6.2 and prove

PROPOSITION 6.3. — *Let (M, g) be a 4d noncompact κ -noncollapsed steady Ricci soliton in Theorem 1.5. Suppose that there exists a sequence of pointed rescaled Ricci flows $(M, g_{p_i}(t); p_i)$, which converge subsequently to a splitting Ricci flow $(h(t) + ds^2; p_\infty)$ for some noncompact ancient κ -solution $h(t)$. Then for any limit flow $(h'(t) + ds^2; q_\infty)$ of rescaled Ricci flows $(M, g_{q_i}(t); q_i)$, $h'(t)$ is a noncompact ancient κ -solution.*

Proof. — We will modify the argument in the proof of Theorem 1.3. By the contradiction, we suppose that there exists a limit flow $(h'(t) + ds^2; q_\infty)$ converged by rescaled flows $(M, g_{q_i}(t); q_i)$, which satisfies (3.1). Then by Proposition 3.8, there exists a uniform constant $C_3(C) > 0$, such that

$$F(p'_i) \leq C_3 \quad (6.8)$$

for all i , and all $p'_i \in f^{-1}(f(q_i))$.

Fix $C' = \max\{100C_{\text{Diam}}, 10C_3\}$ and $T(C')$ as in Proposition 6.2. We choose an $\epsilon > 0$ such that $\epsilon^{-1} > \max\{-10T(C'), 100C'\}$. Thus for the sequence of $(M, g_{p_i}(t); p_i)$ in Proposition 6.3, we can choose a point $p_{i_0} \in \{p_i\}$ such that

$$F(p_{i_0}) > C' \geq 100C_{\text{Diam}}.$$

Let Γ be the integral curve of $\widehat{X}$ passing through p_{i_0} with $\Gamma(0) = p_{i_0}$ as chosen in Section 5.1. We claim:

$$F(\Gamma(s)) > \frac{1}{2}C', \quad \forall s \geq 0. \quad (6.9)$$

Define

$$s_0 = \sup\{s \geq 0 \mid F(\Gamma(s')) \geq C', \forall s' \in [0, s]\}.$$

If $s_0 = \infty$, $F(\Gamma(s)) > C'$ for all $s \geq 0$. Then (6.9) is obvious true in this case. Thus we may consider the case $s_0 < \infty$, i.e., $F(\Gamma(s_0)) = C'$ for some $s' = s_0$. It follows that there exists a $3d$ compact ancient κ -solution $h_{\Gamma(s_0)}(t)$ such that

$$(M, R(\Gamma(s_0))g(R(\Gamma(s_0))^{-1}t); \Gamma(s_0)) \stackrel{\epsilon\text{-close}}{\sim} (N \times \mathbb{R}, h_{\Gamma(s_0)}(t) + ds^2; p_\infty). \quad (6.10)$$

Since the diameter of $h_{\Gamma(s_0)}(0)$ is large, $h_{\Gamma(s_0)}(t)$ can not be a family of shrinking quotient spheres. Hence, by Theorem 6.1, it must be a quotient of Perelman solution after a reparametrization.

By Proposition 6.2, we see that $\text{Diam}(h_{\Gamma(s_0)}(t)) R_h^{\frac{1}{2}}(\Gamma(s_0), t)$ is strictly decreasing for $t \in (-\epsilon^{-1}, 0]$, where R_h denotes the scalar curvature w.r.t. $h_{\Gamma(s_0)}(t)$. Then by the choice of the number s_0 , it follows

$$\text{Diam}(h_{\Gamma(s_0)}(t)) R_h^{\frac{1}{2}}(\Gamma(s_0), t) \geq \frac{2C'}{3}, \quad t \in (-\epsilon^{-1}, 0]. \quad (6.11)$$

Moreover, by the choice of $T(C')$, we have

$$\text{Diam}(h(T(C'))) R_h^{\frac{1}{2}}(\Gamma(s_0), T(C')) = 2C'.$$

Let $t_1 = \min\{-1000, T(C')\} \geq -\frac{\epsilon^{-1}}{2}$. Thus

$$\text{Diam}(h_{\Gamma(s_0)}(t_1)) R_h^{\frac{1}{2}}(\Gamma(s_0), t_1) \geq 2C'. \quad (6.12)$$

As in the proof of Theorem 1.3, we have

$$\phi_t(\Gamma(s)) = \Gamma\left(s - \int_0^t |\nabla f|(\phi_\mu(\Gamma(s))) d\mu\right)$$

Let $T = tR(\Gamma(s_0))^{-1} < 0$ and

$$s = s_0 - \int_0^T |\nabla f|(\phi_\mu(\Gamma(s_0))) d\mu. \quad (6.13)$$

Set

$$s_1 = s_0 - \int_0^{T_1} |\nabla f|(\phi_\mu(\Gamma(s_0))) d\mu,$$

where $T_1 = t_1 R(\Gamma(s_0))^{-1}$, see Figure 6.1. Since the scalar curvature R of (M, g) decays to 0 uniformly by Proposition 3.3, we may assume $|\nabla f| \geq \frac{1}{2}$ along Γ . Thus

$$s_1 - s_0 \geq 500 R^{-1}(\Gamma(s_0)) \geq 500 R_{\max}^{-1} = 500. \quad (6.14)$$

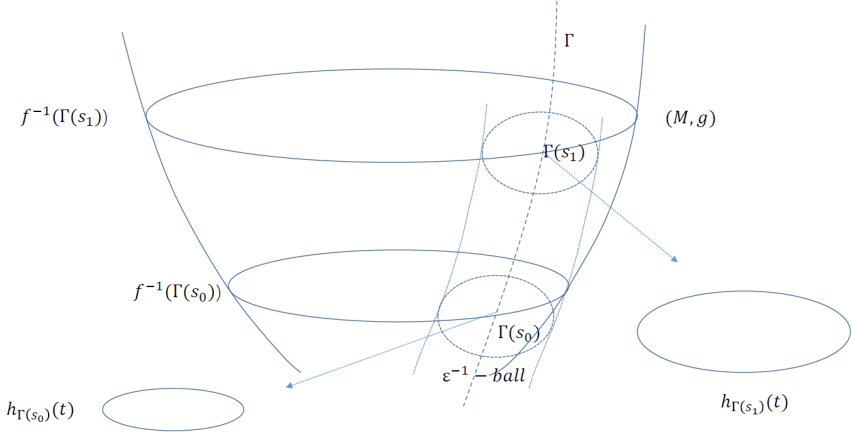


Figure 6.1.

Note that $\phi_T(\Gamma(s_0)) = \Gamma(s)$ and $(g(T), \Gamma(s_0))$ is isometric to $(g(0), \Gamma(s))$, where $g(0) = g$, for all $s \in [s_0, s_1]$. Then

$$\begin{aligned} (M, R(\Gamma(s))g; \Gamma(s)) &\cong (M, R(\Gamma(s_0), T)g(T); \Gamma(s_0)) \\ &\cong \left(M, \frac{R(\Gamma(s_0), T)}{R(\Gamma(s_0))} R(\Gamma(s_0)) g(T); \Gamma(s_0) \right). \end{aligned} \quad (6.15)$$

Since $R(\Gamma(s_0), T) \leq R(\Gamma(s_0))$ by (2.4), we get from (6.10),

$$(M, R(\Gamma(s))g; \Gamma(s)) \stackrel{\epsilon-\text{close}}{\sim} \left(N \times \mathbb{R}, \frac{R(\Gamma(s_0), T)}{R(\Gamma(s_0))} h_{\Gamma(s_0)}(t) + ds^2; p_\infty \right). \quad (6.16)$$

On the other hand, there is another $3d$ compact ancient κ -solution $h_{\Gamma(s_1)}(t)$ corresponding to the point $\Gamma(s)$ such that

$$(M, R(\Gamma(s))g; \Gamma(s)) \stackrel{\epsilon-\text{close}}{\sim} (h_{\Gamma(s_1)}(0) + ds^2, p'_\infty). \quad (6.17)$$

Hence, combining (6.16)–(6.17), we derive

$$h_{\Gamma(s)}(0) \stackrel{\epsilon-\text{close}}{\sim} \frac{R(\Gamma(s_0), T)}{R(\Gamma(s_0))} h_{\Gamma(s_0)}(t), \quad \forall t \in [t_1, 0]. \quad (6.18)$$

By the convergence of $(M, g_p(t); p)$, we know that there exists a constant $C_1 > 0$ such that

$$\left| \frac{R(\Gamma(s_0), T)}{R(\Gamma(s_0))} - R_h(\Gamma(s_0), t) \right| \leq C_1 \epsilon, \quad \forall t \in [t_1, 0],$$

and so,

$$\left| \text{Diam} \left(\frac{R(\Gamma(s_0), T)}{R(\Gamma(s_0))} h_{\Gamma(s_0)}(t) \right) - \text{Diam}(R_h(\Gamma(s_0), t) h_{\Gamma(s_0)}(t)) \right| \leq C_2 \sqrt{\epsilon}$$

for some $C_2 > 0$. Then by (6.18), the monotonicity (6.11) implies that

$$F(\Gamma(s)) \geq \frac{2C'}{3} - 2C_2 \sqrt{\epsilon} > \frac{1}{2} C', \quad \forall s \in [s_0, s_1]. \quad (6.19)$$

Moreover, by (6.12),

$$F(\Gamma(s_1)) > 2C' - 2C_2 \sqrt{\epsilon} > C'. \quad (6.20)$$

By (6.20) together with (6.19) and (6.14), we can repeat the above argument to obtain (6.9). On the other hand, the curve $\Gamma(s)$ passes through the sequence of level sets $f^{-1}(f(q_i))$ because of $\lim_{s \rightarrow \infty} f(\Gamma(s)) = \infty$. Thus for each q_i ($i \gg 1$) there exists $p'_i \in f^{-1}(f(q_i))$ such that $p'_i = \Gamma(s_i)$ for some s_i . By (6.8), $F(p'_i) \leq C_3$, which contradicts to (6.9). Hence, Proposition 6.3 is proved. $\square$

6.3. Proofs of Theorems 1.5 and 1.6

We first use Proposition 4.1 together with Proposition 6.3 to prove Theorem 1.5.

Proof of Theorem 1.5.

Case 1: $\limsup_{p \rightarrow \infty} F_\epsilon(p) < C$ for any $\epsilon < 1$. — (3.1) holds for all split ancient κ -solution $h(t)$ arising from the blow-down of (M, g) , and so they are all of type I by Proposition 4.1. Thus by (4.1) in Lemma 4.2 to the case of $n = 4$, all $h(t)$ must be a family of $3d$ shrinking round quotient spheres.

Case 2: $\limsup_{\epsilon \rightarrow 0} \limsup_{p \rightarrow \infty} F_\epsilon(p) = \infty$. — In this case, by taking a diagonal subsequence, there is a sequence of pointed flows $(M, g_{q_i}(t); q_i)$, which converges subsequently to a splitting Ricci flow $(N' \times \mathbb{R}, h'(t) + ds^2; q_\infty)$ for some noncompact ancient κ -solution $h'(t)$. Then by Proposition 6.3, $(N, h(t))$ is also noncompact from any splitting limit flow $(N \times \mathbb{R}, h(t) + ds^2; p_\infty)$ of $(M, g_{p_i}(t); p_i)$. Theorem 1.5 is proved. $\square$

By Theorem 1.5, we can use the argument in the proof of Theorem 1.2 to get Theorem 1.6 immediately.

Proof of Theorem 1.6. — By Theorem 1.5, we see that any 3d split ancient flow $(N', h'(t))$ arising from the blow-down of (M, g) is a family of shrinking round quotient spheres. Moreover, as in the proof of Theorem 1.2 one can show that $\text{Ric}(g)$ is strictly positive on M and consequently $(N', h'(t))$ is in fact a family of shrinking round spheres. Thus Definition 1.1(ii) is satisfied. Thus by [18, Lemma 6.5], (M, g) is asymptotically cylindrical. It follows that (M, g) is isometric to the Bryant Ricci soliton up to scaling by [7]. Theorem 1.6 is proved. $\square$

Based on Proposition 4.1 we propose the following conjecture for the general dimensional case of Theorem 1.5.

CONJECTURE 6.4. — *Let (M^n, g) ($n \geq 4$) be an n -dimensional noncompact κ -noncollapsed steady gradient Ricci soliton with $\text{Rm} \geq 0$ and $\text{Ric} > 0$ away from a compact set K of M . Then either all split limit ancient Ricci flows $(N, h(t))$ arising from the blow-down of (M, g) are $(n-1)$ -dimensional shrinking Ricci solitons described in (4.1), or all $(N, h(t))$ are noncompact ancient κ -solutions of $(n-1)$ -dimension.*

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