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R. V. SARAYKAR
N. E. JOSHI

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STRUCTURE OF THE SET OF STATIONARY SOLUTIONS OF VISCOUS HYDROMAGNETIC EQUATIONS WITH DIFFUSIVITY

R.V. Saraykar (1) **and N.E. Joshi** (2)

(1)(2) Department of Mathematics, Nagpur University Campus, Nagpur 440010 - India.

Résumé : Dans ce papier, à la suite de C. Foias et de R. Temam [4], nous démontrons les propriétés génériques de l'ensemble des solutions des équations stationnaires de l'hydromagnétisme avec viscosité et diffusivité magnétique. L'outil important utilisé est le théorème de Sard dans sa version en dimension infinie due à Smale.

Summary : In this paper, following C. Foias and R. Temam [4], we have proved generic properties of the set of solutions of stationary hydromagnetic equations with viscosity and magnetic diffusivity. The main tool used is Smale's infinite-dimensional version of Sard's theorem.

INTRODUCTION

In this paper we study the structure of the set of stationary solutions of equations for viscous incompressible conducting fluid with diffusivity in the presence of a magnetic field. The boundary condition for the velocity field is assumed to be homogeneous. We consider the following system of equations governing the stationary hydromagnetic flows :

$$(0.1) \qquad -\nu \Delta u + \sum_{i=1}^{n} u_i D_i u - \sum_{i=1}^{n} B_i D_i B + \operatorname{grad} p = f \text{ in } \Omega$$

$$(0.2) \qquad -\lambda \Delta B + \sum_{i=1}^{n} u_i D_i B - \sum_{i=1}^{n} B_i D_i u = o \text{ in } \Omega$$

(0.3) $$\text{div. } u = o, \text{ div. } B = o \quad \text{in } \Omega$$

(0.4) $$u = o,\ B = \phi, \text{ on } \Gamma, \text{ where } \phi \,.\, n = o \text{ on } \Gamma.$$

Here Ω is an open bounded set of R^n, $n = 2$ or 3, and Γ is its boundary. $D_i = \partial / \partial x_i$ and p denotes the total pressure. u and B are the velocity and magnetic fields.

Without loss of generality, we take $\lambda = \nu$. The problem is reduced to the functional equation in u and B. We denote by $S(f,\phi,\nu)$ the set of solutions $\{u,B\}$ of the problem (0.1) - (0.4).

The methods of proving the properties of $S(f,\phi,\nu)$ rely on those in C. Foias and R. Temam [4].
In section 1, we prove general properties of $S(f,\phi,\nu)$.
In section 2, we prove a generic property. The proof is based on an infinite-dimensional version of Sard's theorem due to Smale and some results developed in Section 1.

The case with nonhomogeneous boundary condition for velocity field can be treated as in [4].

Notations are as in [4].

1. GENERAL PROPERTIES OF THE SET $S(f,\phi,\nu)$

Regarding Γ, the boundary of Ω, we assume that

(1.1) Γ is a manifold of class C^r of dimension $n - 1$ and Ω is locally located on one side of Γ ($r = 2$ unless otherwise specified).

(1.2) Γ has a finite number of connected components.

Let
$$\mathcal{V} = \left\{ u \in \mathscr{D}(\Omega)^n,\ \text{div. } u = o \right\}$$

$$V = \text{closure of } \mathcal{V} \text{ in } H_o^1(\Omega)$$

$$= \left\{ u \in H_o^1(\Omega) \,/\, \text{div. } u = o \right\}$$

$$\mathcal{W} = \left\{ u \in \mathscr{D}(\Omega)^n \,/\, \text{div. } u = o,\ u \,.\, n = o \text{ on } \Gamma \right\}$$

$$W_s = \text{closure of } \mathcal{W} \text{ in } H^s(\Omega)$$

$$W_1 = W, \quad W_o = H.$$

Then $\quad H = \text{closure of } \mathscr{V} \text{ in } L^2(\Omega)$

$$= \{u \,/\, u \in L^2(\Omega)^n, \ \text{div.}\, u = o, \ u\,.\,n = o \ \text{ on } \Gamma\}.$$

We have the following continuous injections :

$$V \subset H \subset V',$$

$$W_s \subset W \subset H \subset W' \subset W_s'.$$

1.1. - The Homogeneous Hydromagnetic Problem.

Given f and ϕ, to find u, B and p such that

$$-\nu\,\Delta u + \sum_{i=1}^{n} u_i D_i u - \sum_{i=1}^{n} B_i D_i B + \text{grad}\, p = f \ \text{ in } \Omega \tag{1.3}$$

$$-\nu\,\Delta B + \sum_{i=1}^{n} u_i D_i B - \sum_{i=1}^{n} B_i D_i u \qquad = o \text{ in } \Omega \tag{1.4}$$

$$\text{div.}\, u = o, \ \ \text{div.}\, B = o \quad \text{in } \Omega \tag{1.5}$$

$$u = o, \ \ B = \phi \ \text{ on } \Gamma \text{ where } B\,.\,n = \phi\,.\,n = o \text{ on } \Gamma. \tag{1.6}$$

We assume that $f \in H^{-1}(\Omega)$, the dual space of $H_o^1(\Omega)$ and ϕ is given in $H^{1/2}(\Gamma)$ (the space of traces of functions in $H^1(\Omega)$) with

$$\int_\Gamma \phi\,.\,\overline{n}\ d\Gamma = o \tag{1.7}$$

$\overline{n}$ being the unit outward normal on Γ. Then (1.7) is implied by (1.6).

One can find Φ ([7] , [11]) with $\Phi \in H^1(\Omega)$, div.$\Phi = o, \Phi = \phi$ on Γ. Hence the problem (1.3) - (1.6) is equivalent to finding $\{u, \overline{B}\}$, $\overline{B} = B - \Phi$, solution of

$$-\nu\,\Delta u + \sum_{i=1}^{n} u_i D_i u - \sum_{i=1}^{n} \overline{B}_i D_i \overline{B} - \sum_{i=1}^{n} \Phi_i\, D_i \overline{B}$$
$$- \sum_{i=1}^{n} \overline{B}_i D_i \Phi + \text{grad}\, p = \overline{f} \text{ in } \Omega \tag{1.8}$$

$$-\nu\,\Delta \overline{B} + \sum_{i=1}^{n} u_i D_i \overline{B} + \sum_{i=1}^{n} u_i D_i \Phi - \sum_{i=1}^{n} \overline{B}_i D_i u$$
$$- \sum_{i=1}^{n} \Phi_i D_i u = \nu\,\Delta \Phi \text{ in } \Omega \tag{1.9}$$

(1.10) $$\text{div. } u = o, \quad \text{div. } \overline{B} = o \quad \text{in } \Omega$$

(1.11) $$u = o, \quad \overline{B} = o \quad \text{on } \Gamma$$

where

(1.12) $$\overline{f} = f + \sum_{i=1}^{n} \Phi_i D_i \Phi$$

In variational form, (1.8)–(1.11) become : to find $u \in V$ and $\overline{B} \in V$ such that

$$\nu((u,v)) + b(u,u,v) - b(\overline{B},\overline{B},v) - b(\Phi,\overline{B},v) - b(\overline{B},\Phi,v) = (\overline{f},v) \quad \text{for all } v \in V$$

and

$$\nu((\overline{B},c)) + b(u,\overline{B},c) + b(u,\Phi,c) - b(\overline{B},u,c) - b(\Phi,u,c) = -\nu((\Phi,c)) \quad \text{for all } c \in V.$$

Equivalently,

(1.13) $$\nu A u + \mathscr{B}(u,u) - \mathscr{B}(\overline{B},\overline{B}) - \mathscr{B}(\Phi,\overline{B}) - \mathscr{B}(\overline{B},\Phi) = \overline{f}$$

(1.14) $$\nu A\overline{B} + \mathscr{B}(u,\overline{B}) + \mathscr{B}(u,\Phi) - \mathscr{B}(\overline{B},u) - \mathscr{B}(\Phi,u) = \nu A \Phi$$

$\mathscr{B}$ is bilinear and A is linear

Let $S(f,\phi,\nu)$ be the set of solutions $\{u,\overline{B}\}$ of the problem (1.8) – (1.11) or (1.13)–(1.14). We now list some properties of this set which are easy consequences of wellknown results.

THEOREM 1.1. *Let (1.1) and (1.2) hold for Ω. Let f be given in $H^{-1}(\Omega)$ and ϕ in $H^{1/2}(\Gamma)$ with*

(1.15) $$\int_{\Gamma_i} \phi \, . \, n \, d\Gamma = o, \quad i = 1,2,\ldots,k$$

which is satisfied by (1.6).

Then

(P_1) $S(f,\phi,\nu)$ *is not empty*

(P_2) $S(f,\phi,\nu)$ *is closed and bounded in* $H^1(\Omega) \times H^1(\Omega)$ *and compact in* $L^2(\Omega) \times L^2(\Omega)$,

(1.16) $$| u,B |_1 \leqslant \sigma_o \text{ for all } \{u,B\} \in S(f,\phi,\nu)$$

where $\sigma_o = \sigma_o(f,\phi,\nu,\Omega)$ *depends on* f,ϕ,ν,Ω.

(P_3) $S(f,\phi,\nu)$ *is reduced to one point (uniqueness) if*

(1.17) $$\nu - C_1 \sigma_o(f,\phi,\nu) > 0$$

C_1 *depends on* Ω

(P_4) *If* $f \in L^2(\Omega)$ *and* $\phi \in H^{3/2}(\Gamma)$, *then*

$$S(f,\phi,\nu) \subset H^2(\Omega) \times H^2(\Omega) \quad \textit{(regularity).}$$

Here $|u,B|_1$ is the product norm on $V \times V$ defined, as usual, as $(|u|_1^2 + |B|_1^2)^{1/2}$ which is equivalent to the norm $\| \{u,B\} \|' = |u|_1 + |B|_1$. (1.18)

Proof. The proof for (P_1), the existence of solution $\{u,B\}$ is given by easily extending the standard proof for existence for a single equation, say, Navier-Stokes equation ([6] , [8]). See also [5] , in which existence is proved assuming (1.15) as the boundary condition.

The crucial step in the proof is to show that choice of above Φ can be made so that

(1.19) $$b(u,\Phi,B) \leqslant \frac{1}{2} \nu \| u \| . \| B \| \quad \text{for all } u, B \in H_0^1(\Omega)$$

H^1-norm of Φ depends on ϕ and ν. (1)

We now prove (1.16) :

We take the scalar product of (1.13) with u and (1.14) with $\overline{B}$. Then we get

(1.20) $$\nu \| u \|^2 - b(\overline{B},\Phi,u) - b(\Phi,\overline{B},u) - b(\overline{B},\overline{B},u) = (\overline{f},u)$$

and

(1.21) $$\nu \| \overline{B} \|^2 - b(\overline{B},u,\overline{B}) + b(u,\Phi,\overline{B}) - b(\Phi,u,\overline{B}) = -\nu((\Phi,\overline{B}))$$

since $b(u,u,u)$ o and $b(u,B,B) = o$.

Also because $b(u,v,w) = -b(u,w,v)$, we get on adding (1.20) and (1.21)

$$\nu(\| u \|^2 + \| \overline{B} \|^2) - b(\overline{B},\Phi,u) + b(u,\Phi,\overline{B}) = (\overline{f},u) - \nu((\Phi,\overline{B}))$$

---

(1) Inequality (1.19) can be proved as follows using arguments as in [6] , p. 109 :
By Theo. 1.1. [5] , $\phi \in H^{1/2}$ and $\int_{\Gamma_i} \phi.\overline{n}\, d\Gamma = o \quad \forall\, i$ implies $\phi = \text{curl } F |_\Gamma$, $F \in H^2(\Omega)$. i.e. $\Phi|_\Gamma = \text{curl } F |_\Gamma$.
Then as in [6] , we can write $\Phi = \text{curl } (b\zeta)$, ζ constructed in [6] .
Thus following [6] , p. 109, we get
$$\left| \int_\Omega u_k . \Phi . B_{x_k} dx \right| \leqslant C_1 \| B \| \left[2\left(\int_\Omega |\text{grad } u|^2 dx \right)^{1/2} + (C_2(\rho) \| u \|^2)^{1/2} \right]$$
(where $C_2(\rho) \to 0$ as $\rho \to 0$)
$$\leqslant \| B \| (\epsilon . C_3 + C_4(\rho)) \| u \|$$
choose ϵ & ρ so small that $\epsilon . C_3 + C_4(\rho) \leqslant \frac{\nu}{2}$ and (1.19) follows.

So, by using (1.19), we have

$$\nu \,\|\{u,\bar{B}\}\|^2 \leqslant |\bar{f}|\ \|u\| + \nu\,\|\Phi\|\,.\,\|\bar{B}\| + \frac{1}{2}\nu\,\|u\|\,.\,\|\bar{B}\| + \frac{1}{2}\nu\,\|u\|\,.\,\|\bar{B}\|$$

$$\leqslant |\bar{f}|\,.\,\|u\| + C_2\nu\,\|\bar{B}\| + \frac{1}{2}\nu\,(\|u\|^2 + \|\bar{B}\|^2)$$

$$\text{since } a.b \leqslant \frac{a^2}{2} + \frac{b^2}{2}$$

Hence $\frac{1}{2}\nu\,\|\{u,\bar{B}\}\|^2 \leqslant (|\bar{f}| + C_3\nu)\,\|\{u,\bar{B}\}\|$ using (1.18)

C_3 depends on ϕ.

So $\|\{u,B-\Phi\}\| = \|\{u,\bar{B}\}\| \leqslant \frac{2}{\nu}(|\bar{f}| + C_3\nu)$ and (1.16) follows.

The closedness of S in $H^1(\Omega) \times H^1(\Omega)$ is elementary. Compactness of S in $L^2(\Omega) \times L^2(\Omega)$ follows by Rellich Lemma applied to product space. For compactness result for Navier-Stokes (stationary) equation, see [6].

Uniqueness (P_3) is standard by using estimates made above. (cf. [8], [11] for single equation).

Lastly, regularity (P_4) follows by the arguments given in [4], by applying them to product-case (cf. [6], [11]).

Hereafter we assume that

(1.22) $\qquad f \in H,\ \phi \in H^{3/2}(\Gamma)$ and satisfies (1.13).

From [5], it follows that the function Φ can be chosen in $H^2(\Omega)$ with

(1.23)

$$\text{div}.\Phi = o,\quad \Phi = \phi \ \text{ on } \Gamma$$

and satisfying (1.19).

Φ can be chosen with

$$|\Phi|_2 \leqslant \sigma\left(|\phi|_{H^{3/2}(\Gamma)}, \nu\right)$$

where $\sigma(s,\nu)$ is increasing with respect to s, decreasing with respect to ν. Thus Φ remains bounded in $H^2(\Omega)$ when ϕ remains bounded in $H^{3/2}(\Gamma)$ (1.24)

Same thing holds for $\sigma_i(f,\phi,\nu)$ which appear in this following sections. They remain bounded when (f,ϕ) remain bounded in $H \times H^{3/2}(\Gamma)$.

1.2. - A Priori Estimates

LEMMA 1.1. $S(f,\phi,\nu)$ *is bounded and closed in*

$$H^2(\Omega) \times H^2(\Omega).$$

Proof. We have, from (1.13) and (1.14), the inequalities

$$\nu\,|Au| \leqslant |\mathscr{B}(u,u)| + |\mathscr{B}(\overline{B},\overline{B})| + |\mathscr{B}(\Phi,\overline{B})| + |\mathscr{B}(\overline{B},\Phi)| + |\overline{f}|,$$

$$\nu\,|A\overline{B}| \leqslant |\mathscr{B}(u,\overline{B})| + |\mathscr{B}(u,\Phi)| + |\mathscr{B}(\overline{B},u)| + |\mathscr{B}(\Phi,u)| + \nu\,|A\Phi|$$

By Lemma 1.1 [4] and remark thereafter,

$$|\mathscr{B}(u,v)| \leqslant c\,|u|_1^{5/4}\,|v|_2^{3/4}$$

$$|\mathscr{B}(u,v)| \leqslant c\,|v|_2\,|u|_1$$

and also

$$\leqslant c\,|u|_2\,|v|_1.$$

Hence

$$\nu\,|Au| \leqslant c_1'\,|u|_1^{5/4}\,.\,|u|_2^{3/4} + c_2'\,|\overline{B}|_1^{5/4}\,|\overline{B}|_2^{3/4} + c_3'\,|\Phi|_2\,|\overline{B}|_1 + |\overline{f}|$$

and

$$\nu\,|A\overline{B}| \leqslant c_4'\,|u|_1^{5/4}\,|\overline{B}|_2^{3/4} + c_5'\,|\Phi|_2\,|u|_1 + c_6'\,|\overline{B}|_1^{5/4}\,|u|_2^{3/4} + c_7'\,\nu\,|\Phi|_2$$

We have

$$C_o^{-1}\,|u|_2 \leqslant |Au| \leqslant C_o\,|u|_2 \quad \forall\, u \in D_A \tag{1.25}$$

Also from (1.16),

$$|\overline{B}|_1 = |B-\Phi|_1 \leqslant |B|_1 + |\Phi|_1 \leqslant \sigma_o + |\Phi|_1 = \sigma_1(f,\phi,\nu)$$

Also

$$|u|_1 \leqslant \sigma_o \leqslant \sigma_1(f,\phi,\nu)$$

Hence we get

$$\nu\,|Au| \leqslant C_1'\,\sigma^{5/4}\,|Au|^{3/4} + C_2'\,\sigma^{5/4}\,|A\overline{B}|^{3/4} + C_3\,\sigma_1\,|\Phi|_2 + |\overline{f}|$$

and

$$\nu\,|A\bar{B}| \leqslant C_4'\,\sigma_1^{5/4}\,|A\bar{B}|^{3/4} + C_5'\,|\Phi|_2\,\sigma_1 + C_6'\,.\,\sigma_1^{5/4}\,|Au|^{3/4} + C_7'\,\nu\,|\Phi|_2$$

Adding these two inequalities,

$$\nu\,(\,|Au| + |A\bar{B}|\,) \leqslant C_4\,\sigma_1^{5/4}\,(\,|Au|^{3/4} + |A\bar{B}|^{3/4}\,) + |\bar{f}| + C_5\,\sigma_1\,|\Phi|_2$$

Applying Young's inequality,

$$C_4\,\sigma_1^{5/4}\,|Au|^{3/4} \leqslant \frac{C_6\,\sigma_1^5}{\nu^3} + \frac{\nu}{2}\,|Au|$$

and

$$C_4\,\sigma_1^{5/4}\,|A\bar{B}|^{3/4} \leqslant \frac{C_7\,\sigma_1^5}{\nu^3} + \frac{\nu}{2}\,|A\bar{B}|$$

Hence

$$\nu\,(\,|Au| + |A\bar{B}|\,) \leqslant \frac{C_8\,\sigma_1^5}{\nu^3} + |\bar{f}| + C_5\,\sigma_1\,|\Phi|_2 + \frac{\nu}{2}\,(\,|Au| + |A\bar{B}|\,)$$

so $\frac{\nu}{2}\,|\{Au, A\bar{B}\}| \leqslant C_9\,(\frac{\sigma_1^5}{\nu^3} + |\bar{f}| + \sigma_1\,.\,|\Phi|_2)$ using (1.18).

Hence

$$|\{Au, A\bar{B}\}| \leqslant \frac{C_{10}}{\nu}\,(\frac{\sigma_1^5}{\nu^3} + |\bar{f}| + \sigma_1\,.\,|\Phi|_2) = \sigma_2(f,\phi,\nu)\,.$$

Using (1.25) again,

$$|\{u,\bar{B}\}|_2 \leqslant C_o\,.\,\sigma_2(f,\phi,\nu)$$

and $S(f,\phi,\nu)$ is bounded in $H^2(\Omega) \times H^2(\Omega)$.

Since the set is closed in $H^1(\Omega) \times H^1(\Omega)$ and injection of $H^2(\Omega) \to H^1(\Omega)$ is dense and continuous, it is closed in $H^2(\Omega) \times H^2(\Omega)$. The lemma is proved:

Before we state next lemma, we introduce, together with P_m of [4], one more projection operator P_m^* as follows : There are eigen vectors c_j in W_s such that

$$((c_j,c))_s = \lambda_j^*(c_j,c) \quad \forall\, c \in W_s.$$

$\{c_j\}$ then form basis of W_s and hence basis of $W \cap (L^n(\Omega))^n$. $W \cap (L^n(\Omega))^n = W$ if $n \leqslant 4$.

Also injection of $W_s \to H$ is compact, so $\{c_j\}$ form basis of H. Then we may assume that

$$o < \lambda_1^* \leqslant \lambda_2^* \leqslant \ldots \quad \text{and} \quad \lambda_m^* \geqslant c.m^{2/n}\,,\ m \geqslant 1\,;$$

c is a constant depending only on Ω.

Then P_m^*, $m \geqslant 1$, denotes the orthogonal projection onto the space spanned by $c_1, c_2, \ldots, c_j$, either in H, W or W'.

LEMMA 1.2. *There exists a constant* σ_3 *depending on* f, ϕ, Ω, ν *such that if*

(1.26) $$\lambda_{m+1} \geqslant \sigma_3, \quad \lambda^*_{m+1} \geqslant \sigma_3$$

(m *to be chosen max. of* m_1, m_2 *with* $\lambda_{m_1+1} \geqslant \sigma_3$ *and* $\lambda^*_{m_2+1} \geqslant \sigma_3$) *then for every pair* $\{u,B\}$, $\{v,c\}$ *belonging to* $S(f, \phi, \nu)$, *we have*

(1.27) $$|\{u,B\} - \{v,c\}| \equiv |u-v| + |B-c| \leqslant \sigma_4 \left(|P_m(u-v)| + |P^*_m(B-c)| \right)$$

(1.28) $$|\{Au,AB\} - \{Av,Ac\}| \leqslant \sigma_4 \left(|AP_m(u-v)| + |AP^*_m(B-c)| \right)$$

where σ_4 *is simply related to* σ_3.

Proof. We have from (0.1),

$$\nu((u,\theta)) + b(u,u,\theta) - b(B,B,\theta) = (f,\theta)$$

$$\nu((v,\theta)) + b(v,v,\theta) - b(c,c,\theta) = (f,\theta) \quad \forall\, \theta \in V$$

or

(1.29) $$\nu((w,\theta)) + b(u,u,\theta) - b(v,v,\theta) - b(B,B,\theta) + b(c,c,\theta) = 0$$

where

$$w = u - v \in D_A.$$

Similarly, from (0.2),

(1.30) $$\nu((F,\psi)) + b(u,B,\psi) - b(v,c,\psi) - b(B,u,\psi) + b(c,v,\psi) = 0 \quad \forall\, \psi \in W$$

where

$$F = B - C \in D_A.$$

Let

$$Q_m = I - P_m, \quad Q^*_m = I - P^*_m$$

and put $\theta = Q_m w$ in (1.29), $\psi = Q^*_m F$ in (1.30).

Then (1.29) gives

$$
\begin{aligned}
\nu \| Q_m w \|^2 &= -b(u,u,Q_m w) + b(v,v,Q_m w) + b(B,B,Q_m w) - b(c,c,Q_m w) \\
&= -b(u,w,Q_m w) - b(w,v,Q_m w) + b(B,F,Q_m w) + b(F,c,Q_m w) \\
&= -b(u,P_m w,Q_m w) - b(P_m w,v,Q_m w) - b(Q_m w,v,Q_m w) \\
&\quad + b(B,P_m^* F,Q_m w) + b(B,Q_m^* F,Q_m w) + b(P_m^* F,c,Q_m w) + b(Q_m^* F,c,Q_m w)
\end{aligned} \tag{1.31}
$$

where we have used properties of b and that

$$(P_m w, Q_m w) = o.$$

Similar arrangements in (1.30) give

$$
\begin{aligned}
\nu \| Q_m^* F \|^2 &= -b(P_m w,B,Q_m^* F) - b(Q_m w,B,Q_m^* F) - b(v,P_m^* F,Q_m^* F) + b(B,P_m w,Q_m^* F) \\
&\quad + b(B, Q_m w,Q_m^* F) + b(P_m^* F,v,Q_m^* F) + b(Q_m^* F,v,Q_m^* F)
\end{aligned} \tag{1.32}
$$

Using Lemma 1.1 [4], (1.31) gives

$$
\begin{aligned}
\nu \| Q_m w \|^2 &\leqslant C_{11} (| u |_2 + | v |_2) \| Q_m w \| . | P_m w | \\
&\quad + C_{12} \| Q_m w \| . | v |_2 . | Q_m w | \\
&\quad + C_{13} (| B |_2 + | c |_2) | P_m^* F | . \| Q_m w \| \\
&\quad + C_{14} (| B |_2 + | c |_2) | Q_m^* F | . \| Q_m w \|
\end{aligned} \tag{1.33}
$$

and (1.32) gives

$$
\begin{aligned}
\nu \| Q_m^* F \|^2 &\leqslant C_{15} | B |_2 | P_m w | . \| Q_m^* F \| + C_{16} | v |_2 | P_m^* F | . \| Q_m^* F \| \\
&\quad + C_{17} | B |_2 | Q_m w | . \| Q_m^* F \| \\
&\quad + C_{18} | v |_2 | Q_m^* F | . \| Q_m^* F \|
\end{aligned} \tag{1.34}
$$

By Lemma 1.1, $| u |_2$, $| v |_2$, $| B |_2$, $| c |_2$ are all bounded by $\sigma_2(f, \phi, \nu)$.

Hence by (1.33),

$$\nu \| Q_m w \|^2 \leqslant C_{19}\, \sigma_2 (| P_m w | + | Q_m w |) \| Q_m w \|$$
$$+ C_{20}\, \sigma_2 (| P_m^* F | + | Q_m^* F |) \| Q_m w \|$$

and by (1.34),

$$\nu \| Q_m^* F \|^2 \leqslant C_{21}\, \sigma_2 (| P_m w | + | Q_m w |) \| Q_m^* F \|$$
$$+ C_{22}\, \sigma_2 (| P_m^* F | + | Q_m^* F |) \| Q_m^* F \|$$

Adding these two inequalities, we get

$$\nu (\| Q_m w \|^2 + \| Q_m^* F \|^2) \leqslant \sigma_2 \,.\, C_{23} (\| Q_m w \| + \| Q_m^* F \|) \,.$$
$$\cdot (| P_m w | + | Q_m w | + | P_m^* F | + | Q_m^* F |)$$

or using equivalent product norm,

$$\nu \| \{ Q_m w, Q_m^* F \} \|^2 \leqslant \sigma_2 \,.\, C_{24} \| \{ Q_m w, Q_m^* F \} \| \times (| \{ P_m w, P_m^* F \} | + | \{ Q_m w, Q_m^* F \} |)$$

Hence

(1.35) $$\nu \| \{ Q_m w, Q_m^* F \} | \leqslant \sigma_2 \,.\, C_{24} (| \{ P_m w, P_m^* F \} | + | \{ Q_m w, Q_m^* F \} |)$$

Also we have

$$| Q_m \theta | \leqslant (\lambda_{m+1})^{-1/2} \| Q_m \theta \| \quad \forall\, \theta \in V$$

and

$$| Q_m^* \psi | \leqslant (\lambda_{m+1}^*)^{-1/2} \| Q_m^* \psi \| \quad \forall\, \psi \in W$$

So,

$$| \{ Q_m \theta, Q_m^* \psi \} | \leqslant \max . \{ \lambda_{m+1}^{-1/2}, \lambda_{m+1}^{*-1/2} \} \,.\, \| \{ Q_m \theta, Q_m^* \psi \} \|$$

Let this max. be $\lambda_{m+1}^{-1/2}$.

Hence,

(1.36) $$| \{ Q_m \theta, Q_m^* \psi \} | \leqslant \lambda_{m+1}^{-1/2} \,.\, \| \{ Q_m \theta, Q_m^* \psi \} \|$$

Then (1.35) gives

$$\left(\nu - \frac{C_{24} \cdot \sigma_2}{\lambda_{m+1}^{1/2}} \right) \| \{ Q_m w, Q_m^* F \} \| \leqslant C_{24} \cdot \sigma_2 \,.\, | \{ P_m w, P_m^* F \} |$$

If (1.26) holds with

$$\sigma_3 > \left(\frac{2\,C_{24}\cdot\sigma_2}{\nu}\right)^2$$

then

$$\lambda_{m+1}^{1/2} \geqslant \sigma_3^{1/2} > \frac{2C_{24}\cdot\sigma_2}{\nu}$$

or

$$\nu - \frac{C_{24}\cdot\sigma_2}{\lambda_{m+1}^{1/2}} > \frac{\nu}{2}$$

So

$$\frac{\nu}{2} \| \{Q_m w, Q_m^* F\} \| \leqslant C_{24}\,.\,\sigma_2\, |\{P_m w, P_m^* F\}|$$

Hence

$$\| \{Q_m w, Q_m^* F\} \| \leqslant \sigma_3^{1/2}\, |\{P_m w, P_m^* F\}|$$

Then (1.36) gives

$$\begin{aligned} |Q_m w, Q_m^* F| &\leqslant \lambda_{m+1}^{-1/2}\,.\,\sigma_3^{1/2}\, |\{P_m w, P_m^* F\}| \\ &\leqslant \sigma_3\, |\{P_m w, P_m^* F\}| \end{aligned}$$

(σ_3 chosen such that $\geqslant 1$)

and so

$$\begin{aligned} |\{w,F\}| &= (\,|\{P_m w, P_m^* F\}|^2 + |\{Q_m w, Q_m^* F\}|^2)^{1/2} \\ &\leqslant (1+\sigma_3^2)^{1/2}\, |\{P_m w, P_m^* F\}| \\ &\leqslant (1+\sigma_3^2)^{1/2}\, |\{P_m(u-v), P_m^*(B-c)\}| \end{aligned}$$

which gives (1.27) by using equivalent norms.

We now prove (1.28) : Replacing θ by $A\,Q_m w$ in (1.29) and ψ by $A\,Q_m^*\,F$ in (1.30), we get

$$\nu\,|A\,Q_m w|^2 = -\,b(u,u,A\,Q_m w) + b(v,v,A\,Q_m w) + b(B,B,A\,Q_m w) - b(c,c,A\,Q_m w)$$

$= $ (as before)

$$\begin{aligned} &= -\,b(P_m w,u,A\,Q_m w) - b(Q_m w,u,A\,Q_m w) - b(v,P_m w,A\,Q_m w) \\ &\quad - b(v,Q_m w,A\,Q_m w) + b(P_m^* F,B,A\,Q_m w) + b(Q_m^* F,B,A\,Q_m w) \\ &\quad + b(c,P_m^* F,A\,Q_m w) + b(c,Q_m^* F,A\,Q_m w) \end{aligned} \tag{1.37}$$

$$\leqslant C_{25} \cdot \sigma_2 \cdot (| A P_m w | + \| Q_m w \| + | A P_m^* F | + \| Q_m^* F \|) \, . \, | A Q_m w |$$

similar to (1.33) ; and similar to (1.32) and (1.34) we get

$$(1.38) \quad \nu \, | A Q_m^* F |^2 \leqslant C_{26} \cdot \sigma_2 (| A P_m w | + \| Q_m w \| + | A P_m^* F | + \| Q_m^* F \|) \, . \, | A Q_m^* F |$$

Here we have used Lemma 1.1 [4] and (1.25).

Hence on adding respective sides of (1.37) and (1.38), and using (1.18),

$$(1.39) \quad \nu \, | \{ A Q_m w, A Q_m^* F \} | \leqslant C_{27} \cdot \sigma_2 \cdot (| \{ A P_m w, A P_m^* F \} | + \| \{ Q_m w, Q_m^* F \} \|)$$

We have similar to (1.36),

$$\| \{ Q_m \theta, Q_m^* \psi \} \| \leqslant \lambda_{m+1}^{-1/2} \, . \, | \{ A\theta, A\psi \} | \quad \forall \, \theta, \psi \in D_A .$$

Hence

$$\left(\nu - \frac{C_{27} \cdot \sigma_2}{\lambda_{m+1}^{1/2}} \right) | \{ A Q_m w, A Q_m^* F \} | \leqslant C_{27} \cdot \sigma_2 \, | \{ A P_m w, A P_m^* F \} |$$

choose

$$\sigma_3 = \left(\frac{2(C_{24} + C_{27}) \cdot \sigma_2}{\nu} \right)^2$$

so

$$\frac{\nu}{2} \, | \{ A Q_m w, A Q_m^* F \} | \leqslant C_{27} \cdot \sigma_2 \, | \{ A P_m w, A P_m^* F \} |$$

or

$$| \{ A Q_m w, A Q_m^* F \} | \leqslant \frac{2 C_{27} \cdot \sigma_2}{\nu} \, | \{ A P_m w, A P_m^* F \} |$$

$$\leqslant \sigma_3^{1/2} \, | \{ A P_m w, A P_m^* F \} |$$

Then

$$| \{ Aw, AF \} | \leqslant (1 + \sigma_3)^{1/2} \, | \{ A P_m w, A P_m^* F \} |$$

and we get (1.28).

1.3. - Other Properties of S(f, ϕ, ν)

We are now in a position to prove

THEOREM 1.2. *(P_5) :* $S(f, \phi, \nu)$ *is a compact subset of*

$$H^2(\Omega) \times H^2(\Omega).$$

(P_6) : $S(f, \phi, \nu)$ *is homeomorphic to a compact subset of* $R^m \times R^m$, m *sufficiently large so that (1.26) is satisfied.*

Proof. By using lemma 1.2 above, proof of [4] can directly be extended in this case, using product spaces.

2. GENERIC PROPERTIES

We set

(2.1) $$E_1 = \{u \in H^2(\Omega), \text{div. } u = o, \; u = o \quad \text{on } \Gamma\}$$

(2.2) $$E_2 = \{B \in H^2(\Omega), \text{div. } B = o, \; B.n = o \quad \text{on } \Gamma\}$$

Let $$E = E_1 \times E_2.$$

Set $$F = F_1 \times F_2,$$

(2.3) $$F_1 = H \times H$$
$$F_2 = \{\phi \in H^{3/2}(\Gamma), \phi \,.\, n = o \quad \text{on } \Gamma\}.$$

Then $\int_{\Gamma_i} \phi \,.\, n \, d\Gamma = o$, $i = 1,\dots,k$ is automatically satisfied.

$$\mathcal{E} : E \to F,$$

(2.4) $$\mathcal{E}(u,B) = \{\nu \mathcal{A} u + \mathcal{B}(u,u) - \mathcal{B}(B,B), \nu \mathcal{A} B + \mathcal{B}(u,B) - \mathcal{B}(B,u) \; ; B \,|_{\Gamma}\}$$

where $\mathcal{A} u = -P_H \Delta u$, P_H the projector in $L^2(\Omega)$ onto H.

We now prove

THEOREM 2.1. *We assume that* $\Omega \subset R^n$, $n = 2,3$ *satisfying (1.1), (1,2). Then, for every* $\nu > o$

(P_7) : There exists a dense open set $\Theta \subset F$, *such that for every* $(f, 0, \phi) \in \Theta$, *the set* $S(f, \phi, \nu)$ *is finite.*

For every connected component Θ_α of Θ, the number of elements in $S(f, \phi, \nu)$ for $(f, 0, \phi) \in \Theta_\alpha$ is constant and every solution is a C^∞-function of $(f, 0, \phi)$.

Proof. We apply Sard-Smale theorem ([4] , [9]) with E, F, $\mathcal{E}$ given as above. Clearly $\mathcal{E}$ is C^∞ and its Fréchet derivative is given by

$$< \mathcal{E}'(u,B) , (v,c) >$$

$$(2.5) \qquad = \left\{ \begin{array}{ll} \nu \mathcal{A} v + \mathcal{B}(u,c) + \mathcal{B}(v,u) & v \mathcal{A} c + \mathcal{B}(u,c) - \mathcal{B}(v,u) + \\ \qquad\qquad , & \qquad\qquad , \; c\,|_\Gamma \\ - \mathcal{B}(B,c) - \mathcal{B}(v,B) & + \mathcal{B}(v,B) - \mathcal{B}(B,c) \end{array} \right.$$

$\mathcal{E}'(u,B) \in \mathcal{L}(E,F)$ has the form $\Lambda + K$,

$$\Lambda(v,c) = \left\{ \nu \mathcal{A} v, \nu \mathcal{A} c, c\,|_\Gamma \right\},$$

$$K(v,c) = \left\{ \mathcal{B}(u,c) + \mathcal{B}(v,u) - \mathcal{B}(B,c) - \mathcal{B}(v,B), \mathcal{B}(u,c) + \mathcal{B}(v,B) - \mathcal{B}(v,u) - \mathcal{B}(B,c), 0 \right\}$$

That Λ is an isomorphism from E to F follows from the classical result on Stoke's problem applied to product space. Also $\mathcal{B}$ is bilinear continuous on $H^2(\Omega) \times H^1(\Omega)$ and on $H^1(\Omega) \times H^2(\Omega)$ with values in $L^2(\Omega)$ or H. Hence for $u \in E_1$, $B \in E_2$,

$$(v,c) \to \mathcal{B}(u,c) + \mathcal{B}(v,u) - \mathcal{B}(B,c) - \mathcal{B}(v,B)$$

and

$$(v,c) \to \mathcal{B}(u,c) + \mathcal{B}(v,B) - \mathcal{B}(v,u) - \mathcal{B}(B,c)$$

are linear continuous mappings from $H^1 \times H^1$ into H. Hence

$$(v,c) \to \left\{ \mathcal{B}(u,c) + \mathcal{B}(v,u) - \mathcal{B}(B,c) - \mathcal{B}(v,B), \mathcal{B}(u,c) + \mathcal{B}(v,B) - \mathcal{B}(v,u) - \mathcal{B}(B,c) \right\}$$

is linear continuous from $H^1 \times H^1$ into $H \times H$.

K is therefore compact. Hence dim. Ker. K and dim. Coker K are both finite and they are equal. For Λ, being isomorphism, both these dimensions are zero. Hence we conclude that $\mathcal{E}'(u,B)$ is a Fredholm operator of index zero.

Hence by Smale's theorem, the set Θ of regular values $\{f, 0, \phi\}$ of $\mathcal{E}$ is dense in F and $S(f, \phi, \nu)$ is descrete in E for all $(f, 0, \phi)$ in Θ. Since by (P_5), $S(f, \phi, \nu)$ is compact in E, it is finite.

The proof of openness of Θ and the last part of the theorem can be given in this case by easily extending the proof given in [4] , p. 161-162.

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