

M. A. HERRERO

J. L. VAZQUEZ

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Annales de la faculté des sciences de Toulouse 5^e série, tome 3, n° 2 (1981), p. 113-127

http://www.numdam.org/item?id=AFST_1981_5_3_2_113_0

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ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF A STRONGLY NONLINEAR PARABOLIC PROBLEM

M.A. Herrero ⁽¹⁾ and J.L. Vazquez ⁽²⁾

(1) *Departamento de Ecuaciones Funcionales, Facultad de Ciencias Matemáticas, Universidad Complutense, Madrid 3 - España.*

(2) *Division de Matematicas, Universidad Autonoma de Madrid, Madrid - 34 - España.*

Résumé : Nous étudions le problème d'évolution $u_t + Au = 0$ dans $(0, T) \times \mathbb{R}^N$, $u(0) = u_0$ dans \mathbb{R}^N , avec $N \geq 1$, $0 < T \leq \infty$, $Au = -\operatorname{div}(|Du|^{p-2} Du)$, Du étant le gradient de u , $1 < p < \infty$ et nous supposons que u_0 appartient à un espace de fonctions intégrables. On prouve l'existence d'un temps fini d'extinction si $N \geq 2$ et $p < \frac{2N}{N+1}$. Dans le cas contraire (si $N = 1$ et $p > 1$ ou si $N \geq 2$ et $p \geq \frac{2N}{N+1}$) on prouve la loi de conservation : $\int_{\mathbb{R}^N} u(t, x) dx = \int_{\mathbb{R}^N} u_0(x) dx$ pour tout $t > 0$. On estime aussi la convergence vers zéro des intégrales $\int_{\mathbb{R}^N} |u(t, x)|^m dx$, $m > 1$ et on obtient certains effets régularisants.

Summary : The evolution problem $u_t + Au = 0$ in $(0, T) \times \mathbb{R}^N$, $u(0) = u_0$ in \mathbb{R}^N is considered where $N \geq 1$, $0 < T \leq \infty$, $Au = -\operatorname{div}(|Du|^{p-2} Du)$, with Du the gradient of u , $1 < p < \infty$ and u is supposed to belong to some integrable space. If $N \geq 2$ and $p < \frac{2N}{N+1}$ the existence of a finite extinction time is shown. On the contrary, if $N = 1$, $p > 1$ or $N \geq 2$, $p \geq \frac{2N}{N+1}$ conservation of total mass holds, i.e. $\int_{\mathbb{R}^N} u(t, x) dx = \int_{\mathbb{R}^N} u_0(x) dx$ for every $t > 0$. We prove also that the integrals $\int_{\mathbb{R}^N} |u(t, x)|^m dx$, $m > 1$ converge to zero as t goes to infinity, and some regularizing effects are shown.

INTRODUCTION AND PRELIMINARIES

We shall consider the asymptotic behaviour in time of the solutions of

$$(P) \quad \begin{cases} u_t + Au = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ u(0) = u_0 & \text{in } \mathbb{R}^N \end{cases}$$

with $N \geq 1$, $1 < p < \infty$ and $Au = - \sum_{i=1}^N \frac{\partial}{\partial x_i} (|Du|^{p-2} Du)$ where $Du = \left(\frac{\partial u}{\partial x_i} \right)_i$ is the gradient of u . The operator A has been widely considered in the literature in P.D.E., and arises in several physical situations, such as one-dimensional non newtonian fluids and glaciology.

This behaviour depends strongly on p and N : in fact, if $p \geq \frac{2N}{N+1}$ we show that the total mass $\int_{\mathbb{R}^N} u(t, x) dx$ is conserved, i.e, is independent of time. On the contrary if $p < \frac{2N}{N+1}$ we show that the solution corresponding to initial data $u_0 \in L^m(\mathbb{R}^N)$, $m = N(\frac{2}{p} - 1)$ vanishes in finite time. The existence of a finite extinction time was found by Bénilan and Crandall [2] for the equation (E) $u_t - \Delta u^m = 0$ in spatial domain \mathbb{R}^N ⁽¹⁾ if and only if $0 < m < \frac{N-2}{N}$, $N \geq 3$. As it is noted in [2], equation (E) in bounded domains with homogeneous Dirichlet conditions has also that property if $0 < m < 1$. The case $N = 1$ was considered by Sabinina [8]. Several properties of solutions of (E) related to the ones we consider here can be found in Evans [5]. Finite extinction times for (E _{β}) $u_t - \Delta \beta(u) = 0$ with β maximal monotone graph and bounded domain are discussed in terms of β in [3].

We also consider the homogeneous Dirichlet problem

$$(P_\Omega) \quad \begin{cases} u_t - \operatorname{div}(|Du|^{p-2} Du) = 0 & \text{in } (0, T) \times \Omega \\ u(x, t) = 0 & \text{in } (0, T) \times \partial\Omega \\ u(x, 0) = u_0(x) & \text{in } \Omega \end{cases}$$

for $\Omega \subset \mathbb{R}^N$ open and bounded. We show the existence of a finite extinction time if $p < 2$, $u_0 \in L^m(\Omega)$, and m as above, completing a result of Bamberger [1]: he showed that effect for $\frac{2N}{N+2} \leq p < 2$ and $u_0 \in L^2(\Omega)$. For $p \geq 2$ it is easy to see that solutions with positive initial data do not vanish.

For $p > \frac{2N}{N+1}$ L. Véron [11] shows a smoothing and decay effect for the solutions

(1) with $u_0 \in L^\beta(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ for a $\beta = \beta(m, N)$.

of (P_Ω) : in fact, if $N(\frac{2}{p}-1) < m_0 < m \leq \infty$ and $u_0 \in L^{m_0}(\Omega)$, then $u(t, \cdot) \in L^m(\Omega)$ and in addition $\|u\| \leq Ct^{-\delta} \cdot \|u_0\|_{m_0}^\sigma$ where δ, σ depend on m, m_0, p and N . We adapt his proof for (P) to get similar results. We know that for $m_0 = N(\frac{2}{p}-1)$ solutions vanish. For $1 < m_0 < N(\frac{2}{p}-1)$ we prove a «backwards» effect : for $t > 0$, $u(t, \cdot) \in L^1(\mathbb{R}^N)$ and $\|u\|_1 \leq Ct^{-\delta} \|u_0\|_{m_0}^\sigma$ with $\delta, \sigma > 0$ as before.

We shall need some facts about the operator A in \mathbb{R}^N and in $\Omega \subset \mathbb{R}^N$ bounded with homogeneous Dirichlet conditions : First, if $J(u) = \frac{1}{p} \int_{\mathbb{R}^N} |Du|^p$ when $u \in L^2(\mathbb{R}^N)$ and $|Du| \in L^p(\mathbb{R}^N)$, $J(u) = +\infty$ otherwise, J is a convex l.s.c. proper functional in $L^2(\mathbb{R}^N)$ whose subdifferential A is defined as $Au = -\operatorname{div}(|Du|^{p-2} Du)$ in the domain $D(A) = \left\{ u \in L^2(\mathbb{R}^N) : |Du| \in L^p(\mathbb{R}^N), \operatorname{div}(|Du|^{p-2} Du) \in L^2(\mathbb{R}^N) \text{ and for every } v \in D(J), \int_{\mathbb{R}^N} Au \cdot v = \int_{\mathbb{R}^N} |Du|^{p-2} Du \cdot Dv \right\}$. If $p \geq 2$, the last condition may be omitted as it follows by density. A is accretive in $L^1(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$, hence in every $L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$: in fact for $t > 0$ and $u_1, u_2 \in D(A) \cap L^p(\mathbb{R}^N)$, $\| [u_1(t, \cdot) - u_2(t, \cdot)]^+ \|_p \leq \| [u_1(0, \cdot) - u_2(0, \cdot)]^+ \|_p$ where $u^+ = \max(u, 0)$. This implies a comparison principle that allows us to consider only nonnegative initial data and solutions ; for nonpositive data we consider $-u$ instead of u . Defining for $p \neq 2$, $A_p = A \cap (L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N))$ we may close A_p to find \overline{A}_p m -accretive in $L^p(\mathbb{R}^N)$. As $\mathcal{D}(\mathbb{R}^N) \subset D(A_p)$, $\overline{D(A_p)}^{L^p} = L^p(\mathbb{R}^N)$.

The corresponding results for Ω bounded and homogeneous Dirichlet conditions are well known ; $Au = -\operatorname{div}(|Du|^{p-2} Du)$ and $D(A) = \{ u \in W_0^{1,p}(\Omega) \cap L^2(\Omega) : Au \in L^2(\Omega) \}$. On the other hand A_p is defined as m -accretive operator in $L^p(\mathbb{R}^N)$ by restriction if $p > 2$ and closure if $p < 2$.

We shall use the following inequality due to Nirenberg and Gagliardo (see [6], Th. 9.3.).

LEMMA 0. Let q, r be any numbers satisfying $1 \leq q, r \leq \infty$ and $u \in C_0^1(\mathbb{R}^N)$. Then

$$\|u\|_p \leq C \|Du\|_r^a \|u\|_q^{1-a}$$

where $\frac{1}{p} = a \cdot \frac{1}{r^*} + (1-a) \frac{1}{q}$ and $\frac{1}{r^*} = \frac{1}{r} - \frac{1}{N}$ for all a in the interval $0 \leq a \leq 1$, with $C = C(N, q, r, a)$, with the following exception : $r = N$ and $a = 1$ (hence $p = \infty$).

We remark that by density the result remains true for $u \in L^q(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ such that $Du \in L^r(\mathbb{R}^N)$ if $r, q < \infty$ and $q \leq r^*$ if r^* is positive. To show this, approach u by u^1 bounded, then convolve u^1 with a regular kernel to get $u^2 \in C^\infty(\mathbb{R}^N)$ and finally cut u^2 with a smooth function ζ_n which vanishes outside $B_{2n}(0)$ and is equal to 1 on $B_n(0)$; let us check this last step.

Assume $u \in C^\infty(\mathbb{R}^N)$ and put $u_n = u \zeta_n$, where $\zeta_n(x) = \zeta_0\left(\frac{|x|}{n}\right)$, $0 \leq \zeta_0 \leq 1$, $\zeta_0(x) = 1$ if $|x| \leq 1$, $\zeta_0(x) = 0$ if $|x| \geq 2$ and $|D\zeta_n(x)| \leq C$. It is clear that $u_n \rightarrow u$ in $L^q(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$. Also $D_{u_n} = Du \cdot \zeta_n + u \cdot D\zeta_n$. $Du \cdot \zeta_n \rightarrow Du$ in $L^r(\mathbb{R}^N)$ and we have to prove that $u \cdot D\zeta_n \rightarrow 0$ in $L^r(\mathbb{R}^N)$. Then, C representing different constants independent of n :

$$\|uD\zeta_n\|_r^r = \int_{\mathbb{R}^N} |u|^r |D\zeta_n|^r \leq \frac{c}{n^r} \int_{n \leq |x| \leq 2n} |u|^r$$

if $q \leq r$, $\int |u|^r \leq \|u\|_\infty^{r-q} \cdot \int |u|^q$, so $\|uD\zeta_n\|_r^r \leq \frac{C \|u\|_\infty^{r-q}}{n^r} \cdot \|u\|_q^q \rightarrow 0$;

if $r < q \leq r^*$, $\int |u|^r \leq \left(\int |u|^q\right)^{r/q} \cdot \left(\int 1\right)^{1-q/q}$, so :

$$\|uD\zeta_n\|_r^r \leq \frac{c}{n^r} \cdot \|u\|_{L^q(n \leq |x| \leq 2n)}^r \cdot n^{N(1-\frac{r}{q})} \rightarrow 0.$$

If $r^* < 0$ the previous proof applies as well for every q , $1 \leq q < \infty$.

Our plan is as follows : Sections 1, 2, 3 are devoted to problem (P). Section 1 studies the existence of a finite extinction time when $p < \frac{2N}{N+1}$, $u_0 \in L^m(\mathbb{R}^N)$, $m = N(\frac{2}{p} - 1)$. Section 2 is devoted to conservation of mass and Section 3 to the regularizing effects and decay of the integral norms $\|u(t, \cdot)\|_m$ as $t \rightarrow \infty$. Finally Section 4 gathers the results on (P_Ω) , Ω open and bounded.

1. - FINITE EXTINCTION TIME

We obtain the following result

THEOREM 1. Let $N \geq 2$, $1 < p < \frac{2N}{N+1}$ and let $u_0 \in L^m(\mathbb{R}^N)$ where $m = N(\frac{2}{p} - 1)$. Then for every $t > 0$ $u(t, \cdot) \in L^\infty(\mathbb{R}^N)$ and there exists $t_0 > 0$ such that $u(t, \cdot) = 0$ a.e. if $t \geq t_0$.

Proof. We may assume that $u_0(x)$, $u(t,x)$ are nonnegative. A formal proof to be justified later by discretization in time runs as follows : As $p < \frac{2N}{N+1}$ if $m = N(\frac{2}{p} - 1)$ we have $m > 1$. Let $p^* = \frac{Np}{N-p}$ and $q = \frac{m+p-2}{p}$: then $m = p^*q$. Also for $k \geq 0$ we write $(u-k)_+ = \max(u-k, 0)$ and $v = v_k = (u-k)_+^q$. Multiply $u_t - \operatorname{div}(|Du|^{p-2} Du) = 0$ by $m(u-k)_+^{m-1}$ and integrate over \mathbb{R}^N to obtain :

$$(1.1) \quad \frac{d}{dt} \int_{\mathbb{R}^N} (u-k)_+^m = m \int_{\mathbb{R}^N} u_t (u-k)_+^{m-1} = m \int_{\mathbb{R}^N} \operatorname{div}(|Du|^{p-2} Du) (u-k)_+^{m-1}$$

Integration by parts and Sobolev's inequality give

$$(1.2) \quad - \int_{\mathbb{R}^N} \operatorname{div}(|Du|^{p-2} Du)(u-k)_+^{m-1} = (m-1)\bar{q}^p \int_{\mathbb{R}^N} |Dv|^p \geq \\ \geq C_p (m-1)\bar{q}^p \left(\int_{\mathbb{R}^N} v^{p^*} \right)^{p/p^*}$$

Write $E_{m,k}(t) = \int_{\mathbb{R}^N} (u-k)_+^m dx$. (1.1) and (1.2) give

$$(1.3) \quad \frac{d}{dt} E_{m,k}(t) + C_p m(m-1)\bar{q}^p E_{m,k}^{p/p^*}(t) \leq 0$$

Integrating (1.3) gives

$$(1.4) \quad \begin{cases} E_{m,k}(t) \leq E_{m,k}(0) \left[1 - \frac{C_p m(m-1)p}{Nq^p (E_{m,k}(0))^{p/N}} \cdot t \right]^{\frac{N}{p}} & \text{for } 0 < t \leq t_{0,k} \\ E_{m,k}(t) = 0 & \text{for } t \geq t_{0,k} \end{cases}$$

where

$$t_{0,k} = \frac{Nq^p}{pC_p m(m-1)} E_{m,k}(0)^{p/N}$$

If we take $k = 0$ the existence of a finite extinction time $t_0 = t_{0,0}$ results. Given $\bar{t} > 0$, if we take $k > 0$ large enough extinction of $E_{m,k}(t)$ in time $t_{0,k} \leq \bar{t}$ may be obtained. Hence $u(t, \cdot) \in L^\infty(\mathbb{R}^N)$ for $t > 0$, a regularizing effect.

This formal proof can be made rigorous by means of the discrete scheme and Crandall-Liggett's results. Assume that $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, let $h > 0$ and define a discrete approximation to the solution of (P) thus : $u_{i+1} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is defined implicitly in terms of u_i by

$$(1.5) \quad \frac{u_{i+1} - u_i}{h} + Au_{i+1} = 0$$

Now repeat the previous argument on (1.5) to obtain a discrete version of (1.3) and pass to the limit as $h \rightarrow 0$. The assumption on u_0 can be weakened by approximation for $t_{0,k}$ depends only on $\|u_0\|_m$. The details repeat those in [2] for $u_t - \Delta u^m = 0$ and we omit them. Only the integration by parts needs some care : if $m \geq 2$, $u_0 \in D(A) \cap L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then

$$(1.6) \quad - \int_{\mathbb{R}^N} Au_i u_i^{m-1} + (m-1) \int_{\mathbb{R}^N} |Du_i|^p u_i^{m-2} = 0$$

by the characterization of $D(A)$. If $m < 2$ we have to linearize the function $\phi(u) = u^{m-1}$ near the

origin to apply integration by parts. Passing to the limit it follows by Fatou in this case that

$$(1.7) \quad - \int_{\mathbb{R}^N} Au_i u_i^{m-1} + m \int_{\mathbb{R}^N} |Du_i|^p u_i^{m-2} \leq 0$$

For u_0 as in the theorem the result follows by density for A is accretive #

2. - MASS CONSERVATION

We say that the mass conservation law (MCL) holds for (P) if for every $t > 0$

$$\int_{\mathbb{R}^N} u(t,x) dx = \int_{\mathbb{R}^N} u_0(x) dx.$$

In this section the validity of MCL is discussed in terms of p :

THEOREM 2. *MCL holds for (P) if and only if $N = 1, p > 1$ or $N \geq 2, p \geq \frac{2N}{N+1}$.*

In order to prove Theorem 2 we need some previous results. A variant of the following Lemma has been used in [10] :

LEMMA 1. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in L^2(\mathbb{R}^N)$ be such that $u \in D(A)$ and $-Au = u$ a.e. in Ω . Let $\eta \in C^\infty(\Omega)$ be such that $\text{supp}(\eta) \subset \subset \Omega$, $\|\eta\|_\infty = 1$ and let χ be the characteristic function of $\text{supp}(D\eta)$. Then*

$$(2.1) \quad \|\eta Du\|_p \leq p \|D\eta\|_\infty \cdot \|\chi u\|_p.$$

Proof. Multiply $u = Au$ by $u\eta^p$, integrate over \mathbb{R}^N , integrate by parts ($u \in D(A)$) and apply Hölder's inequality.

LEMMA 2. *Let $\frac{2N}{N+1} \leq p \leq 2$ and let u be a solution of $Au + u = f$, $f \in L^1(\mathbb{R}^N)$. Then*

$$\int_{\mathbb{R}^N} Au = 0.$$

Proof. By accretivity of A in $L^1(\mathbb{R}^N)$, we may restrict ourselves to consider $f \in L^\infty_0(\mathbb{R}^N)$. We obtain first an estimate for $\|Du\|_p$ over the exterior of a ball : Assume $\text{supp}(f) \subset B_R(0)$ and take $n > R$. Choose $\eta_n \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 0$ if $|x| \leq n$, $\eta_n = 1$ if $|x| \geq 2n$ and $\|D\eta_n\|_\infty \leq \frac{C_1}{n}$, $c_1 > 1$. Put $A_n = \{x \in \mathbb{R}^N : n \leq |x| \leq 2n\}$ and $D_n = \{x \in \mathbb{R}^N : |x| \geq n\}$. Then (2.1) gives in $\Omega = \mathbb{R}^N - B_R(0)$:

$$(2.2) \quad \|Du\|_{L^p(D_n)} \leq \frac{C}{n} \|\chi u\|_{L^p(\mathbb{R}^N)} \leq \frac{C}{n} \|u\|_{L^p(A_n)}$$

Hereafter C denotes several positive constants depending only on p and N and not on n.

By virtue of [9], Corollary 2, the following estimate applies to u(x), for |x| > R:

$$(2.3) \quad u(x) \leq C |x|^{-\frac{p}{2-p}}$$

Also by accretivity $\|u\|_1 \leq \|f\|_1$, so that

$$\|u\|_{L^p(A_n)}^p \leq \|u\|_{L^1(A_n)} \cdot \|u\|_{L^\infty(A_n)}^{p-1} = o(1) \cdot n^{-\frac{p(p-1)}{2-p}} \quad (1)$$

$$\|Du\|_{L^p(D_n)} = o(1) \cdot n^{-\frac{1}{2-p}} \text{ . Putting } \zeta_n(x) = 1 - \eta_n(x) \text{ we have}$$

$$(2.4) \quad \int |Au \cdot \zeta_n| \leq \int |Du|^{p-1} |D\zeta_n| \leq \frac{o(1)}{n} \cdot n^{-\frac{p-1}{2-p}} \cdot n^{N/p} = o(n) \cdot n^{\frac{N}{p} - \frac{1}{2-p}}$$

Since $\int_{\mathbb{R}^N} Au = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} Au \zeta_n$, the desired result follows whenever

$$\frac{N}{p} - \frac{1}{2-p} \leq 0 \text{ i.e. } p \geq \frac{2N}{N+1} \quad \#$$

We say that the finite propagation property (PF) holds for (P) if for every admissible initial datum $u_0(x)$ having compact support in \mathbb{R}^N , the corresponding solution $u(t,x)$ is such that for every $t > 0$ $u(t, \cdot)$ has compact support in \mathbb{R}^N . It is known that (PF) holds for (P) if and only if $p > 2$ (see [4]). There exists a simple relation between (FP) and (MCL):

LEMMA 3. *If $p > 2$, then (MCL) holds.*

Proof. Let $u(x,t)$ be a solution of (P) such that $u(x,0) = u_0(x)$ has compact support. If $t > 0$ we know that there exists n such that $\text{supp } u(t') \subset B_n(0)$ for $0 \leq t' \leq t$. Take ζ_n as before. Then for t' fixed:

$$\int_{\mathbb{R}^N} Au = \int_{\mathbb{R}^N} Au \cdot \zeta_n = \int_{\mathbb{R}^N} |Du|^{p-2} Du \cdot D\zeta_n = 0$$

Hence $\int_{\mathbb{R}^N} u_t dx = 0$ and it follows that $\int_{\mathbb{R}^N} u(t,x) = \int_{\mathbb{R}^N} u_0(x)$. This last assertion can be justified by means of the discrete scheme as before.

If $\text{supp}(u_0)$ is not compact, approximate u_0 by $\{u_{0_n}\}$, a sequence of initial data with compact support #

(1) Here $o(1)$ denotes a quantity that goes to 0 as $n \rightarrow \infty$.

Proof (of Theorem 2). If $N = 1, p > 1$ or $N \geq 2, 2 > p \geq \frac{2N}{N+1}$ the result follows from Lemma 2 applied to the discrete scheme

$$\frac{u_{i+1} - u_i}{h} + Au_{i+1} = 0$$

for then $\int_{\mathbb{R}^N} u_i = \int_{\mathbb{R}^N} u_{i+1}$. If $p > 2$ it follows from Lemma 3 in the same way. The case $p = 2$ is classic (and it falls within the scope of [2]).

For the negative part it is sufficient to remind Theorem 1, for (MCL) is incompatible with extinction #

3. DECAY OF THE INTEGRAL NORMS. REGULARIZING EFFECT

Our first result is the extension to \mathbb{R}^N of the work of L. Véron [11] for the case Ω bounded.

THEOREM 3. Let $p > \frac{2N}{N+m_0}$, $u_0 \in L^{m_0}(\mathbb{R}^N)$ with $m_0 \geq 1$. If $t > 0, u(t, \cdot) \in L^m(\mathbb{R}^N)$ for every m such that $m_0 \leq m \leq \infty$. In addition, the following estimate holds :

(3.1) $\|u(t, \cdot)\|_m \leq \frac{C}{t^\delta} \cdot \|u_0\|_{m_0}^\sigma$ for some constant $C = C(m, m_0, N, p)$, where

(3.2)
$$\left\{ \begin{array}{l} \delta = \frac{N(m - m_0)}{m(m_0 p + N(p-2))} \text{ if } m < +\infty, \delta = \frac{N}{m_0 p + N(p-2)} \text{ if } m = +\infty. \\ \sigma = \frac{m_0 (mp + N(p-2))}{m(m_0 p + N(p-2))} \text{ if } m < +\infty, \sigma = \frac{m_0 p}{m_0 p + N(p-2)} \text{ if } m = +\infty. \end{array} \right.$$

Proof. The case $m = m_0$ follows from the accretivity property ; it suffices to show the case $m = +\infty$, the intermediate cases being obtained from these by interpolation. Assume (for simplicity) that $u \geq 0$; for $p \leq N$ we adapt the iterative procedure of L. Véron [11] as follows. Define the sequences m_n, r_n by :

(3.3) $m_n = \gamma^n \cdot m_0$ with $1 < \gamma < \frac{N}{N-1}, m_0 \left(\frac{\gamma p}{N(\gamma-1)} - 1 \right) > \frac{1}{\gamma-1}$

(3.4) $\frac{r_n + p-2}{m_n} = \frac{r_{n-1}}{m_{n-1}} - \frac{p}{N}$

Note that from (3.3) and (3.4) it follows :

$$(3.5) \quad r_n = \frac{\gamma p}{N(\gamma-1)} m_{n-1} + \frac{p-2}{\gamma-1} = \frac{\gamma^n p}{N(\gamma-1)} m_0 + \frac{p-2}{\gamma-1}.$$

Now we claim that, if we write $v = u^{q_{n-1}}$ with $q_n = \frac{m_n + p - 2}{p}$, Nirenberg-Gagliardo's inequality applies to v . Namely one has :

$$(3.6) \quad \|v\|_{\frac{m_n}{q_{n-1}}}^{\frac{r_n + p - 2}{q_{n-1}}} \leq C \cdot \|Dv\|_p^p \cdot \|v\|_{\frac{m_{n-1}}{q_{n-1}}}^{\frac{r_n - m_{n-1}}{q_{n-1}}}$$

That is a consequence of the following facts : i) As it was pointed out in Theorem 1, we can suppose $u \in D(A) \cap L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ hence $v \in L^{\frac{m_n}{q_{n-1}}}(\mathbb{R}^N) \cap L^{\frac{m_{n-1}}{q_{n-1}}}(\mathbb{R}^N)$ for each $m_{n-1} > 1$, for then $\frac{m_{n-1}}{q_{n-1}}$ is always greater than one, ii) $Dv \in L^p(\mathbb{R}^N)$ as a consequence of the validity of formula (1.7), iii) Nirenberg-Gagliardo's inequality (Lemma 0) applies with the present regularity, as it was observed at the introduction.

We shall give a formal proof, just as at the first part of Theorem 1 (rigorous justification by means of the discrete schema approximation is made in the same way as there). Assume first $p < N$. Multiply the equation $u_t - Au = 0$ by $m_{n-1} u^{m_{n-1}-1}$ and integrate over \mathbb{R}^N to get

$$(3.7) \quad \frac{d}{dt} \left(\int_{\mathbb{R}^N} v^{\frac{m_{n-1}}{q_{n-1}}} \right) + C_{m,n} \left(\int_{\mathbb{R}^N} |Dv|^p \right) \leq 0$$

Next multiply (3.7) by $\|u\|_{\frac{m_{n-1}}{q_{n-1}}}^{r_n - m_{n-1}}$ and use (3.6). It follows that

$$(3.8) \quad \|u\|_{\frac{m_{n-1}}{q_{n-1}}}^{r_n - m_{n-1}} \cdot \frac{d}{dt} \left(\|u\|_{\frac{m_{n-1}}{q_{n-1}}}^{m_{n-1}} \right) + C \|u\|_{\frac{m_n}{q_n}}^{r_n + p - 2} \leq 0$$

where C involves $C_{m,n}$ and the constant in (3.6), which depends only on N and p . Take $t_n = t(1 - \frac{1}{2^n})$ and integrate (3.8) in $[t_{n-1}, t_n]$. In this way we obtain :

$$(3.9) \quad \|u(t_n)\|_{\frac{m_n}{q_n}}^{r_n + p - 2} \leq \frac{2^n}{C \cdot t} \|u(t_{n-1})\|_{\frac{m_{n-1}}{q_{n-1}}}^{r_n}$$

The previous argument remains true if we replace u by $u_k = (u-k)_+$ for some $k > 0$. But then $|\Omega_{k,t}| = \text{meas} \{x : u_k(t) > 0\}$ is finite and

$$\|u_k(t)\|_\infty = \lim_{m_n \rightarrow \infty} \sup \|u_k(t)\|_{m_n} \leq \lim_{m_n \rightarrow \infty} \sup \|u_k(t_n)\|_{m_n}.$$

Now (3.1), (3.2) follow from two facts : a) $\lim_{m_n \rightarrow \infty} \sup \|u_k(t_n)\|_{m_n}$ can be evaluated now just in the same way as in [11], which implies estimates (3.1) (3.2) for u_k . b) These estimates do not depend on k , and consequently we can pass to the limit and obtain the desired results for $k = 0$.

When $p = N$, choose $\{\beta_n\}$ such that

$$(3.10) \quad \beta_n = q_n + m_n \left(1 - \frac{1}{N}\right)$$

Write $w_{n-1} = u^{\beta_{n-1}}$. Then $D(u^{q_{n-1}}) = \frac{q_{n-1}}{\beta_{n-1}} \cdot w_{n-1}^{\frac{q_{n-1} - \beta_{n-1}}{\beta_{n-1}}} \cdot Dw_{n-1}$, i.e.,

$$Dw_{n-1} = \frac{\beta_{n-1}}{q_{n-1}} \cdot D(u^{q_{n-1}}) \cdot w_{n-1}^{\frac{\beta_{n-1} - q_{n-1}}{\beta_{n-1}}}. \text{ Now by Hölder}$$

$$(3.11) \quad \left(\int_{\mathbb{R}^N} |Dw_{n-1}| \right)^N \leq \left(\frac{\beta_{n-1}}{q_{n-1}} \right)^N \cdot \left(\int_{\mathbb{R}^N} |D(u^{q_{n-1}})|^N \right) \cdot \left(\int_{\mathbb{R}^N} u^{m_{n-1}} \right)^{N-1}$$

On the other hand, by Sobolev

$$(3.12) \quad \left(\int_{\mathbb{R}^N} |Dw_{n-1}| \right)^N \geq C_N \left(\int_{\mathbb{R}^N} u^{\frac{N\beta_{n-1}}{N-1}} \right)^{N-1}$$

Now multiply (3.7) by $\|u\|_{m_{n-1}}^{r_n - m_{n-1}}$, use (3.11), (3.12) and a standard interpolation argument to get :

$$(3.13) \quad \|u\|_{m_{n-1}}^{r_n - m_{n-1}} \frac{d}{dt} \left[\|u\|_{m_{n-1}}^{m_{n-1}} \right] + C \cdot \|u\|_{m_n}^{r_n + N - 2} \leq 0$$

where $C = C_{m,n} \cdot \left(\frac{\beta_{n-1}}{q_{n-1}} \right)^N \cdot C_N$. (3.13) is the analogous of (3.8) and we can now argue as in the previous case.

When $p > N$ we do not need to use the iterative procedure. For note that Nirenberg-Gagliardo's inequality reads :

$$(3.14) \quad \|v\|_{\infty} \leq C \|Dv\|_p^a \cdot \|v\|_{m/q}^{1-a} \text{ where } a = \frac{N(m+p-2)}{mp+N(p-2)}, m > 1$$

(3.14) and (3.7) give

$$(3.15) \quad \|u\|_{m^a}^{\frac{(1-a)pq}{a}} \frac{d}{dt} (\|u\|_m^m) + C_m \left(\frac{1}{c}\right)^{\frac{p}{a}} \|u^q\|_{\infty}^{\frac{p}{a}} \leq 0, q = \frac{m+p-2}{p}$$

Now note that from the inequality

$$\phi(t)^\omega \frac{d}{dt} \phi(t) + k \psi(t)^\theta \leq 0$$

it follows, integrating between 0 and t

$$(3.16) \quad \psi(t) \leq \left(\frac{1}{kt}\right)^{1/\theta} \cdot \frac{(\phi(0))^\theta}{\omega+1}$$

Use (3.16) with $\psi(t) = \|u^q\|_\infty$, $\phi(t) = \|u\|_m^m$, $\omega = \frac{(1-a)pq}{a}$, $\theta = \frac{p}{a}$ and (3.1), (3.2)

follow. Note that this argument includes the case $N = 1$ which was discarded in [11] #

When $1 < m_0 < N\left(\frac{2}{p} - 1\right)$ we have the following result, concerning a «backwards regularizing effect».

THEOREM 4. *Let $1 \leq m_0 < N\left(\frac{2}{p} - 1\right)$, $u_0 \in L^{m_0}(\mathbb{R}^N)$. If $t > 0$, $u(t, \cdot) \in L^m(\mathbb{R}^N)$ for every m such that $1 \leq m \leq m_0$. In addition the following estimate holds :*

$$(3.17) \quad \left\{ \begin{array}{l} \|u(t, \cdot)\|_m \leq \frac{C}{t^\delta} \|u_0\|_{m_0}^\sigma \text{ for some constant } C = C(m, m_0, N, p), \text{ where} \\ \delta = \frac{N(m_0 - m)}{m[N(2-p) - m_0 p]}, \quad \sigma = \frac{m_0[N(2-p) - mp]}{m[N(2-p) - m_0 p]} \end{array} \right.$$

Proof. Let us see first that $u(t, \cdot) \in L^m(\mathbb{R}^N)$ for each m such that $1 < m < m_0$ (the case $m = m_0$ follows by accretivity). Remark that

$$(3.18) \quad \|v\|_{m/q} \leq C \|Dv\|_p^a \cdot \|v\|_{m_0/q}^{1-a},$$

where v, q are as in the last part of Theorem 3, the validity of (3.18) is justified as there, and

$a = \frac{N(m_0 - m)(m + p - 2)}{m[m_0(N - p) - N(m + p - 2)]}$. Arguing as in Theorem 1 (with $k = 0$), we arrive at

$$(3.19) \quad \frac{d}{dt} E_m(t) + k E_m(t)^{\frac{pq}{am}} \leq 0, \quad E_m(t) = \int_{\mathbb{R}^N} u^m(t, x) dx.$$

Now notice that solutions of the inequality $f' + \alpha f^\gamma \leq 0$ with $\gamma > 1$ satisfy $f \leq \frac{1}{((\gamma - 1)\alpha t)^{\frac{1}{\gamma - 1}}}$. This gives (3.17).

The case $u(t, \cdot) \in L^1(\mathbb{R}^N)$ is obtained by modifying slightly the previous argument :

instead of (3.18) write

$$(3.20) \quad \|v\|_{1/q} \leq C \|Dv\|_p^a \cdot \|v\|_{m/q}^{1-a} \quad \text{with } 1 < m < 3 - p \left(1 + \frac{1}{N}\right),$$

$a = \frac{N(m-1)(p-1)}{[m(N-p) - N(p-1)]}$. Corresponding to (3.19) we have

$$(3.21) \quad \|u\|_m^{(1-a)\frac{pq}{a}} \cdot \frac{d}{dt} (\|u\|_n^m) + C_m \left(\frac{1}{c}\right)^{\frac{p}{a}} \|u\|_1^{\frac{pq}{a}} \leq 0.$$

Now integrate (3.21) between 0 and t and use the fact that $\|u(t)\|_m$ is not increasing in t to get the result #

4. - BOUNDED DOMAINS

Concerning (P_Ω) with Ω bounded, it is known that there is a finite extinction time if $u_0 \in L^2(\Omega)$ and $\frac{2N}{N+2} \leq p < 2$ ([1]). In that paper, extinction of the L^2 norm of the solution implies this result. The method of the proof of Theorem 1, based on the extinction of the L^m norm of solutions for some $m > 1$, enables us to extend the above mentioned result to get the following complete picture.

THEOREM 5. *Assume that Ω is bounded and regular. Let $u_0 \in L^m(\Omega)$ where $m \geq \max \left\{ N \left(\frac{2}{p} - 1 \right), 1 \right\}$ and $p < 2$. The corresponding solution of (P_Ω) vanishes in a finite time t_0 . If $p \geq 2$ there are, for $u_0 \in C^\infty(\Omega)$ and $u_0 > 0$, solutions which are strictly positive for every $t > 0$.*

Proof. Let $m > N \left(\frac{2}{p} - 1 \right)$ (the case $m = N \left(\frac{2}{p} - 1 \right)$ is an easy modification of the proof in Theorem 1). We write again $q = \frac{m+p-2}{p}$, $v = u^q$. By Hölder

$$(4.1) \quad \left(\int_\Omega u^m \right) \leq \left(\int_\Omega u^{p^*q} \right)^{\frac{m}{p^*q}} \cdot |\Omega|^{\frac{p^*q-m}{p^*q}}, \quad \text{where } p^* = \frac{Np}{N-p}, \quad |\Omega| = \text{meas}(\Omega).$$

Starting as in Theorem 1 (with $k = 0$) we arrive at

$$(4.2) \quad \frac{d}{dt} \left(\int_\Omega u^m \right) + \frac{m(m-1)}{q^p} \left(\int_\Omega |Dv|^p \right) \leq 0$$

Next use Sobolev ($\|Dv\|_p \geq c \|v\|_{p^*}$) and (4.1) to obtain

$$(4.3) \quad \frac{d}{dt} \left(\int_{\Omega} u^m \right) + \frac{cm(m-1)}{p^q |\Omega| \omega} \left(\int_{\Omega} u^m \right)^{\frac{pq}{m}} \leq 0, \text{ with } \omega = \frac{N-p}{N^2} \cdot \frac{N(p-2)+mp}{(m+p-2)}$$

From (4.3) we conclude that u vanishes at most at t_0 , where

$$t_0 = \frac{(2-p)q^p}{cm^2(m-1)} \cdot |\Omega| \omega \cdot \|u_0\|_m^{2-p}$$

Assume now that Ω is connected ⁽¹⁾. When $p = 2$ the fact that for $u_0 \geq 0, u_0 \neq 0$ and $t > 0, u(t, \cdot) > 0$ follows from the strong maximum principle of L. Nirenberg (see [7]). If $p > 2$ take $\Omega = B_R(0)$ and g a positive eigenfunction corresponding to the first eigenvalue λ of $-\Delta$ in $B_R(0)$ with homogeneous Dirichlet conditions; g is radially symmetric, C^∞ and $Ag \leq Cg$ for some $C > 0$. To check this last assertion, note that

$$\begin{aligned} -g'' - \frac{N-1}{r} g' = g \text{ and hence } Ag = -\lambda(p-1) |g'|^{p-2} g'' - \frac{N-1}{r} |g'|^{p-2} g' &= \\ = \lambda(p-1) |g'|^{p-2} g' + (p-1) \frac{N-1}{r} |g'|^{p-2} g' - \frac{N-1}{r} |g'|^{p-2} g' &\leq \lambda(p-1) |g'|^{p-2} g' \leq Cg. \end{aligned}$$

Now try as a subsolution $\bar{v}(t,x) = T(t) g(x)$, where $T(t) = \frac{T_0}{(1+c(p-2)T_0^{p-2} t)^{1/p-2}}$ solves $T'(t) + CT(t)^{p-1} = 0$. It follows from the maximum principle that if $u_0(x) \geq T_0 g(x)$, the corresponding $u(t,x)$ is greater or equal than $\bar{v}(t,x)$ for each $t > 0$ #

Remark. Observe that as a consequence of the decay of some m -norm, $m > 1$ and Ω being bounded, MCL never holds. When $p \geq 2$ we have shown that for smooth initial data there is a retention property: if $u_0 > 0$ in some $\tilde{\Omega} \subset \Omega, u(t,x) > 0$ in $\bar{\Omega}$ for each $t > 0$.

We conclude by noting that the results of this paper are valid when Au is replaced by other similar nonlinear.

$$\Delta_p u = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(| \frac{\partial u}{\partial x_i} |^{p-2} \frac{\partial u}{\partial x_i} \right)$$

As a natural generalization we may consider operators like

$$Bu = \sum_{i=1}^N \frac{\partial}{\partial x_i} \beta_i \left(\frac{\partial u}{\partial x_i} \right)$$

(1) For general Ω argue on each connected component.

where $\sum_{i=1}^N s_i \beta_i(s_i) \geq c |s|^p$ with $s = (s_1, \dots, s_N)$.

Some of the previous results have immediate counterparts. In particular Theorem 1 remains valid unchanged.

Acknowledgements. The authors wish to thank L. Véron for some useful conversations on Section 3.

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(Manuscrit reçu le 6 juillet 1980)