ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

I. PERAL
J. C. PERAL
M. WALIAS

L^p-estimates for hyperbolic operators applications to $\square u = u^k$

Annales de la faculté des sciences de Toulouse 5e série, tome 4, nº 2 (1982), p. 143-152
<http://www.numdam.org/item?id=AFST_1982_5_4_2_143_0>

Annales Faculté des Sciences Toulouse
Vol IV, 1982, P 143 à 152

L^p-ESTIMATES FOR HYPERBOLIC OPERATORS APPLICATIONS TO $\square u = u^k$

I. Peral (1), **J.C. Peral** (2), **M. Walias** (3)

(1)(2)(3) Facultad de Ciencias, Division de Matematicas C-XVI, Universidad Autonoma de Madrid. Cantoblanco, Madrid-34 - Spain.

Résumé : Nous obtenons dans ce travail des estimations $L^p - L^p$ pour l'équation des ondes non homogène. Le problème est complètement résolu en dimension d'espace $n \leqslant 3$. Dans le cas $n \geqslant 4$ nous avons une réponse positive si $\left|\frac{1}{p}-\frac{1}{2}\right| \leqslant \frac{1}{n-1}$. Finalement, nous utilisons ces estimations pour montrer l'existence et unicité de solutions faibles pour certains problèmes non linéaires.

Summary : $L^p - L^p$ estimates are obtained in this paper for the non-homogeneous wave equation. The problem is completely solved for space dimension $n \leqslant 3$. A positive answer is given for dimension $n \geqslant 4$ and $\left|\frac{1}{p}-\frac{1}{2}\right| \leqslant \frac{1}{n-1}$. These estimates are used in order to prove existence and uniqueness of weak solutions for some non-linear problems.

ABSTRACT

We present some results on non-linear hyperbolic equations obtained by means of some information on the linear Cauchy problem. Those results for the linear case are essentially :

THEOREM. *Let*

$$\begin{cases} \square u \equiv u_{tt} - \Delta_x u = 0 \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

where $f \in L^p(R^n)$, $g \in L_1^p(R^n)$, *then for each* $t \in [0,\infty)$ *we have*

(1) $$\| u(t,x) \|_{L^p(R^n)} \leq C_p(t) \left(\| f \|_{L^p(R^n)} + \| g \|_{L_1^p(R^n)} \right)$$

if

$$\left| \frac{1}{p} - \frac{1}{2} \right| \leq \frac{1}{n-1}$$

The result is sharp.

The homogeneity of Fourier multipliers implies $C_p(t) \approx \max(1, |t|)$.

Proof and extension to other kind of hyperbolic operators can be seen in J. C. Peral [3] and for related estimates of the type (L^p, L^q) Littman [1], [2] and Strichartz [8] [9].

In § 1 we will solve partially a problem posed by Littman in [1].

In § 2 we give existence and uniqueness results for certain non-linear hyperbolic equations $\Box u = F(u)$. Those results are in the context of the L^p spaces.

Finally in § 3 we deal with the case $|F(u)| = u^k$.

Everything will be expressed in terms of either the wave equations of the Klein-Gordon equations.

It is not hard to generalize these results to some others more general hyperbolic equations.

1. - A PRIORI L^p-ESTIMATES FOR THE NON-HOMOGENEOUS EQUATIONS

It has been proved by Littman in [1] that an estimates of the type

$$\int_{R^n} \int_t |v(t,x)|^p \, dtdx \leq C_p(T) \int_{R^n} \int_t \Box |v(t,x)|^p \, dtdx$$

with $v \in C_0^\infty(R^{n+1})$ and say support $(v) \subset (0,T) \times R^n$ is false if $p > \frac{2n}{n-3}$.

For the same problem we get the following positive results.

THEOREM. *Let*

$$\begin{cases} \Box u(t,x) = F(t,x) \\ u(0,x) = 0 \\ u_t(0,x) = 0 \end{cases}$$

where $F \in L^p([0,T] \times R^n)$ *and* $|\frac{1}{p} - \frac{1}{2}| \leqslant \frac{1}{n-1}$. *Then*

$$\| u \|_{L^p([0,T] \times R^n)} \leqslant C_p(T) \, \| F \|_{L^p([0,T] \times R^n)}.$$

*Proof. B*y the Duhamel principle we know that the solution of the problem is expressed as

$$u(t,x) = \int_0^t v(t,\tau,x) d\tau$$

where $v(t,\tau,x)$ solves the following problem

$$\begin{cases} \Box v(t,\tau,x) = 0 \\ v(\tau,\tau,x) = 0 \\ v_t(\tau,\tau,x) = F(\tau,x) \end{cases}$$

The estimates (1) gives for each t

$$\left(\int_{R^n} |v(t,\tau,x)|^p \, dx \right)^{1/p} \leqslant C_p(t-\tau) \left(\int_{R^n} |F(\tau,x)|^p \, dx \right)^{1/p}$$

if $|\frac{1}{p} - \frac{1}{2}| \leqslant \frac{1}{n-1}$. But, $C_p(t-\tau) \leqslant C_p(t)$, $0 \leqslant \tau \leqslant t$.

By applying the Minkowski integral inequality, for each t we get

$$\left(\int_{R^n} |u(t,x)|^p \, dx \right)^{1/p} \leqslant \int_0^t \int_{R^n} |v(t,\tau,x)|^p \, dx \Big)^{1/p} d\tau \leqslant$$

$$\leqslant \int_0^t C_p(t) \left(\int_{R^n} |F(\tau,x)|^p \, dx \right)^{1/p} d\tau \leqslant C_p(t) t^{1/g} \left(\int_0^t \int_{R^n} |F(\tau,x)|^p \, dxd \right)^{1/p}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Then we get

$$\int_0^T \int_{R^n} |u(t,x)|^p \, dxdt \leqslant \left(\int_0^T |C_p(t)|^p \, t^{p/q} \, dt \right) \| F \|_{L^p([0,T] \times R^n)}$$

This implies

$$\| u \|_{L^p([0,T] \times R^n)} \leqslant C'_p(T) \, \| F \|_{L^p([0,T] \times R^n)}$$

where

$$C'_p(T) \cong T^2$$

because $C_p(t) \approx |t|$ is the norm of Fourier multiplier $\dfrac{\sin t\,|\xi|}{|\xi|}$.

Remark. For $n \leqslant 3$ our result gives the range $1 \leqslant p \leqslant \infty$. For $n \geqslant 4$ there is a range where the problem is open, that is $\dfrac{2(n-1)}{n-3} \leqslant p \leqslant \dfrac{2n}{n-3}$ and the conjugates.

We can prove the same kind of estimates for more general equations such a Klein-Gordon equation for example. See J.C. Peral [3] .

2. - NON-LINEAR CAUCHY PROBLEM

Assume

$$F : [0,\infty) \times R^n \times R \to R$$

and i) $F(t,x,0) = 0$

ii) $|F(t,x,u) - F(t,x,v)| \leqslant K|u-v|$

if we consider

$$\begin{cases} \square u = F(t,x,u) \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

where $f \in L^p(R^n)$; $g \in L^p_1(R^n)$ and $\left|\dfrac{1}{p} - \dfrac{1}{2}\right| \leqslant \dfrac{1}{n-1}$ then we can give the following theorem.

THEOREM 1. *Let* F *be satisfying i) and ii) then given any* $T > 0$, *The Cauchy problem*

$$\begin{cases} \square u = F(t,x,u) \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

where $g \in L^p_1(R^n)$, $f \in L^p(R^n)$ *and* $\left|\dfrac{1}{p} - \dfrac{1}{2}\right| \leqslant \dfrac{1}{n-1}$, *has unique weak solution* $u \in L^p(]0,T[\times R^n)$ *and*

$$\|u\|_{L^p((0,T) \times R^n)} \leqslant C_p(T) \left(\|g\|_{p,1} + \|f\|_p \right)$$

Proof. With fixed $T > 0$ consider u_1 solution of the problem

$$\begin{cases} \Box u = 0 \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

and, in general for $k \in \mathbb{N}$ u_k solution of the problem

$$\begin{cases} \Box u = F(t,x,u_{k-1}) \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

As a consequence of the result in § 1, for each $k \in N$ we have

$$u_k \in L^\infty((0,T), L^p(R^n)) \text{ and } u_k \in L^p((0,T) \times R^n).$$

we will prove that $\{u_k\}_{k \in N}$ is a Cauchy sequence in the first space.

In fact $u_k - u_{k-1}$ verifies :

$$\begin{cases} \Box (u_k - u_{k-1}) = F(t,x,u_{k-1}) - F(t,x,u_{k-2}) \\ (u_k - u_{k-1})(0,x) = 0 \\ (u_k - u_{k-1})_t (0,x) = 0 \end{cases}$$

then if

$$d_k(t) = \left(\int_{R^n} | u_k(t,x) - u_{k-1}(t,x) |^p \, dx \right)^{1/p}$$

we get

$$d_k(t) \leqslant C_p(t) K \int_0^t d_{k-1}(s) ds \leqslant C_p(T) k \int_0^t d_{k-1}(s) ds = M(T) \int_0^t d_{k-1}(s)$$

Therefore

$$d_k(t) \leqslant \frac{TM(T)^k}{k!} \sup_{s \in [0,T]} d_o(s)$$

and, if $k < \ell$, for each $t \in (0,T)$ we have

$$\| u_k(t.) - u_\ell(t.) \|_{L^p(R^n)} \leqslant \sum_{j=k}^{\ell} d_j(t) \leqslant \left(\sum_{k=\ell} \frac{(M(T)T)^k}{k!} \right) \sup_{s \in (0,T)} d_o(s)$$

Then $u_k \to u \in L^\infty((0,T), L^p(R^n))$ and as A is a continuous operator, u is a solution of our problem.

By using the theorem locally is not hard to prove that u is the unique solution. In others words, we consider the operator

$$L^p((0,T) \times R^n) \to L^p((0,T) \times R^n)$$

such that $Av = u$, is solution of the linear problem

$$\begin{cases} \Box u = F(t,x,v) \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

Then for small T, A is contractive. Also it is clear that the convergence is in the space

$$L^p((0,T) \times R^n)$$

and this finishes the proof.

An important example included in the preceding theorem is the sine-Klein-Gordon equations

$$\Box u + m^2 u = \sin u$$

and for the case $n = 3$, there exists a unique solution for every p, $1 \leqslant p \leqslant \infty$.

The following Gronwall lemma, shows the behaviour of the constant C(T).

LEMMA. *Let* $\alpha(t)$ *and* $\beta(t)$ *be positive and continuous functions on some interval* $[0,T]$. *If*

$$y(t) \leqslant \alpha(t) + \beta(t) \int_0^t y(\tau)\, d\tau$$

then

$$y(t) \leqslant \alpha(t) + \beta(t) \int_0^t \alpha(s) \exp\left(\int_s^t \beta(u) du \right) ds$$

Then if we consider

$$y(t) = \left(\int_{R^n} |u(t,x)|^p\, dx \right)^{1/p}$$

where u is the solution for the non-linear Cauchy problem and

$$\alpha(t) = C_p(t) \left(\|g\|_{p,1} + \|f\|_p \right)$$

$$\beta(t) = KC_p(t)$$

we get for each t

$$\| u(t,.)\|_p \leqslant C_p(t) \exp \int_0^t KC_p(s) \, (\| f\|_p + \| g\|_{p,1})$$

where $C_p(t) \approx \max(1,t)$ as in the linear case. Then

$$\| u\|_{L^p((0,T) \times R^n)} \leqslant (\| f\|_p + \| g\|_{p,1})\left(\int_0^T |C_p(t)|^p \exp \left\{pk \int_0^t C_p(s)ds\right\} dt\right)^{1/p}$$

Observe that in the case $|F(t,x,u)| \leqslant H$, $C_p(t)$ behaves like t^{α} for some $\alpha(p)$.
This is the case of the sine-Klein-Gordon equations.

3. - A RESULT OF EXISTENCE AND UNIQUENESS IN L^{∞} FOR $\Box u + m^2 u = u^k$

For the problem

$$\begin{cases} \Box u + m^2 u = \pm\, u^k \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \quad x \in R^3 \end{cases}$$

existence results are know in the case of finite energy ; see references in Segal [5] and Strauss [6].

Uniqueness is known in the case $k < 5$. See also [7]. We get results of existence and uniqueness based in the estimates of § 1 using the fact that for $n = 3$ such estimates are valid for $1 \leqslant p \leqslant \infty$.

Consider

$$\begin{cases} \Box u + m^2 u = \pm\, u^k \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

where $g \in L_1^{\infty}(R^3)$ and $f \in L^{\infty}(R^3)$ and in addition assume

$$\| f\|_{\infty} + \| g\|_{\infty,1} \leqslant \frac{1}{4}$$

Define u_o as the solution of the linear problem

$$\begin{cases} \Box u + m^2 u = 0 \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

Then for each t we have

$$\| u_o(t,.) \|_{L^\infty(R^3)} \leqslant t \, \| f \|_\infty + \max(t,1) \, \| g \|_{\infty,1}.$$

Besides if $t \in [0,1]$ we get

$$\| u_o(t,.) \|_{L^\infty(R^3)} \leqslant 1/4$$

On the other hand let u_n be the solution of the linear problem

$$\begin{cases} \Box u + m^2 u = \pm u_{n-1}^k \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

The result of § 1 implies

$$\| u_n(t,.) \|_{L^\infty(R^3)} \leqslant \frac{1}{4} + \left(\frac{1}{2}\right)^{k-1} \int_0^t \| u_{n-1}(\tau,.) \|_{L^\infty(R^3)} d\tau$$

if $t \in [0,1]$ and by recursion we deduce :

$$\| u_n(t,.) \|_{L^\infty(R^3)} \leqslant \frac{1}{2} \quad \text{if} \quad t \in [0,1]$$

Let's define

$$d_n(t) = \| u_n(t,.) - u_{n-1}(t,.) \|_{L^\infty(R^3)}, \ t \in [0,1]$$

then we obtain

$$d_n(t) \leqslant C_k \int_0^t d_{n-1}(\tau) d\tau$$

where C_k is the bound of the following expression

$$P_{k-1}(u,v) = u^{k-1} + u^{k-2} v + \ldots + v^{k-1}$$

for

$$u = \| u_{n-1} \|_{L^\infty([0,1] \times R^3)}$$

and

$$v = \| u_{n-2} \|_{L^\infty([0,1] \times R^3)}$$

Therefore we have $C_k \leqslant k(\frac{1}{2})^k < 1$ if $k \geqslant 2$.

Obviously

$$d_n(t) \leqslant \frac{(C_k t)^n}{n} \; \frac{1}{2},$$

and then $\{u_n\}_{n \in N}$ is a Cauchy sequence in

$$L^\infty([0,1], L^\infty(R^3)) \text{ and in } L^\infty([0,1] \times R^3).$$

Since $C_k \leqslant 1$ the following existence and uniqueness theorem has been proved

THEOREM 2. *Let* $f \in L^\infty(R^3)$ *and* $g \in L_1^\infty(R^3)$ *such that* $\| f \|_\infty + \| g \|_{\infty,1} \leqslant \frac{1}{4}$ *then*

$$\begin{cases} \Box u + m^2 u = \pm u^k \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

has unique solution in $B_{1/2}(0) = \left\{ u \in L^\infty([0,1] \times R^3) \; ; \; \| u \|_\infty \leqslant \frac{1}{2} \right\}.$

A classical theorem of uniqueness in the following (see Strauss [6]).

THEOREM 3. *Consider*

$$\begin{cases} \Box u + m^2 u = \pm u^k \\ u(0,x) = g(x) \\ u_t(0,x) = f(x) \end{cases}$$

where $g \in L_1^\infty(R^3)$ *and* $f \in L^\infty(R^3)$. *If the problem has a solution* $u \in L^\infty((0,T) \times R^3)$ *then it is the unique solution in such space.*

Several observations should be made at this point. The theorem 3 gives a result on uniqueness with independence of how big or small are the L^∞-norms of the data. However for the theorem 2 we need the data to be small.

From both theorems we get the following corollary :

COROLLARY. *Under the hipothesis of the Theorem 2, then there is a unique solution in* $L^\infty([0,1] \times R^3)$.

Finally if $\alpha = \| f \|_\infty + \| g \|_{\infty,1} > \frac{1}{4}$ results on existence and uniqueness can be given locally in t.

REFERENCES

[1] W. LITTMAN. *«The wave operator and L^p norms».* J. Math. Mech. 12 (1963) 55-68.

[2] W. LITTMAN. *«Multipliers in L^p and interpolation».* Bull. A.M.S. 71 (1965) 764-766.

[3] J.C. PERAL. *«L^p-estimates for wave equations».* Journal of Functional Analysis, Vol 36, n° 1 (1980), 114-145.

[4] I. SEGAL. *«Non-linear semi-groups».* Annals in Mathematics 78 n° 2 (1963) 339-364.

[5] I. SEGAL. *«The global Cauchy Problem for a realtivistic scalar field with power interaction».* Bull. Soc. Math. France 91 (1963) 129-135.

[6] W. STRAUSS. *«Non-linear invariant wave equations».* Lecture notes in Phisics n° 73, Springer-Verlag, Berlin (1978).

[7] W. STRAUSS, L. VAZQUEZ. *«Numerical solution of a Non-linear Klein-Gordon Equations».* J. Comp. Phys. 28 n°2 (1978) 271-278.

[8] R.S. STRICHARTZ. *«Convolutions with Kernels having singularities on a sphere».* Trans. Amer. Math. Soc. 148 (1970) 461-471.

[9] R.S. STRICHARTZ. *«A priori Estimates for the Wave Equations and some applications».* Jour. Functional Analysis 5 (1970) 218-235.

(Manuscrit reçu le 18 janvier 1981)