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A NOTE ON ELLIPTIC FUNCTIONS AND APPROXIMATION BY ALGEBRAIC NUMBERS OF BOUNDED DEGREE

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Résumé : Soit p une fonction elliptique de Weierstrass d'invariants g_2 et g_3 algébriques. Par un contre-exemple, on montre que pour l'obtention d'une minoration pour l'approximation simultanée de $p(a)$, b et $p(ab)$ par des nombres algébriques de degré borné, une hypothèse supplémentaire sur les nombres β qui approximent b est nécessaire.

Summary : Let p be a Weierstrass elliptic function with algebraic invariants g_2 and g_3. By a counterexample it is shown that lower bounds for the simultaneous approximation of $p(a)$, b and $p(ab)$ by algebraic numbers of bounded degree cannot be given without an added hypothesis on the numbers β approximating b.

Let p be a Weierstrass elliptic function with algebraic invariants g_2, g_3 ; for $a,b \in \mathbb{C}$ such that a and ab are not poles of p, we consider the simultaneous approximation of $p(a)$, b and $p(ab)$ by algebraic numbers. It was shown in [2] , Theorem 2, that lower bounds for the approximation errors in terms of the heights and degrees of these algebraic numbers can only be given if the numbers β used to approximate b do not lie in the field $\mathbb{K}$ of complex multiplication of p. (As this condition is equivalent to the algebraic independence of $p(z)$ and $p(\beta z)$ as functions of z, the result proves the conjecture on admissible sets in Appendix 2 of [3]).

Now consider simultaneous approximation of the same numbers by algebraic numbers of bounded degree. The sequences of algebraic numbers constructed in [2] have rapidly rising degrees, so they do not provide a relevant counterexample. It is the purpose of this note to show how the original example should be modified for the new problem.

Let $\Omega = \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$ denote the period lattice of $\wp$, and $\mathbb{F}$ the field $\mathbb{Q}(g_2,g_3)$. For every $d \in \mathbb{N}$, the set of $z \in \mathbb{C} \setminus \Omega$ such that $\wp(z)$ is algebraic of degree at most d is denoted by A_d. Let B be an open set in $\mathbb{C}$ such that its closure $\overline{B}$ is contained in the interior of the fundamental parallelogram $[0,1]\omega_1 + [0,1]\omega_2$.

LEMMA 1. *For every* $d \geqslant 2$, *the set* A_d *is dense in* $\mathbb{C}$.

Proof. Let $\mathcal{O} \subset \mathbb{C}$ be an arbitrary open set. Take $a \in \mathcal{O} \setminus \Omega$ with $\wp'(a) \neq 0$. According to [1], Chapter 4, Theorem 11, Corollary 2, there exist open sets U, V with $a \in U \subset \mathcal{O}$, $\wp(a) \in V$, such that $\wp$ induces a bijection from U onto V. As $\{z \in \overline{\mathbb{Q}} \mid \mathrm{dg}\, z \leqslant d\}$ is dense in $\mathbb{C}$, we can find $z \in V \cap \overline{\mathbb{Q}}$ with $\mathrm{dg}\, z \leqslant d$. For the unique $u \in U$ with $\wp(u) = z$, we have $u \in \mathcal{O} \cap A_d$. ∎

LEMMA 2. *Assume* $d \geqslant 2[\mathbb{F} : \mathbb{Q}]$. *Then, for every* $g : \mathbb{N} \to \mathbb{R}$, *there exist sequences* $(u_n)_{n=1}^{\infty}$, $(\beta_n)_{n=1}^{\infty}$, $(v_n)_{n=1}^{\infty}$, $(\epsilon_n)_{n=1}^{\infty}$, *such that for all* $n \in \mathbb{N}$ *the following statements are true :*

(1) $$u_n \in A_d \cap B,\ \beta_n \in [0,1] \cap \mathbb{Q},\ v_n \in A_d,\ v_n = \beta_n u_n,\ \epsilon_n \in \,]0,1[\,;$$

(2) $$\epsilon_{n+1} < \exp(-n\,|g(H_n)|), \textit{ where } H_n := \max(H(\wp(u_n)), H(\beta_n), H(\wp(v_n)))\,;$$

(3) $$\epsilon_{n+1} < \epsilon_n^2,\ \epsilon_{n+1} < \frac{1}{4}\,\mathrm{den}^{-4}\,\beta_n\,;$$

(4) $$0 < |\beta_n - \beta_{n+1}| < \epsilon_{n+1},\ |u_n - u_{n+1}| < \epsilon_{n+1}\,.$$

Proof. Take $u_1 \in A_d \cap B$ (the existence of such an u_1 follows from Lemma 1). Define $v_1 := u_1$, $\beta_1 := 1$, $\epsilon_1 := \frac{1}{2}$. Then (1) is true for $n = 1$. Now suppose $u_1,\ldots,u_N$, $\beta_1,\ldots,\beta_N$, $v_1,\ldots,v_N$, $\epsilon_1,\ldots,\epsilon_N$ have been chosen in such a way that (1) holds for $n = 1,\ldots,N$ and (2), (3), (4) hold for $n = 1,\ldots N-1$, and proceed by induction.

Choose $\epsilon_{n+1} \in \,]0,1[$ so small that (2) and (3) hold for $n = N$. Take $r > \epsilon_{N+1}^{-1}$ and consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) := rz$. As f is a continuous bijection, there exists an open set $U \subset \mathbb{C}$ with $fU \subset B \cap B(u_n, \epsilon_{N+1})$. Take $w \in U$ such that $\wp(w) \in \overline{\mathbb{Q}}$ with $\mathrm{dg}\,\wp(w) \leqslant 2$ (the existence of w again follows from Lemma 1). Define $u_{N+1} := rw$. By Lemma 6.1 of [4], $\wp(u_{N+1}) \in \overline{\mathbb{Q}}$ and

$$\mathrm{dg}\,\wp(u_{N+1}) \leqslant [\mathbb{F}(\wp(w)) : \mathbb{Q}] \leqslant 2\,[\mathbb{F} : \mathbb{Q}] \leqslant d,$$

so $u_{N+1} \in A_d$. Furthermore the definition of U gives $u_{N+1} \in B$ and $|u_N - u_{N+1}| < \epsilon_{N+1}$. Take

$s \in \mathbb{N}$ with $0 \leqslant s \leqslant r$ and $0 < |\beta_N - \frac{s}{r}| < \epsilon_{N+1}$; define $\beta_{N+1} := \frac{s}{r}$; then $\beta_{N+1} \in [0,1] \cap \mathbb{Q}$ and (4) holds for $n = N$. Define $v_{N+1} := \beta_{N+1} u_{N+1} = sw$; then as above we find that $v_{N+1} \in A_d$ and (1) holds for $n = N + 1$. ■

THEOREM. *Assume* $d \geqslant 2$ $[\mathbb{F} : \mathbb{Q}]$. *Then, for every* $g : \mathbb{N} \to \mathbb{R}$, *there exist* $a \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{K}$, *such that* a *and* ab *are not poles of* p *and such that for every* $C \in \mathbb{R}$ *there exist infinitely many tuples* $(u,\beta,v) \in \mathbb{C}^3$ *satisfying* $u,v \in A_d$, $\beta \in \mathbb{Q}$ *and*

$$\max(|p(a) - p(u)|, |b - \beta|, |p(ab) - p(v)|) < \exp(-Cg(H))$$

while $\max(H(p(u)), H(\beta), H(p(v))) \leqslant H$.

Proof. According to Lemma 3 of [2], the sequences $(u_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$ constructed in Lemma 2 above are Cauchy sequences and their limits a, b satisfy

$$\max(|a - u_n|, |b - \beta_n|) \leqslant \epsilon_{n+1}^{1/2} \tag{5}$$

for almost all n. Thus $a \in \overline{B}$ and therefore a cannot be a pole of *p*. Formula (4) implies the existence of arbitrarily large n for which $\beta_n \neq b$; as by (3) and (5), every β_n is a convergent of the continued fraction expansion of b and $\lim \beta_n = b$, it follows that b has infinitely many convergents. Thus $b \in \mathbb{R} \setminus \mathbb{Q}$ and therefore $b \notin \mathbb{K}$. On particular, $b \neq 0$; hence ab cannot be a pole of *p* either.

By the continuity of *p* in ab, (5) implies

$$\max(|p(a) - p(u_n)|, |b - \beta_n|, |p(ab) - p(v_n)|) \leqslant c\epsilon_{n+1}^{1/2} \tag{6}$$

for almost all n, where c does not depend on n. In the notation of (2), the right hand member of (6) satisfies

$$c\epsilon_{n+1}^{1/2} < c\exp(-\frac{1}{2} n\,|g(H_n)|) < \exp(-Cg(H_n))$$

if n is sufficiently large in terms of C and c. ■

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