ALEX BIJLSMA

A note on elliptic functions and approximation by algebraic numbers of bounded degree

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A NOTE ON ELLIPTIC FUNCTIONS AND APPROXIMATION
BY ALGEBRAIC NUMBERS OF BOUNDED DEGREE

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Résumé : Soit \( p \) une fonction elliptique de Weierstrass d’invariants \( g_2 \) et \( g_3 \) algébriques. Par un contre-exemple, on montre que pour l’obtention d’une minoration pour l’approximation simultanée de \( p(a) \), \( b \) et \( p(ab) \) par des nombres algébriques de degré borné, une hypothèse supplémentaire sur les nombres \( \beta \) qui approximent \( b \) est nécessaire.

Summary : Let \( p \) be a Weierstrass elliptic function with algebraic invariants \( g_2 \) and \( g_3 \). By a counterexample it is shown that lower bounds for the simultaneous approximation of \( p(a) \), \( b \) and \( p(ab) \) by algebraic numbers of bounded degree cannot be given without an added hypothesis on the numbers \( \beta \) approximating \( b \).

Let \( p \) be a Weierstrass elliptic function with algebraic invariants \( g_2 \), \( g_3 \); for \( a, b \in \mathbb{C} \) such that \( a \) and \( ab \) are not poles of \( p \), we consider the simultaneous approximation of \( p(a) \), \( b \) and \( p(ab) \) by algebraic numbers. It was shown in [2], Theorem 2, that lower bounds for the approximation errors in terms of the heights and degrees of these algebraic numbers can only be given if the numbers \( \beta \) used to approximate \( b \) do not lie in the field \( IK \) of complex multiplication of \( p \). (As this condition is equivalent to the algebraic independence of \( p(z) \) and \( p(\beta z) \) as functions of \( z \), the result proves the conjecture on admissible sets in Appendix 2 of [3]).

Now consider simultaneous approximation of the same numbers by algebraic numbers of bounded degree. The sequences of algebraic numbers constructed in [2] have rapidly rising
degrees, so they do not provide a relevant counterexample. It is the purpose of this note to show how the original example should be modified for the new problem.

Let $\Omega = \mathbb{Z} \omega_1 + \mathbb{Z} \omega_2$ denote the period lattice of $\varphi$, and $\mathbb{Q}$ the field $\mathbb{Q}(\bar{g}_2, \bar{g}_3)$. For every $d \in \mathbb{N}$, the set of $z \in \mathbb{C} \setminus \Omega$ such that $\varphi(z)$ is algebraic of degree at most $d$ is denoted by $A_d$.

Let $B$ be an open set in $\mathbb{C}$ such that its closure $\overline{B}$ is contained in the interior of the fundamental parallelogram $[0,1] \omega_1 + [0,1] \omega_2$.

**LEMMA 1.** For every $d > 2$, the set $A_d$ is dense in $\mathbb{C}$.

**Proof.** Let $\mathcal{O} \subset \mathbb{C}$ be an arbitrary open set. Take $a \in \mathcal{O} \setminus \Omega$ with $\varphi'(a) \neq 0$. According to [1], Chapter 4, Theorem 11, Corollary 2, there exist open sets $U, V$ with $a \in U \subset \mathcal{O}, \varphi(a) \in V$, such that $\varphi$ induces a bijection from $U$ onto $V$. As $\{ z \in \overline{\mathcal{O}} \mid d g z \leq d \}$ is dense in $\mathbb{C}$, we can find $z \in V \cap \overline{\mathcal{O}}$ with $d g z < d$. For the unique $u \in U$ with $\varphi(u) = z$, we have $u \in \mathcal{O} \cap A_d$. \hfill \blacksquare

**LEMMA 2.** Assume $d > 2$. Then, for every $g : \mathbb{N} \rightarrow \mathbb{R}$, there exist sequences $(u_n)_{n=1}^{\infty}, (\beta_n)_{n=1}^{\infty}, (v_n)_{n=1}^{\infty}, (\epsilon_n)_{n=1}^{\infty}$, such that for all $n \in \mathbb{N}$ the following statements are true:

1. $u_n \in A_d \cap B, \beta_n \in [0,1] \cap \mathbb{Q}, v_n \in A_d, v_n = \beta_n u_n, \epsilon_n \in \{ 0, 1 \}$;
2. $e_{n+1} < \exp(-n |g(H_n)|), \text{ where } H_n := \max(\varphi(u_n), H(\beta_n), H(\varphi(v_n)))$;
3. $e_{n+1} < e_n^2, \epsilon_{n+1} \leq \frac{1}{4} \text{ den}^{-4} \beta_n$;
4. $0 < |\beta_n - \epsilon_{n+1}| < \epsilon_{n+1}, |u_n - u_{n+1}| < \epsilon_{n+1}$.

**Proof.** Take $u_1 \in A_d \cap B$ (the existence of such an $u_1$ follows from Lemma 1). Define $v_1 := u_1$, $\beta_1 := 1, \epsilon_1 := \frac{1}{2}$. Then (1) is true for $n = 1$. Now suppose $u_1, ..., u_N, \beta_1, ..., \beta_N, v_1, ..., v_N, \epsilon_1, ..., \epsilon_N$ have been chosen in such a way that (1) holds for $n = 1, ..., N$ and (2), (3), (4) hold for $n = 1, ..., N - 1$, and proceed by induction.

Choose $\epsilon_{n+1} \in \{ 0, 1 \}$ so small that (2) and (3) hold for $n = N$. Take $r > e_{N+1}^{-1}$ and consider the function $f : \mathbb{C} \rightarrow \mathbb{C}$ defined by $f(z) := rz$. As $f$ is a continuous bijection, there exists an open set $U \subset \mathbb{C}$ with $f(U) \subset B \cap B(0_N, \epsilon_{N+1})$. Take $w \in U$ such that $\varphi(w) \in \overline{\mathcal{Q}}$ with $d g \varphi(w) < 2$ (the existence of $w$ again follows from Lemma 1). Define $u_{N+1} := rw$. By Lemma 6.1 of [4], $\varphi(u_{N+1}) \in \overline{\mathcal{Q}}$ and

$$d g \varphi(u_{N+1}) \leq |f(\varphi(w) \cap \mathcal{Q})| \leq 2 |f(\mathcal{Q})| \leq d,$$

so $u_{N+1} \in A_d$. Furthermore the definition of $U$ gives $u_{N+1} \in B$ and $|u_N - u_{N+1}| < \epsilon_{N+1}$. Take
THEOREM. Assume $d \geq 2$ [IF : $\mathbb{Q}$]. Then, for every $g : \mathbb{N} \to \mathbb{R}$, there exist $a \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Q}$, such that $a$ and $ab$ are not poles of $p$ and such that for every $C \in \mathbb{R}$ there exist infinitely many tuples $(u, a, v) \in \mathbb{C}^3$ satisfying $u, v \in A_d$, $\beta \in \mathbb{Q}$ and

$$\max(|p(a) - p(u)|, |b - \beta|, |p(ab) - p(v)|) \leq \exp(-Cg(H))$$

while $\max(H(p(u)), H(\beta), H(p(v))) \leq H.$

Proof. According to Lemma 3 of [2], the sequences $(u_n)_{n=1}^{\infty}$ and $(\beta_n)_{n=1}^{\infty}$ constructed in Lemma 2 above are Cauchy sequences and their limits $a, b$ satisfy

(5) $$\max(|a - u_n|, |b - \beta_n|) \leq e^{1/2}_{n+1}$$

for almost all $n$. Thus $a \in \overline{B}$ and therefore $a$ cannot be a pole of $p$. Formula (4) implies the existence of arbitrarily large $n$ for which $\beta_n \neq b$; as by (3) and (5), every $\beta_n$ is a convergent of the continued fraction expansion of $b$ and $\lim \beta_n = b$, it follows that $b$ has infinitely many convergents. Thus $b \in \mathbb{R} \setminus \mathbb{Q}$ and therefore $b \notin \mathbb{Q}$. On particular, $b \neq 0$; hence $ab$ cannot be a pole of $p$ either.

By the continuity of $p$ in $ab$, (5) implies

(6) $$\max(|p(a) - p(u_n)|, |b - \beta_n|, |p(ab) - p(v_n)|) \leq ce^{1/2}_{n+1}$$

for almost all $n$, where $c$ does not depend on $n$. In the notation of (2), the right hand member of (6) satisfies

$$ce^{1/2}_{n+1} \leq c \exp\left(-\frac{1}{2} n |g(H_n)|\right) \leq \exp(-Cg(H_n))$$

if $n$ is sufficiently large in terms of $C$ and $c$. 

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REFERENCES


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